國立交通大學

土木工程研究所

博士論文

傾斜橫向等向性材料承受三向度點荷重的三維位移與應力基本解

Three-Dimensional Fundamental Solutions of Displacements and Stresses in an Inclined Transversely Isotropic Materials Subjected to three-Dimensional Point Loads

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摘　要

本論文主要探討橫向等向性材料在橫向等向面與水平面呈傾斜狀況下，承受三向度點荷重在無限或半無限空間中，三維位移與應力的基本解。一般而言，在無限空間或半無限空間中，運動或力平衡方程式為偏微分方程式（Partial Differential Equations）。傅立業轉換(Fourier Transform)與拉普拉司轉換(Laplace Transform)，是常用來解決無限空間與半無限空間邊界值問題(Boundary Value Problems)的有效方法。先針對變數x與y部分進行二維傅立業轉換(Double Fourier Transform)，將偏微分方程式轉換成常微分方程式(Ordinary Differential Equations)。本論文提出三種方法來求解前述常微分方程式，以獲得無限空間與半無限空間的應力及位移解析解。第一種方法，利用待定係數法及分離變數法直接求解受點荷重後的非齊性(Nonhomogeneous)常微分方程式，在無限或半無限空間中之齊性解(Homogeneous Solution)及特解(Particular Solution)。第二種方法是將無限空間區分為三個區域 $-\infty < z < 0^-$ (區域2-上半平面)、$0^- < z < 0^+$ (虛擬空間) 及 $0^+ < z < +\infty$ (區域1-下半平面)，或是半無限空間區分為兩個區域 $0 < z < 0^+$ (虛擬空間) 及 $0^+ < z < +\infty$ (區域1-下半平面)，而作用之點荷重在無限空間是作用在 $0^- < z < 0^+$，半無限空間是作用在 $0 < z < 0^+$。在區域1及區域2內，力平衡方程式的右邊並無力量作用，故可視為齊性方程式。接著分別考量無限空間中區域1、區域2及虛擬空間或半無限空間中區域1及虛擬空間的組合邊界值條件。第三種方法，

在無限空間中針對變數z進行傅立業轉換，能將前面所求出之常微分方程式轉換成多項式方程式。這種方法同時針對變數x, y和z進行傅立業轉換，故也可以稱之為三維傅立業轉方法（Triple Fourier Transforms）。換句話說，在無限空間中針對x, y和z進行三維傅立業轉換，可以將偏微分方程式轉換成多項式方程式。在半無限空間中，利用前面二維傅立業轉換得到的常微分方程式，再針對變數z進行拉普拉司轉換，同樣可得到多項式方程式。因此可求得在三維傅立業轉換域的無限空間位移解($\overline{U_i}(\alpha,\beta,\gamma)$)及在二維傅立業及拉普拉司轉換域半無限空間位移解($\overline{U_i}(\alpha,\beta,s)$)。接著將前面所求得不同轉換域的解進行逆轉換分別為三維傅立業逆轉換（Inverse Triple Fourier Transforms）或二維傅立業及拉普拉司逆轉換（Inverse Double Fourier and Laplace Transforms）。利用這種轉換方式可明確的將作用在傾斜的橫向等向性材料三維點荷重的應力及位移解析解求出。本解析解的主要影響參數包括（1）橫向等向面的旋轉角度（2）各個材料參數的異向度（3）幾何位置參數（4）三維的點荷重的形式。

最後本研究比較王承德與廖志中（1991）的解析解，並針對影響參數對位移與應力影響加以探討，發現，在無限空間中，當材料是均質、線彈性及橫向等向面平行水平方向時，所求得的解有一致的結果。在半無限空間中，利用本方法所求出承受點荷重的解與王承德與廖志中的結果有明顯差異。

Three-Dimensional Fundamental Solutions of Displacements and Stresses in an Inclined Transversely Isotropic Materials Subjected to Point Loads

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ABSTRACT

Three-dimensional fundamental solutions of displacements and stresses due to three-dimensional point loads in a transversely isotropic material, where the planes of transverse isotropy are inclined with respect to the horizontal loading surface, are presented in this thesis. Generally, the governing equations for infinite or semi-infinite solids are partial differential equations. The Fourier and Laplace integral transforms are commonly two efficient methods for solving the corresponding boundary value problems of full or half space. Employing the Fourier transform, the partial differential equations can be simplified as ordinary differential equations (ODE). Then, three distinct approaches were used to solve the ODE and the solutions were presented for both infinite and semi-infinite solids in this thesis. Firstly, we solve traditionally the nonhomogeneous ordinary differential equations by the methods of undetermined coefficients and separate variables Secondly, the method of an imaginary space was proposed for deriving the solutions of the problems. Thirdly, the method of algebraic is adopted for deriving the solutions for both full space and half space problems.

Finally, the present fundamental solutions are derived by performing the required triple inverse Fourier transforms, or double inverse Fourier and Laplace transforms. These transformations are powerful to generate the displacements and stresses resulting from the three-dimensional point loads, acting in an inclined transversely isotropic material.

The yielded solutions demonstrate that the displacements and stresses are profoundly influenced by: (1) the rotation of the transversely isotropic planes (ϕ), (2) the type and degree of material anisotropy (E/E', ν/ν', G/G'), (3) the geometric position (r, φ, ξ), and (4). the types of three-dimensional loading (P_x, P_y, P_z). The proposed solutions are exactly the same as those of Wang and Liao (1999) if the full-space is homogeneous, linearly elastic, and the planes of transversely isotropy are parallel to the horizontal loading surface. Additionally, a parametric study is conducted to elucidate the influence of the above-mentioned factors on the displacements and stresses. Computed results reveal that the induced displacements and stresses in the planes of transversely isotropic are parallel to the horizontal loading surface of isotropic/transversely isotropic rocks by a vertical point load are quite different from those from Wang and Liao (1999). Therefore, in the fields of practical engineering, the dip at an angle of inclination should be taken into account in estimating the displacements and stresses in a transversely isotropic rock subjected to applied loads.

Keywords: Displacements; Stresses; Inclined transversely isotropic, full-space; half-space, Triple Fourier transforms; Double Fourier transforms; Laplace transform; Rock anisotropy.

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從大學、碩士班到博士班，合計在交通大學待了 15 個年頭。大學的生涯，成績在 1/2 至 2/3 之間徘徊，每學期在去與留之間掙扎，跌跌撞撞之間，給了我教訓也豐富了我的人生。回想起，真的感激每一位曾經給我機會與孜孜不倦教誨我的師長。從不知是否能順利完成大學學業，到考上研究所，如今完成博士論文，幸蒙我的指導教授　廖志中博士。這十餘年來，指導教授待我，真如師亦如父，亦兄亦為友。在學業與研究工作嚴謹的指導，如師。在待人處世上孜孜不倦地教誨，如父。不計毀譽的給予支持，如兄。適時給予關懷與建議，如友。論文的另一位指導教授　王承德博士，從研究所時一齊年少輕狂，到成為我終身學習的標竿，除了指導我論文的研究，更讓我真實體驗其待人處事的精神，讓我終身受用不盡。協助資質愚鈍的我得以完成本論文，實屬事小。能讓我這些年，貼近在兩位指導教授身邊，接受春風化雨的指導與教化，才是我這輩子最大的福報與資產。最後對於兩位教授，除了感激還是感激。若是沒有兩位指導教授的啟蒙也不會有現在的我。在此致上我最崇高的謝忱;

"兩位老師, 辛苦了, 這份恩情　廷秉永矢不忘"

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"願大家心想事成，百尺竿頭，更進一步"

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"需要感謝的人太多了，就感謝天罷"

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ENGLISH SYMBOLS

$a_i (i = 1 \sim 6)$	elasticity constants defined in Eq.(3.04)
$a_{ij} (i, j = 1 \sim 6)$	elasticity constants defined in Appendix A
$\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$	functions of elasticity constants defined in Eqs.(3.48a)-(3.48c)
$\mathbf{A^i_x}, \mathbf{A^i_y}, \mathbf{A^i_z}$	undetermined coefficients defined in Eqs.(3.49a)-(3.49c)
$\mathbf{A^i_{x1}}, \mathbf{A^i_{y1}}, \mathbf{A^i_{z1}}, \mathbf{A^i_{x2}}, \mathbf{A^i_{y2}}, \mathbf{A^i_{z2}} (i = 1-6)$	undetermined coefficients defined in Eqs.(3.52a)-(3.52b)
$\mathrm{C_{ij}} (\mathrm{i, j} = 1 \sim 6)$	elastic moduli or elasticity constants defined in Eq.(3.04)
$D_{ij} (i, j = 1 \sim 3)$	the cofactors of the third-order determinant $\det[d_{ij}]$
$\mathrm{d_{ij}} (i, j = 1 \sim 3)$	coefficients defined in Eqs.(3.46a)-(3.46f)
$\mathrm{E}, \mathrm{E}', \upsilon, \upsilon', \mathrm{G}'$	elastic constants of a transversely isotropic medium
i	complex number ($= \sqrt{-1}$)
l_{ij}	direction cosines defined in Eq.(3.05)
$\mathrm{P_x}, \mathrm{P_y}, \mathrm{P_z}$	point loads in a Cartesian co-ordinate system
r	radius of a circle
r, θ, z	a cylindrical co-ordinate system
F_x, F_y, F_z	body force components in a Cartesian co-ordinate system
s	integer used in Laplace transforms

$u_1, u_2, \mathrm{u}_3, u_4, u_5, \mathrm{u}_6$	roots of the characteristic equation
$u_i(x, y, z)$	displacement components
$\bar{u}_i(\alpha, \beta, z)$	Double Fourier transforms of $u_i(x, y, z)$
$\overline{U}_i(\alpha, \beta, \gamma)$	Fourier transforms of $\overline{U}_i(\alpha, \beta, z)$
$\overline{U}_i(\alpha, \beta, s)$	Laplace transforms of $\overline{U}_i(\alpha, \beta, z)$
x, y, z	a Cartesian co-ordinate system

GREEK SYMBOLS

α, β, γ	integers used in Fourier transform
δ( )	Dirac delta function
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{\mathrm{zz}}$	normal strain components in a Cartesian co-ordinate system (x, y, z)
$\varepsilon_{x'x'}, \varepsilon_{y'y'}, \varepsilon_{z'z'}$	normal strain components in a Cartesian co-ordinate system (x', y', z')
$\gamma_{xz}, \gamma_{y\mathrm{z}}, \gamma_{\mathrm{xy}}$	shear strain components in a Cartesian co-ordinate system (x, y, z)
$\gamma_{x'\mathrm{z}'}, \gamma_{y'\mathrm{z}'}, \gamma_{\mathrm{x'y'}}$	shear strain components in a Cartesian co-ordinate system (x', y', z')
$\sigma_{xx}, \sigma_{yy}, \sigma_{\mathrm{zz}}$	normal stress components in a Cartesian co-ordinate system (x, y, z)

$\sigma_{x'x'}, \sigma_{y'y'}, \sigma_{z'z'}$	normal stress components in a Cartesian co-ordinate system (x', y', z')
$\bar{\sigma}_{xx}(\alpha,\beta,z), \bar{\sigma}_{yy}(\alpha,\beta,z), \bar{\sigma}_{zz}(\alpha,\beta,z)$	Double Fourier transforms of normal stresses
$\bar{\tau}_{xy}(\alpha,\beta,z), \bar{\tau}_{xz}(\alpha,\beta,z), \bar{\tau}_{yz}(\alpha,\beta,z)$	Double Fourier transforms of shear stresses
$\tau_{xy}, \tau_{yz}, \tau_{xz}$	shear stress components in a Cartesian co-ordinate system (x, y, z)
$\tau_{x'y'}, \tau_{y'z'}, \tau_{x'z'}$	shear stress components in a Cartesian co-ordinate system (x', y', z')
φ	the rotation of the transversely isotropic planes
ω	angular frequency
ρ	density of material

CHAPTER I

INTRODUCTION

The failure of a foundation in soil/rock is often caused by extra strains (deformations) or stresses. This fact is particularly important when structures impose very large loads on the underlying soil/rock. Normally, however, the magnitude and distribution of strains and stresses in soil/rock are predicted using numerical/analytical solutions that model the constituent materials as a linearly elastic, homogeneous and isotropic continuum. These solutions can not count the anisotropy of soils/rocks that are deposited via sedimentation over a long period of time, or rock masses cut by regular discontinuities, such as cleavages, foliations, stratifications, schistosities, and joints. Anisotropic soils/rocks are commonly modeled as transversely isotropic (cross-anisotropic) materials based on the practical engineering considerations. Nevertheless, the effects of inclination of discontinuities on the displacements and stresses are of interest. Hence, this thesis derives the analytical solutions for displacements and stresses due to three-dimensional point loads in a transversely isotropic medium with inclined planes of elastic symmetry.

Briefly, this thesis aims to derive the three-dimensional elastic solutions in a transversely isotropic full space and a half space subject to a three-dimensional point load. Employing the Fourier transform, the governing partial differential equations can be simplified as an ordinary differential equation. Then, three distinct approaches were used to solve the ODE for both infinite and semi-infinite solids in this thesis. The solutions show that there are identical for different approaches. The major deriving procedures are shown in Fig. 1.1. The three approaches are briefly described as follows: Firstly, we solve the nonhomogeneous ordinary differential equations by the methods of undetermined coefficients and separate variables, and obtain the homogeneous and particular solution for both a full-space and a half-space. Secondly, we separate the full-space into three regions of $-\infty < z < 0^-$ (region 2-upper half space), $0^- < z < 0^+$ (imaginary space) and $0^+ < z < +\infty$ (region 1-lower half space) or the half-space into two regions of $0 < z < 0^+$ (imaginary space) and $0^+ < z < +\infty$ (region 1), the point load force is loading in the region of $0^- < z < 0^+$ for full-space and $0 < z < 0^+$ for half-space. The right-hand side of the governing equations are zero in regions of $-\infty < z < 0^-$ or $0^+ < z < +\infty$, the equations are said to be homogeneous. Hence, we can solve the boundary-value problem consisting of the three or two regions for full-space or half-space. Thirdly, the Fourier transform respect to variables of z can reduce the aforementioned ordinary differential equations to algebraic equations. This method, which include three times of Fourier transform respect to variables of x, y and z, is also called Triple Fourier transform method. In the other word, the triple Fourier transforms with respect to *x*, *y*, and *z* could reduce the full-space problem of solving partial differential equations to algebraic equations. However, in the half-space, the double Fourier transforms with respect to *x*, and *y* can reduce the problem of solving partial differential equations to ordinary differential equations. Furthermore, by collocating the Laplace transform can reduce the aforementioned ordinary differential equations to algebraic equations. Hence, the displacement components in the triple Fourier transformed domains ($\overline{U}_i(\alpha,\beta,\gamma)$), or in the double Fourier and Laplace ones ($\overline{U}_i(\alpha,\beta,s)$) can be obtained.

Finally, the present fundamental solutions are derived by performing the required triple inverse Fourier transforms, or double inverse Fourier and Laplace transforms. These transformations are powerful to generate the displacements and stresses resulting from the three-dimensional point loads, acting in an inclined transversely isotropic material.

The content of this thesis, includes that Chapter Ⅱ provides a general literature review on the existing relevant solutions for the transversely isotropic media; Chapter Ⅲ introduction the basic theory of Fourier and Laplace integral transforms in a Cartesian co-ordinate system; Chapter Ⅳ and Ⅴ present the detailed derivations for three dimensional elastic solutions of an anisotropic full-space and half-space subjected to point loads by employing the Fourier and Laplace integral transforms, respectively. A series of parametric study using the present analytical solutions for displacements or stresses is conducted by four illustrative examples; The numerical results are demonstrated in Chapter Ⅵ; Eventually, Chapter Ⅶ includes summary and recommendations for future work.

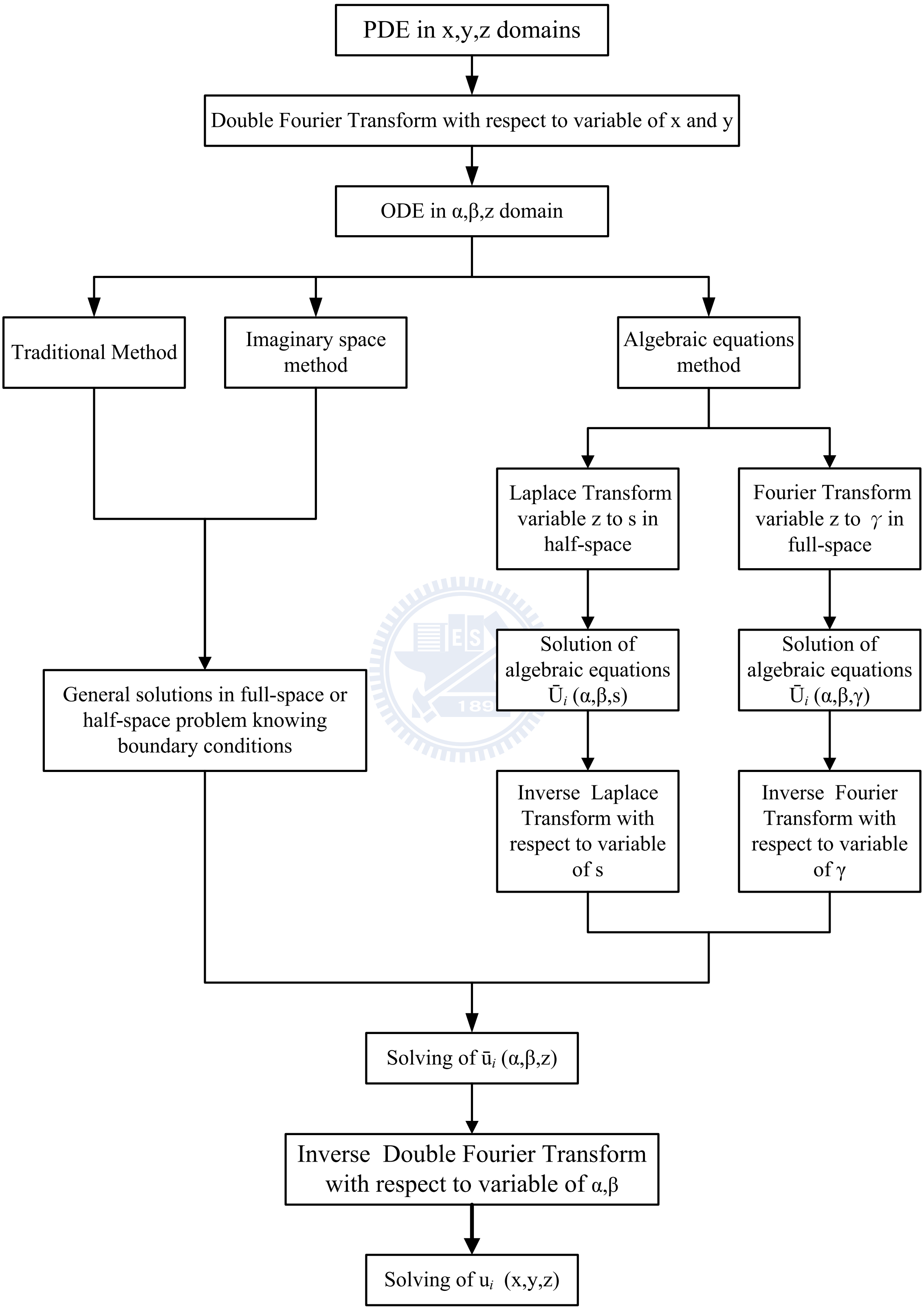


Fig. 1.1 Structure of the thesis

CHAPTER II

LITERATURE REVIEW

The elasticity of transversely isotropic materials is an important field of applied mechanics and engineering science. With the rapid development of modern technologies, the theory of elasticity has become increasingly significant. In addition, the field with which the theory has been typically associated, such as civil engineering and material engineering, also, many emerging technologies demand the development of transversely isotropic elasticity. Some immediate examples are piezo-film technology, anisotropic piezo-electric technology, functionally gradient materials (FGMs), and those involving transversely isotropic and layered microstructures, such as multi-layer systems and tribology mechanics of magnetic recording devices (Pan and Yuan, 2000).

The mathematical details of the basic equations of elasticity can be found in a variety of textbooks (e.g., Sneddon (1951) and Lekhnitskii (1981)). The basic equations of elasticity are geometric equations, constitutive equations, and equations of equilibrium. In tensor form, the geometric equations of strain-displacement relation in a Cartesian co-ordinate system can be written as:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.1}$$

The constitutive equations of stress-strain relation in linear elasticity are represented as:

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} \tag{2.2}$$

Considering the static state, the equations of motion can be reduced to the equations of equilibrium as follows:

$$\sigma_{ij,j} + F_i = 0 \tag{2.3}$$

Eqs. (2.1)-(2.3) will be further discussed in Chapter 3.1.

There are six basic equations of Eq. (2.1), six equations of Eq. (2.2), and three equations of Eq. (2.3). Hence, total fifteen equations of Eqs. (2.1)-(2.3) contain fifteen unknows of three groups of u_i, σ_{ij}, and ε_{ij}. Generally, it is unrealistic to solve the fifteen unknows all together. We often take one group of unknows or some unknows from different groups as the basic variables. The displacements of u_i are frequently adopted as the basic variables to be found. In such a method, the other unknows of two groups of σ_{ij} and ε_{ij} must be eliminated from the equations. More details about the uniqueness and possibility of other similar displacement presentations can be found in the works by Zou *et al*. (1994) and Ding *et al*. (1996). However, if we take the stress components (σ_{ij}) as the basic unknowns, these variables should be satisified with corresponding stress boundary conditions and the compatibility conditions between displacements and strains. Then, the solutions for stresses, and coresponding strains and displacements, can be derived. In anisotropic elasticity, the stress method is usually adopted to solve some relatively simpler problems (Ding *et al*., 2006). A summary of the earlier works in this respect can be found in the monograph of Lekhnitskii (1981). Base on the stress method, the general solutions of axisymmetric problems of transversely isotropic media were derived by Ding (1987), Wang and Wang (1989). In many other cases, it may be easier to find some stresses and some displacements but not all the variables of one single group. The state-space method usually deals with Eqs. (2.1)-(2.3) with two groups of unknowns. It has been utilized to solve the elasticity problems by Tarn (2002) and Georgiadis *et al*. (1999).

The equilibrium equations of Eq. (2.3) for infinite or semi-infinite solids are partial differential equations. Partial differential equations arise in connection with various physical and geometrical problems when the functions involved depending on two or more independent variables. It is fair to say that only the simplest physical systems can be modeled by ordinary differential equations. For infinite or semi-infinite domains, the method of integral techniques is applicable for the partial differential equations to reduce them to ordinary differential equations or algebraic equations. The integral techniques include the Fourier, Laplace, Hankel, and Mellin transforms are often employed to achieve the goal.

Pan and Yuan (2000) obtained the analytical solutions for stresses and strains in anisotropic bimaterials by double Fourier integral transforms. Liao and Wang (1998) presented the solutions of displacements and stresses in a transversely isotropic half-space by using Hankel and finite Fourier exponential transforms. Nevertheless, in transient dynamic problems, Laplace transforms are the most useful tools to transform the variable of time. To the best of the author′s knowledge, no solutions for displacements and stresses have been proposed by employing the Laplace transforms with respect to the spatial co-ordinate (x, y, or z) in a Cartesian co-ordinate system. the Laplace transforms are adopted for solving the half space problem in this thesis.

This chapter reviews the current steate of knowledge with respect to the point loading problem of a transversely isotropic medium. Existing analytical three-dimensional solutions of displacement or stresses subjected to a point load in infinite or semi-infinite space for a transversely isotropic medium are summerized.

2.1 Three-Dimensional Elastic Solution for Displacements and Stresses in a Transversely Isotropic Full Space

Solutions to the problem of a point load acting in the interior of a full-space are called the fundamental solutions or the elastic Green's function solutions (Pan *et al*., 1976 and Tarn *et al*., 1987). In the problems of infinite media, Willis (1965) estimated the elastic interaction energy of two infinitesimal dislocation loops in transversely isotropic magnesium and zinc media. There were two reasons to choice this medium, the first being the case of presentation of the results afforded by the axial symmetry, facilitating a ready comparison with the isotropic results. The second one was that to find the closed-form expressions for fundamental elasticity tensor for such a medium were possible.

Ting and Lee (1996) obtained the solution of Green's function for three-dimensional space of general anisotropic inclined medium subjected to a unit point force. It was expressed in terms of the Stroh eigenvalues p_v (v=1, 2, 3) on the inclined plane, and it remained valid for the degenerate cases when $p_1=p_2$, and $p_1=p_2=p_3$. The Stroh eigenvalues p_v were the roots with positive imaginary part of a sextic algebraic equation. The Green's function was simple when the sextic equation was a cubic equation in p^2. This was the case for any point in a transversely isotropic material and for points on a symmetric plane of cubic, and monoclinic materials.

These solutions in exact closed-form have always played an important role in applied mechanics and in particular numerical formulations of boundary element methods. Many investigators have presented analytical solutions for displacement under a point load in a transversely isotropic full-space, whose the transversely isotropic planes are parallel to the horizontal loading surface. A summary of the existing solutions is given in Table 2.1.

Table 2.1 Existing solutions for a transversely isotropic full-space subjected to a point load

Author	Analytical methods	Type of loading	Presented solutions
Chowdhury (1987)	methods of images and Hankel transforms	vertical	all displacements
Pan (1989)	vector functions	3D	all displacements
Willis (1965)	Fourier transforms	vertical	all displacements
Elliott (1948)	potential functions	vertical	all displacements
Chen (1966)	potential functions	vertical	all displacements
		horizontal	all displacements
Pan and Chou (1976)	potential functions	3D	all displacements
Fabrikant (1989)	potential functions	3D	all displacements
Sveklo (1969)	complex variables	vertical	all displacements
Tarn and Wang (1987)	Fourier and Hankel transforms	3D	all displacements
Lu (1991)	Fourier and Hankel transforms	3D	all displacements
Liao and Wang (1998)	Fourier and Hankel transforms	3D	all displacements

Sheu (2004)	Fourier and Hankel transforms	3D	all displacements

Table I indicates the analytical methods, the type of loading and the presented results. To the best of the authors' knowledge, no closed-form solutions for the displacement have been obtained in cases in which the planes of transverse isotropy inclines to the full-space subjected to 3D point loads (P_x, P_y, P_z), as displayed in Figure 3.1. In this thesis, the methods proposed by Willis (1965) for a transversely isotropic medium are followed. That is, the triple Fourier transforms are adopted to obtain the integral expressions of Green's displacement; then, the triple inverse Fourier transforms and residue calculus are performed to integrate the contours. However, Willis's expressions for Green's function are only valid when the elastic constants fulfill conditions that enable inverse transforms to be carried out (Tarn *et al.*, 1987). Notably, the main difference between Willis's approach (1965) and that proposed herein was the use of orthogonal vectors. In the former, two axes were on the transversely isotropic plane, and the third was parallel to the axis of rotation associated with elastic symmetry. Accordingly, a state of plane strain was assumed in that procedure.

2.2 Three-Dimensional Elastic Solution for Displacements and Stresses in a Transversely Isotropic Half Space

A point load solution is the basis of complex loading problems. For an isotropic solid, it has been studied by Kelvin (Thompson, 1848) for a full-space, Boussinesq

(1885) and Cerruti (1888) for a half-space with a vertical and horizontal point load, respectively. In the case of a single concentrated force acting in the interior of a half-space, Mindlin (1936) proposed closed-form solutions for an isotropic medium using the principle of superposition of eighteen nuclei. Mindlin derived analytical solutions following the Kelvin's (1848) approach and satisfying the condition of vanishing traction on a plane boundary. However, the calculation of nuclei for a half-space is very difficult (Mindlin and Cheng, 1950). Dean et al. (1944) recommended another approach for the same problem by the method of images. Some of their solutions can be extended to anisotropic media, whereas others are difficult.

For the displacements and stresses in transversely isotropic media subjected to a point load, analytical solutions have been presented by several investigators. Some of the solutions were directly derived using the approaches for isotropic solutions (Michell, 1900; Wolf, 1935; Koning, 1957; Barden, 1963; de Urena et al., 1966; Misra and Sen, 1975; Chowdhury, 1987; Pan, 1989). Nevertheless, others employed complex mathematics techniques, such as Fourier transformations (Kröner, 1953; Willis, 1965; Lee, 1979), potential functions (Lekhnitskii, 1940; Elliott, 1948; Shield, 1951; Eubanks and Sternberg, 1954; Lodge, 1955; Hata, 1956; Chen, 1966; Pan and Chou, 1976; Pan and Chou, 1979; Okumura and Dohba, 1989; Fabrikant, 1989; Lin et al., 1991; Hanson and Wang, 1997) and complex variables (Sveklo, 1964, 1969), etc. The summary of the existing solutions is given in Table 2.1. Table 2.1 indicates the type of analytical space, the load, and the results presented in their solutions. Because of mathematical difficulty or oversimplification for solving the problems, these solutions were limited to three-dimensional problems with partial results of displacements (Michell, 1900; Shield, 1951; Barden, 1963) and stresses (Michell, 1900; Shield, 1951; Barden, 1963; Misra and Sen, 1975; Pan and Chou, 1979; Chowdhury, 1987; Fabrikant, 1989), or axially symmetric problems invariant with the tangential co-ordinate, θ (Lekhnitskii, 1940; Elliott, 1948; Koning, 1957; Sveklo, 1964; de Urena et al., 1966; Misra and Sen, 1975). Neglecting the θ, the solutions cannot be extended to solve a half-space problem subjected to asymmetric loads. Pan and Chou (1979) proposed a more general solution using potential functions. In their solution, the buried loads can be vertical or horizontal with respect to the boundary plane. However, only the stress components related to the z-direction were given (i.e., σ_{zz}, σ_{zx}, σ_{zy}), and the expressions for the solution are quite lengthy.

Table 2.2 Existing solutions for transversely isotropic media subjected to a point load

Author	Space	Type of loading	Solutions
Michell (1900)	half	vertical	vertical surface displacement, and partial stresses (inapplicable to boundary value problems)
Wolf (1935)	half	vertical	all displacements and stresses (oversimplified the elastic constants)
Koning (1957)	half	vertical	all displacements and stresses
Barden (1963)	half	vertical	vertical surface displacement, and stresses on load axis
de Urena et al. (1966)	half	vertical	all displacements and stresses
Misra and Sen (1975)	half	vertical	all displacements, and stresses on load axis (oversimplified the elastic constants)
Chowdhury (1987)	full half	vertical buried, vertical	all displacements, and stresses on load axis all displacements, and stresses on load axis
Pan (1989)	full	3-D	all displacements and stresses
Kröner (1953)	half	vertical	all displacements (dimensionally incorrect)
Willis (1965)	full	vertical	all displacements (cumbersome and inaccurate)
Lee (1979)	half	buried, vertical	all stresses (complicated)
Lekhnitskii (1940)	half	vertical	all stresses (incomplete)
Elliott (1948)	full	vertical	all displacements and stresses (incomplete)
Shield (1951)	half	buried, vertical	all displacements and stresses at the surface
Eubanks and Sternberg (1954)	half	vertical	(completeness of Lekhnitskii's method)
Lodge (1955)			(transformed anisotropic problem into isotropic one, inapplicable to general boundary value problems)
Hata (1956)	half	vertical	(rederived the Elliott's and Lodge's solution)
Chen (1966)	full	vertical horizontal	all displacements and stresses all displacements
Pan and Chou (1976)	full	3-D	all displacements and stresses

Table 2.2. Existing solutions for transversely isotropic media subjected to a point load (continued)

Author	Space	Type of loading	Solutions
Pan and Chou (1979)	half	buried, vertical buried, horizontal	all displacements, and stresses on load axis all displacements, and partial stresses (potential functions assumed are lengthy)
Okumura and Dohba (1989)	half	vertical	all displacements
Fabrikant (1989)	full half	3-D 3-D	all displacements, and partial stresses all displacements, and partial stresses (solution of the shear stress is wrong)
Lin et al. (1991)	half	vertical, horizontal	all displacements and stresses
Hanson and Wang (1997)	half	buried, 3-D	(only the potential functions listed)
Sveklo (1964)	half	vertical	all displacements
Sveklo (1969)	full half	vertical buried, vertical	all displacements all displacements

Following the method proposed by Tarn and Wang (1987), Lu (1991) presented analytical solutions for the displacements in a full or half soil space (transverse isotropy) under a long-term consolidation. However, a part of the solutions might be error in handwriting. Utilizing the approaches proposed by Lu (1991), closed-form solutions for displacements and stresses in a transversely isotropic half-space subjected to a point load are rederived and parts of the results are published (Liao and Wang, 1998). However, the solutions are limited to the cases of planes of transverse isotropy parallel to the horizontal loading surface.

The solution of the stresses and displscements in a half-space or a layered solid with transverse isotropy is fundamental to the development of the theory of elasticity and is of importance to many engineering applications. Ding (1987) presented a unified solution for a point force applied on the surface/in the interior of a half-space. The solution could be extended to the problem of layered media using the state-space and Fourier transform methods. Hence, to solve the problems of semi-infinite media, Ding considered a transversely isotropic medium in a Cartesian co-ordinate system (x, y, z), whose z-axis is perpendicular to the isotropic plane of material. Any point force (or concentrated force) applied in the body can be resolved into three components T, Q and P in x-, y-, and z-direction, respectively. Ding assumed that an arbitrary point force was applied at the origin. It can be decomposed the problem into three sub-problems by using the principle of superposition; namely, the solution corresponding to a vertical force, P, in the positive z-direction, the solution to a tangential force, T_x=T, in the x-direction, as well as the solution to a tangential force, T_y=Q, in the y-direction. The last solution can be obtained from the second solution by a co-ordinate transform with x replaced by y, and y by $-x$, respectively. However, it is clear that Ding's solutin has not yielded the analytical solutions of displacements and stresses for an inclined transversely isotropic material owing to three–dimensional point loads. All the fundamental solutions of literature for transversely isotropic materials being the case of axisymmetric problem.

CHAPTER Ⅲ

BASIC THEORY

To derive the solutions for stresses, strains, and displacements in finite domains with simple geometry, the method of separation of variables is usually applicable for the partial differential equations to reduce to the ordinary differential equations (Chiang, 1997). Then the solutions can be constructed by superposition of eigenfunctions. However, considering the domain is infinite or semi-infinite, it is hard to achieve a similar reduction. Thus, the integral techniques include the Fourier, Laplace, Hankel, and Mellin transforms are often utilized to attain the goal. Among them, the Fourier and Laplace transforms are basic and most useful. Generally, the governing equations for infinite or semi-infinite solids are partial differential equations. The Fourier and Laplace integral transforms are efficient methods for solving the partial differential equations and corresponding boundary value problems. Employing the two methods can reduce the problem from solving partial differential equations to ordinary differential equations or algebraic equations.

3.1 Basic Equations for Elastic Boundary Value Problems

<u>Constitutive equations and transformation of elastic constants</u>

The constitutive equations in linear elasticity are represented by the generalized Hooke′s law. If the state of vanishing strain corresponds to zero stress, then in Cartesian co-ordinates, the generalized Hooke′s law can be written as:

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} \quad (i, j, k, l=x, y, z) \tag{3.01}$$

where c_{ijkl} are components of a fourth-rank tensor, repressenting the properties of a material, which generally varies from one point to another in the material. The c_{ijkl} are called elastic stiffness constants. Since Eq. (3.01) contains nine equations (corresponding to all possible combinations of the subscripts *i* and *j* and each equation contains nine strain variables), there are 81 elastic stiffness constants. These are not all independent hower. It will be seen that $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{jilk}$, which reduces the number of independent constants to 36. In addition, the $c_{ijkl} = c_{klij}$, and this means the constants are further reduced to 21. This is the maxium number of constants for any medium.

In a Cartesian co-ordinate system, (x', y', z'), the Eq. (3.01) can then be expressed as:

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \sigma_{z'z'} \\ \tau_{y'z'} \\ \tau_{z'x'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{z'z'} \\ \gamma_{y'z'} \\ \gamma_{z'x'} \\ \gamma_{x'y'} \end{bmatrix} \quad (3.02)$$

The number of elastic constants c_{ijkl} for describing their deformability is 21, 9, 5, and 2 for generally anisotropic, orthotropic, transversely isotropic, and isotropic material, respectively. Thus, for a general anisotropic elastic material, there are 21 independent elastic stiffness constants. If there exist three orthogonal planes of elastic symmetry at any point in a solid, then there are 9 independent elastic stiffness constants, and the material is said to be orthotropic. If at any point there is an axis of symmetry such that the elastic properties in any direction within a plane perpendicular to the axis are all the same, the number of independent elastic stiffness constants will reduce to 5. The plane is called an isotropic plane and the material is called a transversely isotropic material. If any plane in the material is a plane of elastic symmetry, then the material is isotropic, and has only 2 independent elastic stiffness constants.

Fig. 3.1 displays a transversely isotropic medium, in which the z' axis is the rotation axis of elastic symmetry, the x' and y' axes are in the plane of transversely isotropy. In the co-ordinate system (x', y', z'), the corresponding matrix form is $\{\sigma\}_{x'y'z'} = [c]_{x'y'z'}\{\varepsilon\}_{x'y'z'}$.

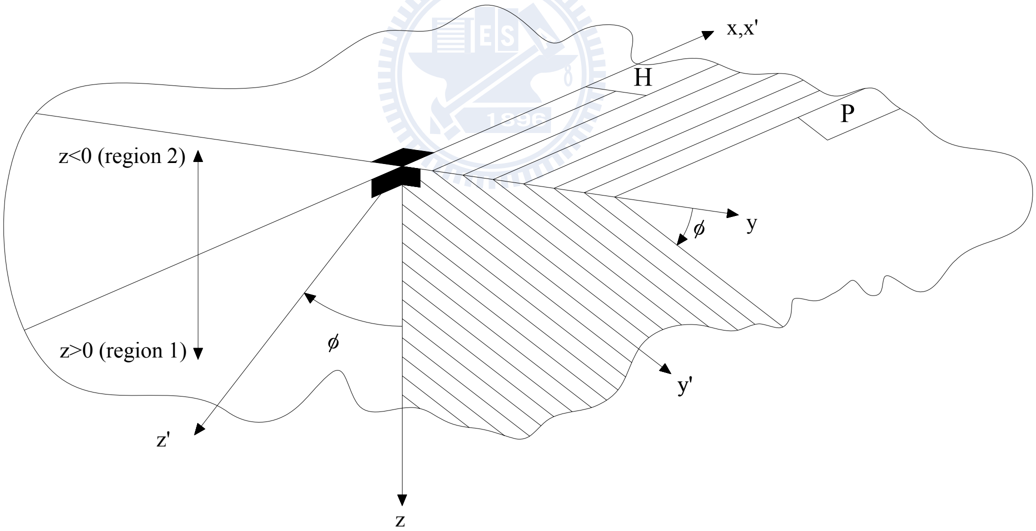


Fig. 3.1 (P_x, P_y, P_z) acting in an inclined transversely isotropic full-space

Regarding the different co-ordinate system (x, y, z), the constitutive equations will have the same form as $\{\sigma\}_{xyz} = [a]_{xyz}\{\varepsilon\}_{xyz}$. Hence, the generalized Hooke′s law for the transversely isotropic material can be expressed as:

$$\begin{bmatrix}\sigma_{x'x'}\\ \sigma_{y'y'}\\ \sigma_{z'z'}\\ \tau_{y'z'}\\ \tau_{z'x'}\\ \tau_{x'y'}\end{bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&0&0&0\\ C_{12}&C_{11}&C_{13}&0&0&0\\ C_{13}&C_{13}&C_{33}&0&0&0\\ 0&0&0&C_{44}&0&0\\ 0&0&0&0&C_{44}&0\\ 0&0&0&0&0&C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{x'x'}\\ \varepsilon_{y'y'}\\ \varepsilon_{z'z'}\\ \gamma_{y'z'}\\ \gamma_{z'x'}\\ \gamma_{x'y'}\end{bmatrix}$$

$$=\begin{bmatrix}a_1&a_1-2a_4&a_3-a_5&0&0&0\\ a_1-2a_4&a_1&a_3-a_5&0&0&0\\ a_3-a_5&a_3-a_5&a_2&0&0&0\\ 0&0&0&a_5&0&0\\ 0&0&0&0&a_5&0\\ 0&0&0&0&0&a_4\end{bmatrix}\begin{bmatrix}\varepsilon_{x'x'}\\ \varepsilon_{y'y'}\\ \varepsilon_{z'z'}\\ \gamma_{y'z'}\\ \gamma_{z'x'}\\ \gamma_{x'y'}\end{bmatrix} \quad (3.03)$$

where $\sigma_{x'x'}$, $\sigma_{y'y'}$, $\sigma_{z'z'}$ are normal stresses; $\varepsilon_{x'x'}$, $\varepsilon_{y'y'}$, $\varepsilon_{z'z'}$ are normal strains; $\tau_{y'z'}$, $\tau_{z'x'}$, $\tau_{x'y'}$ are shear stresses; $\gamma_{y'z'}$, $\gamma_{z'x'}$, $\gamma_{x'y'}$ are shear strains, and C_{11}, C_{12}, C_{13}, C_{33}, C_{44}, C_{66} are elastic moduli or constants. Since $C_{12}=C_{11}-2C_{66}$, hence, only C_{11}, C_{13}, C_{33}, C_{44}, C_{66} are independent for a transversely isotropic material. The relationship between C_{11}, C_{13}, C_{33}, C_{44}, C_{66} and a_1, a_2, a_3, a_4, a_5 can be presented in terms of five elastic constants as:

$$a_1=C_{11}=\frac{E(1-\frac{E}{E'}\upsilon'^2)}{(1+\upsilon)(1-\upsilon-\frac{2E}{E'}\upsilon'^2)}\quad,\quad a_2=C_{33}=\frac{E'(1-\upsilon)}{1-\upsilon-\frac{2E}{E'}\upsilon'^2}\quad,\quad a_5=C_{44}=G'\quad,$$

$$a_3=C_{13}+C_{44}=\frac{E\upsilon'}{1-\upsilon-\frac{2E}{E'}\upsilon'^2}+C_{44},\quad a_4=C_{66}=\frac{C_{11}-C_{12}}{2}=\frac{E}{2(1+\upsilon)} \quad (3.04)$$

where:

1. E and E' are Young′s moduli in the plane of transverse isotropy and in a direction normal to it, respectively.

2. υ and υ' are Poisson′s ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel or normal to it, respectively.

3. G' is the shear modulus in planes normal to the plane of transverse isotropy.

If a new co-ordinate system (x, y, z) is obtained from the original system (x', y', z') by rotation through an angle ϕ about the common axis $x = x'$ (the axis of x and x' parallel to the strike of transverse plane). The matrix of direction cosines l_{ij} for the transformation formulae of the elastic constants are:

$$\left[l_{ij}\right] = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{3.05}$$

where i, j=1-3.

The elastic stiffnesses matrix in the old co-ordinate system (x', y', z') is $\left[c\right]_{x'y'z'}$. Therefore, the elastic stiffnesses matrix $\left[a\right]_{xyz}$ in the new co-ordinate system (x, y, z) can be expressed as:

$$\left[a\right]_{xyz} = \left[q_{ij}\right]^{T}\left[c\right]_{x'y'z'}\left[q_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \tag{3.06}$$

where T is the transpose matrix; $\left[q_{ij}\right]$ are written as follows.

$$\left[q_{ij}\right]=\begin{bmatrix} l_{11}^2 & l_{12}^2 & l_{13}^2 & l_{12}l_{13} & l_{13}l_{11} & l_{12}l_{11} \\ l_{21}^2 & l_{22}^2 & l_{23}^2 & l_{23}l_{22} & l_{23}l_{21} & l_{22}l_{21} \\ l_{31}^2 & l_{32}^2 & l_{33}^2 & l_{33}l_{32} & l_{33}l_{31} & l_{32}l_{31} \\ 2l_{31}l_{21} & 2l_{32}l_{22} & 2l_{33}l_{23} & l_{33}l_{22}+l_{32}l_{23} & l_{33}l_{21}+l_{31}l_{23} & l_{31}l_{22}+l_{32}l_{21} \\ 2l_{31}l_{11} & 2l_{32}l_{12} & 2l_{33}l_{13} & l_{33}l_{12}+l_{32}l_{13} & l_{33}l_{11}+l_{31}l_{13} & l_{31}l_{12}+l_{32}l_{11} \\ 2l_{21}l_{11} & 2l_{12}l_{22} & 2l_{13}l_{23} & l_{13}l_{22}+l_{12}l_{23} & l_{13}l_{21}+l_{11}l_{23} & l_{11}l_{22}+l_{12}l_{21} \end{bmatrix} \tag{3.07}$$

where i, j=1-3.

The new elastic constants of $[a]_{xyz}$ obtained directly from the old elastic conctant a_1-a_5 and ϕ. It is important to note that $[a]_{xyz}$ exist 13 elastic constants under the plane of elastic symmetry system, and the expressions of the elastic constants a_{ij} (i, j=1-6) with respect to a_1-a_5 and ϕ are presented in Appendix A. Appendix A show that the new constants of $a_{15}=a_{16}=0$, $a_{25}=a_{26}=0$, $a_{35}=a_{36}=0$, $a_{45}=a_{46}=0$, $a_{51}=a_{52}=a_{53}=a_{54}=0$ and $a_{61}=a_{62}=a_{63}=a_{64}=0$

Then, the generalized Hooke′s law for a transversely isotropic material is:

$$\{\sigma\}_{xyz}=[a]_{xyz}\{\varepsilon\}_{xyz} \tag{3.08}$$

where $\{\sigma\}_{xyz}$ and $\{\varepsilon\}_{xyz}$ are vectors of stress and engineering strain, respectively. In Cartesian co-ordinates, they are

$$\{\sigma\}_{xyz}=\begin{bmatrix}\sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \tau_{yz} & \tau_{zx} & \tau_{xy}\end{bmatrix}^T \tag{3.09}$$

$$\{\varepsilon\}_{xyz}=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \gamma_{yz} & \gamma_{zx} & \gamma_{xy}\end{bmatrix}^T. \tag{3.10}$$

Strain-displacement relations

When a sign convention for soil and rock problem is required, it is customary to define compressive stresses as positive and tensile stresses as negative. The strain-displacement relationship under small strain condition in a Cartesian coordinate system is:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial u_x}{\partial x} \\ -\dfrac{\partial u_y}{\partial y} \\ -\dfrac{\partial u_z}{\partial z} \\ -\dfrac{\partial u_y}{\partial z} - \dfrac{\partial u_z}{\partial y} \\ -\dfrac{\partial u_x}{\partial z} - \dfrac{\partial u_z}{\partial x} \\ -\dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} \end{bmatrix} \tag{3.11}$$

where u_x, u_y, and u_z are three displacements of a point on the axis of a Cartesian co-ordinate system.

Generalized Hooke′s law in terms of the derivative of displacements

Hence, from Eqs. (3.08), (3.09), (3.10) and (3.11), the generalized Hooke′s law equations for a transversely isotropic medium in a Cartesian coordinate system can be expressed as:

$$\begin{aligned} \sigma_{xx} &= a_{11}\varepsilon_{xx} + a_{12}\varepsilon_{yy} + a_{13}\varepsilon_{zz} + a_{14}\gamma_{yz} \\ &= -a_{11}\frac{\partial u_x}{\partial x} - a_{12}\frac{\partial u_y}{\partial y} - a_{13}\frac{\partial u_z}{\partial z} - a_{14}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \end{aligned} \tag{3.12a}$$

$$\begin{aligned} \sigma_{yy} &= a_{12}\varepsilon_{xx} + a_{22}\varepsilon_{yy} + a_{23}\varepsilon_{zz} + a_{24}\gamma_{yz} \\ &= -a_{12}\frac{\partial u_x}{\partial x} - a_{22}\frac{\partial u_y}{\partial y} - a_{23}\frac{\partial u_z}{\partial z} - a_{24}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \end{aligned} \tag{3.12b}$$

$$\sigma_{zz} = a_{13}\varepsilon_{xx} + a_{23}\varepsilon_{yy} + a_{33}\varepsilon_{zz} + a_{34}\gamma_{yz}$$
$$= -a_{13}\frac{\partial u_x}{\partial x} - a_{23}\frac{\partial u_y}{\partial y} - a_{33}\frac{\partial u_z}{\partial z} - a_{34}(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}) \tag{3.12c}$$

$$\tau_{yz} = a_{14}\varepsilon_{xx} + a_{24}\varepsilon_{yy} + a_{34}\varepsilon_{zz} + a_{44}\gamma_{yz}$$
$$= -a_{14}\frac{\partial u_x}{\partial x} - a_{24}\frac{\partial u_y}{\partial y} - a_{34}\frac{\partial u_z}{\partial z} - a_{44}(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}) \tag{3.12d}$$

$$\tau_{zx} = a_{55}\gamma_{zx} + a_{56}\gamma_{xy}$$
$$= -a_{55}(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}) - a_{56}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) \tag{3.12e}$$

$$\tau_{xy} = a_{56}\gamma_{zx} + a_{66}\gamma_{xy}$$
$$= -a_{56}(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}) - a_{66}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) \tag{3.12f}$$

Equilibrium equations

In Cartesian coordinates, the equations of motion can be expressed by a tensor form as:

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i \quad (i=x, y, z) \tag{3.13}$$

where ρ is the density of material, F_i is the component of the body force per unit volume in *i*-direction, and finally the double dot indicates the second order partial differentiation with respect to time *t*. If the motion of the solid does not involve acceleration, then Eq. (3.13) reduces to the equilibrium equation as:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{bmatrix} + \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.14}$$

where F_x, F_y, and F_z stand for the components of the body forces per unit volume in the co-ordinate directions, x, y and z, respectively. Substituting Eqs. (3.12a)-(3.12f) (σ_{xx}, σ_{yy}, σ_{zz}, τ_{yz}, τ_{zx}, τ_{xy}) into Eq. (3.14) enables the equations to be regrouped as Navier-Cauchy equations for an inclined transversely isotropic material as:

$$\begin{aligned}
&a_{11}\frac{\partial^2 u_x}{\partial x^2}+a_{66}\frac{\partial^2 u_x}{\partial y^2}+a_{55}\frac{\partial^2 u_x}{\partial z^2}+2a_{56}\frac{\partial^2 u_x}{\partial y\partial z}\\
&+(a_{12}+a_{66})\frac{\partial^2 u_y}{\partial x\partial y}+(a_{14}+a_{56})\frac{\partial^2 u_y}{\partial x\partial z}\\
&+(a_{14}+a_{56})\frac{\partial^2 u_z}{\partial x\partial y}+(a_{13}+a_{55})\frac{\partial^2 u_z}{\partial x\partial z}=F_x
\end{aligned} \tag{3.15a}$$

$$\begin{aligned}
&(a_{12}+a_{66})\frac{\partial^2 u_x}{\partial x\partial y}+(a_{14}+a_{56})\frac{\partial^2 u_x}{\partial x\partial z}\\
&+a_{66}\frac{\partial^2 u_y}{\partial x^2}+a_{22}\frac{\partial^2 u_y}{\partial y^2}+a_{44}\frac{\partial^2 u_y}{\partial z^2}+2a_{24}\frac{\partial^2 u_y}{\partial y\partial z}\\
&+a_{56}\frac{\partial^2 u_z}{\partial x^2}+a_{24}\frac{\partial^2 u_z}{\partial y^2}+a_{34}\frac{\partial^2 u_z}{\partial z^2}+(a_{23}+a_{44})\frac{\partial^2 u_z}{\partial y\partial z}=F_y
\end{aligned} \tag{3.15b}$$

$$\begin{aligned}
&(a_{14}+a_{56})\frac{\partial^2 u_x}{\partial x\partial y}+(a_{13}+a_{55})\frac{\partial^2 u_x}{\partial x\partial z}\\
&+a_{56}\frac{\partial^2 u_y}{\partial x^2}+a_{24}\frac{\partial^2 u_y}{\partial y^2}+a_{34}\frac{\partial^2 u_y}{\partial z^2}+(a_{23}+a_{44})\frac{\partial^2 u_y}{\partial y\partial z}\\
&+a_{55}\frac{\partial^2 u_z}{\partial x^2}+a_{44}\frac{\partial^2 u_z}{\partial y^2}+a_{33}\frac{\partial^2 u_z}{\partial z^2}+2a_{34}\frac{\partial^2 u_z}{\partial y\partial z}=F_z
\end{aligned} \tag{3.15c}$$

Point loads

For a dynamic elastic problem, an arbitrary time-harmonic body force in z-direction with angular frequency ω can be expressed as (Eringen and Suhubi, 1975; Rahman, 1995):

$$F_z\,(x,\mathrm{y},\mathrm{z},\mathrm{t})=F_z^*(x,y,\mathrm{z})\,\mathrm{e}^{\mathrm{i}\,\omega\,\mathrm{t}} \tag{3.16}$$

where $F_z^*(x, y, z)$ is the complex amplitude of the body force. Following the suggestions of Eringen and Suhubi (1975) and Rahman (1995), a concentrated force in z-direction (F_z) can be represented as the form of a body force:

$$F_z = p_z \delta(x)\delta(y)\delta(z)\, e^{i\omega t} \tag{3.17}$$

where $\delta(\,)$ is the Dirac delta function.

Nevertheless, for static problems, the terms associated with time t in Eq. (3.17) should be removed. As this research concerning about the static problems, ω in Eq. (3.17) will be zero. Hence, three-dimensional static point loads with components (F_x, F_y, F_z) acting at the origin of the co-ordinate can be expressed as the form of body forces:

$$F_x = P_x \delta(x)\delta(y)\delta(z) \tag{3.18a}$$

$$F_y = P_y \delta(x)\delta(y)\delta(z) \tag{3.18b}$$

$$F_z = P_z \delta(x)\delta(y)\delta(z) \tag{3.18c}$$

Then, the point loads (F_x, F_y, F_z) applied at the point (0, 0, h) of the co-ordinate system can be described as the form of body forces as:

$$F_x = P_x \delta(x)\delta(y)\delta(z-h) \tag{3.19a}$$

$$F_y = P_y \delta(x)\delta(y)\delta(z-h) \tag{3.19b}$$

$$F_z = P_z \delta(x)\delta(y)\delta(z-h) \tag{3.19c}$$

The Dirac detla function is a mathematical artifice for representing an extremely localized function with a finite total area. For example, $\delta(x-\xi)$ is the limit of a spike-like function of x, which is zero almost everywhere expect very near $x = \xi$. That is, the Dirac detla function has an extremely sharp peak in such a way that its area above the x axis is unity, i.e.,

$$\int_a^b \delta(x-\xi)dx = \begin{cases} 1 & if \quad \xi \in (a,b), \\ 0 & if \quad \xi \notin (a,b), \end{cases} \tag{3.20}$$

where (a, b) stands for the open interval between a and b, excluding the end point. As long as the point of concentration ξ lies between a and b, the upper and lower limits can be replaced by $(-\infty,\infty)$. Therefore, an alternative definition of the Dirac detla function is:

$$\int_{-\infty}^{\infty} \delta(x-\xi)dx = 1 \tag{3.21}$$

In mechanics, $\delta(x-\xi)$ may symbolize a concentrated force, i.e., the limit of a pressure distribution with a sharply peaked intensity around $x=\xi$ and a unit body forces per unit volume (Chiang,1997).

3.2 Basic Theory of Fourier and Laplace Transforms

In order to solve elastic solutions for stresses and displacements in an incline transverselyisotropic medium subjected to a point load, the inverse Fourier and Laplace transforms are the most frequently adopted methods since they can be evaluated in a complex plane. One of the mathematical applications of Cauchy′s theorem is to facilitate the explicit evaluation of integrals along a real line. The typical procedures are (1) change the real integration variable to the complex variable, (2) find the singularities of the integrand in the complex plane, (3) connect the original path of integration with an additional path to from a closed contour, (4) apply Cauchy′s integral formula to evaluate the integral along the closed contour, (5) find the integral along the additional path, and (6) subtract from the results of (5) from (4) to get the original integral (Chiang, 1997).

In the first instance, we suppose that the elastic solid is infinite extend and it is deformed by the action of known point loads (F_x, F_y, F_z) at the origin of the co-ordinate. We shall further suppose that, as $|x|$, $|y|$ or $|z| \to \infty$, all the compontents of displacement and of the stress tensor tend to zero. To solve the equations of equilibrium equations, we introduce the Fourier transform (Section 3.2.1) by considering the distribution of displacement and stress in an infinite elastic solid due to the application of point loads acted at the origin of the co-ordinate. It is assumed that the elastic medium is bounded by the space of infinite extend; However, in the case of a semi-infinite domain, the solutions for displacement and stress components must tend to zero as z tends to infinity. Similarly, the method of Laplace transform, as described in Section 3.2.2, is introduced furth to solve the semi-infinite problem.

3.2.1 Fourier Transform

The Fourier transforms can be adopted for reducing an integral function to a simple function, such as $\overline{U}_i(\alpha)$ is the Fourier transform of $u_i(x)$. The definition of Fourier transformation is:

$$\overline{U}_i(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_i(x) e^{-i\alpha x} dx \tag{3.22}$$

Then $u_i(x)$, inverse Fourier transform, is given in terms of $\overline{U}_i(\alpha)$ by the following relation:

$$u_i(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{U}_i(\alpha) e^{i\alpha x} d\alpha \, . \tag{3.23}$$

Let $u_i(x)$ be continuous on the x-axis, and $u_i(x) \to 0$ as $|x| \to \infty$. Furthermore, let $u'_i(x)$ be absolutely integrable on the x-axis. Then,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{du_i(x)}{dx}e^{-i\alpha x}dx = i\alpha\overline{U}_i(\alpha) \tag{3.24}$$

Further applications of Eq. (3.24) give:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d^2u_i(x)}{dx^2}e^{-i\alpha x}dx = i\alpha\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{du_i(x)}{dx}e^{-i\alpha x}dx = (i\alpha)^2\overline{U}_i(\alpha) \tag{3.25}$$

The same are true for higher derivatives.

The theory of Fourier transforms of function of a single variable can be extended to functions of several variabes. Suppose, for instance, that $u_i(x,y)$ is a functions of the two independent variables x and y; then, regarded as a function of x, $u_i(x,y)$ has the Fourier transform:

$$\hat{U}_i(\alpha,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_i(x,y)e^{-i\alpha x}dx \tag{3.26}$$

and this function, regarded as a function of y, has the Fourier transform:

$$\overline{U}_i(\alpha,\beta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{U}_i(\alpha,y)e^{-i\beta y}dy \tag{3.27}$$

Combining Eqs. (3.26) and (3.27), we observe that the relation between the functions $u_i(x,y)$ and $\overline{U}_i(\alpha,\beta)$ is:

$$\overline{U}_i(\alpha,\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_i(x,y)e^{-i(\alpha x+\beta y)}dxdy\ . \tag{3.28}$$

Hence, $\overline{U}_i(\alpha,\beta)$ is the two-dimensional Fourier transforms of the function $u_i(x,y)$. Sequentially, $u_i(x,y)$ may be expressed in terms of $\hat{U}_i(\alpha,y)$ as:

$$u_i(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{U}_i(\alpha, y) e^{i\alpha x} d\alpha \quad . \tag{3.29}$$

Similarly from Eq. (3.29),

$$\hat{U}_i(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{U}_i(\alpha, \beta) e^{i\beta y} d\beta \tag{3.30}$$

and eventually that,

$$u_i(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{U}_i(\alpha, \beta) e^{i(\alpha x + \beta y)} d\alpha d\beta \tag{3.31}$$

giving the inversion formula for the double Fourier transform (Sneddon,1951).

Right now, the generalization to a greater number of variables is obvious. Suppose $u_i(x,y,z)$ is a function of the three independent variables *x*, *y*, *z*; then the triple Fourier transforms of the function $u_i(x,y,z)$ are defined to be:

$$\overline{U}_i(\alpha, \beta, \gamma) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_i(x,y,z) e^{-i(\alpha x + \beta y + \gamma z)} dx dy dz \quad . \tag{3.32}$$

The corresponding inversion formula can then be shown to be:

$$u_i(x,y,z) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{U}_i(\alpha, \beta, \gamma) e^{i(\alpha x + \beta y + \gamma z)} d\alpha d\beta d\gamma \quad . \tag{3.33}$$

3.2.2 Laplace Transform

The function $\overline{U}_i(s)$ of the variable *s* is called the Laplace transform of the original function $u_i(z)$, and will be denoted by $L\{u_i\}$. Thus,

$$\overline{U}_i(s) = L\{u_i\} = \int_{0}^{\infty} u_i(z) e^{-sz} dz \tag{3.34}$$

Let $u_i(z)$ be a function that is piecewise continuous on every finite interval in the range $z \geq 0$ and satisfies $|u_i(z)| \leq Me^{\Psi z}$ for all $z \geq 0$. Then the Laplace transform of $u_i(z)$ exists for all $s > \Psi$.

Since $u_i(z)$ is piecewise continuous, $u_i(z)e^{-sz}$ is integrable over any finite interval on the *z*-axis. From Eq. (3.34), assuming that $s > \Psi$, we obtain:

$$|L\{u_i\}| = \left|\int_0^{\infty} u_i(z)e^{-sz}dz\right| \leq \int_0^{\infty} |u_i(z)|e^{-sz}dz \leq \int_0^{\infty} Me^{\Psi z}e^{-sz}dz = \frac{M}{s-\Psi} \tag{3.35}$$

where the condition $s > \Psi$ is needed for the existence of the last integral.

Furthermore, the original function $u_i(z)$ in Eq. (3.36) is called the inverse transform or inverse of $\overline{U}_i(s)$, and will be denoted by $L^{-1}(\overline{U}_i)$. In another word,

$$u_i(z) = L^{-1}\{\overline{U}_i\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{U}_i(s)e^{sz}ds \tag{3.36}$$

In most physical applications, the variable *z* can be replaced by the time (*t*). In this research, the variable *z* is the position of *z*-axis.

The constant *c* in Eq. (3.36) is chosen to be large enough, say $c > a$, so that the last integral exists. The value of *a* depends on the behavior of $u_i(z)$ at large *z*.

Suppose that $u_i(x)$ is continuous for all $z \geq 0$, and has a derivative $u_i'(x)$ that is piecewise continuous on every finite interval. Then the Laplace transforms of the derivative $u_i'(x)$ exist when $s > \Psi$, and

$$L\{u_i'(z)\} = sL(u_i(z)) - u_i(0) \tag{3.37}$$

Similarly,

$$L\{u_i''(z)\} = s^2 L(u_i(z)) - s u_i(0) - u_i'(0) \tag{3.38}$$

3.3 Reducing the PDE of (3.15a)-(3.15c) to ODE Using Double Fourier Transformation

The equilibrium equations, as expressed in Eqs. (3.15a)-(3.15c) for an inclined transversely isotropic space subjected to three-dimensional point loads are partial differential equations. Double Fourier transforms can be employed for solving the two independent variables of x and y to reduce the problem of solving partial differential equations to ordinary differential equations. We introduce the double Fourier transforms for displacement functions and derivatives as:

$$\frac{1}{(2\pi)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_i(x,y,z) e^{-i(\alpha x + \beta y)} dxdy = \overline{u}_i(\alpha, \beta, z) \tag{3.39a}$$

$$\frac{1}{(2\pi)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial u_i(x,y,z)}{\partial x(y)} e^{-i(\alpha x + \beta y)} dxdy = i\alpha(\beta) \overline{u}_i(\alpha, \beta, z) \tag{3.39b}$$

$$\frac{1}{(2\pi)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 u_i(x,y,z)}{\partial x^2(y^2)} e^{-i(\alpha x + \beta y)} dxdy = -\alpha^2(\beta^2) \overline{u}_i(\alpha, \beta, z) \tag{3.39c}$$

Hence, the double Fourier transforms for point forces F_i are given as:

$$\overline{F}_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_i e^{-i(\alpha x + \beta y)} dxdy \quad (i=x,\ y,\ z) \tag{3.40}$$

Using Eqs. (3.39a)-(3.39c) and Eq. (3.40), the Navier-Cauchy equations (Eqs. (3.15a)-(3.15c)) can be simplified as the following system of linear ordinary differential equations:

$$\{a_{11}\alpha^2 + a_{66}\beta^2 - a_{55}\frac{d^2}{dz^2} + 2ia_{56}\beta\frac{d}{dz}\}\overline{u}_x(\alpha,\beta,z)$$
$$+\{(a_{12}+a_{66})\alpha\beta + i(a_{14}+a_{56})\alpha\frac{d}{dz}\}\overline{u}_y(\alpha,\beta,z) \tag{3.41a}$$
$$+\{(a_{14}+a_{56})\alpha\beta + i(a_{13}+a_{55})\alpha\frac{d}{dz}\}\overline{u}_z(\alpha,\beta,z) = \overline{F}_x$$

$$\{(a_{12}+a_{66})\alpha\beta + i(a_{14}+a_{56})\alpha\frac{d}{dz}\}\overline{u}_x(\alpha,\beta,z)$$
$$+\{a_{66}\alpha^2 + a_{22}\beta^2 - a_{44}\frac{d^2}{dz^2} + 2ia_{24}\beta\frac{d}{dz}\}\overline{u}_y(\alpha,\beta,z) \tag{3.41b}$$
$$+\{a_{56}\alpha^2 + a_{24}\beta^2 - a_{34}\frac{d^2}{dz^2} + i(a_{23}+a_{44})\beta\frac{d}{dz}\}\overline{u}_z(\alpha,\beta,z) = \overline{F}_y$$

$$\{(a_{14}+a_{66})\alpha\beta + i(a_{13}+a_{55})\alpha\frac{d}{dz}\}\overline{u}_x(\alpha,\beta,z)$$
$$+\{a_{56}\alpha^2 + a_{24}\beta^2 - a_{34}\frac{d^2}{dz^2} + i(a_{23}+a_{44})\beta\frac{d}{dz}\}\overline{u}_y(\alpha,\beta,z) \tag{3.41c}$$
$$+\{a_{55}\alpha^2 + a_{44}\beta^2 - a_{33}\frac{d^2}{dz^2} + 2ia_{34}\beta\frac{d}{dz}\}\overline{u}_z(\alpha,\beta,z) = \overline{F}_z$$

The stress components of $\sigma_{xx}(x,y,z)$, $\sigma_{yy}(x,y,z)$, $\sigma_{zz}(x,y,z)$, $\tau_{yz}(x,y,z)$, $\tau_{zx}(x,y,z)$ and $\tau_{xy}(x,y,z)$ (Eqs.(3.12a)-(3.12f)) are performed by the double Fourier transforms with respect to *x* and *y*. Then,

$$\overline{\sigma}_{xx}(\alpha,\beta,z) = -i\alpha a_{11}\overline{u}_x(\alpha,\beta,z) - i\beta a_{12}\overline{u}_y(\alpha,\beta,z) + a_{13}\frac{d\overline{u}_z(\alpha,\beta,z)}{dz}$$
$$+ a_{14}(\frac{d\overline{u}_y(\alpha,\beta,z)}{dz} - i\beta\overline{u}_z(\alpha,\beta,z)) \tag{3.42a}$$

$$\overline{\sigma}_{yy}(\alpha,\beta,z) = -i\alpha a_{12}\overline{u}_x(\alpha,\beta,z) - i\beta a_{22}\overline{u}_y(\alpha,\beta,z) + a_{23}\frac{d\overline{u}_z(\alpha,\beta,z)}{dz}$$
$$+ a_{24}(\frac{d\overline{u}_y(\alpha,\beta,z)}{dz} - i\beta\overline{u}_z(\alpha,\beta,z)) \tag{3.42b}$$

$$\bar{\sigma}_{zz}(\alpha,\beta,z) = -i\alpha a_{13}\bar{u}_x(\alpha,\beta,z) - i\beta a_{23}\bar{u}_y(\alpha,\beta,z) + a_{33}\frac{d\bar{u}_z(\alpha,\beta,z)}{dz} + a_{34}\left(\frac{d\bar{u}_y(\alpha,\beta,z)}{dz} - i\beta\bar{u}_z(\alpha,\beta,z)\right) \tag{3.42c}$$

$$\bar{\tau}_{yz}(\alpha,\beta,z) = -i\alpha a_{14}\bar{u}_x(\alpha,\beta,z) - i\beta a_{24}\bar{u}_y(\alpha,\beta,z) + a_{34}\frac{d\bar{u}_z(\alpha,\beta,z)}{dz} + a_{44}\left(\frac{d\bar{u}_y(\alpha,\beta,z)}{dz} - i\beta\bar{u}_z(\alpha,\beta,z)\right) \tag{3.42d}$$

$$\bar{\tau}_{zx}(\alpha,\beta,z) = a_{55}\left(\frac{d\bar{u}_x(\alpha,\beta,z)}{dz} - i\alpha\bar{u}_z(\alpha,\beta,z)\right) - ia_{56}\left(\beta\bar{u}_x(\alpha,\beta,z) + \alpha\bar{u}_y(\alpha,\beta,z)\right) \tag{3.42e}$$

$$\bar{\tau}_{xy}(\alpha,\beta,z) = a_{56}\left(\frac{d\bar{u}_x(\alpha,\beta,z)}{dz} - i\alpha\bar{u}_z(\alpha,\beta,z)\right) - ia_{66}\left(\beta\bar{u}_x(\alpha,\beta,z) + \alpha\bar{u}_y(\alpha,\beta,z)\right) \tag{3.42f}$$

To obtain the general solution of Eqs. (3.41a)-(3.41c), the homogeneous solution are solved first. Suppose that the solutions of the homogeneous equation can be expressed as the exponential function, namely,

$$\bar{u}_i(\alpha,\beta,z) = A_i(\alpha,\beta)e^{uz}\,. \tag{3.43}$$

We thus set A_i=1. then,

$$\bar{u}_i(\alpha,\beta,z) = e^{u\,z} \tag{3.44}$$

Substituting Eq. (3.44) and it derivatives $\frac{d\bar{u}_i(\alpha,\beta,z)}{dz} = ue^{u\,z}$. and $\frac{d^2\bar{u}_i(\alpha,\beta,z)}{dz^2} = u^2e^{u\,z}$ into Eqs. (3.41a)-(3.41c) and regrouping the system of homogeneous ordinary differential equations, we obtain

$$\det\left[d_{ij}\right] = \det\begin{bmatrix} d_{11}(\alpha,\beta,u) & d_{12}(\alpha,\beta,u) & d_{13}(\alpha,\beta,u) \\ d_{21}(\alpha,\beta,u) & d_{22}(\alpha,\beta,u) & d_{23}(\alpha,\beta,u) \\ d_{31}(\alpha,\beta,u) & d_{32}(\alpha,\beta,u) & d_{33}(\alpha,\beta,u) \end{bmatrix}$$

$$= -a_2 a_5^2 \prod_{i=1}^{3} A_i\{(iu)^2 + \alpha^2 + \beta^2 - [(iu)\cos\phi - \beta\sin\phi]^2\} + [(iu)\cos\phi - \beta\sin\phi]^2 = 0 \tag{3.45}$$

where

$$d_{11}(\alpha,\beta,u) = a_{11}\alpha^2 + a_{66}\beta^2 + a_{55}(iu)^2 + 2a_{56}\beta(iu) \tag{3.46a}$$

$$d_{12}(\alpha,\beta,u) = d_{21}(\alpha,\beta,u) = (a_{12} + a_{66})\alpha\beta + (a_{14} + a_{56})\alpha(iu) \tag{3.46b}$$

$$d_{13}(\alpha,\beta,u) = d_{31}(\alpha,\beta,u) = (a_{14} + a_{56})\alpha\beta + (a_{13} + a_{55})\alpha(iu) \tag{3.46c}$$

$$d_{22}(\alpha,\beta,u) = a_{66}\alpha^2 + a_{22}\beta^2 + a_{44}(iu)^2 + 2a_{24}\beta(iu) \tag{3.46d}$$

$$d_{23}(\alpha,\beta,u) = d_{32}(\alpha,\beta,s) = a_{56}\alpha^2 + a_{24}\beta^2 + a_{34}(iu)^2 + (a_{23} + a_{44})\beta(iu) \tag{3.46e}$$

$$d_{33}(\alpha,\beta,u) = a_{55}\alpha^2 + a_{44}\beta^2 + a_{33}(iu)^2 + 2a_{34}\beta(iu) \tag{3.46f}$$

The characteristic equation of Eq. (3.45) has six roots. The details and the physical basis of which are given in Appendix B. The characteristic roots can be expressed as:

$$u_1 = \frac{-i\beta\sin\phi\cos\phi(-1+A_1) - \sqrt{A_1(\beta^2 + \alpha^2(\cos^2\phi + A_1\sin^2\phi))}}{\cos^2\phi + A_1\sin^2\phi} \tag{3.47a}$$

$$u_2 = \frac{-i\beta\sin\phi\cos\phi(-1+A_2) - \sqrt{A_2(\beta^2 + \alpha^2(\cos^2\phi + A_2\sin^2\phi))}}{\cos^2\phi + A_2\sin^2\phi} \tag{3.47b}$$

$$u_3 = \frac{-i\beta\sin\phi\cos\phi(-1+A_3) - \sqrt{A_3(\beta^2 + \alpha^2(\cos^2\phi + A_3\sin^2\phi))}}{\cos^2\phi + A_3\sin^2\phi} \tag{3.47c}$$

$$u_4 = \frac{-i\beta\sin\phi\cos\phi(-1+A_1) + \sqrt{A_1(\beta^2 + \alpha^2(\cos^2\phi + A_1\sin^2\phi))}}{\cos^2\phi + A_1\sin^2\phi} \tag{3.47d}$$

$$u_5 = \frac{-i\beta\sin\phi\cos\phi(-1+A_2)+\sqrt{A_2(\beta^2+\alpha^2(\cos^2\phi+A_2\sin^2\phi))}}{\cos^2\phi+A_2\sin^2\phi} \tag{3.47e}$$

$$u_6 = \frac{-i\beta\sin\phi\cos\phi(-1+A_3)+\sqrt{A_3(\beta^2+\alpha^2(\cos^2\phi+A_3\sin^2\phi))}}{\cos^2\phi+A_3\sin^2\phi} \tag{3.47f}$$

where the real part of the $\{u_1,\ u_2\ ,\ u_3\}$ are negative and $\{u_4,\ u_5\ ,\ u_6\}$ are positive, and,

$$A_1 = \frac{a_4}{a_5} \tag{3.48a}$$

$$A_2 = \frac{1}{2}\left[\frac{a_5^2+a_1a_2-a_3^2}{a_2a_5}+\left\{\left(\frac{a_5^2+a_1a_2-a_3^2}{a_2a_5}\right)^2-4\frac{a_1}{a_2}\right\}^{\frac{1}{2}}\right] \tag{3.48b}$$

$$A_3 = \frac{1}{2}\left[\frac{a_5^2+a_1a_2-a_3^2}{a_2a_5}-\left\{\left(\frac{a_5^2+a_1a_2-a_3^2}{a_2a_5}\right)^2-4\frac{a_1}{a_2}\right\}^{\frac{1}{2}}\right] \tag{3.48c}$$

By a linear combination of 6 functions $u_1,\ u_2,\ u_3,\ u_4,\ u_5$ and u_6, the expressions for $\bar{u}_{x(H)}(\alpha,\beta,z)$, $\bar{u}_{y(H)}(\alpha,\beta,z)$, and $\bar{u}_{z(H)}(\alpha,\beta,z)$ are:

$$\bar{u}_{x(H)}(\alpha,\beta,z) = \sum_{j=1}^{6} A_x^j e^{u_j z} \tag{3.49a}$$

$$\bar{u}_{y(H)}(\alpha,\beta,z) = \sum_{j=1}^{6} A_y^j e^{u_j z} \tag{3.49b}$$

$$\bar{u}_{z(H)}(\alpha,\beta,z) = \sum_{j=1}^{6} A_z^j e^{u_j z} \tag{3.49c}$$

where A_x^i, A_y^i and A_z^i (i=1-6) are the undetermined coefficients.

In order to derive the solution of the homogeneous equation (Eqs. (3.41a)-(3.41c)), we redefine the three displacement functions (Eqs. (3.49a)-(3.49c)). In the lower half-space, the $\bar{u}_{x(H)}(\alpha,\beta,z)$, $\bar{u}_{y(H)}(\alpha,\beta,z)$, $\bar{u}_{z(H)}(\alpha,\beta,z)$, A_x^j, A_y^j and A_z^j of Eqs. (3.49a)-(3.49c) would express as $\bar{u}_{x1(H)}(\alpha,\beta,z)$, $\bar{u}_{y1(H)}(\alpha,\beta,z)$, $\bar{u}_{z1(H)}(\alpha,\beta,z)$, A_{x1}^j, A_{y1}^j, and A_{z1}^j. Similarly, they can be expressed as $\bar{u}_{x2(H)}(\alpha,\beta,z)$, $\bar{u}_{y2(H)}(\alpha,\beta,z)$, $\bar{u}_{z2(H)}(\alpha,\beta,z)$, A_{x2}^j, A_{y2}^j, and A_{z2}^j in the upper half-space. Hence,

for $z > 0$ (region 1, as shown in Fig.3.1),

$$\bar{u}_{x1(H)}(\alpha,\beta,z) = A_{x1}^1 e^{u_1 z} + A_{x1}^2 e^{u_2 z} + A_{x1}^3 e^{u_3 z} + A_{x1}^4 e^{u_4 z} + A_{x1}^5 e^{u_5 z} + A_{x1}^6 e^{u_6 z} \quad (3.50a)$$

$$\bar{u}_{y1(H)}(\alpha,\beta,z) = A_{y1}^1 e^{u_1 z} + A_{y1}^2 e^{u_2 z} + A_{y1}^3 e^{u_3 z} + A_{y1}^4 e^{u_4 z} + A_{y1}^5 e^{u_5 z} + A_{y1}^6 e^{u_6 z} \quad (3.50b)$$

$$\bar{u}_{z1(H)}(\alpha,\beta,z) = A_{z1}^1 e^{u_1 z} + A_{z1}^2 e^{u_2 z} + A_{z1}^3 e^{u_3 z} + A_{z1}^4 e^{u_4 z} + A_{z1}^5 e^{u_5 z} + A_{z1}^6 e^{u_6 z} \quad (3.50c)$$

and for $z < 0$ (region 2, as also depicted in Fig.3.1),

$$\bar{u}_{x2(H)}(\alpha,\beta,z) = A_{x2}^1 e^{u_1 z} + A_{x2}^2 e^{u_2 z} + A_{x2}^3 e^{u_3 z} + A_{x2}^4 e^{u_4 z} + A_{x2}^5 e^{u_5 z} + A_{x2}^6 e^{u_6 z} \quad (3.51a)$$

$$\bar{u}_{y2(H)}(\alpha,\beta,z) = A_{y2}^1 e^{u_1 z} + A_{y2}^2 e^{u_2 z} + A_{y2}^3 e^{u_3 z} + A_{y2}^4 e^{u_4 z} + A_{y2}^5 e^{u_5 z} + A_{y2}^6 e^{u_6 z} \quad (3.51b)$$

$$\bar{u}_{z2(H)}(\alpha,\beta,z) = A_{z2}^1 e^{u_1 z} + A_{z2}^2 e^{u_2 z} + A_{z2}^3 e^{u_3 z} + A_{z2}^4 e^{u_4 z} + A_{z2}^5 e^{u_5 z} + A_{z2}^6 e^{u_6 z} \quad (3.51c)$$

Now, let:

$$\frac{A_{x1}^j}{D_{11}(\alpha,\beta,u_j)} = \frac{A_{y1}^j}{D_{21}(\alpha,\beta,u_j)} = \frac{A_{z1}^j}{D_{31}(\alpha,\beta,u_j)} \quad (j\text{=1-6}) \quad (3.52a)$$

$$\frac{A_{x2}^j}{D_{11}(\alpha,\beta,u_j)} = \frac{A_{y2}^j}{D_{21}(\alpha,\beta,u_j)} = \frac{A_{z2}^j}{D_{31}(\alpha,\beta,u_j)} \quad (j\text{=1-6}) \quad (3.52b)$$

where D_{ij} (i, j=1-3) are the cofactors of the third–order determinant $\det\lfloor d_{ij} \rfloor$, which are presented in Appendix C.

$$D_{11}(\alpha,\beta,u_j)=det\begin{bmatrix} d_{22}(\alpha,\beta,u_j) & d_{23}(\alpha,\beta,u_j) \\ d_{32}(\alpha,\beta,u_j) & d_{33}(\alpha,\beta,u_j) \end{bmatrix} \tag{3.53a}$$

$$D_{12}(\alpha,\beta,u_j)=D_{21}(\alpha,\beta,u_j)=-det\begin{bmatrix} d_{12}(\alpha,\beta,u_j) & d_{13}(\alpha,\beta,u_j) \\ d_{32}(\alpha,\beta,u_j) & d_{33}(\alpha,\beta,u_j) \end{bmatrix} \tag{3.53b}$$

$$D_{13}(\alpha,\beta,u_j)=D_{31}(\alpha,\beta,u_j)=det\begin{bmatrix} d_{12}(\alpha,\beta,u_j) & d_{13}(\alpha,\beta,u_j) \\ d_{22}(\alpha,\beta,u_j) & d_{23}(\alpha,\beta,u_j) \end{bmatrix} \tag{3.53c}$$

$$D_{22}(\alpha,\beta,u_j)=det\begin{bmatrix} d_{11}(\alpha,\beta,u_j) & d_{13}(\alpha,\beta,u_j) \\ d_{31}(\alpha,\beta,u_j) & d_{33}(\alpha,\beta,u_j) \end{bmatrix} \tag{3.53d}$$

$$D_{23}(\alpha,\beta,u_j)=D_{32}(\alpha,\beta,u_j)=-det\begin{bmatrix} d_{11}(\alpha,\beta,u_j) & d_{13}(\alpha,\beta,u_j) \\ d_{21}(\alpha,\beta,u_j) & d_{23}(\alpha,\beta,u_j) \end{bmatrix} \tag{3.53e}$$

$$D_{33}(\alpha,\beta,u_j)=det\begin{bmatrix} d_{11}(\alpha,\beta,u_j) & d_{12}(\alpha,\beta,u_j) \\ d_{21}(\alpha,\beta,u_j) & d_{22}(\alpha,\beta,u_j) \end{bmatrix} \tag{3.53f}$$

Since only D_{11}, D_{21}, and D_{31} exist in Eqs. (3.52a)-(3.52b), the symbols of them can be simplified as:

$$D_{11}(\alpha,\beta,u_j)=D_{11}^{j},\quad D_{21}(\alpha,\beta,u_j)=D_{21}^{j},\quad D_{31}(\alpha,\beta,u_j)=D_{31}^{j} \tag{3.54}$$

CHAPTER Ⅳ

THREE DIMENSIONAL ELASTIC SOLUTIONS OF A TRANSVERSELY ISOTROPIC FULL-SPACE SUBJECTED TO POINT LOADS

The analytical solutions for displacements and stresses in an inclined transversely isotropic full-space subjected to a point load in the medium are derived in Chapters 4. To present the solutions, the ordinary differential equations reduced from partial differential equations by Double Fourier transform respect to variables of x and y are solved first. Three distinct approaches as shown in figure 1.1 were used to derive the solutions in Sec 4.1-4.3. In Sec 4.1(traditional method), we consider the nonhomogeneous part of the ordinary differential equations and solve the homogeneous and particular solution of Eqs. (3.41a)-(3.41c). In Sec.4.2 (imaginary space method), we separate the full-space into three regions of $-\infty < z < 0^-$ (region 2) , $0^- < z < 0^+$ (imaginary space), and $0^+ < z < +\infty$ (region 1) , the point load force is in the region of $0^- < z < 0^+$. In regions of $-\infty < z < 0^-$ and $0^+ < z < +\infty$, the right-hand side of Eqs. (3.41a)-(3.41c) does not exist, the equilibrium equations are homogeneous linear equations. Hence, we can solve the boundary-value problem consisting of the three regions. In Sec.4.3(algebraic equation method), the Fourier transform respect to variables of z can reduce the aforementioned ordinary differential equations (Eqs. (3.41a)-(3.41c)) to algebraic equations. This method, there are three time of Fourier transform respect to variables of x, y and z, is also called Triple Fourier transform method.

4.1 Traditional Method

In full-space, the coefficients A_{x1}^{j}, A_{y1}^{j}, and A_{z1}^{j} (j=1-6) of Eqs. (3.50a)-(3.50c) can be determined by assuming the displacements in region 1, u_{x1}, u_{y1}, and u_{z1} must be finite when z is approaching to ∞. The real part of the $\{u_4, u_5, u_6\}$ are positive, therefore, $A_{x1}^{4} = A_{x1}^{5} = A_{x1}^{6} = 0$, $A_{y1}^{4} = A_{y1}^{5} = A_{y1}^{6} = 0$, and $A_{z1}^{4} = A_{z1}^{5} = A_{z1}^{6} = 0$. Likewise, in region 2, u_{x2}, u_{y2}, and u_{z2} also must be finite when z is approaching to -∞. The real part of the $\{u_1, u_2, u_3\}$ are negative, hence, $A_{x2}^{1} = A_{x2}^{2} = A_{x2}^{3} = 0$, $A_{y2}^{1} = A_{y2}^{2} = A_{y2}^{3} = 0$, and $A_{z2}^{1} = A_{z2}^{2} = A_{z2}^{3} = 0$.

In order to solve the homogeneous solutions, Eqs. (3.41a)-(3.41c), we define the costants of C_{d2}^{j} and C_{u2}^{j} from Eqs. (3.52a)-(3.52c).

$$\frac{A_{x1}^{j}}{D_{11}(\alpha,\beta,u_j)} = \frac{A_{y1}^{j}}{D_{21}(\alpha,\beta,u_j)} = \frac{A_{z1}^{j}}{D_{31}(\alpha,\beta,u_j)} = C_{d2}^{j} \quad (j\text{=1-3}) \tag{4.01a}$$

$$\frac{A_{x2}^{j}}{D_{11}(\alpha,\beta,u_j)} = \frac{A_{y2}^{j}}{D_{21}(\alpha,\beta,u_j)} = \frac{A_{z2}^{j}}{D_{31}(\alpha,\beta,u_j)} = C_{u2}^{j} \quad (j\text{=4-6}) \tag{4.01b}$$

Adopting the costants of C_{d2}^{j} and C_{u2}^{j}, Eqs. (3.50a)-(3.50c) and Eqs. (3.51a)-(3.51c) can be re-written as follows:

for $z > 0$ (region 1, as shown in Fig.3.1),

$$\bar{u}_{x1(H)}(\alpha,\beta,z) = C_{d2}^{1} D_{11}^{1} e^{u_1 z} + C_{d2}^{2} D_{11}^{2} e^{u_2 z} + C_{d2}^{3} D_{11}^{3} e^{u_3 z} \tag{4.02a}$$

$$\bar{u}_{y1(H)}(\alpha,\beta,z) = C_{d2}^{1} D_{21}^{1} e^{u_1 z} + C_{d2}^{2} D_{21}^{2} e^{u_2 z} + C_{d2}^{3} D_{21}^{3} e^{u_3 z} \tag{4.02b}$$

$$\bar{u}_{z1(H)}(\alpha,\beta,z) = C_{d2}^{1} D_{31}^{1} e^{u_1 z} + C_{d2}^{2} D_{31}^{2} e^{u_2 z} + C_{d2}^{3} D_{31}^{3} e^{u_3 z} \tag{4.02c}$$

, and for $z < 0$ (region 2, as also depicted in Fig.3.1),

$$\bar{u}_{x2(H)}(\alpha,\beta,z) = C_{u2}^{4} D_{11}^{4} e^{u_4 z} + C_{u2}^{5} D_{11}^{5} e^{u_5 z} + C_{u2}^{6} D_{11}^{6} e^{u_6 z} \tag{4.03a}$$

$$\bar{u}_{y2(H)}(\alpha,\beta,z) = C_{u2}^{4} D_{21}^{4} e^{u_4 z} + C_{u2}^{5} D_{21}^{5} e^{u_5 z} + C_{u2}^{6} D_{21}^{6} e^{u_6 z} \tag{4.03b}$$

$$\bar{u}_{z2(H)}(\alpha,\beta,z) = C_{u2}^{4} D_{31}^{4} e^{u_4 z} + C_{u2}^{5} D_{31}^{5} e^{u_5 z} + C_{u2}^{6} D_{31}^{6} e^{u_6 z} \tag{4.03c}$$

Based on the homogeneous solutions, we can assume the particular solutions with the forms as:

$$\bar{u}_{x(P)}(\alpha,\beta,z) = \sum_{j=1}^{6} B_x^j(z) e^{u_j z} \tag{4.04a}$$

$$\bar{u}_{y(P)}(\alpha,\beta,z) = \sum_{j=1}^{6} B_y^j(z) e^{u_j z} \tag{4.04b}$$

$$\bar{u}_{z(P)}(\alpha,\beta,z) = \sum_{j=1}^{6} B_z^j(z) e^{u_j z} \tag{4.04c}$$

The coefficients B_x^j, B_y^j, and B_z^j $(\mathrm{j} = 1 \sim 6)$ can be determined by the method of variation of parameters (Hildebrand, 1976). The point loads (F_x, F_y, F_z) acting at the origin point of the co-ordinate system can be described as Eqs. (3.18a)-(3.18c). Following the approaches for homogeneous solutions, Eqs. (4.04a)-(4.04c) are expanded as:

for $z > 0$ (region 1, as shown in Fig. 3.1),

$$\bar{u}_{x1(P)}(\alpha,\beta,z) = B_x^1 e^{u_1 z} + B_x^2 e^{u_2 z} + B_x^3 e^{u_3 z} \tag{4.05a}$$

$$\bar{u}_{y1(P)}(\alpha,\beta,z) = B_y^1 e^{u_1 z} + B_y^2 e^{u_2 z} + B_y^3 e^{u_3 z} \tag{4.05b}$$

$$\bar{u}_{z1(P)}(\alpha,\beta,z) = B_z^1 e^{u_1 z} + B_z^2 e^{u_2 z} + B_z^3 e^{u_3 z} \tag{4.05c}$$

and for $z<0$ (region 2, as also depicted in Fig. 3.1),

$$\bar{u}_{x2(P)}(\alpha,\beta,z)=-B_x^4e^{u_4z}-B_x^5e^{u_5z}-B_x^6e^{u_6z} \tag{4.06a}$$

$$\bar{u}_{y2(P)}(\alpha,\beta,z)=-B_y^4e^{u_4z}-B_y^5e^{u_5z}-B_y^6e^{u_6z} \tag{4.06b}$$

$$\bar{u}_{z2(P)}(\alpha,\beta,z)=-B_z^4e^{u_4z}-B_z^5e^{u_5z}-B_z^6e^{u_6z} \tag{4.06c}$$

where

$$B_x^j=\frac{P_xD_{11}(u_j)+P_yD_{12}(u_j)+P_zD_{13}(u_j)}{2\pi m_tU_j} \tag{4.07a}$$

$$B_y^j=\frac{P_xD_{21}(u_j)+P_yD_{22}(u_j)+P_zD_{23}(u_j)}{2\pi m_tU_j} \tag{4.07b}$$

$$B_z^j=\frac{P_xD_{31}(u_j)+P_yD_{32}(u_j)+P_zD_{33}(u_j)}{2\pi m_tU_j} \tag{4.07c}$$

and

$$\begin{aligned} m_t &= a_{55}(a_{33}a_{44}-a_{34}^2) \\ &= a_2a_5^2(\cos^2\phi+A_1\sin^2\phi)(\cos^2\phi+A_2\sin^2\phi)(\cos^2\phi+A_3\sin^2\phi) \end{aligned} \tag{4.08}$$

$$U_j(\alpha,\beta)=\frac{\partial[(u-u_1)(u-u_2)(u-u_3)(u-u_4)(u-u_5)(u-u_6)]}{\partial u},u=u_j \tag{4.09}$$

Introducing an imaginary plane along $z=0$, to separate the full-space into two half-spaces, one is $0^+<z<\infty$ (region 1 in Fig. 3.1), and the other is $-\infty<z<0^-$ (region 2 in Fig. 3.1). In the lower half-space, the Eqs. (3.42a)-(3.42f) would express as $\bar{\sigma}_{xx1}(\alpha,\beta,z)$, $\bar{\sigma}_{yy1}(\alpha,\beta,z)$, $\bar{\sigma}_{zz1}(\alpha,\beta,z)$, $\bar{\tau}_{yz1}(\alpha,\beta,z)$, $\bar{\tau}_{zx1}(\alpha,\beta,z)$ and $\bar{\tau}_{xy1}(\alpha,\beta,z)$. Similarly, Eqs. (3.42a)-(3.42f) can be expressed as $\bar{\sigma}_{xx2}(\alpha,\beta,z)$, $\bar{\sigma}_{yy2}(\alpha,\beta,z)$,

$\bar{\sigma}_{zz2}(\alpha,\beta,z)$, $\bar{\tau}_{yz2}(\alpha,\beta,z)$, $\bar{\tau}_{zx2}(\alpha,\beta,z)$ and $\bar{\tau}_{xy2}(\alpha,\beta,z)$ in the upper half-space. In this case, we may rewrite Eqs. (3.42a)-(3.42f) as

$$\begin{aligned}\bar{\sigma}_{xxj}(\alpha,\beta,z) &= -i\alpha a_{11}\bar{u}_{xj}(\alpha,\beta,z) - i\beta a_{12}\bar{u}_{yj}(\alpha,\beta,z) + a_{13}\frac{d\bar{u}_{zj}(\alpha,\beta,z)}{dz} \\ &\quad + a_{14}\left(\frac{d\bar{u}_{yj}(\alpha,\beta,z)}{dz} - i\beta\bar{u}_{zj}(\alpha,\beta,z)\right)\end{aligned} \tag{4.10a}$$

$$\begin{aligned}\bar{\sigma}_{yyj}(\alpha,\beta,z) &= -i\alpha a_{12}\bar{u}_{xj}(\alpha,\beta,z) - i\beta a_{22}\bar{u}_{yj}(\alpha,\beta,z) + a_{23}\frac{d\bar{u}_{zj}(\alpha,\beta,z)}{dz} \\ &\quad + a_{24}\left(\frac{d\bar{u}_{yj}(\alpha,\beta,z)}{dz} - i\beta\bar{u}_{zj}(\alpha,\beta,z)\right)\end{aligned} \tag{4.10b}$$

$$\begin{aligned}\bar{\sigma}_{zzj}(\alpha,\beta,z) &= -i\alpha a_{13}\bar{u}_{xj}(\alpha,\beta,z) - i\beta a_{23}\bar{u}_{yj}(\alpha,\beta,z) + a_{33}\frac{d\bar{u}_{zj}(\alpha,\beta,z)}{dz} \\ &\quad + a_{34}\left(\frac{d\bar{u}_{yj}(\alpha,\beta,z)}{dz} - i\beta\bar{u}_{zj}(\alpha,\beta,z)\right)\end{aligned} \tag{4.10c}$$

$$\begin{aligned}\bar{\tau}_{yzj}(\alpha,\beta,z) &= -i\alpha a_{14}\bar{u}_{xj}(\alpha,\beta,z) - i\beta a_{24}\bar{u}_{yj}(\alpha,\beta,z) + a_{34}\frac{d\bar{u}_{zj}(\alpha,\beta,z)}{dz} \\ &\quad + a_{44}\left(\frac{d\bar{u}_{yj}(\alpha,\beta,z)}{dz} - i\beta\bar{u}_{zj}(\alpha,\beta,z)\right)\end{aligned} \tag{4.10d}$$

$$\begin{aligned}\bar{\tau}_{zxj}(\alpha,\beta,z) &= a_{55}\left(\frac{d\bar{u}_{xj}(\alpha,\beta,z)}{dz} - i\alpha\bar{u}_{zj}(\alpha,\beta,z)\right) \\ &\quad - ia_{56}\left(\beta\bar{u}_{xj}(\alpha,\beta,z) + \alpha\bar{u}_{yj}(\alpha,\beta,z)\right)\end{aligned} \tag{4.10e}$$

$$\begin{aligned}\bar{\tau}_{xyj}(\alpha,\beta,z) &= a_{56}\left(\frac{d\bar{u}_{xj}(\alpha,\beta,z)}{dz} - i\alpha\bar{u}_{zj}(\alpha,\beta,z)\right) \\ &\quad - ia_{66}\left(\beta\bar{u}_{xj}(\alpha,\beta,z) + \alpha\bar{u}_{yj}(\alpha,\beta,z)\right)\end{aligned} \tag{4.10f}$$

where j=1, 2.

When the two half-space are ideally bonded at the interface $z=0$ such that the material becomes continuous across the interface, we have:

$$\tau_{zx1}(x,y,0^+)-\tau_{zx2}(x,y,0^-)=P_x\delta(x)\delta(y) \tag{4.11a}$$

$$\tau_{zy1}(x,y,0^+)-\tau_{zy2}(x,y,0^-)=P_y\delta(x)\delta(y) \tag{4.11b}$$

$$\sigma_{zz1}(x,y,0^+)-\sigma_{zz2}(x,y,0^-)=P_z\delta(x)\delta(y) \tag{4.11c}$$

$$u_{x1}(x,y,0^+)-u_{x2}(x,y,0^-)=0 \tag{4.11d}$$

$$u_{y1}(x,y,0^+)-u_{y2}(x,y,0^-)=0 \tag{4.11e}$$

$$u_{z1}(x,y,0^+)-u_{z2}(x,y,0^-)=0 \tag{4.11f}$$

where $-\infty<x<\infty$, $-\infty<y<\infty$. The subscripts 1 and 2 mean that $z=0$ plane is approaching to 0^+ and 0^-, respectively.

Following, the double Fourier transforms of Eqs. (4.11a)-(4.11f) can be expressed as:

$$\bar{\tau}_{zx1}(\alpha,\beta,0^+)-\bar{\tau}_{zx2}(\alpha,\beta,0^-)=\frac{P_x}{2\pi} \tag{4.12a}$$

$$\bar{\tau}_{zy1}(\alpha,\beta,0^+)-\bar{\tau}_{zy2}(\alpha,\beta,0^-)=\frac{P_y}{2\pi} \tag{4.12b}$$

$$\bar{\sigma}_{zz1}(\alpha,\beta,0^+)-\bar{\sigma}_{zz2}(\alpha,\beta,0^-)=\frac{P_z}{2\pi} \tag{4.12c}$$

$$\bar{u}_{x1}(\alpha,\beta,0^+)-\bar{u}_{x2}(\alpha,\beta,0^-)=0 \tag{4.12d}$$

$$\bar{u}_{y1}(\alpha,\beta,0^+)-\bar{u}_{y2}(\alpha,\beta,0^-)=0 \tag{4.12e}$$

$$\bar{u}_{z1}(\alpha,\beta,0^+)-\bar{u}_{z2}(\alpha,\beta,0^-)=0 \tag{4.12f}$$

The general solutions of nonhomogeneous equations are the sum of the forms as:

$$\bar{\tau}_{zx}(\alpha,\beta,z)=\bar{\tau}_{zx(H)}(\alpha,\beta,z)+\bar{\tau}_{zx(P)}(\alpha,\beta,z) \quad (4.13a)$$

$$\bar{\tau}_{zy}(\alpha,\beta,z)=\bar{\tau}_{zy(H)}(\alpha,\beta,z)+\bar{\tau}_{zy(P)}(\alpha,\beta,z) \quad (4.13b)$$

$$\bar{\sigma}_{zz}(\alpha,\beta,z)=\bar{\sigma}_{zz(H)}(\alpha,\beta,z)+\bar{\sigma}_{zz(P)}(\alpha,\beta,z) \quad (4.13c)$$

$$\bar{u}_{x}(\alpha,\beta,z)=\bar{u}_{x(H)}(\alpha,\beta,z)+\bar{u}_{x(P)}(\alpha,\beta,z), \quad (4.13d)$$

$$\bar{u}_{y}(\alpha,\beta,z)=\bar{u}_{y(H)}(\alpha,\beta,z)+\bar{u}_{y(P)}(\alpha,\beta,z), \quad (4.13e)$$

$$\bar{u}_{z}(\alpha,\beta,z)=\bar{u}_{z(H)}(\alpha,\beta,z)+\bar{u}_{z(P)}(\alpha,\beta,z), \quad (4.13f)$$

where $\bar{u}_{i(H)}(\alpha,\beta,z)$ are the corresponding homogeneous solutions, and $\bar{u}_{i(P)}(\alpha,\beta,z)$ are particular solutions.

Hence, from Eqs. (4.02a)-(4.02c), Eqs. (4.03a)-(4.03c), Eqs. (4.05a)-(4.05c), Eqs. (4.06a)-(4.06c), Eqs. (4.10a)-(4.10f) and Eqs. (4.13a)-(4.13f), the system of six linear equations (Eqs. (4.12a)-(4.12f)) has six undetermined coefficients C_{d2}^{1}, C_{d2}^{2}, C_{d2}^{3}, C_{u2}^{4}, C_{u2}^{5}, C_{u2}^{6}. These coefficients can be associated with $\lfloor f_{ij} \rfloor$ (i, j =1-6) as follows:

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \end{bmatrix} \begin{bmatrix} C_{d2}^{1} \\ C_{d2}^{2} \\ C_{d2}^{3} \\ C_{u2}^{4} \\ C_{u2}^{5} \\ C_{u2}^{6} \end{bmatrix} + \begin{bmatrix} \bar{\tau}_{zx2(P)}(\alpha,\beta,0^{+})-\bar{\tau}_{zx2(P)}(\alpha,\beta,0^{-}) \\ \bar{\tau}_{zy2(P)}(\alpha,\beta,0^{+})-\bar{\tau}_{zy2(P)}(\alpha,\beta,0^{-}) \\ \bar{\sigma}_{zz2(P)}(\alpha,\beta,0^{+})-\bar{\sigma}_{zz2(P)}(\alpha,\beta,0^{-}) \\ \bar{u}_{x2(P)}(\alpha,\beta,0^{+})-\bar{u}_{x2(P)}(\alpha,\beta,0^{-}) \\ \bar{u}_{y2(P)}(\alpha,\beta,0^{+})-\bar{u}_{y2(P)}(\alpha,\beta,0^{-}) \\ \bar{u}_{z2(P)}(\alpha,\beta,0^{+})-\bar{u}_{z2(P)}(\alpha,\beta,0^{-}) \end{bmatrix} = \frac{1}{2\pi} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4.14)$$

where f_{ij} (i, j=1-6) are presented in Appendix D.

Then, the six undetermined coefficients C_{d2}^{1}, C_{d2}^{2}, C_{d2}^{3}, C_{u2}^{4}, C_{u2}^{5}, C_{u2}^{6} can be determined. The boundary conditions of the full space consists of two parts, the first term and the second one in the left-hand side of Eq. (4.14) are the components of homogenous solutions and particular solutions, respectively. At the region of $0^{-} < z < 0^{+}$, the second term in the left–hand side of Eq. (4.14) can be solved as follows:

$$\begin{bmatrix} \bar{\tau}_{zx2(P)}(\alpha,\beta,0^{+}) - \bar{\tau}_{zx2(P)}(\alpha,\beta,0^{-}) \\ \bar{\tau}_{zy2(P)}(\alpha,\beta,0^{+}) - \bar{\tau}_{zy2(P)}(\alpha,\beta,0^{-}) \\ \bar{\sigma}_{zz2(P)}(\alpha,\beta,0^{+}) - \bar{\sigma}_{zz2(P)}(\alpha,\beta,0^{-}) \\ \bar{u}_{x2(P)}(\alpha,\beta,0^{+}) - \bar{u}_{x2(P)}(\alpha,\beta,0^{-}) \\ \bar{u}_{y2(P)}(\alpha,\beta,0^{+}) - \bar{u}_{y2(P)}(\alpha,\beta,0^{-}) \\ \bar{u}_{z2(P)}(\alpha,\beta,0^{+}) - \bar{u}_{z2(P)}(\alpha,\beta,0^{-}) \end{bmatrix} = \frac{1}{2\pi} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.15}$$

where Eq. (4.15) will be demonstrated in Appendix E.

Due to the determinant of f_{ij} (i, j =1-6) does not equal to zero, the undetermined coefficients C_{d2}^{1}, C_{d2}^{2}, C_{d2}^{3}, C_{u2}^{4}, C_{u2}^{5} and C_{u2}^{6} must be zero. Hence, $A_{x1}^{1} = A_{x1}^{2} = A_{x1}^{3} = 0$, $A_{y1}^{1} = A_{y1}^{2} = A_{y1}^{3} = 0$, $A_{z1}^{1} = A_{z1}^{2} = A_{z1}^{3} = 0$. $A_{x2}^{4} = A_{x2}^{5} = A_{x2}^{6} = 0$, $A_{y2}^{4} = A_{y2}^{5} = A_{y2}^{6} = 0$, and $A_{z2}^{4} = A_{z2}^{5} = A_{z2}^{6} = 0$.

In the method of particular solution for full-space, it is clear that the homogenous solutions do not contribute to the general solution of Eqs. (3.41a)-(3.41c). In the other word, the displacement and stress functions can be obtained alone from the particular solutions in the form:

for $z > 0$ (region 1, as shown in Fig. 3.1),

$$\bar{u}_{x1}(\alpha,\beta,z) = \bar{u}_{x1(P)}(\alpha,\beta,z) = B_x^{1} e^{u_1 z} + B_x^{2} e^{u_2 z} + B_x^{3} e^{u_3 z} \tag{4.16a}$$

$$\bar{u}_{y1}(\alpha,\beta,z)=\bar{u}_{y1(P)}(\alpha,\beta,z)=B_y^1e^{u_1z}+B_y^2e^{u_2z}+B_y^3e^{u_3z} \tag{4.16b}$$

$$\bar{u}_{z1}(\alpha,\beta,z)=\bar{u}_{z1(P)}(\alpha,\beta,z)=B_z^1e^{u_1z}+B_z^2e^{u_2z}+B_z^3e^{u_3z} \tag{4.16c}$$

$$\bar{\sigma}_{xx1}(\alpha,\beta,z)=\bar{\sigma}_{xx1(P)}(\alpha,\beta,z) \tag{4.16d}$$

$$\bar{\sigma}_{yy1}(\alpha,\beta,z)=\bar{\sigma}_{yy1(P)}(\alpha,\beta,z) \tag{4.16e}$$

$$\bar{\sigma}_{zz1}(\alpha,\beta,z)=\bar{\sigma}_{zz1(P)}(\alpha,\beta,z) \tag{4.16f}$$

$$\bar{\tau}_{yz1}(\alpha,\beta,z)=\bar{\tau}_{yz1(P)}(\alpha,\beta,z) \tag{4.16g}$$

$$\bar{\tau}_{zx1}(\alpha,\beta,z)=\bar{\tau}_{zx1(P)}(\alpha,\beta,z) \tag{4.16h}$$

$$\bar{\tau}_{xy1}(\alpha,\beta,z)=\bar{\tau}_{xy1(P)}(\alpha,\beta,z) \tag{4.16i}$$

and for $z<0$ (region 2, as also depicted in Fig. 3.1),

$$\bar{u}_{x2}(\alpha,\beta,z)=\bar{u}_{x2(P)}(\alpha,\beta,z)=-B_x^4e^{u_4z}-B_x^5e^{u_5z}-B_x^6e^{u_6z} \tag{4.17a}$$

$$\bar{u}_{y2}(\alpha,\beta,z)=\bar{u}_{y2(P)}(\alpha,\beta,z)=-B_y^4e^{u_4z}-B_y^5e^{u_5z}-B_y^6e^{u_6z} \tag{4.17b}$$

$$\bar{u}_{z2}(\alpha,\beta,z)=\bar{u}_{z2(P)}(\alpha,\beta,z)=-B_z^4e^{u_4z}-B_z^5e^{u_5z}-B_z^6e^{u_6z} \tag{4.17c}$$

$$\bar{\sigma}_{xx2}(\alpha,\beta,z)=\bar{\sigma}_{xx2(P)}(\alpha,\beta,z) \tag{4.17d}$$

$$\bar{\sigma}_{yy2}(\alpha,\beta,z)=\bar{\sigma}_{yy2(P)}(\alpha,\beta,z) \tag{4.17e}$$

$$\bar{\sigma}_{zz2}(\alpha,\beta,z)=\bar{\sigma}_{zz2(P)}(\alpha,\beta,z) \tag{4.17f}$$

$$\bar{\tau}_{yz2}(\alpha,\beta,z)=\bar{\tau}_{yz2(P)}(\alpha,\beta,z) \tag{4.17g}$$

$$\bar{\tau}_{zx2}(\alpha,\beta,z)=\bar{\tau}_{zx2(P)}(\alpha,\beta,z) \quad (4.17h)$$

$$\bar{\tau}_{xy2}(\alpha,\beta,z)=\bar{\tau}_{xy2(P)}(\alpha,\beta,z) \quad (4.17i)$$

4.2 Imaginary Space Method

In Fig. 4.1, we separate the full-space into three imaginary regions of $-\infty<z<0^-$, $0^-<z<0^+$, and $0^+<z<+\infty$. The point load force is in the region of $0^-<z<0^+$. In region of $-\infty<z<0^-$ and $0^+<z<+\infty$, the right-hand side of Eqs. (3.41a)-(3.41c) are not existed, the equilibrium equations are homogeneous linear equations. In the method of undetermined coefficients for full-space, it is clear that the particular solutions do not contribute to the general solution of Eqs. (3.41a)-(3.41c). In the other word, the displacement and stress functions can be obtained alone from the homogeneous solutions.

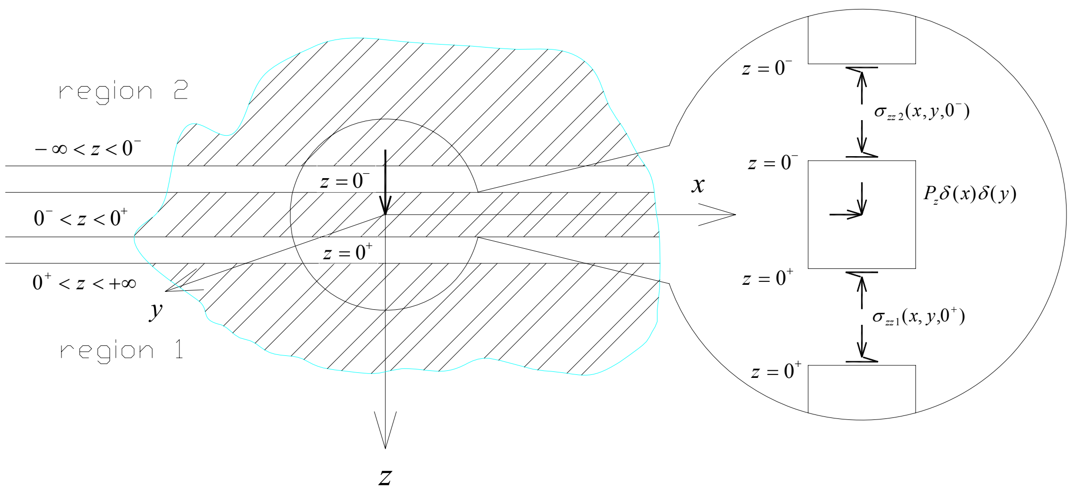


Fig. 4.1 Separate the full-space into three imaginary regions of $-\infty<z<0^-$, $0^-<z<0^+$ and $0^+<z<+\infty$

In order to differentiate varieties of homogeneous solutions of Eqs. (3.50a)-(3.50c) and Eqs. (3.51a)-(3.51c) in Sec 4.1.1 and Sec 4.1.2, we define the costants of C_d^j and C_u^j from the Eqs. (3.52a)-(3.52b).

$$\frac{A_{x1}^j}{D_{11}(\alpha,\beta,u_j)} = \frac{A_{y1}^j}{D_{21}(\alpha,\beta,u_j)} = \frac{A_{z1}^j}{D_{31}(\alpha,\beta,u_j)} = C_d^j \quad (j\text{=1-3}) \tag{4.18a}$$

$$\frac{A_{x2}^j}{D_{11}(\alpha,\beta,u_j)} = \frac{A_{y2}^j}{D_{21}(\alpha,\beta,u_j)} = \frac{A_{z2}^j}{D_{31}(\alpha,\beta,u_j)} = C_u^j \quad (j\text{=4-6}) \tag{4.18b}$$

By using the costants of C_d^j and C_u^j, Eqs. (3.50a)-(3.50c) and Eqs. (3.51a)-(3.51c) can obtain as follow:

for $z > 0^+$ (region 1, as shown in Fig. 3.2),

$$\bar{u}_{x1}(\alpha,\beta,z) = C_d^1 D_{11}^1 e^{u_1 z} + C_d^2 D_{11}^2 e^{u_2 z} + C_d^3 D_{11}^3 e^{u_3 z} \tag{4.19a}$$

$$\bar{u}_{y1}(\alpha,\beta,z) = C_d^1 D_{21}^1 e^{u_1 z} + C_d^2 D_{21}^2 e^{u_2 z} + C_d^3 D_{21}^3 e^{u_3 z} \tag{4.19b}$$

$$\bar{u}_{z1}(\alpha,\beta,z) = C_d^1 D_{31}^1 e^{u_1 z} + C_d^2 D_{31}^2 e^{u_2 z} + C_d^3 D_{31}^3 e^{u_3 z} \tag{4.19c}$$

and for $z < 0^-$ (region 2, as also depicted in Fig. 3.2),

$$\bar{u}_{x2}(\alpha,\beta,z) = C_u^4 D_{11}^4 e^{u_4 z} + C_u^5 D_{11}^5 e^{u_5 z} + C_u^6 D_{11}^6 e^{u_6 z} \tag{4.20a}$$

$$\bar{u}_{y2}(\alpha,\beta,z) = C_u^4 D_{21}^4 e^{u_4 z} + C_u^5 D_{21}^5 e^{u_5 z} + C_u^6 D_{21}^6 e^{u_6 z} \tag{4.20b}$$

$$\bar{u}_{z2}(\alpha,\beta,z) = C_u^4 D_{31}^4 e^{u_4 z} + C_u^5 D_{31}^5 e^{u_5 z} + C_u^6 D_{31}^6 e^{u_6 z} \tag{4.20c}$$

It is note that the difference between Eqs. (4.02a)-(4.02c), Eqs. (4.03a)-(4.03c) and Eqs. (4.19a)-(4.19c), Eqs. (4.20a)-(4.20c), Eqs. (4.02a)-(4.02c) and Eqs. (4.03a)-(4.03c) are the homogeneous solutions of the nonhomogeneous linear equations of the equilibrium equations, and Eqs. (4.19a)-(4.19c) and Eqs. (4.20a)-(4.20c) are the homogeneous solutions of the homogeneous linear equations part of the equilibrium equations.

Furthermore, considering the following pertinent continuity and discontinuity conditions at $z=0$ in the *(x,y,z)* and (α,β,z) domain are as same as the Eqs. (4.11a)-(4.11f) and Eqs. (4.12a)-(4.12f), respectively.

Once Eqs. (4.19a)-(4.19c) and Eqs. (4.20a)-(4.20c) are substituted into Eqs. (4.12a)-(4.12f), the six undetermined coefficients, C_d^1, C_d^2, C_d^3, C_u^4, C_u^5, C_u^6, can be associated with $\lfloor f_{ij} \rfloor$ $(i,j$ =1-6) as follows:

$$\left[f_{ij}\right]\begin{bmatrix} C_d^1 \\ C_d^2 \\ C_d^3 \\ C_u^4 \\ C_u^5 \\ C_u^6 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \end{bmatrix}\begin{bmatrix} C_d^1 \\ C_d^2 \\ C_d^3 \\ C_u^4 \\ C_u^5 \\ C_u^6 \end{bmatrix} = \frac{1}{2\pi}\begin{bmatrix} P_x \\ P_y \\ P_z \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.21}$$

where f_{ij} $(i,j$ =1-6) are presented in Appendix D.

The P_x, P_y, and P_z can be expressed as:

$$P_x = 2\pi\sum_{j=1}^{6} -i(a_{56}(\beta B_x^j + \alpha B_y^j) + a_{55}(\alpha B_z^j - iB_x^j u_j)) \tag{4.22a}$$

$$P_y = 2\pi\sum_{j=1}^{6} -i(\alpha a_{14} B_x^j + \beta a_{24} B_y^j + \beta a_{44} B_z^j) - (a_{44} B_y^j + a_{34} B_z^j) u_j \tag{4.22b}$$

$$P_z = 2\pi\sum_{j=1}^{6} -i(\alpha a_{13} B_x^j + \beta a_{23} B_y^j + \beta a_{34} B_z^j) - (a_{34} B_y^j + a_{33} B_z^j) u_j \tag{4.22c}$$

where B_x^j, B_y^j, B_z^j are same as the Eqs. (4.07a)-(4.07c) and the Eqs. (4.22a)-(4.22c) will be demonstrated in Appendix E.

In order to find the undetermined coefficients, C_d^1, C_d^2, C_d^3, C_u^4, C_u^5, and C_u^6. We can solve by the Cramer's rule, that is, C_d^1 can be obtained from $\lfloor f_{ij} \rfloor$ by replaceing the first column by the term in the right–hand side of Eq. (4.21). Only the first term (j=1) of Eqs. (4.22a)-(4.22c) make the determinant exist, and the term of j=2-6 must let the determinant equal to zero. So we find:

$$C_d^1 = \frac{P_x D_{11}(u_1) + P_y D_{12}(u_1) + P_z D_{13}(u_1)}{2\pi m_t U_1 D_{11}(u_1)} \tag{4.23a}$$

Identically, the aforementioned equations contain C_d^2, C_d^3, C_u^4, C_u^5 and C_u^6 may be written as:

$$C_d^2 = \frac{P_x D_{11}(u_2) + P_y D_{12}(u_2) + P_z D_{13}(u_2)}{2\pi m_t U_2 D_{11}(u_2)} \tag{4.23b}$$

$$C_d^3 = \frac{P_x D_{11}(u_3) + P_y D_{12}(u_3) + P_z D_{13}(u_3)}{2\pi m_t U_3 D_{11}(u_3)} \tag{4.23c}$$

$$C_u^4 = \frac{P_x D_{11}(u_4) + P_y D_{12}(u_4) + P_z D_{13}(u_4)}{2\pi m_t U_4 D_{11}(u_4)} \tag{4.23d}$$

$$C_u^5 = \frac{P_x D_{11}(u_5) + P_y D_{12}(u_5) + P_z D_{13}(u_5)}{2\pi m_t U_5 D_{11}(u_5)} \tag{4.23e}$$

$$C_u^6 = \frac{P_x D_{11}(u_6) + P_y D_{12}(u_6) + P_z D_{13}(u_6)}{2\pi m_t U_6 D_{11}(u_6)} \tag{4.23f}$$

The desired solutions, $\bar{\tau}_{zx}(\alpha,\beta,z)$, $\bar{\tau}_{zy}(\alpha,\beta,z)$, $\bar{\sigma}_{zz}(\alpha,\beta,z)$, $\bar{u}_x(\alpha,\beta,z)$, $\bar{u}_y(\alpha,\beta,z)$, and $\bar{u}_z(\alpha,\beta,z)$ can be obtained by substituting Eqs. (4.23a)-(4.23f) into

Eqs. (4.19a)-(4.19c) and Eqs. (4.20a)-(4.20c). They are the same as those from Eqs. (4.16a)-(4.16i) and Eqs. (4.17a)-(4.17i).

4.3 Algebraic Equation Method (Triple Fourier Transform)

In a full-space problem, $u_i(x,y,z)$ are set as the displacements in a homogeneous and linearly elastic continuum with domain $-\infty < x,y,z < \infty$. Then the triple Fourier transforms of the displacement components are applied to solve the equilibrium equations (Eqs. (3.15a)-(3.15c)). The step by step Fourier transforms for u_i (i=x, y, z) are written as:

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_i(x,y,z)e^{-i(\alpha x+\beta y+\gamma z)}dxdydz = \overline{U}_i(\alpha,\beta,\gamma) \tag{4.24a}$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial u_i(x,y,z)}{\partial x(y,z)}e^{-i(\alpha x+\beta y+\gamma z)}dxdydz = i\alpha(\beta,\gamma)\overline{U}_i(\alpha,\beta,\gamma) \tag{4.24b}$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 u_i(x,y,z)}{\partial x^2(y^2,z^2)}e^{-i(\alpha x+\beta y+\gamma z)}dxdydz = -\alpha^2(\beta^2,\gamma^2)\overline{U}_i(\alpha,\beta,\gamma) \tag{4.24c}$$

In addition, the Dirac delta function has the following property:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1 \tag{4.25}$$

,also the triple Fourier transforms of F_x, F_y, F_z (Eqs. (3.18a)-(3.18c)) can be obtained as:

$$\overline{F}_x = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_x\delta(x)\delta(y)\delta(z)e^{-i(\alpha x+\beta y+\gamma z)}dxdydz = \frac{P_x}{(2\pi)^{\frac{3}{2}}} \tag{4.26a}$$

$$\overline{F}_y = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_y\delta(x)\delta(y)\delta(z)e^{-i(\alpha x+\beta y+\gamma z)}dxdydz = \frac{P_y}{(2\pi)^{\frac{3}{2}}} \tag{4.26b}$$

$$\overline{F}_z = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_z \delta(x)\delta(y)\delta(z) e^{-i(\alpha x+\beta y+\gamma z)} dxdydz = \frac{P_z}{(2\pi)^{\frac{3}{2}}} \tag{4.26c}$$

According to Eqs. (4.24a)-(4.24c) and Eqs. (4.26a)-(4.26c), Eqs. (3.15a)-(3.15c) can be expressed as:

$$\left[d_{ij}\right]\begin{bmatrix}\overline{U}_x\\ \overline{U}_y\\ \overline{U}_z\end{bmatrix} = \begin{bmatrix} d_{11}(\alpha,\beta,\gamma) & d_{12}(\alpha,\beta,\gamma) & d_{13}(\alpha,\beta,\gamma)\\ d_{21}(\alpha,\beta,\gamma) & d_{22}(\alpha,\beta,\gamma) & d_{23}(\alpha,\beta,\gamma)\\ d_{31}(\alpha,\beta,\gamma) & d_{32}(\alpha,\beta,\gamma) & d_{33}(\alpha,\beta,\gamma)\end{bmatrix}\begin{bmatrix}\overline{U}_x(\alpha,\beta,\gamma)\\ \overline{U}_y(\alpha,\beta,\gamma)\\ \overline{U}_z(\alpha,\beta,\gamma)\end{bmatrix} = -\begin{bmatrix}\overline{F}_x\\ \overline{F}_y\\ \overline{F}_z\end{bmatrix} \tag{4.27}$$

where

$$d_{11}(\alpha,\beta,\gamma) = a_{11}\alpha^2 + a_{66}\beta^2 + a_{55}\gamma^2 + 2a_{56}\beta\gamma \tag{4.28a}$$

$$d_{12}(\alpha,\beta,\gamma) = d_{21}(\alpha,\beta,\gamma) = (a_{12}+a_{66})\alpha\beta + (a_{14}+a_{56})\alpha\gamma \tag{4.28b}$$

$$d_{13}(\alpha,\beta,\gamma) = d_{31}(\alpha,\beta,\gamma) = (a_{14}+a_{56})\alpha\beta + (a_{13}+a_{55})\alpha\gamma \tag{4.28c}$$

$$d_{22}(\alpha,\beta,\gamma) = a_{66}\alpha^2 + a_{22}\beta^2 + a_{44}\gamma^2 + 2a_{24}\beta\gamma \tag{4.28d}$$

$$d_{23}(\alpha,\beta,\gamma) = d_{32}(\alpha,\beta,\gamma) = a_{56}\alpha^2 + a_{24}\beta^2 + a_{34}\gamma^2 + (a_{23}+a_{44})\beta\gamma \tag{4.28e}$$

$$d_{33}(\alpha,\beta,\gamma) = a_{55}\alpha^2 + a_{44}\beta^2 + a_{33}\gamma^2 + 2a_{34}\beta\gamma \tag{4.28f}$$

Then, $\overline{U}_x(\alpha,\beta,\gamma)$, $\overline{U}_y(\alpha,\beta,\gamma)$ and $\overline{U}_z(\alpha,\beta,\gamma)$ can be simply obtained from Eq. (4.27) by matrix operations as:

$$\begin{aligned}\begin{bmatrix}\overline{U}_x(\alpha,\beta,\gamma)\\ \overline{U}_y(\alpha,\beta,\gamma)\\ \overline{U}_z(\alpha,\beta,\gamma)\end{bmatrix} &= -\frac{\left[D_{mn}(\alpha,\beta,\gamma)\right]}{D(\alpha,\beta,\gamma)} \times \begin{bmatrix}\overline{F}_x\\ \overline{F}_y\\ \overline{F}_z\end{bmatrix}\\ &= -\frac{1}{D(\alpha,\beta,\gamma)} \times \begin{bmatrix} D_{11}(\alpha,\beta,\gamma) & D_{12}(\alpha,\beta,\gamma) & D_{13}(\alpha,\beta,\gamma)\\ D_{21}(\alpha,\beta,\gamma) & D_{22}(\alpha,\beta,\gamma) & D_{23}(\alpha,\beta,\gamma)\\ D_{31}(\alpha,\beta,\gamma) & D_{32}(\alpha,\beta,\gamma) & D_{33}(\alpha,\beta,\gamma)\end{bmatrix} \times \begin{bmatrix}\overline{F}_x\\ \overline{F}_y\\ \overline{F}_z\end{bmatrix}\end{aligned} \tag{4.29}$$

From Eqs. (4.26a)-(4.26c) and Eq. (4.29), the three algebraic equations $\overline{U}_x(\alpha,\beta,\gamma)$, $\overline{U}_y(\alpha,\beta,\gamma)$ and $\overline{U}_z(\alpha,\beta,\gamma)$ can be obtained as follows:

$$\begin{aligned}\overline{U}_x(\alpha,\beta,\gamma) &= -\frac{\overline{F_x}D_{11}(\alpha,\beta,\gamma)+\overline{F_y}D_{12}(\alpha,\beta,\gamma)+\overline{F_z}D_{13}(\alpha,\beta,\gamma)}{D(\alpha,\beta,\gamma)} \\ &= -\frac{P_xD_{11}(\alpha,\beta,\gamma)+P_yD_{12}(\alpha,\beta,\gamma)+P_zD_{13}(\alpha,\beta,\gamma)}{(2\pi)^{\frac{3}{2}}D(\alpha,\beta,\gamma)}\end{aligned} \tag{4.30a}$$

$$\begin{aligned}\overline{U}_y(\alpha,\beta,\gamma) &= -\frac{\overline{F_x}D_{21}(\alpha,\beta,\gamma)+\overline{F_y}D_{22}(\alpha,\beta,\gamma)+\overline{F_z}D_{23}(\alpha,\beta,\gamma)}{D(\alpha,\beta,\gamma)} \\ &= -\frac{P_xD_{21}(\alpha,\beta,\gamma)+P_yD_{22}(\alpha,\beta,\gamma)+P_zD_{23}(\alpha,\beta,\gamma)}{(2\pi)^{\frac{3}{2}}D(\alpha,\beta,\gamma)}\end{aligned} \tag{4.30b}$$

$$\begin{aligned}\overline{U}_z(\alpha,\beta,\gamma) &= -\frac{\overline{F_x}D_{31}(\alpha,\beta,\gamma)+\overline{F_y}D_{32}(\alpha,\beta,\gamma)+\overline{F_z}D_{33}(\alpha,\beta,\gamma)}{D(\alpha,\beta,\gamma)} \\ &= -\frac{P_xD_{31}(\alpha,\beta,\gamma)+P_yD_{32}(\alpha,\beta,\gamma)+P_zD_{33}(\alpha,\beta,\gamma)}{(2\pi)^{\frac{3}{2}}D(\alpha,\beta,\gamma)}\end{aligned} \tag{4.30c}$$

$$\begin{aligned}D(\alpha,\beta,\gamma) &= \det\left[d_{ij}\right] = a_2a_5^2\prod_{i=1}^{3}A_i\{\alpha^2+\beta^2+\gamma^2-(\gamma\cos\phi-\beta\sin\phi)^2\}+(\gamma\cos\phi-\beta\sin\phi)^2 \\ &= -m_i(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma-\gamma_3)(\gamma-\gamma_4)(\gamma-\gamma_5)(\gamma-\gamma_6)\end{aligned} \tag{4.31}$$

The mean of $\prod_{i=1}^{3} f_i$ is the product f_i function start with *i*=1 to 3.

The characteristic equation (Eq. (4.31)) has six roots, and can be expressed as:

$$\gamma_1(\alpha,\beta) = \frac{-\beta\sin\phi\cos\phi(-1+A_1)+i\sqrt{A_1(\beta^2+\alpha^2(\cos^2\phi+A_1\sin^2\phi))}}{\cos^2\phi+A_1\sin^2\phi} = -iu_1(\alpha,\beta) \tag{4.32a}$$

$$\gamma_2(\alpha,\beta) = \frac{-\beta\sin\phi\cos\phi(-1+A_2)+i\sqrt{A_2(\beta^2+\alpha^2(\cos^2\phi+A_2\sin^2\phi))}}{\cos^2\phi+A_2\sin^2\phi} = -iu_2(\alpha,\beta) \tag{4.32b}$$

$$\gamma_3(\alpha,\beta) = \frac{-\beta\sin\phi\cos\phi(-1+A_3)+i\sqrt{A_3(\beta^2+\alpha^2(\cos^2\phi+A_3\sin^2\phi))}}{\cos^2\phi+A_3\sin^2\phi} = -iu_3(\alpha,\beta) \tag{4.32c}$$

$$\gamma_4(\alpha,\beta)=\frac{-\beta\sin\phi\cos\phi(-1+A_1)-i\sqrt{A_1(\beta^2+\alpha^2(\cos^2\phi+A_1\sin^2\phi))}}{\cos^2\phi+A_1\sin^2\phi}=-iu_4(\alpha,\beta) \tag{4.32d}$$

$$\gamma_5(\alpha,\beta)=\frac{-\beta\sin\phi\cos\phi(-1+A_2)-i\sqrt{A_2(\beta^2+\alpha^2(\cos^2\phi+A_2\sin^2\phi))}}{\cos^2\phi+A_2\sin^2\phi}=-iu_5(\alpha,\beta) \tag{4.32e}$$

$$\gamma_6(\alpha,\beta)=\frac{-\beta\sin\phi\cos\phi(-1+A_3)-i\sqrt{A_3(\beta^2+\alpha^2(\cos^2\phi+A_3\sin^2\phi))}}{\cos^2\phi+A_3\sin^2\phi}=-iu_6(\alpha,\beta) \tag{4.32f}$$

The $D_{mn}(\alpha,\beta,\gamma)$ can be obtained from $D_{mn}(\alpha,\beta,u)$, the relations of γ and u to take the place of $\gamma = iu$, which is expressed in Appendix B.

To obtain the solutions of displacement, $\bar{u}_x(\alpha,\beta,z)$, $\bar{u}_y(\alpha,\beta,z)$, and $\bar{u}_z(\alpha,\beta,z)$ from α, β, and γ transformed domain to *x*, *y*, and *z* physical one, the following inverse Fourier transforms should be addressed to solve Eqs. (4.30a)-(4.30c) as:

$$\bar{u}_x(\alpha,\beta,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{P_xD_{11}(\alpha,\beta,\gamma)+P_yD_{21}(\alpha,\beta,\gamma)+P_zD_{31}(\alpha,\beta,\gamma)}{m_t(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma-\gamma_3)(\gamma-\gamma_4)(\gamma-\gamma_5)(\gamma-\gamma_6)}e^{i\gamma z}d\gamma \tag{4.33a}$$

$$\bar{u}_y(\alpha,\beta,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{P_xD_{12}(\alpha,\beta,\gamma)+P_yD_{22}(\alpha,\beta,\gamma)+P_zD_{32}(\alpha,\beta,\gamma)}{m_t(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma-\gamma_3)(\gamma-\gamma_4)(\gamma-\gamma_5)(\gamma-\gamma_6)}e^{i\gamma z}d\gamma \tag{4.33b}$$

$$\bar{u}_z(\alpha,\beta,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{P_xD_{13}(\alpha,\beta,\gamma)+P_yD_{23}(\alpha,\beta,\gamma)+P_zD_{33}(\alpha,\beta,\gamma)}{m_t(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma-\gamma_3)(\gamma-\gamma_4)(\gamma-\gamma_5)(\gamma-\gamma_6)}e^{i\gamma z}d\gamma \tag{4.33c}$$

To evaluate the integral in Eqs. (4.33a)-(4.33c), the complex γ-plane is first encountered. Since there are six poles in the integrand, we assume the imaginary part of γ_1, γ_2, γ_3 are positive, and that of γ_4, γ_5, γ_6 are negative.

For $z > 0$, a closed contour by adding a large semicirle on the upper γ-plane, as depicted in Fig. 4.2 is employed.

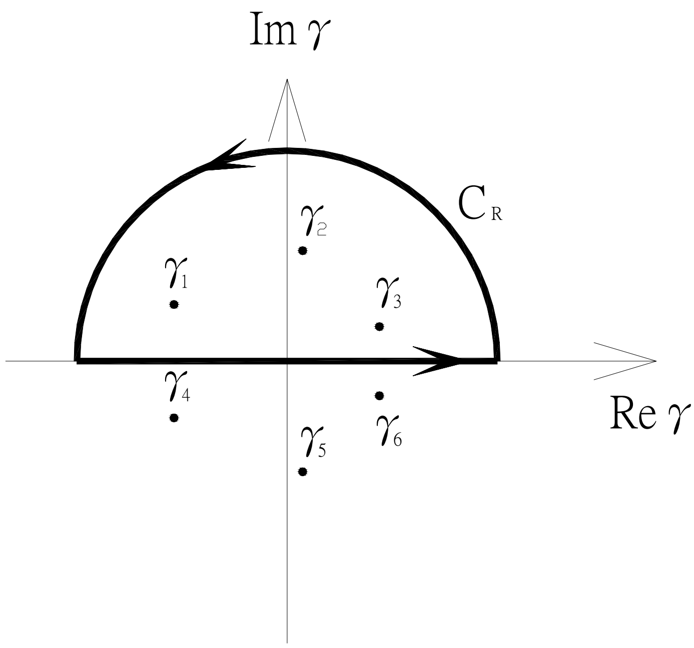


Fig. 4.2 A closed contour on the upper γ-plane.

The solutions of Eqs. (4.33a)-(4.33c), as defined by the three displacement functions are:

$$\bar{u}_{x1}(\alpha,\beta,z) = B^{1}_{x\gamma}e^{i\gamma_1 z} + B^{2}_{x\gamma}e^{i\gamma_2 z} + B^{3}_{x\gamma}e^{i\gamma_3 z} \tag{4.34a}$$

$$\bar{u}_{y1}(\alpha,\beta,z) = B^{1}_{y\gamma}e^{i\gamma_1 z} + B^{2}_{y\gamma}e^{i\gamma_2 z} + B^{3}_{y\gamma}e^{i\gamma_3 z} \tag{4.34b}$$

$$\bar{u}_{z1}(\alpha,\beta,z) = B^{1}_{z\gamma}e^{i\gamma_1 z} + B^{2}_{z\gamma}e^{i\gamma_2 z} + B^{3}_{z\gamma}e^{i\gamma_3 z} \tag{4.34c}$$

, and for $z < 0$, the other closed contour in the lower γ-plane is shown in Fig. 4.3.

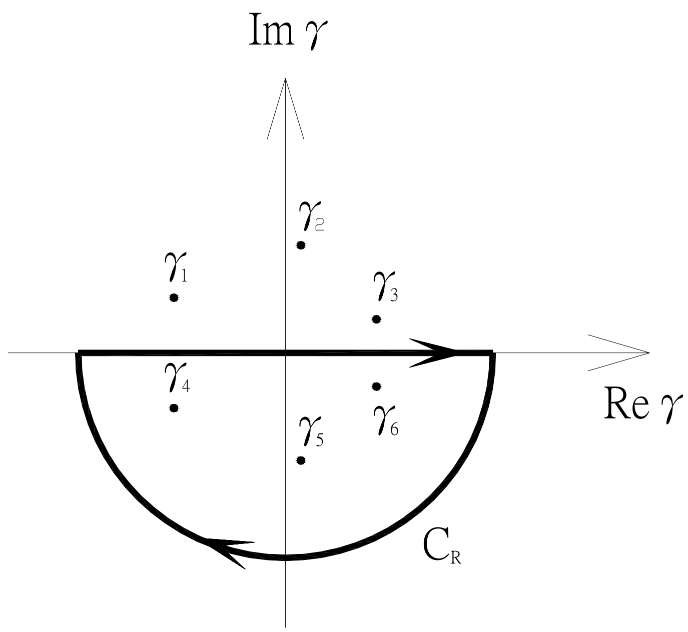


Fig. 4.3 A closed contour on the lower γ-plane.

$$\bar{u}_{x2}(\alpha,\beta,z) = -B_{x\gamma}^{4} e^{i\gamma_4 z} - B_{x\gamma}^{5} e^{i\gamma_5 z} - B_{x\gamma}^{6} e^{i\gamma_6 z} \tag{4.35a}$$

$$\bar{u}_{y2}(\alpha,\beta,z) = -B_{y\gamma}^{4} e^{i\gamma_4 z} - B_{y\gamma}^{5} e^{i\gamma_5 z} - B_{y\gamma}^{6} e^{i\gamma_6 z} \tag{4.35b}$$

$$\bar{u}_{z2}(\alpha,\beta,z) = -B_{z\gamma}^{4} e^{i\gamma_4 z} - B_{z\gamma}^{5} e^{i\gamma_5 z} - B_{z\gamma}^{6} e^{i\gamma_6 z} \tag{4.35c}$$

where

$$B_{x\gamma}^{j}(\alpha,\beta) = i\frac{P_x D_{11}(\alpha,\beta,\gamma_j) + P_y D_{21}(\alpha,\beta,\gamma_j) + P_z D_{31}(\alpha,\beta,\gamma_j)}{2\pi m_t R_j} \tag{4.36a}$$

$$B_{y\gamma}^{j}(\alpha,\beta) = i\frac{P_x D_{12}(\alpha,\beta,\gamma_j) + P_y D_{22}(\alpha,\beta,\gamma_j) + P_z D_{32}(\alpha,\beta,\gamma_j)}{2\pi m_t R_j} \tag{4.36b}$$

$$B_{z\gamma}^{j}(\alpha,\beta) = i\frac{P_x D_{13}(\alpha,\beta,\gamma_j) + P_y D_{23}(\alpha,\beta,\gamma_j) + P_z D_{33}(\alpha,\beta,\gamma_j)}{2\pi m_t R_j} \tag{4.36c}$$

$$R_j(\alpha,\beta)=\frac{\partial[(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma-\gamma_3)(\gamma-\gamma_4)(\gamma-\gamma_5)(\gamma-\gamma_6)]}{\partial\gamma},\gamma=\gamma_j \tag{4.36d}$$

Suppose that $u_j = i\gamma_j$, $\bar{u}_x(\alpha,\beta,z)$, $\bar{u}_y(\alpha,\beta,z)$, and $\bar{u}_z(\alpha,\beta,z)$ from Eqs. (4.16a)-(4.16c) and Eqs. (4.17a)-(4.17c) have the same results as Eqs. (4.34a)-(4.34c) and Eqs. (4.35a)-(4.35c).

4.4 Solutions for Displacements and Stresses by Inverse Fourier Transforms

The displacements $u_x(x,y,z)$, $u_y(x,y,z)$, and $u_z(x,y,z)$ in Eqs. (4.16a)-(4.16c) and Eqs. (4.17a)-(4.17c) also can be solved by the following inverse double Fourier transforms as:

for $z>0$ (region 1, as shown in Fig. 3.1),

$$\bar{u}_{x1}(\alpha,\beta,z)=B_x^1e^{u_1z}+B_x^2e^{u_2z}+B_x^3e^{u_3z} \tag{4.37a}$$

$$\bar{u}_{y1}(\alpha,\beta,z)=B_y^1e^{u_1z}+B_y^2e^{u_2z}+B_y^3e^{u_3z} \tag{4.37b}$$

$$\bar{u}_{z1}(\alpha,\beta,z)=B_z^1e^{u_1z}+B_z^2e^{u_2z}+B_z^3e^{u_3z} \tag{4.37c}$$

$$\bar{\sigma}_{xx1}(\alpha,\beta,z)=\bar{\sigma}_{xx1(P)}(\alpha,\beta,z) \tag{4.37d}$$

$$\bar{\sigma}_{yy1}(\alpha,\beta,z)=\bar{\sigma}_{yy1(P)}(\alpha,\beta,z) \tag{4.37e}$$

$$\bar{\sigma}_{zz1}(\alpha,\beta,z)=\bar{\sigma}_{zz1(P)}(\alpha,\beta,z) \tag{4.37f}$$

$$\bar{\tau}_{yz1}(\alpha,\beta,z)=\bar{\tau}_{yz1(P)}(\alpha,\beta,z) \tag{4.37g}$$

$$\bar{\tau}_{zx1}(\alpha,\beta,z)=\bar{\tau}_{zx1(P)}(\alpha,\beta,z) \tag{4.37h}$$

$$\bar{\tau}_{xy1}(\alpha,\beta,z)=\bar{\tau}_{xy1(P)}(\alpha,\beta,z) \tag{4.37i}$$

and for $z<0$ (region 2, as also depicted in Fig. 3.1),

$$\bar{u}_{x2}(\alpha,\beta,z)=-B_x^4 e^{u_4 z}-B_x^5 e^{u_5 z}-B_x^6 e^{u_6 z} \tag{4.38a}$$

$$\bar{u}_{y2}(\alpha,\beta,z)=-B_y^4 e^{u_4 z}-B_y^5 e^{u_5 z}-B_y^6 e^{u_6 z} \tag{4.38b}$$

$$\bar{u}_{z2}(\alpha,\beta,z)=-B_z^4 e^{u_4 z}-B_z^5 e^{u_5 z}-B_z^6 e^{u_6 z} \tag{4.38c}$$

$$\bar{\sigma}_{xx2}(\alpha,\beta,z)=\bar{\sigma}_{xx2(P)}(\alpha,\beta,z) \tag{4.38d}$$

$$\bar{\sigma}_{yy2}(\alpha,\beta,z)=\bar{\sigma}_{yy2(P)}(\alpha,\beta,z) \tag{4.38e}$$

$$\bar{\sigma}_{zz2}(\alpha,\beta,z)=\bar{\sigma}_{zz2(P)}(\alpha,\beta,z) \tag{4.38f}$$

$$\bar{\tau}_{yz2}(\alpha,\beta,z)=\bar{\tau}_{yz2(P)}(\alpha,\beta,z) \tag{4.38g}$$

$$\bar{\tau}_{zx2}(\alpha,\beta,z)=\bar{\tau}_{zx2(P)}(\alpha,\beta,z) \tag{4.38h}$$

$$\bar{\tau}_{xy2}(\alpha,\beta,z)=\bar{\tau}_{xy2(P)}(\alpha,\beta,z) \tag{4.38i}$$

The desired displacements of $u_x(x,y,z)$, $u_y(x,y,z)$, and $u_z(x,y,z)$ can be obtained by the double inverse Fourier transforms of Eqs. (4.37a)-(4.37c) and Eqs. (4.38a)-(4.38c) as:

for $z>0$ (region 1),

$$u_{x1}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{B_x^1 e^{i(\alpha x+\beta y)+u_1 z}+B_x^2 e^{i(\alpha x+\beta y)+u_2 z}+B_x^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{4.39a}$$

$$u_{y1}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{B_y^1 e^{i(\alpha x+\beta y)+u_1 z}+B_y^2 e^{i(\alpha x+\beta y)+u_2 z}+B_y^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{4.39b}$$

$$u_{z1}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{B_z^1 e^{i(\alpha x+\beta y)+u_1 z}+B_z^2 e^{i(\alpha x+\beta y)+u_2 z}+B_z^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{4.39c}$$

, and for $z<0$ (region 2),

$$u_{x2}(x,y,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{B_x^4 e^{i(\alpha x+\beta y)+u_4 z}+B_x^5 e^{i(\alpha x+\beta y)+u_5 z}+B_x^6 e^{i(\alpha x+\beta y)+u_6 z}\}d\alpha d\beta \tag{4.40a}$$

$$u_{y2}(x,y,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{B_y^4 e^{i(\alpha x+\beta y)+u_4 z}+B_y^5 e^{i(\alpha x+\beta y)+u_5 z}+B_y^6 e^{i(\alpha x+\beta y)+u_6 z}\}d\alpha d\beta \tag{4.40b}$$

$$u_{z2}(x,y,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{B_z^4 e^{i(\alpha x+\beta y)+u_4 z}+B_z^5 e^{i(\alpha x+\beta y)+u_5 z}+B_z^6 e^{i(\alpha x+\beta y)+u_6 z}\}d\alpha d\beta \tag{4.40c}$$

In addition, the stress components of Eqs. (3.12a)-(3.12f) are performed by the double Fourier transforms expressed as Eqs. (3.42a)-(3.42f). The desired $\sigma_{xx}(x,y,z)$, $\sigma_{yy}(x,y,z)$, $\sigma_{zz}(x,y,z)$, $\tau_{yz}(x,y,z)$, $\tau_{zx}(x,y,z)$, and $\tau_{xy}(x,y,z)$ also can be acquired by the double inverse Fourier transforms as:

for $z>0$ (region 1),

$$\sigma_{xx1}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\sigma}_{xx}^1 e^{i(\alpha x+\beta y)+u_1 z}+\bar{\sigma}_{xx}^2 e^{i(\alpha x+\beta y)+u_2 z}+\bar{\sigma}_{xx}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{4.41a}$$

$$\sigma_{yy1}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\sigma}_{yy}^1 e^{i(\alpha x+\beta y)+u_1 z}+\bar{\sigma}_{yy}^2 e^{i(\alpha x+\beta y)+u_2 z}+\bar{\sigma}_{yy}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{4.41b}$$

$$\sigma_{zz1}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\sigma}_{zz}^1 e^{i(\alpha x+\beta y)+u_1 z}+\bar{\sigma}_{zz}^2 e^{i(\alpha x+\beta y)+u_2 z}+\bar{\sigma}_{zz}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{4.41c}$$

$$\tau_{yz1}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\tau}_{yz}^1 e^{i(\alpha x+\beta y)+u_1 z}+\bar{\tau}_{yz}^2 e^{i(\alpha x+\beta y)+u_2 z}+\bar{\tau}_{yz}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{4.41d}$$

$$\tau_{zx1}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\tau}_{zx}^{1}e^{i(\alpha x+\beta y)+u_1 z}+\bar{\tau}_{zx}^{2}e^{i(\alpha x+\beta y)+u_2 z}+\bar{\tau}_{zx}^{3}e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{4.41e}$$

$$\tau_{xy1}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\tau}_{xy}^{1}e^{i(\alpha x+\beta y)+u_1 z)}+\bar{\tau}_{xy}^{2}e^{i(\alpha x+\beta y)+u_2 z}+\bar{\tau}_{xy}^{3}e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{4.41f}$$

, and for $z<0$ (region 2),

$$\sigma_{xx2}(x,y,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\sigma}_{xx}^{4}e^{i(\alpha x+\beta y)+u_4 z}+\bar{\sigma}_{xx}^{5}e^{i(\alpha x+\beta y)+u_5 z}+\bar{\sigma}_{xx}^{6}e^{i(\alpha x+\beta y)+u_6 z}\}d\alpha d\beta \tag{4.42a}$$

$$\sigma_{yy2}(x,y,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\sigma}_{yy}^{4}e^{i(\alpha x+\beta y)+u_4 z}+\bar{\sigma}_{yy}^{5}e^{i(\alpha x+\beta y)+u_5 z}+\bar{\sigma}_{yy}^{6}e^{i(\alpha x+\beta y)+u_6 z}\}d\alpha d\beta \tag{4.42b}$$

$$\sigma_{zz2}(x,y,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\sigma}_{zz}^{4}e^{i(\alpha x+\beta y)+u_4 z}+\bar{\sigma}_{zz}^{5}e^{i(\alpha x+\beta y)+u_5 z}+\bar{\sigma}_{zz}^{6}e^{i(\alpha x+\beta y)+u_6 z}\}d\alpha d\beta \tag{4.42c}$$

$$\tau_{yz2}(x,y,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\tau}_{yz}^{4}e^{i(\alpha x+\beta y)+u_4 z}+\bar{\tau}_{yz}^{5}e^{i(\alpha x+\beta y)+u_5 z}+\bar{\tau}_{yz}^{6}e^{i(\alpha x+\beta y)+u_6 z}\}d\alpha d\beta \tag{4.42d}$$

$$\tau_{zx2}(x,y,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\tau}_{zx}^{4}e^{i(\alpha x+\beta y)+u_4 z}+\bar{\tau}_{zx}^{5}e^{i(\alpha x+\beta y)+u_5 z}+\bar{\tau}_{zx}^{6}e^{i(\alpha x+\beta y)+u_6 z)}\}d\alpha d\beta \tag{4.42e}$$

$$\tau_{xy2}(x,y,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\bar{\tau}_{xy}^{4}e^{i(\alpha x+\beta y)+u_4 z}+\bar{\tau}_{xy}^{5}e^{i(\alpha x+\beta y)+u_5 z}+\bar{\tau}_{xy}^{6}e^{i(\alpha x+\beta y)+u_6 z}\}d\alpha d\beta \tag{4.42f}$$

where:

$$\bar{\sigma}_{xx}^{j}(\alpha,\beta)=-i(\alpha a_{11}B_x^j+\beta a_{12}B_y^j+\beta a_{14}B_z^j)+i(a_{14}B_y^j+a_{13}B_z^j)u_j \tag{4.43a}$$

$$\bar{\sigma}_{yy}^{j}(\alpha,\beta)=-i(\alpha a_{12}B_x^j+\beta a_{22}B_y^j+\beta a_{24}B_z^j)+i(a_{24}B_y^j+a_{23}B_z^j)u_j \tag{4.43b}$$

$$\bar{\sigma}_{zz}^{j}(\alpha,\beta)=-i(\alpha a_{13}B_x^j+\beta a_{23}B_y^j+\beta a_{34}B_z^j)+i(a_{34}B_y^j+a_{33}B_z^j)u_j \tag{4.43c}$$

$$\bar{\tau}_{yz}^{j}(\alpha,\beta)=-i(\alpha a_{14}B_x^j+\beta a_{24}B_y^j+\beta a_{44}B_z^j)+i(a_{44}B_y^j+a_{34}B_z^j)u_j \tag{4.43d}$$

$$\bar{\tau}_{zx}^{j}(\alpha,\beta)=-i(a_{56}\beta B_x^j+a_{56}\alpha B_y^j+a_{55}\alpha B_z^j)+ia_{55}B_x^ju_j \tag{4.43e}$$

$$\bar{\tau}_{xy}^{j}(\alpha,\beta) = -i(a_{66}\beta B_x^j + a_{66}\alpha B_y^j + a_{56}\alpha B_z^j) + ia_{56}B_x^j u_j \qquad (4.43f)$$

In Eqs. (4.43a)-(4.43f), j=1-3 for $z>0$ (region 1), and j=4-6 for $z<0$ (region 2).

If we employe a spherical co-ordinate system, which is shown in Fig. 4.4, the variables α, β, and u_i can be expressed in terms of (k, θ_x) as:

$$\alpha = k \times cos\,\theta_x \qquad (4.44a)$$

$$\beta = k \times sin\,\theta_x \qquad (4.44b)$$

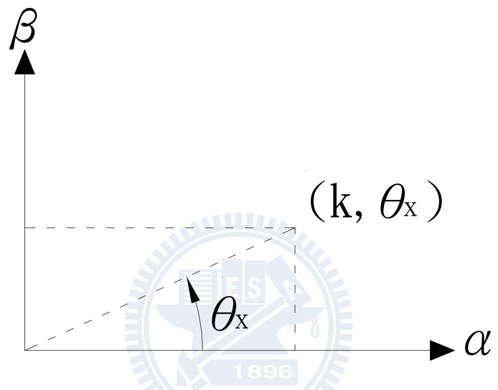


Fig. 4.4 Spherical co-ordinate system (k, θx)

By substiting Eqs. (4.44a)-(4.44b) into Eqs. (3.47a)-(3.47f), we get

$$u_j(\alpha,\beta) = k \times \frac{i\sin\theta_x \sin\phi\cos\phi(-1+A_j) - \sqrt{A_j(\sin^2\theta_x + \cos^2\theta_x(\cos^2\phi + A_j\sin^2\phi))}}{\cos^2\phi + A_j\sin^2\phi} \quad (j\text{=1-3}) \qquad (4.45a)$$

$$u_j(\alpha,\beta) = k \times \frac{i\sin\theta_x \sin\phi\cos\phi(-1+A_j) + i\sqrt{A_j(\sin^2\theta_x + \cos^2\theta_x(\cos^2\phi + A_j\sin^2\phi))}}{\cos^2\phi + A_j\sin^2\phi} \quad (\text{j=4-6}) \qquad (4.45b)$$

where $0<k<\infty$, and $0<\theta_x<2\pi$.

According to Eqs. (4.44a)-(4.44b) and Eqs. (4.45a)-(4.45b), $D_{mn}(\alpha,\beta,\gamma_j)$ (m, n=1-3) in Eqs. (3.53a)-(3.53f), $B_x^j(\alpha,\beta,u_j)$, $B_y^j(\alpha,\beta,u_j)$, $B_z^j(\alpha,\beta,u_j)$ (j=1-6) in Eqs. (4.04a)-(4.04c), and $U_j(\alpha,\beta)$ in Eq. (4.09) can be expressed in terms of k and θ_x as:

$$u_j(\alpha,\beta) = u_j(k,\theta) = ku_j(\theta_x) \tag{4.46a}$$

$$D_{mn}(\alpha,\beta,u_j) = D_{mn}(\alpha,\beta) = D_{mn}(k,\theta_x) = k^4 D_{mn}(\theta_x) \tag{4.46b}$$

$$U_j(\alpha,\beta) = U_j(k,\theta_x) = k^5 U_j(\theta_x) \tag{4.46c}$$

$$B_x^j(\alpha,\beta) = B_x^j(k,\theta_x) = k^{-1} B_x^j(\theta_x) \tag{4.46d}$$

$$B_y^j(\alpha,\beta) = B_y^j(k,\theta_x) = k^{-1} B_y^j(\theta_x) \tag{4.46e}$$

$$B_x^j(\alpha,\beta) = B_x^j(k,\theta_x) = k^{-1} B_x^j(\theta_x) \tag{4.46f}$$

Hence, we can obtain:

$$\overline{u}_{xj}(\alpha,\beta) = \overline{u}_{xj}(k,\theta_x) = k^{-1}\overline{u}_{xj}(\theta_x) \tag{4.47a}$$

$$\overline{u}_{yj}(\alpha,\beta) = \overline{u}_{yj}(k,\theta_x) = k^{-1}\overline{u}_{yj}(\theta_x) \tag{4.47b}$$

$$\overline{u}_{zj}(\alpha,\beta) = \overline{u}_{zj}(k,\theta_x) = k^{-1}\overline{u}_{zj}(\theta_x) \tag{4.47c}$$

Furthermore, Eqs. (4.43a)-(4.43f) can be rewritten as:

$$\overline{\sigma}_{xx}^j(\alpha,\beta) = \overline{\sigma}_{xx}^j(k,\theta_x) = \overline{\sigma}_{xx}^j(\theta_x) \tag{4.48a}$$

$$\overline{\sigma}_{yy}^j(\alpha,\beta) = \overline{\sigma}_{yy}^j(k,\theta_x) = \overline{\sigma}_{yy}^j(\theta_x) \tag{4.48b}$$

$$\bar{\sigma}_{zz}^{j}(\alpha,\beta)=\bar{\sigma}_{zz}^{j}(k,\theta_x)=\bar{\sigma}_{zz}^{j}(\theta_x) \tag{4.48c}$$

$$\bar{\tau}_{yz}^{j}(\alpha,\beta)=\bar{\tau}_{yz}^{j}(k,\theta_x)=\bar{\tau}_{yz}^{j}(\theta_x) \tag{4.48d}$$

$$\bar{\tau}_{zx}^{j}(\alpha,\beta)=\bar{\tau}_{zx}^{j}(k,\theta_x)=\bar{\tau}_{zx}^{j}(\theta_x) \tag{4.48e}$$

$$\bar{\tau}_{zx}^{j}(\alpha,\beta)=\bar{\tau}_{xy}^{j}(k,\theta_x)=\bar{\tau}_{xy}^{j}(\theta_x) \tag{4.48f}$$

The exponential terms in Eqs. (4.41a)-(4.41c), Eqs. (4.42a)-(4.42c), Eqs. (4.43a)-(4.43f) and Eqs. (4.44a)-(4.44f) can be expressed as:

$$i(\alpha x+\beta y)+u_j z=k\times\psi_j(\theta_x) \tag{4.49}$$

Additionally, $d\alpha d\beta$ can be replaced by $dkd\theta_x$ times the absolute value of Jacobian J as:

$$d\alpha d\beta=|J|dkd\theta_x \tag{4.50}$$

where:

$$|J|=\left|\frac{\partial(\alpha,\beta)}{\partial(k,\theta_x)}\right|=\begin{vmatrix}\frac{\partial\alpha}{\partial k} & \frac{\partial\alpha}{\partial\theta_x}\\ \frac{\partial\beta}{\partial k} & \frac{\partial\beta}{\partial\theta_x}\end{vmatrix}=\begin{vmatrix}\cos\theta_x & -k\sin\theta_x\\ \sin\theta_x & k\cos\theta_x\end{vmatrix}=k \tag{4.51}$$

and therefore,

$$d\alpha d\beta=kdkd\theta_x \tag{4.52}$$

Based on Eqs. (4.46a)-(4.46f), Eqs. (4.47a)-(4.47c), Eq.(4.49) and Eq. (4.50), then Eqs. (4.39a)-(4.39c) and Eqs. (4.40a)-(4.40c) can be presented as:

For $z>0$ (region 1),

$$u_{x1}(x,y,z)=\frac{1}{2\pi}\int_0^{2\pi}\{\frac{B_x^1(\theta_x)}{\psi_1(\theta_x)}+\frac{B_x^2(\theta_x)}{\psi_2(\theta_x)}+\frac{B_x^3(\theta_x)}{\psi_3(\theta_x)}\}d\theta_x \tag{4.53a}$$

$$u_{y1}(x,y,z)=\frac{1}{2\pi}\int_0^{2\pi}\{\frac{B_y^1(\theta_x)}{\psi_1(\theta_x)}+\frac{B_y^2(\theta_x)}{\psi_2(\theta_x)}+\frac{B_y^3(\theta_x)}{\psi_3(\theta_x)}\}d\theta_x \tag{4.53b}$$

$$u_{z1}(x,y,z)=\frac{1}{2\pi}\int_0^{2\pi}\{\frac{B_z^1(\theta_x)}{\psi_1(\theta_x)}+\frac{B_z^2(\theta_x)}{\psi_2(\theta_x)}+\frac{B_z^3(\theta_x)}{\psi_3(\theta_x)}\}d\theta_x \tag{4.53c}$$

, and for $z<0$ (region 2),

$$u_{x2}(x,y,z)=-\frac{1}{2\pi}\int_0^{2\pi}\{\frac{B_x^4(\theta_x)}{\psi_4(\theta_x)}+\frac{B_x^5(\theta_x)}{\psi_5(\theta_x)}+\frac{B_x^6(\theta_x)}{\psi_6(\theta_x)}\}d\theta_x \tag{4.54a}$$

$$u_{y2}(x,y,z)=-\frac{1}{2\pi}\int_0^{2\pi}\{\frac{B_y^4(\theta_x)}{\psi_4(\theta_x)}+\frac{B_y^5(\theta_x)}{\psi_5(\theta_x)}+\frac{B_y^6(\theta_x)}{\psi_6(\theta_x)}\}d\theta_x \tag{4.54b}$$

$$u_{z2}(x,y,z)=-\frac{1}{2\pi}\int_0^{2\pi}\{\frac{B_z^4(\theta_x)}{\psi_4(\theta_x)}+\frac{B_z^5(\theta_x)}{\psi_5(\theta_x)}+\frac{B_z^6(\theta_x)}{\psi_6(\theta_x)}\}d\theta_x \tag{4.54c}$$

Also, from Eqs. (4.46a)-(4.46f), Eqs. (4.48a)-(4.48f), Eqs. (4.49) and Eq. (4.52), Eqs. (4.41a)-(4.41f) and Eqs. (4.42a)-(4.42f) can then be written as:

For $z>0$ (region 1),

$$\sigma_{xx1}(x,y,z)=-\frac{1}{2\pi}\int_0^{2\pi}\{\bar{\sigma}_{xx}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2+\bar{\sigma}_{xx}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2+\bar{\sigma}_{xx}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\}d\theta_x \tag{4.55a}$$

$$\sigma_{yy1}(x,y,z)=-\frac{1}{2\pi}\int_0^{2\pi}\{\bar{\sigma}_{yy}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2+\bar{\sigma}_{yy}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2+\bar{\sigma}_{yy}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\}d\theta_x \tag{4.55b}$$

$$\sigma_{zz1}(x,y,z) = -\frac{1}{2\pi}\int_0^{2\pi} \{\bar{\sigma}_{zz}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2 + \bar{\sigma}_{zz}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2 + \bar{\sigma}_{zz}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\} d\theta_x \tag{4.55c}$$

$$\tau_{yz1}(x,y,z) = -\frac{1}{2\pi}\int_0^{2\pi} \{\bar{\tau}_{yz}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2 + \bar{\tau}_{yz}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2 + \bar{\tau}_{yz}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\} d\theta_x \tag{4.55d}$$

$$\tau_{zx1}(x,y,z) = -\frac{1}{2\pi}\int_0^{2\pi} \{\bar{\tau}_{zx}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2 + \bar{\tau}_{zx}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2 + \bar{\tau}_{zx}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\} d\theta_x \tag{4.55e}$$

$$\tau_{xy1}(x,y,z) = -\frac{1}{2\pi}\int_0^{2\pi} \{\bar{\tau}_{xy}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2 + \bar{\tau}_{xy}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2 + \bar{\tau}_{xy}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\} d\theta_x \tag{4.55f}$$

,and for $z < 0$ (region 2),

$$\sigma_{xx2}(x,y,z) = \frac{1}{2\pi}\int_0^{2\pi} \{\bar{\sigma}_{xx}^4(\theta_x)(\frac{1}{\psi_4(\theta_x)})^2 + \bar{\sigma}_{xx}^5(\theta_x)(\frac{1}{\psi_5(\theta_x)})^2 + \bar{\sigma}_{xx}^6(\theta_x)(\frac{1}{\psi_6(\theta_x)})^2\} d\theta_x \tag{4.56a}$$

$$\sigma_{yy2}(x,y,z) = \frac{1}{2\pi}\int_0^{2\pi} \{\bar{\sigma}_{yy}^4(\theta_x)(\frac{1}{\psi_4(\theta_x)})^2 + \bar{\sigma}_{yy}^5(\theta_x)(\frac{1}{\psi_5(\theta_x)})^2 + \bar{\sigma}_{yy}^6(\theta_x)(\frac{1}{\psi_6(\theta_x)})^2\} d\theta_x \tag{4.56b}$$

$$\sigma_{zz2}(x,y,z) = \frac{1}{2\pi}\int_0^{2\pi} \{\bar{\sigma}_{zz}^4(\theta_x)(\frac{1}{\psi_4(\theta_x)})^2 + \bar{\sigma}_{zz}^5(\theta_x)(\frac{1}{\psi_5(\theta_x)})^2 + \bar{\sigma}_{zz}^6(\theta_x)(\frac{1}{\psi_6(\theta_x)})^2\} d\theta_x \tag{4.56c}$$

$$\tau_{yz2}(x,y,z)=\frac{1}{2\pi}\int_{0}^{2\pi}\{\bar{\tau}_{yz}^{4}(\theta_x)(\frac{1}{\psi_4(\theta_x)})^2+\bar{\tau}_{yz}^{5}(\theta_x)(\frac{1}{\psi_5(\theta_x)})^2+\bar{\tau}_{yz}^{6}(\theta_x)(\frac{1}{\psi_6(\theta_x)})^2\}d\theta_x \tag{4.56d}$$

$$\tau_{zx2}(x,y,z)=\frac{1}{2\pi}\int_{0}^{2\pi}\{\bar{\tau}_{zx}^{4}(\theta_x)(\frac{1}{\psi_4(\theta_x)})^2+\bar{\tau}_{zx}^{5}(\theta_x)(\frac{1}{\psi_5(\theta_x)})^2+\bar{\tau}_{zx}^{6}(\theta_x)(\frac{1}{\psi_6(\theta_x)})^2\}d\theta_x \tag{4.56e}$$

$$\tau_{xy2}(x,y,z)=\frac{1}{2\pi}\int_{0}^{2\pi}\{\bar{\tau}_{xy}^{4}(\theta_x)(\frac{1}{\psi_4(\theta_x)})^2+\bar{\tau}_{xy}^{5}(\theta_x)(\frac{1}{\psi_5(\theta_x)})^2+\bar{\tau}_{xy}^{6}(\theta_x)(\frac{1}{\psi_6(\theta_x)})^2\}d\theta_x \tag{4.56f}$$

Let $\omega=e^{i\theta_x}$, and hence, $sin\,\theta_x=\frac{\omega-\omega^{-1}}{2i}$, $cos\,\theta_x=\frac{\omega+\omega^{-1}}{2}$, $d\omega=i\omega d\theta_x$. Eqs. (4.53a)-(4.53c) and Eqs. (4.54a)-(4.54c) can be expressed as:

For $z>0$ (region 1),

$$u_{x1}(x,y,z)=\frac{1}{2\pi}\oint_C\frac{1}{i\omega}\{\frac{B_x^1(\omega)}{\psi_1(\omega)}+\frac{B_x^2(\omega)}{\psi_2(\omega)}+\frac{B_x^3(\omega)}{\psi_3(\omega)}\}d\omega \tag{4.57a}$$

$$u_{y1}(x,y,z)=\frac{1}{2\pi}\oint_C\frac{1}{i\omega}\{\frac{B_y^1(\omega)}{\psi_1(\omega)}+\frac{B_y^2(\omega)}{\psi_2(\omega)}+\frac{B_y^3(\omega)}{\psi_3(\omega)}\}d\omega \tag{4.57b}$$

$$u_{z1}(x,y,z)=\frac{1}{2\pi}\oint_C\frac{1}{i\omega}\{\frac{B_z^1(\omega)}{\psi_1(\omega)}+\frac{B_z^2(\omega)}{\psi_2(\omega)}+\frac{B_z^3(\omega)}{\psi_3(\omega)}\}d\omega \tag{4.57c}$$

for $z<0$ (region 2),

$$u_{x2}(x,y,z)=-\frac{1}{2\pi}\oint_C\frac{1}{i\omega}\{\frac{B_x^4(\omega)}{\psi_4(\omega)}+\frac{B_x^5(\omega)}{\psi_5(\omega)}+\frac{B_x^6(\omega)}{\psi_6(\omega)}\}d\omega \tag{4.58a}$$

$$u_{y2}(x,y,z)=-\frac{1}{2\pi}\oint_C\frac{1}{i\omega}\{\frac{B_y^4(\omega)}{\psi_4(\omega)}+\frac{B_y^5(\omega)}{\psi_5(\omega)}+\frac{B_y^6(\omega)}{\psi_6(\omega)}\}d\omega \tag{4.58b}$$

$$u_{z2}(x,y,z) = -\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\frac{B_z^4(\omega)}{\psi_4(\omega)} + \frac{B_z^5(\omega)}{\psi_5(\omega)} + \frac{B_z^6(\omega)}{\psi_6(\omega)}\}d\omega \tag{4.58c}$$

Besides, Eqs. (4.55a)-(4.55f) and Eqs. (4.56a)-(4.56f) can be represented as:

For $z > 0$ (region 1),

$$\sigma_{xx1}(x,y,z) = -\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{xx}^1(\omega)(\frac{1}{\psi_1(\omega)})^2 + \bar{\sigma}_{xx}^2(\omega)(\frac{1}{\psi_2(\omega)})^2 + \bar{\sigma}_{xx}^3(\omega)(\frac{1}{\psi_3(\omega)})^2\}d\omega \tag{4.59a}$$

$$\sigma_{yy1}(x,y,z) = -\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{yy}^1(\omega)(\frac{1}{\psi_1(\omega)})^2 + \bar{\sigma}_{yy}^2(\omega)(\frac{1}{\psi_2(\omega)})^2 + \bar{\sigma}_{yy}^3(\omega)(\frac{1}{\psi_3(\omega)})^2\}d\omega \tag{4.59b}$$

$$\sigma_{zz1}(x,y,z) = -\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{zz}^1(\omega)(\frac{1}{\psi_1(\omega)})^2 + \bar{\sigma}_{zz}^2(\omega)(\frac{1}{\psi_2(\omega)})^2 + \bar{\sigma}_{zz}^3(\omega)(\frac{1}{\psi_3(\omega)})^2\}d\omega \tag{4.59c}$$

$$\sigma_{yz1}(x,y,z) = -\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{yz}^1(\omega)(\frac{1}{\psi_1(\omega)})^2 + \bar{\sigma}_{yz}^2(\omega)(\frac{1}{\psi_2(\omega)})^2 + \bar{\sigma}_{yz}^3(\omega)(\frac{1}{\psi_3(\omega)})^2\}d\omega \tag{4.59d}$$

$$\sigma_{zx1}(x,y,z) = -\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{zx}^1(\omega)(\frac{1}{\psi_1(\omega)})^2 + \bar{\sigma}_{zx}^2(\omega)(\frac{1}{\psi_2(\omega)})^2 + \bar{\sigma}_{zx}^3(\omega)(\frac{1}{\psi_3(\omega)})^2\}d\omega \tag{4.59e}$$

$$\sigma_{xy1}(x,y,z) = -\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{xy}^1(\omega)(\frac{1}{\psi_1(\omega)})^2 + \bar{\sigma}_{xy}^2(\omega)(\frac{1}{\psi_2(\omega)})^2 + \bar{\sigma}_{xy}^3(\omega)(\frac{1}{\psi_3(\omega)})^2\}d\omega \tag{4.59f}$$

,and for $z < 0$ (region 2),

$$\sigma_{xx2}(x,y,z)=\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{xx}^{4}(\omega)(\frac{1}{\psi_4(\omega)})^2+\bar{\sigma}_{xx}^{5}(\omega)(\frac{1}{\psi_5(\omega)})^2 +\bar{\sigma}_{xx}^{6}(\omega)(\frac{1}{\psi_6(\omega)})^2\}d\omega \tag{4.60a}$$

$$\sigma_{yy2}(x,y,z)=\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{yy}^{4}(\omega)(\frac{1}{\psi_4(\omega)})^2+\bar{\sigma}_{yy}^{5}(\omega)(\frac{1}{\psi_5(\omega)})^2 +\bar{\sigma}_{yy}^{6}(\omega)(\frac{1}{\psi_6(\omega)})^2\}d\omega \tag{4.60b}$$

$$\sigma_{zz2}(x,y,z)=\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{zz}^{4}(\omega)(\frac{1}{\psi_4(\omega)})^2+\bar{\sigma}_{zz}^{5}(\omega)(\frac{1}{\psi_5(\omega)})^2 +\bar{\sigma}_{zz}^{6}(\omega)(\frac{1}{\psi_6(\omega)})^2\}d\omega \tag{4.60c}$$

$$\sigma_{yz2}(x,y,z)=\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{yz}^{4}(\omega)(\frac{1}{\psi_4(\omega)})^2+\bar{\sigma}_{yz}^{5}(\omega)(\frac{1}{\psi_5(\omega)})^2 +\bar{\sigma}_{yz}^{6}(\omega)(\frac{1}{\psi_6(\omega)})^2\}d\omega \tag{4.60d}$$

$$\sigma_{zx2}(x,y,z)=\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{zx}^{4}(\omega)(\frac{1}{\psi_4(\omega)})^2+\bar{\sigma}_{zx}^{5}(\omega)(\frac{1}{\psi_5(\omega)})^2 +\bar{\sigma}_{zx}^{6}(\omega)(\frac{1}{\psi_6(\omega)})^2\}d\omega \tag{4.60e}$$

$$\sigma_{xy2}(x,y,z)=\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{\bar{\sigma}_{xy}^{4}(\omega)(\frac{1}{\psi_4(\omega)})^2+\bar{\sigma}_{xy}^{5}(\omega)(\frac{1}{\psi_5(\omega)})^2 +\bar{\sigma}_{xy}^{6}(\omega)(\frac{1}{\psi_6(\omega)})^2\}d\omega \tag{4.60f}$$

CHAPTER V

THREE DIMENSIONAL ELASTIC SOLUTIONS OF A TRANSVERSELY ISOTROPIC HALF-SPACE SUBJECTED TO POINT LOADS

The analytical solutions for displacements and stresses in an transversely isotropic half-space subjected to a point load in the medium are derived in Chapters 5. Three distinct approaches as same as Chapters 4 were used to derive the solutions in Sec 5.1-5.3. In Sec 5.1, we consider the nonhomogeneous part of the ordinary differential equations and solve the homogeneous and particular solution of Eqs. (3.41a)-(3.41c). In Sec.5.2, we separate the half-space into two regions of $0^- < z < 0^+$ (imaginary space) and $0^+ < z < +\infty$ (region 1) , the point load force is in the region of $0^- < z < 0^+$. In regions of $0^+ < z < +\infty$, the right-hand side of Eqs. (3.41a)-(3.41c) does not exist, the equilibrium equations are homogeneous linear equations. Hence, we can solve the boundary-value problem consisting of the two regions. In Sec.5.3, the Laplace transform respect to variables of z can reduce the aforementioned ordinary differential equations (Eqs. (3.41a)-(3.41c)) to algebraic equations..

5.1 Traditional Method

The thesis aims to determine the distribution of stress and displacement in a semi-infinite elastic solid due to the application of an external point load to its surface analytically. In order to derive the homogenous solutions of Eqs. (3.41a)-(3.41c), we redefine the three displacement functions of Eqs. (3.50a)-(3.50c) and Eqs. (3.51a)-(3.51c) as follows:

for $z > 0$ (region 1 and 2, as shown in Fig. 5.1),

$$\bar{u}_{x1(H)}(\alpha,\beta,z) = A_{x1}^{1}e^{u_1 z} + A_{x1}^{2}e^{u_2 z} + A_{x1}^{3}e^{u_3 z} + A_{x1}^{4}e^{u_4 z} + A_{x1}^{5}e^{u_5 z} + A_{x1}^{6}e^{u_6 z} \quad (5.01a)$$

$$\bar{u}_{y1(H)}(\alpha,\beta,z) = A_{y1}^{1}e^{u_1 z} + A_{y1}^{2}e^{u_2 z} + A_{y1}^{3}e^{u_3 z} + A_{y1}^{4}e^{u_4 z} + A_{y1}^{5}e^{u_5 z} + A_{y1}^{6}e^{u_6 z} \quad (5.01b)$$

$$\bar{u}_{z1(H)}(\alpha,\beta,z) = A_{z1}^{1}e^{u_1 z} + A_{z1}^{2}e^{u_2 z} + A_{z1}^{3}e^{u_3 z} + A_{z1}^{4}e^{u_4 z} + A_{z1}^{5}e^{u_5 z} + A_{z1}^{6}e^{u_6 z} \quad (5.01c)$$

and for $z < 0$ (region 3, as also depicted in Fig. 5.1),

$$\bar{u}_{x2(H)}(\alpha,\beta,z) = 0 \quad (5.02a)$$

$$\bar{u}_{y2(H)}(\alpha,\beta,z) = 0 \quad (5.02b)$$

$$\bar{u}_{z2(H)}(\alpha,\beta,z) = 0 \quad (5.02c)$$

In Eqs. (5.01a)-(5.01c), the undetermined coefficients A_{x1}^{j}, A_{y1}^{j}, and A_{z1}^{j} (j=1-6) can be obtained by assuming the displacements in region 1, u_{x1}, u_{y1}, and u_{z1} must be finite when z is approaching to ∞. Hence, $A_{x1}^{4} = A_{x1}^{5} = A_{x1}^{6} = 0$, $A_{y1}^{4} = A_{y1}^{5} = A_{y1}^{6} = 0$, and $A_{z1}^{4} = A_{z1}^{5} = A_{z1}^{6} = 0$.

Now, let:

$$\frac{A_{x1}^{j}}{D_{11}(\alpha,\beta,u_j)} = \frac{A_{y1}^{j}}{D_{21}(\alpha,\beta,u_j)} = \frac{A_{z1}^{j}}{D_{31}(\alpha,\beta,u_j)} = C_{d2}^{j} \quad (j\text{=1-3}) \quad (5.03)$$

where D_{ij} (i, j=1-3) can be written as the second order derterminants in the following, and the complete forms of D_{ij} are presented in Appendix C.

Similar to the full space problem, the general solition of the homogeneous linear equation of Eqs.(3.41a)-(3.41c) can express as follow:

$$\bar{u}_{x1(H)}(\alpha,\beta,z)=C_{d2}^{1}D_{11}^{1}e^{u_1 z}+C_{d2}^{2}D_{11}^{2}e^{u_2 z}+C_{d2}^{3}D_{11}^{3}e^{u_3 z} \tag{5.04a}$$

$$\bar{u}_{y1(H)}(\alpha,\beta,z)=C_{d2}^{1}D_{21}^{1}e^{u_1 z}+C_{d2}^{2}D_{21}^{2}e^{u_2 z}+C_{d2}^{3}D_{21}^{3}e^{u_3 z} \tag{5.04b}$$

$$\bar{u}_{z1(H)}(\alpha,\beta,z)=C_{d2}^{1}D_{31}^{1}e^{u_1 z}+C_{d2}^{2}D_{31}^{2}e^{u_2 z}+C_{d2}^{3}D_{31}^{3}e^{u_3 z} \tag{5.04c}$$

The point loads (F_x, F_y, F_z) acts at the point (0, 0, h) of the co-ordinate system can be described as Eqs. (3.19a)-(3.19c). Hence, the three displacement functions of the particular solutions can be gained from those of full space case (Sec. 4.1.1):

for $z>h$ (region 1, as shown in Fig. 5.1),

$$\bar{u}_{x1(P)}(\alpha,\beta,z)=B_{x}^{1}e^{u_1(z-h)}+B_{x}^{2}e^{u_2(z-h)}+B_{x}^{3}e^{u_3(z-h)} \tag{5.05a}$$

$$\bar{u}_{y1(P)}(\alpha,\beta,z)=B_{y}^{1}e^{u_1(z-h)}+B_{y}^{2}e^{u_2(z-h)}+B_{y}^{3}e^{u_3(z-h)} \tag{5.05b}$$

$$\bar{u}_{z1(P)}(\alpha,\beta,z)=B_{z}^{1}e^{u_1(z-h)}+B_{z}^{2}e^{u_2(z-h)}+B_{z}^{3}e^{u_3(z-h)} \tag{5.05c}$$

and for $0<z<h$ (region 2, as also depicted in Fig. 5.1),

$$\bar{u}_{x2(P)}(\alpha,\beta,z)=-B_{x}^{4}e^{u_4(z-h)}-B_{x}^{5}e^{u_5(z-h)}-B_{x}^{6}e^{u_6(z-h)} \tag{5.06a}$$

$$\bar{u}_{y2(P)}(\alpha,\beta,z)=-B_{y}^{4}e^{u_4(z-h)}-B_{y}^{5}e^{u_5(z-h)}-B_{y}^{6}e^{u_6(z-h)} \tag{5.06b}$$

$$\bar{u}_{z2(P)}(\alpha,\beta,z)=-B_{z}^{4}e^{u_4(z-h)}-B_{z}^{5}e^{u_5(z-h)}-B_{z}^{6}e^{u_6(z-h)} \tag{5.06c}$$

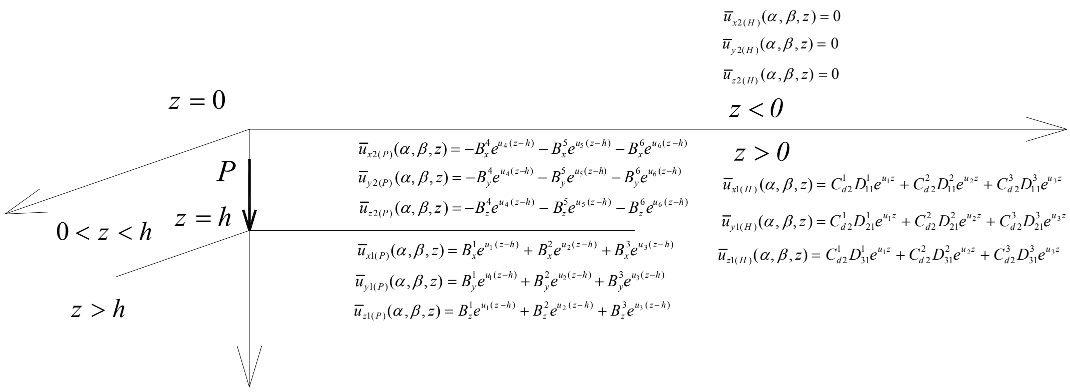


Fig. 5.1 Displacement function for point loads (P_x, P_y, P_z) acting at (0, 0, h) of a half-space

When the half-space ($z>h$) and the strip-space ($0<z<h$) are ideally bonded at the interface $z=h$ such that the material becomes continuous across the interface, we Consider the following boundary conditions:

$$\tau_{zx1}(x,y,h^+)-\tau_{zx2}(x,y,h^-)=P_x\delta(x)\delta(y) \tag{5.07a}$$

$$\tau_{zy1}(x,y,h^+)-\tau_{zy2}(x,y,h^-)=P_y\delta(x)\delta(y) \tag{5.07b}$$

$$\sigma_{zz1}(x,y,h^+)-\sigma_{zz2}(x,y,h^-)=P_z\delta(x)\delta(y) \tag{5.07c}$$

Hence, the double Fourier transforms of Eqs. (5.07a)-(5.07c) can be obtained as:

$$\bar{\tau}_{zx1}(\alpha,\beta,h^+)-\bar{\tau}_{zx2}(\alpha,\beta,h^-)=\frac{P_x}{2\pi} \tag{5.08a}$$

$$\bar{\tau}_{zy1}(\alpha,\beta,h^+)-\bar{\tau}_{zy2}(\alpha,\beta,h^-)=\frac{P_y}{2\pi} \tag{5.08b}$$

$$\bar{\sigma}_{zz1}(\alpha,\beta,h^+)-\bar{\sigma}_{zz2}(\alpha,\beta,h^-)=\frac{P_z}{2\pi} \tag{5.08c}$$

When $h\rightarrow 0$,

$$\bar{\tau}_{zx1}(\alpha,\beta,0^{+})-\bar{\tau}_{zx2}(\alpha,\beta,0^{-})=\frac{P_x}{2\pi} \tag{5.09a}$$

$$\bar{\tau}_{zy1}(\alpha,\beta,0^{+})-\bar{\tau}_{zy2}(\alpha,\beta,0^{-})=\frac{P_y}{2\pi} \tag{5.09b}$$

$$\bar{\sigma}_{zz1}(\alpha,\beta,0^{+})-\bar{\sigma}_{zz2}(\alpha,\beta,0^{-})=\frac{P_z}{2\pi} \tag{5.09c}$$

where $\bar{\tau}_{zx2}(\alpha,\beta,0^{-})$, $\bar{\tau}_{zy2}(\alpha,\beta,0^{-})$, $\bar{\sigma}_{zz2}(\alpha,\beta,0^{-})$, in the region of $z<0$, hence, $\bar{\tau}_{zx2}(\alpha,\beta,0^{-})=\bar{\tau}_{zy2}(\alpha,\beta,0^{-})=\bar{\sigma}_{zz2}(\alpha,\beta,0^{-})=0$.

The Eqs. (5.09a)-(5.09c) can be simplifed as follow:

$$\bar{\tau}_{zx1(H)}(\alpha,\beta,0^{+})+\bar{\tau}_{zx1(P)}(\alpha,\beta,0^{+})=\frac{P_x}{2\pi} \tag{5.10a}$$

$$\bar{\tau}_{zy1(H)}(\alpha,\beta,0^{+})+\bar{\tau}_{zy1(P)}(\alpha,\beta,0^{+})=\frac{P_y}{2\pi} \tag{5.10b}$$

$$\bar{\sigma}_{zz1(H)}(\alpha,\beta,0^{+})+\bar{\sigma}_{zz1(P)}(\alpha,\beta,0^{+})=\frac{P_z}{2\pi} \tag{5.10c}$$

The system of three linear equations (Eqs. (5.10a)-(5.10c)) has three undetermined coefficients C_{d2}^{1}, C_{d2}^{2} and C_{d2}^{3}. These coefficients can be associated with $\lfloor f_{ij} \rfloor$ (*i*, *j* =1-6) as follows:

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}\begin{bmatrix} C_{d2}^{1} \\ C_{d2}^{2} \\ C_{d2}^{3} \end{bmatrix}=\frac{1}{2\pi}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}-\begin{bmatrix} \bar{\tau}_{zx1(P)}(\alpha,\beta,0^{+}) \\ \bar{\tau}_{zy1(P)}(\alpha,\beta,0^{+}) \\ \bar{\sigma}_{zz1(P)}(\alpha,\beta,0^{+}) \end{bmatrix} \tag{5.11}$$

Eq. (5.11) can be separated into two parts as:

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}\begin{bmatrix} C_{d}^{1} \\ C_{d}^{2} \\ C_{d}^{3} \end{bmatrix}=\frac{1}{2\pi}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \tag{5.12a}$$

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} C_{d1}^1 \\ C_{d1}^2 \\ C_{d1}^3 \end{bmatrix} = \begin{bmatrix} \bar{\tau}_{zx1(P)}(\alpha,\beta,0^+) \\ \bar{\tau}_{zy1(P)}(\alpha,\beta,0^+) \\ \bar{\sigma}_{zz1(P)}(\alpha,\beta,0^+) \end{bmatrix} \tag{5.12b}$$

Base on the Eqs. (5.12a)-(5.12b), the relation of C_d^j, C_{di}^j and C_{di}^j can be expressed as::

$$C_d^1 = C_{d1}^1 + C_{d2}^1 \tag{5.13a}$$

$$C_d^2 = C_{d1}^2 + C_{d2}^2 \tag{5.13b}$$

$$C_d^3 = C_{d1}^3 + C_{d2}^3 \tag{5.13c}$$

In region 1 ($h \le z$),considering $z = 0$ and $h = 0$, the Eqs. (3.42c)-(3.42e) can express as:

$$\bar{\tau}_{zx1(P)}(\alpha,\beta,0) = \sum_{j=1}^{3} -i(a_{56}(\beta B_x^j + \alpha B_y^j) + a_{55}(\alpha B_z^j - iB_x^j u_j)) \tag{5.14a}$$

$$\bar{\tau}_{zy1(P)}(\alpha,\beta,0) = \sum_{j=1}^{3} -i(\alpha a_{14} B_x^j + \beta a_{24} B_y^j + \beta a_{44} B_z^j) - (a_{44} B_y^j + a_{34} B_z^j) u_j \tag{5.14b}$$

$$\bar{\sigma}_{zz1(P)}(\alpha,\beta,0) = \sum_{j=1}^{3} -i(\alpha a_{13} B_x^j + \beta a_{23} B_y^j + \beta a_{34} B_z^j) - (a_{34} B_y^j + a_{33} B_z^j) u_j \tag{5.14c}$$

By substiting Eqs. (5.14a)-(5.14b) into Eq. (5.12b), we get

$$C_{d1}^1 = \frac{P_x D_{11}(u_1) + P_y D_{12}(u_1) + P_z D_{13}(u_1)}{2\pi m_t U_1 D_{11}(u_1)} = \frac{B_x^1}{D_{11}(u_1)} = \frac{B_y^1}{D_{21}(u_1)} = \frac{B_z^1}{D_{31}(u_1)} \tag{5.15a}$$

$$C_{d1}^2 = \frac{P_x D_{11}(u_2) + P_y D_{12}(u_2) + P_z D_{13}(u_2)}{2\pi m_t U_2 D_{11}(u_2)} = \frac{B_x^2}{D_{11}(u_2)} = \frac{B_y^2}{D_{21}(u_2)} = \frac{B_z^2}{D_{31}(u_2)} \tag{5.15b}$$

$$C_{d1}^3 = \frac{P_x D_{11}(u_3) + P_y D_{12}(u_3) + P_z D_{13}(u_3)}{2\pi m_t U_3 D_{11}(u_3)} = \frac{B_x^3}{D_{11}(u_3)} = \frac{B_y^3}{D_{21}(u_3)} = \frac{B_z^3}{D_{31}(u_3)} \tag{5.15c}$$

The general solution of nonhomogeneous equation are the sum of the general solution of homogeneous equation and particular solution.

$$\begin{aligned}\bar{u}_x(\alpha,\beta,z) &= C_{d2}^1 D_{11}(\alpha,\beta,u_1)e^{u_1 z} + C_{d2}^2 D_{11}(\alpha,\beta,u_2)e^{u_2 z} + C_{d2}^3 D_{11}(\alpha,\beta,u_3)e^{u_3 z} \\ &\quad + B_x^1 e^{u_1 z} + B_x^2 e^{u_2 z} + B_x^3 e^{u_3 z} \\ &= (C_d^1 - C_{d1}^1) D_{11}(\alpha,\beta,u_1)e^{u_1 z} + (C_d^2 - C_{d1}^2) D_{11}(\alpha,\beta,u_2)e^{u_2 z} \\ &\quad + (C_d^3 - C_{d1}^3) D_{11}(\alpha,\beta,u_3)e^{u_3 z} + B_x^1 e^{u_1 z} + B_x^2 e^{u_2 z} + B_x^3 e^{u_3 z} \\ &= C_d^1 D_{11}(\alpha,\beta,u_1)e^{u_1 z} + C_d^2 D_{11}(\alpha,\beta,u_2)e^{u_2 z} + C_d^3 D_{11}(\alpha,\beta,u_3)e^{u_3 z}\end{aligned} \tag{5.16a}$$

Similarly,

$$\bar{u}_y(\alpha,\beta,z) = C_d^1 D_{21}(\alpha,\beta,u_1)e^{u_1 z} + C_d^2 D_{21}(\alpha,\beta,u_2)e^{u_2 z} + C_d^3 D_{21}(\alpha,\beta,u_3)e^{u_3 z} \tag{5.16b}$$

$$\bar{u}_z(\alpha,\beta,z) = C_d^1 D_{31}(\alpha,\beta,u_1)e^{u_1 z} + C_d^2 D_{31}(\alpha,\beta,u_2)e^{u_2 z} + C_d^3 D_{31}(\alpha,\beta,u_3)e^{u_3 z} \tag{5.16c}$$

5.2 Imaginary Space Method

In order to derive the solutions of Eqs. (3.15a)-(3.15c) in half-space, the Eqs. (4.19a)-(4.19c) and Eqs. (4.20a)-(4.20c) can be rewritten as:

for $z > 0^+$ (region 1, as shown in Fig. 5.2),

$$\bar{u}_{x1}(\alpha,\beta,z) = C_d^1 D_{11}^1 e^{u_1 z} + C_d^2 D_{11}^2 e^{u_2 z} + C_d^3 D_{11}^3 e^{u_3 z} \tag{5.17a}$$

$$\bar{u}_{y1}(\alpha,\beta,z) = C_d^1 D_{21}^1 e^{u_1 z} + C_d^2 D_{21}^2 e^{u_2 z} + C_d^3 D_{21}^3 e^{u_3 z} \tag{5.17b}$$

$$\bar{u}_{z1}(\alpha,\beta,z) = C_d^1 D_{31}^1 e^{u_1 z} + C_d^2 D_{31}^2 e^{u_2 z} + C_d^3 D_{31}^3 e^{u_3 z} \tag{5.17c}$$

and for $z < 0^-$,

$$\bar{u}_{x2}(\alpha,\beta,z) = 0 \tag{5.18a}$$

$$\bar{u}_{y2}(\alpha,\beta,z) = 0 \tag{5.18b}$$

$$\bar{u}_{z2}(\alpha,\beta,z) = 0 \tag{5.18c}$$

Introducing an imaginary plane along $z = 0$, to separate the full-space into two half-spaces, one is $0^+ < z < \infty$ (region 1 in Fig. 5.2), and the other is $0 < z < 0^+$.

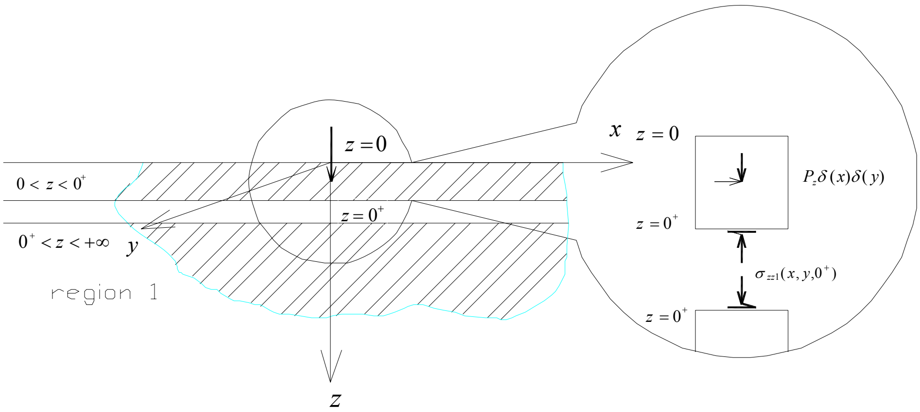


Fig. 5.2 Separate the half-space into two imaginary region of $0 < z < 0^{+}$ and $0^{+} < z < +\infty$.

Furthermore, considering the following pertinent continuity and discontinuity conditions at $z = 0$ are:

$$\tau_{zx}(x, y, 0^{+}) = P_x \delta(x)\delta(y) \qquad (5.19a)$$

$$\tau_{zy}(x, y, 0^{+}) = P_y \delta(x)\delta(y) \qquad (5.19b)$$

$$\sigma_{zz}(x, y, 0^{+}) = P_z \delta(x)\delta(y) \qquad (5.19c)$$

where $-\infty < x < \infty$, $-\infty < y < \infty$. The subscripts 1 and 2 mean that $z = 0$ plane is approaching to 0^{+} and 0^{-}, respectively.

C_d^1, C_d^2, C_d^3.can be determined from the following system of three linear equations These coefficients can be associated with $\lfloor f_{ij} \rfloor$ $(i, j$ =1-3) as:

$$\lfloor f_{ij} \rfloor \begin{bmatrix} C_d^1 \\ C_d^2 \\ C_d^3 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} C_d^1 \\ C_d^2 \\ C_d^3 \end{bmatrix} = \frac{1}{2\pi} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad (5.20)$$

where f_{ij} $(i, j$ =1-3) are presented in Appendix D.

5.3 Algebraic Equation Method (Double Fourier and Laplace Transforms)

Double Fourier transforms with respect to *x* and *y* could reduce the desired problem of solving partial differential equations to ordinary differential equations. Sequentiallt, the Laplace transform could reduce the ordinary differential equations to algebraic equations.

In the half-space problem, $u_i(x,y,z)$ are the displacement components in a homogeneous and linearly elastic continuum with the domain of $-\infty < x, y < \infty$ and $0 < z < \infty$. To solve the equilibrium equations (Eqs. (3.41a)-(3.41c)), the technique of double Fourier transforms for the internal force F_i of Eqs. (3.19a)-(3.19c) are utilized. Hence, the double Fourier transforms of Eqs. (3.19a)-(3.19c) can be obtained as:

$$\overline{F}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_x\delta(x)\delta(y)\delta(z-h)e^{-i(\alpha x+\beta y)}dxdy = \frac{P_x}{2\pi}\delta(z-h) \tag{5.21a}$$

$$\overline{F}_y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_y\delta(x)\delta(y)\delta(z-h)e^{-i(\alpha x+\beta y)}dxdy = \frac{P_y}{2\pi}\delta(z-h) \tag{5.21b}$$

$$\overline{F}_z = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_z\delta(x)\delta(y)\delta(z-h)e^{-i(\alpha x+\beta y)}dxdy = \frac{P_z}{2\pi}\delta(z-h) \tag{5.21c}$$

Based on Eqs. (5.21a)-(5.21c), the Navier-Cauchy equations (Eqs. (3.41a)-(3.41c)) can be rewritten as the following system of linear ordinary differential equations:

$$\begin{aligned}
&\{a_{11}\alpha^2 + a_{66}\beta^2 - a_{55}\frac{d^2}{dz^2} + 2ia_{56}\beta\frac{d}{dz}\}\overline{u}_x(\alpha,\beta,z) \\
&+\{(a_{12}+a_{66})\alpha\beta + i(a_{14}+a_{56})\alpha\frac{d}{dz}\}\overline{u}_y(\alpha,\beta,z) \\
&+\{(a_{14}+a_{56})\alpha\beta + i(a_{13}+a_{55})\alpha\frac{d}{dz}\}\overline{u}_z(\alpha,\beta,z) = -\frac{P_x}{2\pi}\delta(z-h)
\end{aligned} \tag{5.22a}$$

$$\{(a_{12}+a_{66})\alpha\beta+i(a_{14}+a_{56})\alpha\frac{d}{dz}\}\overline{u}_x(\alpha,\beta,z)$$
$$+\{a_{66}\alpha^2+a_{22}\beta^2-a_{44}\frac{d^2}{dz^2}+2ia_{24}\beta\frac{d}{dz}\}\overline{u}_y(\alpha,\beta,z) \tag{5.22b}$$
$$+\{a_{56}\alpha^2+a_{24}\beta^2-a_{34}\frac{d^2}{dz^2}+i(a_{23}+a_{44})\beta\frac{d}{dz}\}\overline{u}_z(\alpha,\beta,z)=-\frac{P_y}{2\pi}\delta(z-h)$$

$$\{(a_{14}+a_{66})\alpha\beta+i(a_{13}+a_{55})\alpha\frac{d}{dz}\}\overline{u}_x(\alpha,\beta,z)$$
$$+\{a_{56}\alpha^2+a_{24}\beta^2-a_{34}\frac{d^2}{dz^2}+i(a_{23}+a_{44})\beta\frac{d}{dz}\}\overline{u}_y(\alpha,\beta,z) \tag{5.22c}$$
$$+\{a_{55}\alpha^2+a_{44}\beta^2-a_{33}\frac{d^2}{dz^2}+2ia_{34}\beta\frac{d}{dz}\}\overline{u}_z(\alpha,\beta,z)=-\frac{P_y}{2\pi}\delta(z-h)$$

The boundary-value problem for the lower half-space $z<0$ with the normal and shear stresses applied on the ground surface. When $z=0$ and $-\infty<x,y<\infty$, the stresses of $\sigma_{zz}(x,y,z)$, $\tau_{yz}(x,y,z)$, and $\tau_{zx}(x,y,z)$ are equal to zero. Utilizing the boundary conditions mentioned abov, we obtain $\overline{\sigma}_{zz}(\alpha,\beta,0)=\overline{\tau}_{yz}(\alpha,\beta,0)=\overline{\tau}_{zx}(\alpha,\beta,0)=0$ after performing the double Fourier transforms. Additionally, Eqs. (3.42c)-(3.42e) can be rewritten as:

$$\frac{\partial\overline{u}_x(\alpha,\beta,0)}{\partial z}=\frac{i\beta a_{56}\overline{u}_x(\alpha,\beta,0)+i\alpha a_{56}\overline{u}_y(\alpha,\beta,0)+i\alpha a_{55}\overline{u}_z(\alpha,\beta,0)}{a_{55}} \tag{5.23a}$$

$$\frac{\partial\overline{u}_y(\alpha,\beta,0)}{\partial z}=\frac{i\alpha(a_{14}a_{33}-a_{13}a_{34})\overline{u}_x(\alpha,\beta,0)+i\beta(a_{24}a_{33}-a_{23}a_{34})\overline{u}_y(\alpha,\beta,0)}{-a_{34}^2+a_{33}a_{44}}+i\beta\overline{u}_z(\alpha,\beta,0) \tag{5.23b}$$

$$\frac{\partial\overline{u}_z(\alpha,\beta,0)}{\partial z}=\frac{i\alpha(a_{14}a_{34}-a_{13}a_{44})\overline{u}_x(\alpha,\beta,0)+i\beta(a_{24}a_{34}-a_{23}a_{44})\overline{u}_y(\alpha,\beta,0)}{a_{34}^2-a_{33}a_{44}} \tag{5.23c}$$

The Laplace transform for the component of *z*-axial displacement is adopted. The step by step Laplace transforms for $\overline{u}_i(\alpha,\beta,z)$ (i=x, y, z) are written as:

$$L\{\overline{u}_i(\alpha,\beta,z)\} = \int_0^\infty \overline{u}_i(\alpha,\beta,z)e^{-sz}dz = \overline{U}_i(\alpha,\beta,s) \tag{5.24a}$$

$$L\{\frac{d\overline{u}_i(\alpha,\beta,z)}{dz}\} = \int_0^\infty \frac{d\overline{u}_i(\alpha,\beta,z)}{dz}e^{-sz}dz = s\overline{U}_i(\alpha,\beta,s) - \overline{u}_i(\alpha,\beta,0) \tag{5.24b}$$

$$L\{\frac{d^2\overline{u}_i(\alpha,\beta,z)}{dz^2}\} = \int_0^\infty \frac{d^2\overline{u}_i(\alpha,\beta,z)}{dz^2}e^{-sz}dz = s^2\overline{U}_i(\alpha,\beta,s) - s\overline{u}_i(\alpha,\beta,0) - \frac{d\overline{u}_i(\alpha,\beta,0)}{dz} \tag{5.24c}$$

Based on the aforemenioned transforms, the Eqs. (5.22a)-(5.22c) can be expressed as:

$$\left[f_{ij}\right]\begin{bmatrix}\overline{U}_x\\ \overline{U}_y\\ \overline{U}_z\end{bmatrix} = \begin{bmatrix} f_{11}(\alpha,\beta,s) & f_{12}(\alpha,\beta,s) & f_{13}(\alpha,\beta,s)\\ f_{21}(\alpha,\beta,s) & f_{22}(\alpha,\beta,s) & f_{23}(\alpha,\beta,s)\\ f_{31}(\alpha,\beta,s) & f_{32}(\alpha,\beta,s) & f_{33}(\alpha,\beta,s)\end{bmatrix}\begin{bmatrix}\overline{U}_x(\alpha,\beta,s)\\ \overline{U}_y(\alpha,\beta,s)\\ \overline{U}_z(\alpha,\beta,s)\end{bmatrix} = -\begin{bmatrix}\frac{p_x e^{-sh}}{2\pi}+f_{10}\\ \frac{p_y e^{-sh}}{2\pi}+f_{20}\\ \frac{p_z e^{-sh}}{2\pi}+f_{30}\end{bmatrix} \tag{5.25}$$

where

$$f_{10} = (-a_{55}s + ia_{56}\beta)\overline{u}_x(\alpha,\beta,0) + ia_{14}\alpha\overline{u}_y(\alpha,\beta,0) + ia_{13}\alpha\overline{u}_z(\alpha,\beta,0) \tag{5.26a}$$

$$f_{20} = ia_{56}\alpha\overline{u}_x(\alpha,\beta,0) + (ia_{24}\beta - a_{44}s)\overline{u}_y(\alpha,\beta,0) + (ia_{23}\beta - a_{34}s)\overline{u}_z(\alpha,\beta,0) \tag{5.26b}$$

$$f_{30} = ia_{55}\alpha\overline{u}_x(\alpha,\beta,0) + (-a_{34}s + ia_{44}\beta)\overline{u}_y(\alpha,\beta,0) + (-a_{33}s + ia_{34}\beta)\overline{u}_z(\alpha,\beta,0) \tag{5.26c}$$

and

$$f_{11}(\alpha,\beta,s) = a_{11}\alpha^2 + a_{66}\beta^2 + a_{55}(is)^2 + 2a_{56}\beta(is) \tag{5.27a}$$

$$f_{12}(\alpha,\beta,s) = f_{21}(\alpha,\beta,s) = (a_{12}+a_{66})\alpha\beta + (a_{14}+a_{56})\alpha(is) \tag{5.27b}$$

$$f_{13}(\alpha,\beta,s) = f_{31}(\alpha,\beta,s) = (a_{14}+a_{56})\alpha\beta + (a_{13}+a_{55})\alpha(is) \tag{5.27c}$$

$$f_{22}(\alpha,\beta,s)=a_{66}\alpha^2+a_{22}\beta^2+a_{44}(is)^2+2a_{24}\beta(is) \tag{5.27d}$$

$$f_{23}(\alpha,\beta,s)=f_{32}(\alpha,\beta,s)=a_{56}\alpha^2+a_{24}\beta^2+a_{34}(is)^2+(a_{23}+a_{44})\beta(is) \tag{5.27e}$$

$$f_{33}(\alpha,\beta,s)=a_{55}\alpha^2+a_{44}\beta^2+a_{33}(is)^2+2a_{34}\beta(is) \tag{5.27f}$$

$$\begin{aligned}\det\left[f_{ij}\right]&=\\&=a_2a_5^2\prod_{i=1}^{3}A_i\{(is)^2+\alpha^2+\beta^2-[(is)\cos\phi-\beta\sin\phi]^2\}+[(is)\cos\phi-\beta\sin\phi]^2\\&=-m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)\end{aligned} \tag{5.28}$$

The $\bar{u}_x(\alpha,\beta,0)$, $\bar{u}_x(\alpha,\beta,0)$, and $\bar{u}_x(\alpha,\beta,0)$ are the undetermined boundary conditions in Eqs. (5.26a)-(5.26c).

The characteristic equation (Eq. (5.28)) with six roots can be expressed as:

$$s_1(\alpha,\beta)=\frac{-i\beta\sin\phi\cos\phi(-1+A_1)-\sqrt{A_1(\beta^2+\alpha^2(\cos^2\phi+A_1\sin^2\phi))}}{\cos^2\phi+A_1\sin^2\phi}=u_1(\alpha,\beta)=i\gamma_1(\alpha,\beta) \tag{5.29a}$$

$$s_2(\alpha,\beta)=\frac{-i\beta\sin\phi\cos\phi(-1+A_2)-\sqrt{A_2(\beta^2+\alpha^2(\cos^2\phi+A_2\sin^2\phi))}}{\cos^2\phi+A_2\sin^2\phi}=u_2(\alpha,\beta)=i\gamma_2(\alpha,\beta) \tag{5.29b}$$

$$s_3(\alpha,\beta)=\frac{-i\beta\sin\phi\cos\phi(-1+A_3)-\sqrt{A_3(\beta^2+\alpha^2(\cos^2\phi+A_3\sin^2\phi))}}{\cos^2\phi+A_3\sin^2\phi}=u_3(\alpha,\beta)=i\gamma_3(\alpha,\beta) \tag{5.29c}$$

$$s_4(\alpha,\beta)=\frac{-i\beta\sin\phi\cos\phi(-1+A_1)+\sqrt{A_1(\beta^2+\alpha^2(\cos^2\phi+A_1\sin^2\phi))}}{\cos^2\phi+A_1\sin^2\phi}=u_4(\alpha,\beta)=i\gamma_4(\alpha,\beta) \tag{5.29d}$$

$$s_5(\alpha,\beta)=\frac{-i\beta\sin\phi\cos\phi(-1+A_2)+\sqrt{A_2(\beta^2+\alpha^2(\cos^2\phi+A_2\sin^2\phi))}}{\cos^2\phi+A_2\sin^2\phi}=u_5(\alpha,\beta)=i\gamma_5(\alpha,\beta) \tag{5.29e}$$

$$s_6(\alpha,\beta)=\frac{-i\beta\sin\phi\cos\phi(-1+A_3)+\sqrt{A_3(\beta^2+\alpha^2(\cos^2\phi+A_3\sin^2\phi))}}{\cos^2\phi+A_3\sin^2\phi}=u_6(\alpha,\beta)=i\gamma_6(\alpha,\beta) \tag{5.29f}$$

where the real part of the $\{s_1,\ s_2\ ,\ s_3\}$ are negative and $\{s_4,\ s_5\ ,\ s_6\}$ are positive.

The A_1, A_2, and A_3 are the same as presented in Eqs. (3.48a)-(3.48c).

Eq. (5.25) can be rewritten as a system of three equations:

$$\overline{U}_x(\alpha,\beta,s)=\frac{p_x D_{11}(\alpha,\beta,s)+p_y D_{12}(\alpha,\beta,s)+p_z D_{13}(\alpha,\beta,s)}{2\pi m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{-sh} \\ +\frac{f_{10}D_{11}(\alpha,\beta,s)+f_{20}D_{12}(\alpha,\beta,s)+f_{30}D_{13}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)} \tag{5.30a}$$

$$\overline{U}_y(\alpha,\beta,s)=\frac{p_x D_{21}(\alpha,\beta,s)+p_y D_{22}(\alpha,\beta,s)+p_z D_{23}(\alpha,\beta,s)}{2\pi m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{-sh} \\ +\frac{f_{10}D_{21}(\alpha,\beta,s)+f_{20}D_{22}(\alpha,\beta,s)+f_{30}D_{23}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)} \tag{5.30b}$$

$$\overline{U}_z(\alpha,\beta,s)=\frac{p_x D_{31}(\alpha,\beta,s)+p_y D_{32}(\alpha,\beta,s)+p_z D_{33}(\alpha,\beta,s)}{2\pi m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{-sh} \\ +\frac{f_{10}D_{31}(\alpha,\beta,s)+f_{20}D_{32}(\alpha,\beta,s)+f_{30}D_{33}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)} \tag{5.30c}$$

In a half-space, the desired displacements of $\bar{u}_x(\alpha,\beta,z)$, $\bar{u}_y(\alpha,\beta,z)$, and $\bar{u}_z(\alpha,\beta,z)$ can be obtained by the following inverse Laplace transform of Eqs. (5.30a)-(5.30c) as:

$$\bar{u}_x(\alpha,\beta,z)=\frac{1}{(2\pi)^2 i}\int_{c-i\infty}^{c+i\infty}\frac{p_x D_{11}(\alpha,\beta,s)+p_y D_{12}(\alpha,\beta,s)+p_z D_{13}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{s(z-h)}ds \\ +\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{f_{10}D_{11}(\alpha,\beta,s)+f_{20}D_{12}(\alpha,\beta,s)+f_{30}D_{13}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{sz}ds \tag{5.31a}$$

$$\bar{u}_y(\alpha,\beta,z)=\frac{1}{(2\pi)^2 i}\int_{c-i\infty}^{c+i\infty}\frac{p_x D_{21}(\alpha,\beta,s)+p_y D_{22}(\alpha,\beta,s)+p_z D_{23}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{s(z-h)}ds \\ +\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{f_{10}D_{21}(\alpha,\beta,s)+f_{20}D_{22}(\alpha,\beta,s)+f_{30}D_{23}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{sz}ds \tag{5.31b}$$

$$\bar{u}_z(\alpha,\beta,z)=\frac{1}{(2\pi)^2 i}\int_{c-i\infty}^{c+i\infty}\frac{p_x D_{31}(\alpha,\beta,s)+p_y D_{32}(\alpha,\beta,s)+p_z D_{33}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{s(z-h)}ds + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{f_{10}D_{31}(\alpha,\beta,s)+f_{20}D_{32}(\alpha,\beta,s)+f_{30}D_{33}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{sz}ds \tag{5.31c}$$

where $c>0$ and the path of integration with respect to s is a vertical line parallel to and on the right of imaginary axis in the complex s plane.

Substituting Eqs. (5.26a)-(5.26c) into Eqs. (5.31a)-(5.31c), then $\bar{u}_x(\alpha,\beta,z)$, $\bar{u}_y(\alpha,\beta,z)$, and $\bar{u}_z(\alpha,\beta,z)$ can be rewritten as:

$$\bar{u}_x(\alpha,\beta,z)=\frac{1}{(2\pi)^2 i}\int_{c-i\infty}^{c+i\infty}\frac{p_x D_{11}(\alpha,\beta,s)+p_y D_{12}(\alpha,\beta,s)+p_z D_{13}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{s(z-h)}ds + \int_{c-i\infty}^{c+i\infty}\{A_{xx}\bar{u}_x(\alpha,\beta,0)+A_{xy}\bar{u}_y(\alpha,\beta,0)+A_{xz}\bar{u}_z(\alpha,\beta,0)e^{sz}ds \tag{5.32a}$$

$$\bar{u}_y(\alpha,\beta,z)=\frac{1}{(2\pi)^2 i}\int_{c-i\infty}^{c+i\infty}\frac{p_x D_{21}(\alpha,\beta,s)+p_y D_{22}(\alpha,\beta,s)+p_z D_{23}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{s(z-h)}ds + \int_{c-i\infty}^{c+i\infty}\{A_{yx}\bar{u}_x(\alpha,\beta,0)+A_{yy}\bar{u}_y(\alpha,\beta,0)+A_{yz}\bar{u}_z(\alpha,\beta,0)e^{sz}ds \tag{5.32b}$$

$$\bar{u}_z(\alpha,\beta,z)=\frac{1}{(2\pi)^2 i}\int_{c-i\infty}^{c+i\infty}\frac{p_x D_{31}(\alpha,\beta,s)+p_y D_{32}(\alpha,\beta,s)+p_z D_{33}(\alpha,\beta,s)}{m_t(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}e^{s(z-h)}ds + \int_{c-i\infty}^{c+i\infty}\{A_{zx}\bar{u}_x(\alpha,\beta,0)+A_{zy}\bar{u}_y(\alpha,\beta,0)+A_{zz}\bar{u}_z(\alpha,\beta,0)e^{sz}ds \tag{5.32c}$$

where

$$A_{xx}(\alpha,\beta,s)=\frac{(-a_{55}s+ia_{56}\beta)D_{11}(\alpha,\beta,s)+ia_{56}\alpha D_{12}(\alpha,\beta,s)+ia_{55}\alpha D_{13}(\alpha,\beta,s)}{m_t S(\alpha,\beta,s)} \tag{5.33a}$$

$$A_{xy}(\alpha,\beta,s)=\frac{ia_{14}\alpha D_{11}(\alpha,\beta,s)+(ia_{24}\beta-a_{44}s)D_{12}(\alpha,\beta,s)+(-a_{34}s+ia_{44}\beta)D_{13}(\alpha,\beta,s)}{m_t S(\alpha,\beta,s)} \tag{5.33b}$$

$$A_{xz}(\alpha,\beta,s)=\frac{ia_{13}\alpha D_{11}(\alpha,\beta,s)+(ia_{23}\beta-a_{34}s)D_{12}(\alpha,\beta,s)+(-a_{33}s+ia_{34}\beta)D_{13}(\alpha,\beta,s)}{m_t S(\alpha,\beta,s)} \tag{5.33c}$$

$$A_{yx}(\alpha,\beta,s)=\frac{(-a_{55}s+ia_{56}\beta)D_{21}(\alpha,\beta,s)+ia_{56}\alpha D_{22}(\alpha,\beta,s)+ia_{55}\alpha D_{23}(\alpha,\beta,s)}{m_t S(\alpha,\beta,s)} \tag{5.33d}$$

$$A_{yy}(\alpha,\beta,s)=\frac{ia_{14}\alpha D_{21}(\alpha,\beta,s)+(ia_{24}\beta-a_{44}s)D_{22}(\alpha,\beta,s)+(-a_{34}s+ia_{44}\beta)D_{23}(\alpha,\beta,s)}{m_t S(\alpha,\beta,s)} \tag{5.33e}$$

$$A_{yz}(\alpha,\beta,s)=\frac{ia_{13}\alpha D_{21}(\alpha,\beta,s)+(ia_{23}\beta-a_{34}s)D_{22}(\alpha,\beta,s)+(-a_{33}s+ia_{34}\beta)D_{23}(\alpha,\beta,s)}{m_t S(\alpha,\beta,s)} \tag{5.33f}$$

$$A_{zx}(\alpha,\beta,s)=\frac{(-a_{55}s+ia_{56}\beta)D_{31}(\alpha,\beta,s)+ia_{56}\alpha D_{32}(\alpha,\beta,s)+ia_{55}\alpha D_{33}(\alpha,\beta,s)}{m_t S(\alpha,\beta,s)} \tag{5.33g}$$

$$A_{zy}(\alpha,\beta,s)=\frac{ia_{14}\alpha D_{31}(\alpha,\beta,s)+(ia_{24}\beta-a_{44}s)D_{32}(\alpha,\beta,s)+(-a_{34}s+ia_{44}\beta)D_{33}(\alpha,\beta,s)}{m_t S(\alpha,\beta,s)} \tag{5.33h}$$

$$A_{zz}(\alpha,\beta,s)=\frac{ia_{13}\alpha D_{31}(\alpha,\beta,s)+(ia_{23}\beta-a_{34}s)D_{32}(\alpha,\beta,s)+(-a_{33}s+ia_{34}\beta)D_{33}(\alpha,\beta,s)}{m_t S(\alpha,\beta,s)} \tag{5.33i}$$

$$S(\alpha,\beta,s)=(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6) \tag{5.33j}$$

When $0 \le z < h$, the solutions of the first term in the left-hand side of Eqs. (5.32a)-(5.32c) can be integral by the path of the contour in Fig. 5.3 and expressed as:

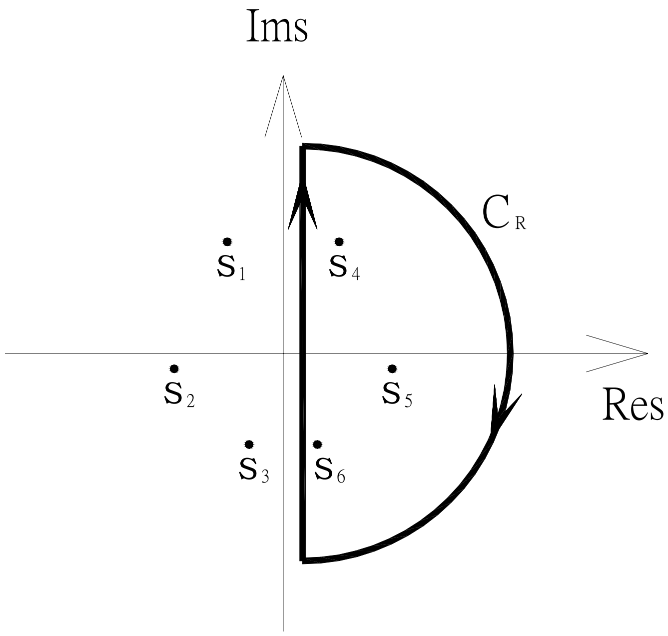


Fig. 5.3 A closed contour on right-hand side of s-plane.

$$\bar{u}_{x(1)}(\alpha,\beta,z) = -B_{xs}^{4}e^{s_4(z-h)} - B_{xs}^{5}e^{s_5(z-h)} - B_{xs}^{6}e^{s_6(z-h)} \tag{5.34a}$$

$$\bar{u}_{y(1)}(\alpha,\beta,z) = -B_{ys}^{4}e^{s_4(z-h)} - B_{ys}^{5}e^{s_5(z-h)} - B_{ys}^{6}e^{s_6(z-h)} \tag{5.34b}$$

$$\bar{u}_{z(1)}(\alpha,\beta,z) = -B_{zs}^{4}e^{s_4(z-h)} - B_{zs}^{5}e^{s_5(z-h)} - B_{zs}^{6}e^{s_6(z-h)} \tag{5.34c}$$

When $z > h$,the solutions can be integral by the path of the contour in Fig. 5.4 and expressed as:

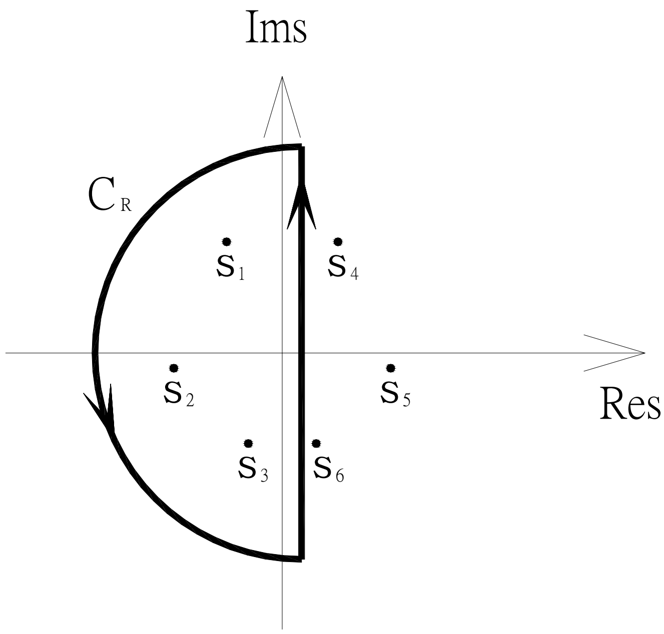


Fig. 5.4. A closed contour on left-hand side of s-plane.

$$\bar{u}_{x(1)}(\alpha,\beta,z) = B_{xs}^{1} e^{s_1(z-h)} + B_{xs}^{2} e^{s_2(z-h)} + B_{xs}^{3} e^{s_3(z-h)} \tag{5.35a}$$

$$\bar{u}_{y(1)}(\alpha,\beta,z) = B_{ys}^{1} e^{s_1(z-h)} + B_{ys}^{2} e^{s_2(z-h)} + B_{ys}^{3} e^{s_3(z-h)} \tag{5.35b}$$

$$\bar{u}_{z(1)}(\alpha,\beta,z) = B_{zs}^{1} e^{s_1(z-h)} + B_{zs}^{2} e^{s_2(z-h)} + B_{zs}^{3} e^{s_3(z-h)} \tag{5.35c}$$

where

$$B_{xs}^{j}(\alpha,\beta) = \frac{P_x D_{11}(\alpha,\beta,s_j) + P_y D_{12}(\alpha,\beta,s_j) + P_z D_{13}(\alpha,\beta,s_j)}{2\pi m_t S_j} \tag{5.36a}$$

$$B_{ys}^{j}(\alpha,\beta) = \frac{P_x D_{21}(\alpha,\beta,s_j) + P_y D_{22}(\alpha,\beta,s_j) + P_z D_{23}(\alpha,\beta,s_j)}{2\pi m_t S_j} \tag{5.36b}$$

$$B_{zs}^{j}(\alpha,\beta) = \frac{P_x D_{31}(\alpha,\beta,s_j) + P_y D_{32}(\alpha,\beta,s_j) + P_z D_{33}(\alpha,\beta,s_j)}{2\pi m_t S_j} \tag{5.36c}$$

Suppose that $u = s$ and $u_j = s_j$, since $B^j_{xs}(\alpha,\beta)$, $B^j_{ys}(\alpha,\beta)$, and $B^j_{zs}(\alpha,\beta)$ in Eqs. (5.36a)-(5.36c) have the same forms as $B^j_x(\alpha,\beta)$, $B^j_y(\alpha,\beta)$, and $B^j_z(\alpha,\beta)$ in Eqs. (4.07a)-(4.07c); hence, $\bar{u}_{x(1)}(\alpha,\beta,z)$, $\bar{u}_{y(1)}(\alpha,\beta,z)$, and $\bar{u}_{z(1)}(\alpha,\beta,z)$ in Eqs. (5.34a)-(5.34c) and Eqs. (5.35a)-(5.35c), and $\bar{u}_{x(P)}(\alpha,\beta,z)$, $\bar{u}_{y(P)}(\alpha,\beta,z)$, and $\bar{u}_{z(P)}(\alpha,\beta,z)$ in Eqs. (5.05a)-(5.05c), and Eqs. (5.06a)-(5.06c) are the particular solutions of the three displacement functions.

Following, we can find the solutions of the second term in the left-hand side of Eqs. (5.32a)-(5.32c) should be the general solution of Eqs. (5.04a)-(5.04c) as:

When $z > 0$, the solutions can be integral by the path of the contour in Fig. 5.4 and expressed as:

$$\bar{u}_{x(H)}(\alpha,\beta,z) = A^1_x e^{s_1 z} + A^2_x e^{s_2 z} + A^3_x e^{s_3 z} \tag{5.37a}$$

$$\bar{u}_{y(H)}(\alpha,\beta,z) = A^1_y e^{s_1 z} + A^2_y e^{s_2 z} + A^3_y e^{s_3 z} \tag{5.37b}$$

$$\bar{u}_{z(H)}(\alpha,\beta,z) = A^1_z e^{s_1 z} + A^2_z e^{s_2 z} + A^3_z e^{s_3 z} \tag{5.37c}$$

where

$$A^i_x = A^i_{xx}\bar{u}_x(\alpha,\beta,0) + A^i_{xy}\bar{u}_y(\alpha,\beta,0) + A^i_{xz}\bar{u}_z(\alpha,\beta,0) \tag{5.38a}$$

$$A^i_y = A^i_{yx}\bar{u}_x(\alpha,\beta,0) + A^i_{yy}\bar{u}_y(\alpha,\beta,0) + A^i_{yz}\bar{u}_z(\alpha,\beta,0) \tag{5.38b}$$

$$A^i_z = A^i_{zx}\bar{u}_x(\alpha,\beta,0) + A^i_{zy}\bar{u}_y(\alpha,\beta,0) + A^i_{zz}\bar{u}_z(\alpha,\beta,0) \tag{5.38c}$$

, and in which:

$$A_{xx}^{j}(\alpha,\beta)=\frac{(-a_{55}s_j+ia_{56}\beta)D_{11}(\alpha,\beta,s_j)+ia_{56}\alpha D_{12}(\alpha,\beta,s_j)+ia_{55}\alpha D_{13}(\alpha,\beta,s_j)}{m_t S_j} \quad (5.39a)$$

$$A_{xy}^{j}(\alpha,\beta)=\frac{ia_{14}\alpha D_{11}(\alpha,\beta,s_j)+(ia_{24}\beta-a_{44}s_j)D_{12}(\alpha,\beta,s_j)+(-a_{34}s_j+ia_{44}\beta)D_{13}(\alpha,\beta,s_j)}{m_t S_j} \quad (5.39b)$$

$$A_{xz}^{j}(\alpha,\beta)=\frac{ia_{13}\alpha D_{11}(\alpha,\beta,s_j)+(ia_{23}\beta-a_{34}s_j)D_{12}(\alpha,\beta,s_j)+(-a_{33}s_j+ia_{34}\beta)D_{13}(\alpha,\beta,s_j)}{m_t S_j} \quad (5.39c)$$

$$A_{yx}^{j}(\alpha,\beta)=\frac{(-a_{55}s_1+ia_{56}\beta)D_{21}(\alpha,\beta,s_j)+ia_{56}\alpha D_{22}(\alpha,\beta,s_j)+ia_{55}\alpha D_{23}(\alpha,\beta,s_j)}{m_t S_j} \quad (5.39d)$$

$$A_{yy}^{j}(\alpha,\beta)=\frac{ia_{14}\alpha D_{21}(\alpha,\beta,s_j)+(ia_{24}\beta-a_{44}s_j)D_{22}(\alpha,\beta,s_j)+(-a_{34}s_j+ia_{44}\beta)D_{23}(\alpha,\beta,s_j)}{m_t S_j} \quad (5.39e)$$

$$A_{yz}^{j}(\alpha,\beta)=\frac{ia_{13}\alpha D_{21}(\alpha,\beta,s_j)+(ia_{23}\beta-a_{34}s_j)D_{22}(\alpha,\beta,s_j)+(-a_{33}s_j+ia_{34}\beta)D_{23}(\alpha,\beta,s_j)}{m_t S_j} \quad (5.39f)$$

$$A_{zx}^{j}(\alpha,\beta)=\frac{(-a_{55}s_j+ia_{56}\beta)D_{31}(\alpha,\beta,s_j)+ia_{56}\alpha D_{32}(\alpha,\beta,s_j)+ia_{55}\alpha D_{33}(\alpha,\beta,s_j)}{m_t S_j} \quad (5.39g)$$

$$A_{zy}^{j}(\alpha,\beta)=\frac{ia_{14}\alpha D_{31}(\alpha,\beta,s_j)+(ia_{24}\beta-a_{44}s_j)D_{32}(\alpha,\beta,s_j)+(-a_{34}s_j+ia_{44}\beta)D_{33}(\alpha,\beta,s_j)}{m_t S_j} \quad (5.39h)$$

$$A_{zz}^{j}(\alpha,\beta)=\frac{ia_{13}\alpha D_{31}(\alpha,\beta,s_j)+(ia_{23}\beta-a_{34}s_j)D_{32}(\alpha,\beta,s_j)+(-a_{33}s_j+ia_{34}\beta)D_{33}(\alpha,\beta,s_j)}{m_t S_j} \quad (5.39i)$$

$$S_j(\alpha,\beta)=\frac{\partial[(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)]}{\partial s}, s=s_j \quad (5.39j)$$

When z=0, Eqs. (5.37a)-(5.37c), and Eqs. (5.04a)-(5.04c) can be stated as:

$$\begin{bmatrix} \bar{u}_{x(H)}(\alpha,\beta,0) \\ \bar{u}_{y(H)}(\alpha,\beta,0) \\ \bar{u}_{z(H)}(\alpha,\beta,0) \end{bmatrix} = \begin{bmatrix} A_{xx}^1 + A_{xx}^2 + A_{xx}^3 & A_{xy}^1 + A_{xy}^2 + A_{xy}^3 & A_{xz}^1 + A_{xz}^2 + A_{xz}^3 \\ A_{yx}^1 + A_{yx}^2 + A_{yx}^3 & A_{yy}^1 + A_{yy}^2 + A_{yy}^3 & A_{yz}^1 + A_{yz}^2 + A_{yz}^3 \\ A_{zx}^1 + A_{zx}^2 + A_{zx}^3 & A_{zy}^1 + A_{zy}^2 + A_{zy}^3 & A_{zz}^1 + A_{zz}^2 + A_{zz}^3 \end{bmatrix} \begin{bmatrix} \bar{u}_x(\alpha,\beta,0) \\ \bar{u}_y(\alpha,\beta,0) \\ \bar{u}_z(\alpha,\beta,0) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{U_1} & \frac{1}{U_2} & \frac{1}{U_3} \\ \frac{D_{21}^1}{D_{11}^1 U_1} & \frac{D_{21}^2}{D_{11}^2 U_2} & \frac{D_{21}^3}{D_{11}^2 U_2} \\ \frac{D_{31}^1}{D_{11}^1 U_1} & \frac{D_{31}^2}{D_{11}^3 U_2} & \frac{D_{31}^3}{D_{11}^3 U_3} \end{bmatrix} \begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_x(\alpha,\beta,0) \\ \bar{u}_y(\alpha,\beta,0) \\ \bar{u}_z(\alpha,\beta,0) \end{bmatrix}$$

$$= \frac{1}{U_1 U_2 U_3 D_{11}^1 D_{11}^2 D_{11}^3} \begin{bmatrix} D_{11}^1 & D_{11}^2 & D_{11}^3 \\ D_{21}^1 & D_{21}^2 & D_{21}^3 \\ D_{31}^1 & D_{31}^2 & D_{31}^3 \end{bmatrix} \begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_x(\alpha,\beta,0) \\ \bar{u}_y(\alpha,\beta,0) \\ \bar{u}_z(\alpha,\beta,0) \end{bmatrix}$$

$$= \begin{bmatrix} D_{11}^1 & D_{11}^2 & D_{11}^3 \\ D_{21}^1 & D_{21}^2 & D_{21}^3 \\ D_{31}^1 & D_{31}^2 & D_{31}^3 \end{bmatrix} \begin{bmatrix} C_{d2}^1 \\ C_{d2}^2 \\ C_{d2}^3 \end{bmatrix} \tag{5.40}$$

Then Eq. (5.40) can be rewritten as:

$$\begin{aligned} \bar{u}_{x(H)}(\alpha,\beta,0) &= (A_{xx}^1 + A_{xx}^2 + A_{xx}^3)\bar{u}_x(\alpha,\beta,0) + (A_{xy}^1 + A_{xy}^2 + A_{xy}^3)\bar{u}_y(\alpha,\beta,0) \\ &\quad + (A_{xz}^1 + A_{xz}^2 + A_{xz}^3)\bar{u}_z(\alpha,\beta,0) \\ &= C_{d2}^1 D_{11}^1 + C_{d2}^2 D_{11}^2 + C_{d2}^1 D_{11}^2 \end{aligned} \tag{5.41a}$$

$$\begin{aligned} \bar{u}_{y(H)}(\alpha,\beta,0) &= (A_{yx}^1 + A_{yx}^2 + A_{yx}^3)\bar{u}_x(\alpha,\beta,0) + (A_{yy}^1 + A_{yy}^2 + A_{yy}^3)\bar{u}_y(\alpha,\beta,0) \\ &\quad + (A_{yz}^1 + A_{yz}^2 + A_{yz}^3)\bar{u}_z(\alpha,\beta,0) \\ &= C_{d2}^1 D_{21}^1 + C_{d2}^2 D_{21}^2 + C_{d2}^1 D_{21}^2 \end{aligned} \tag{5.41b}$$

$$\begin{aligned} \bar{u}_{z(H)}(\alpha,\beta,0) &= (A_{zx}^1 + A_{zx}^2 + A_{zx}^3)\bar{u}_x(\alpha,\beta,0) + (A_{zy}^1 + A_{zy}^2 + A_{zy}^3)\bar{u}_y(\alpha,\beta,0) \\ &\quad + (A_{zz}^1 + A_{zz}^2 + A_{zz}^3)\bar{u}_z(\alpha,\beta,0) \\ &= C_{d2}^1 D_{31}^1 + C_{d2}^2 D_{31}^2 + C_{d2}^1 D_{31}^2 \end{aligned} \tag{5.41c}$$

The general solutions are the sum of the homogeneous solution (Eqs. (5.37a)-(5.37c)) and particular solutions (Eqs. (5.34a)-(5.34c)) as follows:

$$\bar{u}_x(\alpha,\beta,0) = (A_{xx}^1 + A_{xx}^2 + A_{xx}^3)\bar{u}_x(\alpha,\beta,0) + (A_{xy}^1 + A_{xy}^2 + A_{xy}^3)\bar{u}_y(\alpha,\beta,0) + (A_{xz}^1 + A_{xz}^2 + A_{xz}^3)\bar{u}_z(\alpha,\beta,0) - B_{xs}^4 e^{-s_4 h} - B_{xs}^5 e^{-s_5 h} - B_{xs}^6 e^{-s_6 h} \tag{5.42a}$$

$$\bar{u}_y(\alpha,\beta,0) = (A_{yx}^1 + A_{yx}^2 + A_{yx}^3)\bar{u}_x(\alpha,\beta,0) + (A_{yy}^1 + A_{yy}^2 + A_{yy}^3)\bar{u}_y(\alpha,\beta,0) + (A_{yz}^1 + A_{yz}^2 + A_{yz}^3)\bar{u}_z(\alpha,\beta,0) - B_{ys}^4 e^{-s_4 h} - B_{ys}^5 e^{-s_5 h} - B_{ys}^6 e^{-s_6 h} \tag{5.42b}$$

$$\bar{u}_z(\alpha,\beta,0) = (A_{zx}^1 + A_{zx}^2 + A_{zx}^3)\bar{u}_x(\alpha,\beta,0) + (A_{zy}^1 + A_{zy}^2 + A_{zy}^3)\bar{u}_y(\alpha,\beta,0) + (A_{zz}^1 + A_{zz}^2 + A_{zz}^3)\bar{u}_z(\alpha,\beta,0) - B_{zs}^4 e^{-s_4 h} - B_{zs}^5 e^{-s_5 h} - B_{zs}^6 e^{-s_6 h} \tag{5.42c}$$

Since,

$$\sum_j^6 A_{xx}^j = \sum_j^6 A_{yy}^j = \sum_j^6 A_{zz}^j = 1 \tag{5.43a}$$

$$\sum_j^6 A_{xy}^j = \sum_j^6 A_{xz}^j = \sum_j^6 A_{yx}^j = \sum_j^6 A_{yz}^j = \sum_j^6 A_{zx}^j = \sum_j^6 A_{zy}^j = 0 \tag{5.43b}$$

Eqs. (5.41a)-(5.41c) and Eqs.(5.42a)-(5.42c) can be expressed as:

$$\begin{aligned}\bar{u}_{x(P)}(\alpha,\beta,0) &= -B_{xs}^4 e^{-s_4 h} - B_{xs}^5 e^{-s_5 h} - B_{xs}^6 e^{-s_6 h} \\ &= (A_{xx}^4 + A_{xx}^5 + A_{xx}^6)\bar{u}_x(\alpha,\beta,0) + (A_{xy}^4 + A_{xy}^5 + A_{xy}^6)\bar{u}_y(\alpha,\beta,0) \\ &\quad + (A_{xz}^4 + A_{xz}^5 + A_{xz}^6)\bar{u}_z(\alpha,\beta,0)\end{aligned} \tag{5.44a}$$

$$\begin{aligned}\bar{u}_{y(P)}(\alpha,\beta,0) &= -B_{ys}^4 e^{-s_4 h} - B_{ys}^5 e^{-s_5 h} - B_{ys}^6 e^{-s_6 h} \\ &= (A_{yx}^4 + A_{yx}^5 + A_{yx}^6)\bar{u}_x(\alpha,\beta,0) + (A_{yy}^4 + A_{yy}^5 + A_{yy}^6)\bar{u}_y(\alpha,\beta,0) \\ &\quad + (A_{yz}^4 + A_{yz}^5 + A_{yz}^6)\bar{u}_z(\alpha,\beta,0)\end{aligned} \tag{5.44b}$$

$$\begin{aligned}\bar{u}_{z(P)}(\alpha,\beta,0) &= -B_{zs}^4 e^{-s_4 h} - B_{zs}^5 e^{-s_5 h} - B_{zs}^6 e^{-s_6 h} \\ &= (A_{zx}^4 + A_{zx}^5 + A_{zx}^6)\bar{u}_x(\alpha,\beta,0) + (A_{zy}^4 + A_{zy}^5 + A_{zy}^6)\bar{u}_y(\alpha,\beta,0) \\ &\quad + (A_{zz}^4 + A_{zz}^5 + A_{zz}^6)\bar{u}_z(\alpha,\beta,0)\end{aligned} \tag{5.44c}$$

When $z > h$ and $h \to 0$, the particular of Eqs. (5.35a)-(5.35c), and the general solutions of $\bar{u}_x(\alpha,\beta,0)$, $\bar{u}_y(\alpha,\beta,0)$, and $\bar{u}_z(\alpha,\beta,0)$ can be expressed as:

$$\begin{aligned}\bar{u}_x(\alpha,\beta,0) &= (A_{xx}^1 + A_{xx}^2 + A_{xx}^3)\bar{u}_x(\alpha,\beta,0) + (A_{xy}^1 + A_{xy}^2 + A_{xy}^3)\bar{u}_y(\alpha,\beta,0)\\ &\quad + (A_{xz}^1 + A_{xz}^2 + A_{xz}^3)\bar{u}_z(\alpha,\beta,0) + B_{xs}^1 + B_{xs}^2 + B_{xs}^3\end{aligned} \tag{5.45a}$$

$$\begin{aligned}\bar{u}_y(\alpha,\beta,0) &= (A_{yx}^1 + A_{yx}^2 + A_{yx}^3)\bar{u}_x(\alpha,\beta,0) + (A_{yy}^1 + A_{yy}^2 + A_{yy}^3)\bar{u}_y(\alpha,\beta,0)\\ &\quad + (A_{yz}^1 + A_{yz}^2 + A_{yz}^3)\bar{u}_z(\alpha,\beta,0) + B_{ys}^1 + B_{ys}^2 + B_{ys}^3\end{aligned} \tag{5.45b}$$

$$\begin{aligned}\bar{u}_z(\alpha,\beta,0) &= (A_{zx}^1 + A_{zx}^2 + A_{zx}^3)\bar{u}_x(\alpha,\beta,0) + (A_{zy}^1 + A_{zy}^2 + A_{zy}^3)\bar{u}_y(\alpha,\beta,0)\\ &\quad + (A_{zz}^1 + A_{zz}^2 + A_{zz}^3)\bar{u}_z(\alpha,\beta,0) + B_{zs}^1 + B_{zs}^2 + B_{zs}^3\end{aligned} \tag{5.45c}$$

The general solutions of $\bar{u}_{x(P)}(\alpha,\beta,0)$, $\bar{u}_{y(P)}(\alpha,\beta,0)$, and $\bar{u}_{z(P)}(\alpha,\beta,0)$ can present as:

$$\begin{aligned}\begin{bmatrix}\bar{u}_{x(P)}(\alpha,\beta,0)\\ \bar{u}_{y(P)}(\alpha,\beta,0)\\ \bar{u}_{z(P)}(\alpha,\beta,0)\end{bmatrix} &= \begin{bmatrix}B_{xs}^1 + B_{xs}^2 + B_{xs}^3\\ B_{ys}^1 + B_{ys}^2 + B_{ys}^3\\ B_{zs}^1 + B_{zs}^2 + B_{zs}^3\end{bmatrix}\\
&= \begin{bmatrix}\frac{D_{11}^1}{U_1}+\frac{D_{11}^2}{U_2}+\frac{D_{11}^3}{U_3} & \frac{D_{12}^1}{U_1}+\frac{D_{12}^2}{U_2}+\frac{D_{12}^3}{U_3} & \frac{D_{13}^1}{U_1}+\frac{D_{13}^2}{U_2}+\frac{D_{13}^3}{U_3}\\ \frac{D_{21}^1}{U_1}+\frac{D_{21}^2}{U_2}+\frac{D_{21}^3}{U_3} & \frac{D_{22}^1}{U_1}+\frac{D_{22}^2}{U_2}+\frac{D_{22}^3}{U_3} & \frac{D_{23}^1}{U_1}+\frac{D_{23}^2}{U_2}+\frac{D_{23}^3}{U_3}\\ \frac{D_{31}^1}{U_1}+\frac{D_{31}^2}{U_2}+\frac{D_{31}^3}{U_3} & \frac{D_{32}^1}{U_1}+\frac{D_{32}^2}{U_2}+\frac{D_{32}^3}{U_3} & \frac{D_{33}^1}{U_1}+\frac{D_{33}^2}{U_2}+\frac{D_{3}^3}{U_3}\end{bmatrix}\begin{bmatrix}\frac{P_x}{2\pi}\\ \frac{P_y}{2\pi}\\ \frac{P_z}{2\pi}\end{bmatrix}\\
&= \begin{bmatrix}D_{11}^1 & D_{11}^2 & D_{11}^3\\ D_{21}^1 & D_{21}^2 & D_{21}^3\\ D_{31}^1 & D_{31}^2 & D_{31}^3\end{bmatrix}\begin{bmatrix}\frac{1}{U_1} & \frac{D_{12}^1}{D_{11}^1U_1} & \frac{D_{13}^1}{D_{11}^1U_1}\\ \frac{1}{U_2} & \frac{D_{12}^2}{D_{11}^2U_2} & \frac{D_{13}^2}{D_{11}^2U_2}\\ \frac{1}{U_3} & \frac{D_{12}^3}{D_{11}^3U_3} & \frac{D_{13}^3}{D_{11}^3U_3}\end{bmatrix}\begin{bmatrix}\frac{P_x}{2\pi}\\ \frac{P_y}{2\pi}\\ \frac{P_z}{2\pi}\end{bmatrix} = \begin{bmatrix}D_{11}^1 & D_{11}^2 & D_{11}^3\\ D_{21}^1 & D_{21}^2 & D_{21}^3\\ D_{31}^1 & D_{31}^2 & D_{31}^3\end{bmatrix}\begin{bmatrix}C_{d1}^1\\ C_{d1}^2\\ C_{d1}^3\end{bmatrix}\end{aligned} \tag{5.46}$$

We show that:

$$\bar{u}_x(\alpha,\beta,0) = \bar{u}_{x(H)}(\alpha,\beta,0) + \bar{u}_{x(P)}(\alpha,\beta,0) = C_d^1D_{11}^1 + C_d^2D_{11}^2 + C_d^3D_{11}^3 \tag{5.47a}$$

$$\bar{u}_y(\alpha,\beta,0)=\bar{u}_{y(H)}(\alpha,\beta,0)+\bar{u}_{y(P)}(\alpha,\beta,0)=C_d^1 D_{21}^1+C_d^2 D_{21}^2+C_d^3 D_{21}^3 \quad (5.47b)$$

$$\bar{u}_z(\alpha,\beta,0)=\bar{u}_{z(H)}(\alpha,\beta,0)+\bar{u}_{z(P)}(\alpha,\beta,0)=C_d^1 D_{31}^1+C_d^2 D_{31}^2+C_d^3 D_{31}^3 \quad (5.47c)$$

The undetermined coefficients $\bar{u}_x(\alpha,\beta,0)$, $\bar{u}_y(\alpha,\beta,0)$, and $\bar{u}_z(\alpha,\beta,0)$ can be transformed to C_d^1, C_d^2, and C_d^3, in which C_d^1, C_d^2, and C_d^3 satisfy the rules of Eq. (5.12a). Hence, Eqs. (5.31a)-(5.31c) can be rewritten as:

$$\bar{u}_x(\alpha,\beta,z)=C_d^1 D_{11}^1 e^{u_1 z}+C_d^2 D_{11}^2 e^{u_2 z}+C_d^3 D_{11}^3 e^{u_3 z} \quad (5.48a)$$

$$\bar{u}_y(\alpha,\beta,z)=C_d^1 D_{21}^1 e^{u_1 z}+C_d^2 D_{21}^2 e^{u_2 z}+C_d^3 D_{21}^3 e^{u_3 z} \quad (5.48b)$$

$$\bar{u}_z(\alpha,\beta,z)=C_d^1 D_{31}^1 e^{u_1 z}+C_d^2 D_{31}^2 e^{u_2 z}+C_d^3 D_{31}^3 e^{u_3 z} \quad (5.48c)$$

The undetermined coefficients C_d^1, C_d^2, and C_d^3 can be determined using the method of Cramer′s rule.

5.4 Solutions for Displacements and Stresses by Inverse Fourier Transforms

The desired $u_x(x,y,z)$, $u_y(x,y,z)$, and $u_z(x,y,z)$ can be obtained by performing the double inverse Fourier transforms of Eqs. (5.16a)-(5.16c) as follows:

$$u_x(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_d^1 D_{11}(\alpha,\beta,u_1)e^{i(\alpha x+\beta y)+u_1 z}+C_d^2 D_{11}(\alpha,\beta,u_2)e^{i(\alpha x+\beta y)+u_2 z}+C_d^3 D_{11}(\alpha,\beta,u_3)e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \quad (5.49a)$$

$$u_y(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_d^1 D_{21}(\alpha,\beta,u_1)e^{i(\alpha x+\beta y)+u_1 z}+C_d^2 D_{21}(\alpha,\beta,u_2)e^{i(\alpha x+\beta y)+u_2 z}+C_d^3 D_{21}(\alpha,\beta,u_3)e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \quad (5.49b)$$

$$u_z(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_d^1 D_{31}(\alpha,\beta,u_1)e^{i(\alpha x+\beta y)+u_1 z}+C_d^2 D_{31}(\alpha,\beta,u_2)e^{i(\alpha x+\beta y)+u_2 z}+C_d^3 D_{31}(\alpha,\beta,u_3)e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{5.49c}$$

Similarly, $\sigma_{xx}(x,y,z)$, $\sigma_{yy}(x,y,z)$, $\sigma_{zz}(x,y,z)$, $\tau_{yz}(x,y,z)$, $\tau_{zx}(x,y,z)$, and $\tau_{xy}(x,y,z)$ also can be acquired by the double inverse Fourier transforms as:

For $z>0$ (region 1),

$$\sigma_{xx}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_d^1\bar{\sigma}_{xx}^1 e^{i(\alpha x+\beta y)+u_1 z}+C_d^2\bar{\sigma}_{xx}^2 e^{i(\alpha x+\beta y)+u_2 z}+C_d^3\bar{\sigma}_{xx}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{5.50a}$$

$$\sigma_{yy}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_d^1\bar{\sigma}_{yy}^1 e^{i(\alpha x+\beta y)+u_1 z}+C_d^2\bar{\sigma}_{yy}^2 e^{i(\alpha x+\beta y)+u_2 z}+C_d^3\bar{\sigma}_{yy}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{5.50b}$$

$$\sigma_{zz}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_d^1\bar{\sigma}_{zz}^1 e^{i(\alpha x+\beta y)+u_1 z}+C_d^2\bar{\sigma}_{zz}^2 e^{i(\alpha x+\beta y)+u_2 z}+C_d^3\bar{\sigma}_{zz}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{5.50c}$$

$$\tau_{yz}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_d^1\bar{\tau}_{yz}^1 e^{i(\alpha x+\beta y)+u_1 z}+C_d^2\bar{\tau}_{yz}^2 e^{i(\alpha x+\beta y)+u_2 z}+C_d^3\bar{\tau}_{yz}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{5.50d}$$

$$\tau_{zx}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_d^1\bar{\tau}_{zx}^1 e^{i(\alpha x+\beta y)+u_1 z}+C_d^2\bar{\tau}_{zx}^2 e^{i(\alpha x+\beta y)+u_2 z}+C_d^3\bar{\tau}_{zx}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{5.50e}$$

$$\tau_{xy}(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_d^1\bar{\tau}_{xy}^1 e^{i(\alpha x+\beta y)+u_1 z}+C_d^2\bar{\tau}_{xy}^2 e^{i(\alpha x+\beta y)+u_2 z}+C_d^3\bar{\tau}_{xy}^3 e^{i(\alpha x+\beta y)+u_3 z}\}d\alpha d\beta \tag{5.50f}$$

where:

$$\bar{\sigma}_{xx}^j=-i(\alpha a_{11}D_{11}^j+\beta a_{12}D_{21}^j+\beta a_{14}D_{31}^j-i(a_{14}D_{21}^j+a_{13}D_{31}^j)u_j) \tag{5.51a}$$

$$\bar{\sigma}_{yy}^j=-i(\alpha a_{12}D_{11}^j+\beta a_{22}D_{21}^j+\beta a_{24}D_{31}^j-i(a_{24}D_{21}^j+a_{23}D_{31}^j)u_j) \tag{5.51b}$$

$$\bar{\sigma}_{zz}^{j} = -i(\alpha a_{13} D_{11}^{j} + \beta a_{23} D_{21}^{j} + \beta a_{34} D_{31}^{j} - i(a_{34} D_{21}^{j} + a_{33} D_{31}^{j}) u_j) \tag{5.51c}$$

$$\bar{\tau}_{yz}^{j} = -i(\alpha a_{14} D_{11}^{j} + \beta a_{24} D_{21}^{j} + \beta a_{44} D_{31}^{j} - i(a_{44} D_{21}^{j} + a_{34} D_{31}^{j}) u_j) \tag{5.51d}$$

$$\bar{\tau}_{zx}^{j} = -i(a_{56}(\beta D_{11}^{j} + \alpha D_{21}^{j}) + a_{55}(\alpha D_{31}^{j} - i D_{11}^{j} u_j)) \tag{5.51e}$$

$$\bar{\tau}_{xy}^{j} = -i(a_{66}(\beta D_{11}^{j} + \alpha D_{21}^{j}) + a_{56}(\alpha D_{31}^{j} - i D_{11}^{j} u_j)) \tag{5.51f}$$

In Eqs. (5.51a)-(5.51f), j=1-3.

If we take a spherical co-ordinate system, which is shown in Fig. 4.4, the variables α, β, and u_i can be expressed in the Eqs. (4.44a)-(4.44b) and Eqs. (4.45a)-(4.45b).

According to Eqs. (3.53a)-(3.53f), D_{11}^{j}, D_{21}^{j}, D_{31}^{j} (j=1-6) and C_d^{j} (j=1-3) in Eqs. (5.49a)-(5.49c) and Eqs. (5.50a)-(5.50f) can be presented in terms of k and θ_x as:

$$D_{11}^{j}(k,\theta_x) = k^4 D_{11}^{j}(\theta_x) \tag{5.52a}$$

$$D_{21}^{j}(k,\theta_x) = k^4 D_{21}^{j}(\theta_x) \tag{5.52b}$$

$$D_{31}^{j}(k,\theta_x) = k^4 D_{31}^{j}(\theta_x) \tag{5.52c}$$

$$C_d^{j}(k,\theta_x) = k^{-5} C_d^{j}(\theta_x) \tag{5.52d}$$

Hence, Eqs. (5.51a)-(5.51f) also can be rewritten as:

$$\bar{\sigma}_{xx}^{j}(k,\theta_x) = k^5 \bar{\sigma}_{xx}^{j}(\theta_x) \tag{5.53a}$$

$$\bar{\sigma}_{yy}^{j}(k,\theta_x) = k^5 \bar{\sigma}_{yy}^{j}(\theta_x) \tag{5.53b}$$

$$\bar{\sigma}_{zz}^{j}(k,\theta_x) = k^5 \bar{\sigma}_{zz}^{j}(\theta_x) \tag{5.53c}$$

$$\bar{\tau}_{yz}^{j}(k,\theta_x) = k^5 \bar{\tau}_{yz}^{j}(\theta_x) \tag{5.53d}$$

$$\bar{\tau}_{zx}^{j}(k,\theta_x) = k^5 \bar{\tau}_{zx}^{j}(\theta_x) \tag{5.53e}$$

$$\bar{\tau}_{xy}^{j}(k,\theta_x) = k^5 \bar{\tau}_{xy}^{j}(\theta_x) \tag{5.53f}$$

Based on Eqs. (4.44a)-(4.44b), Eqs. (4.45a)-(4.45b), Eq. (4.49), Eq. (4.52), Eqs. (5.52a)-(5.52d) and Eqs. (5.53a)-(5.53f), then Eqs. (5.49a)-(5.49c) and Eqs. (5.50a)-(5.50f) can be represented as:

For $z > 0$ (region 1),

$$u_x(x,y,z) = -\frac{1}{2\pi}\int_0^{2\pi} \{C_d^1(\theta_x)D_{11}^1(\theta_x)\frac{1}{\psi_1(\theta_x)} + C_d^2(\theta_x)D_{11}^2(\theta_x)\frac{1}{\psi_2(\theta_x)} + C_d^3(\theta_x)D_{11}^3(\theta_x)\frac{1}{\psi_3(\theta_x)}\}d\theta_x \tag{5.54a}$$

$$u_y(x,y,z) = -\frac{1}{2\pi}\int_0^{2\pi} \{C_d^1(\theta_x)D_{21}^1(\theta_x)\frac{1}{\psi_1(\theta_x)} + C_d^2(\theta_x)D_{21}^2(\theta_x)\frac{1}{\psi_2(\theta_x)} + C_d^3(\theta_x)D_{21}^3(\theta_x)\frac{1}{\psi_3(\theta_x)}\}d\theta_x \tag{5.54b}$$

$$u_z(x,y,z) = -\frac{1}{2\pi}\int_0^{2\pi} \{C_d^1(\theta_x)D_{31}^1(\theta_x)\frac{1}{\psi_1(\theta_x)} + C_d^2(\theta_x)D_{31}^2(\theta_x)\frac{1}{\psi_2(\theta_x)} + C_d^3(\theta_x)D_{31}^3(\theta_x)\frac{1}{\psi_3(\theta_x)}\}d\theta_x \tag{5.54c}$$

$$\sigma_{xx}(x,y,z) = \frac{1}{2\pi}\int_0^{2\pi} \{C_d^1(\theta_x)\bar{\sigma}_{xx}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2 + C_d^2(\theta_x)\bar{\sigma}_{xx}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2 + C_d^3(\theta_x)\bar{\sigma}_{xx}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\}d\theta_x \tag{5.54d}$$

$$\sigma_{yy}(x,y,z) = \frac{1}{2\pi}\int_0^{2\pi} \{C_d^1(\theta_x)\bar{\sigma}_{yy}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2 + C_d^2(\theta_x)\bar{\sigma}_{yy}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2 + C_d^3(\theta_x)\bar{\sigma}_{yy}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\}d\theta_x \tag{5.54e}$$

$$\sigma_{zz}(x,y,z)=\frac{1}{2\pi}\int_0^{2\pi}\{C_d^1(\theta_x)\bar{\sigma}_{zz}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2+C_d^2(\theta_x)\bar{\sigma}_{zz}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2$$
$$+C_d^3(\theta_x)\bar{\sigma}_{zz}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\}d\theta_x \tag{5.54f}$$

$$\tau_{yz}(x,y,z)=\frac{1}{2\pi}\int_0^{2\pi}\{C_d^1(\theta_x)\bar{\tau}_{yz}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2+C_d^2(\theta_x)\bar{\tau}_{yz}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2$$
$$+C_d^3(\theta_x)\bar{\tau}_{yz}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\}d\theta_x \tag{5.54g}$$

$$\tau_{zx}(x,y,z)=\frac{1}{2\pi}\int_0^{2\pi}\{C_d^1(\theta_x)\bar{\tau}_{zx}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2+C_d^2(\theta_x)\bar{\tau}_{zx}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2$$
$$+C_d^3(\theta_x)\bar{\tau}_{zx}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\}d\theta_x \tag{5.54h}$$

$$\tau_{xy}(x,y,z)=\frac{1}{2\pi}\int_0^{2\pi}\{C_d^1(\theta_x)\bar{\tau}_{xy}^1(\theta_x)(\frac{1}{\psi_1(\theta_x)})^2+C_d^2(\theta_x)\bar{\tau}_{xy}^2(\theta_x)(\frac{1}{\psi_2(\theta_x)})^2$$
$$+C_d^3(\theta_x)\bar{\tau}_{xy}^3(\theta_x)(\frac{1}{\psi_3(\theta_x)})^2\}d\theta_x \tag{5.54i}$$

Again, assuming $\omega=e^{i\theta_x}$, and hence, $sin\,\theta_x=\frac{\omega-\omega^{-1}}{2i}$, $cos\,\theta_x=\frac{\omega+\omega^{-1}}{2}$, $d\omega=i\omega d\theta_x$, Eqs. (5.54a)-(5.54i) can be expressed as:

For $z>0$ (region 1),

$$u_x(x,y,z)=-\frac{1}{2\pi}\oint_C\frac{1}{i\omega}\{C_d^1(\omega)D_{11}^1(\omega)\frac{\psi_4(\omega)}{\psi_7(\omega)}+C_d^2(\omega)D_{11}^2(\omega)\frac{\psi_5(\omega)}{\psi_8(\omega)}$$
$$+C_d^3(\omega)D_{11}^3(\omega)\frac{\psi_6(\omega)}{\psi_9(\omega)}\}d\omega \tag{5.55a}$$

$$u_y(x,y,z)=-\frac{1}{2\pi}\oint_C\frac{1}{i\omega}\{C_d^1(\omega)D_{21}^1(\omega)\frac{\psi_4(\omega)}{\psi_7(\omega)}+C_d^2(\omega)D_{21}^2(\omega)\frac{\psi_5(\omega)}{\psi_8(\omega)}$$
$$+C_d^3(\omega)D_{21}^3(\omega)\frac{\psi_6(\omega)}{\psi_9(\omega)}\}d\omega \tag{5.55b}$$

$$u_z(x,y,z) = -\frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{C_d^1(\omega)D_{31}^1(\omega)\frac{\psi_4(\omega)}{\psi_7(\omega)} + C_d^2(\omega)D_{31}^2(\omega)\frac{\psi_5(\omega)}{\psi_8(\omega)} + C_d^3(\omega)D_{31}^3(\omega)\frac{\psi_6(\omega)}{\psi_9(\omega)}\}d\omega \quad (5.55c)$$

$$\sigma_{xx}(x,y,z) = \frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{C_d^1(\omega)\bar{\sigma}_{xx}^1(\omega)(\frac{\psi_4(\omega)}{\psi_7(\omega)})^2 + C_d^2(\omega)\bar{\sigma}_{xx}^2(\omega)(\frac{\psi_5(\omega)}{\psi_8(\omega)})^2 + C_d^3(\omega)\bar{\sigma}_{xx}^3(\omega)(\frac{\psi_6(\omega)}{\psi_9(\omega)})^2\}d\omega \quad (5.55d)$$

$$\sigma_{yy}(x,y,z) = \frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{C_d^1(\omega)\bar{\sigma}_{yy}^1(\omega)(\frac{\psi_4(\omega)}{\psi_7(\omega)})^2 + C_d^2(\omega)\bar{\sigma}_{yy}^2(\omega)(\frac{\psi_5(\omega)}{\psi_8(\omega)})^2 + C_d^3(\omega)\bar{\sigma}_{yy}^3(\omega)(\frac{\psi_6(\omega)}{\psi_9(\omega)})^2\}d\omega \quad (5.55e)$$

$$\sigma_{zz}(x,y,z) = \frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{C_d^1(\omega)\bar{\sigma}_{zz}^1(\omega)(\frac{\psi_4(\omega)}{\psi_7(\omega)})^2 + C_d^2(\omega)\bar{\sigma}_{zz}^2(\omega)(\frac{\psi_5(\omega)}{\psi_8(\omega)})^2 + C_d^3(\omega)\bar{\sigma}_{zz}^3(\omega)(\frac{\psi_6(\omega)}{\psi_9(\omega)})^2\}d\omega \quad (5.55f)$$

$$\tau_{yz}(x,y,z) = \frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{C_d^1(\omega)\bar{\tau}_{yz}^1(\omega)(\frac{\psi_4(\omega)}{\psi_7(\omega)})^2 + C_d^2(\omega)\bar{\tau}_{yz}^2(\omega)(\frac{\psi_5(\omega)}{\psi_8(\omega)})^2 + C_d^3(\omega)\bar{\tau}_{yz}^3(\omega)(\frac{\psi_6(\omega)}{\psi_9(\omega)})^2\}d\omega \quad (5.55g)$$

$$\tau_{zx}(x,y,z) = \frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{C_d^1(\omega)\bar{\tau}_{zx}^1(\omega)(\frac{\psi_4(\omega)}{\psi_7(\omega)})^2 + C_d^2(\omega)\bar{\tau}_{zx}^2(\omega)(\frac{\psi_5(\omega)}{\psi_8(\omega)})^2 + C_d^3(\omega)\bar{\tau}_{zx}^3(\omega)(\frac{\psi_6(\omega)}{\psi_9(\omega)})^2\}d\omega \quad (5.55h)$$

$$\tau_{xy}(x,y,z) = \frac{1}{2\pi}\oint_C \frac{1}{i\omega}\{C_d^1(\omega)\bar{\tau}_{xy}^1(\omega)(\frac{\psi_4(\omega)}{\psi_7(\omega)})^2 + C_d^2(\omega)\bar{\tau}_{xy}^2(\omega)(\frac{\psi_5(\omega)}{\psi_8(\omega)})^2 + C_d^3(\omega)\bar{\tau}_{xy}^3(\omega)(\frac{\psi_6(\omega)}{\psi_9(\omega)})^2\}d\omega \quad (5.55i)$$

where:

$$\psi_7(\omega) = \psi_1(\omega) \times \psi_4(\omega) \quad (5.56a)$$

$$\psi_8(\omega) = \psi_2(\omega) \times \psi_5(\omega) \tag{5.56b}$$

$$\psi_9(\omega) = \psi_3(\omega) \times \psi_6(\omega) \tag{5.56c}$$

CHAPTER Ⅵ

ILLUSTRATIVE EXAMPLES

The present analytical solutions demonstrate that there are several factors could affect the displacements and stresses in an inclined transversely isotropic material. These factors include (1) the rotation of the transversely isotropic planes (ϕ), (2) the type and degree of material anisotropy (E/E', ν/ν', G/G'), (3) the geometric position (r, φ, ξ), as seen in Fig. 6.1, and (4) the types of three-dimensional loading (P_x, P_y, P_z). Based on Eqs. (4.57a)-(4.57c), (4.59a)-(4.59f) (for z>0, region 1) and Eqs. (4.58a)-(4.58c), (4.60a)-(4.60f) (for z<0, region 2), a Mathematica® (1999) program is written to clarify the effect of aforementioned factors on the induced displacements and stresses in a full space with transversely isotropic medium subjected a point load. (* The Mathematica program is written on the basis of Eqs. (4.57a)-(4.57c), (4.59a)-(4.59f) and Eqs. (4.58a)-(4.58c), (4.60a)-(4.60f), which can be adopted for calculation the displacement and stress components at any point in the full-space subjected a point load.*)

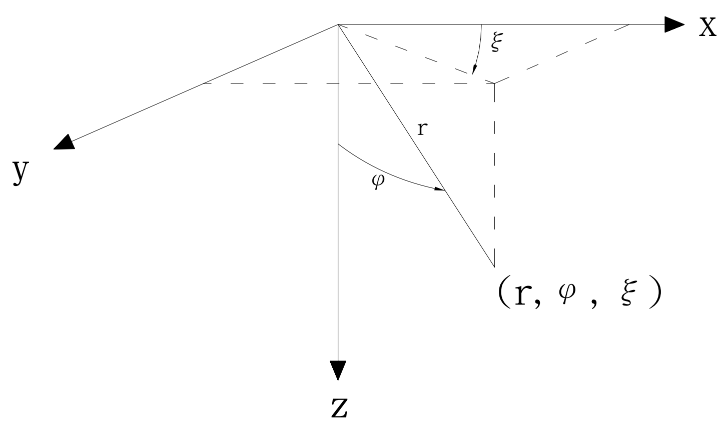


Fig. 6.1 Spherical co-ordinate system (r, φ, ξ)

6.1 Selected Parameters for Calculation

A parametric study is conducted in this chapter to illustrate the generated analytical solutions, and investigate the influence of the rotation of transversely isotropic planes, the geometric position, and the degree and type of rock anisotropy on the displacements and stresses. Two examples, one is to present the effect of ϕ on the displacements and stresses subjected to a vertical point load P_z at x=y=z=1 (as shown in Figs. 6.2(a)-6.2(c) for displacements, and Figs. 6.3(a)-6.3(f) for stresses), and the other is to exhibit the effect of φ on the stresses owning to P_z at ϕ=90° and ξ=45° (as depicted in Figs. 6.4(a)-6.4(f)). Seven hypothetical rocks, including one isotropic and six transversely isotropic rocks are considered to constitute the foundation materials. For typical ranges of transversely isotropic rocks, Gerrard (1975) and Amadei *et al*. (1987) suggested that the ratios E/E' and G/G' ranging from 1.0 to 3.0, and ν/ν' varying between 0.75 and 1.5. Hence, the degree of rock anisotropy, specified by the ratios E/E', ν/ν', and G/G' is accounted for investigating its effect on the displacements and stresses. Table 6.1 lists the rock types, and the elastic properties for the present hypothetical rocks. The values adopted in Table 6.1 of E and ν are 50 GPa and 0.25, respectively.

Table 6.1 Elastic properties for the hypothetical rocks (E=50 GPa, ν=0.25)

Rock type	E/E'	ν/ν'	G/G'
Rock 1. Isotropy	1.0	1.0	1.0
Rock 2. Transversely isotropy	2.0	1.0	1.0
Rock 3. Transversely isotropy	3.0	1.0	1.0
Rock 4. Transversely isotropy	1.0	0.75	1.0
Rock 5. Transversely isotropy	1.0	1.5	1.0
Rock 6. Transversely isotropy	1.0	1.0	2.0
Rock 7. Transversely isotropy	1.0	1.0	3.0

6.2 Example Results for Full-Space Problem

Using the elastic properties of the hypothetical rocks listing in Table 6.1, the effect of dip angle (ϕ) on the displacements and stresses of the position of x=y=z=1 induced by the P_z at origin (0,0,0) are depicted in Figs. 6.2-6.3. In addition, the effect of φ on the stresses resulting from P_z at ϕ=90° and ξ=45° is shown in Fig. 6.4.

Figs. 6.2(a)-6.2(c) show the normalized displacements u_x*r/P_z (Fig. 6.2(a)), u_y*r/P_z (Fig. 6.2(b)), and u_z*r/P_z (Fig. 6.2(c)) at the position x=y=z=1 *vs*. the rotation of the transversely isotropic planes (ϕ), due to a vertical point load (P_z) at the origin, for the constituted isotropic/transversely isotropic rocks (Rock 1/Rocks 2-7, Table 6.1). Fig. 6.2(a) depicts the normalized displacement u_x of the rocks, induced by P_z. It is observed that any value in each curve is symmetric with respect to the origin of the co-ordinates, and the ratios E/E' (Rocks 2 and 3), ν/ν' (Rocks 4 and 5), and G/G' (Rocks 6 and 7) all strongly influence this displacement. This figure also exhibits that the magnitude of the normalized induced displacement (0.00026 m^2/GN) for Rock 1 is independent of the change in ϕ. However, for Rocks 2 and 3, the displacement is maximal at about ϕ=0°-180°, and is minimal at approximately ϕ=60°-240°. As for Rocks 6 and 7, the displacement is maximal at around ϕ=50°-230°, and is minimal at about ϕ=100°-280°. Fig. 6.2(b) presents the normalized displacement u_y of the rocks, due to P_z. This figure clearly reveals that the displacement induced in transversely isotropic rocks is deeply affected by the ratios E/E' (Rocks 2 and 3) and G/G' (Rocks 6 and 7), but is only slightly influenced by ν/ν' (Rocks 4 and 5). Notably, the normalized displacement (0.00026 m^2/GN) of the isotropic rock (Rock 1) is also independent of ϕ. Nevertheless, it is found that the values of induced displacement for Rocks 2 and 3 would be partially within the range of -0.0004 to 0, meaning there could be an opposite-direction displacement occurred in these media. Fig. 6.2(c) displays the normalized displacement u_z of the rocks, subjected to P_z. Clearly, the ratios E/E' (Rocks 2 and 3) and G/G' (Rocks 6 and 7) profoundly impact the induced displacement, but the effect of ν/ν' (Rocks 4 and 5) on it is little. The magnitude of the normalized induced displacement for Rock 1 is always 0.00179 m^2/GN; however, for Rocks 2, 3, 6, and 7, the values of u_z are nearly greater than those of Rock 1. The calculated results for the displacement fields are all in good agreement with Wang and Liao′s solutions (1999) if the full-space is homogeneous, linearly elastic, and the planes of transversely isotropy are parallel to the horizontal axes.

Figs. 6.3(a)-6.3(f) plot the non-dimensional normal stresses $\sigma_{xx}*r^2/P_z$ (Fig. 6.3(a)), $\sigma_{yy}*r^2/P_z$ (Fig. 6.3(b)), $\sigma_{zz}*r^2/P_z$ (Fig. 6.3(c)), and the non-dimensional shear stresses

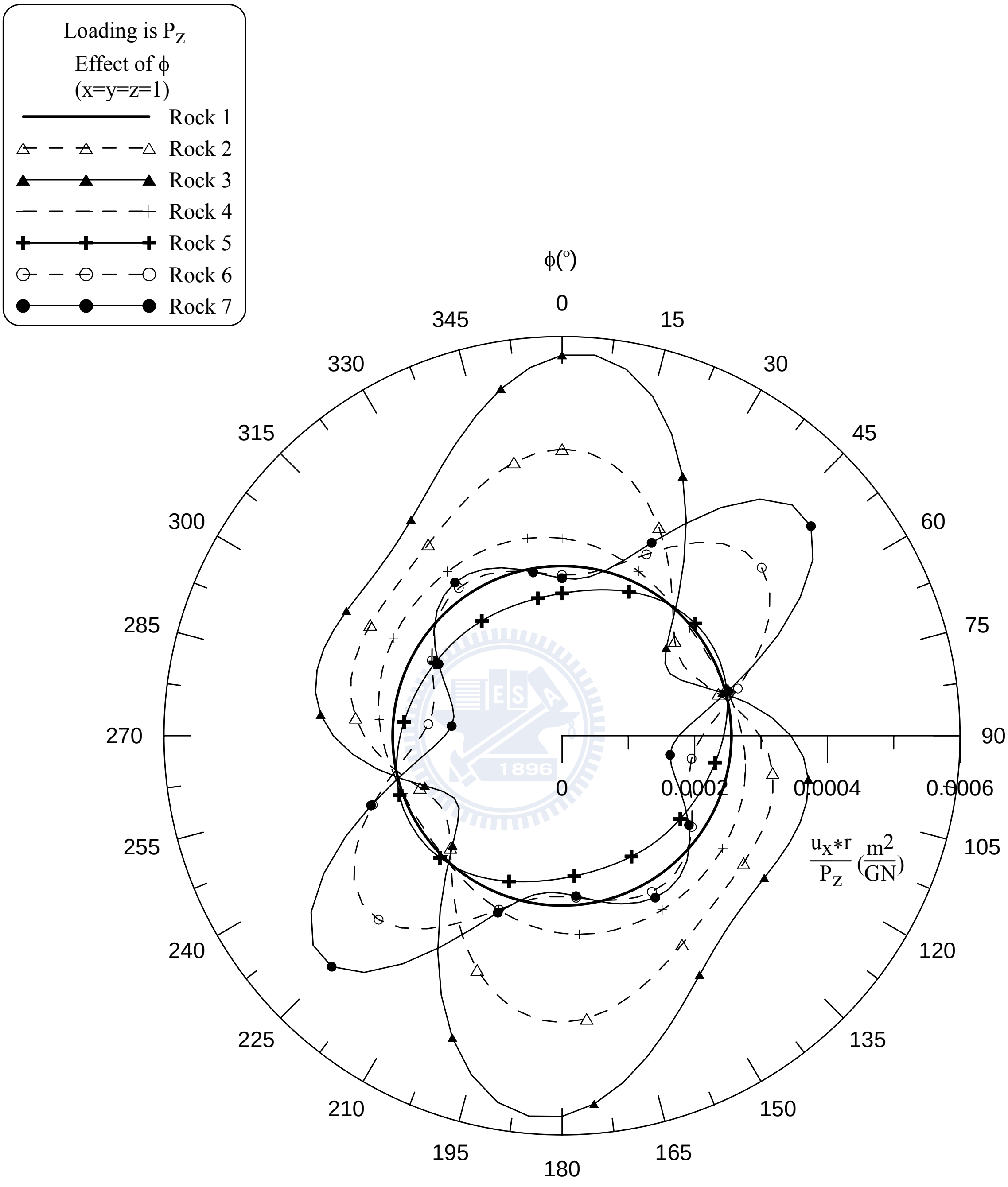
$\tau_{yz}*r^2/P_z$ (Fig. 6.3(d)), $\tau_{zx}*r^2/P_z$ (Fig. 6.3(e)), $\tau_{xy}*r^2/P_z$ (Fig. 6.3(f)), *vs*. the rotation of the transversely isotropic planes (ϕ), subjected to a vertical point load (P_z), at x=y=z=1, for the isotropic (Rock 1) and transversely isotropic rocks (Rocks 2-7). Fig. 6.3(a) illustrates the effect of ϕ on $\sigma_{xx}*r^2/P_z$, for Rocks 1-7. This figure shows the induced stress for the isotropic rock (Rock 1) has the same value (0.005105), that is again independent of ϕ. However, it is found that the values of induced stress for Rocks 1-7 varying between -0.004 and 0.02, namely, there is an obvious tensile stress occurred in Rock 7. In addition, any value in each curve is symmetric with respect to the origin of the co-ordinates. Hence, from this figure, it is apparently revealed that the induced stress is greatly influenced by the rotation of the transversely isotropic planes (ϕ), and the type and degree of rock anisotropy (E/E', ν/ν', G/G'). Fig. 6.3(b) presents the effect of ϕ on $\sigma_{yy}*r^2/P_z$, for Rocks 1-7. Notably, the value in the curves is also symmetric with respect to the origin of the co-ordinates, and the ratios E/E' (Rocks 2 and 3), ν/ν' (Rocks 4 and 5), and G/G' (Rocks 6 and 7) do also have a considerable influence on the stress. This graph exhibits the magnitude of the non-dimensional normal stress ($\sigma_{yy}*r^2/P_z$) for Rock 1 (0.005105) is also independent of ϕ, and the value of the non-dimensional stress is within 0.06. In particular, the computed results of Rock 4/Rock 5 are totally great/less than those of Rock 1. Fig. 6.3(c) depicts the effect of ϕ on $\sigma_{zz}*r^2/P_z$, for Rocks 1-7. This stress depends heavily on the ratios E/E' (Rocks 2 and 3) and G/G' (Rocks 6 and 7); nevertheless, the effect of the ratios ν/ν' (Rocks 4 and 5) on it is slight. The maximum value of the non-dimensional stress approaches 0.026. Fig. 6.3(d) plots the effect of ϕ on $\tau_{yz}*r^2/P_z$, for Rocks 1-7. Evidently, the ratios E/E' (Rocks 2 and 3) and G/G' (Rocks 6 and 7) could intensely affect the induced stress; however, the effect of the ratios ν/ν' (Rocks 4 and 5) on it is still little. The trend of these stress curves in this figure is similar to that in Fig. 6.3(c). Fig. 6.3(e) displays the effect of ϕ on $\tau_{zx}*r^2/P_z$, for Rocks 1-7. The maximum value of the non-dimensional stress is about 0.026. Fig. 6.3(f) shows the effect of ϕ on $\tau_{xy}*r^2/P_z$, for Rocks 1-7. The effect of the ratios ν/ν' (Rocks 4 and 5) in this figure is more explicit than another shear stresses (Figs. 6.3(d) and 6.3(e)). Especially, the calculated results of Rock 4/Rock 5 are great/less than those of Rock 1. The maximum value of the non-dimensional stress is within the range of 0.024. The computed results for the stress fields are exactly identical with those estimated from Wang and Liao′s solutions (1999), in which the planes of transversely isotropic full-space are parallel to the horizontal loading surface.

Figs. 6.4(a)-6.4(f) plot the non-dimensional normal stresses ($\sigma_{xx}*r^2/P_z$, $\sigma_{yy}*r^2/P_z$, $\sigma_{zz}*r^2/P_z$), and the non-dimensional shear stresses ($\tau_{yz}*r^2/P_z$, $\tau_{zx}*r^2/P_z$, $\tau_{xy}*r^2/P_z$), *vs*. the geometric position φ (from 0° to 360°), due to a vertical point load (P_z), at the rotation of the transversely isotropic planes ϕ=90° and the geometric position ξ=45°, for the constituted isotropic/transversely isotropic rocks (Rock 1/Rocks 2-7). Fig. 6.4(a) clarifies the effect of φ on $\sigma_{xx}*r^2/P_z$, for Rocks 1-7. It is observed that the magnitudes of the estimated stress are symmetric with respect to φ=180°. The upper/lower part of this figure denotes the compressive/tensile stress occurred in the rock media. The maximum values of tensile/compressive stress appeared at φ=0°/180° in Rock 7. In addition, the induced stresses are found to be influenced by the ratios E/E' (Rocks 2 and

3), ν/ν' (Rocks 4 and 5), G/G' (Rocks 6 and 7), and they are all zero at φ=90° and 270°. Fig. 6.4(b) demonstrates the effect of φ on $\sigma_{yy}*r^2/P_z$, for Rocks 1-7. Results reveal that the magnitudes of the computed stress are also symmetric with respect to φ=180°, and the tensile and compressive stresses would be occurred in all media. However, the maximum values of tensile/compressive stress approximately appeared at φ=125° and 235°/55° and 305° in Rock 4. That means at a given position (ϕ=90° and ξ=45°), the decrease of the ratio ν/ν' from 1.0 (Rock 1) to 0.75 (Rock 4) could remarkably affect the stress (σ_{yy}). Fig. 6.4(c) shows the induced non-dimensional normal stress $\sigma_{zz}*r^2/P_z$ for Rocks 1-7. The distributions and magnitudes of the calculated stress are quite different from those of Figs. 6.4(a) and 6.4(b). The tensile/compressive stress can be found within φ=0°-90° and 270°-360°/90°-270°. Moreover, the stress (σ_{zz}) is apparently impacted by the ratios G/G' (Rocks 6 and 7); nevertheless, it is little affected by the ratios E/E' (Rocks 2 and 3) and υ/υ' (Rocks 4 and 5). The induced non-dimensional shear stress $\tau_{yz}*r^2/P_z$ for Rocks 1-7 is depicted in Fig. 6.4(d). It is noted that the positive/negative values of τ_{yz} are respectively symmetric with respect to φ=180°. Additionally, the computed stresses are all zero at φ=0°, 180°, and 360°. The results of Rocks 2, 4, 6, 7 are rather distinct from those of Rocks 1, 3, 5. Similarly, the trends can be discovered in Fig. 6.4(e) for $\tau_{zx}*r^2/P_z$. Eventually, the induced non-dimensional shear stress $\tau_{xy}*r^2/P_z$ for Rocks 1-7 is displayed in Fig. 6.4(f). The calculated positive/negative values of τ_{xy} are symmetric with φ=90° and 270°. The zero values for τ_{xy} are found at φ=0°, 90°, 180°, 270°, and 360°. Furthermore, the influences of the type and degree of rock anisotropy in this figure are more explicit than those in Figs.

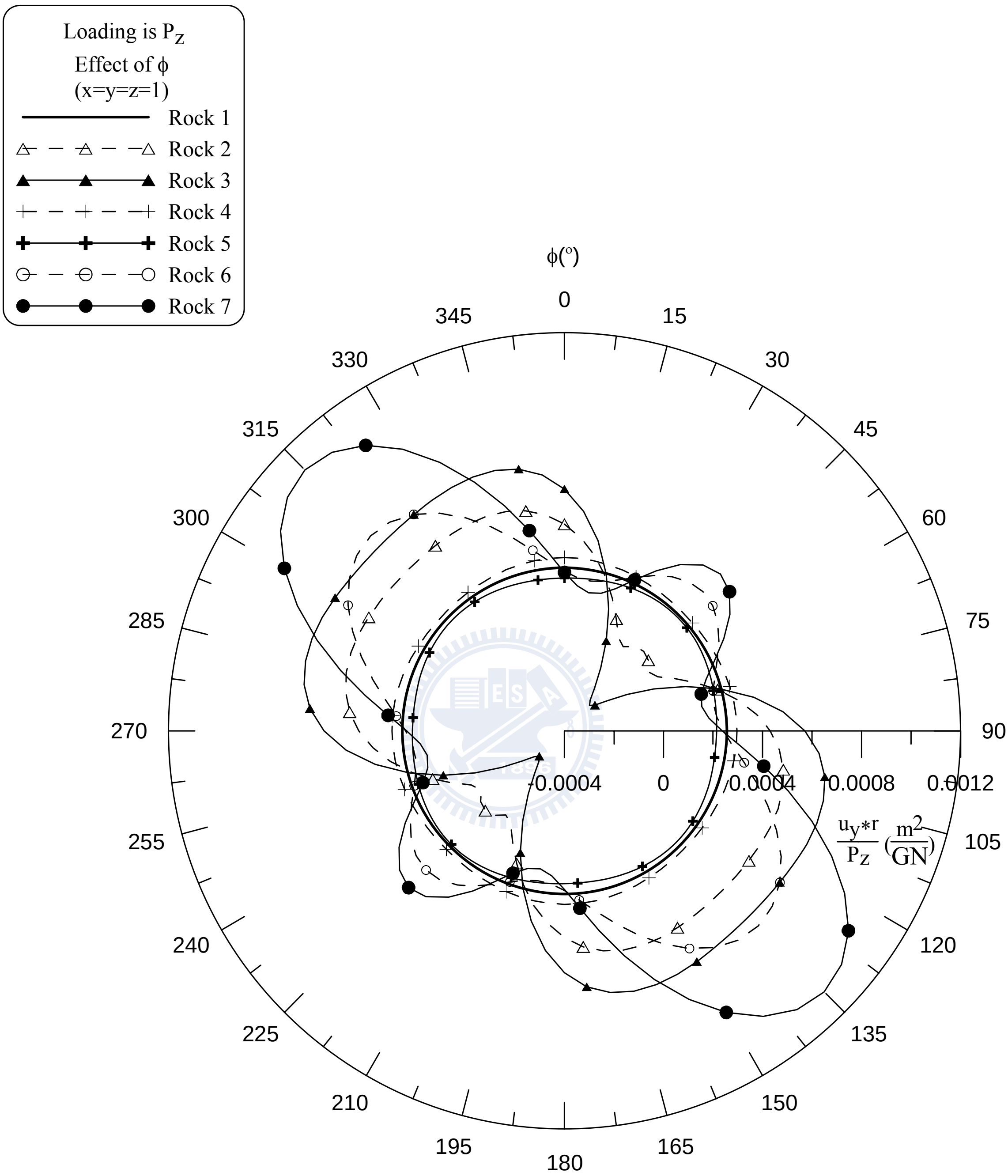
6.4(d) and 6.4(e). That signifies again that at ϕ=90° and ξ=45°, the normal and shear stresses owing to a vertical point load are strongly impacted by the geometric position (φ) and rock anisotropy (E/E', υ/υ', G/G').

The examples are presented to illustrate the derived solutions and demonstrate how the rotation of transversely isotropic planes (ϕ), the geometric position (r, φ, ξ), and the degree and type of material anisotropy (E/E', υ/υ', G/G') would influence the normalized displacements and non-dimensional normal and shear stresses. Results reveal that the displacements and stresses in the inclined isotropic/transversely isotropic rocks (Rock 1/Rocks 2-7) due to a vertical point load are quite different from those solutions by assuming the transversely isotropic planes are parallel to the horizontal surface. Hence, it is imperative to consider the dip at an angle of inclination when calculating the induced displacements and stresses in a transversely isotropic material by applied loads.



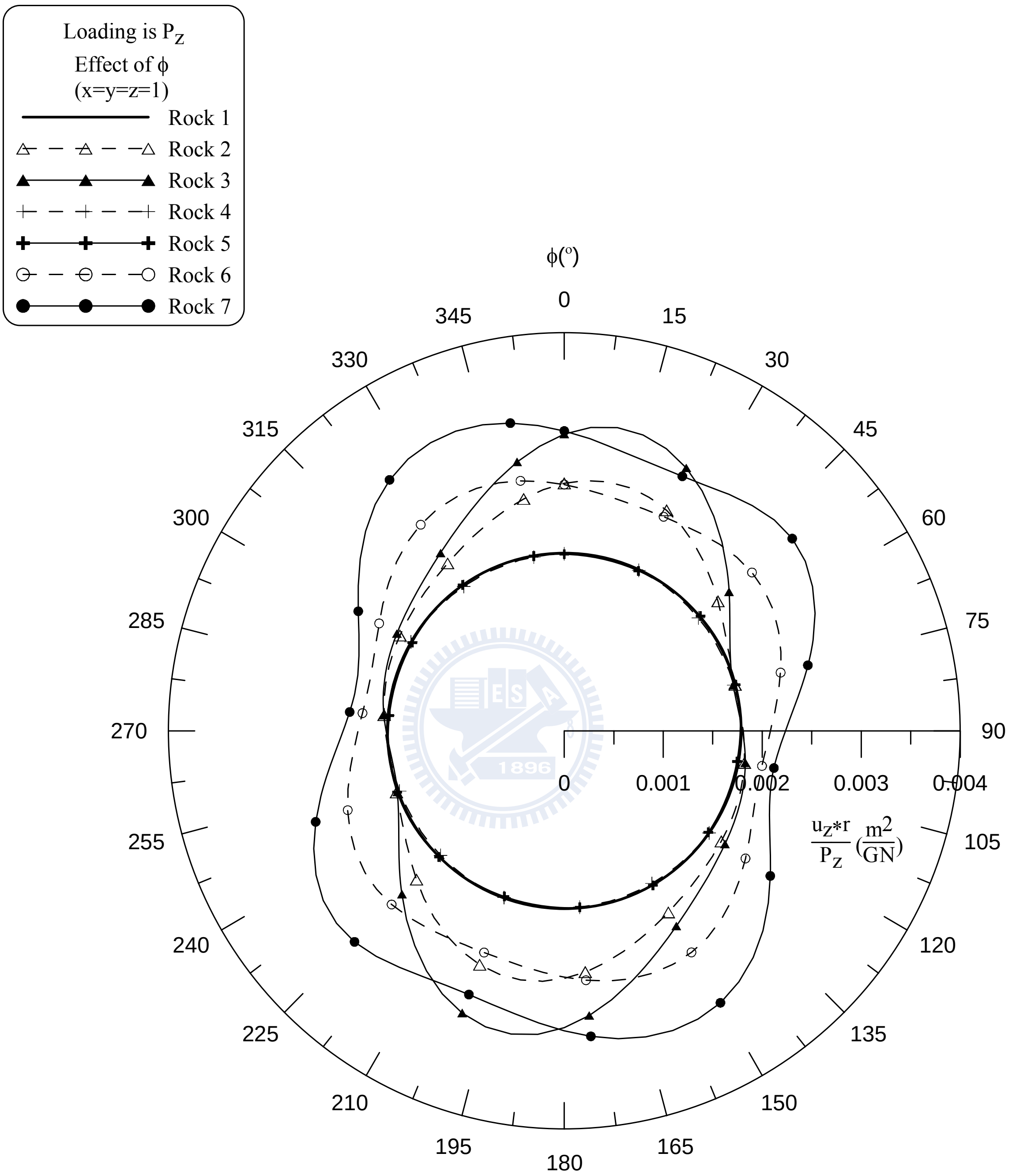
(a)

Fig. 6.2.(a) At the position *x*=*y*=*z*=1, the effect of ϕ on the normalized displacement u_x*r/P_z



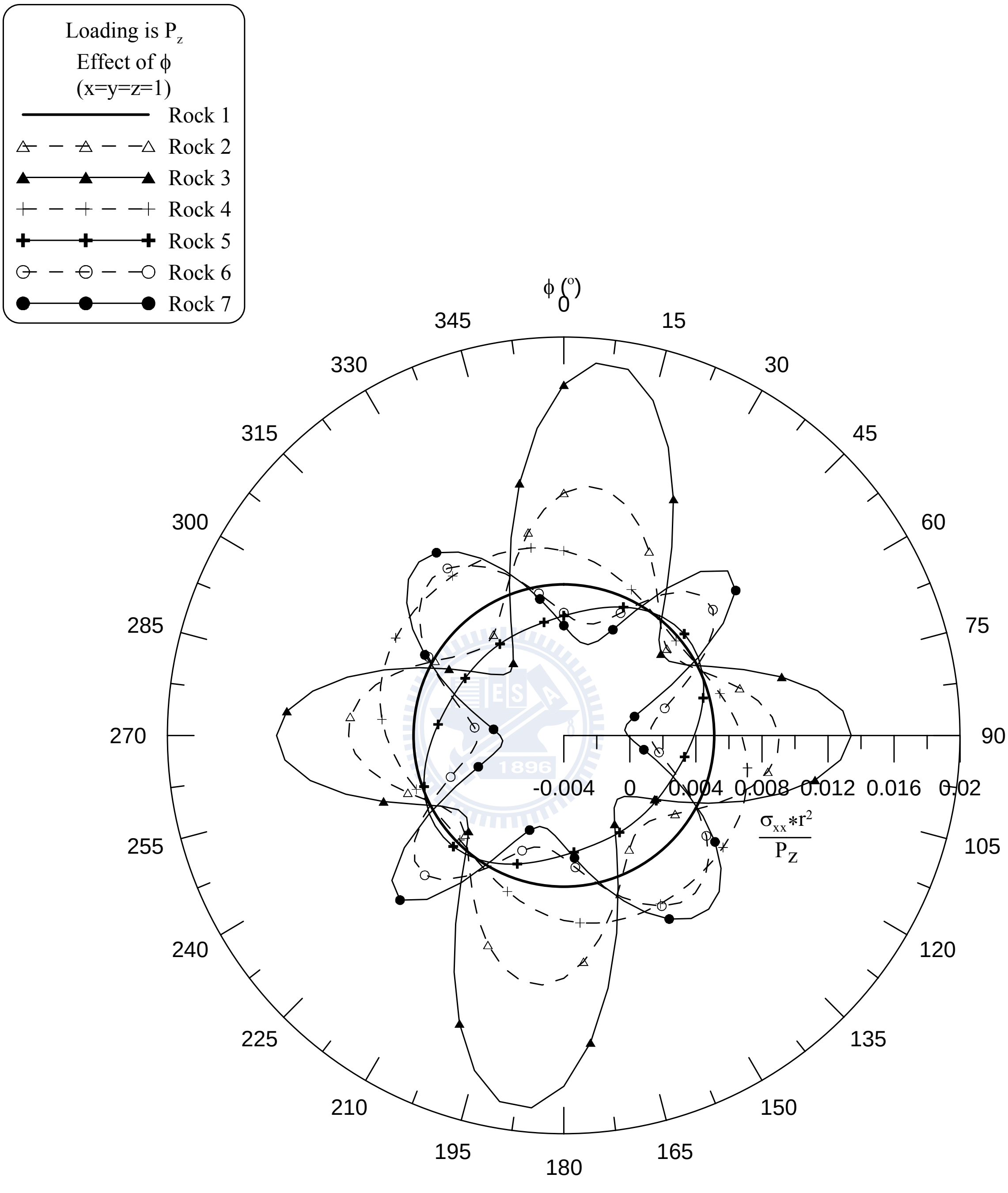
(b)

Fig. 6.2.(b) At the position $x=y=z=1$, the effect of ϕ on the normalized displacement u_y*r/P_z



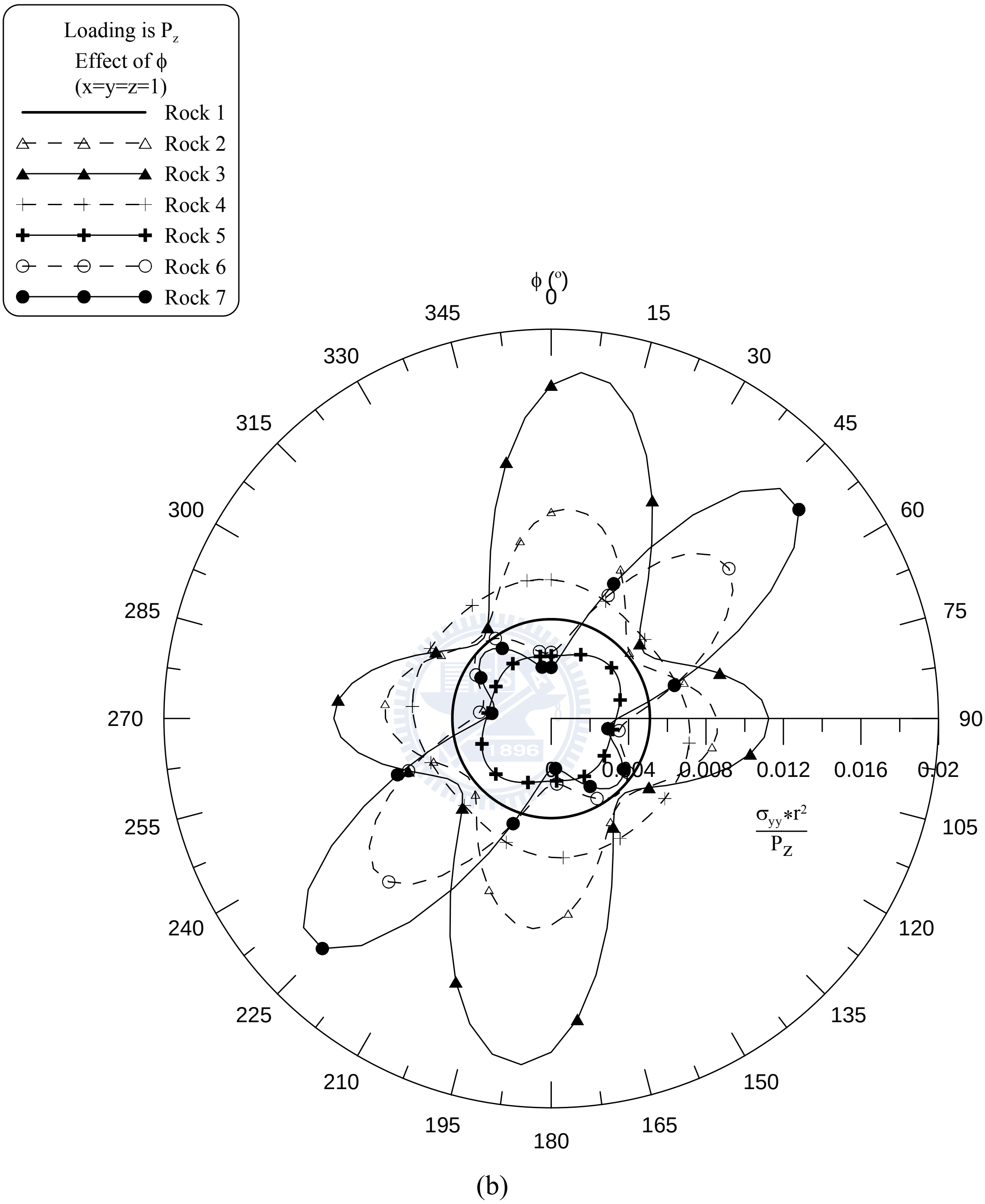
(c)

Fig. 6.2.(c) At the position $x=y=z=1$, the effect of ϕ on the normalized displacement u_z*r/P_z



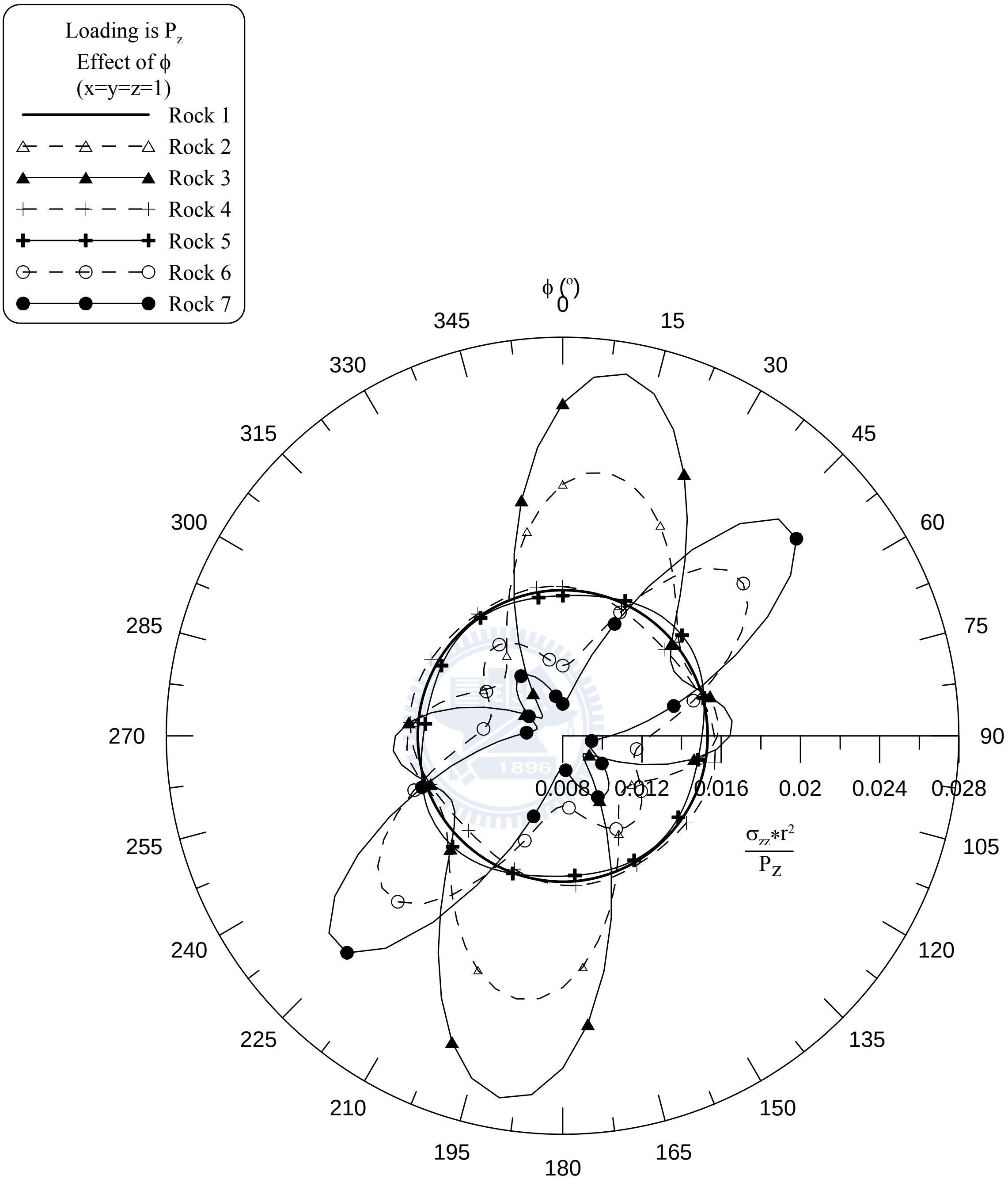
(a)

Fig. 6.3.(a) At the position $x=y=z=1$, the effect of ϕ on the non-dimensional normal stress $\sigma_{xx}*r^2/P_z$



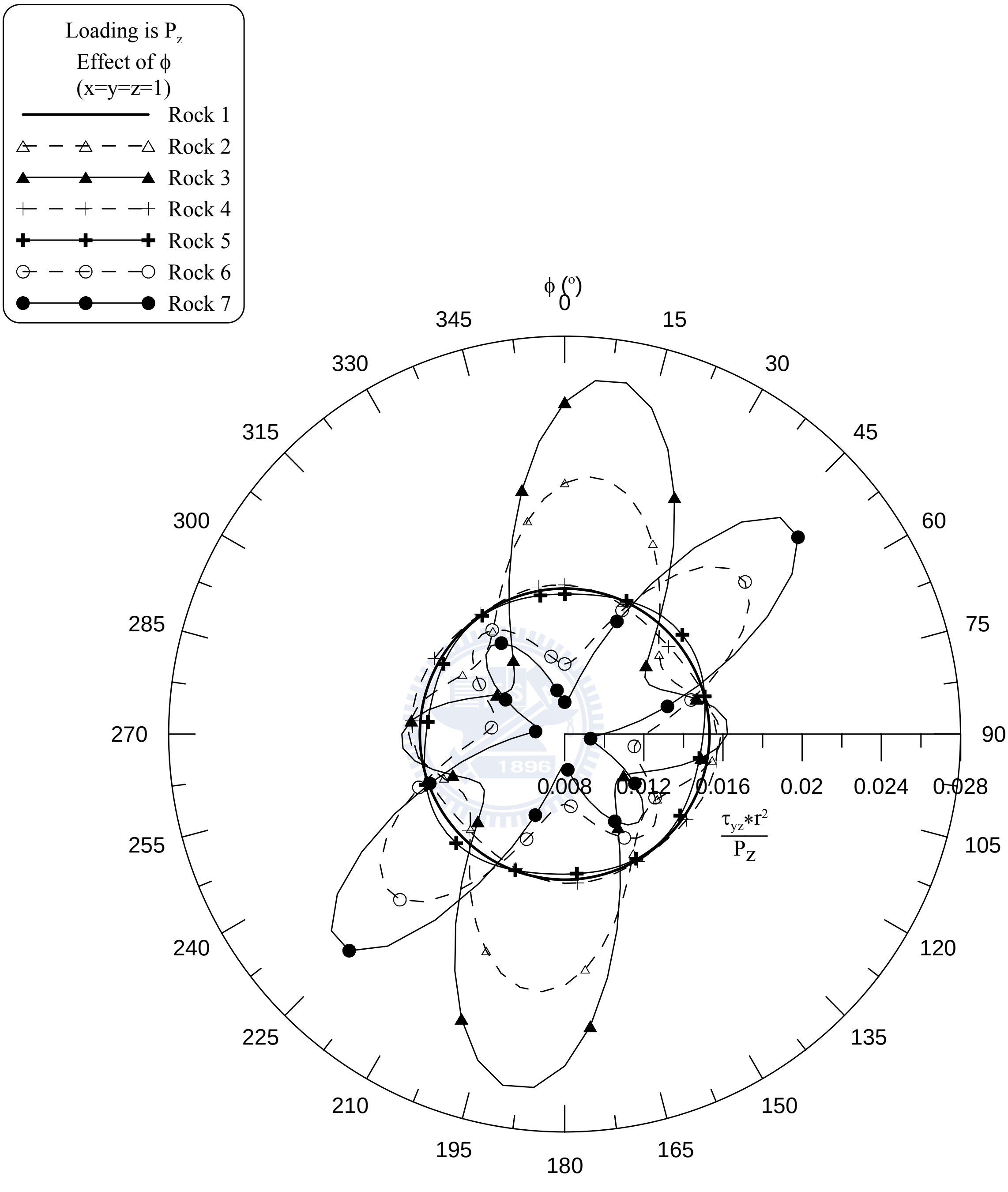
(b)

Fig. 6.3.(b) At the position x=y=z=1, the effect of ϕ on the non-dimensional normal stress $\sigma_{yy}*r^2/P_z$



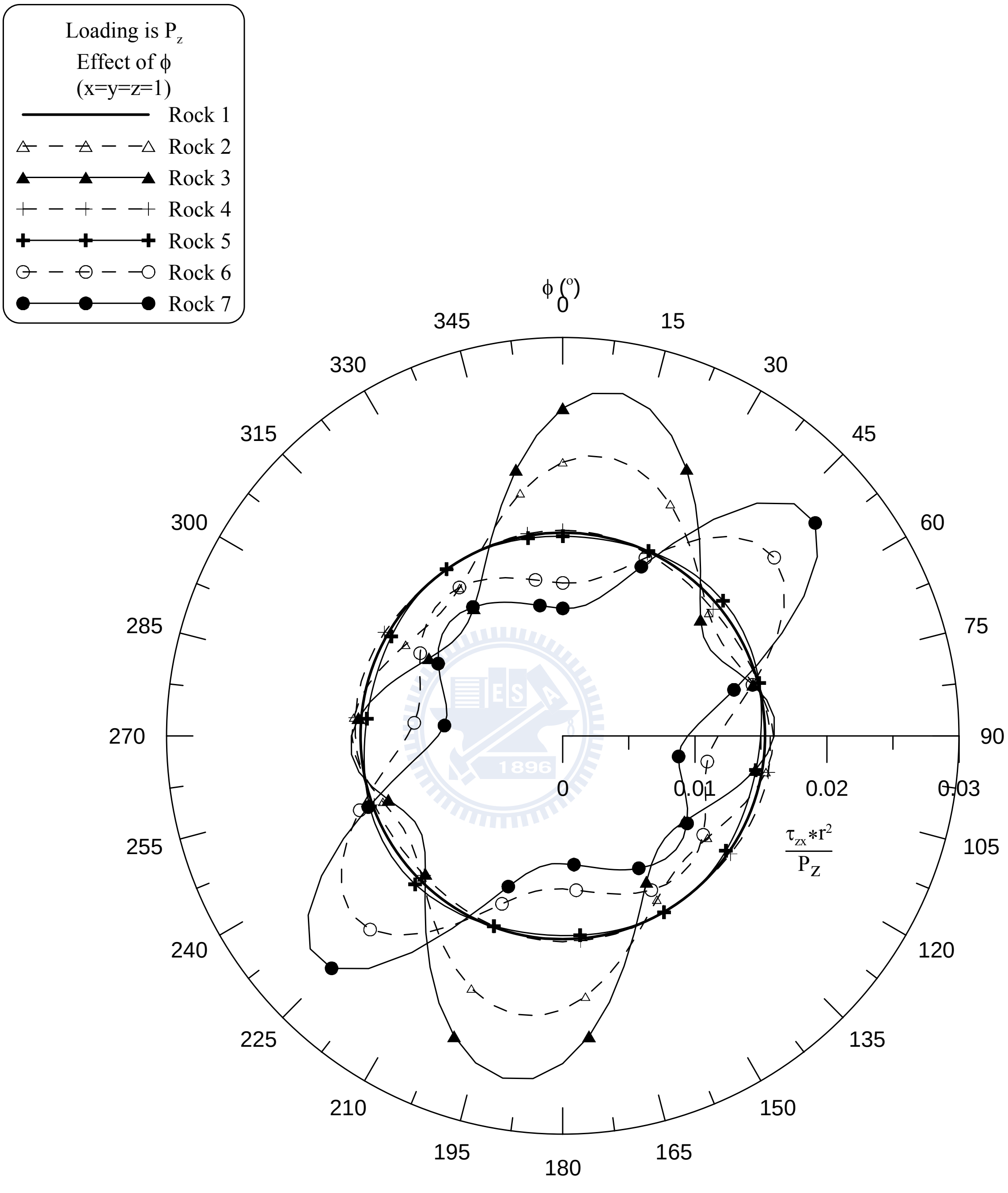
(c)

Fig. 6.3.(c) At the position x=y=z=1, the effect of ϕ on the non-dimensional normal stress $\sigma_{zz}*r^2/P_z$



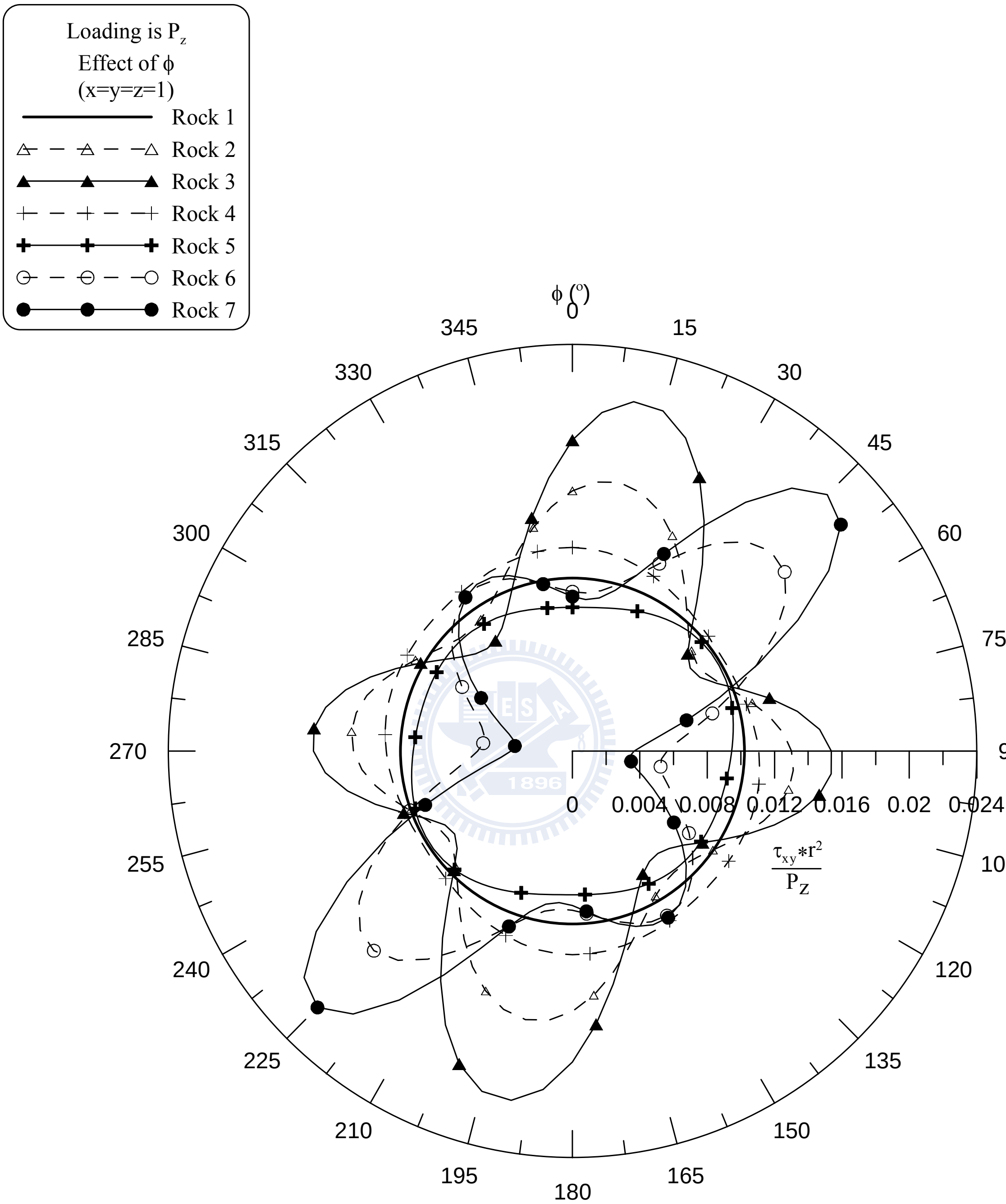
(d)

Fig. 6.3.(d) At the position *x*=*y*=*z*=1, the effect of ϕ on the non-dimensional shear stress $\tau_{yz}*r^2/P_z$



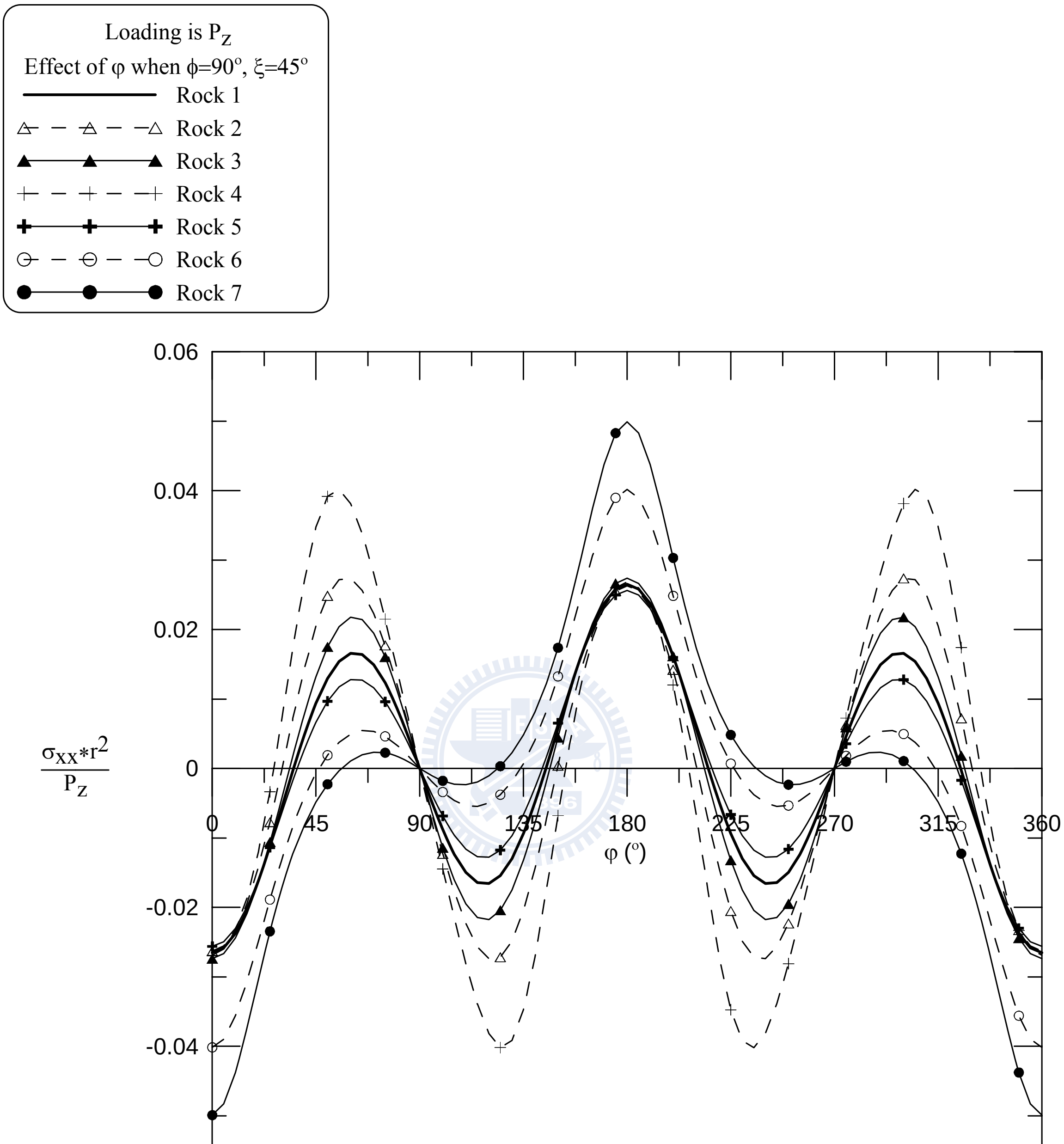
(e)

Fig. 6.3.(e) At the position x=y=z=1, the effect of ϕ on the non-dimensional shear stress $\tau_{zx}*r^2/P_z$



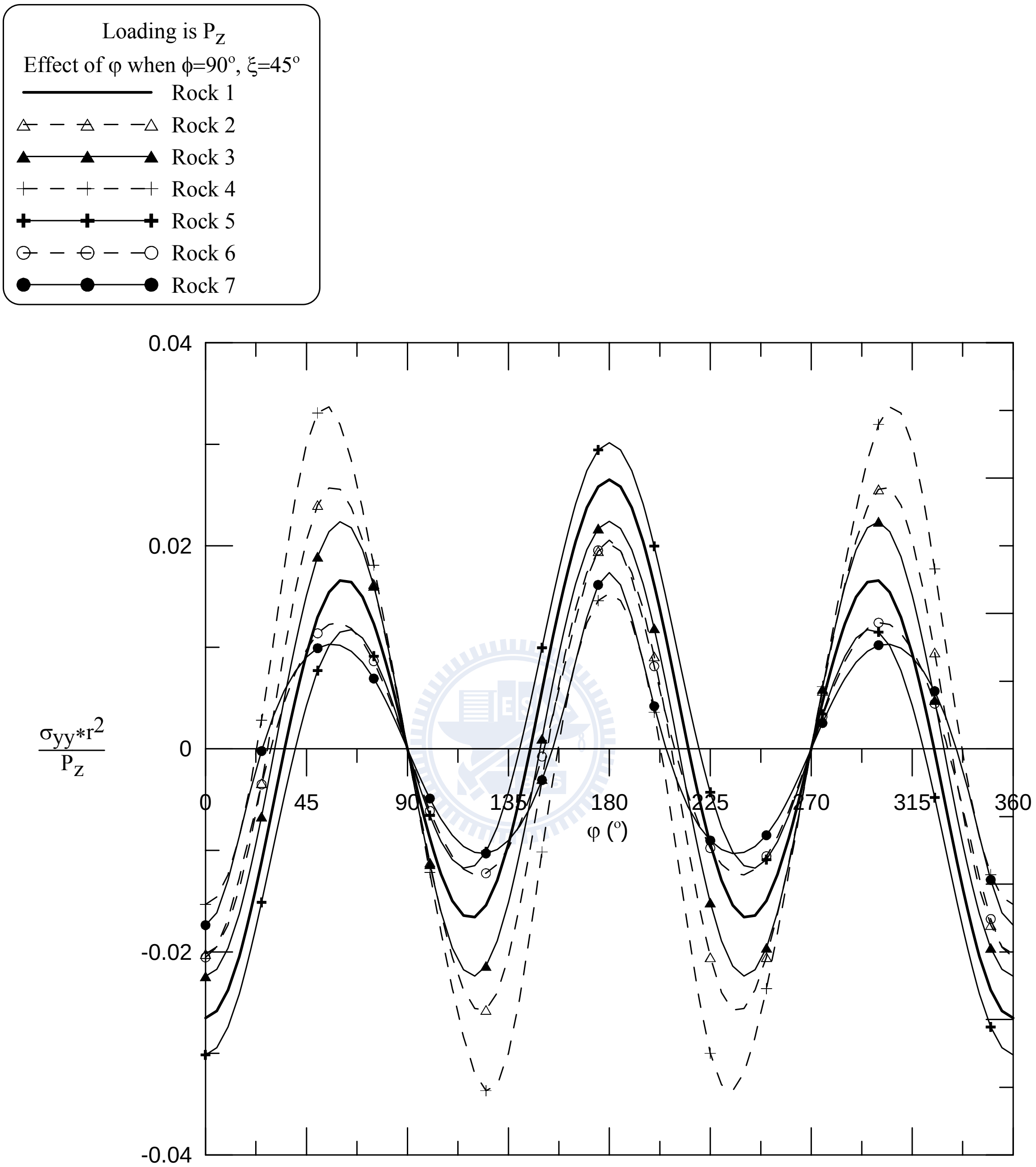
(f)

Fig. 6.3.(f) At the position x=y=z=1, the effect of ϕ on the non-dimensional shear stress $\tau_{xy}*r^2/P_z$



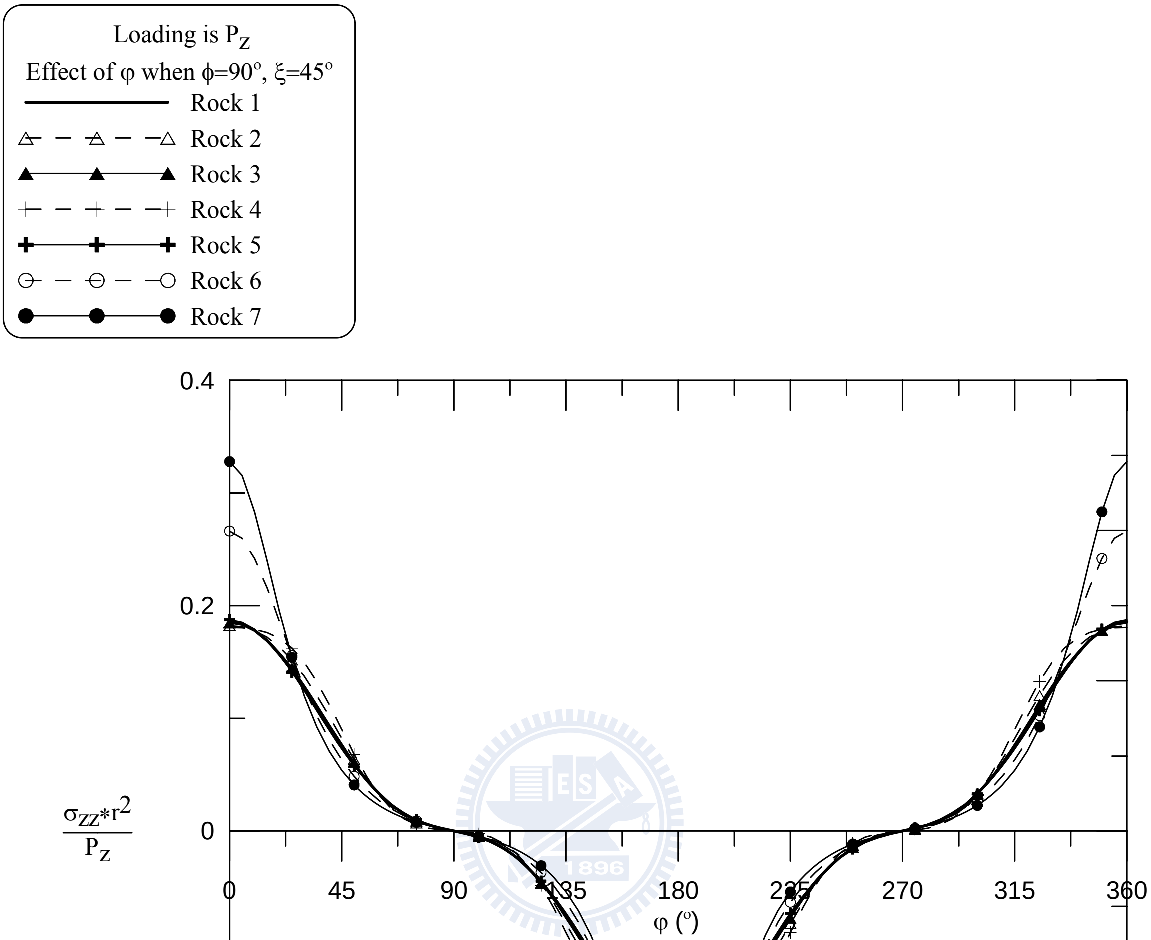
(a)

Fig. 6.4.(a) At the position ϕ=90°, ξ=45°, the effect of φ on the non-dimensional normal stress $\sigma_{xx}*r^2/P_z$



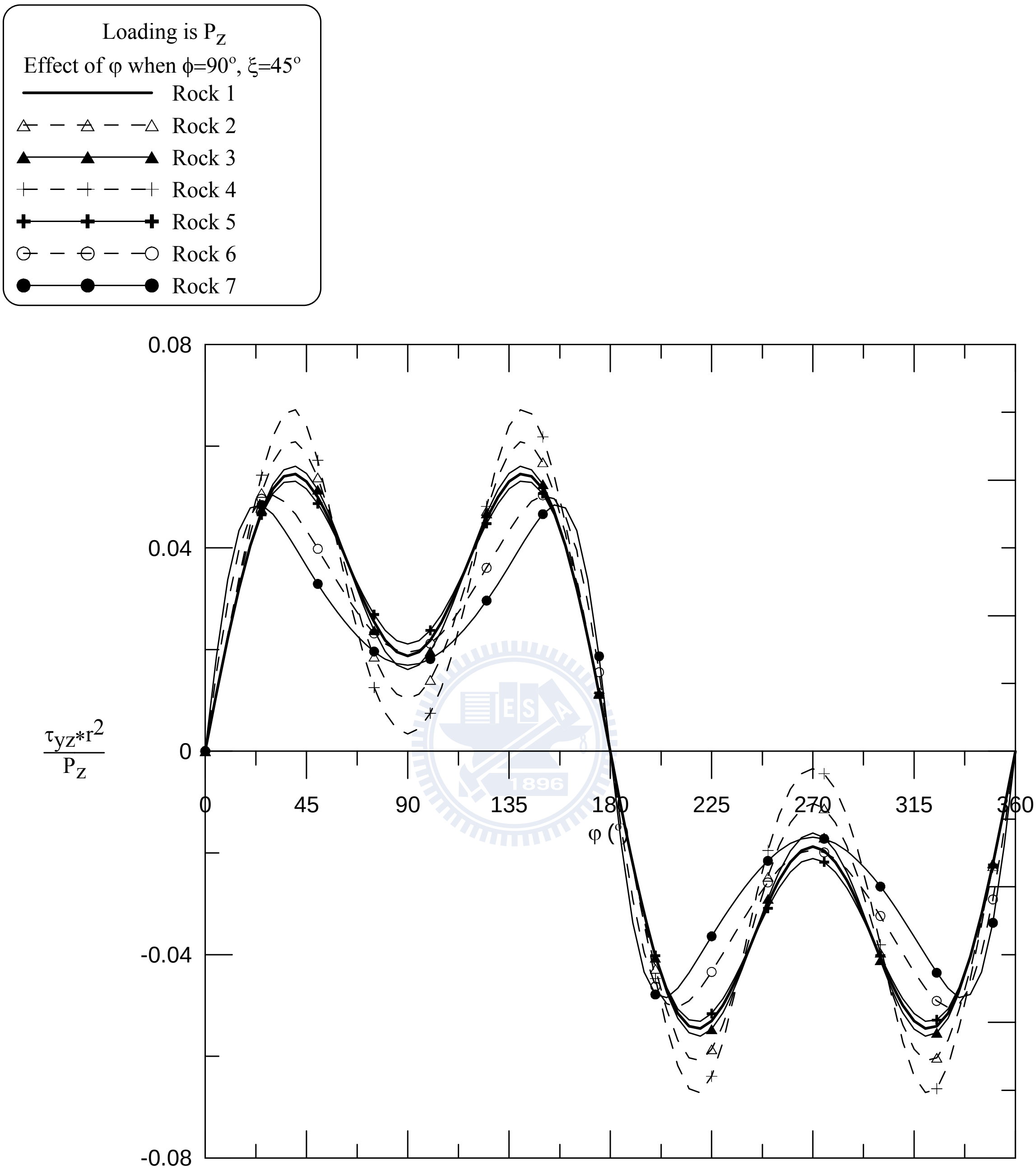
(b)

Fig. 6.4.(b) At the position ϕ=90°, ξ=45°, the effect of φ on non-dimensional normal stress $\sigma_{yy}*r^2/P_z$



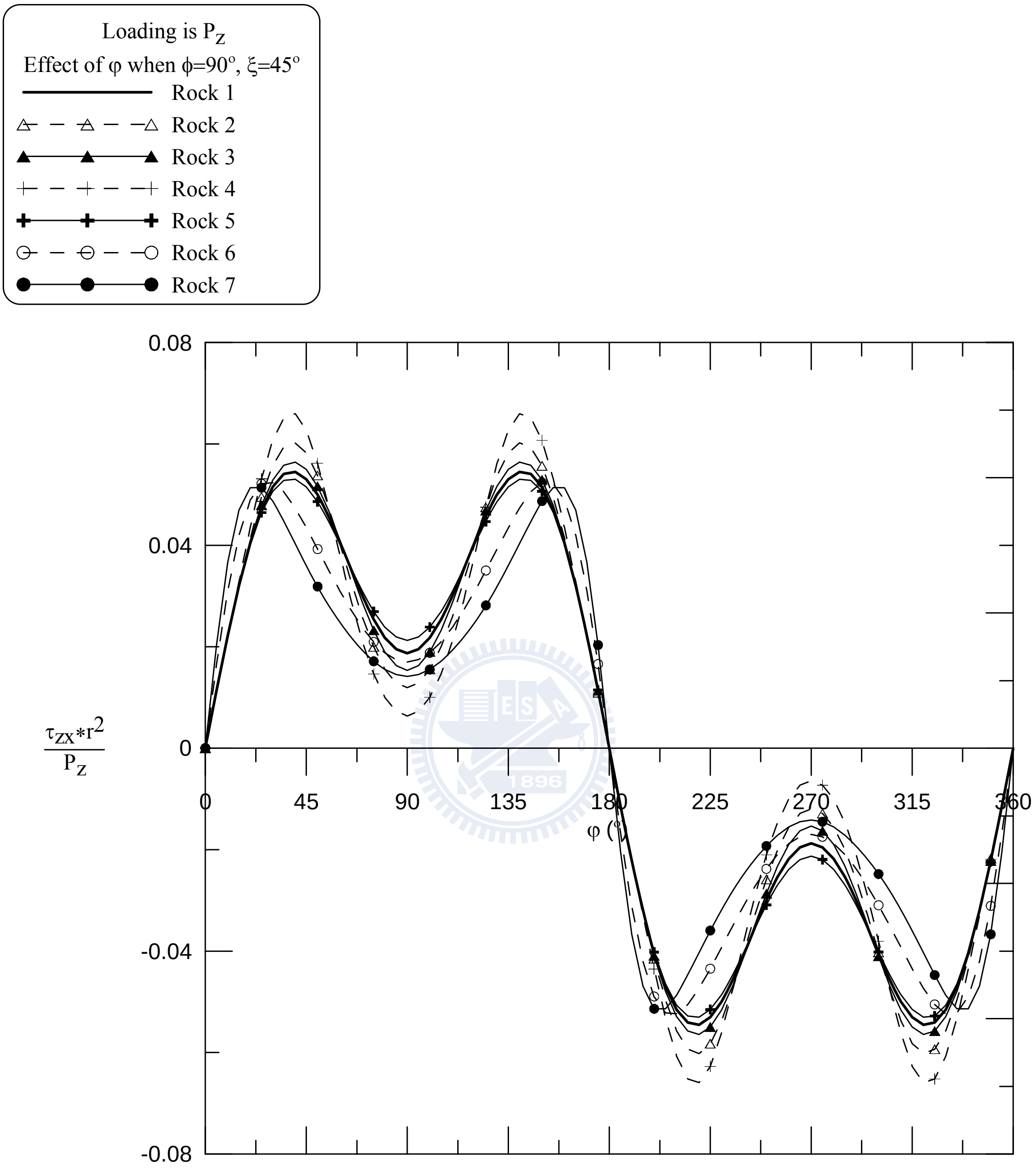
(c)

Fig. 6.4.(c) At the position ϕ=90°, ξ=45°, the effect of φ on the non-dimensional normal stress $\sigma_{zz}*r^2/P_z$



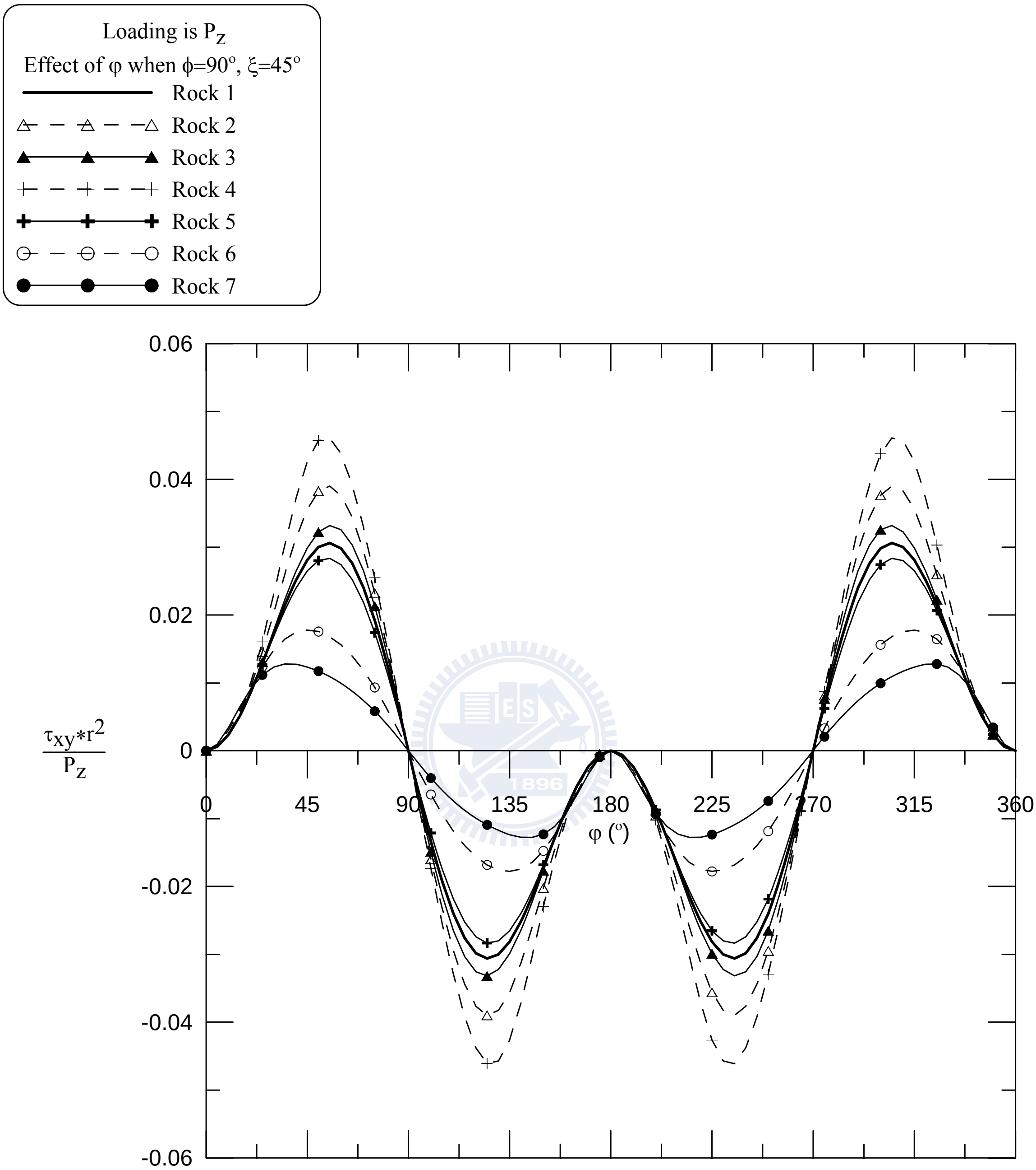
(d)

Fig. 6.4.(d) At the position ϕ=90°, ξ=45°, the effect of φ on the non-dimensional shear stress $\tau_{yz}*r^2/P_z$



(e)

Fig. 6.4.(e) At the position ϕ=90°, ξ=45°, the effect of φ on the non-dimensional shear stress $\tau_{zx}*r^2/P_z$



(f)

Fig. 6.4.(f) At the position ϕ=90°, ξ=45°, the effect of φ on the non-dimensional shear stress $\tau_{xy}*r^2/P_z$

6.3 Example Results for Half-Space Problems

Equations (5.55a)-(5.55c) can be utilized to calculate the displacements in a transversely isotropic half-space induced by a point load. A Mathematica® (1999) program based on these solutions was written for conducting a parametric study.

A vertical point load acting on the bounded surface is considered as an example (Figures 6.5 and 6.6) for demonstrating the presented formulations. The hypothetical constituted foundation materials include several types of isotropic and transversely isotropic rocks. Their elastic properties are listed in Table 6.1 with E/E' and G/G' ranging between 1 and 3, and υ/υ' varying between 0.75~1.5. For transversely isotropic rocks, the ratios E/E', υ/υ' and G/G' define the degree of rock anisotropy. The values adopted in Table 6.1 of E and υ are 50 GPa and 0.25, respectively. The chosen domains of anisotropy variation are based on the suggestions of Gerrard (1975) and Amadei et al. (1987).

A parametric study is conducted for looking at the effect of the ratios E/E', υ/υ' and G/G' on the vertical or lateral displacements. Briefly, only parts of the results, including the vertical displacement (U_z) on the line (-10<x<10, y=0 and z=1), and the lateral displacement (U_x) on the line (x=1, y=0 and 0<z<3) are presented and discussed in the following.

Firstly, the influence of the degree and type of rock anisotropy on the vertical displacement is investigated. Figure 6.5 presents the effect of ratios E/E', υ/υ' and G/G' on the normalized vertical displacement. This figure indicates that the normalized vertical displacement is less than the value of 0.03 on the line (x=1, y=0 and 0<z<3) for all the constituted foundation materials. However, the magnitude of displacement is influenced by rock anisotropy. Figure 6.1 shows that the vertical displacement increases with the increase $G/G^{'}$ with $E/E^{'}=\upsilon/\upsilon^{'}=1$. However, the variation of $\upsilon/\upsilon^{'}$ and $E/E^{'}$ on the vertical displacement is little for this cases.

Secondly, the effect of rock anisotropy on the lateral displacement in the medium is studied. Figure 6.6 presents the effect of ratios $E/E^{'}$, $\upsilon/\upsilon^{'}$ and $G/G^{'}$ on the normalized lateral displacement. This figure indicates that the normalized lateral displacement is minus sign near the surface. It is mean that the direction of the displacement turn toward the point load. And the normalized lateral displacement is positive sign when the area distant from the surface. However, the magnitude of displacement is influenced by rock anisotropy. Figure 6.6 shows that the lateral displacement increases with the increase $E/E^{'}$ with $G/G^{'}=\upsilon/\upsilon^{'}=1$. However, the variation of $\upsilon/\upsilon^{'}$ and $G/G^{'}$ on the lateral displacement is little for this cases.

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This example was utilized to examine the closed-form solutions and investigate the effect of rock anisotropy on the displacement distributions in the medium. The results show that the displacement in the medium subjected to a point load on the surface are easy and correct to calculate by the presented solutions. Also, the results indicate that the displacement accounted for rock anisotropy are quite different for the displacement calculated from isotropic solutions.

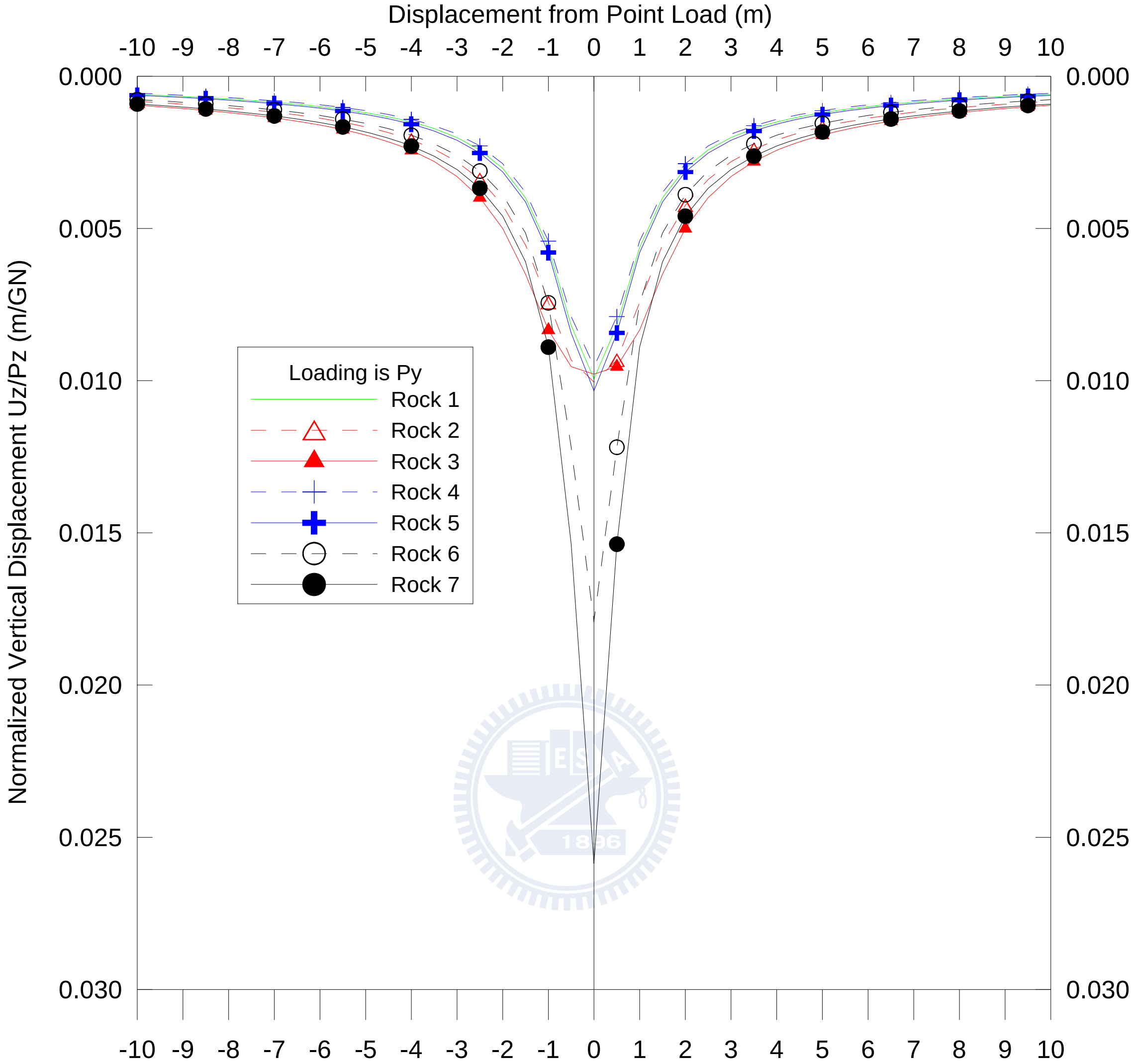

Fig. 6.5 Effect of ratios of E/E', υ/υ' and G/G' on normalized vertical displacement

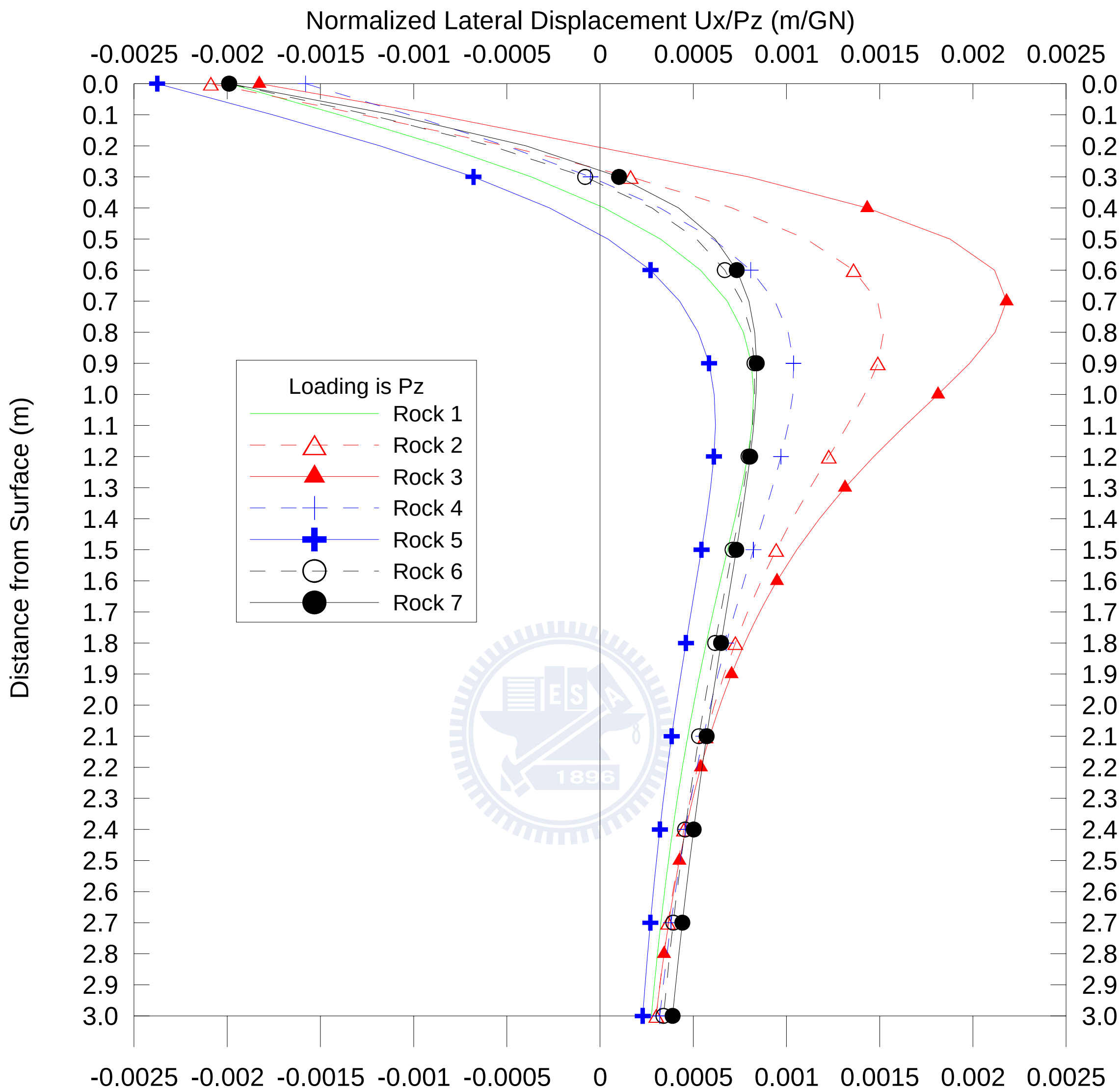

Fig. 6.6 Effect of ratios of E/E', υ/υ' and G/G' on non-dimensional lateral displacement

CHAPTER Ⅶ

SUMMARY AND RECOMMENDATIONS

7.1 Summary

In this work, analytical solutions are presented for displacements and stresses induced by three-dimensional point loads in a transversely isotropic, where the transversely isotropic planes are arbitrarily oriented with respect to the horizontal axis. The solutions for the full-space and half-space are identical to those of Wang and Liao (1999), when the transversely isotropic planes are parallel to the horizontal axis. The approaches adopted herein can be conveniently applied to analyze the strains and stresses by using the strain-displacement, and stress-strain relations. These solutions could realistically imitate the actual stratum of loading situations in numerous areas of engineering. In addition, they provide a mathematical model for solving problems in soil/rock mechanics in which the transversely isotropic planes dip at an angle of inclination from the horizontal surface. Analyses based on the proposed solutions could be helpful in benchmarking modern computational analysis techniques, such as FEM, FDM and others.

The summary of this dissertation can be concluded with the following points:

- The strata of transversely isotropic rock masses usually incline with respect to the horizontal ground surface. In this thesis, the point loading problem for a transversely isotropic rock mass with strata dipping at an angle to the horizontal surface are conscientiously and carefully studied.

- There are six eigenvalues in the characteristic equation of Eq. (3.45), then, the general solutions of Eqs. (3.49a)-(3.49c) in the transformed domain can superpose the six eigensolutions of Eqs. (3.50a)-(3.50c), and Eqs. (3.51a)-(3.51c). It is very important but rather difficult to find the eigenvalues of the present characteristic equations. Fortunately, by discovering the relations of the determinant of $\lfloor d_{ij} \rfloor$ and the velocity of body waves, the six eigenvalues can be obtained.
- Three distinct approaches are introduced to derive the partial differential equations of governing equations in infinite or semi-infinite solids. The present methods are proved to be have the identically results.

7.2 Recommendations for Future Work

The work accomplished in this thesis can be improved and enhanced by further research on the following points:

- By using the method of undetermined coefficients, it is possible to expand the approach to solve the multi-layer or multi-material problems.

- If the singular points can be completely found in the process of inverse transforms, then the proposed solutions could be easily extended to determine the displacements, strains, and stresses resulting from three-dimensional point loads in an inclined transversely isotropic half-space. The interesting results would be addressed in the near future.
- Using the relation of the determinant of $\lfloor d_{ij} \rfloor$, and the velocity of body waves, it is possible to solve the loading problem for the cubic and orthotropic materials.

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APPENDIX A THE EXPRESSIONS OF a_{ij} (i, j=1-6)

The elastic constants in Eq. (3.06) can be expressed as:

$$a_{11} = a_1$$

$$a_{12} = a_{21} = (a_1 - 2a_4)\cos^2\phi + (a_3 - a_5)\sin^2\phi$$

$$a_{13} = a_{31} = (a_3 - a_5)\cos^2\phi + (a_1 - 2a_4)\sin^2\phi$$

$$a_{14} = a_{41} = (a_1 - a_3 - 2a_4 + a_5)\cos\phi\sin\phi$$

$$a_{22} = a_1\cos^4\phi + 2(a_3 + a_5)\cos^2\phi\sin^2\phi + a_2\sin^4\phi$$

$$a_{23} = a_{32} = \frac{1}{8}\{a_1 + a_2 + 6a_3 - 10a_5 - [a_1 + a_2 - 2(a_3 + a_5)]\cos 4\phi\}$$

$$a_{24} = a_{42} = \frac{1}{4}\{a_1 - a_2 + [a_1 + a_2 - 2(a_3 + a_5)]\cos 2\phi\}\sin 2\phi$$

$$a_{33} = a_2\cos^4\phi + 2(a_3 + a_5)\cos^2\phi\sin^2\phi + a_1\sin^4\phi$$

$$a_{34} = a_{43} = -\frac{1}{4}\{-a_1 + a_2 + [a_1 + a_2 - 2(a_3 + a_5)]\cos 2\phi\}\sin 2\phi$$

$$a_{44} = \frac{1}{8}\{a_1 + a_2 - 2a_3 + 6a_5 - [a_1 + a_2 - 2(a_3 + a_5)]\cos 4\phi\}$$

$$a_{55} = a_5\cos^2\phi + a_4\sin^2\phi$$

$$a_{56} = a_{65} = (a_4 - a_5)\cos\phi\sin\phi$$

$$a_{66} = a_4\cos^2\phi + a_5\sin^2\phi$$

$$a_{15} = a_{16} = a_{25} = a_{26} = a_{35} = a_{36} = a_{45} = a_{46} = a_{51} = a_{52} = a_{53} = a_{54} = a_{61} = a_{62} = a_{63} = a_{64} = 0$$

APPENDIX B THE DERIVATION OF THE CHARACTERISTIC EQUATION

According to Eq. (3.02), the x' and y' axes are in the plane of transversely isotropy. The generalized Hooke's law for a transversely isotropic material can be expressed as:

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \sigma_{z'z'} \\ \tau_{y'z'} \\ \tau_{z'x'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} a_1 & a_1-2a_4 & a_3-a_5 & 0 & 0 & 0 \\ a_1-2a_4 & a_1 & a_3-a_5 & 0 & 0 & 0 \\ a_3-a_5 & a_3-a_5 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_4 \end{bmatrix} \begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{z'z'} \\ \gamma_{y'z'} \\ \gamma_{z'x'} \\ \gamma_{x'y'} \end{bmatrix} \qquad \text{(B.1)}$$

The strain-displacement relationship for small strain condition is:

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{z'z'} \\ \gamma_{y'z'} \\ \gamma_{z'x'} \\ \gamma_{x'y'} \end{bmatrix} = \begin{bmatrix} -\frac{\partial u_{x'}}{\partial x'} \\ -\frac{\partial u_{y'}}{\partial y'} \\ -\frac{\partial u_{z'}}{\partial z'} \\ -\frac{\partial u_{y'}}{\partial z'} - \frac{\partial u_{z'}}{\partial y'} \\ -\frac{\partial u_{x'}}{\partial z'} - \frac{\partial u_{z'}}{\partial x'} \\ -\frac{\partial u_{y'}}{\partial x'} - \frac{\partial u_{x'}}{\partial y'} \end{bmatrix} \qquad \text{(B.2)}$$

where $u_{x'}$, $u_{y'}$, and $u_{z'}$ are three displacements of a point on the axis of a Cartesian co-ordinate system.

The equation of force equilibrium is:

$$\begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} & \tau_{z'x'} \\ \tau_{x'y'} & \sigma_{y'y'} & \tau_{y'z'} \\ \tau_{z'x'} & \tau_{y'z'} & \sigma_{z'z'} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{bmatrix} - \begin{bmatrix} F_{x'} \\ F_{y'} \\ F_{z'} \end{bmatrix} = \rho \begin{bmatrix} \frac{\partial^2 u_{x'}}{\partial t^2} \\ \frac{\partial^2 u_{y'}}{\partial t^2} \\ \frac{\partial^2 u_{z'}}{\partial t^2} \end{bmatrix} \tag{B.3}$$

If we set ($F_{x'}$, $F_{y'}$, $F_{z'}$)=(0, 0, 0), then, Eq. (B.3) can be expressed as:

$$a_1 \frac{\partial^2 u_{x'}}{\partial x'^2} + a_4 \frac{\partial^2 u_{x'}}{\partial y'^2} + a_5 \frac{\partial^2 u_{x'}}{\partial z'^2} + (a_1 - a_4) \frac{\partial^2 u_{y'}}{\partial x' \partial y'} + a_3 \frac{\partial^2 u_{z'}}{\partial x' \partial z'} = \rho \frac{\partial^2 u_{x'}}{\partial t^2} \tag{B.4a}$$

$$(a_1 - a_4) \frac{\partial^2 u_{x'}}{\partial x' \partial y'} + a_4 \frac{\partial^2 u_{y'}}{\partial x'^2} + a_1 \frac{\partial^2 u_{y'}}{\partial y'^2} + a_5 \frac{\partial^2 u_{y'}}{\partial z'^2} + a_3 \frac{\partial^2 u_{z'}}{\partial y' \partial z'} = \rho \frac{\partial^2 u_{y'}}{\partial t^2} \tag{B.4b}$$

$$a_3 \frac{\partial^2 u_{x'}}{\partial x' \partial z'} + a_3 \frac{\partial^2 u_{y'}}{\partial y' \partial z'} + a_5 \frac{\partial^2 u_{z'}}{\partial x'^2} + a_5 \frac{\partial^2 u_{z'}}{\partial y'^2} + a_2 \frac{\partial^2 u_{z'}}{\partial z'^2} = \rho \frac{\partial^2 u_{z'}}{\partial t^2} \tag{B.4c}$$

For the elastic dynamic problem, an arbitrary time-harmonic body force in x', y', and z' direction with angular frequency ω can be written as:

$$u_{x'}(x', y', z', t) = u^*_{x'}(x', y', z') \exp(-i\omega t) \tag{B.5a}$$

$$u_{y'}(x', y', z', t) = u^*_{y'}(x', y', z') \exp(-i\omega t) \tag{B.5b}$$

$$u_{z'}(x', y', z', t) = u^*_{z'}(x', y', z') \exp(-i\omega t) \tag{B.5c}$$

where $u^*_{x'}$, $u^*_{y'}$ and $u^*_{z'}$ are the complex amplitude of the body force.

Further, Eqs. (B.5a)-(B.5c) are preformed the triple Fourier transforms as:

$$\bar{u}_{x'}(\alpha,\beta,\gamma)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_{x'}(x',y',z')e^{-i(\alpha x'+\beta y'+\gamma z')}dx'dy'dz' \quad \text{(B.6a)}$$

$$\bar{u}_{y'}(\alpha,\beta,\gamma)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_{x'}(x',y',z')e^{-i(\alpha x'+\beta y'+\gamma z')}dx'dy'dz' \quad \text{(B.6b)}$$

$$\bar{u}_{z'}(\alpha,\beta,\gamma)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_{x'}(x',y',z')e^{-i(\alpha x'+\beta y'+\gamma z')}dx'dy'dz' \quad \text{(B.6c)}$$

Substituting Eqs. (B.5a)-(B.5c) and Eqs. (B.6a)-(B.6c) into Eqs. (B.4a)-(B.4c), we have the triple Fourier-type integrals as:

$$a_1\alpha^2\bar{u}^*_{x'}+a_4\beta^2\bar{u}^*_{x'}+a_5\gamma^2\bar{u}^*_{x'}+(a_1-a_4)\alpha\beta\bar{u}^*_{y'}+a_3\alpha\gamma\bar{u}^*_{z'}=\rho\omega^2\bar{u}^*_{x'} \quad \text{(B.7a)}$$

$$(a_1-a_4)\alpha\beta\bar{u}^*_{x'}+a_4\alpha^2\bar{u}^*_{y'}+a_1\beta^2\bar{u}^*_{y'}+a_5\gamma^2\bar{u}^*_{y'}+a_3\beta\gamma\bar{u}^*_{z'}=\rho\omega^2\bar{u}^*_{y'} \quad \text{(B.7b)}$$

$$a_3\alpha\gamma\bar{u}^*_{x'}+a_3\beta\gamma\bar{u}^*_{y'}+a_5\alpha^2\bar{u}^*_{z'}+a_5\beta^2\bar{u}^*_{z'}+a_2\gamma^2\bar{u}^*_{z'}=\rho\omega^2\bar{u}^*_{z'} \quad \text{(B.7c)}$$

Rearranging Eqs. (B.7a)-(B.7c) as:

$$\begin{bmatrix} d_{11}-\rho\omega^2 & d_{12} & d_{13} \\ d_{21} & d_{22}-\rho\omega^2 & d_{23} \\ d_{31} & d_{32} & d_{33}-\rho\omega^2 \end{bmatrix}\begin{bmatrix}\bar{u}^*_{x'}\\ \bar{u}^*_{y'}\\ \bar{u}^*_{z'}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{(B.8)}$$

where $\left[d_{ij}\right]=\begin{bmatrix} a_1\alpha^2+a_4\beta^2+a_5\gamma^2 & (a_1-a_4)\alpha\beta & a_3\alpha\gamma \\ (a_1-a_4)\alpha\beta & a_4\alpha^2+a_1\beta^2+a_5\gamma^2 & a_3\beta\gamma \\ a_3\alpha\gamma & a_3\beta\gamma & a_5\alpha^2+a_5\beta^2+a_2\gamma^2 \end{bmatrix}$ (i, j=1-3).

It is very clear that the zero vector $\left[\bar{u}^*_{x'} \quad \bar{u}^*_{y'} \quad \bar{u}^*_{z'}\right]=\left[0 \quad 0 \quad 0\right]$ is the solution of Eq. (B.8) for any value of $\rho\omega^2$. However, the value of $\rho\omega^2$ for Eq. (B.8) when

$\lfloor \bar{u}_{x'}^* \quad \bar{u}_{y'}^* \quad \bar{u}_{z'}^* \rfloor \neq [0 \quad 0 \quad 0]$ is called the eigenvalue of this matrix.

$$\begin{bmatrix} d_{11}-\rho\omega^2 & d_{12} & d_{13} \\ d_{21} & d_{22}-\rho\omega^2 & d_{23} \\ d_{31} & d_{32} & d_{33}-\rho\omega^2 \end{bmatrix}$$
$$= -\{\rho\omega^2 - [a_5\gamma^2 + a_4(\alpha^2+\beta^2)]\} \cdot \{\rho\omega^2 - \frac{a_1(\alpha^2+\beta^2)+a_2\gamma^2+a_5(\alpha^2+\beta^2+\gamma^2)+\Delta}{2}\} \cdot$$
$$\{\rho\omega^2 - \frac{a_1(\alpha^2+\beta^2)+a_2\gamma^2+a_5(\alpha^2+\beta^2+\gamma^2)-\Delta}{2}\} \qquad \text{(B.9)}$$

where $\Delta = \sqrt{[(a_1-a_5)(\alpha^2+\beta^2)-(a_2-a_5)\gamma^2]^2 + 4a_3^2(\alpha^2+\beta^2)\gamma^2}$.

In addition, the variables α, β, γ can be expressed in terms of k, $\theta_{x'}$, $\theta_{z'}$ as:

$\alpha = k\sin\theta_{z'}\cos\theta_{x'}$,

$\beta = k\sin\theta_{z'}\sin\theta_{x'}$,

$\gamma = k\cos\theta_{z'}$.

Hence, the relationships between α, β, γ, and k, $\theta_{x'}$, $\theta_{z'}$ are:

$(\alpha^2+\beta^2) = k^2\sin^2\theta_{z'}$,

$\gamma^2 = k^2\cos^2\theta_{z'}$.

Moreover, by introducing $V^2 = \frac{\omega^2}{k^2}$ (V denotes the body-wave velocity), the final results for the three body-wave velocities are identical with those in a transversely isotropic medium, as follows:

$$V_{SH,\theta,z'} = \sqrt{\frac{a_5 \cos^2\theta_{z'} + a_4 \sin^2\theta_{z'}}{\rho}} \qquad \text{(B.10a)}$$

$$V_{P,\theta,z'} = \sqrt{\frac{a_1 \sin^2\theta_{z'} + a_2 \cos^2\theta_{z'} + a_5 + \Delta'}{2\rho}} \qquad \text{(B.10b)}$$

$$V_{SV,\theta,z'} = \sqrt{\frac{a_1 \sin^2\theta_{z'} + a_2 \cos^2\theta_{z'} + a_5 - \Delta'}{2\rho}} \qquad \text{(B.10c)}$$

where $\Delta' = \sqrt{[(a_1 - a_5)\sin^2\theta_{z'} - (a_2 - a_5)\cos^2\theta_{z'}]^2 + 4a_3^2 \sin^2\theta_{z'} \cos^2\theta_{z'}}$.

The angle $\theta_{z'}$ is between the direction of travel wave and the z' axis. Therefore, the determinant of $[d_{ij}]$, is witten as:

$$\begin{aligned}
D &= \det[d_{ij}] \\
&= \rho^3 k^6 (V_{SH,\theta,z'} \cdot V_{P,\theta,z'} \cdot V_{SV,\theta,z'})^2 \\
&= \left[a_5\gamma^2 + a_4(\alpha^2+\beta^2)\right]\{[a_1(\alpha^2+\beta^2) + a_5\gamma^2][a_5(\alpha^2+\beta^2) + a_2\gamma^2] - a_3{}^2(\alpha^2+\beta^2)\gamma^2\} \\
&= a_2 a_5{}^2 \prod_{i=1}^{3}\left[A_i(\alpha^2+\beta^2) + \gamma^2\right] \\
&= a_2 a_5{}^2 k^6 \prod_{i=1}^{3}\left[A_i \sin^2\theta_{z'} + \cos^2\theta_{z'}\right]
\end{aligned} \qquad \text{(B.11)}$$

where

$$A_1 = \frac{a_4}{a_5},$$

$$A_2 = \frac{1}{2}\left[\frac{a_5{}^2 + a_1 a_2 - a_3{}^3}{a_2 a_5} + \left\{\left(\frac{a_5{}^2 + a_1 a_2 - a_3{}^2}{a_2 a_5}\right)^2 - 4\frac{a_1}{a_2}\right\}^{\frac{1}{2}}\right],$$

$$A_3 = \frac{1}{2}\left[\frac{a_5^2 + a_1a_2 - a_3^3}{a_2a_5} - \left\{\left(\frac{a_5^2 + a_1a_2 - a_3^2}{a_2a_5}\right)^2 - 4\frac{a_1}{a_2}\right\}^{\frac{1}{2}}\right].$$

As depicted in Fig. B1, considering a new co-ordinate system x, y, z and the original system x', y', z' with a common origin point be taken so that x and x' axes lie in the xy plane (basic plane). The angle between the x and x' axes is taken as one co ordinate, φ, and the angle between the y' axis and the xy plane as the other co-ordinate, ϕ. We assign the cosines of the angles between the axes of the old and new co-ordinate system by Table B1.

Table B1 the cosines of the angles between the old and new axes

	x'	y'	z'
x	$\cos\varphi$	$\sin\varphi$	0
y	$-\cos\phi \ \sin\varphi$	$\cos\phi \ \cos\varphi$	$\sin\phi$
z	$\sin\phi \ \sin\varphi$	$-\sin\phi \ \cos\varphi$	$\cos\phi$

Then, the value of D in Eq. (B.11) can be presented as:

$$\begin{aligned} D &= \rho^3 k^6 (V_{SH,\theta,t} \cdot V_{P,\theta,t} \cdot V_{SV,\theta,t})^2 \\ &= a_2 a_5^2 k^6 \prod_{i=1}^{3}\left[A_i \sin^2\theta_t + \cos^2\theta_t\right] \end{aligned} \quad \text{(B.12)}$$

where θ_t is the angle between the vector (α, β, γ) and the z axis, and it can be expressed in terms of α, β, γ, and ϕ as:

$$\cos\theta_t = \frac{\alpha\sin\phi\sin\varphi - \beta\sin\phi\cos\varphi + \gamma\cos\phi}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}} \quad \text{(B.13)}$$

Additionally, from Eq. (B.13), $\cos^2\theta_t = \dfrac{(\alpha\sin\phi\sin\varphi - \beta\sin\phi\cos\varphi + \gamma\cos\phi)^2}{\alpha^2+\beta^2+\gamma^2}$,

and thus, $\sin^2\theta_t = \dfrac{\alpha^2+\beta^2+\gamma^2-(\alpha\sin\phi\sin\varphi - \beta\sin\phi\cos\varphi + \gamma\cos\phi)^2}{\alpha^2+\beta^2+\gamma^2}$.

Hence, Eq. (B.12) can be rearranged as:

$$
\begin{aligned}
D &= a_2 a_5{}^2 k^6 \prod_{i=1}^{3}\left[A_i\sin^2\theta_t + \cos^2\theta_t\right] \\
&= \frac{a_2 a_5{}^2 k^6}{(\alpha^2+\beta^2+\gamma^2)^3}\prod_{i=1}^{3}\left\{\begin{array}{l} A_i\left[\alpha^2+\beta^2+\gamma^2-(\alpha\sin\phi\sin\varphi-\beta\sin\phi\cos\varphi+\gamma\cos\phi)^2\right] \\ +(\alpha\sin\phi\sin\varphi-\beta\sin\phi\cos\varphi+\gamma\cos\phi)^2 \end{array}\right\} \\
&= a_2 a_5{}^2 \prod_{i=1}^{3}\left\{\begin{array}{l} A_i\left[\alpha^2+\beta^2+\gamma^2-(\alpha\sin\phi\sin\varphi-\beta\sin\phi\cos\varphi+\gamma\cos\phi)^2\right] \\ +(\alpha\sin\phi\sin\varphi-\beta\sin\phi\cos\varphi+\gamma\cos\phi)^2 \end{array}\right\}
\end{aligned}
\tag{B.14}
$$

Eventually, the six eigenroots can be generated by setting D=0 in Eq. (B.14) (also in Eq. (3.45)). They are respectively expressed as follow:

$$
\gamma_j(\alpha,\beta) = -\frac{\left\{\begin{array}{l} \beta\cos\phi\sin\phi\cos\varphi(-1+A_j)-\alpha\cos\phi\sin\phi\sin\varphi(-1+A_j) \\ +i\sqrt{A_j\{(\alpha^2+\beta^2)\cos^2\phi+\sin^2\phi(\beta\cos\varphi-\alpha\sin\varphi)^2+\sin^2\phi(\alpha\cos\varphi+\beta\sin\varphi)^2A_j\}} \end{array}\right\}}{\cos^2\phi+A_j\sin^2\phi}
$$

(j=1-3)(B.15a)

$$
\gamma_j(\alpha,\beta) = \frac{\left\{\begin{array}{l} -\beta\cos\phi\sin\phi\cos\varphi(-1+A_j)+\alpha\cos\phi\sin\phi\sin\varphi(-1+A_j) \\ +i\sqrt{A_j\{(\alpha^2+\beta^2)\cos^2\phi+\sin^2\phi(\beta\cos\varphi-\alpha\sin\varphi)^2+\sin^2\phi(\alpha\cos\varphi+\beta\sin\varphi)^2A_j\}} \end{array}\right\}}{\cos^2\phi+A_j\sin^2\phi}
$$

(j=4-6)(B.15b)

As depicted in Fig. 1, considering a new co-ordinate system x, y, z is obtained from the original system x', y', z' by rotation an angle ϕ about the x=x' axis, $\varphi = 0$. Then, the eigenvalue, γ of Eqs. (B.15a)-(B.15b), can be presented as:

$$\gamma_j(\alpha,\beta)=\frac{-\beta\cos\phi\sin\phi(-1+A_j)-i\sqrt{A_j\{\beta^2+\alpha^2(\cos^2\phi+A_j\sin^2\phi)\}}}{\cos^2\phi+A_j\sin^2\phi}\ \text{(j=1-3)} \tag{B.16a}$$

$$\gamma_j(\alpha,\beta)=\frac{-\beta\cos\phi\sin\phi(-1+A_j)+i\sqrt{A_j\{\beta^2+\alpha^2(\cos^2\phi+A_j\sin^2\phi)\}}}{\cos^2\phi+A_j\sin^2\phi}\ \text{(j=4-6)} \tag{B.16b}$$

If we further set $i\gamma = u_j$, then, Eq. (B.16a)-Eq.(B.16b) can be respectively expressed in Eqs. (3.47a)-(3.47f).

APPENDIX C THE EXPRESSIONS OF D_{ij} (i, j=1-3)

The complete expressions for D_{ij} (i, j=1-3) in Eqs. (3.53a)-(3.53f) are presented as:

$$
\begin{aligned}
D_{11}(u_j) = {} & (a_{33}a_{44} - a_{34}^2)u_j^4 \\
& + 2i\beta(a_{23}a_{34} - a_{24}a_{33})u_j^3 \\
& + (\alpha^2(-a_{33}a_{66} + 2a_{34}a_{56} - a_{44}a_{55}) + \beta^2(-a_{22}a_{33} + a_{23}^2 + 2a_{23}a_{44} - 2a_{24}a_{34}))u_j^2 \\
& + 2i\beta(\alpha^2(a_{24}a_{55} + a_{34}a_{66} - a_{56}(a_{23} + a_{44})) + \beta^2(a_{22}a_{34} - a_{23}a_{24}))u_j \\
& + (\alpha^4(a_{55}a_{66} - a_{56}^2) + \alpha^2\beta^2(a_{22}a_{55} - 2a_{24}a_{56} + a_{44}a_{66}) + \beta^4(a_{22}a_{44} - a_{24}^2))
\end{aligned}
$$

$$
\begin{aligned}
D_{12}(u_j) = {} & i\alpha(a_{33}(a_{14} + a_{56}) - a_{34}(a_{13} + a_{55}))u_j^3 \\
& + \alpha\beta(a_{33}(a_{12} + a_{66}) + a_{34}(a_{14} + a_{56}) - (a_{13} + a_{55})(a_{23} + a_{44}))u_j^2 \\
& + i\alpha(\alpha^2(a_{13}a_{56} - a_{14}a_{55}) + \beta^2(a_{23}(a_{14} + a_{56}) + a_{24}(a_{13} + a_{55}) - 2a_{34}(a_{12} + a_{66}))u_j \\
& - \alpha\beta(\alpha^2(a_{55}(a_{12} + a_{66}) - a_{56}(a_{14} + a_{56})) + \beta^2(-a_{24}(a_{14} + a_{56}) + a_{44}(a_{12} + a_{66})))
\end{aligned}
$$

$$
\begin{aligned}
D_{13}(u_j) = {} & i\alpha(-a_{34}(a_{14} + a_{56}) + a_{44}(a_{13} + a_{55}))u_j^3 \\
& - \alpha\beta(a_{23}(a_{14} + a_{56}) - 2a_{24}(a_{13} + a_{55}) + a_{34}(a_{12} + a_{66}))u_j^2 \\
& - i\alpha(\alpha^2(-a_{56}(a_{14} + a_{56}) + a_{66}(a_{13} + a_{55})) \\
& + \beta^2(-a_{12}(a_{23} + a_{44}) + a_{22}(a_{13} + a_{55}) + a_{24}(a_{14} + a_{56}) - a_{66}(a_{23} + a_{44})))u_j \\
& + \alpha\beta(\alpha^2(a_{12}a_{56} - a_{14}a_{66}) + \beta^2(a_{24}(a_{12} + a_{66}) - a_{22}(a_{14} + a_{56})))
\end{aligned}
$$

$$
\begin{aligned}
D_{22}(u_j) = {} & a_{33}a_{55}u_j^4 \\
& - 2i\beta(a_{33}a_{56} + a_{34}a_{55})u_j^3 \\
& + (\alpha^2(-a_{11}a_{33} + a_{13}(a_{13} + 2a_{55})) - \beta^2(a_{33}a_{66} + 4a_{34}a_{56} + a_{44}a_{55}))u_j^2 \\
& + 2i\beta(\alpha^2(a_{11}a_{34} - a_{13}(a_{14} + a_{56}) - a_{14}a_{55}) + \beta^2(a_{34}a_{66} + a_{44}a_{56}))u_j \\
& + \alpha^4 a_{11}a_{55} + \alpha^2\beta^2(a_{11}a_{44} + a_{55}a_{66} - (a_{14} + a_{56})^2) + \beta^4 a_{44}a_{66}
\end{aligned}
$$

$$
\begin{aligned}
D_{23}(u_j) = {} & -a_{34}a_{55}u_j^4 \\
& + i\beta(2a_{34}a_{56} + a_{55}(a_{23} + a_{44}))u_j^3 \\
& - (\alpha^2(-a_{11}a_{34} + a_{14}a_{55} + a_{13}(a_{14} + a_{56})) - \beta^2(a_{24}a_{55} + a_{34}a_{66} + 2a_{56}(a_{23} + a_{44})))u_j^2 \\
& - i\beta(\alpha^2(a_{11}(a_{23} + a_{44}) - a_{14}(a_{14} + a_{56}) - a_{56}(a_{14} - a_{56}) - (a_{12} - a_{66})(a_{13} + a_{55})) \\
& + \beta^2(2a_{24}a_{56} + a_{66}(a_{23} + a_{44})))u_j \\
& - (\alpha^4 a_{11}a_{56} + \alpha^2\beta^2(a_{11}a_{24} - a_{12}(a_{14} + a_{56}) - a_{14}a_{66}) + \beta^4 a_{24}a_{66})
\end{aligned}
$$

$$
\begin{aligned}
D_{33}(u_j) = &(a_{44}a_{55})u_j^4 \\
&-2i\beta(a_{24}a_{55}+a_{44}a_{56})u_j^3 \\
&+(\alpha^2(-a_{11}a_{44}-a_{55}a_{66}+(a_{14}+a_{56})^2)-\beta^2(a_{22}a_{55}+4a_{24}a_{56}+a_{44}a_{66})))u_j^2 \\
&+(2i\beta(\alpha^2(a_{11}a_{24}-a_{14}a_{66}-a_{12}(a_{14}+a_{56}))+\beta^2(a_{22}a_{56}+a_{24}a_{66})))u_j \\
&+\alpha^4 a_{11}a_{66}+\alpha^2\beta^2(a_{11}a_{22}-a_{12}(a_{12}+2a_{66}))+\beta^4 a_{22}a_{66}
\end{aligned}
$$

APPENDIX D THE EXPRESSIONS OF f_{ij} (i, j=1-6)

The expressions of f_{ij} (i, j=1-6) can be presented as:

$$f_{11} = -i(a_{56}(\beta D_{11}^1 + \alpha D_{21}^1) + a_{55}(\alpha D_{31}^1 - iD_{11}^1 u_1)),$$

$$f_{12} = -i(a_{56}(\beta D_{11}^2 + \alpha D_{21}^2) + a_{55}(\alpha D_{31}^2 - iD_{11}^2 u_2)),$$

$$f_{13} = -i(a_{56}(\beta D_{11}^3 + \alpha D_{21}^3) + a_{55}(\alpha D_{31}^3 - iD_{11}^3 u_3)),$$

$$f_{14} = i(a_{56}(\beta D_{11}^4 + \alpha D_{21}^4) + a_{55}(\alpha D_{31}^4 - iD_{11}^4 u_4)),$$

$$f_{15} = i(a_{56}(\beta D_{11}^5 + \alpha D_{21}^5) + a_{55}(\alpha D_{31}^5 - iD_{11}^5 u_5)),$$

$$f_{16} = i(a_{56}(\beta D_{11}^6 + \alpha D_{21}^6) + a_{55}(\alpha D_{31}^6 - iD_{11}^6 u_6)),$$

$$f_{21} = -i(\alpha a_{14} D_{11}^1 + \beta a_{24} D_{21}^1 + \beta a_{44} D_{31}^1) - (a_{44} D_{21}^1 + a_{34} D_{31}^1) u_1,$$

$$f_{22} = -i(\alpha a_{14} D_{11}^2 + \beta a_{24} D_{21}^2 + \beta a_{44} D_{31}^2) - (a_{44} D_{21}^2 + a_{34} D_{31}^2) u_2,$$

$$f_{23} = -i(\alpha a_{14} D_{11}^3 + \beta a_{24} D_{21}^3 + \beta a_{44} D_{31}^3) - (a_{44} D_{21}^3 + a_{34} D_{31}^3) u_3,$$

$$f_{24} = i(\alpha a_{14} D_{11}^4 + \beta a_{24} D_{21}^4 + \beta a_{44} D_{31}^4) + (a_{44} D_{21}^4 + a_{34} D_{31}^4) u_4,$$

$$f_{25} = i(\alpha a_{14} D_{11}^5 + \beta a_{24} D_{21}^5 + \beta a_{44} D_{31}^5) + (a_{44} D_{21}^5 + a_{34} D_{31}^5) u_5,$$

$$f_{26} = i(\alpha a_{14} D_{11}^6 + \beta a_{24} D_{21}^6 + \beta a_{44} D_{31}^6) + (a_{44} D_{21}^6 + a_{34} D_{31}^6) u_6,$$

$$f_{31} = -i(\alpha a_{13} D_{11}^1 + \beta a_{23} D_{21}^1 + \beta a_{34} D_{31}^1) - (a_{34} D_{21}^1 + a_{33} D_{31}^1) u_1,$$

$$f_{32} = -(\alpha a_{13} D_{11}^2 + \beta a_{23} D_{21}^2 + \beta a_{34} D_{31}^2) - (a_{34} D_{21}^2 + a_{33} D_{31}^2) u_2 ,$$

$$f_{33} = -i(\alpha a_{13} D_{11}^3 + \beta a_{23} D_{21}^3 + \beta a_{34} D_{31}^3) - (a_{34} D_{21}^3 + a_{33} D_{31}^3) u_3 ,$$

$$f_{34} = i(\alpha a_{13} D_{11}^4 + \beta a_{23} D_{21}^4 + \beta a_{34} D_{31}^4) + (a_{34} D_{21}^4 + a_{33} D_{31}^4) u_4 ,$$

$$f_{35} = i(\alpha a_{13} D_{11}^5 + \beta a_{23} D_{21}^5 + \beta a_{34} D_{31}^5) + (a_{34} D_{21}^5 + a_{33} D_{31}^5) u_5 ,$$

$$f_{36} = i(\alpha a_{13} D_{11}^6 + \beta a_{23} D_{21}^6 + \beta a_{34} D_{31}^6) + (a_{34} D_{21}^6 + a_{33} D_{31}^6) u_6 ,$$

$$f_{41} = D_{11}^1 ,\quad f_{42} = D_{11}^2 ,\quad f_{43} = D_{11}^3 ,\quad f_{44} = -D_{11}^4 ,\quad f_{45} = -D_{11}^5 ,\quad f_{46} = -D_{11}^6 ,$$

$$f_{51} = D_{21}^1 ,\quad f_{52} = D_{21}^2 ,\quad f_{53} = D_{21}^3 ,\quad f_{54} = -D_{21}^4 ,\quad f_{55} = -D_{21}^5 ,\quad f_{56} = -D_{21}^6 ,$$

$$f_{61} = D_{31}^1 ,\quad f_{62} = D_{31}^2 ,\quad f_{63} = D_{31}^3 ,\quad f_{64} = -D_{31}^4 ,\quad f_{65} = -D_{31}^5 ,\quad f_{66} = -D_{31}^6 .$$

APPENDIX E BOUNDARY CONDITIONS

In order to demonstrate the Eq. (4.15), considering the pertinent continuity and discontinuity conditions at $z = 0$:

In region I of Fig. 3.1, for $z = 0^+$, the Cauchy′s theory of residues are utilized to integrate the contours, and a closed contour by adding a large semicirle on the upper γ-plane, is shown in Fig. E.1

$$\bar{u}_{x1}(\alpha,\beta,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma$$
$$= \frac{1}{\sqrt{2\pi}}\left(\oint_{C1}\bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma - \int_{C_{S1}} \bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma\right) \tag{E.01a}$$

$$\bar{u}_{y1}(\alpha,\beta,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma$$
$$= \frac{1}{\sqrt{2\pi}}\left(\oint_{C1}\bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma - \int_{C_{S1}} \bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma\right) \tag{E.01b}$$

$$\bar{u}_{z1}(\alpha,\beta,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{U}_z(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma$$
$$= \frac{1}{\sqrt{2\pi}}\left(\oint_{C1}\bar{U}_z(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma - \int_{C_{S1}} \bar{U}_z(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma\right) \tag{E.01c}$$

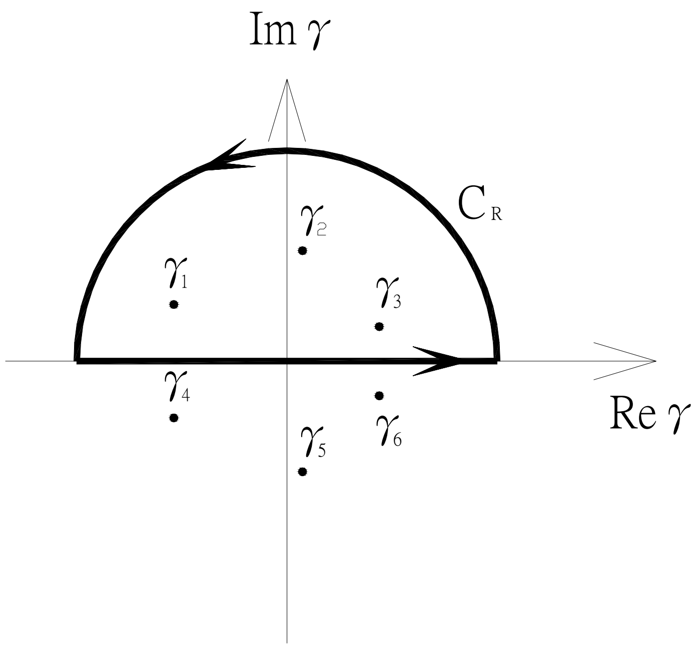

Fig. E.1 Path of integration for Eqs. (E.01a)-(E.01c)

Similarly, as depicted in Fig. 3.1, the region Ⅱ, for $z = 0^-$, a closed contour by adding a large semicirle on the lower γ-plane is utilized.

$$\bar{u}_{x2}(\alpha,\beta,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma$$
$$= \frac{1}{\sqrt{2\pi}}(\oint_{C2}\bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma - \int_{C_{S2}} \bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma) \tag{E.02a}$$

$$\bar{u}_{y2}(\alpha,\beta,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma$$
$$= \frac{1}{\sqrt{2\pi}}(\oint_{C2}\bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma - \int_{C_{S2}} \bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma) \tag{E.02b}$$

$$\bar{u}_{z2}(\alpha,\beta,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{U}_z(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma$$
$$= \frac{1}{\sqrt{2\pi}}(\oint_{C2}\bar{U}_z(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma - \int_{C_{S2}} \bar{U}_z(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma) \tag{E.02c}$$

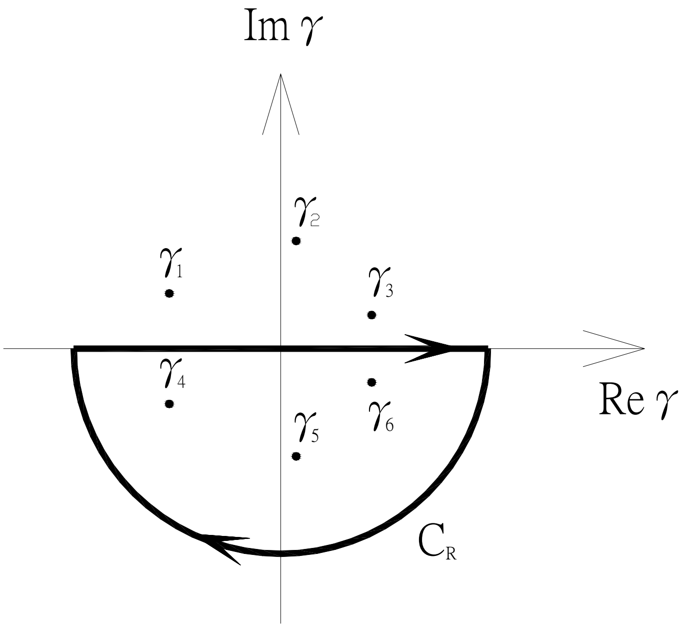

Fig. E.2 Path of integration for Eqs. (E.02a)-(E.02c)

From Eqs. (E.01a)-(E.01c), Eqs. (E.02a)-(E.02c), and Fig. E.3, we can find that:

$$\bar{u}_{x1}(\alpha,\beta,z)-\bar{u}_{x2}(\alpha,\beta,z)=\frac{1}{\sqrt{2\pi}}\left(\int_{C_{S1}}\bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma-\int_{C_{S2}}\bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma\right)=0 \tag{E.03a}$$

$$\bar{u}_{y1}(\alpha,\beta,z)-\bar{u}_{y2}(\alpha,\beta,z)=\frac{1}{\sqrt{2\pi}}\left(\int_{C_{S1}}\bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma-\int_{C_{S2}}\bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma\right)=0 \tag{E.03b}$$

$$\bar{u}_{z1}(\alpha,\beta,z)-\bar{u}_{z2}(\alpha,\beta,z)=\frac{1}{\sqrt{2\pi}}\left(\int_{C_{S1}}\bar{U}_z(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma-\int_{C_{S2}}\bar{U}_z(\alpha,\beta,\gamma)e^{i\gamma z}d\gamma\right)=0 \tag{E.03c}$$

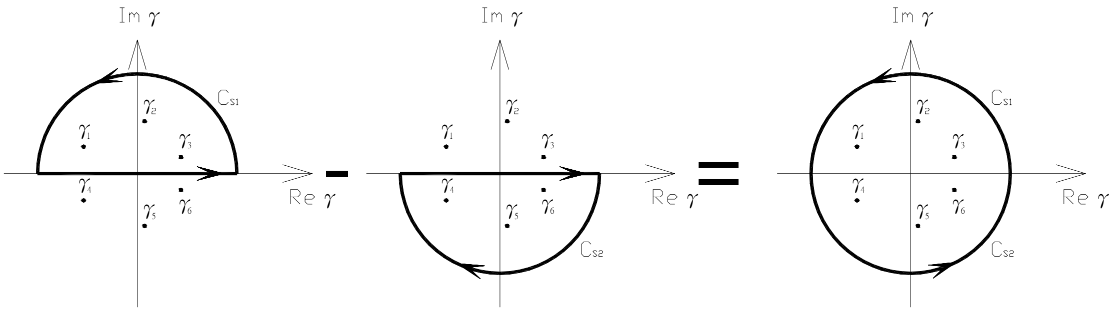

Fig. E.3. Path of integration for Eqs. (E.03a)-(E.03c)

From Eqs. (4.16a)-(4.16c), Eqs. (4.17a)-(4.17c), and the relations of Eqs. (E.03a)-(E.03c), we observe that:

$$\bar{u}_{x1}(\alpha,\beta,0)-\bar{u}_{x2}(\alpha,\beta,0)=B_x^1+B_x^2+B_x^3+B_x^4+B_x^5+B_x^6=0 \qquad \text{(E.04a)}$$

$$\bar{u}_{y1}(\alpha,\beta,0)-\bar{u}_{y2}(\alpha,\beta,0)=B_y^1+B_y^2+B_y^3+B_y^4+B_y^5+B_y^6=0 \qquad \text{(E.04b)}$$

$$\bar{u}_{z1}(\alpha,\beta,0)-\bar{u}_{z2}(\alpha,\beta,0)=B_z^1+B_z^2+B_z^3+B_z^4+B_z^5+B_z^6=0 \qquad \text{(E.04c)}$$

From Eqs. (4.16f)-(4.16h) and Eqs. (4.17f)-(4.17h), $\bar{\tau}_{zx1}-\bar{\tau}_{zx2}$, $\bar{\tau}_{yz1}-\bar{\tau}_{yz1}$, and $\bar{\sigma}_{zz1}-\bar{\sigma}_{zz2}$ can express as follow:

$$\bar{\tau}_{zx1}-\bar{\tau}_{zx2}=a_{55}\{(\frac{\partial\bar{u}_{x1}}{\partial z}-\frac{\partial\bar{u}_{x2}}{\partial z})-i\alpha(\bar{u}_{z1}-\bar{u}_{z2})\}-ia_{56}\{\beta(\bar{u}_{x1}-\bar{u}_{x2})+\alpha(\bar{u}_{y1}-\bar{u}_{y2})\} \qquad \text{(E.05a)}$$

$$\begin{aligned}\bar{\tau}_{yz1}-\bar{\tau}_{yz1}=&-i\alpha a_{14}(\bar{u}_{x1}-\bar{u}_{x2})-i\beta a_{24}(\bar{u}_{y1}-\bar{u}_{y2})+a_{34}(\frac{\partial\bar{u}_{z1}}{\partial z}-\frac{\partial\bar{u}_{z2}}{\partial z})\\&+a_{44}\{(\frac{\partial\bar{u}_{y1}}{\partial z}-\frac{\partial\bar{u}_{y2}}{\partial z})-i\beta(\bar{u}_{z1}-\bar{u}_{z2})\}\end{aligned} \qquad \text{(E.05b)}$$

$$\begin{aligned}\bar{\sigma}_{zz1}-\bar{\sigma}_{zz2}=&-i\alpha a_{13}(\bar{u}_{x1}-\bar{u}_{x2})-i\beta a_{23}(\bar{u}_{y1}-\bar{u}_{y2})+a_{33}(\frac{\partial\bar{u}_{z1}}{\partial z}-\frac{\partial\bar{u}_{z2}}{\partial z})\\&+a_{34}\{(\frac{\partial\bar{u}_{y1}}{\partial z}-\frac{\partial\bar{u}_{y2}}{\partial z})-i\beta(\bar{u}_{z1}-\bar{u}_{z1})\}\end{aligned} \qquad \text{(E.05c)}$$

when $z=0$, substituting Eqs. (E.04a)-(E.04c) into Eqs. (E.05a)-(E.05c) can obtain the following expressions:

$$\bar{\tau}_{zx1}-\bar{\tau}_{zx2}=a_{55}(\frac{\partial\bar{u}_{x1}}{\partial z}-\frac{\partial\bar{u}_{x2}}{\partial z}) \tag{E.06a}$$

$$\bar{\tau}_{yz1}-\bar{\tau}_{yz1}=a_{34}(\frac{\partial\bar{u}_{z1}}{\partial z}-\frac{\partial\bar{u}_{z2}}{\partial z})+a_{44}(\frac{\partial\bar{u}_{y1}}{\partial z}-\frac{\partial\bar{u}_{y2}}{\partial z}) \tag{E.06b}$$

$$\bar{\sigma}_{zz1}-\bar{\sigma}_{zz2}=a_{33}(\frac{\partial\bar{u}_{z1}}{\partial z}-\frac{\partial\bar{u}_{z2}}{\partial z})+a_{34}(\frac{\partial\bar{u}_{y1}}{\partial z}-\frac{\partial\bar{u}_{y2}}{\partial z}) \tag{E.06c}$$

By using the method of Laurent′s theorem, we find that:

$$\begin{aligned}
&\frac{\partial\bar{u}_{x1}}{\partial z}-\frac{\partial\bar{u}_{x2}}{\partial z}\\
&=\frac{1}{\sqrt{2\pi}}(\oint_{C1}\frac{\partial\bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}}{\partial z}d\gamma-\oint_{C2}\frac{\partial\bar{U}_x(\alpha,\beta,\gamma)e^{i\gamma z}}{\partial z}d\gamma)\\
&=-\frac{i}{(2\pi)^2}(\oint_C\frac{P_x\gamma D_{11}+P_y\gamma D_{12}+P_z\gamma D_{13}}{m_t(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma-\gamma_3)(\gamma-\gamma_4)(\gamma-\gamma_5)(\gamma-\gamma_6)}e^{i\gamma z}d\gamma\\
&=-\frac{1}{2\pi a_{55}}P_x
\end{aligned} \tag{E.07a}$$

$$\begin{aligned}
&\frac{\partial\bar{u}_{y1}}{\partial z}-\frac{\partial\bar{u}_{y2}}{\partial z}\\
&=\frac{1}{\sqrt{2\pi}}(\oint_{C1}\frac{\partial\bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}}{\partial z}d\gamma-\oint_{C2}\frac{\partial\bar{U}_y(\alpha,\beta,\gamma)e^{i\gamma z}}{\partial z}d\gamma)\\
&=-\frac{i}{(2\pi)^2}(\oint_C\frac{P_x\gamma D_{21}+P_y\gamma D_{22}+P_z\gamma D_{23}}{m_t(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma-\gamma_3)(\gamma-\gamma_4)(\gamma-\gamma_5)(\gamma-\gamma_6)}e^{i\gamma z}d\gamma\\
&=\frac{a_{33}a_{55}}{2\pi a_{55}(a_{33}a_{44}-a_{34}^2)}P_y-\frac{a_{34}a_{55}}{2\pi a_{55}(a_{33}a_{44}-a_{34}^2)}P_z
\end{aligned} \tag{E.07b}$$

$$\frac{\partial \overline{u}_{z1}}{\partial z} - \frac{\partial \overline{u}_{z2}}{\partial z}$$

$$= \frac{1}{\sqrt{2\pi}} (\oint_{C1} \frac{\partial \overline{U}_z(\alpha,\beta,\gamma) e^{i\gamma z}}{\partial z} d\gamma - \oint_{C2} \frac{\partial \overline{U}_z(\alpha,\beta,\gamma) e^{i\gamma z}}{\partial z} d\gamma)$$

$$= -\frac{i}{(2\pi)^2} (\oint_C \frac{P_x \gamma D_{31} + P_y \gamma D_{32} + P_z \gamma D_{33}}{m_t(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma-\gamma_3)(\gamma-\gamma_4)(\gamma-\gamma_5)(\gamma-\gamma_6)} e^{i\gamma z} d\gamma \qquad \text{(E.07c)}$$

$$= -\frac{a_{34} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_y + \frac{a_{44} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_z$$

Replacing Eqs. (E.07a)-(E.07c) into Eqs. (E.06a)-(E.06c), we prove that:

$$\overline{\tau}_{zx1} - \overline{\tau}_{zx2} = a_{55} (\frac{\partial \overline{u}_{x1}}{\partial z} - \frac{\partial \overline{u}_{x2}}{\partial z}) = \frac{P_x}{2\pi} \qquad \text{(E.08a)}$$

$$\overline{\tau}_{yz1} - \overline{\tau}_{yz1} = a_{34} (\frac{\partial \overline{u}_{z1}}{\partial z} - \frac{\partial \overline{u}_{z2}}{\partial z}) + a_{44} (\frac{\partial \overline{u}_{y1}}{\partial z} - \frac{\partial \overline{u}_{y2}}{\partial z})$$

$$= a_{34} (-\frac{a_{34} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_y + \frac{a_{44} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_z)$$

$$+ a_{44} (\frac{a_{33} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_y - \frac{a_{34} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_z)$$

$$= \frac{P_y}{2\pi} \qquad \text{(E.08b)}$$

$$\overline{\sigma}_{zz1} - \overline{\sigma}_{zz2} = a_{33} (\frac{\partial \overline{u}_{z1}}{\partial z} - \frac{\partial \overline{u}_{z2}}{\partial z}) + a_{34} (\frac{\partial \overline{u}_{y1}}{\partial z} - \frac{\partial \overline{u}_{y2}}{\partial z})$$

$$= a_{33} (-\frac{a_{34} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_y + \frac{a_{44} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_z)$$

$$+ a_{34} (\frac{a_{33} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_y - \frac{a_{34} a_{55}}{2\pi a_{55}(a_{33}a_{44} - a_{34}^2)} P_z)$$

$$= \frac{P_z}{2\pi} \qquad \text{(E.08c)}$$

According to the Cauchy′s theory of residues and the method of Laurent′s theorem, we already accomplish the purpose desired and demonstrate the Eq. (4.15).

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