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The exact gossiping problem[1]

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Abstract

This paper studies a variation of the gossiping problem, where there are n persons, each of whom initially has a message. A pair of persons can pass all messages they have by making one telephone call. The exact gossiping problem is to determine the minimum number of calls for each person to know exactly k messages. This paper gives solution to the problem for $k \leqslant 4$ or $i + 2^{k-i-2} \leqslant n \leqslant i - 2 + 2^{k-i-1}$ with $k/2 - 1 \leqslant i \leqslant k - 4$.

Keywords: Gossip; Broadcast; Call; Multigraph; Tree; Component

1. Introduction

Gossiping and broadcasting problems have been extensively studied for several decades; see [2] for a survey. In these problems, there are n persons, initially each of whom knows a unique message and is ignorant of the messages of the other persons. Messages are then spread by telephone calls. In each call, two persons exchange all information they had. The *gossiping problem* is to find the minimum number of calls required for all persons to know all messages. It has been proven that the solution to the problem is $2n - 4$ for $n \geqslant 4$.

Many variations of the gossiping problem have been studied. Examples include restricting the calls to certain pairs of persons, allowing conference calls, allowing only one-way calls, partial gossiping, and set-to-set broadcasting. The *partial gossiping problem*, introduced by Richards and Liestman [4], is to determine, for a given k, the minimum number $P(n, k)$ of calls required for each person to know *at least* k messages. For the case of $k = n$, the well-known result is

$$P(n, n) = 2n - 4 \quad \text{for } n \geqslant 4.$$

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Richards and Liestman [4] determined $P(n,k)$ for $k \leqslant 3$ and gave upper bounds for $k \geqslant 4$. Chang and Tsay [1] gave a complete solution to $P(n,k)$:

$$P(n,k) = \begin{cases} \left\lceil \frac{2^{k-1}-1}{2^{k-1}} n \right\rceil & \text{for } n \geqslant 2^{k-1}-1, \\ n+i & \text{for } 0 \leqslant i \leqslant k-4 \text{ and } i+2^{k-i-2} \leqslant n \leqslant i-2+2^{k-i-1}. \end{cases} \tag{1}$$

Richards and Liestman [4] also considered the *exact gossiping problem*, which is to determine, for a given k, the minimum number $E(n,k)$ of calls required for each person to know *exactly* k messages, i.e., the n persons can *k-gossip exactly*. Define

$$S(k) = \{n: n \text{ persons can } k\text{-gossip exactly}\}.$$

Richards and Liestman [4] determined that $S(2)$ is the set of positive even integers and, for $k \geqslant 3$, $S(k) = \{n: n \geqslant k\}$ with the single exception that 5 is not in $S(3)$. They also gave upper bounds for $E(n,k)$, namely, for $k \geqslant 4$,

$$E(n,k) \leqslant \begin{cases} E(\lceil \frac{n}{2} \rceil, k-1) + \lceil \frac{n}{2} \rceil & \text{for } n \geqslant 2k, \\ 4k-9 & \text{for } n = 2k-1, \\ 3k-7 & \text{for } k \leqslant n < 2k-1. \end{cases}$$

In this paper, we study the exact value of $E(n,k)$. In particular, we determine all values of $E(n,k)$ for $k \leqslant 4$ (see Theorems 3 and 6). For general k, we show that $E(n,k) = P(n,k) = n+i$ for $k/2-1 \leqslant i \leqslant k-4$ and $i+2^{k-i-2} \leqslant n \leqslant i-2+2^{k-i-1}$ (see Theorem 9).

2. Exact gossiping

We represent the n persons by the set $V = \{1, 2, \ldots, n\}$. To any sequence of calls

$$c(1), c(2), \ldots, c(t)$$

between these n persons, there corresponds a multigraph G_c whose vertex set is V and whose edge set contains these t calls. From now on, persons and vertices (respectively, calls and edges) will be treated as interchangeable.

Lemma 1. $P(n,k) \leqslant E(n,k)$.

Proof. The lemma follows from the fact that an exact k-gossiping is a partial k-gossiping. □

It is clear that $E(n,1) = P(n,1) = 0$ for $n \geqslant 1$, $E(n,2) = P(n,2) = n/2$ for even $n \geqslant 2$, and $E(n,2)$ is not defined for odd n.

The following lemma is useful for determining an upper bound of $E(n,k)$ in terms of other $E(n',k)$'s with $n' < n$.

Lemma 2. $E(m+n,k) \leqslant E(m,k) + E(n,k)$.

Proof. An exact k-gossiping for m persons together with an exact k-gossiping for another n persons makes an exact k-gossiping for $m+n$ persons. □

Theorem 3. *If* $n \geqslant 3$ *and* $n \neq 5$, *then* $E(n,3) = 3\lceil n/4 \rceil$.

Proof. It is clear that $E(3,3) \leqslant 3 = 3\lceil \frac{3}{4} \rceil$ and $E(4,3) \leqslant 3 = 3\lceil \frac{4}{4} \rceil$. In general, we can write $n = 4m_1 + 3m_2$ with $0 \leqslant m_2 \leqslant 3$. By Lemma 2,

$$E(n,3) = E(4m_1 + 3m_2, 3) \leqslant m_1 E(4,3) + m_2 E(3,3) \leqslant 3m_1 + 3m_2.$$

On the other hand, suppose the n persons can 3-gossip exactly by a call sequence c, In any component H of G_c, the first call must share with the second (respectively, third) call a vertex otherwise some person in these two calls will eventually know at least four messages. So, at the end of the first three calls in H, 3 or 4 persons in these calls have already known 3 messages. Hence, H has exactly 3 edges and 3 or 4 vertices. Thus, $E(n,3) \geqslant 3a + 3b$, where $n = 4a + 3b$. Since m_1 is the largest non-negative integer a such that we can write $n = 4a + 3b$, where a and b are non-negative integers, $m_1 \geqslant a$. Therefore,

$$E(n,3) \geqslant n - a \geqslant n - m_1 = 3m_1 + 3m_2.$$

Both inequalities imply $E(n,3) = 3m_1 + 3m_2 = 3\lceil n/4 \rceil$. □

Note that, by (1), $P(n,3) = \lceil 3n/4 \rceil$ for $n \geqslant 3$. Compared to Theorem 3, we have $E(n,3) = P(n,3)$ when $n \equiv 0$ or $3 \pmod 4$, $E(n,3) = P(n,3) + 1$ when $n \equiv 2 \pmod 4$, and $E(n,3) = P(n,3) + 2$ when $n \equiv 1 \pmod 4$.

The following two lemmas are useful for establishing the lower bounds of $E(n,4)$.

Lemma 4 (Chang and Tsay [1]). *Suppose* c *is a call sequence on* V *and* T *is a component of* G_c *that is a tree. If every vertex in* T *knows at least* k *messages, then* T *has at least* 2^{k-1} *vertices.*

Lemma 5. *Suppose* c *is a call sequence on* V *and* T *is a component of* G_c *that is a tree. If every vertex of* T *knows exactly* k *messages, then* T *has an even number of vertices.*

Proof. For every vertex x in T, there exists exactly one edge e_x incident to x such that e_x is the first call after which x knows k messages. Suppose $e_x = \{x, y\}$. Since e_x is a bridge of T and c is an exact k-gossip, y knows less than k messages before the call e_x and exactly k messages after e_x, i.e., $e_y = e_x$. Therefore, $\{e_x : x \text{ is a vertex in } T\}$ is a perfect matching of T, which implies that T has an even number of vertices. □

Theorem 6. *If $n \geqslant 4$, then*

$$E(n,4) = \begin{cases} \lceil \frac{7n}{8} \rceil + 1 & \text{if } n \equiv 1,3 \pmod 8, \\ \lceil \frac{7n}{8} \rceil & \text{otherwise.} \end{cases}$$

Proof. Denote by $f(n)$ the right-hand side of the equality. Fig. 1 shows that $E(n,4) \leqslant f(n)$ for $4 \leqslant n \leqslant 11$.

In general, we can write $n = 8m_1 + m_2$ with $4 \leqslant m_2 \leqslant 11$. By Lemma 2,

$$E(n,4) \leqslant m_1 E(8,4) + E(m_2,4) \leqslant 7m_1 + f(m_2) = f(n).$$

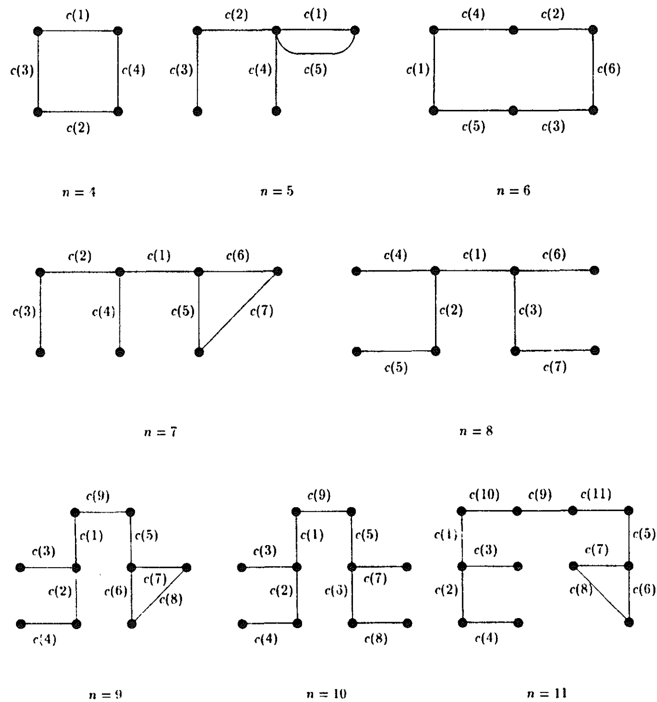

Fig. 1. Call sequences.

Suppose c is an optimal call sequence for $E(n,4)$ and G_c has n_i components of i vertices for $i \geqslant 4$. It is clear that

$$\sum_{i \geqslant 4} i n_i = n. \tag{2}$$

Note that every component of i vertices has at least $i-1$ edges, and at least i edges for $i \in \{4, 5, 6, 7, 9, 11\}$ by Lemmas 4 and 5. This, together with (2), implies

$$E(n,4) \geqslant \sum_{4 \leqslant i \leqslant 7} i n_i + 7n_8 + 9n_9 + 9n_{10} + 11n_{11} + \sum_{i \geqslant 12} (i-1)n_i$$

$$\geqslant n - n_8 - n_{10} - \sum_{i \geqslant 12} n_i. \tag{3}$$

By the choice of m_1 and m_2, $1 + m_1 \geqslant n_8 + n_{10} + \sum_{n \geqslant 12} n_i$ and the strict inequality holds when $m_2 \in \{8, 10\}$. Thus, by (3), $E(n,4) \geqslant f(n)$. □

Note that, by (1), $P(n,4) = \lceil 7n/8 \rceil$ for $n \geqslant 4$. Compared to Theorem 6, we have $E(n,4) = P(n,4)$ except $E(n,4) = P(n,4) + 1$ when $n \equiv 1, 3 \pmod 8$.

For the case of $k \geqslant 4$, it becomes harder to determine $E(n,k)$ in general. We shall establish results for some cases where $E(n,k) = P(n,k)$. The following lemmas are useful in subdividing vertices in order to construct exact k-gossiping for these results.

Lemma 7. *If m and i are integers such that $0 \leqslant m \leqslant 2^i - 2$, then we can write*

$$m = \sum_{r=1}^{i} (2^{j_r} - 1)$$

where $0 \leqslant j_r \leqslant i-1$ for $1 \leqslant r \leqslant i$.

Proof. The lemma is obvious for $i = 1$. Suppose the lemma is true for all $i' < i$. Now consider the case of $i \geqslant 2$. For the case of $m = 2^i - 2$, we can choose $j_1 = j_2 = i - 1$ and all other $j_r = 0$. For the case of $0 \leqslant m \leqslant 2^i - 3$, let

$$j_1 = \begin{cases} 0 & \text{if } 0 \leqslant m \leqslant 2^{i-1} - 2, \\ i-1 & \text{if } 2^{i-1} - 1 \leqslant m \leqslant 2^i - 3. \end{cases}$$

Then $0 \leqslant m - (2^{j_1} - 1) \leqslant 2^{i-1} - 2$. By the induction hypothesis,

$$m - (2^{j_1} - 1) = \sum_{r=1}^{i-1} (2^{j_r} - 1),$$

where $0 \leqslant j_r \leqslant i-2$ for $1 \leqslant r \leqslant i-1$. So

$$m = \sum_{r=1}^{i} (2^{j_r} - 1),$$

where $0 \leqslant j_r \leqslant i-1$ for $1 \leqslant r \leqslant i$. □

Lemma 8. *Suppose $Z = \{z_1, z_2, \ldots, z_{2^j}\}$ is a set of 2^j persons such that z_1 knows exactly j' messages and every other person knows a unique message and every one is ignorant of the messages of the other persons. Then there is a calling scheme using $2^j - 1$ calls such that each person knows exactly $j' + j$ messages at the end.*

Proof. Consider the following calls in j iterations. In iteration r, $0 \leqslant r \leqslant j-1$, z_s calls z_{s+2^r} for $1 \leqslant s \leqslant 2^r$. In this iteration, 2^r calls are made and at the completion of this iteration the first 2^{r+1} persons all know exactly $j' + r + 1$ messages. So at the completion of these j iterations, totally $2^j - 1$ calls have been made and all persons know exactly $j' + j$ messages. □

Note that the above proof is similar to the construction for the gossiping time on a complete graph of n vertices given by Knödel [3].

Theorem 9. $E(n,k) = P(n,k) = n + i$ if $k/2 - 1 \leqslant i \leqslant k - 4$ *and* $i + 2^{k-i-2} \leqslant n \leqslant i - 2 + 2^{k-i-1}$.

Proof. $E(n,k) \geqslant P(n,k) = n + i$ by (1) and Lemma 1. For the proof of $E(n,k) \leqslant n + i$, consider the following construction. Choose two disjoint subsets X and Y of V as follows:

$$X = \{x_1, x_2, \ldots, x_i\} \quad \text{and} \quad Y = \{y_1, y_2, \ldots, y_{2^{k-i-2}}\}.$$

Then

$$|V - (X \cup Y)| = n - 2^{k-i-2} - i \leqslant 2^{k-i-2} - 2.$$

By Lemma 7, we can write

$$|V - (X \cup Y)| = \sum_{r=1}^{k-i-2} (2^{j_r} - 1),$$

where $0 \leqslant j_r \leqslant k - i - 3$ for $1 \leqslant r \leqslant k - i - 2$. Note that $k - i - 2 \leqslant i$. Let $j_r = 0$ for $k - i - 2 < r \leqslant i$. Without loss of generality, we may assume that $0 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_i$. Then we can write $V - (X \cup Y)$ into disjoint union of $V_1, V_2, \ldots, V_i$ such that $|V_r| = 2^{j_r} - 1$ for $1 \leqslant r \leqslant i$.

Since $i \leqslant k - 4$, $|Y| \geqslant 4$. Make the following calls in $k - i - 2$ iterations, where each iteration contains two phases.

In phase one of the 0th iteration, each person of X calls y_1 in the order $x_1, x_2, \ldots, x_i$, and then y_1 calls y_2, y_3 calls y_4. In this phase $i + 2$ calls are made and upon the completion of this phase y_1 and y_2 know $i + 2$ messages, y_3 and y_4 know 2 messages, x_r knows $r + 1$ messages for $1 \leqslant r \leqslant i$. In phase two, if $j_i = k - i - 3$, then make the following calls otherwise make no calls. First y_3 calls x_i and then y_3 calls all other x_r with $j_r = k - i - 3$. Then each x_r, including x_i, with $j_r = k - i - 3$ together with V_r forms a set of 2^{j_r} persons in which x_r knows $i + 3$ messages and every other person knows only one message. Make $2^{j_r}-1$ calls among $\{x_r\}\cup V_r$ as described in the proof of Lemma 8 so that each person knows exactly $(i+3)+(k-i-3)=k$ messages.

In phase one of iteration t, $1\leqslant t\leqslant k-i-3$, y_s calls y_{s+2^t} for $1\leqslant s\leqslant 2^t$. In this phase, 2^t calls are made and at the completion of this phase the first 2^{t+1} persons of Y all know exactly $i+3+t$ messages. In phase two, if there is some $j_r=k-i-3-t$, then make the following calls, otherwise make no calls. y_1 calls each x_r with $j_r=k-i-3-t$ so that x_r learns all $i+3+t$ messages from y_1 but y_1 knows only the original messages. Then each x_r with $j_r=k-i-3-t$ together with V_r forms a set of 2^{j_r} persons in which x_r knows $i+3+t$ messages and each other person knows one message. Make $2^{j_r}-1$ calls among $\{x_r\}\cup V_r$ as described in the proof of Lemma 8 so that each person knows exactly $(i+3+t)+(k-i-3-t)=k$ messages.

At the end of these $k-i-2$ iterations, each person knows exactly k messages. The number of calls in phase one of all iterations is

$$(i+2)+\sum_{t=1}^{k-i-3}2^t=i+2^{k-i-2}.$$

The number of calls in phase two of all iterations is

$$\sum_{r=1}^{i}|\{x_r\}\cup V_r|=|V-Y|=n-2^{k-i-2}.$$

Thus, totally $n+i$ calls are made, i.e., $E(n,k)\leqslant n+i$. □

3. Conclusion

This paper studies the exact gossip problem. In particular, it determines the minimum number $E(n,k)$ of calls required for each person of n persons to know exactly k messages for $k\leqslant 4$ or $i+2^{k-i-2}\leqslant n\leqslant i-2+2^{k-i-1}$ with $k/2-1\leqslant i\leqslant k-4$. The results are

$$E(n,k)=\begin{cases}0 & \text{if } n\geqslant k=1,\\ \frac{n}{2} & \text{if } n\geqslant k=2 \text{ and } n \text{ is even},\\ \text{undefined} & \text{if } n\geqslant k=2 \text{ and } n \text{ is odd},\\ 3\lceil\frac{n}{4}\rceil & \text{if } n\geqslant k=3,\\ \lceil\frac{7n}{8}\rceil+1 & \text{if } n\geqslant k=4 \text{ and } n\equiv 1,3 \pmod 8,\\ \lceil\frac{7n}{8}\rceil & \text{if } n\geqslant k=4 \text{ and } n\equiv 0,2,4,5,6,7 \pmod 8,\\ n+i & \text{if } \frac{k}{2}-1\leqslant i\leqslant k-4 \text{ and } i+2^{k-i-2}\leqslant n\leqslant i-2+2^{k-i-1}.\end{cases}$$

Note that in these results, $E(n,k) = P(n,k)$ except $E(n,2)$ is undefined but $P(n,2) = \lceil n/2 \rceil$ for odd $n \geqslant 2$, $E(n,3) = P(n,3) + 1$ for $n \equiv 2 (\text{mod} 4)$, $E(n,3) = P(n,3) + 2$ for $n \equiv 1 (\text{mod} 4)$, and $E(n,4) = P(n,4) + 1$ for $n \equiv 1, 3 (\text{mod} 8)$. We have not yet determined the values of $E(n,k)$ for $k \geqslant 5$ and $0 \leqslant i < k/2 - 1$ and $i + 2^{k-i-2} \leqslant n \leqslant i - 2 + 2^{k-i-1}$. We suspect that $E(n,k)$ is larger than $P(n,k)$ for some cases in this range. The complete solution to $E(n,k)$ is desirable.

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