國 立 交 通 大 學

電信工程學系

碩 士 論 文

適用於多重輸出入正交分頻多工調變之空頻碼設計

Design of Space-Frequency Codes for MIMO-OFDM Systems

研究生：賴俊池

指導教授：王忠炫博士

中 華 民 國 九十六 年 一 月

適用於多重輸出入正交分頻多工調變之空頻碼設計

Design of Space-Frequency Codes for MIMO-OFDM Systems

研究生　：賴俊池　　Student：C. C. Lai

指導教授：王忠炫　　Advisor：C. H. Wang

國立交通大學

電信工程學系碩士班

碩士論文

A Thesis
Submitted to Department of Communication Engineering
College of Electrical and Computer Engineering
National Chiao Tung University
in partial Fulfillment of the Requirements
for the Degree of
Master of Science
in
Electrical Engineering
January 2007
Hsinchu, Taiwan, Republic of China

中華民國九十六年一月

適用於多重輸出入正交分頻多工調變之空頻碼設計

學生：賴俊池　　　　　　　　　　　指導教授：王忠炫

國立交通大學電信工程學系碩士班

摘要

本論文主要探討在頻率選擇性衰減通道 (frequency-selective fading channel) 環境下，針對多重輸出入正交分頻多工調變 (MIMO-OFDM) 系統設計空頻碼 (space-frequency codes)。藉由使用中央極限定理，亦即當隨機變數個數達到特定數量時，機率密度函數將趨近高斯分布之特性，計算空頻碼之成對錯誤機率並提供對應之碼字設計準則，亦即秩 (rank) 準則和距離 (distance) 準則；吾人可依通道變化速度快慢之差異，將通道狀況區分為快速衰減通道 (rapid fading channel) 和類靜止通道 (quasi-static channel) 兩種，各自通道所對應之設計準則也將被討論。在詳細分析下，可知利用中央極限定理分析空頻碼所得之碼字設計法則和空時碼之設計法則相似。在系統模擬部分，將比較在討論之通道環境裡，不同設計方式所得之籬柵碼位元錯誤率的比較情形。

Design of Space-Frequency Codes for MIMO-OFDM Systems

Student: C. C. Lai **Advisor: C. H. Wang**

Department of Communication Engineering

National Chiao Tung University

Abstract

A space-frequency coded multiple-input multiple-output orthogonal frequency division multiplexing (MIMO-OFDM) system on frequency-selective fading channel is considered. By exploiting the central limit theory that the random variables can approach Gaussian distribution, which is used when the number of random variables is large enough, we will analyze the pairwise error probability of space-frequency codes and provide the corresponding design criteria, which mean rank criterion and distance criterion. According to the channel variation speed, rapid fading channel and quasi-static fading channel are distinguished, the individual design criteria are also presented. From the code design of these case, it is deduced that the code construction of space-frequency codes is similar to the code construction of space-time codes by applying the central limit theory. The simulation results are presented to compare the various trellis code designed from various criteria in the channel environments.

Acknowledgement

I am deeply grateful to my advisor Dr. C. H. Wang for his enthusiastic guidance and great patience in research. I also appreciate my friends for their ardent assistance. Finally, I feel an immense gratitude to my parents for their encouragement and love.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In future wireless communication, the demand for safer and higher high data rate is rapidly increasing. Recently there has been much interest in applying multiple transmitter and receiver antennas in broadband wireless communication techniques, and this is also called multiple-input multiple-output (MIMO) system, since the unfavorable effects of the wireless propagation environments can be largely reduced by using multiple transmitter and receiver antennas. In order to raise security of communication, numbers of coding and modulation are adopted. One popular method of them is space-time coding. It is a coding technique executed in space and time domains to introduce related transmitted signals between various antennas at various time slots. There are two kinds of design standards of space-time codes commonly [1]. When the product of the codeword distance matrix and the number of the receiver antennas does not exceed 4 [2], the rank and determinant criteria are used. Oppositely, the rank and trace criteria will be applied.

In wireless environments, the channel is almost frequency-selective, so intersymbol interference (ISI) would exist. Orthogonal frequency division multiplexing (OFDM) is one of the techniques used to lower the effect of ISI and can transform the frequency-selective fading channel into the set of correlated flat fading channels. It is clear that the correlation between the different subcarriers in one OFDM symbol for the same transmitter and receiver antenna pair is connection with variances of the channel taps and the difference of the frequencies. OFDM not only provides high data transmission rate and owns high bandwidth efficiency, but it also can effectively solves ISI by adding sufficient cyclic prefix (CP). In order to obtain the benefits of MIMO system and OFDM modulation, MIMO-OFDM has been acquired lots of attention, and coding for MIMO-OFDM system on frequency-selective fading channel will be required.

Generally speaking, coding across OFDM subcarriers and multiple transmitter antennas is called space-frequency coding, both space and frequency diversity are expected in frequency-selective fading channels. [3] has not only told us how to design the full diversity in the space-frequency scheme, but it has also shown that most of space-time codes designed to achieve full spatial diversity over frequency-nonselective channels are not validly applied in MIMO-OFDM system, since these codes do not exploit the frequency diversity. By the design criteria of [3], one construction of space-frequency codes was presented, the codewords were generated by multiplying particular columns of DFT-matrix with the modulated sample vectors [4]. Even though the method can achieve the maximum diversity, it is not easy to construct the codewords and the code rate is a little low. Later, many papers designed to achieve maximum diversity gain and own high code rate were submitted, such as [5]-[7].

In my thesis, I will exploit the spirit of the central limit theory presented in [1] to gain the design criteria of space-frequency codes. The organization of the thesis is shown below. In chapter II, the common space-frequency coded system, it's pairwise error probability and the corresponding design criteria will be introduced. Chapter III describes how to get the design criteria by using the central limit theory in great detail. In chapter IV, some simulation results are presented. Finally, we give some conclusions.

Notation: Superscripts T and H denote transpose and conjugate transpose respectly.

Chapter 2

The Space-Frequency Codes

2.1 Space-Frequency Coded Systems

In this section, the MIMO-OFDM system, the channel model and the relation between the transmittd and received signals will be introduced.

The space-frequency coded OFDM-based multi-antenna system is represented in Fig 2.1. This system consists of M_T transmitter antennas and M_R receiver antennas, and N subcarriers is considered in each OFDM scheme. Suppose that frequency-selective fading channels have L independent delay paths and the same power delay profile for each transmitter and receiver antenna link. The fading gains are constant over each OFDM symbol.

The impulse response of the channel can be represented as the tap matrix $\mathbf{H}_l$, which is given by

$$\mathbf{H}_l = \begin{bmatrix} h_{0,0}(l) & h_{0,1}(l) & \cdots & h_{0,M_T-1}(l) \\ h_{1,0}(l) & h_{1,1}(l) & \cdots & h_{1,M_T-1}(l) \\ \vdots & \vdots & \ddots & \vdots \\ h_{M_R-1,0}(l) & h_{M_R-1,1}(l) & \cdots & h_{M_R-1,M_T-1}(l) \end{bmatrix} \tag{2.1}$$

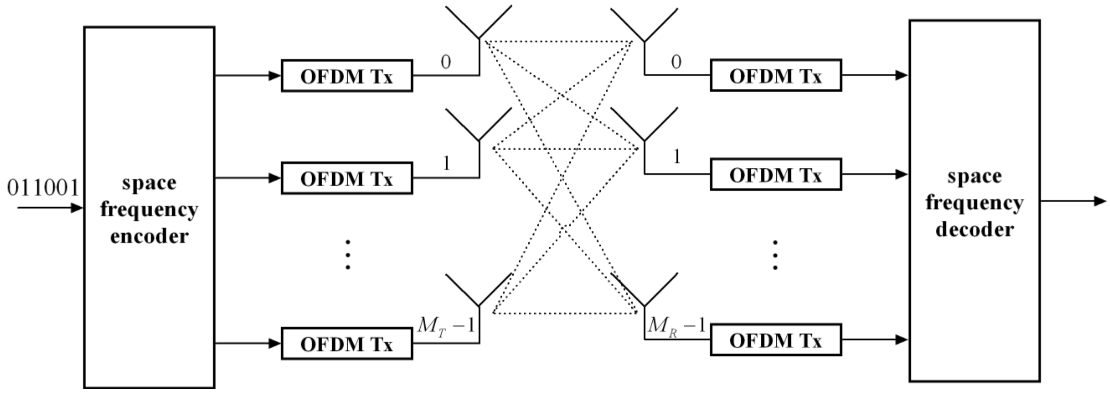

Figure 2.1: The space-frequency coded MIMO-OFDM system.

where $h_{j,i}(l)$ is the value of the impulse response from the i-th transmitter antenna to the j-th receiver antenna at time l ($l = 0, 1, \cdots, L-1$). Each $h_{j,i}(l)$ is modeled as the circularly symmetric complex Gaussian random variable with zero mean and variance one. Assume that the MIMO channel is spatially uncorrelated, i.e., $h_{j,i}(l)$ is independent for different indices i and j. After defining the time matrix of the channel, we can know that the frequency response matrix of the MIMO channel $\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)$ is

$$\begin{aligned}
\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right) &= \sum_{l=0}^{L-1} \mathbf{H}_l e^{-j\frac{2\pi}{N}kl} \\
&= \begin{bmatrix} H_{0,0}(\frac{k}{N}) & H_{0,1}(\frac{k}{N}) & \cdots & H_{0,M_T-1}(\frac{k}{N}) \\ H_{1,0}(\frac{k}{N}) & H_{1,1}(\frac{k}{N}) & \cdots & H_{1,M_T-1}(\frac{k}{N}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M_R-1,0}(\frac{k}{N}) & H_{M_R-1,1}(\frac{k}{N}) & \cdots & H_{M_R-1,M_T-1}(\frac{k}{N}) \end{bmatrix}
\end{aligned} \tag{2.2}$$

where the element $H_{j,i}(\frac{k}{N})$ ($k = 0, 1, \cdots, N-1$) denotes the frequency response of the channel from the i-th transmitter antenna to the j-th receiver antenna.

In the MIMO-OFDM system, the data streams are all OFDM-modulated and OFDM-demodulated. The OFDM modulator adopts an N-point IFFT to the consecutive source samples and then adds sufficient CP. After passing the channel, the OFDM demodulator eliminates CP and applies an N-point

FFT to recovery the consecutive source samples. Assume that the transmitted codeword matrix in one OFDM symbol is

$$\begin{aligned}\mathbf{C} &= \begin{bmatrix} c_0^{(0)} & c_1^{(0)} & \cdots & c_{N-1}^{(0)} \\ c_0^{(1)} & c_1^{(1)} & \cdots & c_{N-1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_0^{(M_T-1)} & c_1^{(M_T-1)} & \cdots & c_{N-1}^{(M_T-1)} \end{bmatrix} \\ &= \begin{bmatrix} \underline{c}_0 & \underline{c}_1 & \cdots & \underline{c}_{N-1} \end{bmatrix}\end{aligned} \tag{2.3}$$

where each element of the transmitted matrix $c_k^{(i)}$ denotes the data transmitted from the i-th antenna on the k-th tone and is taken from a finite complex alphabet set in which the power of each alphabet is unity. In the second equality, $\underline{c}_k = [c_k^{(0)} \; c_k^{(1)} \; \cdots \; c_k^{(M_T-1)}]^T$ represents the k-th frequency vector, and all of the M_t samples are sent to the channel at the same time. On the basis of the described framework, the reconstructed data vector at the k-th tone, i.e., $\underline{r}_k = [r_k^{(0)} \; r_k^{(1)} \; \cdots \; r_k^{(M_R-1)}]^T$, is given by

$$\underline{r}_k = \sqrt{E_s}\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\underline{c}_k + \underline{z}_k, \quad k = 0, 1, \cdots, N-1 \tag{2.4}$$

where $\underline{z}_k = [z_k^{(0)} \; z_k^{(1)} \; \cdots \; z_k^{(M_R-1)}]^T$ is an addtive complex-valued Gaussian noise vector and satisfies

$$E[\underline{z}_k \, \underline{z}_{k'}^H] = \begin{cases} \sigma_n^2 \mathbf{I}_{M_R} & k = k' \\ 0 & k \neq k' \end{cases} \tag{2.5}$$

with $\mathbf{I}_{M_R}$ denoting the M_R-by-M_R identity matrix and σ_n^2 is the noise power. We also assume that the noise is uncorrelated for different receiver antennas. From (2.4), the space-frequency system is therefore equivalent to Fig2.2.

If we assume that the channel state information is only perfectly known at the receiver, the maximum likelihood decoder of the space-frequency system is

$$\hat{\mathbf{C}} = \arg\min_{\mathbf{C}} \sum_{k=0}^{N-1} ||\underline{r}_k - \sqrt{E_s}\mathbf{H}(e^{j\frac{2\pi}{N}k})\underline{c}_k||^2 \tag{2.6}$$

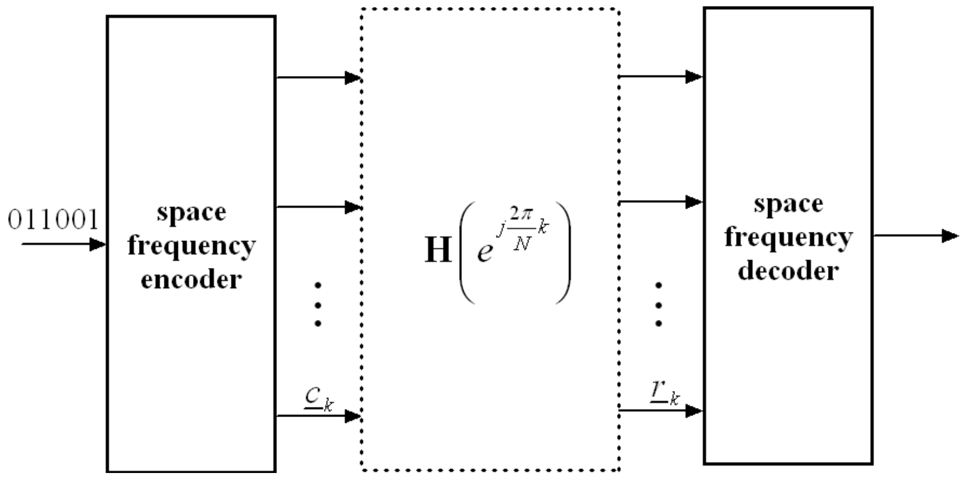

Figure 2.2: The equivalent space-frequency MIMO-OFDM system.

where E_s is the energy of every transmitted signal. (2.6) tells us that the decoder deals with that the minimization which is over all the likely codewords.

2.2 Pairwise Error Probability and Design criteria

Based on the previously mentioned system, the pairwise error probability and the corresponding design criteria of the space-frequency system are derived in this section.

It is assumed that the codeword $\mathbf{C}$ is transmitted, and $\mathbf{E} = [\underline{e}_0 \underline{e}_1 \cdots \underline{e}_{N-1}]$ is another legal codeword. Given that the channel state information is known at the receiver, the pairwise error probability that the codeword $\mathbf{E}$ is decoded is

$$P\left(\mathbf{C} \rightarrow \mathbf{E}|\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\right) = Q\left(\sqrt{\frac{E_s}{2\sigma_n^2} d^2\left(\mathbf{C}, \mathbf{E}|\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\right)}\right) \tag{2.7}$$

where $Q(x)$ is the complementary error function and

$$d^2\left(\mathbf{C},\mathbf{E}|\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\right)=\sum_{k=0}^{N-1}\parallel \mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\left(\underline{c}_k-\underline{e}_k\right)\parallel^2 \tag{2.8}$$

represents the squared Euclidean distance between $\mathbf{C}$ and $\mathbf{E}$. Then using the approximation

$$Q(x)\leq\frac{1}{2}exp\{-\frac{x^2}{2}\},x\geq 0 \tag{2.9}$$

to (2.7), so the upper bound of the pairwise error probability conditioned on the channel condition is

$$P\left(\mathbf{C}\rightarrow\mathbf{E}|\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\right)\leq\frac{1}{2}e^{-\frac{E_s}{4\sigma_n^2}d^2\left(\mathbf{C},\mathbf{E}|\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\right)}. \tag{2.10}$$

We proceed to define $\underline{y}_k=\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\left(\underline{c}_k-\underline{e}_k\right)$ for $k=0,\cdots,N-1$ and the vector $\underline{y}$ is constituted as

$$\underline{y}=\left[\underline{y}_0^T\ \underline{y}_1^T\ \cdots\ \underline{y}_{N-1}^T\right]^T. \tag{2.11}$$

From (2.11), we can gain $d^2\left(\mathbf{C},\mathbf{E}|\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\right)=|\ \underline{y}\ |^2$ and (2.10) can be rewrited as

$$P\left(\mathbf{C}\rightarrow\mathbf{E}|\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\right)\leq\frac{1}{2}e^{-\frac{E_s}{4\sigma_n^2}|\ \underline{y}\ |^2}. \tag{2.12}$$

Because all the channel tap matrices $\mathbf{H}_l$ are i.i.d. complex gaussian random variables, all the $\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)$ are jointly gaussian random variables. After averaging all the random variables, we can get the upper bound of the pairwise error probability is

$$P\left(\mathbf{C}\rightarrow\mathbf{E}\right)\leq\prod_{i=0}^{rank(\mathbf{C}_{\underline{y}})-1}\left(1+\lambda_i(\mathbf{C}_{\underline{y}})\frac{E_s}{4\sigma_n^2}\right)^{-1} \tag{2.13}$$

in which $\lambda_i(\mathbf{C}_{\underline{y}})$ and $rank(\mathbf{C}_{\underline{y}})$ respectively denote the i-th nonzero eigenvalue and the rank of $\mathbf{C}_{\underline{y}}$ which represents the covariance matrix of $\underline{y}$ and is

$$\mathbf{C}_{\underline{y}}=\mathbf{I}_{M_R}\otimes\mathbf{R}_{\underline{y}} \tag{2.14}$$

where $\otimes$ denotes the Kronecker product and

$$\mathbf{R}_{\underline{y}} = F(\mathbf{C},\mathbf{E})\,F^H(\mathbf{C},\mathbf{E}) = \sum_{l=0}^{L-1}\left[\mathbf{D}^l(\mathbf{C}-\mathbf{E})^T(\mathbf{C}-\mathbf{E})^*\mathbf{D}^{l^H}\right]. \tag{2.15}$$

In (2.15), the matrix $\mathbf{D}$ and $F(\mathbf{C},\mathbf{E})$ respectively indicate

$$\mathbf{D} = diag\left\{e^{-j\frac{2\pi}{N}k}\right\}_{k=0}^{N-1} \tag{2.16}$$

and

$$F(\mathbf{C},\mathbf{E}) = \left[(\mathbf{C}-\mathbf{E})^T \;\; \mathbf{D}^1(\mathbf{C}-\mathbf{E})^T \;\; \cdots \;\; \mathbf{D}^{L-1}(\mathbf{C}-\mathbf{E})^T\right]. \tag{2.17}$$

We can easily know that at high SNR, the upper bound of the pairwise error probability of the space-frequency coding (2.13) can be simplified as

$$P(\mathbf{C}\rightarrow\mathbf{E}) \le \left(\frac{E_s}{4\sigma_n^2}\right)^{-rank(\mathbf{C}_{\underline{y}})}\left(\prod_{i=0}^{rank(\mathbf{C}_{\underline{y}})-1}\lambda_i\left(\mathbf{C}_{\underline{y}}\right)\right)^{-1}. \tag{2.18}$$

From (2.18), we can define the performance criteria as follows.

- **Rank (diversity gain) criterion**: Maximize the minimum rank over all pairs of any two different codewords as large as possible.

- **Product (coding gain) criterion**: The minimum value of the product $\left(\prod_{i=0}^{rank(\mathbf{C}_{\underline{y}})-1}\lambda_i\left(\mathbf{C}_{\underline{y}}\right)\right)$ for any pairs of distinct codewords should be maximized.

Generally speaking, $N > M_T L$ is assumed, which is common in the space-frequency coded MIMO-OFDM system. According to the property of Kronecker product, we can know

$$\begin{aligned} rank(\mathbf{C}_{\underline{y}}) &= rank(\mathbf{I}_{M_R})\cdot rank(\mathbf{R}_{\underline{y}}) \\ &\le M_R M_T L. \end{aligned} \tag{2.19}$$

So the maximum diversity gain of the method for this design is the product of number of transmitter antennas, receiver antennas and channel taps. The proposed rank criterion can efficiently take advantage of both the spatial diversity and the channel diversity. The coding gain can be obtained by means of the theorem of the Kronecker product, which is that each eigenvalue of $\mathbf{C}_{\underline{y}}$ is one of the eigenvalues $\mathbf{R}_{\underline{y}}$ with multiplicity M_R.

Chapter 3

Proposed Design Criteria of Space-Frequency Codes

3.1 System Model

In order to easily state the following content, we must define newly the space-frequency system, so the detailed architecture is established in this section. Because we want to encode and decode during several OFDM symbols, the channel environments and the corresponding codeword matrices will be expanded in the suitable form.

It is assumed that frequency-selective fading channel is constant over every OFDM symbol. In terms of the fading amplitude variation speed, rapid fading channel and quasi-static fading channel are distinguished, which are called general cases. For rapid fading channel, the channel gains change randomly from one OFDM symbol to another. It is referred to as quasi-static fading if the fading gains are maintained during several symbols. Specifically, coding for one OFDM symbol is an exception, which is special case, for it has nothing to do with the fading gain variation speed. At each OFDM symbol, the L independent channel taps and the delay power profile are the same in each link of transmitter and receiver antenna. Let $h_{j,i}^{m}(l)$ be the l-th channel tap of the link transmitted from transmitter antenna i to receiver antenna j during the m-th OFDM symbol, so the corresponding impulse response of the channel is modeled as

$$h_{j,i}^{m}(\tau) = \sum_{l=0}^{L-1} h_{j,i}^{m}(l)\delta(\tau - \tau_l) \tag{3.1}$$

where τ_l is the l-th delay time in the real channel environment. All the fading gain $h_{j,i}^{m}(l)$'s are modeled as circularly symmetric complex Gaussian random variables with zero mean and variance δ_l^2, i.e., $E\left\{|h_{j,i}^{m}(l)|^2\right\} = \delta_l^2$. In the following contents, it is assumed that the powers of the L coefficients in each link are equal powers and $\sum_{l=0}^{L-1} \delta_l^2 = 1$. The MIMO channel is also spatially uncorrelated as Chapter 2, so the channel taps are independent for different transmitter antenna index i and receiver antenna index j. After (3.1) is well defined, the frequency response of the channel is

$$H_{j,i}^{m}(f) = \sum_{l=0}^{L-1} h_{j,i}^{m}(l)e^{-\mathbf{j}2\pi f\tau_l} \tag{3.2}$$

where $\mathbf{j} = \sqrt{-1}$.

In the MIMO-OFDM system, each transmitted codeword during M symbols can be defined as

$$\mathbf{C} = \left[\underline{c}_0^1 \cdots \underline{c}_{N-1}^1 \; \underline{c}_0^2 \cdots \underline{c}_{N-1}^2 \cdots \underline{c}_k^m \cdots \underline{c}_0^M \cdots \underline{c}_{N-1}^M\right] \tag{3.3}$$

where $\underline{c}_k^m$ denotes the M_T-by-1 vector transmitted over the k-th subcarrier in the m-th OFDM symbol. At the receiver, the reconstructed signal vector $\underline{r}_k^m$ is given by

$$\underline{r}_k^m = \sqrt{E_s}\hat{\mathbf{H}}^m(k)\underline{c}_k^m + \underline{z}_k^m, k = 0, 1, \cdots, N-1; m = 1, 2, \cdots, M \tag{3.4}$$

where

$$\hat{\mathbf{H}}^m(k) = \begin{bmatrix} H^m_{0,0}(k) & H^m_{0,1}(k) & \cdots & H^m_{0,M_T-1}(k) \\ H^m_{1,0}(k) & H^m_{1,1}(k) & \cdots & H^m_{1,M_T-1}(k) \\ \vdots & \vdots & \ddots & \vdots \\ H^m_{M_R-1,0}(k) & H^m_{M_R-1,1}(k) & \cdots & H^m_{M_R-1,M_T-1}(k) \end{bmatrix}. \tag{3.5}$$

Each element of the matrix $\hat{\mathbf{H}}^m(k)$ is the corresponding frequency response at the k-th subcarrier transmitted from the i-th transmitter antenna to the j-th receiver antenna during the m-th symbol, which is expressed as

$$H^m_{j,i}(k) = \sum_{l=0}^{L-1} h^m_{j,i}(l)\, e^{-\mathbf{j}2\pi k \Delta f \tau_l} \tag{3.6}$$

where $\Delta f = 1/T$ is the subcarrier spacing and T is the OFDM symbol duration.

It is assumed that the channel state information is also correctly known at the receiver but not at the transmitter. Therefore, when we encode the codewords over M OFDM symbols and decode them, the ML decoder can be described as

$$\hat{\mathbf{C}} = \arg\min_{\mathbf{C}} \sum_{m=1}^{M} \sum_{k=0}^{N-1} ||\underline{r}^m_k - \sqrt{E_s}\hat{\mathbf{H}}^m(k)\underline{c}^m_k||^2 \tag{3.7}$$

3.2 Proposed Design Criteria of Space-Frequency Codes for Special Case: One OFDM Symbol

In this section, coding over one OFDM symbol is discussed, and the design criteria will be presented. The time index m will be omitted in order to make the analysis convenient, so the corresponding codeword form is (2.3).

The ML decoder select as its estimate an erroneous codeword $\mathbf{E}$ when the codeword $\mathbf{C}$ is transmitted, and this holds if

$$\sum_{k=0}^{N-1} ||\underline{r}_k - \sqrt{E_s}\hat{\mathbf{H}}(k)\underline{c}_k||^2 \geq \sum_{k=0}^{N-1} ||\underline{r}_k - \sqrt{E_s}\hat{\mathbf{H}}(k)\underline{e}_k||^2 \tag{3.8}$$

where

$$\hat{\mathbf{H}}(k) = \begin{bmatrix} H_{0,0}(k) & H_{0,1}(k) & \cdots & H_{0,M_T-1}(k) \\ H_{1,0}(k) & H_{1,1}(k) & \cdots & H_{1,M_T-1}(k) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M_R-1,0}(k) & H_{M_R-1,1}(k) & \cdots & H_{M_R-1,M_T-1}(k) \end{bmatrix}. \tag{3.9}$$

Then (3.8) is rewrited to

$$\sum_{k=0}^{N-1} 2\text{Re}\left\{\sqrt{E_S}\left(\underline{z}_k\right)^H \hat{\mathbf{H}}\left(k\right)\left(\underline{c}_k - \underline{e}_k\right)\right\} \geq \sum_{k=0}^{N-1} |\sqrt{E_S}\hat{\mathbf{H}}\left(k\right)\left(\underline{c}_k - \underline{e}_k\right)|^2 \tag{3.10}$$

where $Re\{\cdot\}$ represents the real part of a complex value. So the pairwise error probability conditioned on the channel state information is

$$P\left(\mathbf{C} \rightarrow \mathbf{E}|\text{all } \hat{\mathbf{H}}\left(k\right)\right) \leq Q\left(\sqrt{\frac{E_s}{2\sigma_n^2}\sum_{k=0}^{N-1}|\sqrt{E_S}\hat{\mathbf{H}}\left(k\right)\left(\underline{c}_k - \underline{e}_k\right)|^2}\right). \tag{3.11}$$

Then using the property (2.9), the conditional pairwise error probability becomes

$$P\left(\mathbf{C} \rightarrow \mathbf{E}|\mathbf{H}\left(e^{j\frac{2\pi}{N}k}\right)\right) \leq \frac{1}{2}e^{-\frac{E_s}{4\sigma_n^2}\sum_{k=0}^{N-1}\sum_{j=0}^{M_R-1}|\sum_{i=0}^{M_T-1}H_{j,i}(k)\left(c_k^{(i)}-e_k^{(i)}\right)|^2}. \tag{3.12}$$

Let us adapt $H_{j,i}(k)$ for

$$H_{j,i}(k) = \sum_{l=0}^{L-1} h_{j,i}(l)e^{-\mathbf{j}2\pi k\Delta f\tau_l} = \underline{h}_{j,i}^T\underline{G}(k) \tag{3.13}$$

where

$$\underline{h}_{j,i} = \left[h_{j,i}(0)\ h_{j,i}(1)\ \cdots\ h_{j,i}(L-1)\right]^T \tag{3.14}$$

and

$$\begin{aligned} \underline{G}(k) &= \left[e^{-\mathbf{j}2\pi k\Delta f\tau_0}\ e^{-\mathbf{j}2\pi k\Delta f\tau_1}\ \cdots\ e^{-\mathbf{j}2\pi k\Delta f\tau_{L-1}}\right]^T \\ &= \left[w^{k\tau_0}\ w^{k\tau_1}\ \cdots\ w^{k\tau_{L-1}}\right]^T, w = e^{-\mathbf{j}2\pi\Delta f}. \end{aligned} \tag{3.15}$$

It is apparent that

$$\begin{aligned} \left|\sum_{i=0}^{M_T-1} H_{j,i}(k)\left(c_k^{(i)} - e_k^{(i)}\right)\right|^2 &= \left|\sum_{i=0}^{M_T-1} \underline{h}_{j,i}^T \underline{G}(k)\left(c_k^{(i)} - e_k^{(i)}\right)\right|^2 \\ &= \left|\sum_{i=0}^{M_T-1} \underline{h}_{j,i}^T \underline{\hat{G}}(k,i)\right|^2 \end{aligned} \tag{3.16}$$

where

$$\underline{\hat{G}}(k,i) = \underline{G}(k)\left(c_k^{(i)} - e_k^{(i)}\right), \tag{3.17}$$

so

$$\begin{aligned} \left|\sum_{i=0}^{M_{T-1}} H_{j,i}(k)\left(c_k^{(i)} - e_k^{(i)}\right)\right|^2 &= \left[\sum_{i=0}^{M_T-1} \underline{h}_{j,i}^T \underline{\hat{G}}(k,i)\right]\left[\sum_{i=0}^{M_T-1} \underline{h}_{j,i}^T \underline{\hat{G}}(k,i)\right]^H \\ &= \left[\underline{\tilde{h}}_j^{\mathrm{T}} \underline{\tilde{G}}(k)\right]\left[\underline{\tilde{h}}_j^{\mathrm{T}} \underline{\tilde{G}}(k)\right]^H \end{aligned} \tag{3.18}$$

where

$$\underline{\tilde{h}}_j = \left[\underline{h}_{j,0}^T\ \underline{h}_{j,1}^T\ \cdots\ \underline{h}_{j,M_T-1}^T\right]^T \tag{3.19}$$

and

$$\underline{\tilde{G}}(k) = \left[\underline{\hat{G}}(k,0)^T\ \underline{\hat{G}}(k,1)^T\ \cdots\ \underline{\hat{G}}(k,M_T-1)^T\right]^T \tag{3.20}$$

are two M_T-by-1 vectors. Therefore, the pairwise error probability conditioned on the channel is

$$P\left(\mathbf{C} \to \mathbf{E} | \text{all } \underline{h}_{j,i}\right) \le \frac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{k=0}^{N-1} \sum_{j=0}^{M_R-1} \underline{\tilde{h}}_j^T \underline{\tilde{G}}(k) \underline{\tilde{G}}(k)^H \left(\underline{\tilde{h}}_j^T\right)^H}. \tag{3.21}$$

In order to use Gaussian approximation easily, we must modify the exponent of (3.21) as

$$P\left(\mathbf{C} \to \mathbf{E} | \text{all } \underline{h}_{j,i}\right) \le \frac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{j=0}^{M_R-1} \underline{\tilde{h}}_j^T \left[\sum_{k=0}^{N-1} \underline{\tilde{G}}(k) \underline{\tilde{G}}(k)^H\right] \left(\underline{\tilde{h}}_j^T\right)^H}. \tag{3.22}$$

Because $\underline{\tilde{G}}(k)\underline{\tilde{G}}(k)^H$ is Hermitian, so $\sum_{k=0}^{N-1} \underline{\tilde{G}}(k)\underline{\tilde{G}}(k)^H$ is also Hermitian and nonegative definite [8]. Hence, there exists a unitary matrix $\mathbf{V}$ that diagonalizes $\sum_{k=0}^{N-1} \underline{\tilde{G}}(k)\underline{\tilde{G}}(k)^H$, which is also called Spectral Theorem [9], and we can gain

$$\sum_{k=0}^{N-1} \underline{\tilde{G}}(k)\underline{\tilde{G}}(k)^H = \sum_{n=0}^{M_T L-1} \lambda_n \underline{v}_n \underline{v}_n^H \tag{3.23}$$

where $\lambda_n,\ n = 0, \ldots, M_T L-1$ are the eigenvalues of the matrix $\sum_{k=0}^{N-1} \underline{\tilde{G}}(k)\underline{\tilde{G}}(k)^H$ and assumed that $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{M_T L-1} \geq 0$. The corresponding eigenvectors $\left\{\underline{v}_0, \underline{v}_1, \cdots, \underline{v}_{M_T L-1}\right\}$ develop an orthonormal basis of $M_T L$-dimensional vector space, i.e.,

$$\underline{v}_n \cdot \underline{v}_{n'} = \begin{cases} 1, \text{ for } n = n' \\ 0, \text{ for } n \neq n' \end{cases}. \tag{3.24}$$

Then, the pairwise error probability can be replaced by

$$\begin{aligned} P\left(\mathbf{C} \rightarrow \mathbf{E} | \text{all } \underline{h}_{j,i}\right) &\leq \frac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{j=0}^{M_R-1} \underline{\tilde{h}}_j^T \left[\sum_{n=0}^{M_T L-1} \lambda_n \underline{v}_n \underline{v}_n^H\right] \left(\underline{\tilde{h}}_j^T\right)^H} \\ &= \frac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{j=0}^{M_R-1} \sum_{n=0}^{M_T L-1} \lambda_n \underline{\tilde{h}}_j^T \underline{v}_n \underline{v}_n^H \left(\underline{\tilde{h}}_j^T\right)^H} \\ &= \frac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{j=0}^{M_R-1} \sum_{n=0}^{M_T L-1} \lambda_n |\beta_{j,n}|^2} \end{aligned} \tag{3.25}$$

where $\beta_{j,n} = \underline{\tilde{h}}_j^T \underline{v}_n$ are independent zero mean complex Gaussian random variables with variance $\frac{1}{2L}$ per dimension. Let r denote the rank of the matrix $\sum_{k=0}^{N-1} \underline{\tilde{G}}(k)\underline{\tilde{G}}(k)^H$. $A = \frac{E_s}{4\sigma_n^2}$ and $d = \sum_{j=0}^{M_R-1} \sum_{n=0}^{r-1} \lambda_n |\beta_{j,n}|^2$ are also considered. If rM_R is large enough to use Gaussian approximation, d can be close to Gaussian random variable with mean

$$\begin{aligned} \mu_d &= \sum_{j=0}^{M_R-1} \sum_{n=0}^{r-1} \lambda_n E\left[|\beta_{j,n}|^2\right] \\ &= \sum_{j=0}^{M_R-1} \sum_{n=0}^{r-1} \lambda_n \frac{1}{L} \\ &= \frac{M_R}{L} \sum_{n=0}^{r-1} \lambda_n \\ &\triangleq \frac{M_R}{L} x_1 \end{aligned} \tag{3.26}$$

and variance

$$
\begin{aligned}
\sigma_d^2 &= \sum_{j=0}^{M_R-1} \sum_{n=0}^{r-1} \lambda_n^2 \mathrm{var}\left[|\beta_{j,n}|^2\right] \\
&= \sum_{j=0}^{M_R-1} \sum_{n=0}^{r-1} \lambda_n^2 \tfrac{1}{L^2} \\
&= \tfrac{M_R}{L^2} \sum_{n=0}^{r-1} \lambda_n^2 \\
&\triangleq \tfrac{M_R}{L^2} x_2 .
\end{aligned}
\tag{3.27}
$$

Using the equation

$$
\int_{d=0}^{\infty} e^{-Ad} p(d) d(d) = e^{\frac{1}{2}A^2\sigma_d^2 - A\mu_d} Q\left(\frac{A\sigma_d^2 - \mu_d}{\sigma_d}\right), \tag{3.28}
$$

the whole pairwise error probability can be upper-bound by

$$
\begin{aligned}
P\left(\mathbf{C} \to \mathbf{E}\right) \leq\ & \tfrac{1}{2} \exp\left\{ \tfrac{1}{2}\left(\tfrac{E_s}{4\sigma_n^2}\right)^2 \tfrac{M_R}{L^2} x_2 - \left(\tfrac{E_s}{4\sigma_n^2}\right) \tfrac{M_R}{L} x_1 \right\} \cdot \\
& Q\left(\left(\tfrac{E_s}{4\sigma_n^2}\right) \sqrt{\tfrac{M_R}{L^2} x_2} - \frac{\frac{M_R}{L} x_1}{\sqrt{\frac{M_R}{L^2} x_2}} \right)
\end{aligned}
\tag{3.29}
$$

For reasonably high SNR, which satisfies

$$
\left(\frac{E_s}{4\sigma_n^2}\right) \sqrt{\frac{M_R}{L^2} x_2} - \frac{\frac{M_R}{L} x_1}{\sqrt{\frac{M_R}{L^2} x_2}} \geq 0 \Rightarrow \frac{E_s}{4\sigma_n^2} \geq \frac{x_1}{x_2} L, \tag{3.30}
$$

we use the inequality (2.9) to gain

$$
\begin{aligned}
P\left(\mathbf{C} \to \mathbf{E}\right) &\leq \tfrac{1}{4} \exp\left\{ -\tfrac{E_s}{8\sigma_n^2} \tfrac{M_R}{L} x_1 \right\} \\
&= \tfrac{1}{4} \exp\left\{ -\tfrac{E_s}{8\sigma_n^2} \tfrac{M_R}{L} \sum_{n=0}^{r-1} \lambda_n \right\} .
\end{aligned}
\tag{3.31}
$$

Assume that $\rho\left(\mathbf{C}, \mathbf{E}\right)$ is the set of frequency indices such taht $|\underline{c}_k - \underline{e}_k|^2 \neq 0$ and δ is the corresponding number of the frequency indices in which the two codewords differ, it can be easily known that

$$
r \triangleq \min\left\{M_T L, \delta\right\}. \tag{3.32}
$$

Further, the sum of the eigenvalues x_1 is expressed as

$$\begin{aligned}
\sum_{n=0}^{r-1} \lambda_n &= tr\left\{\sum_{k=0}^{N-1} \underline{\tilde{G}}(k)\underline{\tilde{G}}(k)^H\right\} \\
&= \sum_{k=0}^{N-1} tr\left\{\underline{\tilde{G}}(k)\underline{\tilde{G}}(k)^H\right\} \\
&= \sum_{k=0}^{N-1} tr\left\{\left[(\underline{c}_k-\underline{e}_k)\otimes \underline{G}(k)\right]\left[(\underline{c}_k-\underline{e}_k)\otimes \underline{G}(k)\right]^H\right\} \\
&= \sum_{k=0}^{N-1} tr\left\{(\underline{c}_k-\underline{e}_k)(\underline{c}_k-\underline{e}_k)^H\right\}\cdot tr\left\{\underline{G}(k)\underline{G}(k)^H\right\} \\
&= \sum_{k=0}^{N-1} L|\underline{c}_k-\underline{e}_k|^2.
\end{aligned} \tag{3.33}$$

The upper bound of the pairwise error probability can be finally approximated by

$$P(\mathbf{C}\to\mathbf{E}) \le \frac{1}{4}\exp\left\{-\frac{E_s}{8\sigma_n^2}M_R\sum_{k=0}^{N-1}|\underline{c}_k-\underline{e}_k|^2\right\}. \tag{3.34}$$

From (3.34), the Euclidean distance dominates the pairwise error probability at high SNR, so the code design are listed below

- **Rank criterion**: Maximize the minimum rank r over all pairs of any two different codewords as large as possible for Gaussian approximation.

- **Distance criterion**: The minimum value of the Euclidean distance for any pairs of distinct codewords should be maximized.

From the design criteria, we can recognize that the method of encoding and decoding one OFDM symbol is similar to the codeword design for space-time codes when the channel distribution can approach a Gaussian random variable.

3.3 Proposed Design Criteria of Space-Frequency Codes for General Cases: Rapid Fading Channel and Quasi-Static Fading Channel

3.3.1 Rapid Fading Channel

In this section, the code design of space-frequency codes over several OFDM symbols will be derived on rapid fading channel, which means that the fading gains are different during the various OFDM symbols. The system model of this case is organized in section 3.1 without any correction.

From (3.7), when the codeword $\mathbf{C}$ is transmitted, another codeword $\mathbf{E}$ is mistaken for the correct codeword by the ML decoder if

$$\sum_{m=1}^{M}\sum_{k=0}^{N-1}||\underline{r}_k^m-\sqrt{E_s}\hat{\mathbf{H}}^m(k)\underline{c}_k^m||^2\geq\sum_{m=1}^{M}\sum_{k=0}^{N-1}||\underline{r}_k^m-\sqrt{E_s}\hat{\mathbf{H}}^m(k)\underline{e}_k^m||^2. \quad (3.35)$$

Then, the pairwise error probability is

$$P\left(\mathbf{C}\rightarrow\mathbf{E}|\text{all }\hat{\mathbf{H}}^m(k)\right)=Q\left(\sqrt{\frac{E_s}{2\sigma_n^2}d^2\left(\mathbf{C},\mathbf{E}||\text{all }\hat{\mathbf{H}}^m(k)\right)}\right) \quad (3.36)$$

where

$$d^2\left(\mathbf{C},\mathbf{E}|\text{all }\hat{\mathbf{H}}^m(k)\right)=\sum_{m=1}^{M}\sum_{k=0}^{N-1}|\hat{\mathbf{H}}^m(k)\left(\underline{c}_k^m-\underline{e}_k^m\right)|^2. \quad (3.37)$$

Substituting (2.9) into (3.36), we obtain the upper bound of the pairwise error probability on rapid fading channel as

$$\begin{aligned}P\left(\mathbf{C}\rightarrow\mathbf{E}|\text{all }\hat{\mathbf{H}}^m(k)\right) &\leq \tfrac{1}{2}e^{-\frac{E_s}{4\sigma_n^2}\sum_{m=1}^{M}\sum_{k=0}^{N-1}|\hat{\mathbf{H}}^m(k)\left(\underline{c}_k^m-\underline{e}_k^m\right)|^2}\\ &= \tfrac{1}{2}e^{-\frac{E_s}{4\sigma_n^2}\sum_{m=1}^{M}\sum_{k=0}^{N-1}\sum_{j=0}^{M_R-1}|\sum_{i=0}^{M_T-1}H_{j,i}^m(k)\left(c_k^{m,(i)}-e_k^{m,(i)}\right)|^2}\end{aligned} \quad (3.38)$$

where $\underline{c}_k^m - \underline{e}_k^m = \left[c_k^{m,(0)} - e_k^{m,(0)} \;\; c_k^{m,(1)} - e_k^{m,(1)} \;\; \cdots \;\; c_k^{m,(M_T-1)} - e_k^{m,(M_T-1)}\right]^T$. Similar to (3.13), $H_{j,i}^m(k)$ is modified to

$$H_{j,i}^m(k) = \sum_{l=0}^{L-1} h_{j,i}^m(l) e^{-\mathbf{j}2\pi k \Delta f \tau_l} = \left(\underline{h}_{j,i}^m\right)^T \underline{G}(k) \tag{3.39}$$

where

$$\underline{h}_{j,i}^m = \left[h_{j,i}^m(0) \; h_{j,i}^m(1) \; \cdots \; h_{j,i}^m(L-1)\right]^T. \tag{3.40}$$

So we can consequently rewriten $|\sum_{i=0}^{M_T-1} H_{j,i}^m(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right)|^2$ as follows,

$$\begin{aligned} |\sum_{i=0}^{M_T-1} H_{j,i}^m(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right)|^2 &= |\sum_{i=0}^{M_T-1} \left(\underline{h}_{j,i}^m\right)^T \underline{G}(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right)|^2 \\ &= |\sum_{i=0}^{M_T-1} \left(\underline{h}_{j,i}^m\right)^T \underline{\hat{G}}(k,m,i)|^2 \end{aligned} \tag{3.41}$$

where

$$\underline{\hat{G}}(k,m,i) = \underline{G}(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right). \tag{3.42}$$

As a result, we can get

$$\begin{aligned} |\sum_{i=0}^{M_T-1} H_{j,i}^m(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right)|^2 &= \left[\sum_{i=0}^{M_T-1} \left(\underline{h}_{j,i}^m\right)^T \underline{\hat{G}}(k,m,i)\right]\left[\sum_{i=0}^{M_T-1} \left(\underline{h}_{j,i}^m\right)^T \underline{\hat{G}}(k,m,i)\right]^H \\ &= \left[\left(\underline{\tilde{h}}_j^m\right)^T \underline{\tilde{G}}(k,m)\right]\left[\left(\underline{\tilde{h}}_j^m\right)^T \underline{\tilde{G}}(k,m)\right]^H \\ &= \left(\underline{\tilde{h}}_j^m\right)^T \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H \left[\left(\underline{\tilde{h}}_j^m\right)^T\right]^H \end{aligned} \tag{3.43}$$

where

$$\underline{\tilde{h}}_j^m = \left[\left(\underline{h}_{j,0}^m\right)^T \; \left(\underline{h}_{j,1}^m\right)^T \; \cdots \; \left(\underline{h}_{j,M_T-1}^m\right)^T\right]^T \tag{3.44}$$

and

$$\underline{\tilde{G}}(k,m) = \left[\underline{\hat{G}}(k,m,0)^T \; \underline{\hat{G}}(k,m,1)^T \; \cdots \; \underline{\hat{G}}(k,m,M_T-1)^T\right]^T \tag{3.45}$$

are all $M_T L$-by-1 vectors. So the pairwise error probability conditioned on the channel is

$$P\left(\mathbf{C} \to \mathbf{E} | \text{all } \underline{h}_{j,i}^m\right) \le \frac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{m=1}^{M} \sum_{k=0}^{N-1} \sum_{j=0}^{M_R-1} \left(\underline{\tilde{h}}_j^m\right)^T \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H \left[\left(\underline{\tilde{h}}_j^m\right)^T\right]^H}, \tag{3.46}$$

and $\underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H$ can be expressed as

$$
\begin{aligned}
\underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H &= \left[\left(\underline{c}_k^m - \underline{e}_k^m\right) \otimes \underline{G}(k)\right]\left[\left(\underline{c}_k^m - \underline{e}_k^m\right) \otimes \underline{G}(k)\right]^H \\
&= \left[\left(\underline{c}_k^m - \underline{e}_k^m\right) \otimes \underline{G}(k)\right]\left[\left(\underline{c}_k^m - \underline{e}_k^m\right)^H \otimes \underline{G}(k)^H\right] \\
&= \left[\left(\underline{c}_k^m - \underline{e}_k^m\right)\left(\underline{c}_k^m - \underline{e}_k^m\right)^H\right] \otimes \underline{G}(k)\underline{G}(k)^H.
\end{aligned}
\tag{3.47}
$$

According to (3.22), the right side of (3.46) can be modified as

$$
P\left(\mathbf{C} \rightarrow \mathbf{E} | \text{all } \underline{h}_{j,i}^m\right) \leq \frac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum\limits_{m=1}^{M} \sum\limits_{j=0}^{M_R-1} \left(\underline{\tilde{h}}_j^m\right)^T \left\{\sum\limits_{k=0}^{N-1} \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H\right\} \left[\left(\underline{\tilde{h}}_j^m\right)^T\right]^H} \tag{3.48}
$$

As $\sum\limits_{k=0}^{N-1} \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H$ is Hermitian and nonegative definite, it can be equal to

$$
\sum_{k=0}^{N-1} \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H = \sum_{n=0}^{M_T L-1} \lambda_n^m \underline{v}_n^m \left[\underline{v}_n^m\right]^H \tag{3.49}
$$

where λ_n^m and $\underline{v}_n^m$ are the n-th eigenvalue and the corresponding eigenvector of the m-th OFDM symbol respectively. It is also assumed that $\lambda_0^m \geq \lambda_1^m \geq \cdots \geq \lambda_{M_T L-1}^m \geq 0$ and all the eigenvectors of the m-th transmitted symbol satifies

$$
\underline{v}_n^m \cdot \underline{v}_{n'}^m = \begin{cases} 1, & \text{for } n = n' \\ 0, & \text{for } n \neq n' \end{cases}. \tag{3.50}
$$

Afterwards, substituting (3.49) into (3.48), so (3.48) becomes

$$
\begin{aligned}
P\left(\mathbf{C} \rightarrow \mathbf{E} | \text{all } \underline{h}_{j,i}^m\right) &\leq \tfrac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum\limits_{m=1}^{M} \sum\limits_{j=0}^{M_R-1} \sum\limits_{n=0}^{M_T L-1} \lambda_n^m \left(\underline{\tilde{h}}_j^m\right)^T \underline{v}_n^m \left[\underline{v}_n^m\right]^H \left[\left(\underline{\tilde{h}}_j^m\right)^T\right]^H} \\
&= \tfrac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum\limits_{m=1}^{M} \sum\limits_{j=0}^{M_R-1} \sum\limits_{n=0}^{M_T L-1} \lambda_n^m |\beta_{j,n}^m|^2}
\end{aligned}
\tag{3.51}
$$

where $\beta_{j,n}^m = \left(\underline{\tilde{h}}_j^m\right)^T \underline{v}_n^m$ are i.i.d. zero mean complex Gaussian random variables with variance $\frac{1}{L}$. If r_r is the rank of $\sum\limits_{k=0}^{N-1} \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H$ and $d_r = \sum\limits_{m=1}^{M} \sum\limits_{j=0}^{M_R-1} \sum\limits_{n=0}^{r_r-1} \lambda_n^m |\beta_{j,n}^m|^2$, d_r can be approximated by a Gaussian random variable with mean μ_{d_r}

$$
\begin{aligned}
\mu_{d_r} &= \sum_{m=1}^{M} \sum_{j=0}^{M_R-1} \sum_{n=0}^{r_r-1} \lambda_n^m E\left\{|\beta_{j,n}^m|^2\right\} \\
&= \sum_{m=1}^{M} \sum_{j=0}^{M_R-1} \sum_{n=0}^{r_r-1} \lambda_n^m \tfrac{1}{L} \\
&= \tfrac{M_R}{L} \sum_{m=1}^{M} \sum_{n=0}^{r_r-1} \lambda_n^m \\
&\triangleq \tfrac{M_R}{L} x_{1,r}
\end{aligned} \tag{3.52}
$$

and variance $\sigma_{d_r}^2$

$$
\begin{aligned}
\sigma_{d_r}^2 &= \sum_{m=1}^{M} \sum_{j=0}^{M_R-1} \sum_{n=0}^{r_r-1} (\lambda_n^m)^2 \operatorname{var}\left\{|\beta_{j,n}^m|^2\right\} \\
&= \sum_{m=1}^{M} \sum_{j=0}^{M_R-1} \sum_{n=0}^{r_r-1} (\lambda_n^m)^2 \tfrac{1}{L^2} \\
&= \tfrac{M_R}{L^2} \sum_{m=1}^{M} \sum_{n=0}^{r_r-1} (\lambda_n^m)^2 \\
&\triangleq \tfrac{M_R}{L^2} x_{2,r}.
\end{aligned} \tag{3.53}
$$

Using the equation similar to (3.28), the pairwise error probability on rapid fading channel is

$$
\begin{aligned}
P(\mathbf{C} \to \mathbf{E}) \leq\ & \tfrac{1}{2} \exp\left\{ \tfrac{1}{2} \left(\tfrac{E_s}{4\sigma_n^2}\right)^2 \tfrac{M_R}{L^2} x_{2,r} - \left(\tfrac{E_s}{4\sigma_n^2}\right) \tfrac{M_R}{L} x_{1,r} \right\} \cdot \\
& Q\left(\left(\tfrac{E_s}{4\sigma_n^2}\right) \sqrt{\tfrac{M_R}{L^2} x_{2,r}} - \frac{\frac{M_R}{L} x_{1,r}}{\sqrt{\frac{M_R}{L^2} x_{2,r}}} \right)
\end{aligned} \tag{3.54}
$$

At high SNR, which means

$$
\left(\frac{E_s}{4\sigma_n^2}\right) \sqrt{\frac{M_R}{L^2} x_{2,r}} - \frac{\frac{M_R}{L} x_{1,r}}{\sqrt{\frac{M_R}{L^2} x_{2,r}}} \geq 0 \Rightarrow \frac{E_s}{4\sigma_n^2} \geq \frac{x_{1,r}}{x_{2,r}} L, \tag{3.55}
$$

the pairwise error probability can be further gained by (2.9)

$$
\begin{aligned}
P(\mathbf{C} \to \mathbf{E}) &\leq \tfrac{1}{4} \exp\left\{ -\tfrac{E_s}{8\sigma_n^2} \tfrac{M_R}{L} x_{1,r} \right\} \\
&= \tfrac{1}{4} \exp\left\{ -\tfrac{E_s}{8\sigma_n^2} \tfrac{M_R}{L} \sum_{m=1}^{M} \sum_{n=0}^{r_r-1} \lambda_n^m \right\}.
\end{aligned} \tag{3.56}
$$

The sum of the eigenvalues is farther equal to

$$\begin{aligned}
\sum_{m=1}^{M}\sum_{n=0}^{r_r-1}\lambda_n^m &= tr\left\{\sum_{m=1}^{M}\sum_{k=0}^{N-1}\tilde{\underline{G}}(k,m)\tilde{\underline{G}}(k,m)^H\right\} \\
&= \sum_{m=1}^{M}\sum_{k=0}^{N-1} tr\left\{\tilde{\underline{G}}(k,m)\tilde{\underline{G}}(k,m)^H\right\} \\
&= \sum_{m=1}^{M}\sum_{k=0}^{N-1} tr\left\{\left[\left(\underline{c}_k^m-\underline{e}_k^m\right)\otimes\underline{G}(k)\right]\left[\left(\underline{c}_k^m-\underline{e}_k^m\right)\otimes\underline{G}(k)\right]^H\right\} \quad (3.57)\\
&= \sum_{m=1}^{M}\sum_{k=0}^{N-1} tr\left\{\left(\underline{c}_k^m-\underline{c}_k^m\right)\left(\underline{c}_k^m-\underline{e}_k^m\right)^H\right\}\cdot tr\left\{\underline{G}(k)\underline{G}(k)^H\right\} \\
&= \sum_{m=1}^{M}\sum_{k=0}^{N-1} L\left|\underline{c}_k^m-\underline{e}_k^m\right|^2.
\end{aligned}$$

Assume that $\rho_r\left(\mathbf{C},\mathbf{E}\right)$ is the set of all symbol and frequency indices such taht $|\underline{c}_k^m-\underline{e}_k^m|^2\neq 0$ and δ_r is the corresponding number of the set $\rho_r\left(\mathbf{C},\mathbf{E}\right)$ in which the two codewords differ, it can be easily known that

$$r_r \triangleq \min\left\{M_T L, \delta_r\right\}. \tag{3.58}$$

Finally, the pairwise error probability is upper-bound by

$$P\left(\mathbf{C}\rightarrow\mathbf{E}\right)\leq\frac{1}{4}\exp\left\{-\frac{E_s}{8\sigma_n^2}M_R\sum_{m=1}^{M}\sum_{k=0}^{N-1}\left|\underline{c}_k^m-\underline{e}_k^m\right|^2\right\}. \tag{3.59}$$

Similar to (3.34), the pairwise error probability is dominated by the Eucldean distance and the analogous code design are

- **Rank criterion**: The minimum rank r_r over all pairs of any two different codewords is maximized as large as possible.

- **Distance criterion**: Maximize the minimum value of the Euclidean distance for any pairs of distinct codewords.

We can conclude that the design criteria on rapid fading channel is identical to the design criteria for one OFDM symbol.

3.3.2 Quasi-Static Fading Channel

Quasi-static fading, which means that the fading gains are the same during the various OFDM symbols, will be discussed. The system model of this channel environment is analogous to section 3.1, but all channel matrices $\hat{\mathbf{H}}^m(k)$ must be replaced by (3.9).

When M OFDM symbols are encoded and transmitted on quasi-static fading channel, the ML decoder can be described as

$$\hat{\mathbf{C}} = \arg\min_{\mathbf{C}} \sum_{m=1}^{M} \sum_{k=0}^{N-1} ||\underline{r}_k^m - \sqrt{E_s}\hat{\mathbf{H}}(k)\underline{c}_k^m||^2 \tag{3.60}$$

The wrong codeword $\mathbf{E}$ is decided by the decoder when the correct codeword is $\mathbf{C}$ if

$$\sum_{m=1}^{M} \sum_{k=0}^{N-1} ||\underline{r}_k^m - \sqrt{E_s}\hat{\mathbf{H}}(k)\underline{c}_k^m||^2 \geq \sum_{m=1}^{M} \sum_{k=0}^{N-1} ||\underline{r}_k^m - \sqrt{E_s}\hat{\mathbf{H}}(k)\underline{e}_k^m||^2. \tag{3.61}$$

Therefore, the specific pairwise error probability conditioned on quasi-static fading channel is

$$P\left(\mathbf{C} \to \mathbf{E}|\text{all } \hat{\mathbf{H}}(k)\right) = Q\left(\sqrt{\frac{E_s}{2\sigma_n^2} d^2\left(\mathbf{C}, \mathbf{E}|\hat{\mathbf{H}}(k)\right)}\right) \tag{3.62}$$

where

$$d^2\left(\mathbf{C}, \mathbf{E}|\text{all } \hat{\mathbf{H}}(k)\right) = \sum_{m=1}^{M} \sum_{k=0}^{N-1} |\hat{\mathbf{H}}(k)\left(\underline{c}_k^m - \underline{e}_k^m\right)|^2. \tag{3.63}$$

We also use (2.9) to get

$$\begin{aligned} P\left(\mathbf{C} \to \mathbf{E}|\hat{\mathbf{H}}(k)\right) &\leq \tfrac{1}{2} e^{-\frac{1}{2}\left\{\frac{E_s}{2\sigma_n^2} d^2\left(\mathbf{C}, \mathbf{E}|\hat{\mathbf{H}}(k)\right)\right\}} \\ &= \tfrac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{m=1}^{M} \sum_{k=0}^{N-1} |\hat{\mathbf{H}}(k)\left(\underline{c}_k^m - \underline{e}_k^m\right)|^2} \\ &= \tfrac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{m=1}^{M} \sum_{k=0}^{N-1} \sum_{j=0}^{M_R-1} |\sum_{i=0}^{M_T-1} H_{j,i}(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right)|^2} \end{aligned} \tag{3.64}$$

So we can rewriten $|\sum_{i=0}^{M_T-1} H_{j,i}(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right)|^2$ as follows,

$$\begin{aligned} |\sum_{i=0}^{M_T-1} H_{j,i}(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right)|^2 &= |\sum_{i=0}^{M_T-1} \left(\underline{h}_{j,i}\right)^T \underline{G}(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right)|^2 \\ &= |\sum_{i=0}^{M_{T-1}} \left(\underline{h}_{j,i}\right)^T \underline{\hat{G}}(k,m,i)|^2. \end{aligned} \tag{3.65}$$

Then,

$$\begin{aligned} |\sum_{i=0}^{M_T-1} H_{j,i}(k)\left(c_k^{m,(i)} - e_k^{m,(i)}\right)|^2 &= \left[\left(\underline{\tilde{h}}_j\right)^T \underline{\tilde{G}}(k,m)\right]\left[\left(\underline{\tilde{h}}_j\right)^T \underline{\tilde{G}}(k,m)\right]^H \\ &= \left(\underline{\tilde{h}}_j\right)^T \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H \left[\left(\underline{\tilde{h}}_j\right)^T\right]^H \end{aligned} \tag{3.66}$$

Therefore, the pairwise error probability can be amended to

$$P\left(\mathbf{C} \to \mathbf{E}|\text{all } \underline{h}_{j,i}\right) \leq \frac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{m=1}^{M} \sum_{k=0}^{N-1} \sum_{j=0}^{M_R-1} \left(\underline{\tilde{h}}_j\right)^T \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H \left[\left(\underline{\tilde{h}}_j\right)^T\right]^H}. \tag{3.67}$$

In order to simplify the complexity of computation and easily design, assume that λ_n and $\underline{v}_n$ are the n-th eigenvalue and the corresponding eigenvector of the matrix $\sum_{m=1}^{M}\sum_{k=0}^{N-1} \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H$. Note that these eigenvectors also form an orthonormal set. Accordingly, we can decompose $\sum_{m=1}^{M}\sum_{k=0}^{N-1} \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H$ as

$$\sum_{m=1}^{M}\sum_{k=0}^{N-1} \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H = \sum_{n=0}^{M_T L-1} \lambda_n \underline{v}_n \left[\underline{v}_n\right]^H \tag{3.68}$$

Hence, the pairwise error probability can be further expressed as

$$\begin{aligned} P\left(\mathbf{C} \to \mathbf{E}|\text{all}_{j,i}\right) &\leq \tfrac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{j=0}^{M_R-1} \sum_{n=0}^{M_T L-1} \lambda_n \left(\underline{\tilde{h}}_j\right)^T \underline{v}_n [\underline{v}_n]^H \left[\left(\underline{\tilde{h}}_j\right)^T\right]^H} \\ &= \tfrac{1}{2} e^{-\frac{E_s}{4\sigma_n^2} \sum_{j=0}^{M_R-1} \sum_{n=0}^{M_T L-1} \lambda_n |\beta_{j,n}|^2} \end{aligned} \tag{3.69}$$

where $\beta_{j,n} = \left(\underline{\tilde{h}}_j\right)^T \underline{v}_n$ and all $\beta_{j,n}$ are i.i.d. complex Gaussian random variables with zero mean and variance $\frac{1}{L}$ per dimension. If r_q is the rank of the matrix $\sum_{m=1}^{M}\sum_{k=0}^{N-1} \underline{\tilde{G}}(k,m)\underline{\tilde{G}}(k,m)^H$ and $d_q = \sum_{j=0}^{M_R-1}\sum_{n=0}^{M_T L-1} \lambda_n |\beta_{j,n}|^2 =$

$\sum_{j=0}^{M_R-1}\sum_{n=0}^{r_q-1}\lambda_n|\beta_{j,n}|^2$, when $r_q M_R$ is large enough to approach to Gaussian distribution, d_q owns mean μ_{d_q}

$$\begin{aligned}\mu_{d_q} &= \sum_{j=0}^{M_R-1}\sum_{n=0}^{r_q-1}\lambda_n E\left\{|\beta_{j,n}|^2\right\} \\ &= \tfrac{M_R}{L}\sum_{n=0}^{r_q-1}\lambda_n \\ &\triangleq \tfrac{M_R}{L}x_{1,q}\end{aligned} \tag{3.70}$$

and variance $\sigma_{d_q}^2$

$$\begin{aligned}\sigma_{d_q}^2 &= \sum_{j=0}^{M_R-1}\sum_{n=0}^{r_q-1}\left(\lambda_n\right)^2 \mathrm{var}\left\{|\beta_{j,n}|^2\right\} \\ &= \tfrac{M_R}{L^2}\sum_{n=0}^{r_q-1}\left(\lambda_n\right)^2 \\ &\triangleq \tfrac{M_R}{L^2}x_{2,q}.\end{aligned} \tag{3.71}$$

We also use (3.28) to compute the pairwise error probability, and the pairwise error probability becomes

$$\begin{aligned}P\left(\mathbf{C}\to\mathbf{E}\right) \le\ & \tfrac{1}{2}\exp\left\{\tfrac{1}{2}\left(\tfrac{E_s}{4\sigma_n^2}\right)^2\tfrac{M_R}{L^2}x_{2,q}-\left(\tfrac{E_s}{4\sigma_n^2}\right)\tfrac{M_R}{L}x_{1,q}\right\}\cdot \\ & Q\left(\left(\tfrac{E_s}{4\sigma_n^2}\right)\sqrt{\tfrac{M_R}{L^2}x_{2,q}}-\frac{\frac{M_R}{L}x_{1,q}}{\sqrt{\frac{M_R}{L^2}x_{2,q}}}\right).\end{aligned} \tag{3.72}$$

It is assumed that the space-frequency codes are operated in a somewhat high SNR case, which means

$$\left(\frac{E_s}{4\sigma_n^2}\right)\sqrt{\frac{M_R}{L^2}x_{2,q}}-\frac{\frac{M_R}{L}x_{1,q}}{\sqrt{\frac{M_R}{L^2}x_{2,q}}}\ge 0 \Rightarrow \frac{E_s}{4\sigma_n^2}\ge\frac{x_{1,q}}{x_{2,q}}L. \tag{3.73}$$

By using (2.9) again, the bound can be approximated as

$$\begin{aligned}P\left(\mathbf{C}\to\mathbf{E}\right) &\le \tfrac{1}{4}\exp\left\{-\tfrac{E_s}{8\sigma_n^2}\tfrac{M_R}{L}x_{1,q}\right\} \\ &= \tfrac{1}{4}\exp\left\{-\tfrac{E_s}{8\sigma_n^2}\tfrac{M_R}{L}\sum_{m=1}^{M}\sum_{n=0}^{r_q-1}\lambda_n^m\right\}.\end{aligned} \tag{3.74}$$

If $\rho_q\left(\mathbf{C},\mathbf{E}\right)$ and δ_q represent the set of the indices fulfilling $|\underline{c}_k^m-\underline{e}_k^m|^2\neq 0$ and the number of this set elements respectively, the possible maximum value of r_q is

$$r_q \triangleq \min\left\{M_T L,\delta_q\right\}. \tag{3.75}$$

Using the trace property, we can turn the sum of the eigenvalues into

$$
\begin{aligned}
\sum_{n=0}^{r_q-1} \lambda_n &= tr\left\{ \sum_{m=1}^{M} \sum_{k=0}^{N-1} \tilde{\underline{G}}(k,m)\tilde{\underline{G}}(k,m)^H \right\} \\
&= \sum_{m=1}^{M} \sum_{k=0}^{N-1} tr\left\{ \tilde{\underline{G}}(k,m)\tilde{\underline{G}}(k,m)^H \right\} \\
&= \sum_{k=0}^{N-1} tr\left\{ \left[(\underline{c}_k^m - \underline{e}_k^m) \otimes \underline{G}(k) \right] \left[(\underline{c}_k^m - \underline{e}_k^m) \otimes \underline{G}(k) \right]^H \right\} \\
&= \sum_{k=0}^{N-1} tr\left\{ (\underline{c}_k^m - \underline{e}_k^m) (\underline{c}_k^m - \underline{e}_k^m)^H \right\} \cdot tr\left\{ \underline{G}(k)\underline{G}(k)^H \right\} \\
&= \sum_{m=1}^{M} \sum_{k=0}^{N-1} L|\underline{c}_k^m - \underline{e}_k^m|^2,
\end{aligned}
\tag{3.76}
$$

and in consequence the pairwise error probability would be built into

$$
P\left(\mathbf{C} \rightarrow \mathbf{E}\right) \leq \frac{1}{4} \exp\left\{ -\frac{E_s}{8\sigma_n^2} M_R \sum_{m=1}^{M} \sum_{k=0}^{N-1} |\underline{c}_k^m - \underline{e}_k^m|^2 \right\}. \tag{3.77}
$$

Finally, the corresponding design criteria on quasi-static fading channels are arranged to

- **Rank criterion**: Achieve the largest rank r_q between two unlike codewords as possible.

- **Distance criterion**: The minimum value of the Euclidean distance for any sets of any two distinct codewords should be maximized.

We can easily know that all the discussed channel environments have the same methods to design codewords when the central limit theory is exploited.

Chapter 4

Simulation

In this section, the arguments will be evidenced by way of the simulation results. We apadt an MIMO-OFDM system with two transmitter antennas and three receiver antenna, and we assume that the OFDM bandwidth is $800kHz$ and divided into 128 subcarriers. So the corresponding subcarrier spacing is $6.25kHz$ and the symbol duration is $160\mu s$. A CP of $40\mu s$ is appended to each OFDM symbol. A two equal-power tap fading channel with the delay spread of $5\mu s$ is used, and the sum of two tap powers is normalized to 1. QPSK modulation is used, and the adapted generator sequences for simulation is listed in table 4.1.

Table 4.1: The simulated generator sequences.

code	M_T	generator sequences	rank	distance
BBH[11]	2	$g^1 = [(2,2),(1,0)],\ g^2 = [(0,2),(3,0)]$	2	6
TSC[10]	2	$g^1 = [(0,2),(2,0)],\ g^2 = [(0,1),(1,0)]$	2	4
YT[12]	2	$g^1 = [(2,1),(2,0)],\ g^2 = [(0,2),(3,2)]$	2	10
VY[1]	2	$g^1 = [(0,2),(1,2)],\ g^2 = [(2,3),(2,0)]$	2	10

The simulation results of designing for one OFDM symbol in Figure 4.1. Figure 4.2 and Figure 4.3 are the simulation results of the various design criteria for two OFDM symbols on rapid fading channel and quasi-static fading channel, respectively. The performance of VY is the same to YT in all simulation environment, and both methods are better than BBH and TSC.

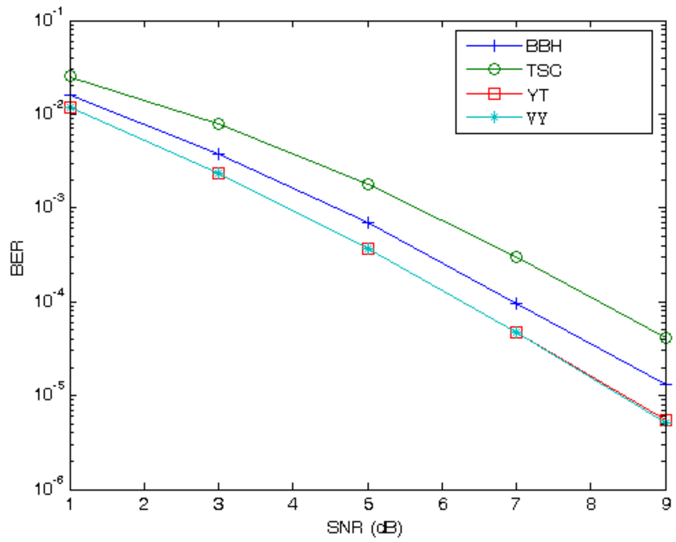

Figure 4.1: Simulation results of the various design criteria for one OFDM symbol.

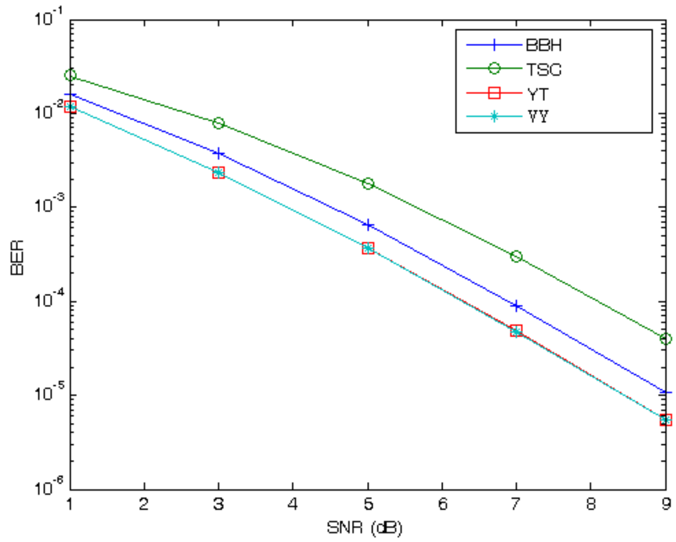

Figure 4.2: Simulation results of the various design criteria for several OFDM symbols on rapid fading channel.

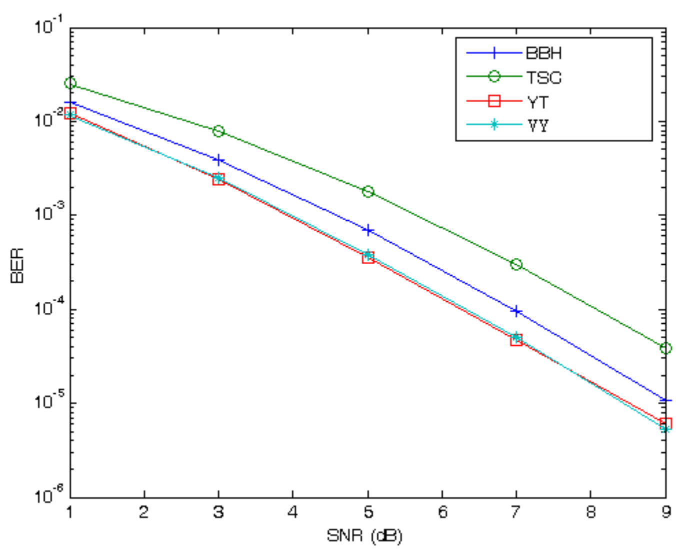

Figure 4.3: Simulation results of the various design criteria for several OFDM symbols on rapid fading channel.

Chapter 5

Conclusion

By making use of the central limit theory, the pairwise error probability and the corresponding design criteria of space-frequency coded MIMO-OFDM system are presented in this thesis. It is easily known that the method for codeword construction of space-frequency codes is similar to the method of space-time codes by way of the same analysis, and the corresponding trellis codes are taken to compare with the other trellis codes designed from various methods. Most space-frequency trellis codes are designed for one OFDM symbol, and therefore we discuss how to encode and decode during several OFDM symbols. Based on the channel variation speed, rapid fading channel and quasi-static fading channel are recognized, and it is proved that the specific design criteria of two distinct channel environments are also similar to the way designed for one OFDM symbol by applying the central limit theory. Simulation results are presented to support the arguments.

Bibliography

[1] B. Vucetic, J. Yuan, *Space-Time Coding*, Wilsey, 2003.

[2] A. Papoulis, Probability, *Random Variables, and Stochastic Process*, Third Edition, McGraw Hill, 1991.

[3] H. Bölcskei, A.J. Paulraj; "Space-frequency coded broadband OFDM system," in *Proc. IEEE WCNC 2000*, pp. 1-6, September 2000.

[4] H. Bölcskei, A.J. Paulraj; "Space-frequency codes for broadband fading channel," in *Proc. ISIT*, p. 219, June 24-29, 2001.

[5] L. Shao, S. Roy, S. Sandhu; "Rate-one space frequency block codes with maximum diversity gain for MIMO-OFDM," in *GLOCOM*, p. 809-813 volume 2, December 2003.

[6] L. Shao, S. Roy, S. Sandhu; "Rate-one space-frequency block codes with maximum diversity for MIMO-OFDM," *trans. on IEEE Wireless Comm.*, p. 1674-1687 volume 4, July 2005.

[7] Wei Zhang, Xiang-Gen Xia, P.C. Ching; "A Design of High-rate Space-Frequency Codes for MIMO-OFDM Systems," in *Proc. Global Telecomm.*, p.209-213 volume 1, November-December, 2004.

[8] R. A. Horn and C. R. Johnson, *Matrix Analysis*, New York, Cambridge Univ. Press, 1988.

[9] S. J. Leon, *Linear Algebra with Applications*, NJ, Prentice Hall, 1998

[10] V. Tarkh, N. Seshadri, and A. R. Calderbank; “Space-time codes for high data rate wireless communication: Performance criterion and code construction,” in *IEEE Trans. Inform. Theory*, p.744-765 volume 44, March 1998.

[11] S.Baro, G. Bauch, and A. Hansmann; “Improved codes for space-time trellis coded modulation,” in *IEEE Commun. Lett.*, p.20-22 Volume 4, January 2000.

[12] Y. Sasazaki and T. Ohtsuki; “Improved design criteria and new codes on space-frequency trellis coding over frequency selective fading channels,” p. 2187 - 2191 volume 4, September 2002.