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Tian-Shyr Dai [a]

[a] Department of Information and Finance Management and Institute of Finance , National Chiao-Tung University , 1001 Ta Hsueh Road, Hsinchu, Taiwan 300, ROC

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Efficient option pricing on stocks paying discrete or path-dependent dividends with the stair tree

TIAN-SHYR DAI*

Department of Information and Finance Management and Institute of Finance,
National Chiao-Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan 300, ROC

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Pricing options on a stock that pays discrete dividends has not been satisfactorily settled because of the conflicting demands of computational tractability and realistic modelling of the stock price process. Many papers assume that the stock price minus the present value of future dividends or the stock price plus the forward value of future dividends follows a lognormal diffusion process; however, these assumptions might produce unreasonable prices for some exotic options and American options. It is more realistic to assume that the stock price decreases by the amount of the dividend payout at the ex-dividend date and follows a lognormal diffusion process between adjacent ex-dividend dates, but analytical pricing formulas and efficient numerical methods are hard to develop. This paper introduces a new tree, the stair tree, that faithfully implements the aforementioned dividend model without approximations. The stair tree uses extra nodes only when it needs to simulate the price jumps due to dividend payouts and return to a more economical, simple structure at all other times. Thus it is simple to construct, easy to understand, and efficient. Numerous numerical calculations confirm the stair tree's superior performance to existing methods in terms of accuracy, speed, and/or generality. Besides, the stair tree can be extended to more general cases when future dividends are completely determined by past stock prices and dividends, making the stair tree able to model sophisticated dividend processes.

Keywords: Option pricing; Stair tree; Dividends

By assuming that the stock price process follows a lognormal diffusion process, Black and Scholes (1973) arrive at their ground-breaking option pricing formula for non-dividend-paying stocks. Merton (1973) extends the model to the case where the underlying stock pays a non-stochastic continuous dividend yield. The resulting formula is often called the Black–Scholes–Merton formula. In reality, however, almost all stock dividends are paid at discrete time points rather than continuously. Pricing options on a stock that pays discrete dividends with known amounts seems to be investigated first in Black (1975). This dividend setting is called the discrete dividend for simplicity.

The discrete-dividend option pricing problem has drawn a lot of attention in the literature. According to Frishling (2002), the stock price with discrete dividends has been modelled by three following ways.

Model 1: This model, crystallized under the discussions of Roll (1977), Geske (1979), and Whaley (1981), assumes that the stock price is divided into two parts: the stock price minus the present value of future dividends over the life of the option and the present value of future dividends. The former part (the net-of-dividend stock price) is assumed to follow a lognormal diffusion process, whereas the latter part is assumed to grow at the risk-free rate. Thus vanilla options can be computed by applying the Black–Scholes–Merton formula with the stock price replaced by the net-of-dividend stock price. Cox and Rubinstein (1985) call it 'ad hoc adjustment.'

Model 2: Musiela and Rutkowski (1997), following Heath and Jarrow (1988), suggest that the cum-forward-dividend stock price, defined as the stock price plus the forward values of the dividends paid from the prevailing time up to maturity, follows a lognormal diffusion process. Thus vanilla options can be computed by applying the Black–Scholes–Merton formula by replacing

*Email: d88006@csie.ntu.edu.tw

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the stock price with the cum-forward-dividend stock price and by adding the forward values of the dividends prior to maturity to the exercise price.

Model 3: The stock price decreases by the amount of the dividend paid at the ex-dividend date and follows a lognormal price process between adjacent ex-dividend dates.

Although the above three models attempt to solve the discrete-dividend option pricing problem, Frishling (2002) shows that they generate very different option prices. Roughly speaking, assume the volatility input to these three models is σ. Model 1 sets the volatility of the net-of-dividend stock price at σ, while Model 3 sets the volatility of the stock price at σ. The volatility of the stock price in Model 1 is lower than that in Model 3 because the volatility of the present value of future dividends, a component of the stock price, is assumed to be 0 in Model 1. Model 1 therefore produces lower option prices, and the difference becomes larger as σ becomes larger. Similarly, Model 2 produces higher option prices than Model 3 since Model 2 assigns the volatility of the forward values of the dividends, which is not a part of stock price, to be σ.

Although Model 1 and Model 2 are widely accepted in the literature (see Whaley 1982, Carr 1998, Chance *et al.* 2002) in solving the discrete-dividend problem, they suffer from many problems. For example, Frishling (2002) shows that Model 1 and Model 2 could incorrectly price barrier options. Bender and Vorst (2001) show that arbitrage opportunities exist in Model 1 if the volatility surface is continuously interpolated around ex-dividend dates. Bos and Vandermark (2002) show that both Model 1 and Model 2 violate a perfectly reasonable continuity requirement.

Although Model 3 is much closer to reality than the other two models, there is no exact pricing formula for European options. Hull (2000) recommends an approximate pricing formula by adjusting the volatility input to Model 1 using a simple formula. However, this paper shows that the performance of Hull's volatility adjustment is mixed. Bos and Vandermark (2002) present an approach that is a mixture of the stock and exercise price adjustments (in other words, Model 1 and Model 2). Bos and Shepeleva (2002) claim that this approach results in some inaccuracies, especially for in- and out-of-the-money options. They suggest a different pricing formula by adjusting the volatility input to Model 1 using a complex formula. But their approach can not be easily extended for pricing American options. Besides, the numerical results in this paper also suggest that my approach provides more accurate option values than the aforementioned approaches for pricing European options.

Model 3 can be implemented by the tree or the related PDE method.† But a naive application of these methods results in combinatorial explosion. Take the well-known CRR binomial tree proposed by Cox *et al.* (1979) as an example. Assume that the tree starts at time step 0 and ends at time step n. Let R stand for the gross risk-free return per time step. When the stock does not pay dividends, in one time step the price S becomes Su (the up move) with probability p_u and Sd (the down move) with probability $p_d \equiv 1 - p_u$, where $p_u \equiv (R-d)/(u-d)$. The relation $ud = 1$ is enforced by the CRR binomial tree. The black nodes at the first two time steps of the bushy tree in figure 1 forms a 2-time-step CRR binomial tree. The CRR binomial tree recombines; thus the size of the tree is only quadratic in n. Unfortunately, the recombination property disappears if the stock pays discrete dividends. Assume that a dividend D is paid at time step 2. The bushy tree splits into three trees after the ex-dividend date. Each such tree will be split further at each subsequent ex-dividend date. As a result, the tree size grows exponentially with the number of ex-dividend dates. The bushy tree implements Model 3 faithfully, but the exponential complexity renders it impractical.

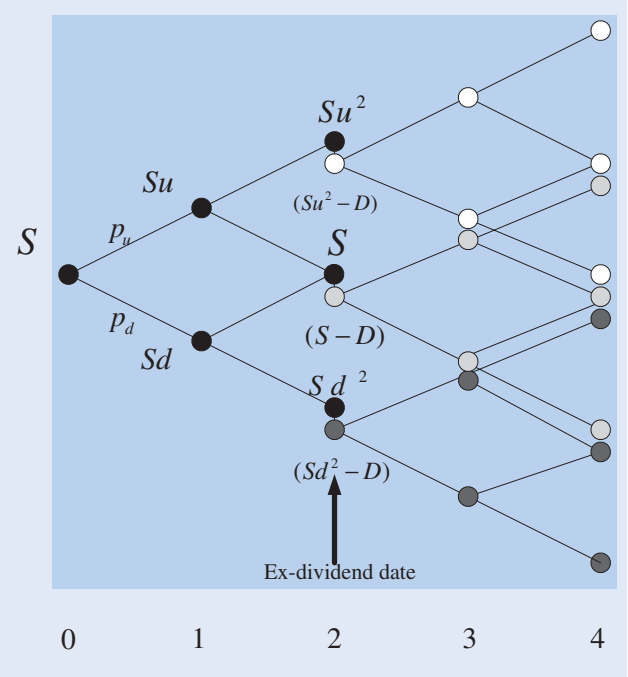

Figure 1. The bushy tree. The initial stock price is S. The upward and the downward multiplicative factors for the stock price are u and d, respectively. The upward and the downward branching probabilities are p_u and p_d, respectively. The black nodes in the first two time steps form a CRR tree. A dividend D is paid out at time step 2. The values in parenthesis at time step 2 denote the stock prices immediately after dividend payout. Three separate trees beginning at time step 2 are coloured in white, light grey, and dark grey, respectively.

In addition to Model 1 and Model 2, efficient numerical algorithms and simple formulas can also result by approximating the discrete dividend with either (1) a fixed dividend yield on each ex-dividend date or (2) a fixed continuous dividend yield. The first approach is

†Basically, the trinomial tree is analogous to an explicit finite-difference model (see Lyuu 2002). Thus my method for handling known dividends or path-dependent dividends can be extended to an explicit finite-difference model.

Table 1. Sizes of the stair tree and the bushy tree. The stair tree (`Stair`) and the bushy tree (`Bushy`) are compared in terms of numbers of nodes. The stock price is 100, the volatility is 30%, the risk-free interest rate is 10%, and the time to maturity is 0.75 year. The number of time steps for both the stair and bushy trees is 300. The number of ex-dividend dates is in the first column. The exdividend dates divide the 0.75-year time span into equal-length time intervals. A 1-dollar dividend is paid at each ex-dividend date. For example, 2 ex-dividend dates means that a 1-dollar dividend is paid at year 0.25 and year 0.5.

# Ex-dividend dates	`Stair`	*Bushy*
1	46,051	1,744,201
2	46,851	53,060,451
3	47,851	1,301,124,826
4	48,960	26,604,783,451
5	51,475	466,301,626,701

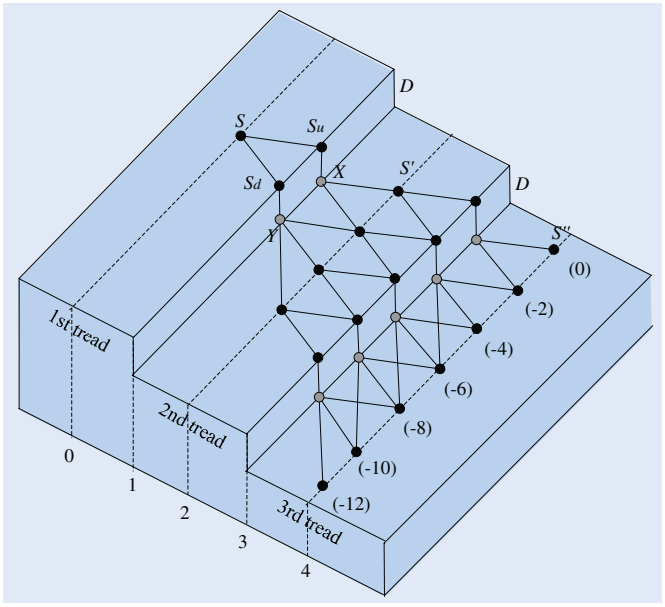

Figure 2. The structure of the stair tree. The initial stock price is S. The upward and the downward multiplicative factors for the stock price are u and d, respectively. The grey nodes are the nodes right after the dividend is paid. S' and S'' denote the largest stock price at time step 2 and time step 4, respectively. The stock price for each node on the third tread is represented as $S''u^k = S''e^{k\sigma\sqrt{\Delta t}}$, where k is parenthesized.

followed by Geske and Shastri (1985). They replace the discrete dividends with fixed dividend yields. The resulting tree hence recombines and is efficient. My paper will show that this approach works well for American options but poorly for European options. Chiras and Manaster (1978), following Merton's (1973) idea, adopt the second approach. They transform the discrete dividends into a fixed continuous dividend yield and then apply the Black–Scholes–Merton formula. As this approach is equivalent to the first approach in pricing European options, it shares the same faults.

The major contribution of this paper is a novel tree model, the stair tree, that faithfully implements Model 3 without combinatorial explosion. Numerical results in the paper will show that the prices calculated by the stair tree are extremely close to those generated by the Monte Carlo simulation for European options and those generated by the bushy tree for American options. The stair tree is furthermore efficient and general. In contrast, the Monte Carlo simulation cannot handle American options easily, and the bushy tree grows exponentially. Table 1 compares the sizes of the bushy tree and the stair tree. The size difference grows with the number of ex-dividend dates. Compared with the stair tree, existing schemes that implement Model 3 are less accurate, less efficient, and/or less general.

The idea behind the stair tree is straightforward. The stair tree limits the stock prices at each time step t to be of the form Pu^k. Here P denotes the stock price of some specific node at time step t, u denotes the upward multiplicative factor for the stock price in the CRR binomial tree, and k is some even integer. It therefore preserves the CRR tree structure at each time step. Consider a 4-time-step stair tree with a dividend payout D at time step 1 and time step 3 as illustrated in figure 2. The price drops due to the dividend payouts (at time step 1 and time step 3) resemble the riser. Note that the ex-dividend stock prices at nodes X and Y are $Su - X$ and $Sd - X$, respectively. The time interval between time step 0 and time step 1 (an ex-dividend date), and the time interval between time step 2 and time step 3 resemble treads. This tree is therefore called the stair tree.

Assume S' denotes the largest stock price at time step 2. Because the stock prices at time step 2 are restricted to be $S'u^k$ for nonpositive even integers k, the stair tree remains recombining at time step 3 and so on until the next ex-dividend date. In general, the stair tree follows the CRR tree structure between ex-dividend dates. This idea greatly reduces the number of tree nodes. For the nodes at the ex-dividend dates (like the grey nodes in figure 2), trinomial branching schemes are devised to connect the two adjacent CRR tree structures. The theoretical guarantee that simple and efficient branching schemes exist constitutes a major contribution of the paper. The adaptive mesh model proposed by Figlewski and Gao (1999) and Gao *et al.* (1999) also adjusts the tree structure by adding trinomial branches at certain points in the tree. The adaptive mesh model focuses on suppressing the nonlinearity error which makes the pricing results oscillate, while the stair tree model focuses on implementing Model 3 faithfully without combinatorial explosion.

Pricing options whose underlying stock pays stochastic dividends is discussed in Cox and Rubinstein (1985), Miltersen and Schwartz (1998), and Chance *et al.* (2002). This setting is important since the dividend payout is in practice not perfectly predictable, especially when the ex-dividend dates are far into the future. For example, even the so-called widow-and-orphan AT&T stock valued for its stable dividend payouts cut its quarterly dividend from 22 cents per share to 3.75 cents per share in the 4th quarter of 2000. Miltersen and Schwartz (1998) discuss pricing options on commodity futures with stochastic convenience yields. Chance *et al.* (2002) show that the Black–Scholes–Merton model is upheld by assuming that

the discretely stochastic dividends are uncorrelated with the stock price. Cox and Rubinstein (1985) argue that pricing options on dividend-paying stocks can be handled by the arbitrage-based pricing theory when the future dividends are known exogenously or completely determined by past stock prices and dividends. I call their setting the path-dependent dividends as the dividends depend solely on the past history of the stock price path. In this setting, the dividend paid at time τ could be written as a function of stock prices and the dividends prior to time τ. This is more general and realistic than the discrete dividend setting in many ways. First, it can solve the negative stock price problem that occurred under the discrete-dividend setting. This problem happens as the stock price drop due to the discrete dividend payment is larger than the cum-dividend stock price at the ex-dividend date. The problem can be avoided by choosing a proper dividend-paying function so that the dividend payment is always less than the cum-dividend stock price. Second, the path-dependent dividends setting can fit the real world phenomenon by choosing a proper dividend function from empirical studies. Although it is well known that dividends can be explained by a variety of factors such as the net operating profits and long-run sustainable (or permanent) earnings, a dividend function that fits the path-dependent dividends setting can still be constructed if the stock prices and the dividends paid previously serve as good proxies of these factors. I will review one of such dividend models proposed by Marsh and Merton (1987). The stair tree can incorporate such dividend models by adding extra states to keep the information necessary for computing future dividends. A simple numerical example will be given to explain how that is done.

The paper is organized as follows. The mathematical model is briefly covered in section 1. The stair tree for the dividend-paying stock is discussed in section 2. A sample stair tree is given in section 3 to convey the main ideas. Experimental results given in section 4 verify the superiority of the stair tree to other models. In section 5, I will first introduce the path-dependent dividends settings before going on to review Marsh and Merton's dividend model and show how the stair tree incorporate their dividend model. Section 6 concludes the paper.

1. The models

In Model 3, the stock price under the risk-neutral probability is assumed to follow the lognormal diffusion process:

$$S(t+\tau) = S(t)\mathrm{e}^{(r-0.5\sigma^2)\tau+\sigma\omega_\tau}, \tag{1}$$

where $S(t)$ denotes the stock price at year t, r denotes the annual risk-free interest rate, σ denotes the volatility, and ω_τ denotes the standard Brownian motion. In the discrete-time tree model, it is assumed that there are n equal time steps between year 0 and year T. The length of each time step Δt is equal to T/n. Thus, time step i in the discrete-time model corresponds to year $i\Delta t$ in the continuous-time model. The upward and downward multiplicative factors u and d for the stock price equal $\mathrm{e}^{\sigma\sqrt{\Delta t}}$ and $\mathrm{e}^{-\sigma\sqrt{\Delta t}}$, respectively, for the CRR and stair trees. S_i denotes the stock price at year $i\Delta t$ (or, equivalently, time step i for a tree). The stock is assumed to pay m dividends $D_{t_1}, D_{t_2}, \ldots, D_{t_m}$, where D_{t_i} is paid out at time step t_i. I further assume $t_1 < t_2 < \cdots < t_m$ for convenience. Under the discrete dividend assumption, any arbitrary dividend D_{t_i} is already known at time step 0. In general, D_{t_i} can be determined by a function of stock prices and/or the dividends paid up to time step t_i under the path-dependent dividends assumption. The stock price simultaneously falls by the amount αD_{t_i}. For simplicity, α is assumed to be 1 throughout the paper, but a general α poses no difficulties to the stair tree. When the ex-dividend stock price becomes negative, it is assumed to stay at zero from that point onward. Harvey and Whaley (1992), in contrast, assume that the dividend is not paid if its amount exceeds the prevailing stock price. The stair tree can easily incorporate their assumption, too.

The option is assumed to start at time step 0 and mature at time step n. The exercise price for this option is K. Define $(A)^+$ to denote $\max(A, 0)$ for simplicity. The payoff for a European option at maturity is

$$\text{final payoff} = \begin{cases} (S_n - K)^+, & \text{for a call,} \\ (K - S_n)^+, & \text{for a put.} \end{cases}$$

An American option gives the holder the right to exercise the option before maturity. The exercise value for an American option at a non-dividend-paying time step i is

$$\text{exercise value} = \begin{cases} S_i - K, & \text{for a call,} \\ K - S_i, & \text{for a put.} \end{cases}$$

The exercise strategy for an American option at an ex-dividend date is only slightly more complicated. It is never optimal to exercise an American call immediately after the underlying stock pays a dividend because it is dominated by the strategy of exercising the call immediately before. Similarly, it is never optimal to exercise a put before the stock pays a dividend. Consequently, the exercise value for an option at a dividend-paying time step i is

$$\text{exercise value} = \begin{cases} S_i^*; -K, & \text{for a call,} \\ K - S_i, & \text{for a put,} \end{cases} \tag{2}$$

where S_i^* and S_i denote the cum-dividend stock price and the net-of-dividend stock price at time step i, respectively. An option will be exercised early by the owner if the option's continuation value (i.e. the value to hold the option) is smaller than its exercise value.

2. Construction of the stair tree

I illustrate the main ideas by the 4-time-step tree in figure 2. This 4-time-step stair tree contains two ex-dividend dates: one at time step 1 and the other at time step 3. For simplicity, the same D-dollar dividend is

paid at each ex-dividend date. The price drop due to the dividend payout is represented by a riser. Each tread covers a time interval between two adjacent ex-dividend dates except the first tread, which covers the time interval between time step 0 and the first ex-dividend date. The branches follow the CRR tree structure except those from the nodes at the ex-dividend dates. For example, the stock price at the root node is S. The stock prices for its two successor nodes are Su and Sd, where $ud = 1$. Because of the CRR tree structure, the stock prices at the same time step are Pu^k, where P is the stock price of some specific node at that time step and k is an even integer. For example, the stock price for each node at time step 4 can be represented as $S''u^k$, where S'' denotes the largest stock price at time step 4 and k is parenthesized. Technically, any node's stock price can be picked for P because the stock prices at the same time step are part of the geometric sequence

$$\ldots, Pu^{-4}, Pu^{-2}, P, Pu^2, Pu^4, \ldots$$

Note that the first tread contains a single, complete CRR tree. The tree structure on each subsequent tread is composed of a CRR binomial tree with the initial section truncated.

I next construct the branches out of the grey nodes at an ex-dividend date to complete the stair tree. Figure 3 illustrates what happens at an ex-dividend date by zooming in the first three time steps of the stair tree in figure 2. Nodes X and Y are from the first ex-dividend date. The ex-dividend stock price at node X is $S_X = Su - D$. The two branches from X follow the CRR tree structure. S', the stock price for the top node at time step 2, therefore equals $S_X u$. Define the V-log-price of stock price V' as $\ln(V'/V)$; a V-log-price of z implies a stock price of Ve^z. Since the stock price for each node on the second tread can be expressed in terms of $S'u^k$ for some even integer k, the S'-log-prices for nodes at time step 2 in figure 3 are integral multiples of $2\sigma\sqrt{\Delta t}$.

The branches from node Y are constructed as follows. Let the ex-dividend stock price for node Y be S_Y. At least three branches are required for node Y so it has enough degrees of freedom to match the first two moments of the logarithmic stock price process and to satisfy the constraint that the sum of branching probabilities is 1. Three nodes at time step 2 follow node Y. By the log-normality of the stock price, the mean and the variance of the S_Y-log-prices of these nodes (under the risk-neutral probability) equal

$$\mu \equiv (r - \sigma^2/2)\,\Delta t,$$
$$\text{Var} \equiv \sigma^2 \Delta t,$$

which can be obtained by substituting Δt for τ into equation (1). Note that the distance between two adjacent nodes' S_Y-log-prices at time step 2 is $2\sigma\sqrt{\Delta t}$. Thus there exists a unique node Z at time step 2 whose S_Y-log-price $\hat{\mu}$ lies in the interval

$$\left[\mu - \sigma\sqrt{\Delta t}, \mu + \sigma\sqrt{\Delta t}\right). \tag{3}$$

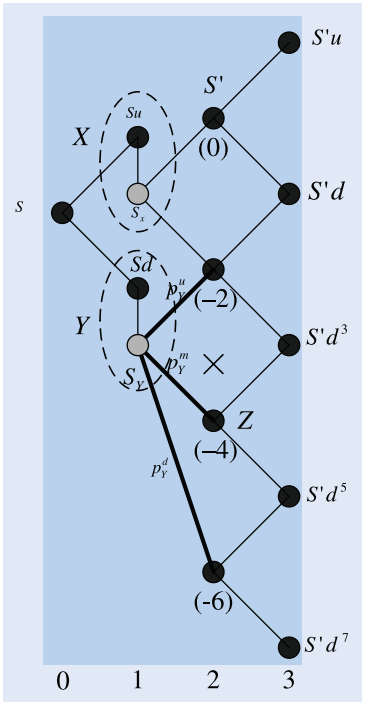

Figure 3. Branching scheme at the ex-dividend date. Nodes X and Y are at the first ex-dividend date (time step 1). Both nodes are represented by dotted ellipses. The cum-dividend stock prices at X and Y are Su and Sd, respectively, whereas the net-of-dividend stock prices at X and Y are $S_X(\equiv Su - D)$ and $S_Y(\equiv Sd - D)$, respectively. The stock price for the top node at time step 2 is S' $(= S_X u)$. The integer k in parentheses for each node at time step 2 means the stock price equals $S'e^{k\sigma\sqrt{\Delta t}}$. The cross right above Z denotes the point with S_Y-log-price μ at time step 2. The three branches of Y are marked with thick solid lines. p_Y^u, p_Y^m, and p_Y^d denote the probabilities for the upper, middle, and lower branches from node Y, respectively.

In other words, the S_Y-log-price of node Z, i.e. $\hat{\mu}$, is closest to μ among the S_Y-log-prices of the nodes at time step 2. I call $\hat{\mu}$ the mean tracker of node Y. The middle branch from node Y will be connected to node Z. Figure 3 illustrates the case where $\hat{\mu} = \ln(S'/S_Y) - 4\sigma\sqrt{\Delta t}$ (or, two nodes below S').

In general, the S_Y-log-prices of the two nodes connected by the upper and lower branches from node Y can be expressed as $\hat{\mu} + \ell_u \sigma\sqrt{\Delta t}$ and $\hat{\mu} - \ell_d \sigma\sqrt{\Delta t}$ for some even positive integers ℓ_u and ℓ_d. It is clear that the jump sizes ℓ_u and ℓ_d should be as small as possible to minimize the size of the stair tree. And ℓ_u and ℓ_d should also be properly selected to make the branching probabilities of node Y valid. Let p_Y^u, p_Y^m, and p_Y^d denote the risk-neutral probabilities for the upper, middle, and lower branches from node Y, respectively. Define β, α, and γ as the S_Y-log-prices minus the mean μ of the nodes connected by the middle, the upper, and the lower branches as follows:

$$\beta \equiv \hat{\mu} - \mu,$$
$$\alpha \equiv \beta + \ell_u \sigma\sqrt{\Delta t},$$
$$\gamma \equiv \beta - \ell_d \sigma\sqrt{\Delta t}.$$

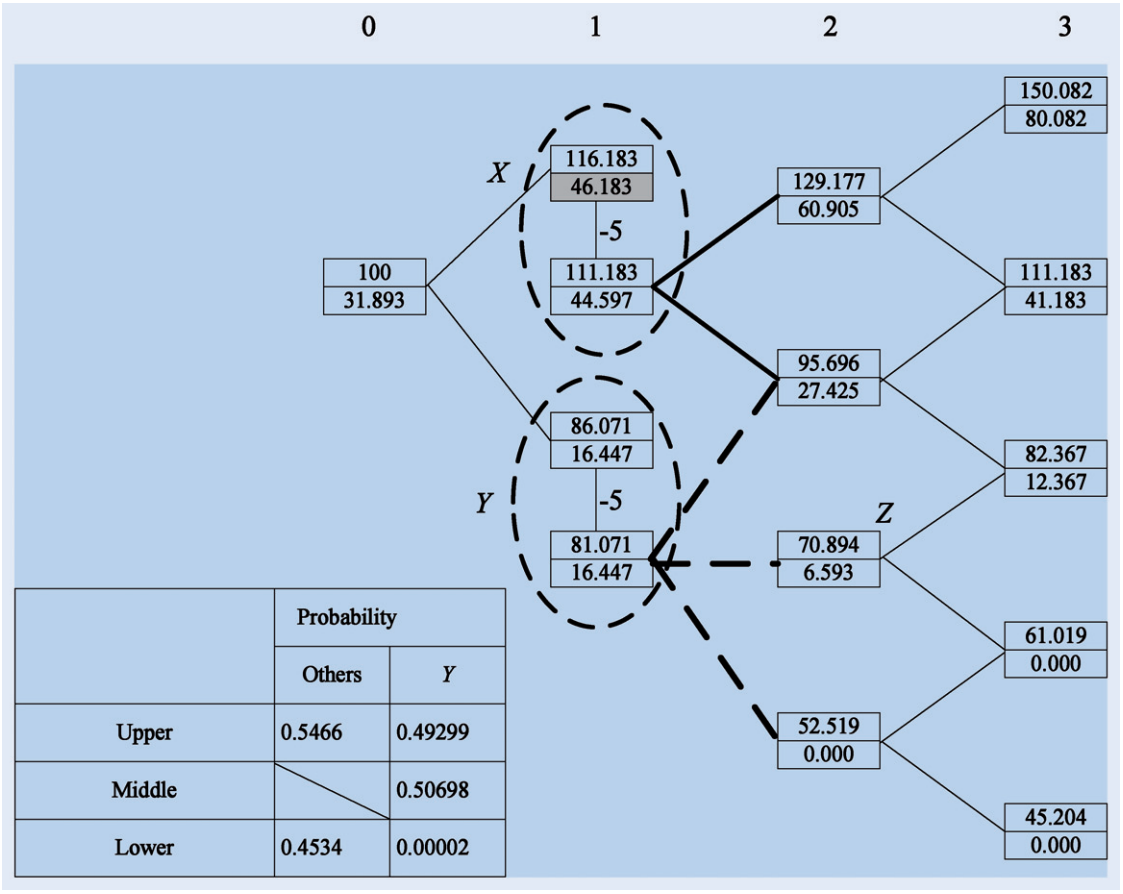

Figure 4. A 3-time-step tree for pricing an American vanilla call. The number at the upper cell of a node denotes the stock price at that node. The number at the lower cell denotes the call value. The grey cell denotes that the American call is exercised early. The two branches of X are marked with thick solid lines, whereas the three branches of Y are marked with thick dotted lines. The branching probabilities are listed in the lower-left table.

Note that the first equation implies that $\beta \in [-\sigma\sqrt{\Delta t}, \sigma\sqrt{\Delta t})$. Note also that $\alpha > \beta > \gamma$. The probabilities can be derived by solving

$$p_Y^u \alpha + p_Y^m \beta + p_Y^d \gamma = 0, \tag{4}$$

$$p_Y^u \alpha^2 + p_Y^m \beta^2 + p_Y^d \gamma^2 = \text{Var}, \tag{5}$$

$$p_Y^u + p_Y^m + p_Y^d = 1. \tag{6}$$

Equations (4) and (5) match the first two moments of the logarithmic stock price, and equation (6) ensures that p_Y^u, p_Y^m, p_Y^d as probabilities sum to one. The three equations do not automatically guarantee $0 \le p_Y^u$, p_Y^m, $p_Y^d \le 1$. A proof to show that they actually do with $\ell_u = \ell_d = 2$ is given in Appendix A. The stair tree hence does not lead to branches with huge jump sizes. This finding is essential to the efficiency of the algorithm. The same procedure can be repeated for nodes below Y.† To handle multiple dividends, just apply the procedure to each ex-dividend date.

Because the first and the second moments are matched via Equations (4)–(6), the stair tree converges to Model 3. Unlike the bushy tree illustrated in figure 1, the stair tree faithfully implements Model 3 without combinational explosion.

3. A sample stair tree

Consider an American vanilla call with an exercise price of 70 that initiates at year 0 and matures at year 0.75. A 3-time-step stair tree is constructed in figure 4 to price this call. Thus the length of each time step Δt is 0.25 year. The initial stock price is 100, the risk-free interest rate is $r = 10\%$, and the volatility of the stock price is $\sigma = 30\%$. The multiplicative factors for the CRR binomial tree are $u = e^{0.3\times\sqrt{0.25}} \approx 1.162$ and $d = e^{-0.3\times\sqrt{0.25}} \approx 0.861$. The branching probabilities are $p_u = (R - d)/(u - d) \approx 0.5466$ and $p_d = 1 - p_u \approx 0.4534$, where $R = e^{0.1\times 0.25} \approx 1.02532$ denotes the gross risk-free return per time step. In the figure, the number at the upper cell of a node denotes the stock price at that node, whereas the number at the lower cell denotes the call option price.

Assume a five-dollar dividend per share is paid at year 0.25 (time step 1). Note that nodes X and Y (marked by

†The aforementioned method can also be done by first adding CRR binomial branches to the bottom node (like node Y), and then inserting trinomial branches to other nodes without efficiency and accuracy penalties.

Table 2. Pricing European call options with single discrete dividend. The initial stock price is 100, the risk-free rate is 3%, the time to maturity is 1 year, and a 5-dollar-dividend is paid at year 0.6. The volatilities of the stock price are shown in the first row. The exercise prices are listed in the first column. `FDY` denotes the fixed dividend yield approach of Geske and Shastri (1985). `Model1` and `Model2` denote the option prices generated by Model 1 and Model 2, respectively. `Hull` denotes volatility adjustment approach of `Hull` (2000). `Stair` denotes the stair tree model in this paper. `Model3` denotes the prices generated by Model 3 that based on Monte Carlo simulation with 100,000 trials. Option prices that deviate from `Model3` by 0.3 are marked by asterisks.

	0.4						0.5					
X	FDY	Model1	Hull	Model2	Stair	Model3	FDY	Model1	Hull	Model2	Stair	Model3
95	*16.263	*16.336	17.090	17.112	16.821	16.933	*19.890	*19.969	20.901	20.937	20.570	20.843
100	*14.214	*14.270	15.044	15.048	14.758	14.754	*17.964	*18.003	*18.959	*18.971	18.591	18.584
105	*12.400	*12.439	*13.222	*13.206	12.924	12.989	*16.194	*16.222	17.194	17.182	16.829	16.929

dotted ellipses) are at time step 1. The stock prices at X and Y before the dividend is paid are $100 \times u \approx 116.183$ and $100 \times d \approx 86.071$, respectively. The ex-dividend prices at X and Y are therefore 111.183 and 81.071, respectively. The stock price for the top node at time step 2 is then $111.183 \times u \approx 129.177$. The stock prices at time step 2 can be represented as $129.177 \times u^k$ for nonpositive even integers k.

Let us move on to the branching scheme of node Y. Node Z at time step 2 has a stock price of 70.894. Hence the S_Y-log-price of node Z equals $\ln(70.894/81.071) \approx -0.13414$. It is the mean tracker (i.e. $\hat{\mu}$) of Y because

$$-0.13414 \in [\mu - \sigma\sqrt{\Delta t}, \mu + \sigma\sqrt{\Delta t}),$$

where $\mu = (r - \sigma^2/2)\Delta t = (0.1 - 0.3^2/2) \times 0.25 = 0.01375$ and $\sigma\sqrt{\Delta t} = 0.3 \times \sqrt{0.25} = 0.15$. Thus the S_Y-log-prices of the nodes at time step 2 that will be connected to Y are $\hat{\mu} + 2\sigma\sqrt{\Delta t} \approx 0.1659$, $\hat{\mu} \approx -0.1341$, and $\hat{\mu} - 2\sigma\sqrt{\Delta t} \approx -0.4341$. To compute the branching probabilities from node Y, I substitute $0.1659 - \mu$, $-0.1341 - \mu$, and $-0.4341 - \mu$ into α, β, and γ, respectively, in Equations (4)–(6). The branching probabilities are illustrated in the lower-left table of the figure. The value of the vanilla call is obtained by backward induction on the tree. For example, the continuation option value at node Y is

$$e^{-0.1 \times 0.25} \times (0.49299 \times 27.425 + 0.50698 \times 6.593 + 0.00002 \times 0) \approx 16.447,$$

and the continuation option value at node X is

$$e^{-0.1 \times 0.25} \times (0.5466 \times 60.905 + 0.4534 \times 27.425) \approx 44.597.$$

Note that an American call will be exercised early only at an ex-dividend date. Note also that it is more beneficial for an option holder to exercise a call immediately before the underlying stock pays a dividend than immediately after (see equation (2)). The call will be exercised early at node X since the exercise value 46.183 ($=116.183 - 70$) is larger than the continuation value 44.597. The call value computed by the stair tree is 31.893.

4. Numerical evaluations

I first compare Geske and Shastri's fixed dividend yield model, Hull's volatility adjustment model, the stair tree model, Model 1, Model 2, and Model 3 for pricing European options. Geske and Shastri (1985) use fixed dividend yields to approximate discrete dividends. The fixed dividend yield is defined as the discrete dividend amount divided by the initial stock price. For example, the dividend yield is 5% if the initial stock price is 100 and the discrete dividend is 5. I use `FDY` to denote their approach. Note that Chiras and Manaster (1978) approximate the discrete-dividend problem by transforming the discrete dividends into a fixed continuous dividend yield. This approach is equivalent to the `FDY` model in pricing a European option. Frishling (2002) argues that Model 1 generates lower option prices than Model 3. To remove this difference, Hull (2000) recommends that the volatility of the net-of-dividend stock price be adjusted by the volatility of the stock price multiplied by $S(0)/(S(0) - D)$, where D denotes the present value of future dividends over the life of the option. I use `Hull` to denote Hull's volatility adjustment approach. Besides, I use `Model1` and `Model2` to denote the option prices generated by Model 1 and Model 2, respectively. `Stair` denotes the prices generated by the stair tree model. *Model 3* denotes the prices generated by Model 3 that based on the Monte Carlo simulation with 100, 000 trials.

The numerical results for these models are listed in tables 2 and 3, where table 2 focuses on the single-discrete-dividend case and table 3 focuses on the two-discrete-dividend case. All the prices that deviate from `Model3` by 0.3 are marked by asterisks. Frishling (2002) claims that Model 1, Model 2, and Model 3 generate very different option prices. This can be verified in tables 2 and 3 that the option prices generated by Model 2 are higher than the prices generated by Model 3. On the other hand, Model 1 generates lower option prices than Model 3. The difference among these three models becomes larger as volatility increases. `FDY` does not approximate Model 3 well as it produces lower option prices than Model 1. The option prices generated by Hull's volatility adjustment approach do not approximate the prices generated by Model 3 well. It can be observed that only the stair tree model produces options prices that are close to Model 3.

Note that Model 3 seems to produce lower option price (generated by the Monte Carlo simulation) in each two-discrete-dividend case (except one case) in table 3 than that in the corresponding case in table 2. The stair tree model successfully captures this trend, but all other

Table 3. Pricing European call options with two discrete dividends. The numerical settings are the same as those settings in table 2 except that a 2.5-dollar-dividend is paid at year 0.4 and year 0.8. Option prices that deviate from Model3 by 0.3 are marked by asterisks.

	0.4						0.5					
X	FDY	Model1	Hull	Model2	Stair	Model3	FDY	Model1	Hull	Model2	Stair	Model3
95	*16.303	*16.336	17.090	17.112	16.806	16.836	*19.931	*19.969	*20.901	*20.937	20.568	20.549
100	*14.250	*14.270	*15.044	*15.048	14.733	14.733	*18.001	*18.003	*18.959	*18.971	18.583	18.621
105	*12.433	*12.439	*13.222	*13.206	12.904	12.883	*16.228	*16.222	*17.194	*17.182	16.826	16.829

Table 4. Pricing European call options with single discrete dividend. The numerical settings are the same as those settings in table 2. Mix denotes the mixture approach of Bos and Vandermark (2002). Vol denotes the volatility adjustment approach of Bos and Shepeleva (2002). Model3 denotes the prices generated by Model 3 that based on Monte Carlo simulation with 100,000 trials. **RMSE** denotes the root mean squared error. **MAE** denotes the maximum absolute error.

σ	X	Mix	Vol	Stair	Model3
0.4	95	16.802	16.792	16.821	16.933
	100	14.737	14.732	14.758	14.754
	105	12.899	12.899	12.924	12.989
0.5	95	20.550	20.537	20.570	20.843
	100	18.584	18.578	18.591	18.584
	105	16.798	16.798	16.829	16.929
RMSE		0.147	0.152	0.130	
MAE		0.293	0.306	0.272	

Table 5. Pricing European call options with two discrete dividends. The numerical settings are the same as those in table 3.

σ	X	Mix	Vol	Stair	Model3
0.4	95	16.801	16.795	16.806	16.836
	100	14.736	14.734	14.733	14.733
	105	12.898	12.901	12.904	12.883
0.5	95	20.548	20.541	20.568	20.549
	100	18.583	18.581	18.583	18.621
	105	16.797	16.800	16.826	16.829
RMSE		0.026	0.027	0.023	
MAE		0.038	0.041	0.038	

models fail. Note that both Model 1 and the Hull's volatility adjustment approach produce similar option prices in the single-discrete-dividend case and the two-discrete-dividend case. This is because the net-of-dividend stock price in the single-discrete-dividend case ($=100-5e^{-0.03\times0.6}$) is almost equal to that in the two-discrete-dividend case ($=100-2.5e^{-0.03\times0.4}-2.5e^{-0.03\times0.8}$). Model 2 also produces similar option prices in both cases since the cum-forward-dividend stock prices for both cases are almost equal.

To derive approximation analytical formulas for Model 3, Bos and Vandermark (2002) present an approach (denoted as Mix) that is a mixture between the stock and the exercise price adjustment or, in other words, Model 1 and Model 2. Bos and Shepeleva (2002) suggest that the volatility of the net-of-dividend stock price can be adjusted by a complex formula. I use Vol to denote their approach. These two approaches and the stair tree approach are compared in tables 4 and 5. I use the Monte Carlo simulation that prices Model 3 (denoted as Model3) to serve as a benchmark to compute the root mean squared error and the maximum absolute error. Since both these two errors of the stair tree model are lower than the errors of Mix and Vol, I conclude that the stair tree provides more accurate values than these two approaches. Note that Model 3 seems to produce lower option price in each two-discrete-dividend case in table 5 than that in the corresponding case in table 4 as I mentioned before. Bos and Vandermark's approach successfully catches this trend, but Bos and Shepeleva's approach fails.

For American calls with discrete dividends, I compare the stair tree with the popular analytical pricing formula of Roll (1977), Geske (1979), and Whaley (1981) (abbreviated as RGW), and the FDY model of Geske and Shastri (1985) in table 6. The parameters are from Cox *et al.* (1979). The benchmark option prices (B) are from Geske and Shastri (1985). Note that RGW is based on Model 1 and thus underprices the options. RGW focuses on single-dividend cases. Welch and Chen (1988) and Stephan and Whaley (1990) extend RGW for two-dividend cases. But it is hard to extend RGW for three or more dividends because this would have required RGW to evaluate a multivariate cumulative normal density function, whose deterministic computational cost is prohibitive. This phenomenon is known as the curse of dimensionality (see Lyuu 2002). Of course, even if the multivariate integral can be computed efficiently, there is no guarantee that the price is numerically accurate. Geske and Shastri (1985) claim that the FDY model performs well for pricing American calls. Numerical results in table 6 show that the stair tree outperforms the FDY model.

The delta of a call with respect to the stock price is illustrated in figure 5. I use a 140-time-step stair tree to evaluate a call option with 7 months to maturity, and the length of each time step is 0.004167($=(7/12)/140$) year. 301 tree evaluations are performed by setting the initial stock price as $20+0.1x$, where $0 \le x \le 300$. The resulting delta curve is very smooth. The stair tree's quick convergence is verified in table 7, where the prices remain unchanged up to pennies when the number of

Table 6. Pricing American call options. The initial stock price is 40, the risk-free interest rate is 5%, and the volatility is 30%. The ex-dividend dates for the stock are 0.5, 3.5, and 6.5 months. The dividends to be paid at each ex-dividend date are shown in the first row. The exercise prices X are listed in the first column. The times to maturity T (in months) are in the second column. The values of American calls priced by the FDY model and the benchmark value are from Geske and Shastri (1985). RGW denotes the analytical pricing formula of Roll (1977), Geske (1979), and Whaley (1981) (for single-dividend cases) and the extended formula of Stephan and Whaley (1990) (for two-dividend cases). Stair denotes the stair tree model with 140 time steps. Option prices which deviate from the benchmark values by 0.02 are marked by asterisks.

		\$1.0				\$2.0				\$3.0				\$4.0			
X	T	RGW	FDY	Stair	B	RGW	FDY	Stair	B	RGW	FDY	Stair	B	RGW	FDY	Stair	B
35	1	*5.07	5.09	5.09	5.09	*5.05	5.09	5.09	5.08	*5.05	5.08	5.09	5.08	*5.05	5.08	5.09	5.08
	4	*5.38	5.41	5.40	5.40	5.15	5.19	5.18	5.17	*5.08	5.12	5.12	5.11	*5.07	5.10	5.10	5.10
	7	–	*5.79	5.77	5.76	–	*5.29	5.26	5.24	–	*5.15	5.14	5.12	–	5.11	5.11	5.10
40	1	*1.14	1.17	1.17	1.17	*1.03	1.08	1.08	1.07	1.03	1.04	1.04	1.04	*0.93	1.02	1.03	1.02
	4	*2.36	2.38	2.40	2.39	*1.89	1.91	1.93	1.92	1.57	1.60	1.60	1.58	*1.35	1.39	1.40	1.38
	7	–	3.05	3.08	3.06	–	2.31	2.33	2.32	–	1.83	1.83	1.81	–	*1.51	*1.51	1.48
45	1	0.08	0.09	0.09	0.09	0.04	0.05	0.06	0.05	0.04	0.04	0.04	0.04	0.02	0.03	0.03	0.03
	4	0.87	0.87	0.88	0.88	0.62	0.62	0.64	0.64	*0.43	0.44	0.46	0.46	0.31	0.31	0.33	0.32
	7	–	*1.47	1.51	1.50	–	*0.99	1.03	1.02	–	*0.66	0.70	0.69	–	*0.43	0.46	0.46

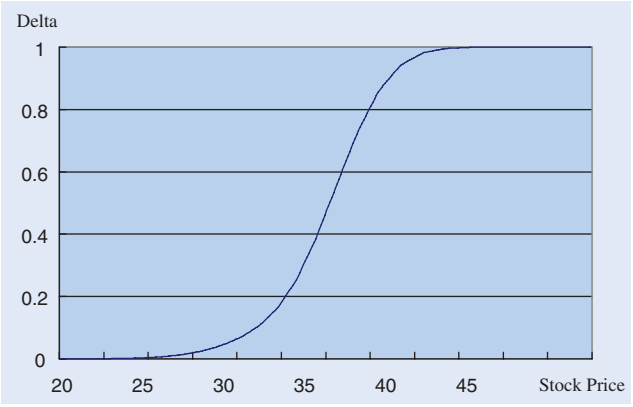

Figure 5. Delta: the x-axis denotes the initial stock price, and the y-axis denotes the delta of the vanilla call. The exercise price is 35, the risk-free interest rate is 5%, the volatility is 30%, and the time to maturity is 7 months. A 4-dollar dividend is paid at months 0.5, 3.5, and 6.5.

Table 7. Convergence of delta. The settings are identical to those in figure 5 except that the initial stock price is 40. The number of time steps n is selected to be a multiple of 14 so that each ex-dividend date coincides with a time step in the stair tree. Price and Delta denote the option price and the delta computed by the stair tree, respectively. The numerical values remain unchanged (up to pennies) for $n \geq 140$.

n	Price	Delta
56	5.13	0.97
70	5.13	0.97
84	5.13	0.96
98	5.12	0.97
112	5.12	0.97
126	5.12	0.97
140	5.11	0.97
154	5.11	0.97

time steps is at least 140. These experiments confirm the reliability of the stair tree.

The discrete dividend assumption is not so realistic since the dividend might not be perfectly predictable especially when the ex-dividend date is far into the future. A more realistic and generalized assumption, the path-dependent dividends assumption, is discussed in the next section. I will also show how the stair tree model can incorporate this assumption.

5. Path-dependent dividends

It is more general and realistic to assume that a stock pays a stochastic dividend rather than a dividend with known amounts at a future ex-dividend date. However, the option can only be hedged if the dividend is known exogenously or completely determined by the stock price process prior to the ex-dividend date as argued in Cox and Rubinstein (1985) unless one adds nonstandard derivatives such as the forward contracts on dividends in Chance *et al.* (2002). I call Cox and Rubinstein's assumption the path-dependent dividends assumption since the future dividend, says D_{t_i}, completely depends on the stock prices and the dividends prior to time step t_i. To be more precise, D_{t_i} can be expressed as

$$D_{t_i} \equiv f(S_0, S_1, S_2, \ldots, S_{t_i}, D_{t_{i-1}}, D_{t_{i-2}}, \ldots)$$

for some function f. In reality, dividends can be explained by a variety of factors such as the net operating profits, long-run sustainable (or permanent) earnings, and so on. If the stock prices and the dividends paid previously serve as good proxies for these factors, a dividend function that fits the path-dependent dividends assumption can be constructed. Indeed, some empirical dividend models can fit path-dependent dividends assumptions with slight modifications. I will first review one of such dividend models proposed by Marsh and Merton (1987).

Then I will show how the stair tree can incorporate their dividend model.

Marsh and Merton (1987) derive a dividend model by following Linter's (1962) stylized facts established by Linter in a classic set of interviews with managers about their dividend policies. Their dividend model can be expressed by a regression formula of the permanent earnings and the dividends paid previously. They argue that their formula can not be directly estimated because management assessments of changes in a firm's permanent earnings are not observable. Thus they assume that the permanent earning to cum-dividend stock price ratio is a positive constant. Under this assumption, a future dividend in their dividend model can be expressed by a regression formula in terms of stock prices and dividends prior to the ex-dividend date. To illustrate how the stair tree incorporates the Marsh and Merton's dividend model, I express their dividend formula by a discrete time model and assume that the length between two ex-dividend dates is two time steps:

$$\log\left[\frac{D_{t+2}}{D_t}\right] + \frac{D_t}{S_{t-2}} = a_0 + a_1 \log\left[\frac{S_t + D_t}{S_{t-2}}\right] + a_2 \log\left[\frac{D_t}{S_{t-2}}\right] + u(t+2), \quad (7)$$

where the dividends are paid at time step t and $t+2$, D_τ denotes the dividend amounts paid at time step τ, S_τ denotes the net-of-dividend stock price at time step τ, and $u(t+2)$ denotes the disturbance term at time step $t+2$. By assuming that the disturbance term $u(t+2) = 0$, equation (7) can be rewritten as

$$D_{t+2} = 10^{a_0 + a_1 \log\left[\frac{S_t + D_t}{S_{t-2}}\right] + a_2 \log\left[\frac{D_t}{S_{t-2}}\right] - \frac{D_t}{S_{t-2}} + \log D_t}. \quad (8)$$

Note that D_{t+2} can be expressed as a function of S_{t-2}, S_t, and D_t. One of their empirical studies focuses on the value-weighted NYSE index over the period 1926–81 and they estimate that $a_0 = -0.101$, $a_1 = 0.437$, and $a_2 = -0.042$ by ordinary least squares method. A simple numerical example is then given to demonstrate how the stair tree can incorporate the dividend model in equation (8) with aforementioned numerical settings.

A 4-time-step stair tree that prices a European vanilla call option with an exercise price of 50 is illustrated in figure 6. The underlying stock price at time step 0 is 100, the length of each time step of the stair tree is 0.25 year, the risk-free interest rate is 10%, and the volatility of the stock price is 30% . Note that the upward multiplication factor $u = e^{0.3\times\sqrt{0.25}} \approx 1.162$ and the downward one $d = e^{-0.3\times\sqrt{0.25}} \approx 0.861$. I further assume that the historical net-of-dividend stock prices S_{-1} and S_{-3}, and the historical dividend D_{-1} to be 110, 80, and 5, respectively. The underlying stock is assumed to pay two dividends (D_1 and D_3) at time steps 1 and 3, respectively. The top cell of each node denotes the stock price (at a non-dividend paying date) or the cum-dividend stock price (at an ex-dividend date) of that node. Each node contains at least one state (denoted by the cell following the top cell) to keep the option price. The nodes enclosed by dotted ellipses contain two states to keep required information for computing D_3 by equation (8) (to be discussed later). Note that the net-of-dividend stock prices and the branching probabilities for the states at ex-dividend dates (time step 1 and time step 3) are illustrated in table 8.

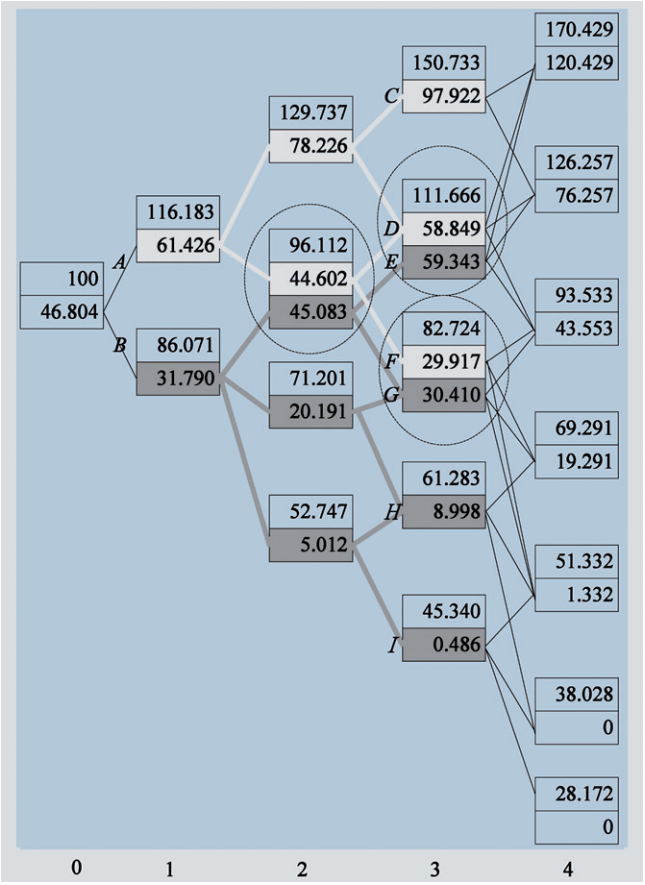

Figure 6. A 4-time-step stair tree that incorporates the Marsh and Merton dividend model. The number at the top cell of each node denotes the stock price (at a non-dividend paying date) or the cum-dividend stock price (at a ex-dividend date) of that node. The number at the following cell(s) denotes the option price(s). Additional states are added to the nodes enclosed by dotted ellipses to keep the required information for computing D_3. The net-of-dividend stock prices and the branching probabilities for all the states at ex-dividend dates are in table 8.

Now I proceed to show how this 4-time-step stair tree is constructed. The cum-dividend stock prices at time step 1 are $100 \times u \approx 116.183$ and $100 \times d \approx 86.071$, respectively. The dividend D_1 is obtained by substituting D_{-1} (=5), S_{-1} (=110), and S_{-3} (=80) into equation (8) to get 4.518. Thus the net-of-dividend stock prices for states A and B are $116.183 - 4.518 \approx 111.666$ and $86.071 - 4.518 \approx 81.553$, respectively. The stock price for the top node at time step 2 is then $111.666 \times u \approx 129.737$. Thus the stock prices at time step 2 can be represented as $129.737 \times u^k$ for nonpositive even integers k. The branches of state A follow the CRR tree structure. The mean tracker of state B can be found by equation (3) to be $\ln(71.201/86.071) \times (\equiv \hat{\mu})$ (expressed in S_B-log-price). Thus the stock prices of the nodes connected to state B are 96.112 (with S_B-log-price $\hat{\mu} + 2\sigma\sqrt{\Delta t}$), 71.201 (with S_B-log-price $\hat{\mu}$),

Table 8. The net-of-dividend stock prices and the branching probabilities for the states at ex-dividend dates in figure 6.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Price	111.666	81.553	146.690	107.622	108.119	78.681	79.177	57.737	41.853
Upper	0.5466	0.4983	0.5466	0.0001	0.0004	0.4699	0.4905	0.4397	0.3744
Middle	–	0.5017	–	0.5133	0.5280	0.5296	0.5095	0.5584	0.6165
Lower	0.4534	1.4×10^{-6}	0.4534	0.4866	0.4715	0.0005	4.6×10^{-5}	0.0019	0.0091

and 52.747 (with S_B-log-price $\hat{\mu} - 2\sigma\sqrt{\Delta t}$). The trinomial branching probabilities of state B can be computed by equations (4)–(6). The net-of-dividend stock prices and the branch probabilities for states A and B are illustrated in table 8.

To compute D_3 by equation (8), S_{-1}, D_1, and S_1 are required. While S_{-1} and D_1 are known to be 110 and 4.518, respectively, there are two possible S_1 (111.666 and 81.553) in this stair tree. Additional states are added to the nodes enclosed by dotted ellipses to keep the information about S_1. I colour all the cells and corresponding branches from time step 1 to time step 3 in light-grey and dark-grey to denote the cases that $S_1 = 111.666$ and $S_1 = 81.553$, respectively. For example, state F denotes the case that $S_1 = 111.666$ and the cum-dividend stock price at time step 3 is 82.724, while state G denotes the case that $S_1 = 81.553$ and the cum-dividend stock price at time step 3 is 82.724. Note that all the branches from the states at time step 2 follow the CRR tree structure.

Now I focus on time step 3. The dividend paid at state C is obtained by substituting $D_1 = 4.518$, $S_1 = 111.666$, and $S_{-1} = 110$ into equation (8) to get 4.043. Thus the net-of-dividend stock price for state C is $150.733 - 4.043 = 146.690$. Similarly, the net-of-dividend stock prices for states D and F are $111.666 - 4.043 = 107.622$ and $82.724 - 4.043 = 78.681$, respectively. The dividend paid at state E is obtained by substituting $D_1 = 4.518$, $S_1 = 81.553$, and $S_{-1} = 110$ into equation (8) to get 3.547. Thus the net-of-dividend stock price for state E is $111.666 - 3.547 = 108.119$. Similarly, the net-of-dividend stock prices for states G, H, and I are $82.724 - 3.547 = 79.177$, $61.283 - 3.547 = 57.737$, and $45.340 - 3.547 = 41.853$, respectively. The stock price for the top node at time step 4 is $146.690 \times u \approx 170.249$. All the stock prices at time step 4 can be represented as $170.249 \times u^k$ for nonpositive even integers k. The branches for state C follow the CRR tree structure. The trinomial branching schemes for states D, E, F, G, H, and I are constructed by following the method for constructing the branches for state B. The trinomial branching probabilities for these states are listed in table 8.

The value for the European vanilla call option can be obtained by backward induction. Note that some nodes have two different option prices due to different historical stock price paths. For example, the option price for state F is

$$e^{-0.1\times0.25} \times (43.553 \times 0.4699 + 19.291 \times 0.5296 + 1.332 \times 0.0005) \approx 29.917,$$

while the option price for state G is

$$e^{-0.1\times0.25} \times (43.553 \times 0.4905 + 19.291 \times 0.5095 + 1.332 \times 4.6 \times 10^{-5}) \approx 30.410.$$

The call value computed by the stair tree is 46.804.

6. Conclusions

Pricing stock options with discrete dividend payouts has not been satisfactorily settled because of the conflicting demands of computational tractability and realistic modelling of the stock price process. It is realistic to assume that the stock price jumps down at an ex-divided date. However, pricing options under this stock price model can not be efficiently and/or accurately implemented by analytical formulas and numerical methods. This paper suggests a recombining tree, the stair tree, that efficiently and faithfully implements this model. Numerical results confirm that the stair tree is both efficient and accurate. Moreover, the stair tree can be extended to more general cases when future dividends are completely determined by past stock prices and dividends. This extension, which is called 'path-dependent dividends assumption' in this paper, makes the stair tree model more realistic and flexible.

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Appendix A: Proof of valid risk-neutral probabilities

Define

$$\begin{aligned}\det &= (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta),\\ \det_u &= (\beta\gamma+\mathrm{Var})(\gamma-\beta),\\ \det_m &= (\alpha\gamma+\mathrm{Var})(\alpha-\gamma),\\ \det_d &= (\alpha\beta+\mathrm{Var})(\beta-\alpha).\end{aligned}$$

Then Cramer's rule applied to Equations (4)–(6) gives $p_Y^u = \det_u/\det$, $p_Y^m = \det_m/\det$, and $p_Y^d = \det_d/\det$. Note that $\det < 0$ because $\alpha > \beta > \gamma$. To ensure that the branching probabilities are valid, it suffices to show that $p_Y^u, p_Y^m, p_Y^d \geq 0$. As $\det < 0$, it is sufficient to show $\det_u, \det_m, \det_d \leq 0$ instead. Finally, as $\alpha > \beta > \gamma$, it suffices to show that $\beta\gamma + \mathrm{Var} \geq 0$, $\alpha\gamma + \mathrm{Var} \leq 0$, and $\alpha\beta + \mathrm{Var} \geq 0$ under the premise $\beta \in [-\sigma\sqrt{\Delta t}, \sigma\sqrt{\Delta t})$. Indeed,

$$\begin{aligned}\beta\gamma + \mathrm{Var} &= \beta^2 - 2\beta\sigma\sqrt{\Delta t} + \sigma^2\Delta t = (\beta - \sigma\sqrt{\Delta t})^2 \geq 0,\\ \alpha\gamma + \mathrm{Var} &= \beta^2 - 4\sigma^2\Delta t + \sigma^2\Delta t = \beta^2 - 3\sigma^2\Delta t < 0,\\ \alpha\beta + \mathrm{Var} &= \beta^2 + 2\beta\sigma\sqrt{\Delta t} + \sigma^2\Delta t = (\beta + \sigma\sqrt{\Delta t})^2 \geq 0,\end{aligned}$$

as desired.