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# Select better suppliers based on manufacturing precision for processes with multivariate data

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Process capability indices have been widely used in the manufacturing industry for measuring process reproduction capability according to manufacturing specifications. Processes with univariate data have been investigated extensively, but are comparatively neglected for processes with multivariate data. Chou (Chou, Y.M., 1994. Selecting a better supplier by testing process capability indices. *Quality Engineering*, 6, 427–438) developed a procedure using univariate $C_p$ to determine whether or not two processes are equally capable, which allows one to select the supplier with better quality. However, for processes with multiple characteristics, no methods are available for comparing two processes with multivariate data. In this paper, we consider the supplier selection problem based on manufacturing precision in which the processes involve multiple quality characteristics. We derive the distribution of the corresponding test statistic, and calculate critical values required for the comparison purpose. A real-world application is presented for justification.

**Keywords:** process capability; multiple quality characteristics; critical value



## 1. Introduction

Process capability indices have been widely used in the manufacturing industry for measuring process reproduction capability conforming to the manufacturing specifications. In current industry practice customers often require their suppliers to provide process capability for certain product characteristics in the supply chain partnership. Process capability indices also can be used as a benchmark for quality improvement activities. $C_p$, $C_{pk}$ and $C_{pm}$, are some well-known indices used in the industry for evaluating process performance, but limited to cases with single engineering specification. Most research work has dealt with statistical properties estimating/testing the univariate indices. Kotz and Lovelace (1998), Kotz and Johnson (2002) and Spiring *et al.* (2003) provided a compact survey and comprehensive discussions on process capability indices over recent years.

Process capability indices, which establish the relationships between the actual process performance and the manufacturing specifications, have been the focus of recent research

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in quality assurance and process capability analysis. The precision index $C_p$ is the first process capability index which appeared in the literature, which is defined in Kane (1986) as:

$$C_p = \frac{USL - LSL}{6\sigma},$$

where $USL$ and $LSL$ are the upper and lower specification limits, and $\sigma$ is the process standard deviation. The index $C_p$ was designed to measure the magnitude of the overall process variation relative to the manufacturing tolerance, which is used for controlled normal processes. Clearly, the index measures manufacturing precision, which reflects product quality consistency (uniformity), an important criterion for judging product quality. A small value of $C_p$ implies that the product quality is not consistent causing complaints from the customers not only damaging marketing potentials but also incurring more repair cost.

The use of capability indices was first explored within the automotive industry. Ford Motor Company (1984) has used $C_p$ to keep track of the process performance and to reduce process variation. Recently, the manufacturing industry has been making extensive efforts to implement statistical process control in their plants and supply bases. Capability indices have received increasing usage not only in capability assessments, but also in the evaluation of purchasing decisions, which are becoming the standard tools for quality reporting. Proper understanding and use are essential for the company to maintain capable product supplies. Process precision measures using $C_p$ for normal, truncated normal, contaminated normal processes based on one single, multiple, (X, R), or (X, S) control chart samples, have been investigated extensively. Examples include Kane (1986), Cheng and Spiring (1989), Chou and Owen (1989), Chan *et al*. (1991), Kirmani *et al*. (1991), Kocherlakota (1992), Pearn *et al*. (1992, 1998, 2004, 2006), Pearn and Wu (2004), Pearn and Chang (2005). Clearly, in using the $C_p$ index for measuring process precision, product uniformity would be the primary concern rather than the process yield.

Although some multivariate capability indices have been proposed, no statistical properties of those indices are discussed. In this paper, we develop a statistical test procedure using the estimator of $MC_p$ (multivariate extension of $C_p$) to judge whether the capability of one process is superior to another process. A real-world application is presented to justify the proposed methodology. Practitioners can use the proposed procedure in making reliable decisions for their in-plant applications.



## 2. Process with multiple characteristics

Most research has been devoted to capability measures with a single process/quality characteristic. However, it is quite common that the manufactured product involves more than one quality characteristic. That is, it requires several different characteristics for adequate product description. Each of those characteristics must satisfy certain specifications. The assessed quality of a product depends on the combined effects of those characteristics rather than on their individual values. For example, automobile paint usually has a range of light reflective abilities and a range of adhesion abilities (see Taam *et al*. 1993). A paint that satisfies one criterion but not the other is considered undesirable. Those characteristics are related to each other through the composition of the paint. It is therefore natural to consider a bivariate characterisation of this paint.

For tolerance region of processes with multiple characteristics, most researchers take an ellipsoidal or a rectangular region. For more complex engineering specifications, the tolerance region would be rather complicated. For instance, a drawing of a connecting rod in a combustion engine consists of crank-bore inner diameter, pin-bore inner diameter, rod length, bore true-location and so on (see Taam *et al.* 1993). In multivariate processes, we usually assume that the process characteristics $X$ follows the multivariate normal distribution $N_v(\mu, \sum)$, where $v$ is the characteristic dimensions, $\mu$ is the mean vector, and $\sum$ is the variance-covariance matrix of $X$. Also $T$ is the target vector, $\overline{X}$ is the sample mean vector and $S$ is the sample covariance matrix. For processes with multivariate data, Taam *et al.* (1993) defined the multivariate capability index $MC_p$

$$MC_p = \frac{vol.(\text{modified tolerance region})}{vol.[(X-\mu)'\Sigma^{-1}(X-\mu) \le k_v(q)]} = \frac{vol.(\text{modified tolerance region})}{(\pi\chi^2_{v,0.9973})^{v/2}|\Sigma|^{1/2}[\Gamma(v/2+1)]^{-1}}, \quad (1)$$

where $k_v(q)$ is the 99.73th percentile of the $\chi^2$ distribution with $v$ degrees of freedom, $|\sum|$ is the determinant of $\sum$, and $\Gamma(\cdot)$ is the gamma function. Note that, if $MC_p$ is less than 1, then the process variation is greater than the specified range of variation. It indicates that the process precision is not adequate with respect to the specifications (product quality is not uniform/consistent).

## 3. Comparing two multivariate processes using $MC_p$

An estimator of $MC_p$ can be expressed as

$$\hat{MC}_p = \frac{vol.(\text{modified tolerance region})}{vol.(\text{estimated 99.73\% process region})} = \frac{vol.(\text{modified tolerance region})}{(\pi\chi^2_{v,0.9973})^{v/2}|S|^{1/2}[\Gamma(v/2+1)]^{-1}} \quad (2)$$

where $S$ is the sample variance-covariance matrix, and $|S|$ is the determinant of $S$. From Equation (1), $\hat{MC}_p$ can be rewritten as $MC_p(|S|/|\Sigma|)^{-1/2}$. Let $\tilde{X} = (X_1, X_2, \ldots, X_n)'$ be an $n$-dimensional vector of measurements taken from a multivariate normal distribution with mean vector $\mu = (\mu_1, \mu_2, \ldots, \mu_v)'$, target vector $T$, process variance-covariance matrix $\hat{\sum}$. Using the following theorem (Theorem 1), we may obtain the distribution of $\hat{MC}_p$. Chou (1994) developed a procedure using univariate $C_p$ to determine whether or not two processes are equally capable, which allows one to select the supplier with better quality. However, for processes with multiple characteristics (multivariate data), no methods are available for comparing two processes with multivariate data. For this purpose, we consider the problem of comparing two multivariate processes using $MC_p$. The hypothesis testing would be as follows: $H_0$: $MC_{p1} \le MC_{p2}$ (process I is not better than process II) versus $H_1$: $MC_{p1} > MC_{p2}$ (process I is better than process II). The critical value $c$ can be determined as:

$$P\left\{\frac{\hat{MC}_{p1}}{\hat{MC}_{p2}} > c | MC_{p1} = MC_{p2}\right\} = \alpha \Rightarrow P\left\{\frac{MC_{p1}(|S_1|/|\Sigma_1|)^{-1/2}}{MC_{p2}(|S_2|/|\Sigma_2|)^{-1/2}} > c | MC_{p1} = MC_{p2}\right\} = \alpha$$

$$\Rightarrow P\left\{\frac{(|S_1|/|\Sigma_1|)^{-1/2}}{(|S_2|/|\Sigma_2|)^{-1/2}} > c\right\} = \alpha \Rightarrow P\left\{\frac{(|S_2|/|\Sigma_2|)}{(|S_1|/|\Sigma_1|)} > c^2\right\} = \alpha \quad (3)$$

**Theorem 1:** *The distribution of the generalised variance $|S|$ of a sample $X_1, X_2, \ldots, X_n$ from $N_v(\mu, \sum)$ is the same as the distribution of $|\sum|/(n-1)^v$ times the product of v independent factors, the distribution of the ith factor being the $\chi^2$ distribution with $n-i$ degrees of freedom.*

**Proof:** See Anderson (2003), p. 268.

**Lemma 1:** *Let $x \sim \chi^2_n$ and $y \sim \chi^2_{n-1}$ be independently distributed. Let $z^2 = 4xy$. Then $z \sim \chi^2_{2n-2}$.*

**Proof:** See Srivastava and Khatri (1979), p. 82.

From the above theorem, $|S|/|\sum|$ is distributed as $\chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v}/(n-1)^v$. For $v = 2$, Equation (3) becomes

$$P\left\{\frac{\chi^2_{n_2-1} \times \chi^2_{n_2-2}}{(n_2-1)^2} \frac{(n_1-1)^2}{\chi^2_{n_1-1} \times \chi^2_{n_1-2}} > c^2\right\} = \alpha \Rightarrow P\left\{\frac{(\chi^2_{2n_2-4})^2/4}{(n_2-1)^2} \frac{(n_1-1)^2}{(\chi^2_{2n_1-4})^2/4} > c^2\right\} = \alpha$$

(Referring to Lemma 1, we can derive $\chi^2_{n-1} \times \chi^2_{n-2} \sim (\chi^2_{2n-4})^2/4$, see Corollary 1 in the Appendix.)

$$\Rightarrow P\left\{\frac{(\chi^2_{2n_2-4})^2}{(n_2-1)^2} \frac{(n_1-1)^2}{(\chi^2_{2n_1-4})^2} > c^2\right\} = \alpha \Rightarrow P\left\{\frac{\left(\chi^2_{2n_2-4}/2n_2-4\right)^2}{\left(\chi^2_{2n_1-4}/2n_1-4\right)^2} \frac{(n_1-1)^2}{(n_2-1)^2} \frac{(2n_2-4)^2}{(2n_1-4)^2} > c^2\right\} = \alpha$$

$$\Rightarrow P\left\{\left(F_{2n_2-4,2n_1-4}\right)^2 > c^2 \frac{(n_2-1)^2}{(n_1-1)^2} \frac{(2n_1-4)^2}{(2n_2-4)^2}\right\} = \alpha \Rightarrow F_{2n_2-4,2n_1-4,1-\alpha} = c\frac{(n_2-1)}{(n_1-1)} \frac{(2n_1-4)}{(2n_2-4)}.$$

Thus, the critical value can be expressed as

$$c = F_{2n_2-4,2n_1-4,1-\alpha} \frac{(n_1-1)}{(n_2-1)} \frac{(2n_2-4)}{(2n_1-4)}. \tag{4}$$

For $v = 3$, Equation (3) can be expressed as

$$P\left\{\frac{\chi^2_{n_2-1} \times \chi^2_{n_2-2} \times \chi^2_{n_2-3}}{(n_2-1)^3} \frac{(n_1-1)^3}{\chi^2_{n_1-1} \times \chi^2_{n_1-2} \times \chi^2_{n_1-3}} > c^2\right\}$$
$$= \alpha \Rightarrow P\left\{\left(\frac{\chi^2_{2n_2-4}}{\chi^2_{2n_1-4}}\right)^2 \frac{\chi^2_{n_2-3}}{\chi^2_{n_1-3}} \frac{(n_1-1)^3}{(n_2-1)^3} > c^2\right\} = \alpha$$

$$\Rightarrow P\left\{\left(\frac{\frac{\chi^2_{2n_2-4}}{2n_2-4}}{\frac{\chi^2_{2n_1-4}}{2n_1-4}}\right)^2 \frac{\frac{\chi^2_{n_2-3}}{n_2-3}}{\frac{\chi^2_{n_1-3}}{n_1-3}} \frac{(2n_2-4)^2}{(2n_1-4)^2} \frac{(n_2-3)}{(n_1-3)} \frac{(n_1-1)^3}{(n_2-1)^3} > c^2\right\} = \alpha$$

$$\Rightarrow P\left\{\left(F_{2n_2-4,2n_1-4}\right)^2 * F_{n_2-3,n_1-3} > c^2 \frac{(2n_1-4)^2}{(2n_2-4)^2} \frac{(n_1-3)}{(n_2-3)} \frac{(n_2-1)^3}{(n_1-1)^3}\right\} = \alpha. \tag{5}$$

Let $z=xy$, where $x \sim (F_{2n_2-4,\, 2n_1-4})^2$, $y \sim F_{n_2-3,\, n_1-3}$, then Equation (5) can be expressed as $\int_0^{c^2(2n_1-4)^2(n_1-3)(n_2-1)^3/(2n_2-4)^2(n_2-3)(n_1-1)^3} f_z(z)dz = 1-\alpha$. Thus, the critical value can be expressed as

$$c = \sqrt{F_z^{-1}(1-\alpha)\frac{(2n_2-4)^2}{(2n_1-4)^2}\frac{(n_2-3)}{(n_1-3)}\frac{(n_1-1)^3}{(n_2-1)^3}}, \tag{6}$$

where

$$f_Z(z) = \int_0^\infty k(n_1,n_2)\frac{x^{((2n_2-4)/4-2)}\left(\frac{z}{x}\right)^{(n_2-3)/2-1}}{\left(1+\frac{(2n_2-4)}{(2n_1-4)}\sqrt{x}\right)^{(2n_2+2n_1-8)/2}\left(1+\frac{(n_2-3)}{(n_1-3)(z/x)}\right)^{(n_2+n_1-6)/2}}dx, \text{ for } x,\, z \geq 0$$

$$k(n_1,n_2) = \frac{\Gamma[(2n_2+2n_1-8)/2]\Gamma((n_2+n_1-6)/2)\left(\frac{2n_2-4}{2n_1-4}\right)^{(2n_2-4)/2}\left(\frac{n_2-3}{n_1-3}\right)^{(n_2-3)/2}}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]\Gamma[(n_2-3)/2]\Gamma[(2n_1-3)/2]},$$

$F_z(t) = \int_0^t f_z(z)dz = 1-\alpha$, and $F_z^{-1}(\cdot)$ is the inverse function of $F_z(\cdot)$; see Corollary 2 in the Appendix.

Tables 1 to 3 display the critical values for various sample sizes in the case with $v=2$ under test levels $\alpha=0.05$, $\alpha=0.025$ and $\alpha=0.01$. Tables 4 to 6 display the critical values for various sample sizes in the case of $v=3$ under $\alpha=0.05$, $\alpha=0.025$ and $\alpha=0.01$. For practical and convenient purpose, a step-by-step procedure is provided below:

*Step 1*: Determine the sample size $n_i$ for each supplier and the $\alpha$-risk (the chance of incorrectly rejecting a better supplier).

*Step 2*: Take a random sample from each process and calculate the sample covariance matrix.

*Step 3*: Calculate the test statistic $\hat{MC}_{p1}/\hat{MC}_{p2}$ and the critical value $c$.

Table 1. Critical value for testing $H_0$: $MC_{p1} \leq MC_{p2}$ under $\alpha=0.05$ ($v=2$).

| | *n*2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *n*1 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| 10 | 2.33 | 2.31 | 2.29 | 2.29 | 2.28 | 2.28 | 2.28 | 2.27 | 2.27 | 2.27 |
| 20 | 1.81 | 1.74 | 1.71 | 1.69 | 1.68 | 1.68 | 1.67 | 1.67 | 1.65 | 1.66 |
| 30 | 1.68 | 1.60 | 1.56 | 1.54 | 1.52 | 1.51 | 1.51 | 1.50 | 1.50 | 1.49 |
| 40 | 1.62 | 1.53 | 1.49 | 1.46 | 1.45 | 1.44 | 1.43 | 1.42 | 1.42 | 1.41 |
| 50 | 1.59 | 1.49 | 1.44 | 1.42 | 1.40 | 1.39 | 1.38 | 1.37 | 1.37 | 1.36 |
| 60 | 1.57 | 1.46 | 1.42 | 1.39 | 1.37 | 1.36 | 1.35 | 1.34 | 1.34 | 1.33 |
| 70 | 1.55 | 1.45 | 1.40 | 1.37 | 1.35 | 1.34 | 1.33 | 1.32 | 1.31 | 1.31 |
| 80 | 1.54 | 1.43 | 1.38 | 1.35 | 1.33 | 1.32 | 1.31 | 1.30 | 1.30 | 1.29 |
| 90 | 1.53 | 1.42 | 1.37 | 1.34 | 1.32 | 1.31 | 1.30 | 1.29 | 1.28 | 1.28 |
| 100 | 1.52 | 1.41 | 1.36 | 1.33 | 1.31 | 1.30 | 1.29 | 1.28 | 1.27 | 1.27 |

Table 2. Critical value for testing $H_0$: $MC_{p1} \leq MC_{p2}$ under $\alpha = 0.025$ ($v = 2$).

| | n2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n1 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| 10 | 2.76 | 2.69 | 2.67 | 2.65 | 2.64 | 2.64 | 2.63 | 2.63 | 2.63 | 2.63 |
| 20 | 2.06 | 1.94 | 1.89 | 1.87 | 1.85 | 1.84 | 1.83 | 1.83 | 1.82 | 1.82 |
| 30 | 1.89 | 1.75 | 1.70 | 1.67 | 1.65 | 1.63 | 1.62 | 1.62 | 1.61 | 1.61 |
| 40 | 1.81 | 1.67 | 1.61 | 1.57 | 1.55 | 1.54 | 1.53 | 1.52 | 1.51 | 1.50 |
| 50 | 1.76 | 1.62 | 1.55 | 1.52 | 1.50 | 1.48 | 1.47 | 1.46 | 1.45 | 1.44 |
| 60 | 1.74 | 1.58 | 1.52 | 1.48 | 1.46 | 1.44 | 1.43 | 1.42 | 1.41 | 1.40 |
| 70 | 1.72 | 1.56 | 1.50 | 1.46 | 1.43 | 1.41 | 1.40 | 1.39 | 1.38 | 1.38 |
| 80 | 1.70 | 1.55 | 1.48 | 1.44 | 1.41 | 1.39 | 1.38 | 1.37 | 1.36 | 1.35 |
| 90 | 1.69 | 1.55 | 1.46 | 1.42 | 1.40 | 1.38 | 1.36 | 1.35 | 1.35 | 1.34 |
| 100 | 1.68 | 1.52 | 1.45 | 1.41 | 1.39 | 1.37 | 1.35 | 1.34 | 1.33 | 1.32 |

Table 3. Critical value for testing $H_0$: $MC_{p1} \leq MC_{p2}$ under $\alpha = 0.01$ ($v = 2$).

| | n2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n1 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| 10 | 3.37 | 3.25 | 3.20 | 3.17 | 3.16 | 3.15 | 3.14 | 3.14 | 3.13 | 3.13 |
| 20 | 2.39 | 2.21 | 2.14 | 2.10 | 2.08 | 2.06 | 2.05 | 2.04 | 2.03 | 2.03 |
| 30 | 2.15 | 1.96 | 1.88 | 1.83 | 1.81 | 1.79 | 1.78 | 1.77 | 1.76 | 1.75 |
| 40 | 2.05 | 1.84 | 1.76 | 1.71 | 1.68 | 1.66 | 1.65 | 1.64 | 1.63 | 1.62 |
| 50 | 1.99 | 1.78 | 1.69 | 1.64 | 1.61 | 1.59 | 1.58 | 1.56 | 1.55 | 1.55 |
| 60 | 1.95 | 1.74 | 1.65 | 1.60 | 1.57 | 1.54 | 1.53 | 1.51 | 1.50 | 1.50 |
| 70 | 1.93 | 1.71 | 1.62 | 1.57 | 1.53 | 1.51 | 1.49 | 1.48 | 1.47 | 1.46 |
| 80 | 1.91 | 1.69 | 1.59 | 1.54 | 1.51 | 1.49 | 1.47 | 1.45 | 1.44 | 1.43 |
| 90 | 1.89 | 1.67 | 1.58 | 1.52 | 1.49 | 1.47 | 1.45 | 1.43 | 1.42 | 1.41 |
| 100 | 1.88 | 1.66 | 1.56 | 1.51 | 1.47 | 1.45 | 1.43 | 1.42 | 1.41 | 1.40 |

Table 4. Critical value for testing $H_0$: $MC_{p1} \leq MC_{p2}$ under $\alpha = 0.05$ ($v = 3$).

| | n2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n1 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| 10 | 2.94 | 3.14 | 3.19 | 3.21 | 3.22 | 3.23 | 3.24 | 3.24 | 3.24 | 3.25 |
| 20 | 1.95 | 2.00 | 1.99 | 1.99 | 1.99 | 1.98 | 1.98 | 1.98 | 1.98 | 1.97 |
| 30 | 1.73 | 1.75 | 1.73 | 1.72 | 1.71 | 1.70 | 1.70 | 1.69 | 1.69 | 1.69 |
| 40 | 1.64 | 1.64 | 1.61 | 1.60 | 1.59 | 1.58 | 1.57 | 1.56 | 1.56 | 1.56 |
| 50 | 1.59 | 1.58 | 1.55 | 1.53 | 1.52 | 1.50 | 1.50 | 1.49 | 1.49 | 1.48 |
| 60 | 1.56 | 1.54 | 1.51 | 1.48 | 1.47 | 1.46 | 1.45 | 1.44 | 1.44 | 1.43 |
| 70 | 1.53 | 1.51 | 1.48 | 1.45 | 1.44 | 1.43 | 1.42 | 1.41 | 1.40 | 1.40 |
| 80 | 1.51 | 1.49 | 1.45 | 1.43 | 1.41 | 1.40 | 1.39 | 1.38 | 1.38 | 1.37 |
| 90 | 1.50 | 1.47 | 1.44 | 1.41 | 1.40 | 1.38 | 1.37 | 1.36 | 1.36 | 1.35 |
| 100 | 1.49 | 1.46 | 1.42 | 1.40 | 1.38 | 1.37 | 1.36 | 1.35 | 1.34 | 1.34 |

Table 5. Critical value for testing $H_0$: $MC_{p1} \leq MC_{p2}$ under $\alpha = 0.025$ ($v = 3$).

| | n2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n1 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| 10 | 3.64 | 3.81 | 3.84 | 3.86 | 3.87 | 3.87 | 3.88 | 3.88 | 3.88 | 3.88 |
| 20 | 2.29 | 2.28 | 2.26 | 2.24 | 2.23 | 2.22 | 2.22 | 2.21 | 2.21 | 2.21 |
| 30 | 2.01 | 1.96 | 1.92 | 1.90 | 1.88 | 1.87 | 1.86 | 1.85 | 1.85 | 1.84 |
| 40 | 1.89 | 1.82 | 1.78 | 1.75 | 1.73 | 1.71 | 1.70 | 1.70 | 1.69 | 1.68 |
| 50 | 1.82 | 1.75 | 1.70 | 1.66 | 1.64 | 1.63 | 1.61 | 1.60 | 1.60 | 1.59 |
| 60 | 1.78 | 1.70 | 1.64 | 1.61 | 1.59 | 1.57 | 1.56 | 1.55 | 1.54 | 1.53 |
| 70 | 1.75 | 1.66 | 1.61 | 1.57 | 1.55 | 1.53 | 1.51 | 1.50 | 1.49 | 1.49 |
| 80 | 1.73 | 1.64 | 1.58 | 1.54 | 1.52 | 1.50 | 1.48 | 1.47 | 1.46 | 1.46 |
| 90 | 1.71 | 1.62 | 1.56 | 1.52 | 1.49 | 1.48 | 1.46 | 1.45 | 1.44 | 1.43 |
| 100 | 1.70 | 1.60 | 1.54 | 1.50 | 1.48 | 1.46 | 1.44 | 1.43 | 1.42 | 1.41 |

Table 6. Critical value for testing $H_0$: $MC_{p1} \leq MC_{p2}$ under $\alpha = 0.01$ ($v = 3$).

| | n2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n1 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| 10 | 4.68 | 4.79 | 4.81 | 4.81 | 4.82 | 4.82 | 4.82 | 4.82 | 4.82 | 4.82 |
| 20 | 2.76 | 2.67 | 2.62 | 2.58 | 2.56 | 2.55 | 2.54 | 2.53 | 2.52 | 2.52 |
| 30 | 2.38 | 2.25 | 2.18 | 2.14 | 2.11 | 2.09 | 2.08 | 2.06 | 2.06 | 2.05 |
| 40 | 2.22 | 2.07 | 1.99 | 1.94 | 1.91 | 1.89 | 1.87 | 1.86 | 1.85 | 1.84 |
| 50 | 2.13 | 1.97 | 1.88 | 1.83 | 1.80 | 1.78 | 1.76 | 1.75 | 1.74 | 1.73 |
| 60 | 2.07 | 1.90 | 1.82 | 1.77 | 1.73 | 1.71 | 1.69 | 1.67 | 1.66 | 1.65 |
| 70 | 2.03 | 1.86 | 1.77 | 1.72 | 1.68 | 1.66 | 1.64 | 1.62 | 1.61 | 1.60 |
| 80 | 2.00 | 1.83 | 1.74 | 1.68 | 1.65 | 1.62 | 1.60 | 1.58 | 1.57 | 1.56 |
| 90 | 1.98 | 1.80 | 1.71 | 1.65 | 1.62 | 1.59 | 1.57 | 1.55 | 1.54 | 1.53 |
| 100 | 1.96 | 1.78 | 1.69 | 1.63 | 1.60 | 1.57 | 1.55 | 1.53 | 1.52 | 1.51 |

*Step 4*: If $\hat{MC}_{p1}/\hat{MC}_{p2} > c$, then we reject $H_0$ and conclude that $MC_{p1} > MC_{p2}$. From the definition of $c$, it is clear that the value of $\hat{MC}_{p1}/\hat{MC}_{p2}$ must be higher than the original target value for the true $MC_{p1}/MC_{p2}$. The power of the test can be also computed below: The power of the test, $\beta$, is given by

$$\beta\left(\frac{MC_{p1}}{MC_{p2}}\right) = P\left\{\frac{\hat{MC}_{p1}}{\hat{MC}_{p2}} > c \mid \frac{MC_{p1}}{MC_{p2}} = k\right\}. \quad (7)$$

Take $v = 2$, and 3, Equation (7) can be expressed as

$$P\left\{F_{2n_2-4,2n_1-4} > c\frac{(n_2-1)}{(n_1-1)}\frac{(2n_1-4)}{(2n_2-4)}\frac{(MC_{p2})}{(MC_{p1})}\right\},$$

$$P\left\{\left(F_{2n_2-4,2n_1-4}\right)^2 * F_{n_2-3,n_1-3} > c^2\frac{(2n_1-4)^2}{(2n_2-4)^2}\frac{(n_1-3)}{(n_2-3)}\frac{(n_2-1)^3}{(n_1-1)^3}\frac{(MC_{p2})^2}{(MC_{p1})^2}\right\}.$$

## 4. A real-world application

To illustrate how the proposed method can be applied to the actual data collected from the factories, we present a real-world example of an electronic component manufacturer making ceramic multilayer capacitors applicable to consumer electronics, telecommunications, automotive parts, and data processing devices. The capacitor consists of a rectangular ceramic block in which a number of interleaved electrodes are contained. A cross section of the ceramic multi-layer capacitor structure is depicted in Figure 1. For a detailed model of the ceramic multilayer capacitor investigated, all the electrical characteristics are displayed in Table 7.

Consider a production process with multiple characteristics following a multivariate normal distribution, which is taken from a multilayer capacitor factory in Taiwan adapting the six-sigma quality improvement program. To compare product quality between a supplier versus another, 50 random samples are taken from the two processes. The quality control of the process involves the simultaneous control of the layer thickness, the layer length, and the layer width. The lower and upper specification limits for layer thickness, layer length, and layer width have been set to [1.45, 1.75], [3.0, 3.4] and [1.45, 1.75], respectively. The sample covariance matrices are summarised below, where $S_1$ represents the data from supplier I, and $S_2$ represents the data from supplier II.

$$S_1 = \begin{bmatrix} 0.00193 & 0.00046 & 0.00086 \\ 0.00046 & 0.00097 & 0.00075 \\ 0.00086 & 0.00075 & 0.00167 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.00236 & 0.00029 & 0.00003 \\ 0.00029 & 0.00176 & 0.00097 \\ 0.00003 & 0.00097 & 0.00161 \end{bmatrix}.$$

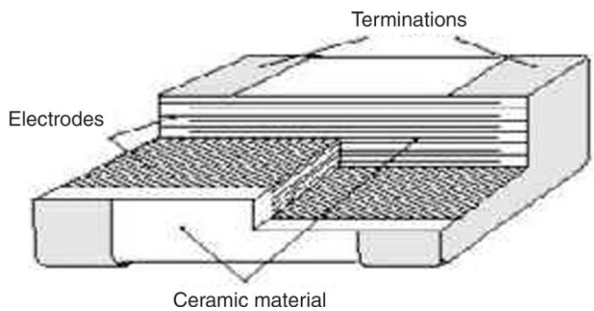


Figure 1. Structure of a ceramic multi-layer capacitor.

Table 7. Specification of Y5V/BME/1206/22uF/6.3V.

| | |
|---|---|
| Capacitance range | 22uF, Size 1206 |
| Tolerance on capacitance after 1000 hours | $-20\%$ to $+80\%$ |
| Rated voltage $U_R$ (DC) | 6.3 V |
| Test voltage (DC) for 1 minute | $2.5 \times U_R$ |
| Tan D (Note 1) | $\leq 12.50\%$ |
| Insulation resistance after 1 minute at $U_R$ (DC) | $R_{\text{ins.}} \times C \geq 500$ s |
| Maximum capacitance change as a function of temperature | $+30\%$ to $-80\%$ |
| Ageing | Typically, 12.5% per time decade |
| Terminations | NiSn plated |
| Resistance to soldering heat | 260°C, 10 sec |

To compare the two processes, we consider a test with null hypothesis $H_0$: $MC_{p1} \leq MC_{p2}$ against the alternative hypothesis $H_1$: $MC_{p1} > MC_{p2}$, where $MC_{p1}$ and $MC_{p2}$ represent the process capability indices of the two suppliers respectively. The test procedure can be described as:

*Step 1*: For the two suppliers with sample sizes $n_1 = n_2 = 50$, set the confidence level as 0.05.

*Step 2*: Calculate the sample covariance. From the above result, we obtain

$$S_1 = \begin{bmatrix} 0.00193 & 0.00046 & 0.00086 \\ 0.00046 & 0.00097 & 0.00075 \\ 0.00086 & 0.00075 & 0.00167 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.00236 & 0.00029 & 0.00003 \\ 0.00029 & 0.00176 & 0.00097 \\ 0.00003 & 0.00097 & 0.00161 \end{bmatrix}.$$

*Step 3*: Calculate the test statistic $\hat{MC}_{p1}/\hat{MC}_{p2}$ and critical value $c$:

$$\hat{MC}_{p1} = \frac{\frac{4}{3}\pi \times (0.3/2) \times (0.4/2) \times (0.3/2)}{|S_1|^{1/2}(\pi \times \chi^2_{3,0.9973})^{3/2}[\Gamma(2.5)]^{-1}} = 2.13239,$$

$$\hat{MC}_{p2} = \frac{\frac{4}{3}\pi \times (0.3/2) \times (0.4/2) \times (0.3/2)}{|S_2|^{1/2}(\pi \times \chi^2_{3,0.9973})^{3/2}[\Gamma(2.5)]^{-1}} = 1.28415.$$

Therefore, $\hat{MC}_{p1}/\hat{MC}_{p2} = 2.13239/1.28415 = 1.6605$, and $c = 1.52$ (refer to Table 4).

*Step 4*: As $1.6605 > 1.52$, we conclude that with 95% confidence process (supplier) I is superior to process (supplier) II. In order to show the sensitivity of the test procedure, the power curve of the test is depicted in Figure 2 under the ratio value of $MC_{p1}/MC_{p2} = 0.8$ to 2.3.

## 5. Conclusions

Processes with multiple quality characteristics are quite common in the manufacturing industry. Processes with univariate data have been investigated extensively, but are

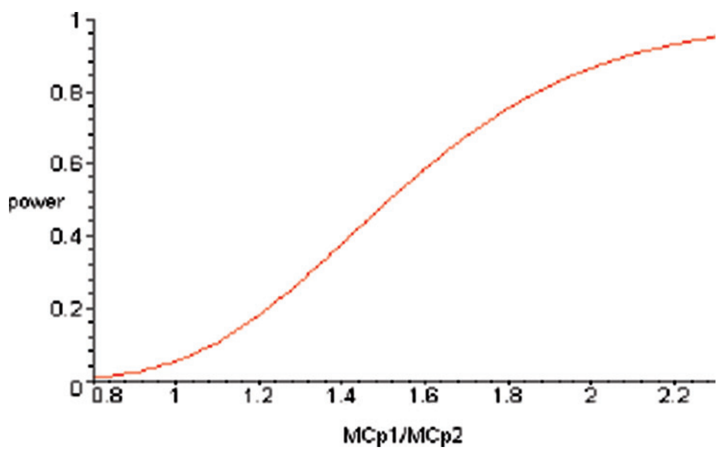


Figure 2. Power curve of testing.

comparatively neglected for processes with multivariate data. Chou (1994) developed a procedure using univariate $C_p$ to determine whether or not two processes are equally capable, which allows one to select the supplier with better quality. However, for processes with multiple characteristics, no methods are available for comparing two processes with multivariate data. In this paper, we considered the supplier selection problem based on manufacturing precision in which the processes involve multiple quality characteristics. We developed an effective test procedure for practitioners to make reliable decisions in their in-plant applications involving supplier selections.



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## Appendix

**Corollary 1:** *If $\chi^2_{n-1}$ and $\chi^2_{n-2}$ are mutually independently distributed, then $\chi^2_{n-1} \times \chi^2_{n-2}$ is distributed as $(\chi^2_{2n-4})^2/4$.*

**Proof:** Let $x_1 \sim \chi^2_{n-1}$ and $x_2 \sim \chi^2_{n-2}$. The joint pdf of $x_1$ and $x_2$ is given by

$$f_{x_1,x_2}(x_1,x_2) = \frac{(1/2)^{(n-1)/2} x_1^{n/2-3/2} e^{-x_1/2}}{\Gamma[(n-1)/2]} \times \frac{(1/2)^{(n-2)/2} x_2^{n/2-2} e^{-x_2/2}}{\Gamma[(n-2)/2]}.$$

Let $z_1 = x_1$ and $z_2 = 2\sqrt{x_1 x_2}$. Using the transformation method, the solution is $x_1 = z_1$ and $x_2 = z_2^2/4z_1$, and the Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 0 \\ -\frac{z_2^2}{4\sqrt{z_1}} & \frac{z_2}{2z_1} \end{vmatrix} = \frac{z_2}{2z_1}.$$

So, we find that the joint p.d.f. of $z_1 z_2$ is

$$f_{z_1,z_2}(z_1,z_2) = \frac{(1/2)^{(n-1)/2} z_1^{n/2-3/2} e^{-z_1/2}}{\Gamma[(n-1)/2]} \times \frac{(1/2)^{(n-2)/2} (z_2^2/4z_1)^{n/2-2} e^{-(z_2^2/4z_1)/2}}{\Gamma[(n-2)/2]} \times \frac{z_2}{2z_1}, 0 \le z_1, z_2 \le \infty.$$

Then, the marginal density function of $z_2$ is obtained as follows:

$$f_{z_2}(z_2) = \int_0^\infty \frac{(1/2)^{(n-1)/2} z_1^{n/2-3/2} e^{-z_1/2}}{\Gamma[(n-1)/2]} \times \frac{(1/2)^{(n-2)/2} (z_2^2/4z_1)^{n/2-2} e^{-(z_2^2/4z_1)/2}}{\Gamma[(n-2)/2]} \times \frac{z_2}{2z_1} dz_1$$
$$= C_1 \times z_2^{n-3} \times \int_0^\infty z_1^{-1/2} \times e^{-z_1/2-(z_2^2/4z_1)/2} dz_1, 0 \le z_2 \le \infty$$

where

$$C_1 = \frac{(1/2)^{2n-9/2}}{\Gamma[(n-1)/2] \times \Gamma[(n-2)/2]}.$$

Let

$$h(z_2) = \int_0^\infty z_1^{-1/2} \times e^{-z_1/2-(z_2^2/4z_1)/2} dz_1.$$

Hence

$$h'(z_2) = \left(-\frac{z_2}{4z_1}\right) \times \int_0^\infty z_1^{-1/2} \times e^{-z_1/2-(z_2^2/4z_1)/2} dz_1.$$

Now, let

$$\frac{z_2^2}{4z_1} = w.$$

Using variable transformation technique, we obtain

$$h'(z_2) = \left(-\frac{1}{2}\right) \times \int_0^\infty w^{-1/2} \times e^{-w/2-(z_2^2/4w)/2} dw = \left(-\frac{1}{2}\right) \times h(z_2).$$

The above equation gives $h(z_2) = e^{(-z_2/2+C_2)}$, where $C_2$ is a constant. Thus, the p.d.f. of $z_2$ can be obtained as below, where $C_3 = C_1 \times e^{-C_2}$. Therefore, we have $z_2 \sim \chi^2_{2n-4}$,

$$f_{z_2}(z_2) = C_1 \times e^{-C_2} \times z_2^{n-3} \times e^{-z_2/2} = C_3 \times z_2^{(2n-4)/2-1} \times e^{-z_2/2},\ 0 \le z_2 \le \infty$$

**Corollary 2:** $x \sim (F_{2n_2-4,\,2n_1-4})^2$, $y \sim F_{n_2-3,\,n_1-3}$, *if* $z = xy$, *then the p.d.f. of* $z$ *is*

$$f_Z(z) = \int_0^\infty k(n_1,n_2) \frac{\left(\frac{1}{2}\right) x^{(2n_2-4)/4-2} \left(\frac{z}{x}\right)^{(n_2-3)/2-1}}{\left(1 + \frac{(2n_2-4)}{(2n_1-4)}\sqrt{x}\right)^{(2n_2+2n_1-8)/2} \left(1 + \frac{(n_2-3)}{(n_1-3)}\frac{z}{x}\right)^{(n_2+n_1-6)/2}} dx, \text{ for } x, z \ge 0$$

*where*

$$k(n_1,n_2) = \frac{\Gamma[(2n_2+2n_1-8)/2]\Gamma[(n_2+n_1-6)/2]\left(\frac{2n_2-4}{2n_2-4}\right)^{(2n_2-4)/2}\left(\frac{n_2-3}{n_1-3}\right)^{(n_2-3)/2}}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]\Gamma[(n_2-3)/2]\Gamma[(2n_1-3)/2]}.$$

**Proof:** Let $t \sim (F_{2n_2-4,2n_1-4})$, $x = t^2$, where

$$f_t(t) = \frac{\Gamma[(2n_2+2n_1-8)/2]}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]} \times \frac{t^{(2n_2-4)/2-1}}{\left(1 + \left(\frac{2n_2-4}{2n_1-4}\right)t\right)^{(2n_2+2n_1-8)/2}}, \text{ for } t \ge 0.$$

Using the transformation method, $t = \sqrt{x}$, and $J = 1/2\sqrt{x}$ (Jacobian), so the p.d.f. of $x$ is

$$f_x(x) = \frac{\Gamma[(2n_2+2n_1-8)/2]}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]} \times \frac{\frac{1}{2}x^{(2n_2-4)/4-1}}{\left(1 + \left(\frac{2n_2-4}{2n_1-4}\right)\sqrt{x}\right)^{(2n_2+2n_1-8)/2}},$$

for $x \ge 0$.

Now, let $z = xy$ given $x \sim (F_{2n_2-4,2n_1-4})^2$, $y \sim F_{n_2-3,n_1-3}$. Using the transformation method, $x = x$, $y = z/x$, and $J = 1/x$ (Jacobian), so the joint pdf of $x$ and $z$ is

$$f_{x,z}(x,z) = k(n_1,n_2) \frac{\frac{1}{2} x^{(2n_2-4)/4-2} \left(\frac{z}{x}\right)^{(n_2-3)/2-1}}{\left(1 + \frac{(2n_2-4)}{(2n_1-4)}\sqrt{x}\right)^{(2n_2+2n_1-8)/2} \left(1 + \frac{(n_2-3)}{(n_1-3)}\frac{z}{x}\right)^{(n_2+n_1-6)/2}}, \text{ for } x, z \ge 0,$$

$$k(n_1,n_2) = \frac{\Gamma[(2n_2+2n_1-8)/2]\Gamma[(n_2+n_1-6)/2]\left(\frac{2n_2-4}{2n_2-4}\right)^{(2n_2-4)/2}\left(\frac{n_2-3}{n_1-3}\right)^{(n_2-3)/2}}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]\Gamma[(n_2-3)/2]\Gamma[(2n_1-3)/2]}.$$

Then the marginal density function of $z$ is obtained as follows:

$$f_Z(Z) = \int_0^{\infty} k(n_1, n_2) \frac{\frac{1}{2} x^{(2n_2-4)/4-2} \left(\frac{z}{x}\right)^{(n_2-3)/2-1}}{\left(1 + \frac{(2n_2-4)}{(2n_1-4)}\sqrt{x}\right)^{(2n_2+2n_1-8)/2} \left(1 + \frac{(n_2-3)}{(n_1-3)}\frac{z}{x}\right)^{(n_2+n_1-6)/2}} dx, \text{ for } x, z \geq 0,$$

where

$$k(n_1, n_2) = \frac{\Gamma[(2n_2+2n_1-8)/2]\Gamma[(n_2+n_1-6)/2]\left(\frac{2n_2-4}{2n_2-4}\right)^{(2n_2-4)/2}\left(\frac{n_2-3}{n_1-3}\right)^{(n_2-3)/2}}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]\Gamma[(n_2-3)/2]\Gamma[(2n_1-3)/2]}.$$

The programs for Table 4, 5 and 6.

≪Statistics
≪Graphics ‘Graphics’
≪Statistics ‘Continuous Distributions’

$n_1 = \text{Table } [k, \{k, 10, 100, 10\}]$

$n_2 = \text{Table } [p, \{p, 10, 100, 10\}]$

$\mathrm{K_1 = \ Table\ [k_1[i,j],\{i,10\},\{j,10\}];}$
$\mathrm{For[i = 1, i \leq 10, i = i+1,}$
$\mathrm{For[\ j = 1, j \leq 10, j = j+1,}$

$$\mathrm{A[i,j] = \frac{Gamma\ [((2*n_2[[j]]*2*n_1[[i]]-8)/2)]*Gamma\ [((n_2[[j]]*n_1[[i]]-6)/2)]*(((2*n_2[[j]]-4)/2*n_1[[i]]-4))^{(2*n_2[[j]]-4)/2}*(((n_2[[j]]-3)/n_1[[i]]-3))^{(n_2[[j]]-3)/2}}{Gamma[((2*n_2[[j]]-4)/2)]*Gamma[((2*n_1[[i]]-4)/2)]*Gamma[((n_2[[j]]-3))/2]*Gamma[((n_1[[i]]-3)/2)]};}$$

$$\mathrm{P[i,j] = Find\ Root[N\ Integrate\ [A[i,j]*\left(\frac{0.5*x^{(2*n_2[[j]]-4)/4)-2}*(z/x)^{(n_2[[j]]-3)/2-1}}{(1+((2*n_2[[j]]-4)/2*n_1[[i]]-4)*\sqrt{x})^{(2*n_2[[j]]*2*n_1[[i]]-8)/2}(1+((n_2[[j]]-3)/n_1[[i]]-3))*(z/x))^{\frac{((n_2[[j]]*n_1[[i]]-6)/2)}{2}}}\right),}$$

$\mathrm{\{z,0,t\},\{x,0,\infty\}] == 0.95,\{t,0,100\}];}$
$\mathrm{PP_1[i,j] = t/.P[i,j];}$

$$\mathrm{k_1[i,j] = Sqrt\left|\ PP_1[i,j]*\left(\frac{2*n_2[[j]]-4}{2*n_1[[i]]-4}\right)^2*\left(\frac{n_2[[j]]-3}{n_1[[i]]-3}\right)*\left(\frac{n_1[[i]]-1}{n_2[[j]]-1}\right)^3\ \right|;}$$

]]
$\mathrm{K_1}$

(Table 4 result)

$\mathrm{K_2 = Table[k_2[i,j],\{i,10\},\{j,10\}];}$
$\mathrm{For\ [i = 1, i \leq 10, i = i+1,}$
$\mathrm{For\ [j = 1, j \leq 10, j = j+1,}$

$$\mathrm{A[i,j] = \frac{Gamma[((2*n_2[[j]]*2*n_1[[i]]-8)/2)]*Gamma[((n_2[[j]]*n_1[[i]]-6)/2)]*(((2*n_2[[j]]-4)/2*n_1[[i]]-4))^{(2*n_2[[j]]-4)/2}*\left(\frac{n_2[[j]]-3}{n_1[[i]]-3}\right)^{(n_2[[j]]-3)/2}}{Gamma[((2*n_2[[j]]-4)/2)]*Gamma[((2*n_1[[i]]-4)/2)]*Gamma[((n_2[[j]]-3)/2)]*Gamma[((n_1[[i]]-3)/2)]};}$$

$$\mathrm{P_2[i,j] = FindRoot[NIntegrate[A[i,j]*\left(\frac{0.5*x^{((2*n_2[[j]]-4)/4)-2}*(z/x)^{((n_2[[j]]-3)/2)-1}}{(1+((2*n_2[[j]]-4)/2*n_1[[i]]-4)*\sqrt{x})^{(2*n_2[[j]]*2*n_1[[i]]-8)/2}(1+((n_2[[j]]-3)/n_1[[i]]-3)*(z/x))^{\frac{n_2[[j]]*n_1[[i]]-6}{2}}}\right),}$$

$\mathrm{\{z,0,t\},\{x,0,\infty\}] == 0.975,\{t,0,100\}];}$
$\mathrm{PP_2[i,j] = t/.P_2[i,j];}$
$\mathrm{k_2[i,j] = Sqrt\lfloor PP_2[i,j]*(((2*n_2[[j]]-4)/2*n_1[[i]]-4))^2*(((n_2[[j]]-3)/n_1[[i]]-3))*(((n_1[[i]]-1)/n_2[[j]]-1))^3\rfloor;}$
]]
$\mathrm{K_2}$

(Table 5 result)

$K_3 = \mathrm{Table}[k_3[i,j],\{i,10\},\{j,10\}];$

$\mathrm{For}\ [i = 1, i \leq 10, i = i + 1,$

$\mathrm{For}\ [j = 1, j \leq 10, j = j + 1,$

$$A[i,j] = \frac{\mathrm{Gamma}[((2 * n_2[[j]] * 2 * n_1[[i]] - 8)/2)] * \mathrm{Gamma}[((n_2[[j]] * n_1[[i]] - 6)/2)] * (((2 * n_2[[j]] - 4)/2 * n_1[[i]] - 4))^{(2*n_2[[j]]-4)/2} * (((n_2[[j]] - 3)/n_1[[i]] - 3))^{(n_2[[j]]-3)/2}}{\mathrm{Gamma}[((2 * n_2[[j]] - 4)/2)] * \mathrm{Gamma}[((2 * n_1[[i]] - 4)/2)] * \mathrm{Gamma}[((n_2[[j]] - 3)/2)] * \mathrm{Gamma}[((n_1[[i]] - 3)/2)]};$$

$$P_3[i,j] = \mathrm{FindRoot}[\mathrm{NIntegrate}[A[i,j] * \left(\frac{0.5 * x^{((2*n_2[[j]]-4)/4)-2} * (z/x)^{((n_2[[j]]-3)/2)-1}}{(1 + ((2 * n_2[[j]] - 4)/2 * n_1[[i]] - 4) * \sqrt{x})^{(2*n_2[[j]]*2*n_1[[i]]-8)/2}(1 + ((n_2[[j]] - 3)/n_1[[i]] - 3) * (z/x))^{n_2[[j]]*n_1[[i]]-6/2}}\right),$$

$\{z,0,t\},\{x,0,\infty\}] == 0.99,\{t,0,100\}];$

$PP_3[i,j] = t/.P_3[i,j];$

$k_3[i,j] = \mathrm{Sqrt}\lfloor PP_3[i,j] * (((2 * n_2[[j]] - 4)/2 * n_1[[i]] - 4))^2 * (((n_2[[j]] - 3)/n_1[[i]] - 3)) * (((n_1[[i]] - 1)/n_2[[j]] - 1))^3 \rfloor;$

]]

$K_3$

(Table 6 result)