國 立 交 通 大 學

機 械 工 程 學 系

博 士 論 文

量子胞神經網路奈米系統之超渾沌、渾沌控制、渾沌化與同步研究

Hyperchaos, Chaos Control, Chaotization and Synchronization for a Quantum Cellular Neural Networks Nano System

研 究 生：楊振雄

指導教授：戈正銘　榮譽教授

中 華 民 國 九 十 七 年 六 月

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摘 要

渾沌現象普遍存在於宇宙之間，大至星際，小至原子。本文研究耦合之量子點胞。將它們用於量子點胞自動機，以構成神經胞網路。即使只有兩個量子點胞，亦可因胞間極化與基底之些許差異而產生渾沌運動。研究此種奈米尺寸之渾沌運動對於未來超小型非線性胞網路之製成及用標準的非線性胞網路硬體有效地進行量子計算都是非常重要的。作為前瞻性的研究，本文將對此系統作全面之研究，研究重點為：

1. 系統渾沌行為之研究。用相圖、功率頻譜圖、參數圖及李亞普諾夫指數分析超渾沌之行為。
2. 系統之渾沌控制。利用外加常值項、GYC 控制理論、實用適應控制、可變結構控制、脈衝控制及最佳控制將渾沌運動控制到週期運動或平衡態，同樣地可將規則運動渾沌化。
3. 系統之渾沌同步。利用非線性控制、線性耦合、GYC 控制理論、可變參數線性耦合實用適應控制、實用適應控制、可變結構控制、最佳控制及脈衝控制方法研究渾沌同步。

Hyperchaos, Chaos Control, Chaotization and Synchronization for a Quantum Cellular Neural Networks Nano System

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Abstract

Chaos exists everywhere in the universe, from interstellar space to interatomic interval. In this thesis, coupled quantum-dot cells will be studied. They are used for Quantum-dots Cellular Automata (QCA), to build Cellular Neural Networks (CNN). It is shown that the connection of even only two quantum-dot cells can cause the onset of chaotic motion by small difference of polarizations and template between cells. This study could be very important for future ultra-small realization of CNNs and, on the other hand, for performing quantum computation efficiently via hardware by using standard CNN. As a foresighted study, a detailed study of this system will be developed. The main parts of our study are:

1. The study of chaotic system: By phase portraits, power spectra, parameter diagram and Lyapunov exponents, the various chaotic behaviors of this hyperchaos system are studied.
2. Chaos control and chaotization for the system: By addition of constant term, by GYC control, by pragmatical adaptive control, by variable structure control, by impulsive control and by optimal control, the chaotic motions of the system can be controlled to

periodic motions or to equilibrium states, as well as the regular motions of the system can be chaotized.

3. Chaos synchronization for the system: By nonlinear control, by linear coupling, by GYC control, by variable parameters linear couple pragmatical adaptive control, by pragmatical adaptive control, by variable structure control, by impulsive control and by optimal control, the synchronization of the chaos motions of the two systems are studied.

誌 謝

本論文得以順利完成，首先要由衷地感謝我的指導老師 戈正銘榮譽教授在學術研究上的細心及辛苦的指導。在兩年碩士班與四年博士班的學生生涯中，學生深切地感受到老師對學術研究的熱情與願景，潛移默化間也成為激勵學生本身不斷追求進步的動力；在待人處事方面，恩師諸多啟迪，使我面對任何事情都可以更圓融態度來處理。這種無形智慧的累積，已踏入社會的我，自信能排除萬難，解決所有難題。浩瀚師恩永銘於心，謹誌於此，以表由衷之感謝與敬意。

我也要感謝幾位口試委員張家歐教授、張江南教授、邵錦昌教授與陳献庚教授。諸位老師給我許多精闢的建議，使本論文更臻完善。『聽君一席話，勝讀十年書』，正代表我對每位口試委員的萬分謝意。

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楊振雄 謹誌

戊子年于新竹交通大學

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Chapter 1

Introduction

In nature, most of dynamic systems are nonlinear and can be described by the nonlinear equations of motion. If the nonlinear term can be ignored, it is possible to be linearized and easily be solved by the already known methods. For many nonlineared systems, the linearization process is reasonable, whereas for some nonlinear systems linearization is unevailable. Hence the researches of nonlinear systems spread quickly today. For the nonlinear system, the study of the types of periodic solutions, the effects to the solutions caused by different parameters and initial conditions, the stability analysis of the solutions, consist of the major tasks. Besides, a substantial understanding of the complicated phenomena raised from nonlinearity is also what we are interested in. The central characteristics are that a process like randomization happens in the deterministic system and small differences in the initial conditions produce very great ones in the final phenomena. The irregular and unpredictable motions of many nonlinear systems have been labeled "chaotic". In the end of nineteen century Poncaré first pointed out some important concepts of chaos theory like homoclinic, bifurcation, etc. Lorenz researched the strange changes in the atmosphere that is the first example to study chaos in 1963. Chaotic phenomena is quite useful in many applications such as fluid mixing [15], human brain study [16], and heart beat regulation [17], etc. Since Ott, Grebogi, and Yorke proposed the OGY method [78], a method of controlling chaos, "controlling of chaos" is receiving increasing attention within the area of non-linear dynamics. Chaos has many applications in various systems while it is unfavorable in many other cases due to its irregular behavior. Therefore, both chaos control and chaotization are important depending on the specific applications. They are effective method for both chaos elimination and utilization and have been thoroughly studied in various fields of science.

Chaotic systems exhibit sensitive dependence on initial conditions. Because of this property, chaotic systems are thought difficult to be synchronized or controlled. From the earlier works

[1-3], especially after Pecora and Carroll [3], the researchers have realized that synchronization of chaotic motions is possible. From then on, synchronization of chaos was of great interest in these years [4-14]. In particular, it was pointed out that chaos synchronization has the potential in secure communication [18, 19], chemical and biological systems [20, 22], etc. Many engineers and scientists were attracted to this discipline [23-36].

Synchronization means that the states of slave system approach eventually to the ones of master system. Two kinds of chaos synchronization are discussed the most often. (1) Duplication: the first kind introduced by Pecora and Carroll [3] consists of a master system and a slave system. The former one evolves chaotic orbits and the latter is identical to the master system except some partial states replaced by that of the master one. (2) Coupling: the second kind consists of two identical chaotic systems except coupling term. Coupled systems can be unidirectional or mutual. Under certain conditions (appropriate coupling functions and/or system parameters with enough evolution time) the slave system will behave the same orbit with the master system.

There are many control methods to synchronize chaotic systems such as observer-based design methods [37-44], adaptive control [45-54], sliding mode control (or variable structure control) [41, 43, 44, 55-58], impulsive control [59-65] and other control methods [66-72]. A another kind of more general synchronism called generalized synchronization (GS) is studied in [73-77], this means that there is a functional relation between state variables of master and slave systems as time evolves. This function needs not to be defined on the whole phase space but on the attractor only. Three methods were proposed to detect GS in [73-75] respectively while another method measuring the smooth degree of this function in [77].

As numerical examples, recently developed Quantum Cellular Neural Networks (Quantum-CNN) model, based on Schrödinger equation in which cells composed of interacting quantum dots are employed in CNN architecture. One of their peculiarities lies in the further degree of freedom possessed by each cell due to the quantum interaction between dots (quantum mechanical phase difference). This fact allows obtaining complex dynamics even in a network with only two cells. Our aim is therefore to investigate their dynamical behavior by suitable variation of coupling parameters and initial conditions. The study could be very important for future ultra-small realization of CNNs and, on the other hand, for performing quantum computation efficiently via hardware by using standard CNN as in literature [23].

In Fig. 1 shows the organization of this dissertation with the following eight chapters:

1. Chapter 1 is an introduction to this dissertation including the backgrounds, motivation and objectives, and the organization of this work.
2. In Chapter 2, hyper chaotic Quantum-CNN oscillator system is studied.
3. In Chapter 3, chaos control and chaotization for the Quantum-CNN oscillator system are studied. By addition of constant term, by variable structure control, by impulsive control and by optimal control, the chaotic motions of the system can be controlled to periodic motions as well as to equilibrium states or the regular motions of the system can be chaotized.
4. In Chapter 4, chaos synchronization for the Quantum-CNN oscillator system is studied by nonlinear control, and by variable structure control.
5. In Chapter 5, chaos synchronization for the Quantum-CNN oscillator system is studied by linear coupling, by impulsive control and by optimal control.
6. In Chapter 6, a pragmatical adaptive control method is applied to the chaos control and chaos synchronization.
7. In Chapter 7, a GYC partial region stability theorem is applied to the chaos control and chaos synchronization.
8. In Chapter 8, conclusions are drawn.

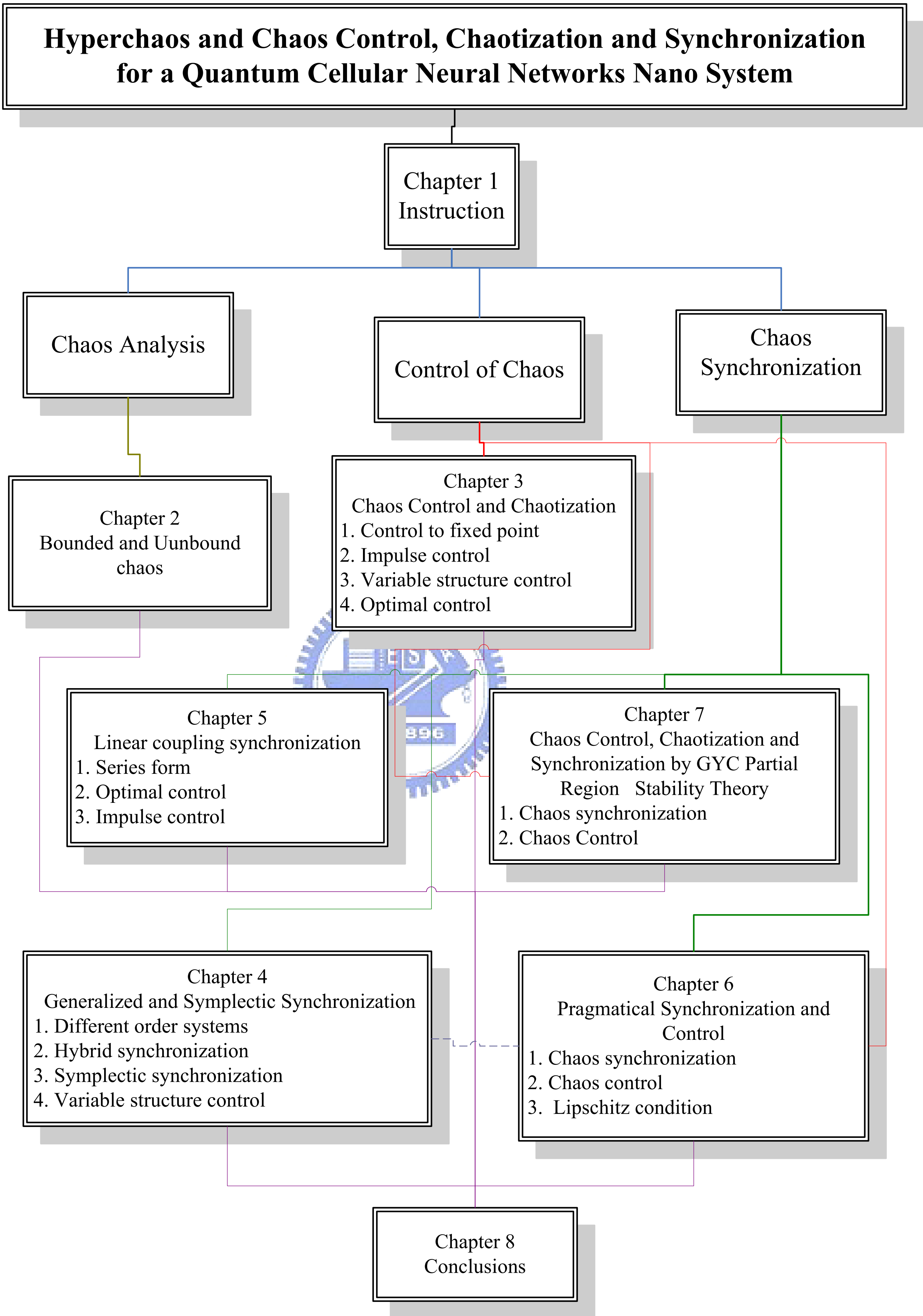


Fig. 1. Schema of the dissertation.

Chapter 2

Bounded and Unbounded Chaos

2-1 Preliminary

In order for a system to exhibit hyperchaos, the minimum dimension of its state space is four. This is because one Lyapunov exponent is always zero and the sum of the exponents must be negative in order for an attractor to form. A hyperchaotic system is characterized by the presence of two or more PLEs in its Lyapunov spectrum, indicating that it is unstable in more than one direction. Besides the theoretical interest in the dynamics of such nonlinear systems, there has been a practical interest in chaos and hyperchaos as means for secure communication. Hyperchaos was first reported from computer simulations of hypothetical ordinary differential equations in [106].

2-2 Bounded Chaos

For a two-cell Quantum-CNN, the following differential equations are obtained [23]:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \qquad (2\text{-}1\text{-}1)$$

where x_1 and x_3 are polarizations, x_2 and x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs.

The evolution of a set of trajectories emanating from various initial conditions is presented in the phase space. When the solution becomes stable, the asymptotic behaviors of the phase trajectories are particularly interested and the transient behaviors in the system are neglected. The phase portraits of the Quantum-CNN system, equation (2-1-1), are plotted in Fig. 2-1.

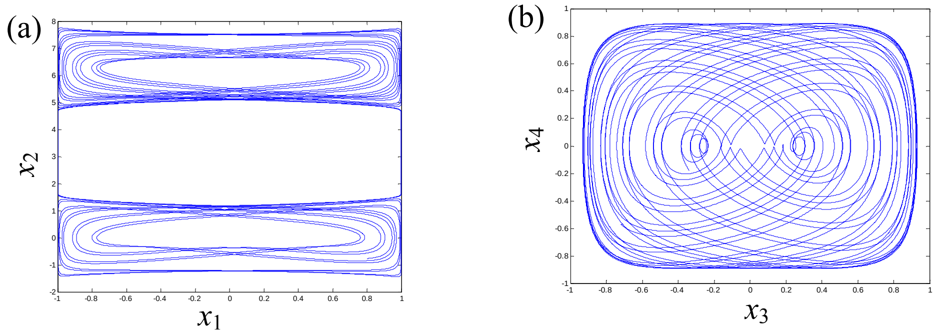


Fig. 2-1. Phase portraits of Quantum-CNN with a_1=0.11, a_2=0.13, ω_1=0.11 and ω_2=0.08.

If the states are not periodic, their spectrum must be in terms of oscillations with a continuum of frequencies. Such a representation of the spectrum is called Fourier integral. The power spectrum analysis of the nonlinear dynamical system, equation (2-1-1) is shown in Fig. 2-2. Apparently, the spectrum of the periodic motion only consists of discrete frequencies. The noise-like spectrum is the characteristics of chaotic dynamical system.

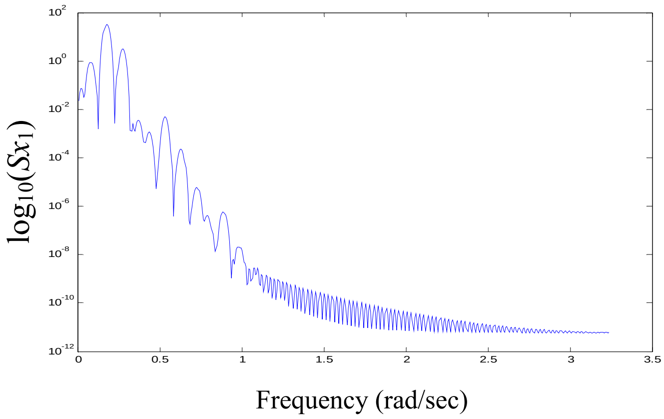


Fig. 2-2. Power spectrum of x_1 for Quantum-CNN with a_1=0.11, a_2=0.13, ω_1=0.11 and ω_2=0.08.

The Lyapunov exponents of the solutions of the nonlinear dynamical system, equation (2-1-1), is plotted in Fig. 2-3. Since there exist two positive Lyapunov exponents, hyperchaos is obtained.

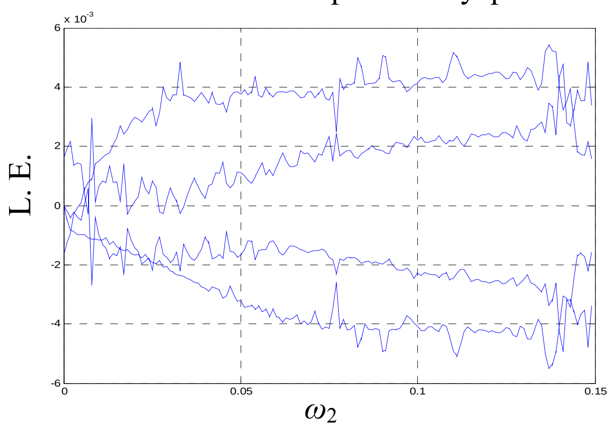


Fig. 2-3. Lyapunov exponents of Quantum-CNN with a_1=0.11, a_2=0.13 and ω_1=0.11.

2-3 Unbounded Chaos

In [101], $\Lambda \subset R^n$ is a compact set invariant under flow $\phi(t, x)$. Then the definition is:

Definition (Chaotic Invariant Set) Λ is said to be chaotic if

1. The flow $\phi(t, x)$ generated by vector field $\dot{x} = f(x)$ has sensitive dependence on initial conditions on Λ.
2. $\phi(t, x)$ is topologically transitive on Λ.

The compact invariant set Λ is *bounded* [102]. Other definitions are, for instance: (a) Chaos is defined as the phenomenon of occurrence of *bounded* nonperiodic evolution in completely deterministic nonlinear dynamical system with high sensitive dependence on initial conditions [103]. (b) Chaos is recurrent motion in simple systems or low-dimensional behavior that has some random aspect as well as a certain order. Exponential divergence from adjacent starts while remaining in a *bounded* region of phase space is a signature of chaotic motion [104]. (c) Chaos is the aperiodic, long-term behavior of a *bounded*, deterministic system that exhibits sensitive dependence on initial conditions [105].

Unbounded chaos is found for a two-cell Quantum-CNN oscillator chaotic system [23]:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{2-2-1}$$

where x_1 and x_3 are polarizations, x_2 and x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs.

We choose ω_2 as abscissa, a_1=3x10^{-5}, a_2=2.3x10^{-5} and ω_1=21x10^{-5}. Lyapunov exponent diagram is shown in Fig. 2-4 with two positive Lyapunov exponents (L. E.). The Lyapunov exponents are smooth curves on unbounded chaos states and they are uneven curves on bouned chaos. They are shown in Fig. 2-3 and Fig. 2-4.

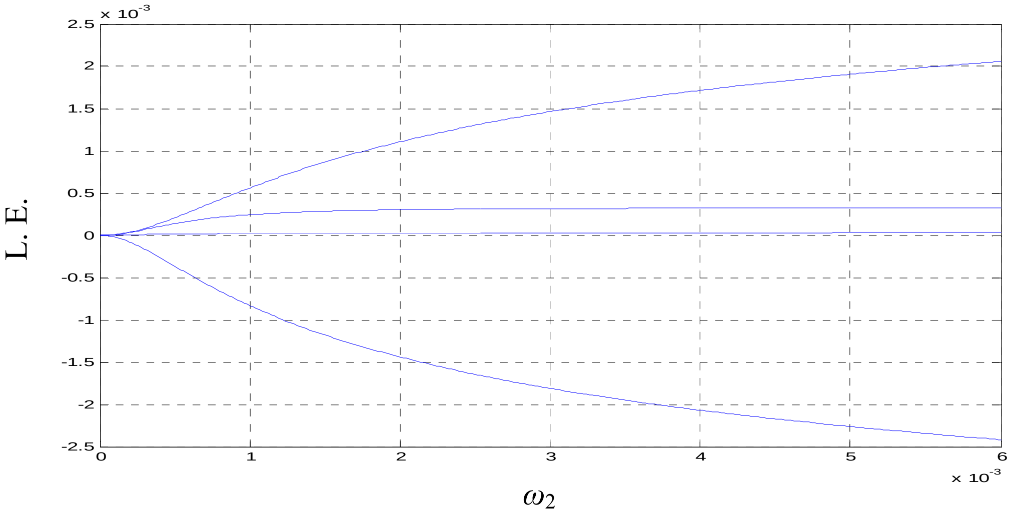


Fig. 2-4. Lyapunov exponents diagram with a_1=3x10^{-5}, a_2=2.3x10^{-5} and ω_1=21x10^{-5}.

Then choose a_1=3x10^{-5}, a_2=2.3x10^{-5}, ω_1=21x10^{-5} and ω_2=10x10^{-5}, the power spectrum is shown in Fig. 2-5.

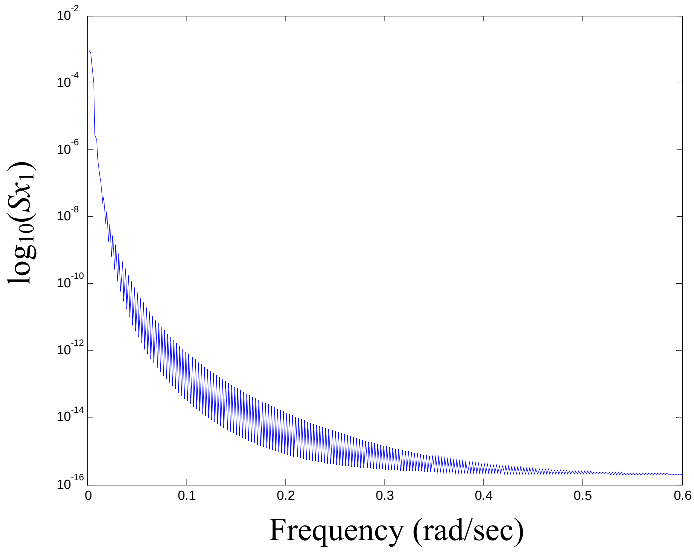


Fig. 2-5. Power spectrum for a_1=3x10^{-5}, a_2=2.3x10^{-5}, ω_1=21x10^{-5} and ω_2=10x10^{-5}.

Time histories of x_1, x_2, x_3 and x_4 are shown in Fig. 2-6, while x_2 is unbounded.

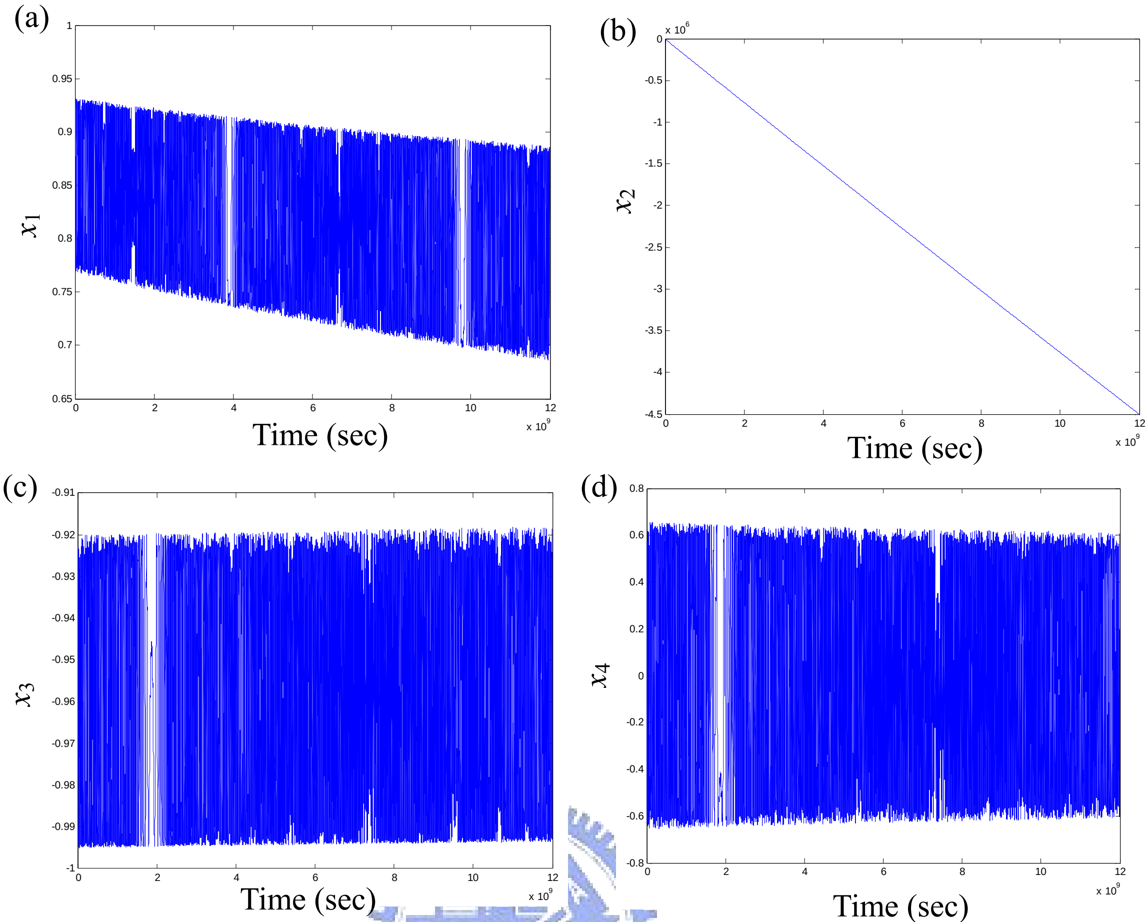


Fig. 2-6. Time histories of x_1, x_2, x_3 and x_4 with parameters a_1=3x10^{-5}, a_2=2.3x10^{-5}, ω_1=21x10^{-5} and ω_2=10x10^{-5}.

The phase portrait of bounded states x_1, x_3 and x_4 is shown in Fig. 2-7.

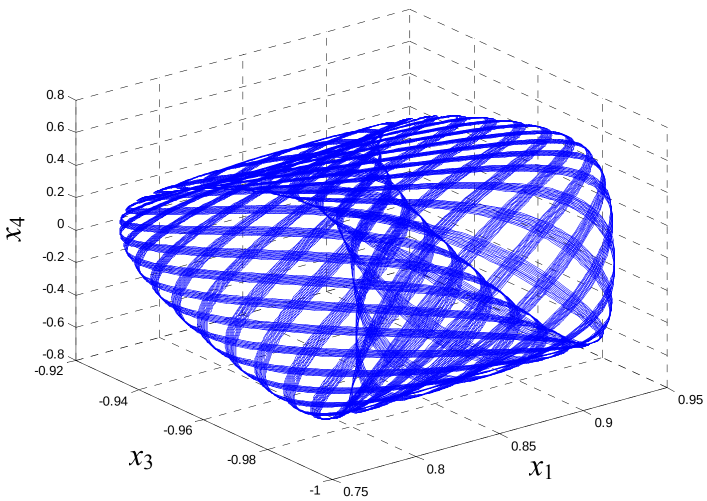


Fig. 2-7. Phase portrait of x_1, x_3 and x_4 for a_1=3x10^{-5}, a_2=2.3x10^{-5}, ω_1=21x10^{-5} and ω_2=10x10^{-5}.

Choose a_1=3x10^{-5}, a_2=2.3x10^{-5}, ω_1=21x10^{-5} and ω_2=40x10^{-5}. The power spectrum is shown in Fig. 2-8.

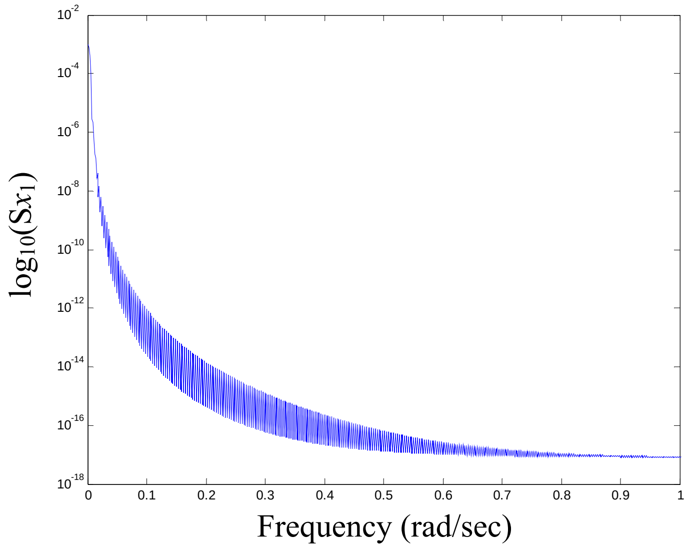


Fig. 2-8. Power spectrum for a_1=3x10^{-5}, a_2=2.3x10^{-5}, ω_1=21x10^{-5} and ω_2=40x10^{-5}.

Time histories of x_1, x_2, x_3 and x_4 are shown in Fig. 2-9, while x_2 and x_4 are unbounded.

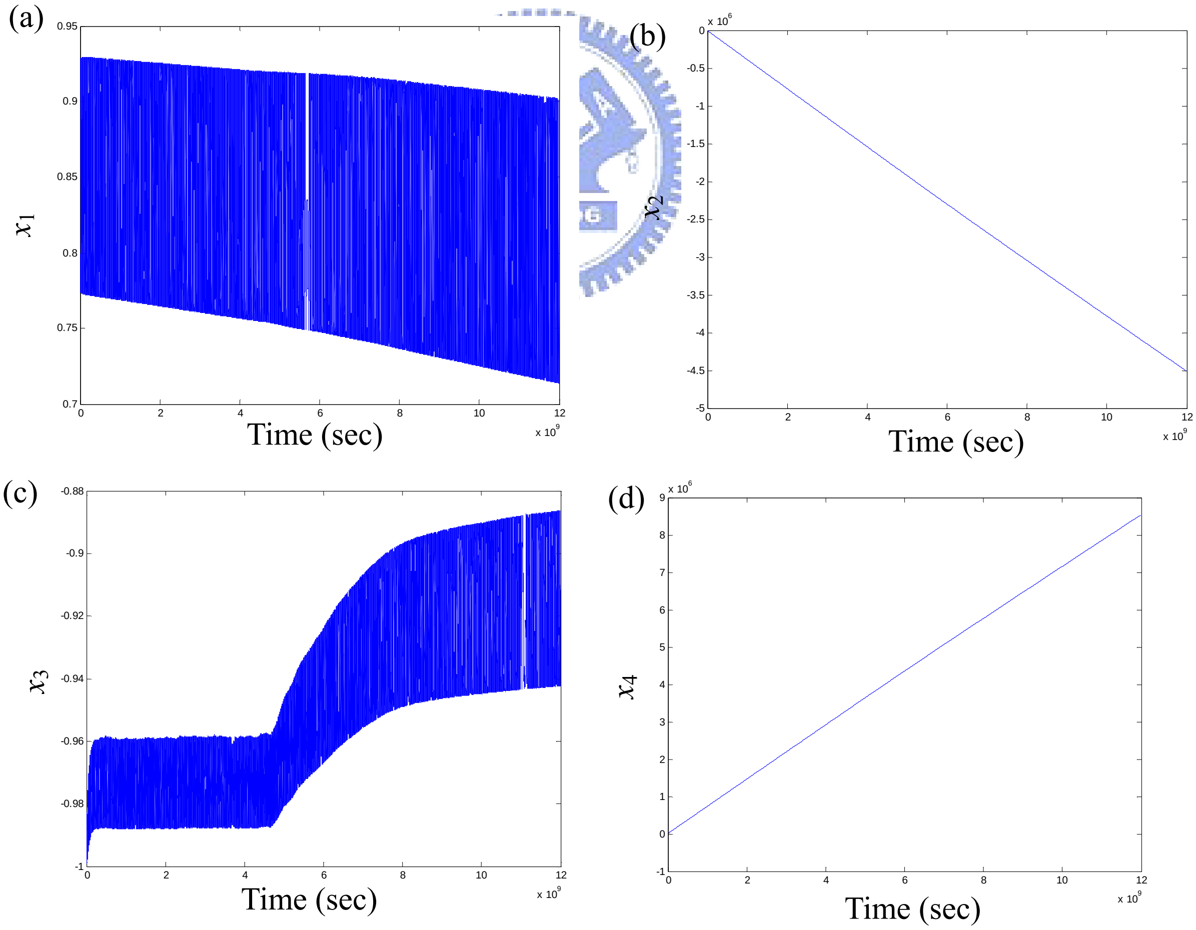


Fig. 2-9. Time histories of x_1, x_2, x_3 and x_4 with parameters a_1=3x10^{-5}, a_2=2.3x10^{-5}, ω_1=21x10^{-5} and ω_2=40x10^{-5}.

The phase portrait of bounded states x_1, x_3 is shown in Fig. 2-10.

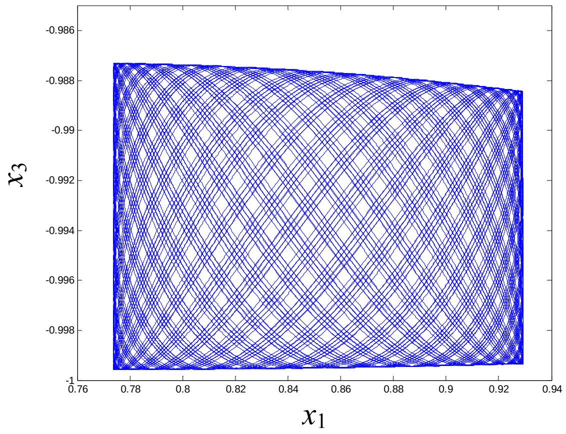


Fig. 2-10. Phase portrait of x_1, x_3 for a_1=3x10^{-5}, a_2=2.3x10^{-5}, ω_1=21x10^{-5} and ω_2=40x10^{-5}.

Table 2-1. Sensitivity to initial conditions. One unbounded state x_2. Time passage 1.2x10^{10}s.

	Initial condition of states				Difference between final states of (a) and (b) case second final states			
Case	x_1	x_2	x_3	x_4	Df_1	Df_2	Df_3	Df_4
(a)	0.29	-0.47	-0.99	0.77	0.2045	-304.67	-0.133	-0.9875
(b)	0.29-1x10^{-7}	-0.47+1x10^{-7}	-0.99+1x10^{-7}	0.77-1x10^{-7}				
(a)	0.8	-0.77	-0.99	-0.57	0.036	8.967	-0.004	-0.232
(b)	0.8-1x10^{-7}	-0.77+1x10^{-7}	-0.99+1x10^{-7}	-0.57-1x10^{-7}				
(a)	0.19	-0.27	-0.98	0.57	-0.068	-202.46	-0.11	0.738
(b)	0.19-1x10^{-7}	-0.27+1x10^{-7}	-0.98+1x10^{-7}	0.57-1x10^{-7}				
(a)	0.29	-0.27	-0.98^{7}	0.27	0.062	45.41	-0.0393	0.337
(b)	0.29-1x10^{-7}	-0.27+1x10^{-7}	-0.98+1x10^{-7}	0.27-1x10^{-7}				
(a)	0.29	-0.47	-0.99	0.27	-0.342	254.6	-0.109	-0.189
(b)	0.29-1x10^{-7}	-0.74+1x10^{-7}	-0.99+1x10^{-7}	0.27-1x10^{-7}				

Table 2-2. Sensitivity to initial conditions. Two unbounded states x_2 and x_4. Time passage 1.2×10^{10}s.

	Initial condition of states				Difference between final states of (a) and (b) case second final states			
Case	x_1	x_2	x_3	x_4	Df_1	Df_2	Df_3	Df_4
(a)	0.8	-0.87	-0.29	-0.42	-0.036	49.36	-0.0017	-63.95
(b)	$0.8+1\times10^{-7}$	$-0.87+1\times10^{-7}$	$-0.29+1\times10^{-7}$	$-0.42+1\times10^{-7}$				
(a)	0.75	-0.57	-0.85	-0.42	0.109	246.67	0.0184	-467.04
(b)	$0.75+1\times10^{-7}$	$-0.57+1\times10^{-7}$	$-0.85+1\times10^{-7}$	$-0.42+1\times10^{-7}$				
(a)	0.09	-0.27	-0.45	-0.57	0.271	557.57	0.336	788.61
(b)	$0.09+1\times10^{-7}$	$-0.27+1\times10^{-7}$	$-0.45+1\times10^{-7}$	$-0.57+1\times10^{-7}$				
(a)	0.19	-0.27	-0.95	-0.75	0.286	-374.79	0.0199	778.72
(b)	$0.19+1\times10^{-7}$	$-0.27+1\times10^{-7}$	$-0.95+1\times10^{-7}$	$-0.75+1\times10^{-7}$				
(a)	0.39	-0.27	-0.98	-0.75	-0.135	-1.49	-0.0285	2.91

Table 2-1 and Table 2-2 show the sensitivity to initial conditions.

The current chaos theory cannot explain that the unbounded chaos violates the fundamental definition of chaos.

Chapter 3

Chaos Control and Chaotization

3-1 Preliminary

Chaos control exploits the sensitivity to initial conditions and to perturbations that is inherent in chaos as a means to stabilize unstable periodic orbits within a chaotic attractor. The control can operate by altering system variables or system parameters, and either by discrete corrections or by continuous feedback. Many methods of chaos control have been derived and tested.

A commonly applied method for control of chaotic dynamical systems was discovered by Ott, Grebogi and Yorke (OGY) [78]. The OGY method requires an analytical description of the linear map describing the behavior near the fixed point. This map is used to determine small perturbations that, when periodically applied, use the system's own dynamics to send the system towards an unstable fixed point. Continued application of these perturbations keeps the system near the fixed point, thereby stabilizing the unstable fixed point even in a noisy system. The OGY method generally is inadequate when the system is far from the fixed point and the linear map is no longer valid. The original OGY method is also limited to controlling only one- or two-dimensional maps. However, the method has been extended to higher dimensions and to cases where multiple control parameters are available.

In some nontraditional applications chaos can be useful and beneficial which led to research on the task of purposely making a nonchaotic dynamical system chaotic and this process is called "anti-control of chaos or chaotization". The chaotization technique is investigated which can be applied to generate chaos irrespective of all the parameter values and states of the system.

3-2 Chaos Control of the Quantum-CNN Systems

Numerical results are studied for the control of chaos of Quantum-CNN oscillators with linear feedback, nonlinear sine function feedback, and nonlinear state cross cosine function feedback. The Routh-Hurwitz theorem is used to derive the conditions of stability of controlled Quantum-CNN systems.

3-2-1 Model of Two Quantum-CNN Oscillator Systems

Recently developed Quantum Cellular Neural Networks (Quantum-CNN) chaotic oscillator is used. Quantum-CNN oscillator equations are derived from a Schrödinger equation taking into account quantum dots cellular automata structures to which in the last decade a wide interest has been devoted with particular attention towards quantum computing. For a two-cell Quantum-CNN, the following differential equations are obtained:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{3-2-1}$$

where x_1 and x_3 are polarizations, x_2 and x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let $a_1 = a_2 = 0.3$, $\omega_1 = 0.33$ and $\omega_2 = 0.31$. The initial values of linear coupled Quantum-CNN systems are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$ and $x_4(0) = 0.57$ respectively. Figs. 3-1~3-2 are phase portrait, and power spectra density diagram respectively.

3-2-2 Controlling Quantum-CNN attractor to equilibrium *O* (0,0,0,0)

Case I Linear feedback control

We design controller is $u_i = k_i x_i$

$$
\begin{cases}
\dot{x}_1 = -2a_1\sqrt{1-x_1^2}\,\sin x_2 + k_1 x_1 \\
\dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1 \dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + k_2 x_2 \\
\dot{x}_3 = -2a_2\sqrt{1-x_3^2}\,\sin x_4 + k_3 x_3 \\
\dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2 \dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + k_4 x_4
\end{cases}
\tag{3-2-2}
$$

Expand the right hand sides of Eq. (3-2-1) into power series:

$$
\begin{cases}
\dot{x}_1 = -2a_1(-\frac{1}{2}x_1^2 x_2 + x_2 - \frac{1}{6}x_2^3 + \cdots) + k_1 x_1 \\
\dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1(x_1 - \frac{1}{2}x_1 x_2^2 + \frac{1}{2}x_1^3 + \cdots) + k_2 x_2 \\
\dot{x}_3 = -2a_2(-\frac{1}{2}x_3^2 x_4 + x_4 - \frac{1}{6}x_4^3 + \cdots) + k_3 x_3 \\
\dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2(x_3 - \frac{1}{2}x_3 x_4^2 + \frac{1}{2}x_3^3 + \cdots) + k_4 x_4
\end{cases}
\tag{3-2-3}
$$

whose Jacobian matrix is

$$
\begin{bmatrix}
-k_1 & -2a_1 & 0 & 0 \\
2a_1 - \omega_1 & -k_2 & \omega_1 & 0 \\
0 & 0 & -k_3 & -2a_2 \\
\omega_2 & 0 & 2a_2 - \omega_2 & -k_4
\end{bmatrix}.
$$

The characteristic equation of J is

$$
\begin{bmatrix}
-k_1 - \lambda & -2a_1 & 0 & 0 \\
2a_1 - \omega_1 & -k_2 - \lambda & \omega_1 & 0 \\
0 & 0 & -k_3 - \lambda & -2a_2 \\
\omega_2 & 0 & 2a_2 - \omega_2 & -k_4 - \lambda
\end{bmatrix}
$$

$$
= (-k_4 - \lambda)\begin{bmatrix}
-k_1 - \lambda & -2a_1 & 0 \\
2a_1 - \omega_1 & -k_2 - \lambda & \omega_1 \\
0 & 0 & -k_3 - \lambda
\end{bmatrix} + 2a_2 \begin{bmatrix}
-k_1 - \lambda & -2a_1 & 0 \\
2a_1 - \omega_1 & -k_2 - \lambda & \omega_1 \\
\omega_2 & 0 & 2a_2 - \omega_2
\end{bmatrix}
$$

$$
\begin{aligned}
&= \lambda^4 + (k_1 + k_2 + k_3 + k_4)\lambda^3 + ((k_1 + k_2)(k_3 + k_4) + k_3 k_4 + k_1 k_2 + 4(a_1^2 + a_2^2) - 2(a_1\omega_1 \\
&+ a_2\omega_2))\lambda^2 + ((k_1 + k_2)(k_3 k_4 + 4a_2^2 - 2a_2\omega_2) + (k_3 + k_4)(k_1 k_2 + 4a_1^2 - 2a_1\omega_1))\lambda + (k_1 k_2 k_3 k_4 \\
&+ 4(a_1^2 k_3 k_4 + a_2^2 k_1 k_2) - 2(a_1\omega_1 k_3 k_4 - a_2\omega_2 k_1 k_2) + 16a_1^2 a_2^2 + 8a_1 a_2(a_1\omega_2 + a_2\omega_1)).
\end{aligned}
\tag{3-2-4}
$$

Take

$$
B = (k_1 + k_2 + k_3 + k_4), \quad C = (k_1 + k_2)(k_3 + k_4) + k_3 k_4 + k_1 k_2 + 4(a_1^2 + a_2^2) - 2(a_1\omega_1 + a_2\omega_2),
$$

$$D=(k_1+k_2)(k_3k_4+4a_2^2-2a_2\omega_2)+(k_3+k_4)(k_1k_2+4a_1^2-2a_1\omega_1),$$
$$E=(k_1k_2k_3k_4+4(a_1^2k_3k_4+a_2^2k_1k_2)-2(a_1\omega_1k_3k_4-a_2\omega_2k_1k_2)+16a_1^2a_2^2+8a_1a_2(a_1\omega_2+a_2\omega_1)).$$

If

$$B>0\,,\ \frac{BC-D}{B}>0\,,\ \frac{B(CD-BE)-D^2}{BC-D}>0\ \text{ and }\ E>0 \tag{3-2-5}$$

then all the eigenvalues of the Jacobian matrix have negative real parts. So we choose k_1=1, k_2=1, k_3=1 and k_4=1, and find B=4, $\frac{BC-D}{B}=4.18$, $\frac{B(CD-BE)-D^2}{BC-D}=4.22$ and E=4.62.

Proposition 1. According to Routh-Hurwitz theorem and Lyapunov asymptotically stability theorem, when k_i's satisfy (3-2-5), the controlled Quantum-CNN system (3-2-2) is locally asymptotically stable at the equilibrium O (0, 0, 0, 0) in spite of the higher order terms.

Numerical simulations are used to investigate the controlled Quantum-CNN system (3-2-1) using fourth-order Runge-Kutta scheme with time step 0.005. After 100 second, the states (x_1, x_2, x_3, x_4) of the controlled Quantum-CNN system (3-2-2) approach zeros in Fig. 3-3.

Case II Nonlinear sine function feedback control

We design controller is $u_i=k_i\sin x_i$

$$\begin{cases}\dot{x}_1=-2a_1\sqrt{1-x_1^2}\sin x_2+k_1\sin x_1\\ \dot{x}_2=-\omega_1(x_1-x_3)+2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2+k_2\sin x_2\\ \dot{x}_3=-2a_2\sqrt{1-x_3^2}\sin x_4+k_3\sin x_3\\ \dot{x}_4=-\omega_2(x_3-x_1)+2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4+k_4\sin x_4\end{cases} \tag{3-2-6}$$

Expand the right hand sides of Eq. (3-2-6) into power series:

$$\begin{cases}\dot{x}_1=-2a_1(-\frac{1}{2}x_1^2x_2++x_2-\frac{1}{6}x_2^3+\cdots)+k_1(x_1-\frac{1}{6}x_1^3+\cdots)\\ \dot{x}_2=-\omega_1(x_1-x_3)+2a_1(x_1-\frac{1}{2}x_1x_2^2+\frac{1}{2}x_1^3+\cdots)+k_2(x_2-\frac{1}{6}x_2^3+\cdots)\\ \dot{x}_3=-2a_2(-\frac{1}{2}x_3^2x_4++x_4-\frac{1}{6}x_4^3+\cdots)+k_3(x_3-\frac{1}{6}x_4^3+\cdots)\\ \dot{x}_4=-\omega_2(x_3-x_1)+2a_2(x_3-\frac{1}{2}x_3x_4^2+\frac{1}{2}x_3^3+\cdots)+k_4(x_4-\frac{1}{6}x_4^3+\cdots)\end{cases} \tag{3-2-7}$$

whose Jacobian matrix is

$$\begin{bmatrix} -k_1 & -2a_1 & 0 & 0 \\ 2a_1-\omega_1 & -k_2 & \omega_1 & 0 \\ 0 & 0 & -k_3 & -2a_2 \\ \omega_2 & 0 & 2a_2-\omega_2 & -k_4 \end{bmatrix}.$$

The characteristic equation of J is

$$\begin{bmatrix} -k_1-\lambda & -2a_1 & 0 & 0 \\ 2a_1-\omega_1 & -k_2-\lambda & \omega_1 & 0 \\ 0 & 0 & -k_3-\lambda & -2a_2 \\ \omega_2 & 0 & 2a_2-\omega_2 & -k_4-\lambda \end{bmatrix}$$

$$= (-k_4-\lambda)\begin{bmatrix} -k_1-\lambda & -2a_1 & 0 \\ 2a_1-\omega_1 & -k_2-\lambda & \omega_1 \\ 0 & 0 & -k_3-\lambda \end{bmatrix} + 2a_2\begin{bmatrix} -k_1-\lambda & -2a_1 & 0 \\ 2a_1-\omega_1 & -k_2-\lambda & \omega_1 \\ \omega_2 & 0 & 2a_2-\omega_2 \end{bmatrix}$$

$$\begin{aligned} &= \lambda^4 + (k_1+k_2+k_3+k_4)\lambda^3 + ((k_1+k_2)(k_3+k_4) + k_3k_4 + k_1k_2 + 4(a_1^2+a_2^2) - 2(a_1\omega_1 \\ &+((k_1+k_2)(k_3k_4+4a_2^2-2a_2\omega_2) + (k_3+k_4)(k_1k_2+4a_1^2-2a_1\omega_1))\lambda + a_2\omega_2))\lambda^2 + (k_1k_2k_3k_4 \\ &+4(a_1^2k_3k_4 + a_2^2k_1k_2) - 2(a_1\omega_1k_3k_4 - a_2\omega_2k_1k_2) + 16a_1^2a_2^2 + 8a_1a_2(a_1\omega_2 + a_2\omega_1)). \end{aligned} \quad (3\text{-}2\text{-}8)$$

Take

$$\begin{aligned} &B = (k_1+k_2+k_3+k_4),\ C = (k_1+k_2)(k_3+k_4) + k_3k_4 + k_1k_2 + 4(a_1^2+a_2^2) - 2(a_1\omega_1 + a_2\omega_2) \\ &D = (k_1+k_2)(k_3k_4+4a_2^2-2a_2\omega_2) + (k_3+k_4)(k_1k_2+4a_1^2-2a_1\omega_1), \\ &E = (k_1k_2k_3k_4 + 4(a_1^2k_3k_4 + a_2^2k_1k_2) - 2(a_1\omega_1k_3k_4 - a_2\omega_2k_1k_2) + 16a_1^2a_2^2 + 8a_1a_2(a_1\omega_2 + a_2\omega_1)). \end{aligned}$$

If

$$B > 0,\ \frac{BC-D}{B} > 0,\ \frac{B(CD-BE)-D^2}{BC-D} > 0 \ \text{ and } \ E > 0 \quad (3\text{-}2\text{-}9)$$

then all the eigenvalues of the Jacobian matrix have negative real parts. So we choose k_1=1, k_2=1, k_3=1 and k_4=1, and find B=4, $\frac{BC-D}{B} = 4.18$, $\frac{B(CD-BE)-D^2}{BC-D} = 4.22$ and E=4.62.

Proposition 2. According to Routh-Hurwitz theorem and Lyapunov asymptotically stability theorem, when k_i's satisfy (3-2-9), the controlled Quantum-CNN system (3-2-6) is locally asymptotically stable at the equilibrium O (0, 0, 0, 0) in spite of the higher order terms.

Numerical simulations are used to investigate the controlled Quantum-CNN system (3-2-1) using fourth-order Runge-Kutta scheme with time step 0.005. After 100 second, the states (x_1, x_2, x_3, x_4) of the controlled Quantum-CNN system (3-2-6) approach zeros in Fig. 3-4.

Case III Nonlinear state cross cosine nonlinear function feedback control

We design controller is $u_i = k_i x_i \cos x_i$

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 + k_1 x_1 \cos x_1 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + k_2 x_2 \cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 + k_3 x_3 \cos x_3 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + k_4 x_4 \cos x_4 \end{cases} \tag{3-2-10}$$

Expand the right hand sides of Eq. (3-2-10) into power series:

$$\begin{cases} \dot{x}_1 = -2a_1(-\frac{1}{2}x_1^2 x_2 + +x_2 - \frac{1}{6}x_2^3 + \cdots) + k_1(x_1 - \frac{1}{6}x_1^3 + \cdots) \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1(x_1 - \frac{1}{2}x_1 x_2^2 + \frac{1}{2}x_1^3 + \cdots) + k_2(x_2 - \frac{1}{6}x_2^3 + \cdots) \\ \dot{x}_3 = -2a_2(-\frac{1}{2}x_3^2 x_4 + +x_4 - \frac{1}{6}x_4^3 + \cdots) + k_3(x_3 - \frac{1}{6}x_4^3 + \cdots) \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2(x_3 - \frac{1}{2}x_3 x_4^2 + \frac{1}{2}x_3^3 + \cdots) + k_4(x_4 - \frac{1}{6}x_4^3 + \cdots) \end{cases} \tag{3-2-11}$$

whose Jacobian matrix is

$$\begin{bmatrix} -k_1 & -2a_1 & 0 & 0 \\ 2a_1 - \omega_1 & -k_2 & \omega_1 & 0 \\ 0 & 0 & -k_3 & -2a_2 \\ \omega_2 & 0 & 2a_2 - \omega_2 & -k_4 \end{bmatrix}.$$

The characteristic equation of J is

$$\begin{bmatrix} -k_1-\lambda & -2a_1 & 0 & 0 \\ 2a_1 - \omega_1 & -k_2-\lambda & \omega_1 & 0 \\ 0 & 0 & -k_3-\lambda & -2a_2 \\ \omega_2 & 0 & 2a_2 - \omega_2 & -k_4-\lambda \end{bmatrix}$$

$$= (-k_4-\lambda)\begin{bmatrix} -k_1-\lambda & -2a_1 & 0 \\ 2a_1-\omega_1 & -k_2-\lambda & \omega_1 \\ 0 & 0 & -k_3-\lambda \end{bmatrix} + 2a_2\begin{bmatrix} -k_1-\lambda & -2a_1 & 0 \\ 2a_1-\omega_1 & -k_2-\lambda & \omega_1 \\ \omega_2 & 0 & 2a_2-\omega_2 \end{bmatrix}$$

$$\begin{aligned} &= \lambda^4 + (k_1+k_2+k_3+k_4)\lambda^3 + ((k_1+k_2)(k_3+k_4) + k_3k_4 + k_1k_2 + 4(a_1^2+a_2^2) - 2(a_1\omega_1 \\ &+ a_2\omega_2))\lambda^2 + ((k_1+k_2)(k_3k_4 + 4a_2^2 - 2a_2\omega_2) + (k_3+k_4)(k_1k_2 + 4a_1^2 - 2a_1\omega_1))\lambda + (k_1k_2k_3k_4 \\ &+ 4(a_1^2k_3k_4 + a_2^2k_1k_2) - 2(a_1\omega_1k_3k_4 - a_2\omega_2k_1k_2) + 16a_1^2a_2^2 + 8a_1a_2(a_1\omega_2 + a_2\omega_1)). \end{aligned} \tag{3-2-12}$$

Take

$B=(k_1+k_2+k_3+k_4)$, $\quad C=(k_1+k_2)(k_3+k_4)+k_3k_4+k_1k_2+4(a_1^2+a_2^2)-2(a_1\omega_1+a_2\omega_2)$,
$D=(k_1+k_2)(k_3k_4+4a_2^2-2a_2\omega_2)+(k_3+k_4)(k_1k_2+4a_1^2-2a_1\omega_1)$,
$E=(k_1k_2k_3k_4+4(a_1^2k_3k_4+a_2^2k_1k_2)-2(a_1\omega_1k_3k_4-a_2\omega_2k_1k_2)+16a_1^2a_2^2+8a_1a_2(a_1\omega_2+a_2\omega_1))$.

If

$$B>0\,,\ \frac{BC-D}{B}>0\,,\ \frac{B(CD-BE)-D^2}{BC-D}>0\ \text{ and }\ E>0 \tag{3-2-13}$$

then all the eigenvalues of the Jacobian matrix have negative real parts. So we choose k_1=1, k_2=1, k_3=1 and k_4=1, and find B=4, $\frac{BC-D}{B}=4.18$, $\frac{B(CD-BE)-D^2}{BC-D}=4.22$ and E=4.62.

Proposition 3. According to Routh-Hurwitz theorem and Lyapunov asymptotically stability theorem, when k_i's satisfy (3-2-13), the controlled Quantum-CNN system (3-2-10) is locally asymptotically stable at the equilibrium $O\,(0, 0, 0, 0)$ in spite of the higher order terms.

Numerical simulations are used to investigate the controlled Quantum-CNN system (3-2-1) using fourth-order Runge-Kutta scheme with time step 0.005. After 100 second, the behaviors of the states (x_1, x_2, x_3, x_4) of the controlled Quantum-CNN system (3-2-10) approach zeros in Fig. 3-5.

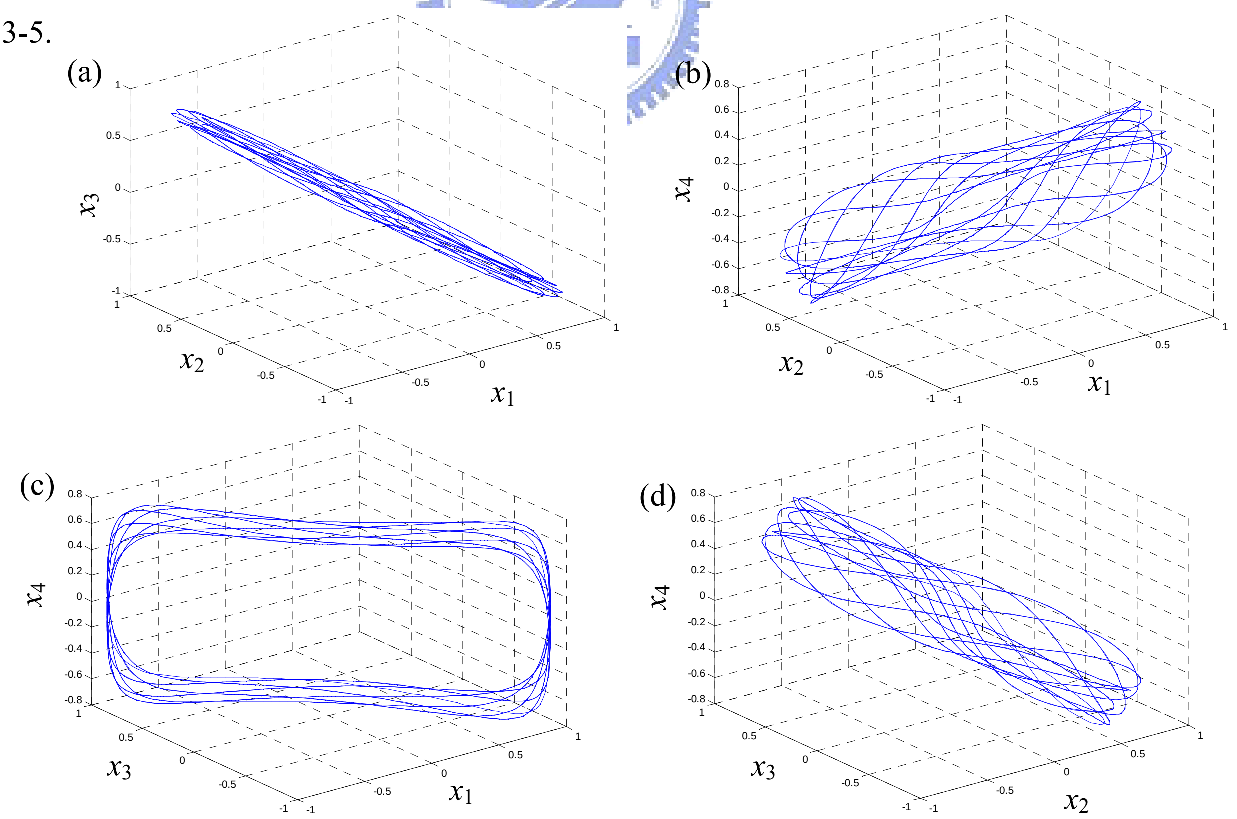


Fig. 3-1. Phase portraits of the Quantum-CNN System.

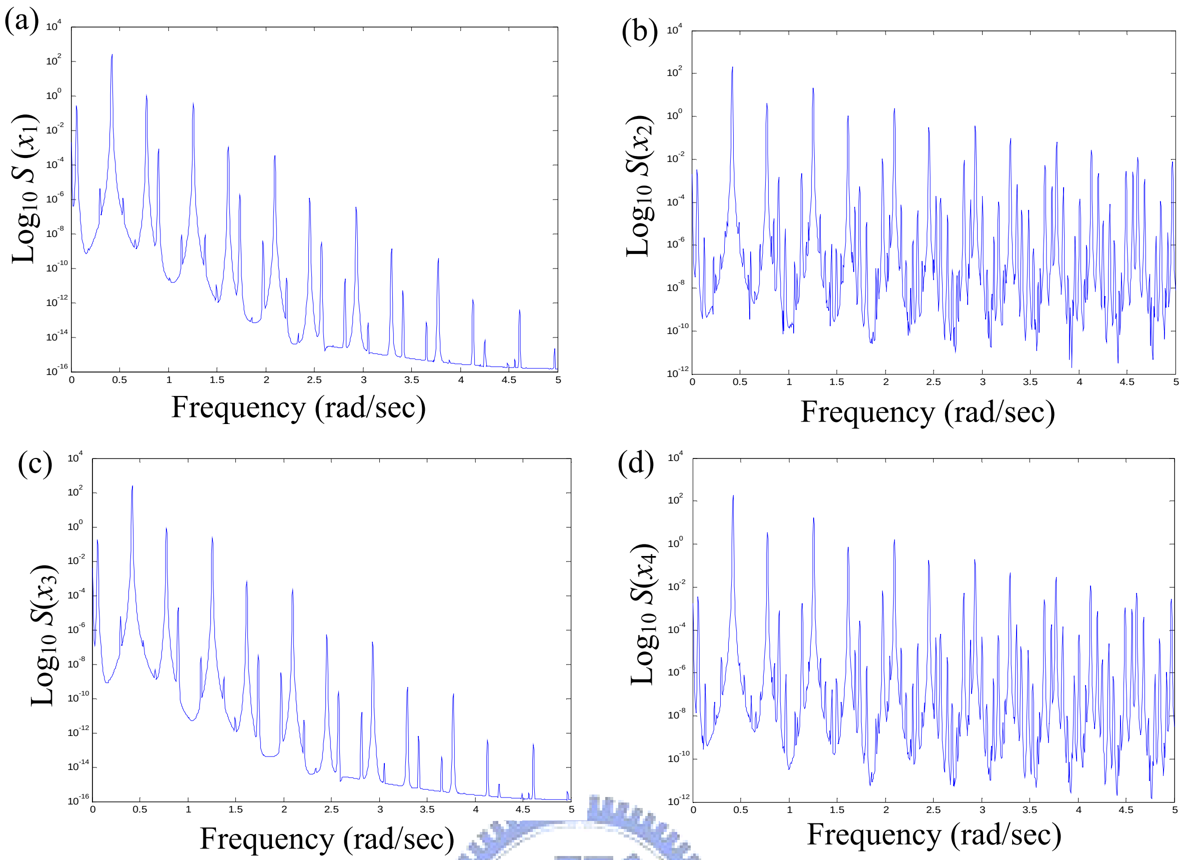


Fig. 3-2. Power Spectrum Density diagram of the Quantum-CNN System.

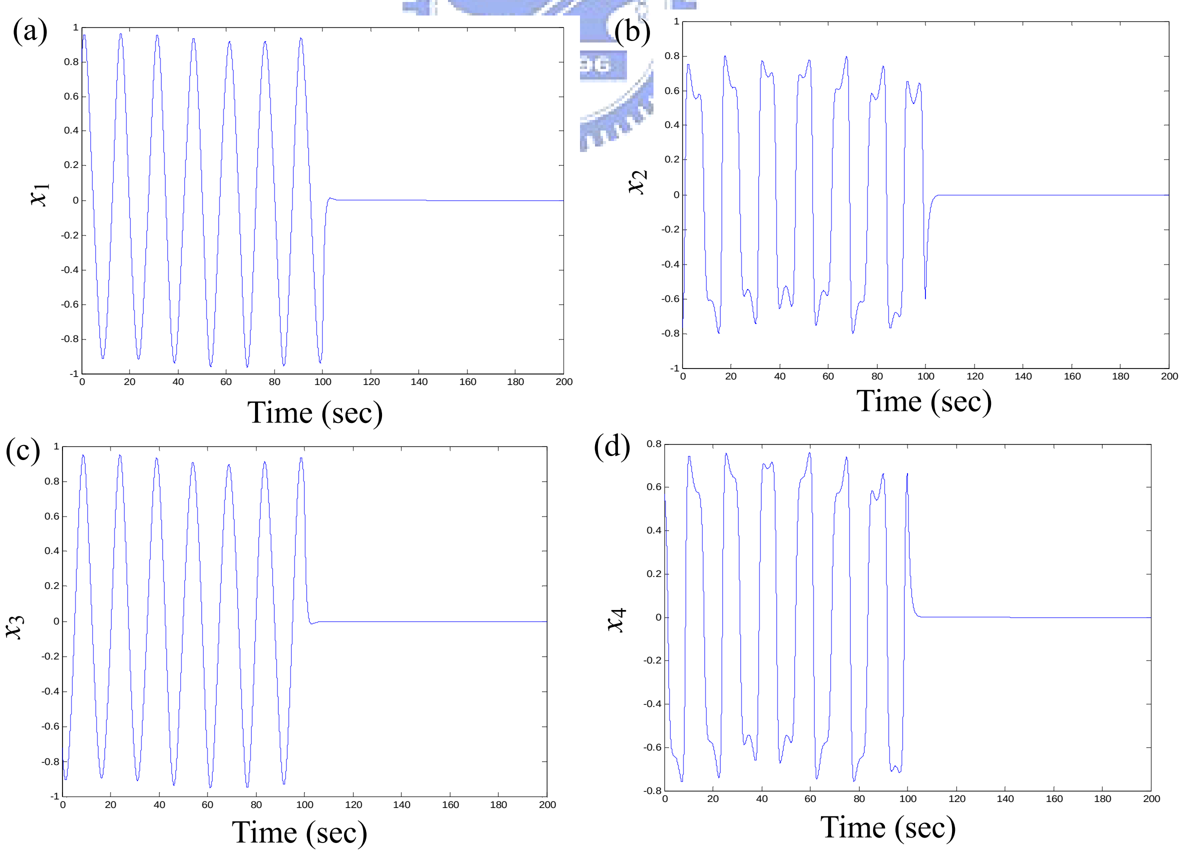


Fig. 3-3. Time histories of the states of the Quantum-CNN System (3-3-2).

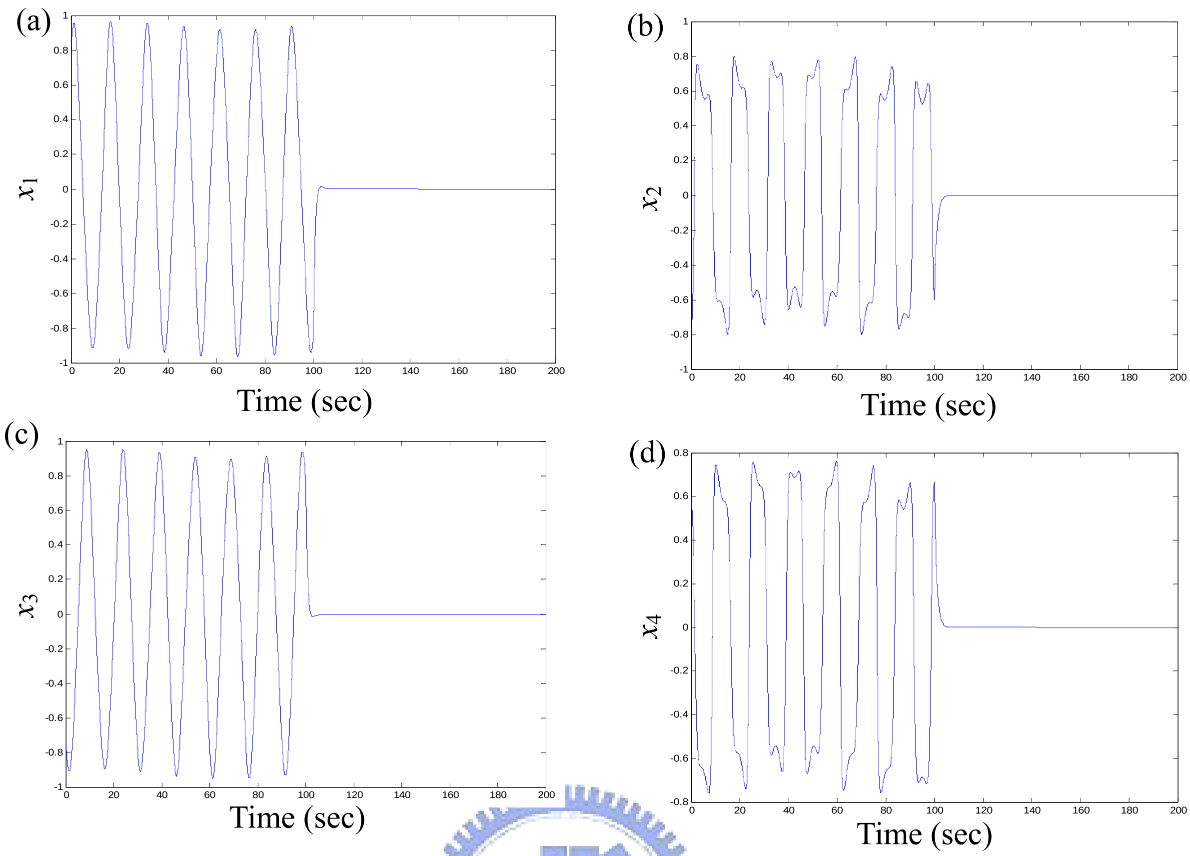


Fig. 3-4. Time histories of the states of the Quantum-CNN System (3-2-6).

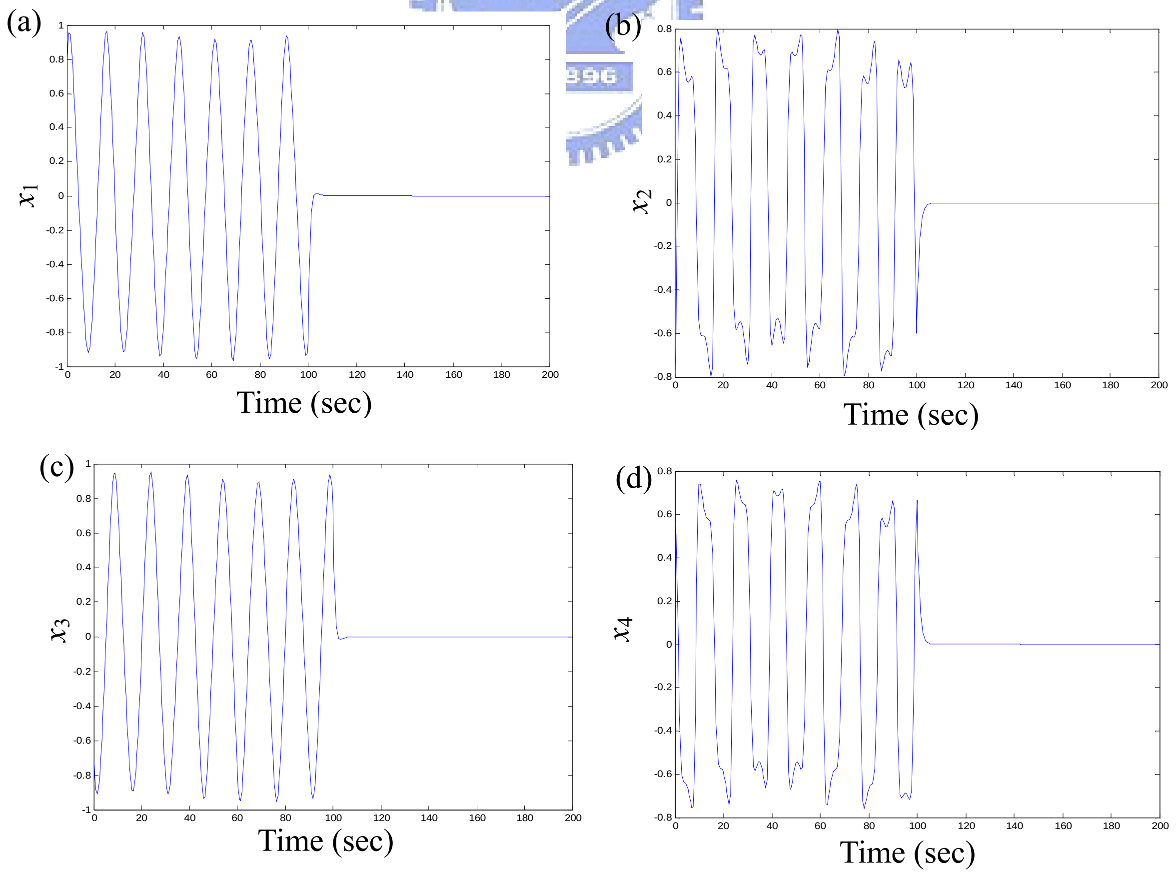


Fig. 3-5. Time histories of the states of the Quantum-CNN System (10).

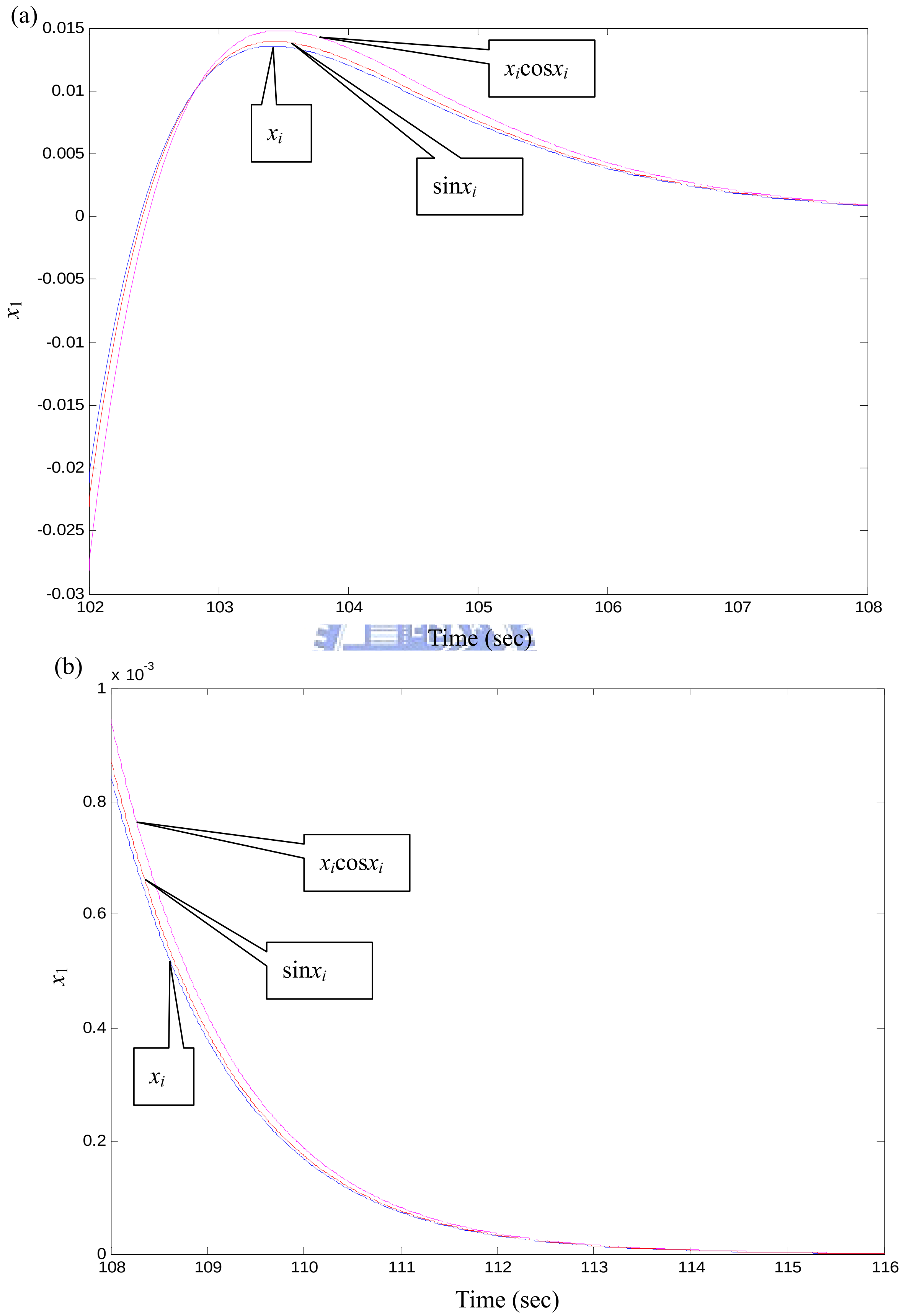


Fig. 3-6. Performances of states of the system with linear, sine or state cross cosine function feedback control.

3-3 Chaos Control of Quantum-CNN Oscillators Chaotic System by Variable Structure Control

Assume that the aim is to control system $\dot{x} = Ax + f(x) + u(t)$ tracking a given desired state vector $y = [y_1, y_2, \cdots, y_n]^T$, where $\dot{y} = Ay$. Let $e=x-y$ be the tracking error vector.

The tracking error dynamics is:

$$\dot{e} = \dot{x} - \dot{y} = Ae + f(x) - u(t) \tag{3-3-1}$$

where $f(x)$ is nonlinear item vector.

The controller is designed as $u(t)=H(\text{t})+f(x)$ in which $H(t)=Kw(t)$. The newly defined control signal $w(t)$ is determined through the sliding mode approach,

$$w(t)\begin{cases} w^+(t) & s(e) > 0 \\ w^-(t) & s(e) < 0 \end{cases} \tag{3-3-2}$$

$s(e)$ is the switching surface and is considered as

$$s(e) = Ce\,. \tag{3-3-3}$$

The reaching law assumed to be $\dot{s} = -q \cdot \text{sgn}(s) - rs$. This design results in the following control signal.

$$w(t) = -(CK)^{-1}\left[C(rI + A)e + q \cdot \text{sgn}(s)\right] \tag{3-3-4}$$

It can be shown that the closed loop system will be stable for positive r and q parameters.

Finally, let us consider our dynamic system Quantum-CNN system. The equation considered is

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\,\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\,\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{3-3-5}$$

Let $a_1 = a_2 = 4.9$, $\omega_1 = 1.13$ and $\omega_2 = 0.85$. The initial values of Quantum-CNN systems are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$ and $x_4(0) = 0.97$ respectively. The result is show in Fig. 3-7 for 1-T periodic motion.

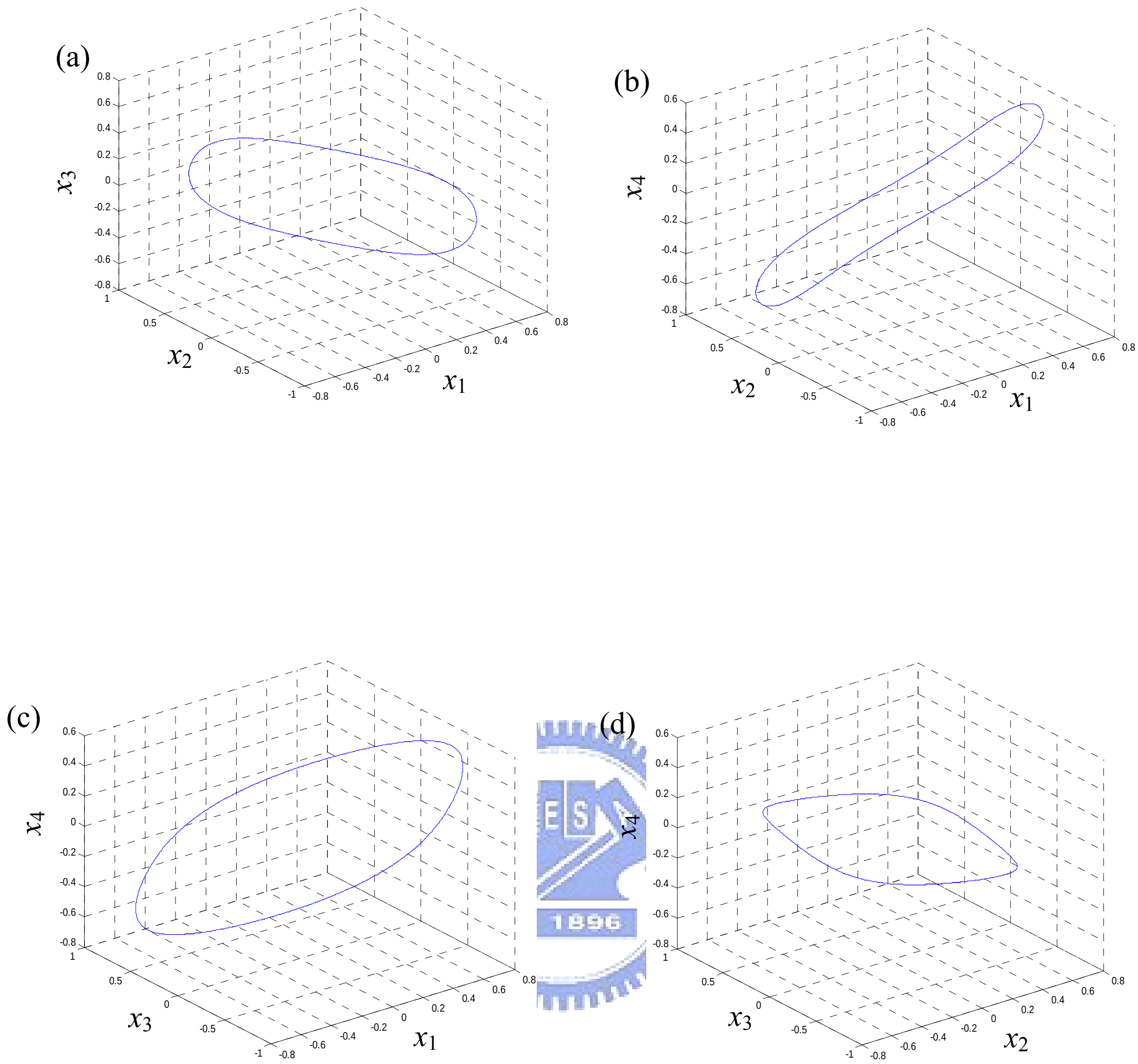


Fig. 3-7. Phase portraits of Quantum-CNN by variable structure control.

3-4 Chaos Control of Quantum-CNN Oscillators Chaotic System by Impulse Control

A technique for suppressing chaos is to apply a periodic impulse input to the system. Consider the system of the form (3-2-1) and assume that the system is controlled by a periodic impulse input

$$u = \rho \sum_{i=0}^{\infty} \delta(\tau - iT_d) \tag{3-4-1}$$

where ρ is a constant impulse intensity, T_d is the periodic between two consecutive impulses, and δ is the standard delta function. With different values of ρ and T_d the controlled system can be stabilized at different periodic orbits or fix point.

Finally, let us consider our dynamic system Quantum-CNN system with periodic impulse of linear feedback. The equation considered is

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 - u_1 x_1, \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - u_2 x_2, \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 - u_3 x_3, \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - u_4 x_4. \end{cases} \tag{3-4-2}$$

Let $a_1 = a_2 = 2.47$, $\omega_1 = 1$ and $\omega_2 = 1$. The initial values of Quantum-CNN systems are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$ and $x_4(0) = 0.57$ respectively.

The result is show in Fig. 3-8.

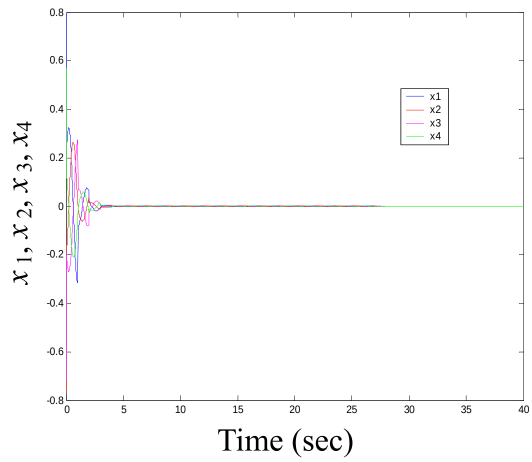


Fig. 3-8. Time histories of states by impulse control.

3-5 Chaotization of Quantum-CNN Chaotic System by Optimum Control

Optimal control is a well-established engineering control strategy, and is useful for both linear and nonlinear system with linear or nonlinear controllers [98]. Now, we use a typical optimal control for the chaotization of chaos of Quantum-CNN system. Let $a_1 = a_2 = 2.47$ and $\omega_1 = \omega_2 = 1$. The initial values of linear coupled Quantum-CNN systems are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$ and $x_4(0) = 0.57$ respectively. Consider the system (3-2-1) with a controller u and define the Hamilton function:

$$H(x_1, x_2, x_3, x_4, u, p) = \boldsymbol{p}^T \boldsymbol{F}(x_1, x_2, x_3, x_4, u, p); \quad \boldsymbol{p}^T = [p_1 \ p_2 \ p_3 \ p_4] \tag{3-5-1}$$

where $\boldsymbol{p}$ is a Lagrange multiplier, called a co-state vector, $\boldsymbol{F}$ is the right hand side of Eq. (3-2-1). Following the variation principle of optimal control, we can obtain

$$\begin{aligned} & p_2(-\omega_1(x_1 - x_3) + 2a_1 \frac{x_1}{\sqrt{1-x_1^2}} \cos x_2) + p_3(-2a_2\sqrt{1-x_3^2} \sin x_4) + p_4(-\omega_2(x_3 - x_1) \\ & \quad + 2a_2 \frac{x_3}{\sqrt{1-x_3^2}} \cos x_4) = 0 \end{aligned} \tag{3-5-2}$$

$$p_2 \frac{-2a_1}{\sqrt{1-x_1^2}} \sin x_2 = 0 \tag{3-5-3}$$

This yield a non-trivial solution for (p_2, p_3, p_4) if and only if

$$\frac{-2a_1}{\sqrt{1-x_1^2}} \sin x_2 = 0 \tag{3-5-4}$$

It gives an optimal surface singularly in the state space. This type of control assumes values on the two allowable boundaries (3-5-3) and (3-5-4) alternatively according to a switching surface. Locating system trajectories on the surface, a typical feedback control in the form

$$u = -k_b \operatorname{sgn}[\frac{-2a_1}{\sqrt{1-x_1^2}} \sin x_2] \tag{3-5-5}$$

can be used. By adjusting the value of k_b from zero initial value to $k_b = 1.6 \times 10^{-4}$ in the above controller with the signum function

$$\text{sgn}[v] = \begin{cases} 1 & \text{if} \quad v > 0 \\ 0 & \text{if} \quad v = 0 \\ -1 & \text{if} \quad v < 0 \end{cases} \tag{3-5-6}$$

the chaotic motion with one positive Lyapunov exponent can be controlled to chaotic motion with two positive Lyapunov exponents as shown by the simulation result in Fig. 3-9.

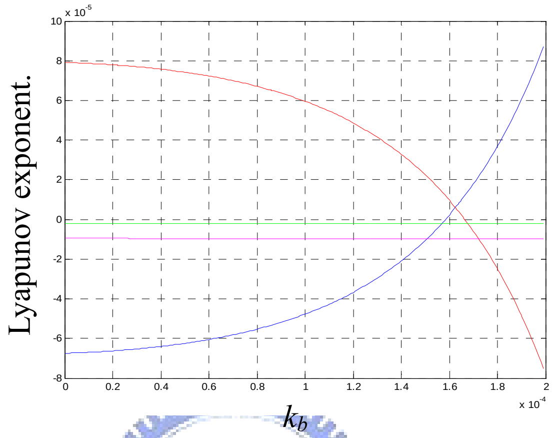


Fig. 3-9. Lyapunov exponent of Quantum-CNN system.

Chapter 4
Generalized and Symplectic Synchronizations

4-1 Preliminary

In recent years, the synchronization of chaotic systems has been studied in various fields. For a particular chaotic system, a master, together with an identical or a different system, a slave system, our goal is to synchronize them via coupling or other methods.

Among many kinds of synchronizations, the generalized synchronization is investigated. It means there exists a functional relationship between the states of the master and those of the slave. In this Section 4-2, a special kind of generalized synchronizations

$$y=x+F(t) \tag{4-1-1}$$

is studied, where x, y are the state vectors of the master and the slave respectively, $F(t)$ is a given vector function of time, which may take various forms, either regular or chaotic functions of time. The generalized synchronization developed may be applied to the design of secure communication. When $F(t)$=0, it reduces to a complete synchronization.

In this Section 4-3, a special kind of generalized synchronizations

$$y_i = \pm x_i + F_i(t), \quad i = 1, 2, \cdots, 2n \tag{4-1-2}$$

is studied, where x_i, y_i are the states of the master and the slave respectively, $F_i(t)$ is a given function of time, which may take various forms, either a regular or a chaotic functions of time. When $F_i(t)$=0, it reduces to a complete synchronization when the signs before x_i are all positive; it reduces to an anti-synchronization when $F_i(t)$=0 and the signs before x_i are all negative; it reduces to a partial complete synchronization and a partial anti-synchronization when the signs before x_i are partly positive and partly negative.

Many approaches have been presented for the synchronization of chaotic systems. There are a chaotic master system and either an identical or a different slave system. Our goal is the synchronization of the chaotic master and the chaotic slave by coupling or by other methods.

Among many kinds of synchronization, generalized synchronization is investigated. There exists a functional relationship between the states of the master and that of the slave. In this Section 4-4, a new synchronization

$$y = H(x, y, t) + F(t) \tag{4-1-3}$$

is studied, where x, y are the state vectors of the "master" and of the "slave", respectively, $F(t)$ is a given function of time in different form, such as a regular or a chaotic function. When $H(x, y, t)=x$, Eq. (4-1-3) reduces to the generalized synchronization given in Section 4-2.

In Eq. (4-1-3), the final desired state y of the "slave" system not only depends upon the "master" system state x but also depends upon the "slave" system state y itself. Therefore the "slave" system is not a traditional pure slave obeying the "master" system completely but plays a role to determine the final desired state of the "slave" system. In other words, it plays an "interwined" role, so we call this kind of synchronization "symplectic synchronization"*, and call the "master" system partner A, the "slave" system partner B.

When $H(x, y, t)=H(x, t)+F(t)$, Eq. (4-1-3) becomes

$$y=H(x, t)+F(t) \tag{4-1-4}$$

which reduces to generalized synchronization. Therefore generalized synchronization is a special case of the symplectic synchronization. There exists great potential of the application of the symplectic synchronization. For instance, when the symplectically synchronized chaotic signal is used as a signal carrier, the secure communication is more difficult to be deciphered.

The variable structure control technique is a discontinuous control strategy that involves, first, selecting a switching surface for the desired dynamics and, secondly, designing a discontinuous control law such that the system trajectory first reaches the surface and then stays in it forever.

4-2 The Generalized Synchronization of a Quantum-CNN Chaotic Oscillator with Different Order Systems

4-2-1 Generalized Synchronization Scheme

*The term **"symplectic"** comes from the Greek for "interwined". H. Weyl first introduced the term in 1939 in his book "The Classical Groups" (P. 165 in both the first edition, 1939, and second edition, 1946, Princeton University Press).

There are two identical nonlinear dynamical systems, and the master system controls the slave system. The master system is given here

$$\dot{x} = Ax + f(x) \tag{4-2-1}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ denotes a state vector, A is a $n\times n$ coefficient matrix, and f is a nonlinear vector function.

The slave system is given here

$$\dot{y} = By + h(y) + u(t) \tag{4-2-2}$$

where $y = [y_1, y_2, \cdots, y_{n-\alpha}]^T \in R^{n-\alpha}$ denotes a state vector, α is a positive integer, $1 \le \alpha \le n$, B is a $(n\text{-}\alpha)\times(n\text{-}\alpha)$ coefficient matrix, h is a nonlinear vector function, and $u(t) = [u_1(t), u_2(t), \cdots$ $u_{n-\alpha}(t)]^T \in R^{n-\alpha}$ is a control input vector.

Our goal is to design a controller $u(t)$ so that the state vector of the slave system (4-2-2) asymptotically approaches the state vector of the master system (4-2-1) plus a given vector function $F(t) = [F_1(t), F_2(t), \cdots, F_{n-\alpha}(t)]^T$, and finally the synchronization will be accomplished in the sense that the limit of the error vector $e(t) = [e_1, e_2, \cdots, e_{n-\alpha}]^T$ approaches zero:

$$\lim_{t\to\infty} e = 0 \tag{4-2-3}$$

where

$$e = x - y + F(t) \tag{4-2-4}$$

From Eq. (4-2-4) we have

$$\dot{e} = \dot{x} - \dot{y} + \dot{F}(t) \tag{4-2-5}$$

$$\dot{e} = (A_{n-\alpha} - B)e + f(x) - h(y) + \dot{F}(t) - u(t) \tag{4-2-6}$$

A Lyapnuov function $V(e)$ is chosen as a positive definite function

$$V(e) = \frac{1}{2}e^T e \tag{4-2-7}$$

Its derivative along any solution of Eq. (4-2-6) is

$$\dot{V}(e) = e^T\{(A_{n-\alpha} - B)[x_i - y_i] + f(x) - h(y) + \dot{F}(t) - u(t)\}, \quad i = 1, 2, \cdots, n-\alpha \tag{4-2-8}$$

where $[x_i - y_i]$ is a $(n\text{-}\alpha)\times 1$ column matrix, $u(t)$ is chosen so that $\dot{V} = e^T C_{n-\alpha} e$, $C_{n\text{-}\alpha}$ is a diagonal negative definite matrix, and $\dot{V}$ is a negative definite function of e. By the Lyapunov theorem of asymptotical stability, we have

$$\lim_{t\to\infty} e = 0\,. \tag{4-2-9}$$

The generalized synchronization is obtained.

4-2-2 Numerical Results of the Generalized Chaos Synchronization of the Quantum-CNN Oscillator with Different Order Systems

Case I A complete synchronization as a special case of the generalized synchronization

For a two-cell Quantum-CNN, the following differential equations are obtained:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{4-2-10}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let $a_1 = 19.4$, $a_2 = 13.1$, $\omega_1 = 9.529$ and $\omega_2 = 7.94$.

A Lorenz system is described by

$$\dot{y}_1 = \sigma(y_2 - y_1),\ \dot{y}_2 = \gamma y_1 - y_2 - y_1 y_3,\ \dot{y}_3 = y_1 y_2 - \beta y_3 \tag{4-2-11}$$

where $\sigma = 10, \gamma = 28, \beta = \dfrac{3}{8}$.

Take α=1. In order to lead (y_1, y_2, y_3) to (x_1, x_2, x_3), we add u_1, u_2 and u_3 to the first, the second and the third equations of Eq. (4-2-11) respectively.

$$\dot{y}_1 = \sigma(y_2 - y_1) + u_1,\ \dot{y}_2 = \gamma y_1 - y_2 - y_1 y_3 + u_2,\ \dot{y}_3 = y_1 y_2 - \beta y_3 + u_3 \tag{4-2-12}$$

Subtracting Eq. (4-2-12) from the first three equations of Eq. (4-2-10), we obtain an error dynamics. The initial values of the Quantum-CNN system and the Lorenz system are taken as $x_1(0)=0.8$, $x_2(0)=-0.77$, $x_3(0)=-0.72$, $x_4(0)=0.57$, $y_1(0)=-0.2$, $y_2(0)=-0.42$ and $y_3(0)=-0.11$.

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \quad i = 1, 2, 3 \tag{4-2-13}$$

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma(y_2 - y_1) - u_1 \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_2 + y_1 y_3 - u_2 \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_1 y_2 + \beta y_3 - u_3
\end{aligned} \tag{4-2-14}$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2$ and $e_3=x_3-y_3$.

Choose a Lyapunov function in the form of the positive definite function:

$$V(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) \tag{4-2-15}$$

Its time derivative along any solution of Eq. (4-2-14) is

$$\begin{aligned}
\dot{V} = {} & e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma(y_2 - y_1) - u_1] + e_2[-\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\
& - \gamma y_1 + y_2 + y_1 y_3 - u_2] + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 - y_1 y_2 + \beta y_3 - u_3]
\end{aligned} \tag{4-2-16}$$

Choose

$$\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma y_2 - \sigma x_1 \\
u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_1 y_3 - x_2 \\
u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_1 y_2 + \beta x_3
\end{aligned}$$

Eq. (4-2-16) can be rewritten as

$$\dot{V} = -\sigma e_1^2 - e_2^2 - \gamma e_3^2 < 0 \tag{4-2-17}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. This means that the complete chaos synchronization of the different order systems, the Quantum-CNN system and the Lorenz system, can be achieved. The numerical results are shown in Fig. 4-1. After 10 second, the motion trajectories will enter a chaotic attractor.

Case II A sine function synchronization

We have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + F_i \sin \omega t) = 0, \quad i = 1, 2, 3 \tag{4-2-18}$$

where $\dot{e} = \dot{x} - \dot{y} + \boldsymbol{F}\,\omega \cos \omega t$.

Let $F_1 = F_2 = F_3 = F$, Eq. (4-2-6) becomes

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma(y_2 - y_1) - u_1 + F\omega\cos\omega t \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_2 + y_1 y_3 - u_2 + F\omega\cos\omega t \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_1 y_2 + \beta y_3 - u_3 + F\omega\cos\omega t
\end{aligned} \tag{4-2-19}$$

where $e_1=x_1-y_1+F\sin\omega t$, $e_2=x_2-y_2+F\sin\omega t$ and $e_3=x_3-y_3+F\sin\omega t$. F and ω are taken as F=0.7, ω=1.

Choose a Lyapunov function in the form of the positive definite function:

$$V(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) \tag{4-2-20}$$

Its time derivative along any solution of Eq. (4-2-19) is

$$\begin{aligned}
\dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma(y_2 - y_1) - u_1 + F\omega\cos\omega t] + e_2[-\omega_1(x_1 - x_3) \\
&+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_2 + y_1 y_3 - u_2 + F\omega\cos\omega t] + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 \\
&-y_1 y_2 + \beta y_3 - u_3 + F\omega\cos\omega t]
\end{aligned} \tag{4-2-21}$$

Choose

$$\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma y_2 + \sigma x_1 + F(\omega\cos\omega t + \sin\omega t) \\
u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_1 y_3 + x_2 + F(\omega\cos\omega t + \sin\omega t) \\
u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_1 y_2 + \beta x_3 + F(\omega\cos\omega t + \sin\omega t)
\end{aligned}$$

Eq. (4-2-21) can be rewritten as

$$\dot{V} = -\sigma e_1^2 - e_2^2 - \gamma e_3^2 < 0 \tag{4-2-22}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. This means that the sine function synchronization of the different order systems, the Quantum-CNN system and the Lorenz system, can be achieved. The numerical results are shown in Fig. 4-2. After 10 second, the motion trajectories will enter a chaotic attractor.

Case III A Chen system state synchronization

The goal system for synchronization is a Chen system

$$\dot{z}_1 = a(z_2 - z_1),\ \dot{z}_2 = (c-a)z_1 - z_1 z_3 - c z_2,\ \dot{z}_3 = z_1 z_2 - b z_3 \tag{4-2-23}$$

where $a = 10, b = \frac{3}{8}, c = 28$. The initial values of the states of the Chen system is taken as $z_1(0)$=0.5, $z_2(0)$=0.5 and $z_3(0)$=0.5. z=(z_1, z_1, z_1) is chosen as $F(t)$. Then

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + z_1) = 0, \quad i = 1, 2, 3 \tag{4-2-24}$$

Eq. (4-2-6) becomes

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma(y_2 - y_1) - u_1 + a(z_2 - z_1) \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_2 + y_1 y_3 - u_2 + a(z_2 - z_1) \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_1 y_2 + \beta y_3 - u_3 + a(z_2 - z_1)
\end{aligned} \tag{4-2-25}$$

where $e_1 = x_1 - y_1 + z_1$, $e_2 = x_2 - y_2 + z_2$ and $e_3 = x_3 - y_3 + z_3$.

Choose a Lyapunov function in the form of the positive definite function:

$$V(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) \tag{4-2-26}$$

Its time derivative along any solution of Eq. (4-2-25) is

$$\begin{aligned}
\dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma(y_2 - y_1) - u_1 + a(z_2 - z_1)] + e_2[-\omega_1(x_1 - x_3) \\
&\quad + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_2 + y_1 y_3 - u_2 + a(z_2 - z_1)] + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 \\
&\quad - y_1 y_2 + \beta y_3 - u_3 + a(z_2 - z_1)]
\end{aligned} \tag{4-2-27}$$

Choose

$$\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma y_2 + \sigma x_1 + a(z_2 - z_1) + z_1 \\
u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_1 y_3 + x_2 + a(z_2 - z_1) + z_1 \\
u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_1 y_2 + \beta x_3 + a(z_2 - z_1) + z_1
\end{aligned}$$

Eq. (4-2-27) can be rewritten as

$$\dot{V} = -\sigma e_1^2 - e_2^2 - \gamma e_3^2 < 0 \tag{4-2-28}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. This means that the Chen system state synchronization of the different order systems, the Quantum-CNN system and the Lorenz system, can be achieved. The numerical results are shown in Fig. 4-3. After 10 second, the motion trajectories will enter a chaotic attractor.

Case IV A Chen system **states** synchronization

We have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + z_i) = 0, \quad i = 1, 2, 3 \tag{4-2-29}$$

and

$$\dot{e} = \dot{x} - \dot{y} + \dot{z} \tag{4-2-30}$$

where $\dot{z} = (\dot{z}_1, \dot{z}_2, \dot{z}_3)$. Eq. (4-2-6) becomes

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma(y_2 - y_1) - u_1 + a(z_2 - z_1) \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_2 + y_1y_3 - u_2 + (c-a)z_1 - z_1z_3 - cz_2 \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_1y_2 + \beta y_3 - u_3 + z_1z_2 - bz_3
\end{aligned} \tag{4-2-31}$$

where $e_1 = x_1 - y_1 + z_1$, $e_2 = x_2 - y_2 + z_2$ and $e_3 = x_3 - y_3 + z_3$.

Choose a Lyapunov function in the form of the positive definite function:

$$V(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) \tag{4-2-32}$$

Its derivative along any solution of Eq. (4-2-30) is

$$\begin{aligned}
\dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma(y_2 - y_1) - u_1 + a(z_2 - z_1)] + e_2[-\omega_1(x_1 - x_3) \\
&\quad + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - \gamma y_1 + y_2 + y_1y_3 - u_2 + (c-a)z_1 - z_1z_3 - cz_2] \\
&\quad + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 - y_1y_2 + \beta y_3 - u_3 + z_1z_2 - bz_3]
\end{aligned} \tag{4-2-33}$$

Choose

$$u_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 - \sigma y_2 + \sigma x_1 + a(z_2 - z_1) + z_1$$

$$u_2 = -\omega_1(x_1 - x_3) + 2a_1 \frac{x_1}{\sqrt{1-x_1^2}} \cos x_2 - \gamma y_1 + y_1 y_3 + x_2 + (c-a)z_1 - z_1 z_3 + z_2(1-c)$$

$$u_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 - y_1 y_2 + \beta x_3 + z_1 z_2 + z_3(1-b)$$

Eq. (4-2-33) can be rewritten as

$$\dot{V} = -\sigma e_1^2 - e_2^2 - \gamma e_3^2 < 0 \qquad (4\text{-}2\text{-}34)$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. This means that the Chen system states synchronization of the different order systems, the Quantum-CNN system and the Lorenz system, can be achieved. The numerical results are shown in Fig. 4-4. After 10 second, the motion trajectories will enter a chaotic attractor.

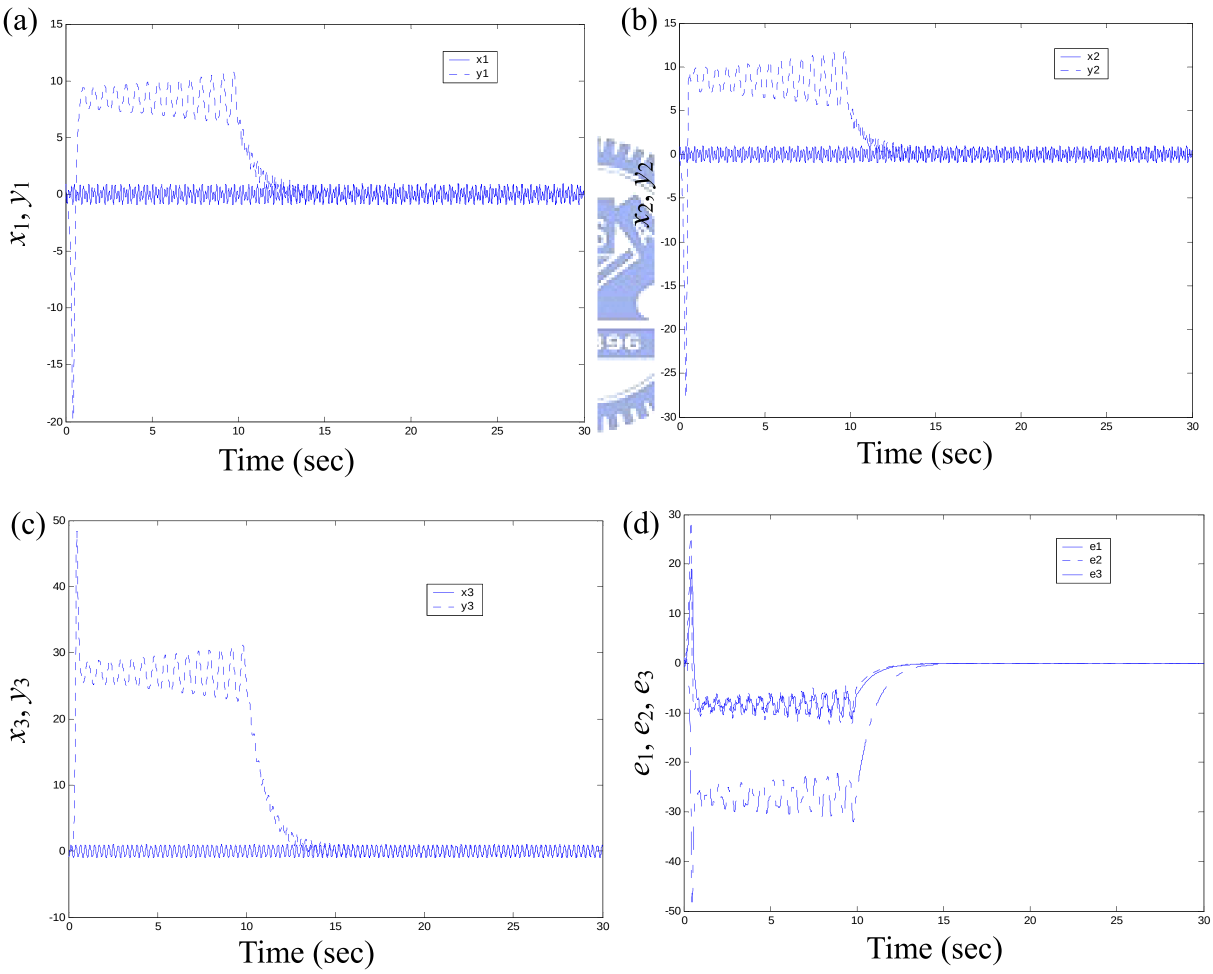


Fig. 4-1. Time histories of the master states, of the slave states, and of the synchronization errors for the Quantum-CNN system and the Lorenz system.

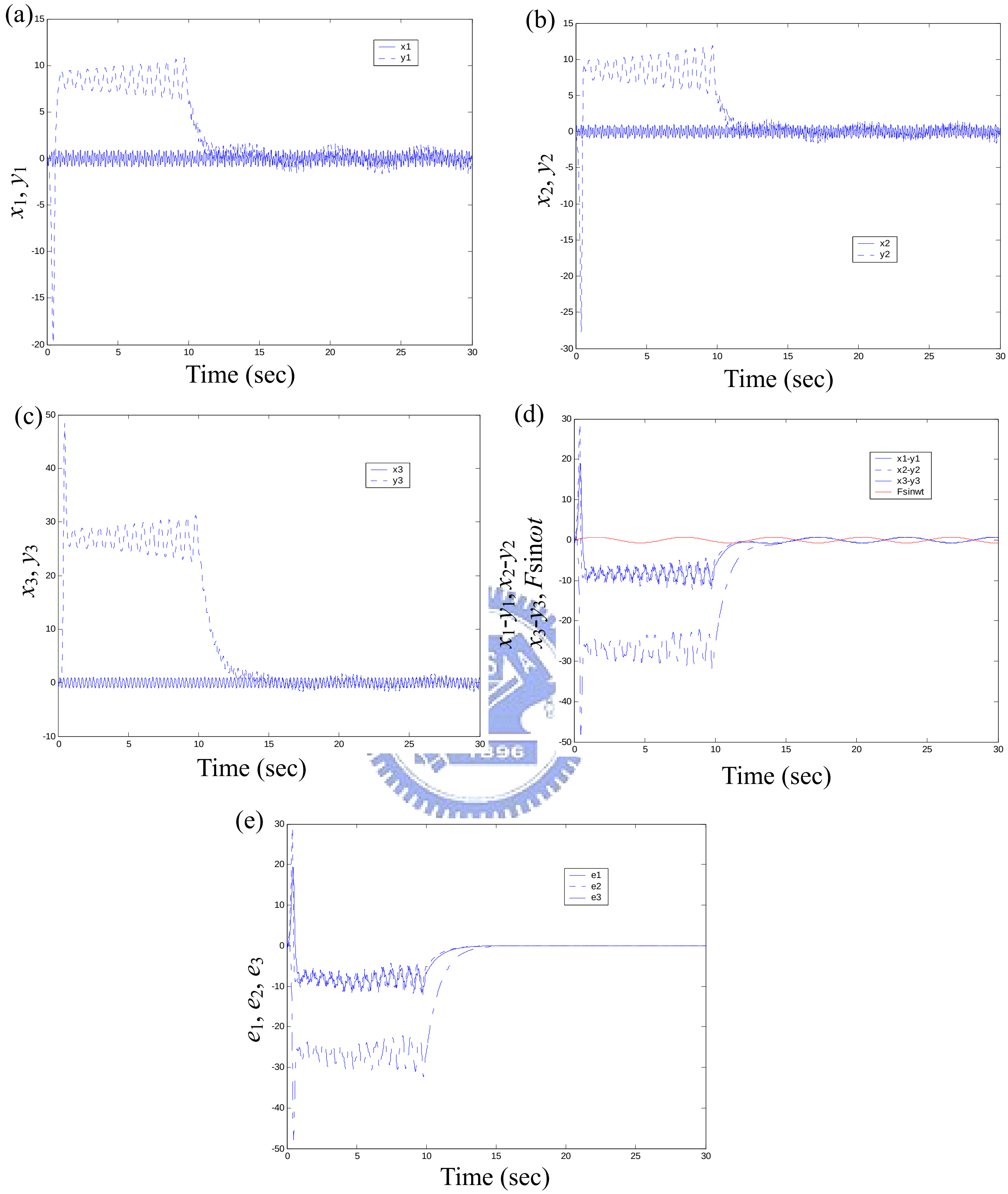


Fig. 4-2. Time histories of the master states, of the slave states, and of the sine function synchronization errors for the Quantum-CNN system and the Lorenz system, where $e_i = x_i - y_i + F\sin\omega t$, i=1, 2, 3.

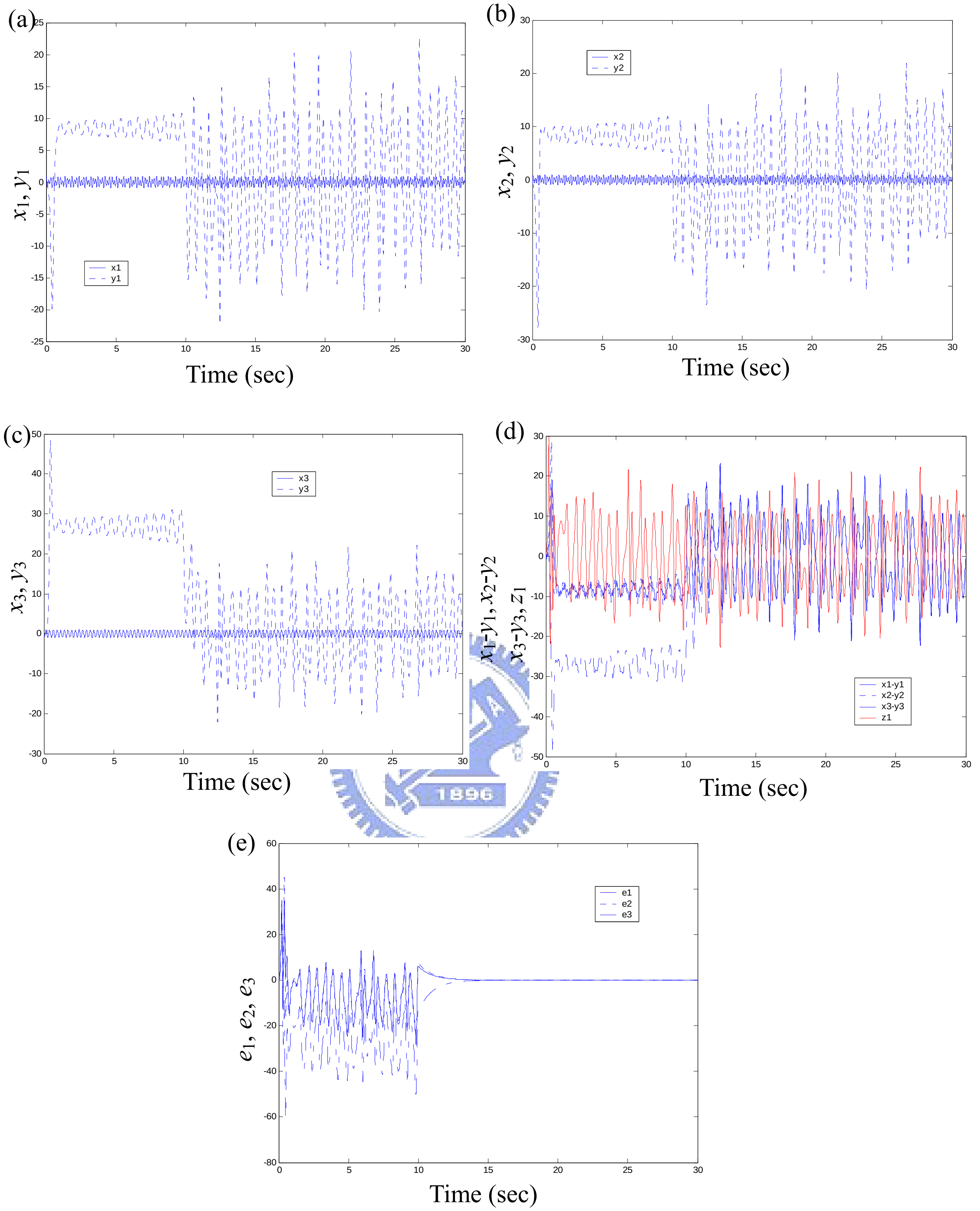


Fig. 4-3. Time histories of the master states, of the slave states, and of the Chen system state synchronization errors for the Quantum-CNN system and the Lorenz system, where $e_i=x_i-y_i+z_1$, i=1, 2, 3.

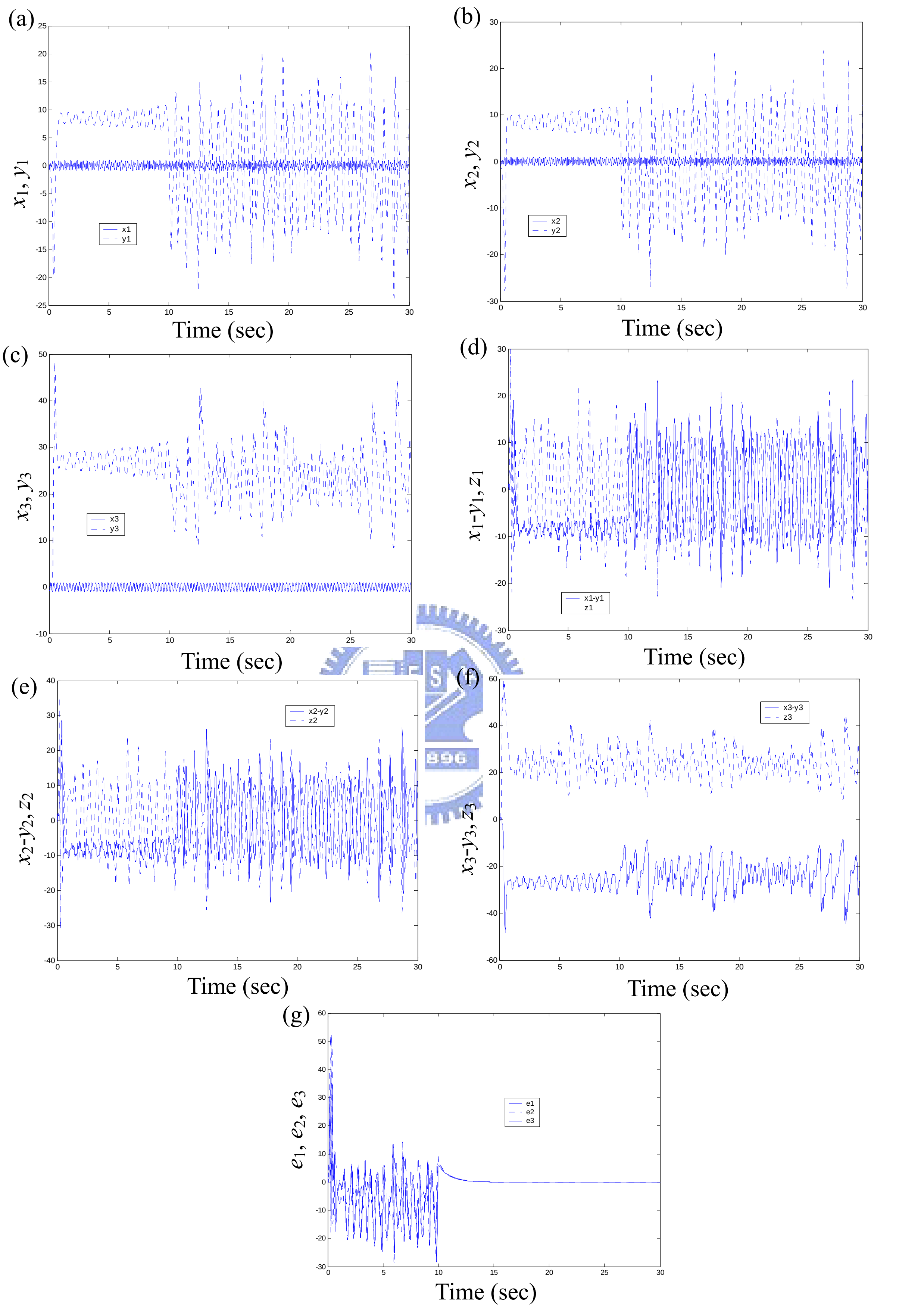


Fig. 4-4. Time histories of the master states, of the slave states, and of the Chen system states synchronization errors for the Quantum-CNN system and the Lorenz system, where $e_i=x_i-y_i+z_i$, i=1, 2, 3.

4-3 The Generalized Synchronization of a Quantum-CNN Chaotic Oscillator with a Double Duffing Chaotic System

4-3-1 Generalized Synchronization Scheme

There are two identical nonlinear dynamical systems and the master system controls the slave system. The master system is given here

$$\dot{x} = Ax + f(x) \tag{4-3-1}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ denotes the state vector, A is a $n \times n$ coefficient matrix, and $f(x)$ is a nonlinear vector function.

The slave system is given here

$$\dot{y} = By + h(y) + u(t) \tag{4-3-2}$$

where $y = [y_1, y_2, \cdots, y_n]^T \in R^n$ denotes a state vector, B is a $n \times n$ coefficient matrix, $h(x)$ is a nonlinear vector function, and $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T \in R^n$ is a control input vector.

Our goal is to design a controller $u(t)$ so that the state vector of the slave system (4-3-2) asymptotically approaches the state vector of the master system (4-3-1) plus a given vector function $F(t) = [F_1(t), F_2(t), \cdots, F_n(t)]^T$ which is either a regular or a chaotic function of time, and finally the synchronization will be accomplished in the sense that the limit of the error vector $e(t) = [e_1, e_2, \cdots, e_n]^T$ approaches zero:

$$\lim_{t \to \infty} e = 0 \tag{4-3-3}$$

where

$$e_i = x_i \mp y_i + F_i(t), \quad i = 1, 2, \cdots, n \tag{4-3-4}$$

From Eq. (4-3-5) we have

$$\dot{e}_i = \dot{x}_i \mp \dot{y}_i + \dot{F}_i(t), \quad i = 1, 2, \cdots, n \tag{4-3-5}$$

Introduce Eq. (4-3-1) and Eq. (4-3-2) in Eq. (4-3-4):

$$\dot{e} = (A - B)[x_i \pm y_i] + f(x) \mp h(y) + \dot{F}(t) \mp u(t), \quad i = 1, 2, \cdots, n \tag{4-3-6}$$

where $[x_i \pm y_i]$ is a $n \times 1$ column matrix.

A Lyapnuov function $V(e)$ is chosen as a positive definite function

$$V(e) = \frac{1}{2} e^T e \tag{4-3-7}$$

Its derivative along the solution of Eq. (4-3-6) is

$$\dot{V}(e) = e^T \{(A-B)[x_i \pm y_i] - f(x) \mp h(y) + \dot{F}(t) \mp u(t)\}, \quad i = 1, 2, \cdots, n \tag{4-3-8}$$

where $u(t)$ is chosen so that $\dot{V} = e^T C_{n\times n} e$, where $C_{n\times n}$ is a diagonal negative definite matrix, and $\dot{V}$ is a negative definite function of e. By the Lyapunov theorem of asymptotical stability

$$\lim_{t\to\infty} e = 0 \tag{4-3-9}$$

The generalized synchronization is obtained.

4-3-2 Numerical Results of Generalized Chaos Synchronization of the Quantum-CNN Oscillator with a Double Duffing Chaotic Systems

Case I A partial complete synchronization and a partial anti-synchronization as the special case of the generalized synchronization

For a two-cell Quantum-CNN, the following differential equations are obtained:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\,\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1 \dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\,\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2 \dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{4-3-10}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let $a_1 = 19.4$, $a_2 = 13.1$, $\omega_1 = 9.529$ and $\omega_2 = 7.94$.

A double Duffing chaotic system is described by

$$\dot{y}_1 = y_2,\ \dot{y}_2 = y_1 - y_1^3 - \rho_1 y_2 + F_1\cos\xi_1 t,\ \dot{y}_3 = y_4,\ \dot{y}_4 = y_3 - y_3^3 - \rho_2 y_4 + F_2\cos\xi_2 t \tag{4-3-11}$$

where $\rho_1 = 0.5, \rho_2 = 1.5, F_1 = 1.9, F_2 = 0.9, \xi_1 = 0.97, \xi_2 = 0.79$.

In order to lead (y_1, y_2, y_3, y_4) to (x_1, x_2, x_3, x_4), we add u_1, u_2, u_3 and u_4 to the first, the second, the third, and the fourth equations of Eq. (4-3-11) respectively.

$$\begin{cases} \dot{y}_1 = y_2 + u_1,\ \dot{y}_2 = y_1 - y_1^3 - \rho_1 y_2 + F_1 \cos\xi_1 t + u_2 \\ \dot{y}_3 = y_4 + u_3,\ \dot{y}_4 = y_3 - y_3^3 - \rho_2 y_4 + F_2 \cos\xi_2 t + u_4 \end{cases} \tag{4-3-12}$$

Subtracting Eq. (4-3-12) from Eq. (4-3-10), we obtain an error dynamics. The initial values of the Quantum-CNN system and double Duffing systems are taken as $x_1(0)=0.8$, $x_2(0)=-0.77$, $x_3(0)=-0.72$, $x_4(0)=0.57$, $y_1(0)=0.75$, $y_2(0)=-0.3$, $y_3(0)=-0.4$ and $y_4(0)=-0.5$.

When i=1, 2, negative signs before y_i in Eq. (4-3-4) are chosen; when i=3, 4, positive signs before y_i in Eq. (4-3-4) are chosen, while $F(t)=0$:

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \quad i=1,2 \quad \text{and} \quad \lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i + y_i) = 0, \quad i=3,4 \tag{4-3-13}$$

$$\begin{aligned} \dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1 \\ \dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2 \\ \dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + y_4 + u_3 \\ \dot{e}_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + y_3 - y_3^3 - \rho_2 y_4 + F_2\cos\xi_2 t + u_4 \end{aligned} \tag{4-3-14}$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2$, $e_3=x_3+y_3$ and $e_4=x_4+y_4$.

Choose a Lyapunov function in the form of the positive definite function:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{4-3-15}$$

Its time derivative along any solution of Eq. (4-3-14) is

$$\begin{aligned} \dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1] + e_2[-\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ &\quad - y_1 + y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2] + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 + y_4 + u_3] \\ &\quad + e_4[-\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + y_3 - y_3^3 - \rho_2 y_4 + F_2\cos\xi_2 t + u_4] \end{aligned} \tag{4-3-16}$$

Choose

$$\begin{aligned} u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 + x_1 - y_1 \\ u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 x_2 - F_1\cos\xi_1 t \\ u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - x_3 - y_3 \end{aligned}$$

$$u_4 = -\omega_2(x_3 - x_1) + 2a_2 \frac{x_3}{\sqrt{1-x_3^2}} \cos x_4 - y_3 + y_3^3 - \rho_2 x_4 - F_2 \cos \xi_2 t$$

Eq. (4-3-16) can be rewritten as

$$\dot{V} = -e_1^2 - \rho_1 e_2^2 - e_3^2 - \rho_2 e_4^2 < 0 \tag{4-3-17}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. This means that the partial complete synchronization and the partial anti-synchronization of Quantum-CNN system with a double Duffing chaotic system can be achieved. The numerical results are shown in Fig. 4-5. After 5 second, the motion trajectories will enter a chaotic attractor.

Case II An anti-synchronization as the special case of the generalized synchronization

When i=1, 2, 3, 4, the positive signs before y_i in Eq. (4-3-4) are chosen, while $F(t)$=0:

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty} (x_i + y_i) = 0, \quad i = 1, 2, 3, 4 \tag{4-3-18}$$

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 + y_2 + u_1 \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1 \frac{x_1}{\sqrt{1-x_1^2}} \cos x_2 + y_1 - y_1^3 - \rho_1 y_2 + F_1 \cos \xi_1 t + u_2 \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + y_4 + u_3 \\
\dot{e}_4 &= -\omega_2(x_3 - x_1) + 2a_2 \frac{x_3}{\sqrt{1-x_3^2}} \cos x_4 + y_3 - y_3^3 - \rho_2 y_4 + F_2 \cos \xi_2 t + u_4
\end{aligned} \tag{4-3-19}$$

where $e_1=x_1+y_1$, $e_2=x_2+y_2$, $e_3=x_3+y_3$ and $e_4=x_4+y_4$.

Choose a Lyapunov function in the form of the positive definite function:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{4-3-20}$$

Its time derivative along any solution of Eq. (4-3-19) is

$$\begin{aligned}
\dot{V} = {} & e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 + y_2 + u_1] + e_2[-\omega_1(x_1 - x_3) + 2a_1 \frac{x_1}{\sqrt{1-x_1^2}} \cos x_2 + y_1 \\
& - y_1^3 - \rho_1 y_2 + F_1 \cos \xi_1 t + u_2] + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 + y_4 + u_3] \\
& + e_4[-\omega_2(x_3 - x_1) + 2a_2 \frac{x_3}{\sqrt{1-x_3^2}} \cos x_4 + y_3 - y_3^3 - \rho_2 y_4 + F_2 \cos \xi_2 t + u_4]
\end{aligned} \tag{4-3-21}$$

Choose

$$
\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - x_1 - y_1 \\
u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 - \rho_1 x_2 - F_1\cos\xi_1 t \\
u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - x_3 - y_3 \\
u_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 - \rho_2 x_4 - F_2\cos\xi_2 t
\end{aligned}
$$

Eq. (4-3-21) can be rewritten as

$$
\dot{V} = -e_1^2 - \rho_1 e_2^2 - e_3^2 - \rho_2 e_4^2 < 0 \qquad (22)
$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. This means that anti-synchronization of the Quantum-CNN system with a double Duffing chaotic system can be achieved. The numerical results are shown in Fig. 4-6. After 5 second, the motion trajectories will enter a chaotic attractor.

Case III A partial complete synchronization and a partial slave self-synchronization

When $i = 1, 2, 3, 4$, negative signs are chosen before y_i in Eq. (4). For i=1,3, $F_1(t)= F_3(t)=0$ and for i=2,4, $F_1(t)=y_1(t)$, $F_3(t)=y_3(t)$ are chosen, where $y_1(t)$ and $y_3(t)$ are chaotic functions of time. Then we have

$$
\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \;\; i = 1, 3 \qquad (4\text{-}3\text{-}23)
$$

$$
\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + y_j) = 0, \;\; i = 2, 4, \;\; j = 1, 3 \qquad (4\text{-}3\text{-}24)
$$

Eq. (4-3-5) becomes

$$
\dot{e}_1 = \dot{x}_1 - \dot{y}_1, \;\; \dot{e}_2 = \dot{x}_2 - \dot{y}_2 + \dot{y}_1, \;\; \dot{e}_3 = \dot{x}_3 - \dot{y}_3, \;\; \dot{e}_4 = \dot{x}_4 - \dot{y}_4 + \dot{y}_3 \qquad (4\text{-}3\text{-}25)
$$

Eq. (4-3-6) becomes

$$
\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1 \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2 + y_2 \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - u_3 \\
\dot{e}_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 + \rho_2 y_4 - F_2\cos\xi_2 t - u_4 + y_4
\end{aligned}
\qquad (4\text{-}3\text{-}26)
$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2+y_1$, $e_3=x_3-y_3$ and $e_4=x_4-y_4+y_3$.

Choose a Lyapunov function in the form of the positive definite function:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{4-3-27}$$

Its time derivative along any solution of Eq. (4-3-26) is

$$\begin{aligned}\dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1] + e_2[-\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 \\ &+ y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2 + y_2] + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - u_3] + e_4 \\ &[-\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 + \rho_2 y_4 - F_2\cos\xi_2 t - u_4 + y_4]\end{aligned} \tag{4-3-28}$$

Choose

$$\begin{aligned}u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 + x_1 - y_1 \\ u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 x_2 - F_1\cos\xi_1 t + y_2 + \rho_1 y_1 \\ u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 + x_3 - y_3 \\ u_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 + \rho_2 x_4 - F_2\cos\xi_2 t + y_4 + \rho_2 y_3\end{aligned}$$

Eq. (4-3-28) can be rewritten as

$$\dot{V} = -e_1^2 - \rho_1 e_2^2 - e_3^2 - \rho_2 e_4^2 < 0 \tag{4-3-29}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. This means that the partial complete synchronization and the partial slave self-synchronization of the Quantum-CNN system with a double Duffing chaotic system can be achieved. The numerical results are shown in Fig. 4-7. After 5 second, the motion trajectories will enter a chaotic attractor.

Case IV A partial sine/cosine function synchronization and a partial slave self- synchronization

When $i = 1, 2, 3, 4$, negative signs are chosen before y_i in Eq. (4-3-4). For $i = 1, 3$, $F_1(t) = G_1\sin\varpi_1 t, F_3(t) = G_3\cos\varpi_3 t$, and for $i = 2, 4$, $F_2(t) = y_1(t), F_4(t) = y_3(t)$ are chosen, where $y_1(t)$ and $y_3(t)$ are chaotic functions of time. We have

$$\lim_{t\to\infty} e_1 = \lim_{t\to\infty}(x_1 - y_1 + G_1\sin\varpi_1 t) = 0, \quad \lim_{t\to\infty} e_3 = \lim_{t\to\infty}(x_3 - y_3 + G_3\cos\varpi_3 t) = 0 \tag{4-3-30}$$

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + y_j) = 0, \quad i = 2, 4, \quad j = 1, 3 \tag{4-3-31}$$

Eq. (4-3-5) becomes

$$\dot{e}_1 = \dot{x}_1 - \dot{y}_1 + \dot{F}_1(t),\ \dot{e}_2 = \dot{x}_2 - \dot{y}_2 + \dot{F}_2(t),\ \dot{e}_3 = \dot{x}_3 - \dot{y}_3 + \dot{F}_3(t),\ \dot{e}_4 = \dot{x}_4 - \dot{y}_4 + \dot{F}_4(t) \quad (4\text{-}3\text{-}32)$$

Eq. (4-3-6) becomes

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1 + G_1\varpi_1\cos\varpi_1 t \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2 + y_2 \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - u_3 - G_3\varpi_3\sin\varpi_3 t \\
\dot{e}_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 + \rho_2 y_4 - F_2\cos\xi_2 t - u_4 + y_4
\end{aligned} \quad (4\text{-}3\text{-}33)$$

where $e_1 = x_1 - y_1 + G_1\sin\varpi_1 t$, $e_2 = x_2 - y_2 + y_1$, $e_3 = x_3 - y_3 + G_3\cos\varpi_3 t$, $e_4 = x_4 - y_4 + y_3$, $G_1 = 0.37, G_3 = 0.71, \varpi_1 = 0.31, \varpi_3 = 0.19$.

Choose a Lyapunov function in the form of the positive definite function:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \quad (4\text{-}3\text{-}34)$$

Its time derivative along any solution of Eq. (4-3-33) is

$$\begin{aligned}
\dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1 + G_1\varpi_1\cos\varpi_1 t] + e_2[-\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\
&- y_1 + y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2 + y_2] + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - u_3 - G_3\varpi_3\sin\varpi_3 t] \\
&+ e_4[-\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 + \rho_2 y_4 - F_2\cos\xi_2 t - u_4 + y_4]
\end{aligned} \quad (4\text{-}3\text{-}35)$$

Choose

$$\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 + x_1 - y_1 + G_1(\varpi_1\cos\varpi_1 t + \sin\varpi_1 t) \\
u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 x_2 - F_1\cos\xi_1 t + y_2 + \rho_1 y_1 \\
u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 + x_3 - y_3 + G_3(\cos\varpi_3 t - \varpi_3\sin\varpi_3 t) \\
u_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 + \rho_2 x_4 - F_2\cos\xi_2 t + y_4 + \rho_2 y_3
\end{aligned}$$

Eq. (4-3-35) can be rewritten as

$$\dot{V} = -e_1^2 - \rho_1 e_2^2 - e_3^2 - \rho_2 e_4^2 < 0 \quad (4\text{-}3\text{-}36)$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. This means that the partial sine/cosine function synchronization and the partial slave self-synchronization of the Quantum-CNN system with a double Duffing chaotic system can be achieved. The numerical results are shown in Fig. 4-8. After 5 second, the motion trajectories will enter a chaotic attractor.

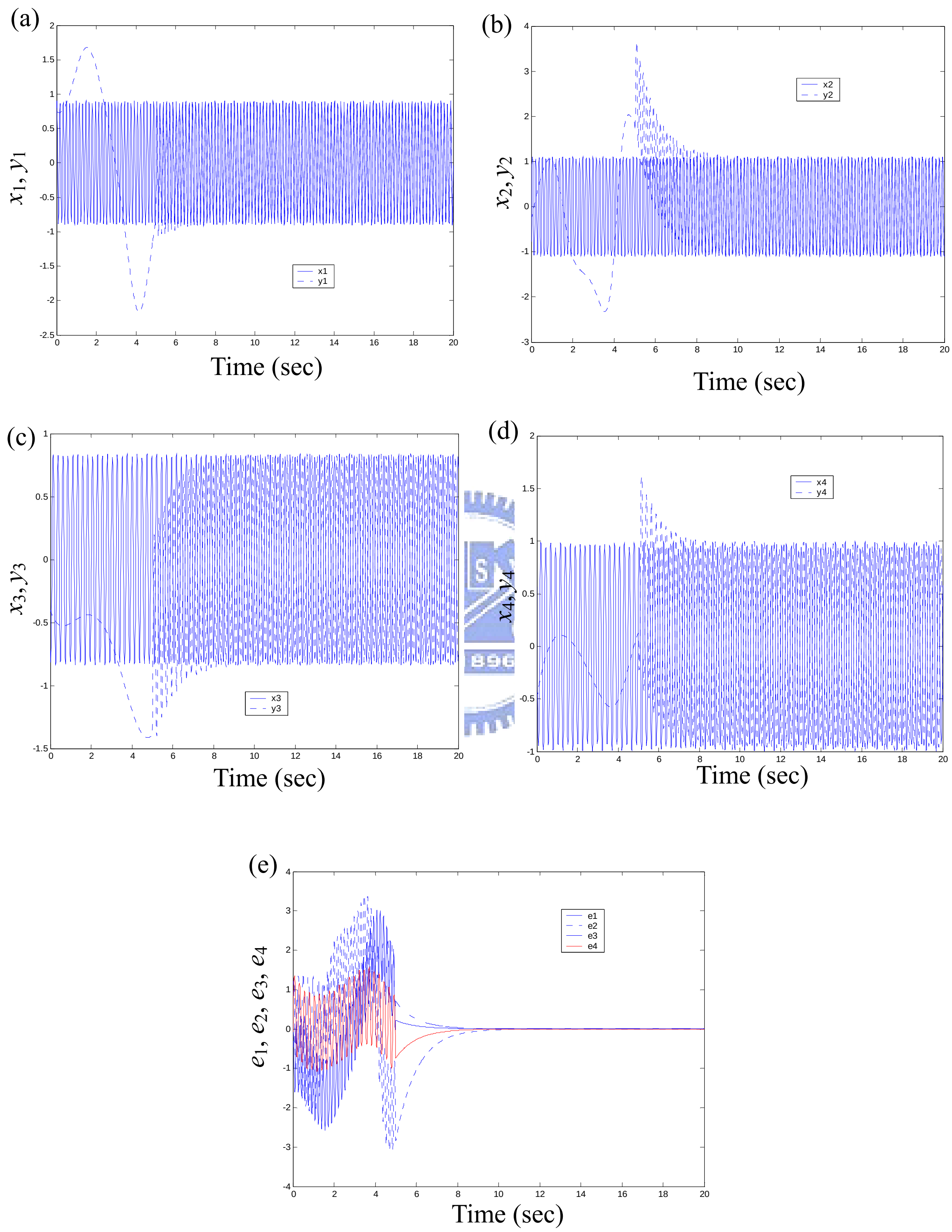


Fig. 4-5. Time histories of the master states, of the slave states, and of the errors for Case I.

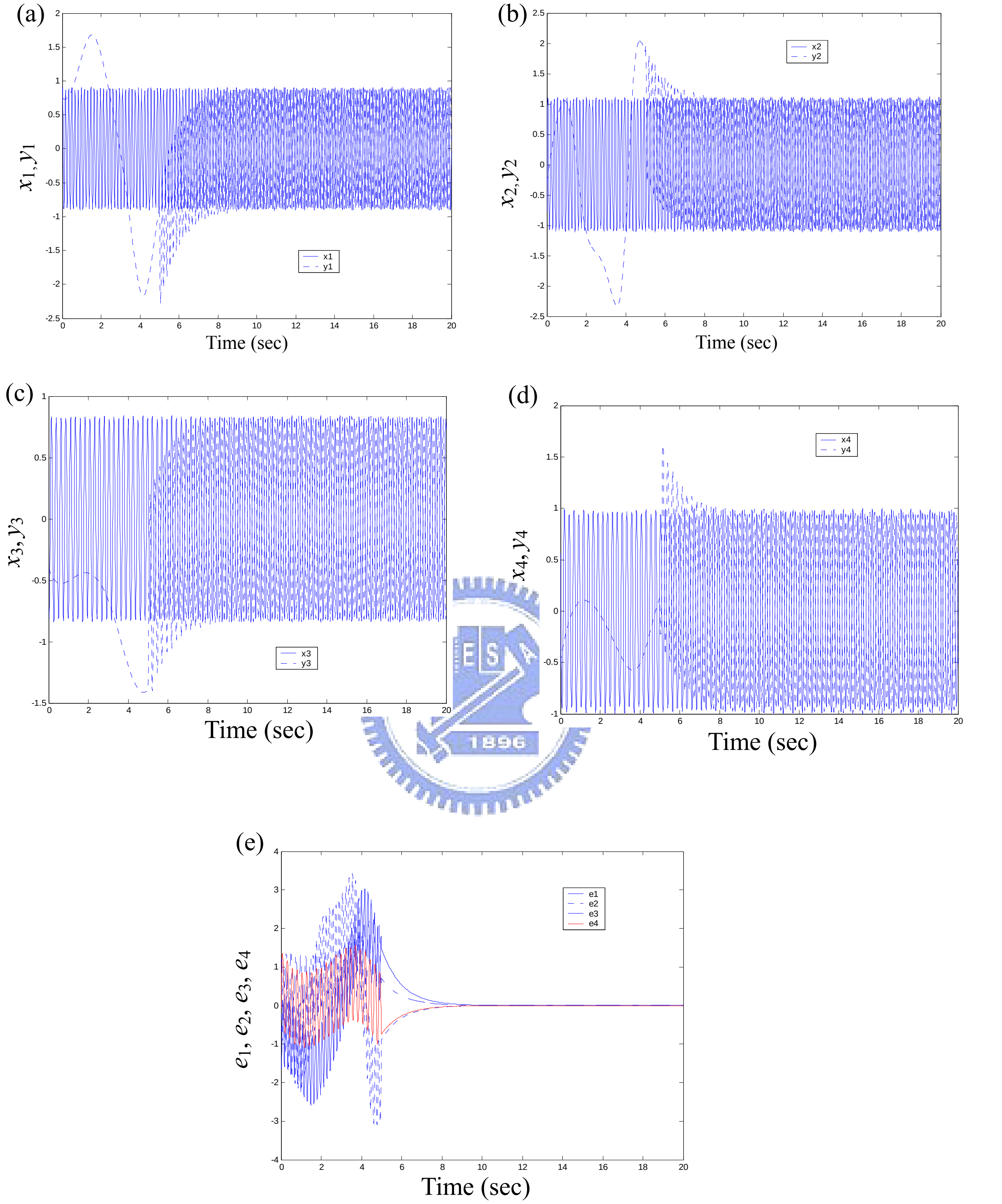


Fig. 4-6. Time histories of the master states, of the slave states, and of the errors for Case II.

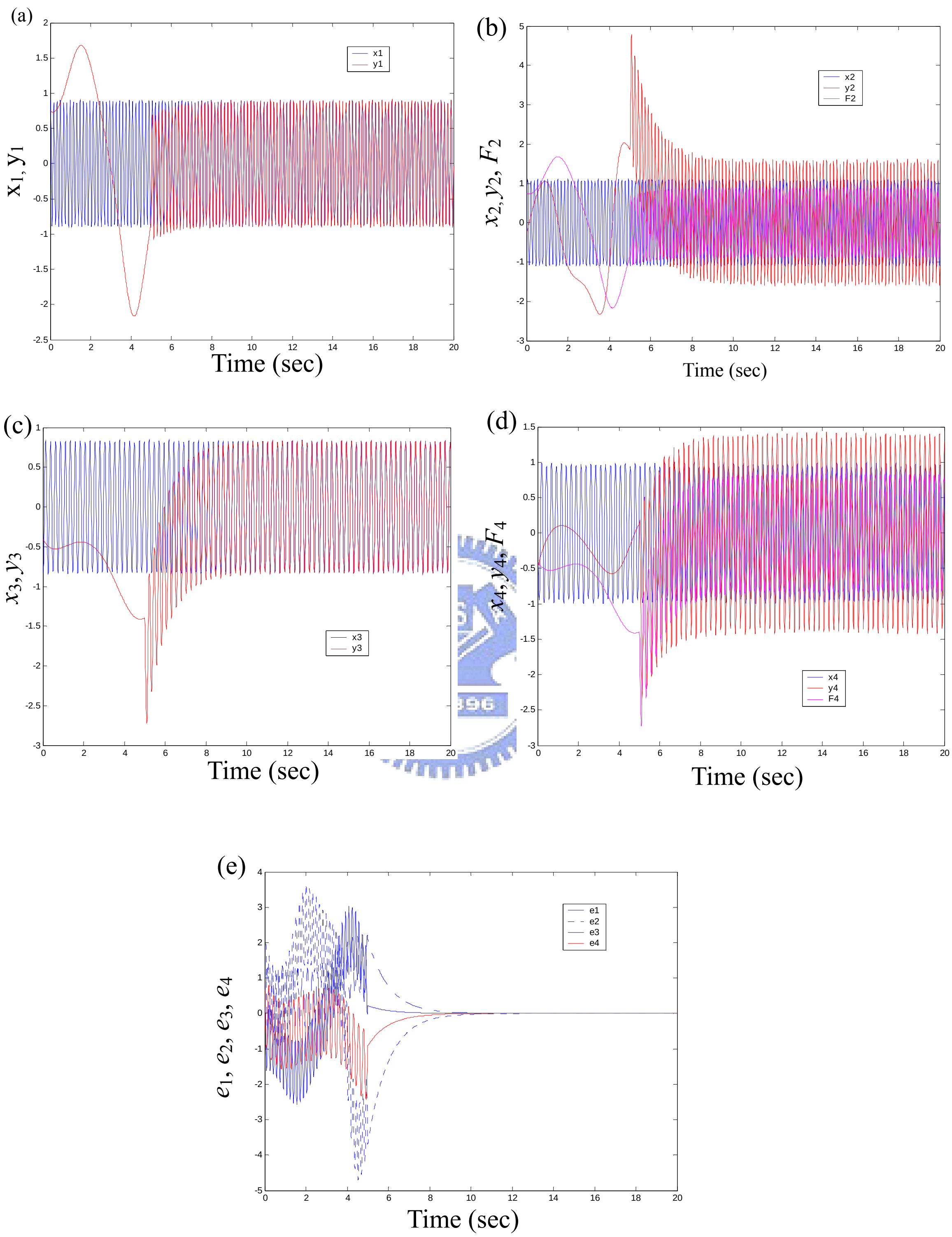


Fig. 4-7. The master state, the slave state, the error, F_2 and F_4 time histories for Case III.

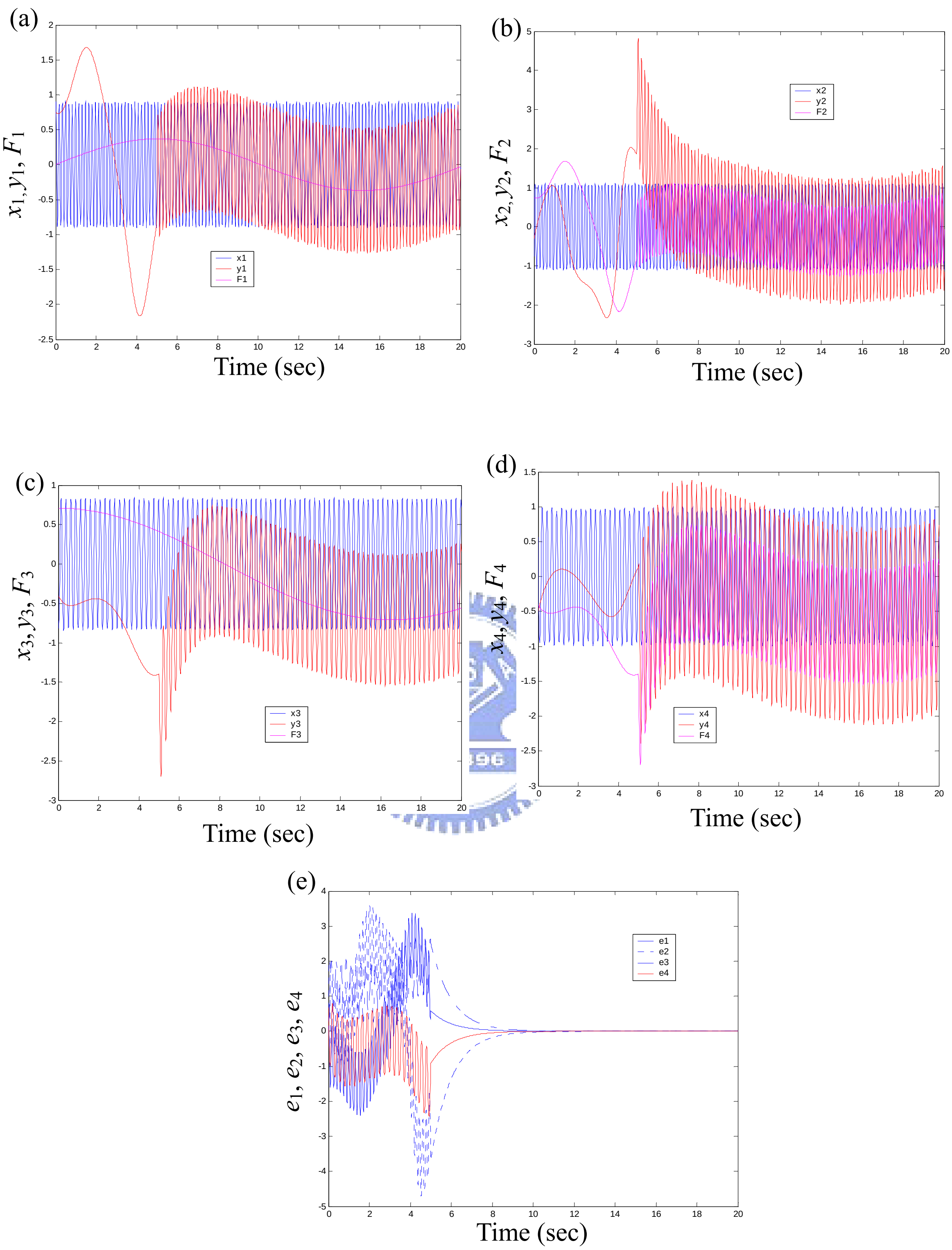


Fig. 4-8. The master state, the slave state, the error, F_1, F_2, F_3 and F_4 time histories for Case IV.

4-4 Symplectic Synchronization of Different Chaotic Systems

In this section, a new symplectic synchronization of chaotic systems is studied. Traditional generalized synchronizations are special cases of the symplectic synchronization. A sufficient condition is given for the asymptotical stability of the null solution of an error dynamics. The symplectic synchronization may be applied to the design of secure communication.

4-4-1 Symplectic Synchronization Scheme

There are two different nonlinear chaotic systems. The partner A controls the partner B partially. The partner A is given by

$$\dot{x} = f(x) \tag{4-4-1}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ is a state vector and *f* is a vector function.

The partner B is given by

$$\dot{y} = g(y) \tag{4-4-2a}$$

where $y = [y_1, y_2, \cdots, y_n]^T \in R^n$ is a state vector, and g is a vector function different from *f*.

After a controller *u*(*t*) is added, partner B becomes

$$\dot{y} = g(y) + u(t) \tag{4-4-2b}$$

where $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T \in R^n$ is the control vector.

Our goal is to design the controller *u*(*t*) so that the state vector *y* of the partner B asymptotically approaches $H(x, y, t) + F(t)$, a given function *H*(*x*, *y*, *t*) plus a given vector function $F(t) = [F_1(t), F_2(t), \cdots, F_n(t)]^T$ which is a regular or a chaotic function of time. Define error vector $e(t) = [e_1, e_2, \cdots, e_n]^T$:

$$e = H(x, y, t) - y + F(t) \tag{4-4-3}$$

$$\lim_{t\to\infty} e = 0 \tag{4-4-4}$$

is demanded.

From Eq. (4-4-3), it is obtained that

$$\dot{e} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial t} - \dot{y} + \dot{F}(t) \tag{4-4-5}$$

By Eq. (4-4-1), Eq. (4-4-2a), and Eq. (4-4-2b), Eq. (4-4-5) becomes

$$\dot{e} = \frac{\partial H}{\partial x} f(x) + \frac{\partial H}{\partial y} g(y) + \frac{\partial H}{\partial t} - g(y) - u(t) + \dot{F}(t) \tag{4-4-6}$$

A positive definite Lyapnuov function $V(e)$ is chosen:

$$V(e) = \frac{1}{2} e^T e \tag{4-4-7}$$

Its derivative along any solution of Eq. (4-4-6) is

$$\dot{V}(e) = e^T \{ \frac{\partial H}{\partial x} f(x) + \frac{\partial H}{\partial y} g(y) + \frac{\partial H}{\partial t} - g(y) + \dot{F}(t) - u(t) \}. \tag{4-4-8}$$

In Eq. (4-4-8), $u(t)$ is designed so that $\dot{V} = e^T C_{n\times n} e$, where $C_{n\times n}$ is a diagonal negative definite matrix. $\dot{V}$ is a negative definite function of e. By Lyapunov theorem of asymptotical stability

$$\lim_{t\to\infty} e = 0 \tag{4-4-9}$$

The symplectic synchronization is obtained.

4-4-2 Numerical Results for the Symplectic Chaos Synchronization of Quantum-CNN Oscillator and Rössler System

Case I A cubic symplectic synchronization

For a two-cell Quantum-CNN, following differential equations are obtained:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2,\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1 \dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4,\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2 \dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{4-4-10}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are the parameters that weigh the effects on the cell of the difference of polarization of the neighboring cells, like the cloning templates in traditional CNNs. When $a_1 = 19.4$, $a_2 = 13.1$, $\omega_1 = 9.529$ and $\omega_2 = 7.94$, the system is chaotic.

A chaotic Rössler system is described by

$$\dot{y}_1 = -y_2 - y_3,\ \dot{y}_2 = y_1 - \alpha y_2 + y_4,\ \dot{y}_3 = y_1 y_3 + \beta,\ \dot{y}_4 = \gamma y_3 + \sigma y_4 \tag{4-4-11}$$

where $\alpha = 0.5, \beta = 0.52, \gamma = 0.5, \sigma = 0.05$.

For symplectic synchronization of these two systems, u_1, u_2, u_3 and u_4 are added to the four equations of Eq. (4-4-11), respectively:

$$\dot{y}_1 = -y_2 - y_3 + u_1,\ \dot{y}_2 = y_1 - \alpha y_2 + y_4 + u_2,\ \dot{y}_3 = y_1 y_3 + \beta + u_3,\ \dot{y}_4 = \gamma y_3 + \sigma y_4 + u_4 \quad (4\text{-}4\text{-}12)$$

The initial values of the states of the Quantum-CNN system and of the Rössler system are taken as $x_1(0)=0.8$, $x_2(0)=-0.77$, $x_3(0)=-0.72$, $x_4(0)=0.57$, $y_1(0)=0.3$, $y_2(0)=-0.4$, $y_3(0)=-0.7$ and $y_4(0)=0.15$.

We take $F_1(t)=x_4^3(t)$, $F_2(t)=x_1^3(t)$, $F_3(t)=x_2^3(t)$ and $F_4(t)=x_3^3(t)$. They are chaotic functions of time. $H_i(x,y,t)=-x_i^2 y_i$ $(i=1, 2, 3, 4)$ are given. By Eq. (4-4-4) we have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty} (-x_i^2 y_i - y_i + x_j^3) = 0, \quad i=1,2,3,4; \quad j=\begin{cases} 4 & ,i=1 \\ i-1 & ,i\neq 1 \end{cases} \quad (4\text{-}4\text{-}13)$$

From Eq. (4-4-5) we have

$$\dot{e}_i = -2\dot{x}_i x_i y_i - x_i^2 \dot{y}_i - \dot{y}_i + 3\dot{x}_j x_j^2, \quad i=1,2,3,4; \quad j=\begin{cases} 4 & ,i=1 \\ i-1 & ,i\neq 1 \end{cases} \quad (4\text{-}4\text{-}14)$$

Eq. (4-4-6) can be expressed as

$$\begin{aligned}
\dot{e}_1 &= 2y_1x_1(2a_1\sqrt{1-x_1^2}\sin x_2) + (y_2+y_3)x_1^2 + 3x_4^2(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4) \\
&+ y_2 + y_3 - u_1 \\
\dot{e}_2 &= -2y_2x_2(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2) + 3x_1^2(-2a_1\sqrt{1-x_1^2}\sin x_2) \\
&-(y_1 - \alpha y_2 + y_4)x_2^2 - y_1 + \alpha y_2 - y_4 - u_2 \\
\dot{e}_3 &= 2y_3x_3(2a_2\sqrt{1-x_3^2}\sin x_4) + 3x_2^2(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2) \\
&-(y_1y_3 + \beta)x_2^3 - y_1y_3 - \beta - u_3 \\
\dot{e}_4 &= -2y_4x_4(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4) + 3x_3^2(-2a_2\sqrt{1-x_3^2}\sin x_4) \\
&-(\gamma y_3 + \sigma y_4)x_4^2 - \gamma y_3 - \sigma y_4 - u_4
\end{aligned} \quad (4\text{-}4\text{-}15)$$

where $e_1 = -x_1^2 y_1 - y_1 + x_4^3$, $e_2 = -x_2^2 y_2 - y_2 + x_1^3$, $e_3 = -x_3^2 y_3 - y_3 + x_2^3$ and $e_4 = -x_4^2 y_4 - y_4 + x_3^3$.

Choose a positive definite Lyapunov function:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \quad (4\text{-}4\text{-}16)$$

Its time derivative along any solution of Eq. (4-4-15) is

$$\dot{V} = e_1\{2y_1x_1(2a_1\sqrt{1-x_1^2}\sin x_2)+(y_2+y_3)x_1^2+y_2+y_3+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4+3x_4^2(-\omega_2$$
$$(x_3-x_1))-u_1\}+e_2\{-2y_2x_2(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)-(y_1-\alpha y_2+y_4)x_2^2-y_1$$
$$+\alpha y_2-y_4+3x_1^2(-2a_1\sqrt{1-x_1^2}\sin x_2)-u_2\}+e_3\{2y_3x_3(2a_2\sqrt{1-x_3^2}\sin x_4)-(y_1y_3+\beta)x_2^3 \quad (4\text{-}4\text{-}17)$$
$$-y_1y_3-\beta+3x_2^2(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)-u_3\}+e_4\{-2y_4x_4(-\omega_2(x_3-x_1)$$
$$+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)-(\gamma y_3+\sigma y_4)x_4^2-\gamma y_3-\sigma y_4+3x_3^2(-2a_2\sqrt{1-x_3^2}\sin x_4)-u_4\}$$

Choose

$$u_1 = 2y_1x_1(2a_1\sqrt{1-x_1^2}\sin x_2)+(y_2+y_3)x_1^2+y_2+y_3+3x_4^2(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)$$
$$-y_1x_1^2-y_1+x_4^3$$
$$u_2 = -2y_2x_2(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)-(y_1-\alpha y_2+y_4)x_2^2+3x_1^2(-2a_1\sqrt{1-x_1^2}\sin x_2)$$
$$-y_1-y_4-\alpha(y_2x_2^2-x_1^3)$$
$$u_3 = 2y_3x_3(2a_2\sqrt{1-x_3^2}\sin x_4)-(y_1y_3+\beta)x_2^3-y_1y_3-\beta+3x_2^2(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)$$
$$-y_3x_3^2-y_3+x_2^3$$
$$u_4 = -2y_4x_4(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)-(\gamma y_3+\sigma y_4)x_4^2-\gamma y_3+3x_3^2(-2a_2\sqrt{1-x_3^2}\sin x_4)$$
$$-\sigma(y_4x_4^2+2y_4-x_3^3)$$

Eq. (4-4-17) becomes

$$\dot{V} = -(e_1^2+\alpha e_2^2+e_3^2+\sigma e_4^2)<0 \quad (4\text{-}4\text{-}18)$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. Cubic symplectic synchronization of the Quantum-CNN system and the Rössler system is achieved. The numerical results are shown in Fig. 4-9. After 5 second, the motion trajectories enter a chaotic attractor.

Case II A time delay symplectic synchronization

We take $F_1(t)=x_1(t-T)$, $F_2(t)=x_2(t-T)$, $F_3(t)=x_3(t-T)$ and $F_4(t)=x_4(t-T)$. They are chaotic functions of time, where time delay T=1 sec is a positive constant.

$H_i(x, y, t) = (x_i^2 + y_i)(e^{-t} + 2)$ (i=1, 2, 3, 4) are given. By Eq. (4-4-4) we have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}((x_i^2 + y_i)(e^{-t} + 2) - y_i + x_i(t-T)) = 0, \quad i = 1, 2, 3, 4 \tag{4-4-19}$$

From Eq. (4-4-5) we have

$$\dot{e}_i = (2x_i\dot{x}_i + \dot{y}_i)(e^{-t} + 2) - e^{-t}(x_i^2 + y_i) - \dot{y}_i + \dot{x}_i(t-T), \quad i = 1, 2, 3, 4 \tag{4-4-20}$$

Eq. (4-4-6) is expressed as

$$\begin{aligned}
&\dot{e}_1 = 2x_1(-2a_1\sqrt{1-x_1^2}\sin x_2)(e^{-t}+2) + (-y_2 - y_3)(e^{-t}+2) - (x_1^2 + y_1)e^{-t} + y_2 + y_3 - u_1 \\
&-2a_1\sqrt{1-x_1^2(t-T)}\sin x_2(t-T) \\
&\dot{e}_2 = 2x_2(-\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)(e^{-t}+2) + 2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T) \\
&+(y_1 - \alpha y_2 + y_4)(e^{-t}+2) - (x_2^2 + y_2)e^{-t} - y_1 + \alpha y_2 - y_4 - u_2 - \omega_1(x_1(t-T) - x_3(t-T)) \\
&\dot{e}_3 = 2x_3(-2a_2\sqrt{1-x_3^2}\sin x_4)(e^{-t}+2) + (y_1 y_3 + \beta)(e^{-t}+2) - (x_3^2 + y_3)e^{-t} - y_1 y_3 - \beta \\
&-2a_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) - u_3 \\
&\dot{e}_4 = 2x_4(-\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)(e^{-t}+2) + 2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T) \\
&+(\gamma y_3 + \sigma y_4)(e^{-t}+2) - (x_4^2 + y_4)e^{-t} - \gamma y_3 - \sigma y_4 - u_4 - \omega_2(x_3(t-T) - x_1(t-T))
\end{aligned} \tag{4-4-21}$$

where $e_1 = (x_1^2 + y_1)(e^{-t} + 2) - y_1 + x_1(t-T)$, $e_2 = (x_2^2 + y_2)(e^{-t} + 2) - y_2 + x_2(t-T)$,

$e_3 = (x_3^2 + y_3)(e^{-t} + 2) - y_3 + x_3(t-T)$, $e_4 = (x_4^2 + y_4)(e^{-t} + 2) - y_4 + x_4(t-T)$.

Choose a positive definite Lyapunov function:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{4-4-22}$$

Its time derivative along any solution of Eq. (4-4-21) is

$$
\begin{aligned}
\dot{V} = {} & e_1\{2x_1(-2a_1\sqrt{1-x_1^2}\sin x_2)(e^{-t}+2)+(-y_2-y_3)(e^{-t}+2)-(x_1^2+y_1)e^{-t}+y_2+y_3 \\
& -2a_1\sqrt{1-x_1^2(t-T)}\sin x_2(t-T)-u_1\}+e_2\{2x_2(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2) \\
& (e^{-t}+2)+(y_1-\alpha y_2+y_4)(e^{-t}+2)-(x_2^2+y_2)e^{-t}-y_1+\alpha y_2-y_4-\omega_1(x_1(t-T) \\
& -x_3(t-T))+2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T)-u_2\}+e_3\{2x_3(-2a_2\sqrt{1-x_3^2}\sin x_4)(e^{-t}+2) \\
& +(y_1y_3+\beta)(e^{-t}+2)-(x_3^2+y_3)e^{-t}-y_1y_3-\beta-2a_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T)-u_3\} \\
& +e_4\{2x_4(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)(e^{-t}+2)+2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T) \\
& -\gamma y_3-\sigma y_4-\omega_2(x_3(t-T)-x_1(t-T))+(\gamma y_3+\sigma y_4)(e^{-t}+2)-(x_4^2+y_4)e^{-t}-u_4\}
\end{aligned}
\tag{4-4-23}
$$

Choose

$$
\begin{aligned}
u_1 = {} & 2x_1(-2a_1\sqrt{1-x_1^2}\sin x_2)(e^{-t}+2)+(-y_2-y_3)(e^{-t}+2)-(x_1^2+y_1)e^{-t}+y_2+y_3 \\
& -2a_1\sqrt{1-x_1^2(t-T)}\sin x_2(t-T)+(x_1^2+y_1)(e^{-t}+2)-y_1+x_1(t-T) \\
u_2 = {} & 2x_2(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)(e^{-t}+2)+(y_1-\alpha y_2+y_4)(e^{-t}+2)-(x_2^2+y_2)e^{-t} \\
& -y_1-y_4-\omega_1(x_1(t-T)-x_3(t-T))+2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T) \\
& +\alpha((x_2^2+y_2)(e^{-t}+2)+x_2(t-T)) \\
u_3 = {} & 2x_3(-2a_2\sqrt{1-x_3^2}\sin x_4)(e^{-t}+2)+(y_1y_3+\beta)(e^{-t}+2)-(x_3^2+y_3)e^{-t} \\
& -y_1y_3-\beta-2a_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T)+(x_3^2+y_3)(e^{-t}+2)-y_3+x_3(t-T) \\
u_4 = {} & 2x_4(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)(e^{-t}+2)+(\gamma y_3+\sigma y_4)(e^{-t}+2)-(x_4^2+y_4)e^{-t} \\
& -\gamma y_3-\omega_2(x_3(t-T)-x_1(t-T))+2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T) \\
& +\sigma((x_4^2+y_4)(e^{-t}+2)-2y_4+x_4(t-T))
\end{aligned}
$$

Eq. (4-4-23) becomes

$$
\dot{V} = -(e_1^2+\alpha e_2^2+e_3^2+\sigma e_4^2)<0 \tag{4-4-24}
$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. Time delay symplectic synchronization of the Quantum-CNN system and the Rössler system is achieved. The numerical results are shown in Fig. 4-10. After 5 second, the motion trajectories enter a chaotic attractor.

Case III A cubic time delay symplectic synchronization

We take $F_1(t)=x_4(t)x_1(t-T)$, $F_2(t)=x_1(t)x_2(t-T)$, $F_3(t)=x_2(t)x_3(t-T)$, and $F_4(t)=x_3(t)x_4(t-T)$, where T=1 sec is a positive constant time delay. They are chaotic functions of time. $H_i(x,\ y,\ t)=x_i^3-(y_i^3\sin\varpi_i t-1)\sin\varpi_i t\ \ (i=1,2,3,4)$ are given. By Eq. (4-4-3) we have

$$\lim_{t\to\infty}e_i=\lim_{t\to\infty}(x_i^3-(y_i^3\sin\varpi_i t-1)\sin\varpi_i t-y_i+x_jx_i(t-T))=0, i=1,2,3,4,\ j=\begin{cases}4 & ,i=1\\ i-1, & i\neq 1\end{cases}\tag{4-4-25}$$

From Eq. (4-4-5) we have

$$\begin{aligned}\dot e_i=&(3\dot x_i x_i^2-(3\dot y_i y_i^2\sin\varpi_i t+y_i^3\varpi_i\cos\varpi_i t)\sin\varpi_i t\\ &-(y_i^3\sin\varpi_i t-1)\varpi_i\cos\varpi_i t-\dot y_i+\dot x_j x_i(t-T)+x_j\dot x_i(t-T),\end{aligned}\quad i=1,2,3,4,\ j=\begin{cases}4 & ,i=1\\ i-1, & i\neq 1\end{cases}\tag{4-4-26}$$

Eq. (4-4-6) is expressed as

$$\begin{aligned}
\dot e_1=&3x_1^2(-2a_1\sqrt{1-x_1^2}\sin x_2)-3y_1^2(-y_2-y_3)\sin^2\varpi_1 t-y_1^3\varpi_1\sin 2\varpi_1 t+\varpi_1\cos\varpi_1 t+y_2+\\
&y_3-u_1+(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)x_1(t-T)-2a_1x_4\sqrt{1-x_1^2(t-T)}\sin x_2(t-T)\\
\dot e_2=&3x_2^2(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)-3y_2^2(y_1-\alpha y_2+y_4)\sin^2\varpi_2 t-y_2^3\varpi_2\sin 2\varpi_2 t\\
&+\varpi_2\cos\varpi_2 t-2a_1x_2(t-T)\sqrt{1-x_1^2}\sin x_2+2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T)\\
&+x_1(-\omega_1(x_1(t-T)-x_3(t-T))-y_1+\alpha y_2-y_4-u_2\\
\dot e_3=&3x_3^2(-2a_2\sqrt{1-x_3^2}\sin x_4)-3y_3^2(y_1y_3+\beta)\sin^2\varpi_3 t+y_3^3\varpi_3\sin 2\varpi_3 t+\varpi_3\cos\varpi_3 t-y_1y_3\\
&-\beta-u_3+(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)x_3(t-T)-2a_2x_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T)\\
\dot e_4=&3x_4^2(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)-3y_4^2(\gamma y_3+\sigma y_4)\sin^2\varpi_4 t+y_4^3\varpi_4\sin 2\varpi_4 t\\
&+\varpi_4\cos\varpi_4 t-2a_2x_4(t-T)\sqrt{1-x_3^2}\sin x_4+2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T))\\
&+x_3(-\omega_2(x_3(t-T)-x_1(t-T))-\gamma y_3-\sigma y_4-u_4
\end{aligned}\tag{4-4-27}$$

where

$$e_1=x_1^3-(y_1^3\sin\varpi_1 t-1)\sin\varpi_1 t-y_1+x_4(t)x_1(t-T), e_2=x_2^3-(y_2^3\sin\varpi_2 t-1)\sin\varpi_2 t-y_2+x_1(t)x_2(t-T),$$

$$e_3=x_3^3-(y_3^3\sin\varpi_3 t-1)\sin\varpi_3 t-y_3+x_2(t)x_3(t-T), e_4=x_4^3-(y_4^3\sin\varpi_4 t-1)\sin\varpi_4 t-y_4+x_3(t)x_4(t-T).$$

Choose a positive definite Lyapunov function:

$$V(e_1,e_2,e_3,e_4)=\frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2)\tag{4-4-28}$$

Its time derivative along any solution of Eq. (4-4-27) is

$$\dot{V} = e_1\{3x_1^2(-2a_1\sqrt{1-x_1^2}\sin x_2) - 3y_1^2(-y_2-y_3)\sin^2\varpi_1 t - y_1^3\varpi_1\sin 2\varpi_1 t + \varpi_1\cos\varpi_1 t + y_2$$
$$+y_3 + (-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)x_1(t-T) - 2a_1x_4\sqrt{1-x_1^2(t-T)}\sin x_2(t-T)$$
$$-u_1\} + e_2\{3x_2^2(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2) - 3y_2^2(y_1-\alpha y_2+y_4)\sin^2\varpi_2 t - y_1$$
$$-y_2^3\varpi_2\sin 2\varpi_2 t + \varpi_2\cos\varpi_2 t + \alpha y_2 - y_4 - 2a_1x_2(t-T)\sqrt{1-x_1^2}\sin x_2 + x_1(-\omega_1(x_1(t$$
$$-T) - x_3(t-T)) + 2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T) - u_2\} + e_3\{3x_3^2(-2a_2\sqrt{1-x_3^2}\sin x_4) \tag{4-4-29}$$
$$-3y_3^2(y_1y_3+\beta)\sin^2\varpi_3 t + y_3^3\varpi_3\sin 2\varpi_3 t + \varpi_3\cos\varpi_3 t - y_1y_3 - \beta + (-\omega_1(x_1-x_3)$$
$$+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)x_3(t-T) - 2a_2x_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) - u_3\} + e_4\{3x_4^2$$
$$(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4) - 3y_4^2(\gamma y_3 + \sigma y_4)\sin^2\varpi_4 t + y_4^3\varpi_4\sin 2\varpi_4 t$$
$$+\varpi_4\cos\varpi_4 t - \gamma y_3 - \sigma y_4 - 2a_2x_4(t-T)\sqrt{1-x_3^2}\sin x_4 + x_3(-\omega_2(x_3(t-T) - x_1(t$$
$$-T)) + 2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)) - u_4\}$$

Choose

$$u_1 = 3x_1^2(-2a_1\sqrt{1-x_1^2}\sin x_2) - 3y_1^2(-y_2-y_3)\sin^2\varpi_1 t - y_1^3\varpi_1\sin 2\varpi_1 t + \varpi_1\cos\varpi_1 t + y_2$$
$$+y_3 + (-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4)x_1(t-T) - 2a_1x_4\sqrt{1-x_1^2(t-T)}\sin x_2(t-T)$$
$$+x_1^3 - (y_1^3\sin\varpi_1 t - 1)\sin\varpi_1 t - y_1 + x_4(t)x_1(t-T)$$
$$u_2 = 3x_2^2(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2) - 3y_2^2(y_1-\alpha y_2+y_4)\sin^2\varpi_2 t - y_2^3\varpi_2\sin 2\varpi_2 t$$
$$+\varpi_2\cos\varpi_2 t - y_1 - y_4 - 2a_1x_2(t-T)\sqrt{1-x_1^2}\sin x_2 + 2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T)$$
$$+x_1(-\omega_1(x_1(t-T) - x_3(t-T)) + \alpha(x_2^3 - (y_2^3\sin\varpi_2 t - 1)\sin\varpi_2 t + x_1(t)x_2(t-T))$$
$$u_3 = 3x_3^2(-2a_2\sqrt{1-x_3^2}\sin x_4) - 3y_3^2(y_1y_3+\beta)\sin^2\varpi_3 t + y_3^3\varpi_3\sin 2\varpi_3 t + \varpi_3\cos\varpi_3 t - y_1y_3$$
$$-\beta + (-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2)x_3(t-T) - 2a_2x_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) + x_3^3$$
$$-(y_3^3\sin\varpi_3 t - 1)\sin\varpi_3 t - y_3 + x_2(t)x_3(t-T)$$
$$u_4 = 3x_4^2(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4) - 3y_4^2(\gamma y_3 - \sigma y_4)\sin^2\varpi_4 t + y_4^3\varpi_4\sin 2\varpi_4 t$$
$$+\varpi_4\cos\varpi_4 t - \gamma y_3 - 2a_2x_4(t-T)\sqrt{1-x_3^2}\sin x_4 + 2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T))$$
$$+x_3(-\omega_2(x_3(t-T) - x_1(t-T)) + \sigma(x_4^3 - (y_4^3\sin\varpi_4 t - 1)\sin\varpi_4 t - 2y_4 + x_3(t)x_4(t-T))$$

Eq. (4-4-29) becomes

$$\dot{V} = -(e_1^2 + \alpha e_2^2 + e_3^2 + \sigma e_4^2) < 0 \tag{4-4-30}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. Cubic time delay symplectic synchronization of the Quantum-CNN system and the Rössler system is achieved. The numerical results are shown in Fig. 4-11. After 5 second, the motion trajectories enter a chaotic attractor.

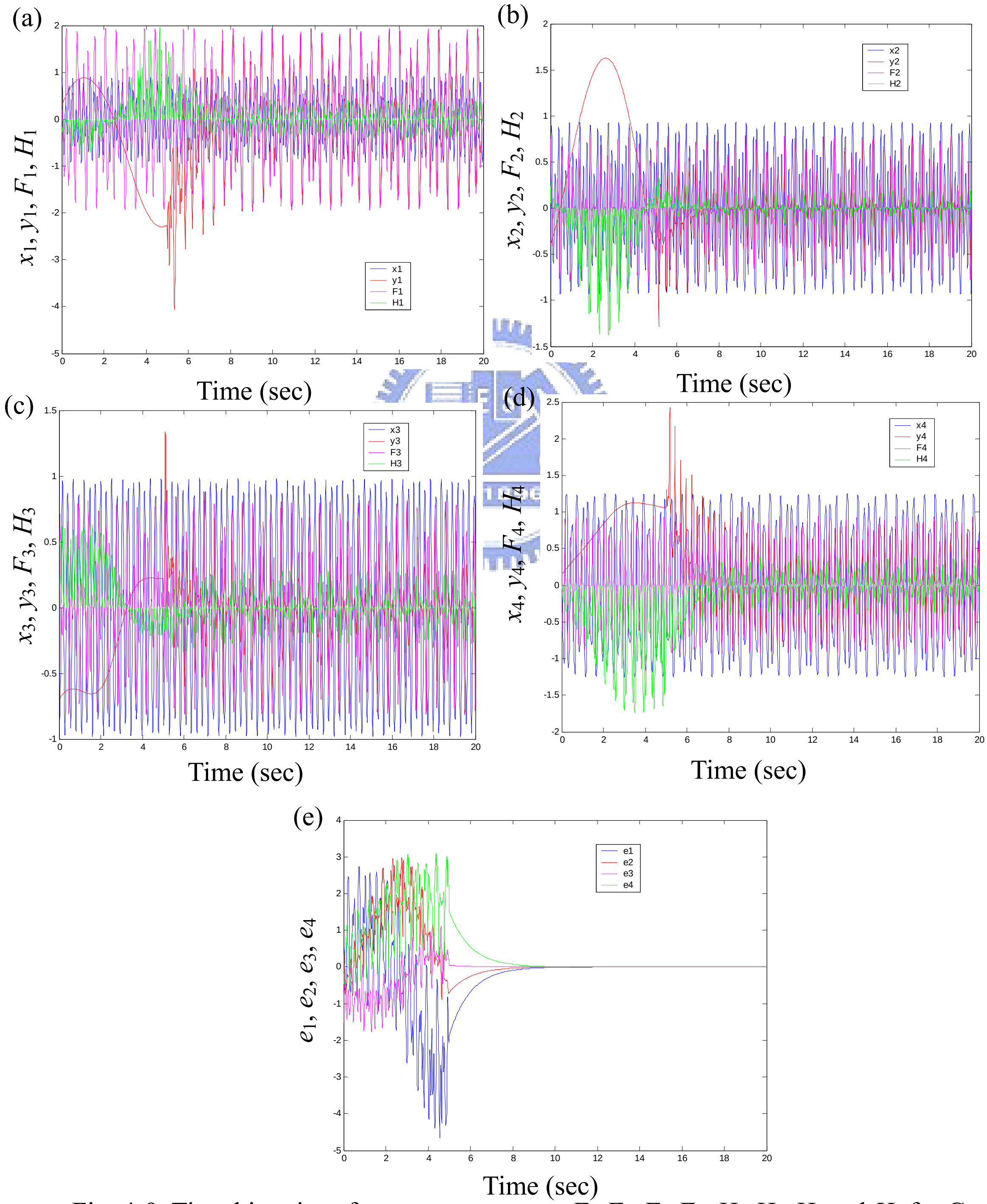


Fig. 4-9. Time histories of states, state errors, F_1, F_2, F_3, F_4, H_1, H_2, H_3 and H_4 for Case I.

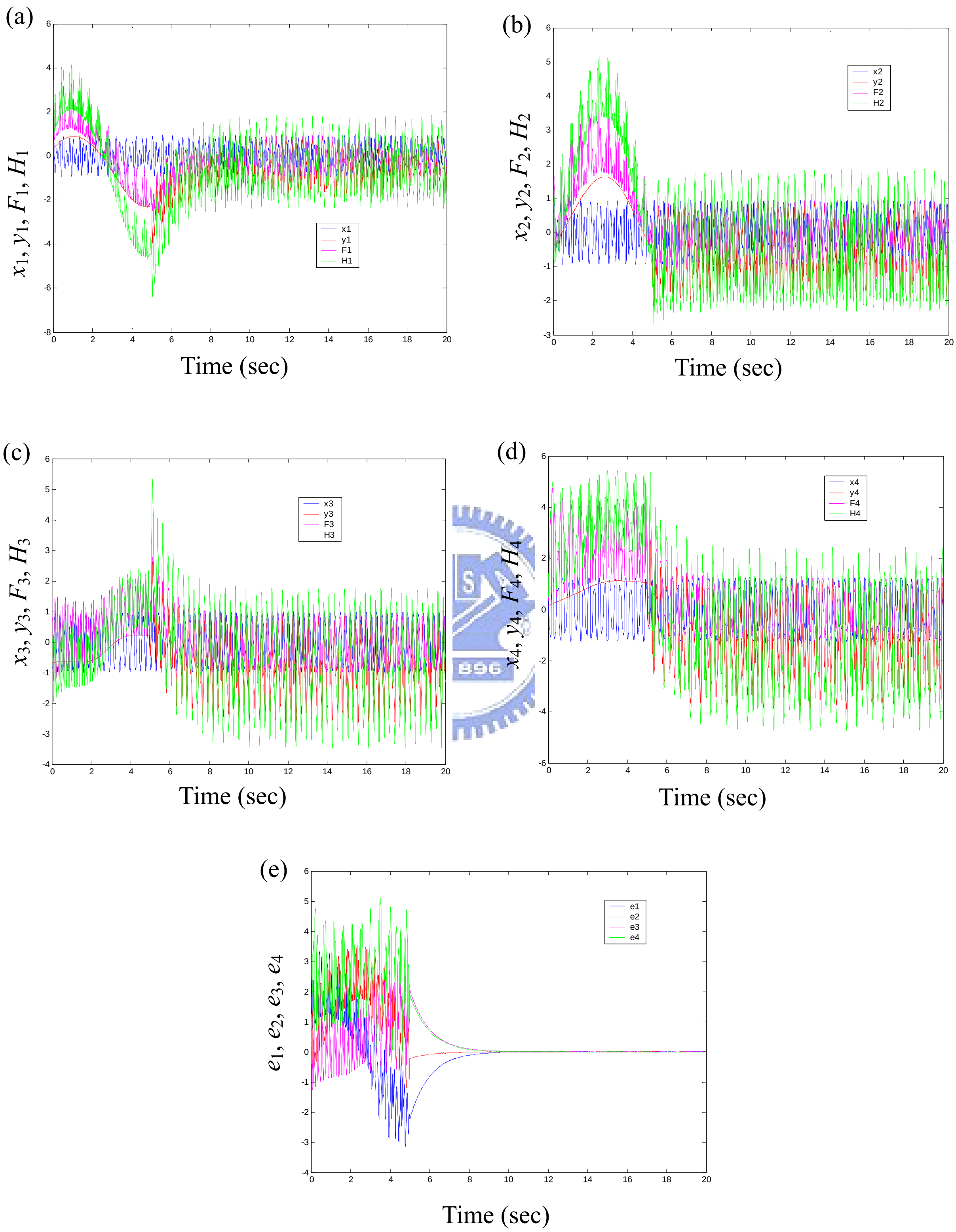


Fig. 4-10. Time histories of states, state errors, F_1, F_2, F_3, F_4, H_1, H_2, H_3 and H_4 for Case II.

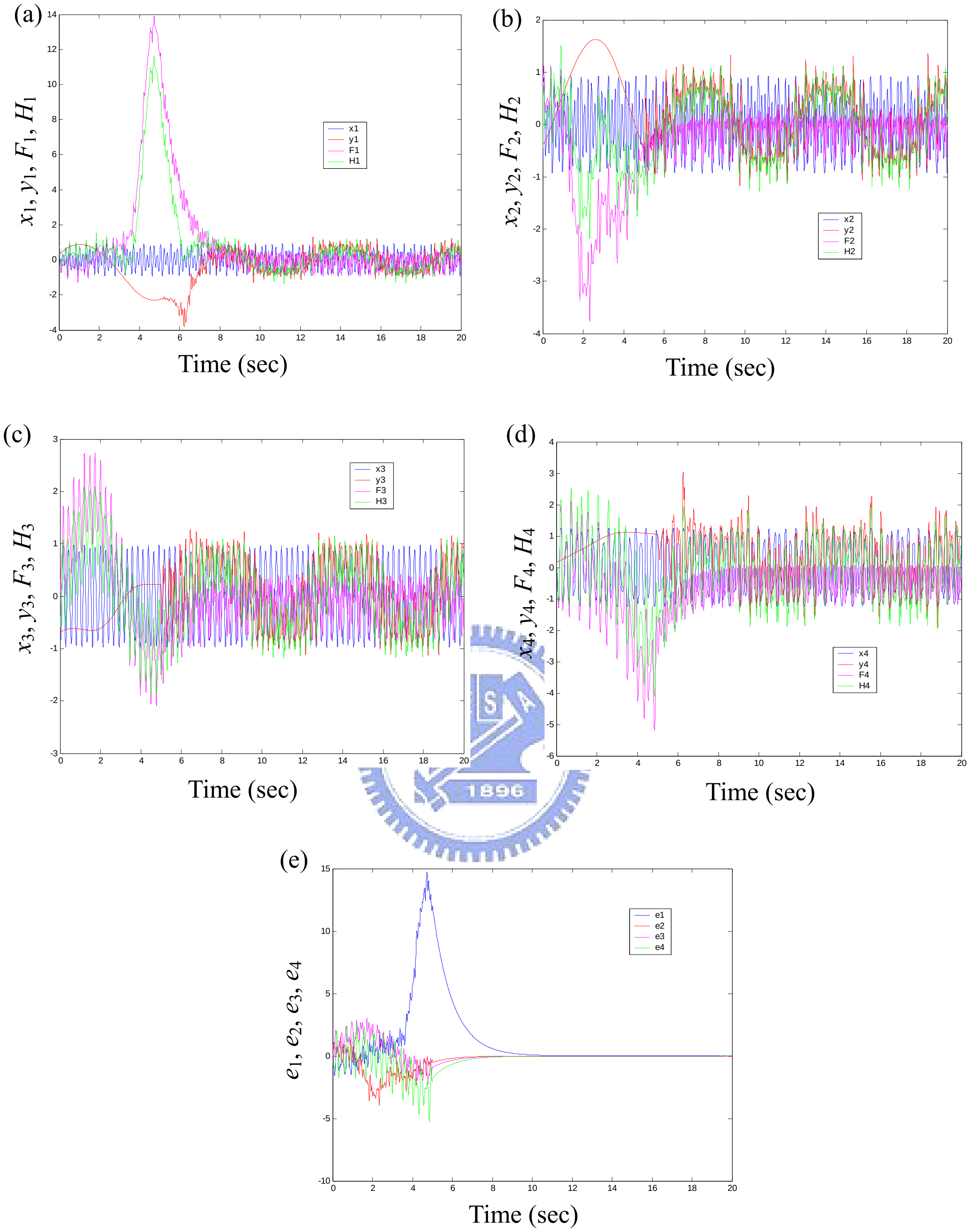


Fig. 4-11. Time histories of states, state errors, $F_1, F_2, F_3, F_4, H_1, H_2, H_3$ and H_4 for Case III.

4-5 Chaos Synchronization of Quantum-CNN Oscillators Chaotic System by Variable Structure Control

There are two identical nonlinear dynamical systems and the master system controls the slave system. The master system is given here

$$\dot{x} = Ax + f(x) \qquad (4\text{-}5\text{-}1)$$

where $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ denotes the state vector, A is a $n \times n$ coefficient matrix, and f is a nonlinear vector function.

The slave system is given here

$$\dot{y} = Ay + f(y) + u(t) \qquad (4\text{-}5\text{-}2)$$

where $y = (y_1, y_2, \cdots, y_n)^T \in R^n$ denotes a state vector and $u(t) = (u_1(t), u_2(t), \cdots, u_n(t))^T \in R^n$ is a control input vector.

Our goal is to design a controller $u(t)$ so that the state vector of the slave system (4-5-2) asymptotically approaches the state vector of the master system (4-5-1) and finally the synchronization will be accomplished in the sense that the limit of the error vector $e(t) = (e_1, e_2, \cdots, e_{2n})^T$ approaches zero:

$$\lim_{t\to\infty} e = 0 \qquad (4\text{-}5\text{-}3)$$

where

$$e = y - x \qquad (4\text{-}5\text{-}4)$$

From Eq. (4) we have

$$\dot{e} = \dot{y} - \dot{x} = Ae + F(x, y) + u(t) \qquad (4\text{-}5\text{-}5)$$

where $F(x, y) = f(y) - f(x)$.

According to the theory of active control, we can use the control input vector-function $u(t)$ to eliminate all items that cannot be shown in the form of the error vector e. By this way, the vector-function $u(t)$ can be determined.

$$u(t) = H(t) - F(x, y) \qquad (4\text{-}5\text{-}6)$$

And Eq. (4-5-5) is rewritten as

$$\dot{e} = Ae + H(t) \tag{4-5-7}$$

Eq. (4-5-7) describes the error dynamics and can be considered in terms of a control problem where the system to be controlled is a linear system with a control input $H(t)$ as the functions of the error vector e. As long as these feedbacks stabilize the system, the error vector e converges to zero as $t\rightarrow\infty$. This implies that the master system (4-5-1) and the salve system (4-5-2) are synchronized finally. There are many possible choices for the control $H(t)$. It is well known that the most distinguished feature of the sliding mode control technique is that, when in sliding mode, the system is robust to parametric uncertainty and external disturbances. Without loss of generality, we choose the sliding mode control law as follows:

$$H(t) = Kw(t) \tag{4-5-8}$$

where $K = (k_1, k_2, \cdots, k_n)^T$ is a constant gain vector, $w(t) \in R$ is the control input and satisfies

$$w(t)\begin{cases} w^+(t) & s(e) > 0 \\ w^-(t) & s(e) < 0 \end{cases} \tag{4-5-9}$$

and $s=s(e)$ is a switching surface that prescribes the desired dynamics.

This results in

$$\dot{e} = Ae + Kw(t). \tag{4-5-10}$$

In what follows, the appropriate sliding mode controller will be designed in terms of the sliding mode control theory.

(1) Sliding surface design: Generally speaking, the sliding surface $s(e)$ can be defined as

$$s(e) = Ce \tag{4-5-11}$$

where $C = (c_1, c_2, \cdots, c_n)$ is a constant vector.

The equivalent control approach is found by recognizing that $\dot{s}(e) = 0$ is a necessary condition for the state trajectory to stay on the switching surface $s(e)=0$. Hence, when in sliding mode, the controlled system satisfies the following conditions

$$s(e) = 0 \tag{4-5-12a}$$

and

$$\dot{s}(e) = 0 \qquad (4\text{-}5\text{-}12b)$$

Substituting Eq. (4-5-10) and Eq. (4-5-11) into Eq. (4-5-12b), we can obtain

$$\dot{s}(e) = \frac{\partial s(e)}{\partial e}\dot{e} = C\left[Ae + Kw(t)\right] = 0. \qquad (4\text{-}5\text{-}13)$$

Solving Eq. (13) for $w(t)$ yields the equivalent control $w_{eq}(t)$

$$w_{eq}(t) = -(CK)^{-1}CAe \qquad (4\text{-}5\text{-}14)$$

where the existence of the matrix inverse $(CK)^{-1}$ is a necessary condition.

Putting Eq. (4-5-14) into Eq. (4-5-10), the state equation in the sliding mode is given as follows:

$$\dot{e} = \left[I - K(CK)^{-1}C\right]Ae. \qquad (4\text{-}5\text{-}15)$$

Suppose that the vector K is selected such that (A, K) is controllable.

(2) Design of the sliding mode controller: Assume that the constant plus proportional rate reaching law is applied. The reaching law can be chosen such that

$$\dot{s} = -q\cdot\mathrm{sgn}(s) - rs \qquad (4\text{-}5\text{-}16)$$

where $sgn(\cdot)$denotes the saturation function, and the gains q>0 and r>0 is determined such that the sliding condition is satisfied and sliding mode motion will occur.

From Eq. (4-5-10) and Eq. (4-5-11), it can be found that

$$\dot{s}(e) = C\left[Ae + Kw(t)\right]. \qquad (4\text{-}5\text{-}17)$$

By Eq. (4-5-16), we have

$$w_{eq}(t) = -(CK)^{-1}\left[C(rI + A)e + q\cdot\mathrm{sgn}(s)\right] \qquad (4\text{-}5\text{-}18)$$

(3) Robust stability analysis: In order to check the stability of the above controlled system, we can construct the following Lyapunov function

$$V = \frac{1}{2}s^2 \qquad (4\text{-}5\text{-}19)$$

and then, differentiation of the above expression Eq. (4-5-19) with respect to time yields

$$\dot{V} = \dot{s}s = \frac{\partial s}{\partial e}\dot{e}s = C\dot{e}s. \qquad (4\text{-}5\text{-}20)$$

Substituting Eq. (10) and Eq. (18) into Eq. (20), then we can obtain

$$\dot{V} = sC\left[Ae + Kw(t)\right] = sC\left\{Ae - K(CK)^{-1}\left[C(rI + A)e + q\cdot\operatorname{sgn}(s)\right]\right\} = -rs^2 - sq\cdot\operatorname{sgn}(s). \quad (4\text{-}5\text{-}21)$$

Since the expression $-sq\cdot\operatorname{sgn}(s)$ is always negative when $e \neq 0$, the inequality $\dot{V} = \dot{s}s < 0$.

As an example, let us consider Quantum-CNN system. For a two-cell Quantum-CNN, the following differential equations are obtained [23]:

The master system is described by:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \quad (4\text{-}5\text{-}22)$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let $a_1 = a_2 = 4.9$, $\omega_1 = 1.13$ and $\omega_2 = 0.85$. The initial values of Quantum-CNN systems are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$, $x_4(0) = 0.97$, $y_1(0) = -0.98$, $y_2(0) = 0.87$, $y_3(0) = 0.92$ and $y_4(0) = -0.93$ respectively.

The slave system is described by:

$$\begin{cases} \dot{y}_1 = -2a_1\sqrt{1-y_1^2}\sin y_2 \\ \dot{y}_2 = -\omega_1(y_1 - y_3) + 2a_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 \\ \dot{y}_3 = -2a_2\sqrt{1-y_3^2}\sin y_4 \\ \dot{y}_4 = -\omega_2(y_3 - y_1) + 2a_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 \end{cases} \quad (4\text{-}5\text{-}23)$$

The result is show in Fig. 4-12.

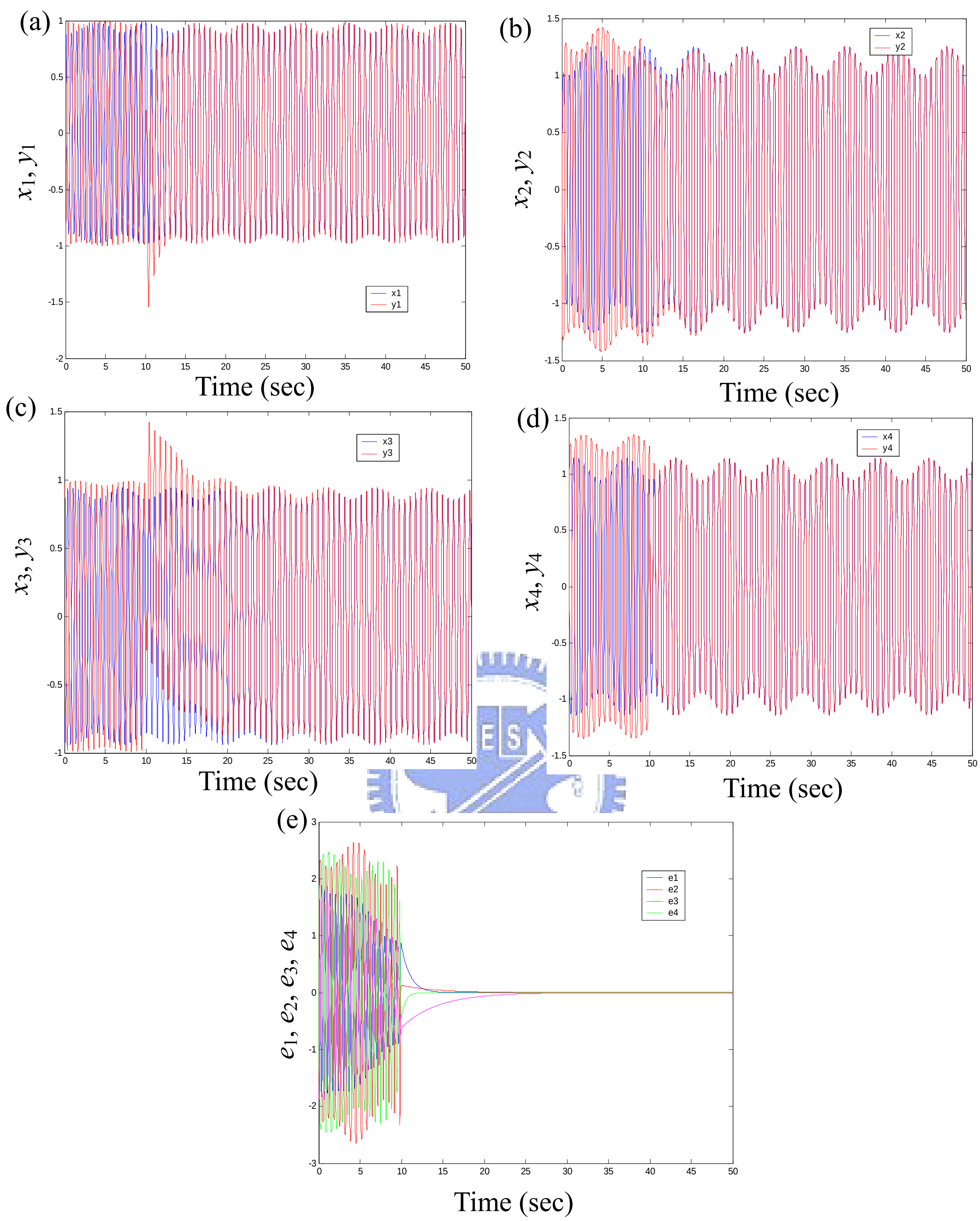


Fig. 4-12. Time histories of states, state errors.

Chapter 5

Linear Coupling Synchronization

5-1 Preliminary

In Section 5-2, a generic criterion of the chaos synchronization is studied for complex chaotic systems in series expansion form with unidirectional/mutual coupling. In Section 5-3, a quadratic optimal regulator is used for the chaos synchronization. As we know in practical systems it is difficult to obtain the precise mathematical model, so in practical applications the investigators would like to employ simple and efficient controllers. Therefore how to design a simple controller with the a few informations of chaotic systems obtained is an open problem.

5-2 Synchronization of the Complex Chaotic Systems in Series Expansion Form

This thesis studies the synchronization of complex chaotic systems in series expansion form by Lyapunov asymptotical stability theorem. A sufficient condition is given for the asymptotical stability of an error dynamics, and is applied to guiding the design of the secure communication. Finally, numerical results are studied for the Quantum-CNN oscillators synchronizing with unidirectional/mutual linear coupling to show the effectiveness of the proposed synchronization strategy.

5-2-1 Synchronization Schemes of Complex Chaotic Systems in Series Expansion Form

In this section, the synchronization by unidirectional/mutual linear coupling is studied. Based on Lyapunov asymptotical stabilization theory, we give a generic situation for judging chaos synchronization of two complex systems in series expansion form. The two chaos systems using mutual/unidirectional linear coupling can be written as

$$\dot{x} = Ax + h(x) + k_1(y - x) \tag{5-2-1}$$

and

$$\dot{y} = Ay + h(y) + k_2(x - y) \tag{5-2-2}$$

where $x, y \in R^n$ represent the state vectors of the chaotic systems, $A \in R^{n\times n}$ is a constant matrix, $h \in R^{n\times n}$ is a continuous nonlinear function of x, y, k_1 and k_2 are constant gains which represent the coupled parameters.

Let

$$e = x - y \tag{5-2-3}$$

then

$$\dot{e} = \left[A + M(x(t), y(t)) - (k_1 + k_2)\right]e + H(x(t), y(t), e) \tag{5-2-4}$$

where $M(x(t), y(t))e + H(x(t), y(t), e) = h(x) - h(y)$. The elements of $M(x(t), y(t))$ depend on state vectors x, y, and all of them are bounded convergent infinite series of x, y. $H(x(t), y(t), e)$ contains nonlinear terms of e.

In order to realize the chaos synchronization among these two chaotic systems, we should choose suitable coupled parameter matrices k_1 and k_2 so that

$$\lim_{t\to\infty} e(t) = 0 \tag{5-2-5}$$

In the following, we give the generic criterion of local chaos synchronization of linear coupling systems using the mutual linear error feedback scheme.

Theorem If there exists a positive definite symmetric constant matrix ***P*** and a constant $\varepsilon>0$, which satisfy

$$B^T P + PB \le -\varepsilon I < 0 \tag{5-2-6}$$

uniformly for any x and y in the phase space, where then the error dynamics system (5-2-4) is locally asymptotical stable.

Proof For the nonautonomous error dynamic system (5-2-4), choose the following Lyapunov function:

$$V(t) = e^T P e \tag{5-2-7}$$

then its derivative is

$$\begin{aligned}\dot{V}(t) &= \dot{e}^T P e + e^T P \dot{e} = e^T B^T(t) P e + e^T P B(t) e + higher\ order\ terms\ of\ e \\ &= e^T (B^T(t) P + P B(t)) e + \cdots \le -\varepsilon e^T e < 0\end{aligned} \tag{5-2-8}$$

for all sufficient small $e \neq 0$. For sufficient small e, its higher order terms are indifferent to the sign of $\dot{V}(t)$. So, the theorem is proved.

5-2-2 Numerical Results of the Synchronization of Two Quantum-CNN Oscillator Systems by Unidirectional and by Mutual Linear Coupling

Case I The synchronization by unidirectional linear coupling

For a two-cell Quantum-CNN, the following differential equations are obtained:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{5-2-9}$$

where x_1 and x_3 are polarizations, x_2 and x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let $a_1 = a_2 = 13.4$, $\omega_1 = 11.9$ and $\omega_2 = 6.04$. The initial values of linear coupled Quantum-CNN systems are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$, $x_4(0) = 0.57$, $y_1(0) = -0.2$, $y_2(0) = 0.41$, $y_3(0) = 0.25$ and $y_4(0) = -0.81$ respectively.

Two Quantum-CNN chaotic systems using the unidirectional linear coupling can be written as

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1-x_3)+2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3-x_1)+2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{5-2-10}$$

and

$$\begin{cases} \dot{y}_1 = -2a_1\sqrt{1-y_1^2}\sin y_2 + k_1(x_1-y_1) \\ \dot{y}_2 = -\omega_1(y_1-y_3)+2a_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + k_2(x_2-y_2) \\ \dot{y}_3 = -2a_2\sqrt{1-y_3^2}\sin y_4 + k_3(x_3-y_3) \\ \dot{y}_4 = -\omega_2(y_3-y_1)+2a_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + k_4(x_4-y_4) \end{cases} \tag{5-2-11}$$

where the values of k_1, k_2, k_3 and k_4 are to be determined.

Expand the right hand sides of Eq. (5-2-10) and Eq.(5-2-11) into power series:

$$\begin{cases} \dot{x}_1 = -2a_1(-\frac{1}{2}x_1^2x_2+\frac{1}{12}x_1^2x_2^3-\frac{1}{8}x_1^4x_2+x_2-\frac{1}{6}x_2^3+\frac{1}{120}x_2^5+\cdots) \\ \dot{x}_2 = -\omega_1(x_1-x_3)+2a_1(x_1-\frac{1}{2}x_1x_2^2+\frac{1}{24}x_1x_2^4+\frac{1}{2}x_1^3-\frac{1}{4}x_1^3x_2^2+\frac{5}{8}x_1^5+\cdots) \\ \dot{x}_3 = -2a_2(-\frac{1}{2}x_3^2x_4+\frac{1}{12}x_3^2x_4^3-\frac{1}{8}x_3^4x_4+x_4-\frac{1}{6}x_4^3+\frac{1}{120}x_4^5+\cdots) \\ \dot{x}_4 = -\omega_2(x_3-x_1)+2a_2(x_3-\frac{1}{2}x_3x_4^2+\frac{1}{24}x_3x_4^4+\frac{1}{2}x_3^3-\frac{1}{4}x_3^3x_4^2+\frac{5}{8}x_3^5+\cdots) \end{cases} \tag{5-2-12}$$

and

$$\begin{cases} \dot{y}_1 = -2a_1(-\frac{1}{2}y_1^2y_2+\frac{1}{12}y_1^2y_2^3-\frac{1}{8}y_1^4y_2+y_2-\frac{1}{6}y_2^3+\frac{1}{120}y_2^5+\cdots)+k_1(x_1-y_1) \\ \dot{y}_2 = -\omega_1(y_1-y_3)+2a_1(y_1-\frac{1}{2}y_1y_2^2+\frac{1}{24}y_1y_2^4+\frac{1}{2}y_1^3-\frac{1}{4}y_1^3y_2^2+\frac{5}{8}y_1^5+\cdots) \\ \quad +k_2(x_2-y_2) \\ \dot{y}_3 = -2a_2(-\frac{1}{2}y_3^2y_4+\frac{1}{12}y_3^2y_4^3-\frac{1}{8}y_3^4y_4+y_4-\frac{1}{6}y_4^3+\frac{1}{120}y_4^5+\cdots)+k_3(x_3-y_3) \\ \dot{y}_4 = -\omega_2(y_3-y_1)+2a_2(y_3-\frac{1}{2}y_3y_4^2+\frac{1}{24}y_3y_4^4+\frac{1}{2}y_3^3-\frac{1}{4}y_3^3y_4^2+\frac{5}{8}y_3^5+\cdots) \\ \quad +k_4(x_4-y_4) \end{cases} \tag{5-2-13}$$

We note that

$$
\begin{aligned}
&x_\alpha x_\beta^2 - y_\alpha y_\beta^2 = y_\beta^2 e_\alpha + x_\alpha e_\beta^2 + 2x_\alpha y_\beta e_\beta,\; x_\alpha^2 x_\beta - y_\alpha^2 y_\beta = -y_\beta e_\alpha^2 + x_\alpha^2 e_\beta + 2x_\alpha y_\beta e_\alpha, \\
&x_\alpha x_\beta^4 - y_\alpha y_\beta^4 = y_\beta^4 e_\alpha + x_\alpha x_\beta^2 e_\beta^2 + x_\alpha x_\beta^2 y_\beta e_\beta + x_\alpha y_\beta (e_\beta^3 + 3x_\beta y_\beta e_\beta), \\
&x_\alpha^2 x_\beta^3 - y_\alpha^2 y_\beta^3 = -y_\beta^3 e_\alpha^2 + x_\alpha^2 x_\beta^2 e_\beta + x_\alpha y_\beta^3 e_\beta + x_\alpha y_\beta (y_\beta^2 e_\alpha + x_\alpha e_\beta^2 + 2x_\alpha y_\beta e_\beta), \\
&x_\alpha^3 x_\beta^2 - y_\alpha^3 y_\beta^2 = y_\alpha^2 y_\beta^2 e_\alpha + x_\alpha^3 e_\beta^2 + x_\alpha^3 y_\beta e_\beta + x_\alpha y_\beta (-y_\beta e_\alpha^2 + x_\alpha^2 e_\beta + 2x_\alpha y_\beta e_\alpha), \\
&x_\alpha^4 x_\beta - y_\alpha^4 y_\beta = -y_\alpha^2 y_\beta e_\alpha^2 + x_\alpha^4 e_\beta + x_\alpha y_\alpha^2 y_\beta e_\alpha + x_\alpha y_\beta (e_\alpha^3 + 3x_\alpha y_\alpha e_\alpha), \\
&x_\alpha^5 - y_\alpha^5 = e_\alpha^5 + 5x_\alpha y_\alpha (e_\alpha^3 + 3x_\alpha y_\alpha e_\alpha) - 10x_\alpha^2 y_\alpha^2 e_\alpha,\; x_\alpha^3 - y_\alpha^3 = e_\alpha^3 + 3x_\alpha y_\alpha e_\alpha.
\end{aligned}
\tag{5-2-14}
$$

From Eq. (5-2-12), Eq.(5-2-13) and Eq. (5-2-14), the error dynamics is:

$$
\begin{aligned}
\dot{e}_1 = {} & -2a_1 e_2 + a_1 (2x_1 y_2 - \frac{1}{6} x_1 y_2^3 + \frac{1}{4} x_1 y_1 y_2 (y_1 + 3x_1) + \cdots) e_1 + a_1 (x_1^2 + x_2 y_2 - \frac{1}{6} x_1 (x_1 x_2^2 + y_2^3 \\
& + 2x_1 y_2^2) + \frac{1}{4} x_1^4 - \frac{1}{12} x_2^2 y_2^2 + \cdots) e_2 - a_1 (y_2 - \frac{1}{6} y_2^3 + \frac{1}{4} y_1^2 y_2 + \cdots) e_1^2 + a_1 (-\frac{1}{6} x_1^2 y_2 + \cdots) e_2^2 \\
& + a_1 (\frac{1}{4} x_1 y_2 + \cdots) e_1^3 + a_1 (\frac{1}{3} + \cdots) e_2^3 + a_1 (-\frac{1}{12} x_2 y_2 + \cdots) e_2^3 + a_1 (-\frac{1}{60} + \cdots) e_2^5 + \cdots - k_1 e_1 \\
\dot{e}_2 = {} & (2a_1 - \omega_1) e_1 + \omega_1 e_3 + a_1 (3x_1 y_1 - y_2^2 + \frac{1}{12} y_2^4 - \frac{1}{2} y_2^2 (y_1^2 + 2x_1^2) + \frac{25}{4} x_1^2 y_1^2 + \cdots) e_1 \\
& + a_1 (2x_1 y_2 + \frac{1}{12} x_1 x_2 y_2 (x_2 + 3y_2) - x_1^3 y_2 + \cdots) e_2 + a_1 (\frac{1}{2} x_1 y_2 y_2 + \cdots) e_1^2 - a_1 (x_1 + \frac{1}{12} x_1 x_2^2 \\
& - \frac{1}{2} x_1^3 + \cdots) e_2^2 + a_1 (1 + \frac{25}{4} x_1 y_1 + \cdots) e_1^3 + a_1 (\frac{1}{12} x_1 y_2 + \cdots) e_2^3 + a_1 (\frac{5}{4} + \cdots) e_1^5 + \cdots - k_2 e_2 \\
\dot{e}_3 = {} & -2a_2 e_4 + a_2 (2x_3 y_4 - \frac{1}{6} x_3 y_4^3 + \frac{1}{4} x_3 y_3 y_4 (y_3 + 3x_3) + \cdots) e_3 + a_2 (x_3^2 + x_4 y_4 - \frac{1}{6} x_3 (x_3 x_4^2 + y_4^3 \\
& + 2x_3 y_4^2) + \frac{1}{4} x_3^4 - \frac{1}{12} x_4^2 y_4^2 + \cdots) e_4 - a_2 (y_4 - \frac{1}{6} y_4^3 + \frac{1}{4} y_3^2 y_4 + \cdots) e_3^2 + a_2 (-\frac{1}{6} x_3^2 y_4 + \cdots) e_4^2 \\
& + a_2 (\frac{1}{4} x_3 y_4 + \cdots) e_3^3 + a_2 (\frac{1}{3} + \cdots) e_4^3 + a_2 (-\frac{1}{12} x_4 y_4 + \cdots) e_4^3 + a_2 (-\frac{1}{60} + \cdots) e_4^5 + \cdots - k_3 e_3 \\
\dot{e}_4 = {} & (2a_2 - \omega_2) e_3 + \omega_2 e_1 + a_2 (3x_3 y_3 - y_4^2 + \frac{1}{12} y_4^4 - \frac{1}{2} y_4^2 (y_3^2 + 2x_3^2) + \frac{25}{4} x_3^2 y_3^2 + \cdots) e_3 \\
& + a_2 (2x_3 y_4 + \frac{1}{12} x_3 x_4 y_4 (x_4 + 3y_4) - x_3^3 y_4 + \cdots) e_4 + a_2 (\frac{1}{2} x_3 y_4 y_4 + \cdots) e_3^2 - a_2 (x_3 + \frac{1}{12} x_3 x_4^2 \\
& - \frac{1}{2} x_3^3 + \cdots) e_4^2 + a_2 (1 + \frac{25}{4} x_3 y_3 + \cdots) e_3^3 + a_2 (\frac{1}{12} x_3 y_4 + \cdots) e_4^3 + a_2 (\frac{5}{4} + \cdots) e_3^5 + \cdots - k_4 e_4
\end{aligned}
$$

where $e = (y_1 - x_1,\ \ y_2 - x_2,\ \ y_3 - x_3,\ \ y_4 - x_4)^T$. We can obtain

$$
B = \begin{pmatrix}
-k_1 + B_{11} & -2a_1 + B_{21} & 0 & 0 \\
2a_1 - \omega_1 + B_{12} & -k_2 + B_{22} & \omega_1 & 0 \\
0 & 0 & -k_3 + B_{33} & -2a_2 + B_{43} \\
\omega_2 & 0 & 2a_2 - \omega_2 + B_{34} & -k_4 + B_{44}
\end{pmatrix}
$$

where

$$B_{11} = a_1[2x_1y_2 - \frac{1}{6}x_1y_2^3 + \frac{1}{4}(x_1y_1^2y_2 + 3x_1^2y_1y_2) + \cdots]$$

$$B_{12} = a_1[3x_1y_1 - y_2^2 + \frac{1}{12}y_2^4 - \frac{1}{2}y_2^2(y_1^2 + 2x_1^2) + \frac{25}{4}x_1^2y_1^2 + \cdots]$$

$$B_{21} = a_1[x_1^2 + x_2y_2 - \frac{1}{6}(x_1^2x_2^2 + x_1y_2^3 + 2x_1^2y_2^2) + \frac{1}{4}x_1^4 - \frac{1}{12}x_2^2y_2^2 + \cdots]$$

$$B_{22} = a_1[2x_1y_2 + \frac{1}{12}x_1x_2y_2(x_2 + 3y_2) - x_1^3y_2 + \cdots]$$

$$B_{33} = a_2[2x_3y_4 - \frac{1}{6}x_3y_4^3 + \frac{1}{4}(x_3y_3^2y_4 + 3x_3^2y_3y_4) + \cdots]$$

$$B_{34} = a_2[3x_3y_3 - y_4^2 + \frac{1}{12}y_4^4 - \frac{1}{2}y_4^2(y_3^2 + 2x_3^2) + \frac{25}{4}x_3^2y_3^2 + \cdots]$$

$$B_{43} = a_2[x_3^2 + x_4y_4 - \frac{1}{6}(x_3^2x_4^2 + x_3y_4^3 + 2x_3^2y_4^2) + \frac{1}{4}x_3^4 - \frac{1}{12}x_4^2y_4^2 + \cdots]$$

$$B_{44} = a_2[2x_3y_4 + \frac{1}{12}x_3x_4y_4(x_4 + 3y_4) - x_3^3y_4 + \cdots]$$

The infinite power series in the first element of B_{11} is

$$2x_1y_2 - \frac{1}{6}x_1y_2^3 + \frac{1}{4}(x_1y_1^2y_2 + 3x_1^2y_1y_2) + \cdots \qquad (5\text{-}2\text{-}15)$$

It is well-known [85-86] that a necessary and sufficient condition for the convergence of the infinite series

$$u_1 + u_2 + \cdots + u_n + \cdots$$

is that for any previously assigned positive ε there exists an N such that, for any $n>N$ and for positive p,

$$\left|u_{n+1} + u_{n+2} + \cdots u_{n+p}\right| < \varepsilon \qquad (5\text{-}2\text{-}16)$$

From Fig. 5-1, we know that

$$|x_i| < 1, \quad |y_i| < 1 \quad (i = 1, 2, 3, 4) \qquad (5\text{-}2\text{-}17)$$

therefore series (5-2-15) which satisfies condition (5-2-16), is convergent and has a bounded sum. For the same reason, the other power series in B_{11}, B_{12}, B_{21}, B_{22}, B_{33}, B_{34}, B_{43} and B_{44} are all convergent and have bounded sums. The time histories of B_{11}, B_{12}, B_{21}, B_{22}, B_{33}, B_{34}, B_{43} and B_{44} are shown in Fig. 5-2.

Choose the positive definite symmetric constant matrix $\boldsymbol{P}$ = diag (p_1, p_2, p_3, p_4),

$p_i > 0,\ i = 1, 2, 3, 4$ and any constant $\varepsilon > 0$, then $C = B^T P + PB + \varepsilon I$ is negative definite if and only if

$$(-1)^{i+1}\Delta_i < 0, \quad i = 1, 2, 3, 4 \tag{5-2-18}$$

where Δ_i represents the ith order sequential subdeterminant of matrix C. Let $\varepsilon = 0.1$, $p_i = 1, i = 1, 2, \cdots, 4$.

$$C = \begin{pmatrix} -2k_1 + 2B_{11} + \varepsilon & -\omega_1 + B_{12} + B_{21} & 0 & \omega_2 \\ -\omega_1 + B_{12} + B_{21} & -2k_2 + 2B_{22} + \varepsilon & \omega_1 & 0 \\ 0 & \omega_1 & -2k_3 + 2B_{33} + \varepsilon & -\omega_2 + B_{34} + B_{43} \\ \omega_2 & 0 & -\omega_2 + B_{34} + B_{43} & -2k_4 + 2B_{44} + \varepsilon \end{pmatrix}$$

$$\begin{aligned}
&\Delta_1 = -2(k_1 - B_{11}) + \varepsilon < 0 \\
&\Delta_2 = -2A_1(k_2 - B_{22}) + A_1\varepsilon - (-\omega_1 + B_{12} + B_{21})^2 > 0 \\
&\Delta_3 = -2A_2(k_3 - B_{33}) + A_2\varepsilon - \omega_1^2 A_1 < 0 \\
&\Delta_4 = -\omega_2 \begin{pmatrix} -\omega_1 + B_{12} + B_{21} & 0 & \omega_2 \\ -2k_2 + 2B_{22} + \varepsilon & \omega_1 & 0 \\ \omega_1 & -2k_3 + 2B_{33} + \varepsilon & -\omega_2 + B_{34} + B_{43} \end{pmatrix} \\
&\quad -(-\omega_2 + B_{34} + B_{43}) \begin{pmatrix} -2k_1 + 2B_{11} & -\omega_1 + B_{12} + B_{21} & \omega_2 \\ -\omega_1 + B_{12} + B_{21} & -2k_2 + 2B_{22} + \varepsilon & 0 \\ 0 & \omega_1 & -\omega_2 + B_{34} + B_{43} \end{pmatrix} \\
&\quad -2A_3(k_4 - B_{44}) + A_3\varepsilon \\
&\quad = -\omega_2[\omega_1(-\omega_1 + B_{12} + B_{21})(-\omega_2 + B_{34} + B_{43}) + \omega_2((-2k_2 + 2B_{22} + \varepsilon) \\
&\quad (-2k_3 + 2B_{33} + \varepsilon) - \omega_1^2)] - (-\omega_2 + B_{34} + B_{43})[\omega_2\omega_1(-\omega_1 + B_{12} + B_{21}) \\
&\quad + (-\omega_2 + B_{34} + B_{43})A_2] - 2A_3(k_4 - B_{44}) + A_3\varepsilon > 0
\end{aligned} \tag{5-2-19}$$

where

$$\begin{cases} A_1 = -2(k_1 - B_{11}) + \varepsilon, \\ A_2 = -2A_1(k_2 - B_{22}) + A_1\varepsilon - (-\omega_1 + B_{12} + B_{21})^2 \\ A_3 = -2A_2(k_3 - B_{33}) + A_2\varepsilon - \omega_1^2 A_1 \end{cases} \tag{5-2-20}$$

i.e.

$$k_1 > \max\{B_{11}(t) + \frac{\varepsilon}{2}\} = 30.07 \tag{5-2-21}$$

k_1 is chosen as 33.07, by Eq.(5-2-20) A_1=6

$$k_2 > \max\{\frac{A_1(2B_{22}(t)+\varepsilon)-(-\omega_1+B_{12}(t)+B_{21}(t))^2}{2A_1}\}=10.9 \tag{5-2-22}$$

k_2 is chosen as 11.8, by Eq.(5-2-20) A_2=11.65

$$k_3 > \max\{\frac{A_2(2B_{33}(t)+\varepsilon)-\omega_1^2 A_1}{2A_2}\}=50.35 \tag{5-2-23}$$

k_3 is chosen as 52, by Eq.(5-2-20) A_3=−38.3

$$\begin{aligned} k_4 > \max\{ & \frac{-\omega_2[\omega_1(-\omega_1+B_{12}(t)+B_{21}(t))(-\omega_2+B_{34}(t)+B_{43}(t))+\omega_2((-2(k_2-B_{22}(t)-\frac{\varepsilon}{2})}{2A_3} \\ & \times\frac{(-2(k_3-B_{33}(t)-\frac{\varepsilon}{2})-\omega_1^2)]+A_3(2B_{44}(t)+\varepsilon)-(-\omega_2+B_{34}(t)+B_{43}(t))}{2A_3} \\ & \times\frac{[\omega_2\omega_1(-\omega_1+B_{12}(t)+B_{21}(t))+(-\omega_2+B_{34}(t)+B_{43}(t))A_2]}{2A_3}\}=3018 \end{aligned} \tag{5-2-24}$$

k_4 is chosen as 3020.

The Lyapunov asymptotical local stability theorem is satisfied. This means that the synchronization of two Quantum-CNN systems by unidirectional linear coupling can be achieved. The numerical results are shown in Fig. 5-3.

Case II The synchronization by mutual linear coupling

Tow Quantum-CNN systems with mutual linear coupling are given:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 + k_{11}(y_1-x_1) \\ \dot{x}_2 = -\omega_1(x_1-x_3)+2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + k_{12}(y_2-x_2) \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 + k_{13}(y_3-x_3) \\ \dot{x}_4 = -\omega_2(x_3-x_1)+2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + k_{14}(y_4-x_4) \end{cases} \tag{5-2-25}$$

and

$$\begin{cases} \dot{y}_1 = -2a_1\sqrt{1-y_1^2}\sin y_2 + k_{21}(x_1-y_1) \\ \dot{y}_2 = -\omega_1(y_1-y_3)+2a_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + k_{22}(x_2-y_2) \\ \dot{y}_3 = -2a_2\sqrt{1-y_3^2}\sin y_4 + k_{23}(x_3-y_3) \\ \dot{y}_4 = -\omega_2(y_3-y_1)+2a_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + k_{24}(x_4-y_4) \end{cases} \tag{5-2-26}$$

where the values of k_{11}, k_{12}, k_{13}, k_{14}, k_{21}, k_{22}, k_{23} and k_{24} are to be determined.

Expand the right hand sides of Eq. (5-2-25) and Eq. (5-2-26) into power series:

$$\begin{cases} \dot{x}_1 = -2a_1(-\frac{1}{2}x_1^2x_2 + \frac{1}{12}x_1^2x_2^3 - \frac{1}{8}x_1^4x_2 + x_2 - \frac{1}{6}x_2^3 + \frac{1}{120}x_2^5 + \cdots) + k_{11}(y_1 - x_1) \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1(x_1 - \frac{1}{2}x_1x_2^2 + \frac{1}{24}x_1x_2^4 + \frac{1}{2}x_1^3 - \frac{1}{4}x_1^3x_2^2 + \frac{5}{8}x_1^5 + \cdots) + k_{12}(y_2 - x_2) \\ \dot{x}_3 = -2a_2(-\frac{1}{2}x_3^2x_4 + \frac{1}{12}x_3^2x_4^3 - \frac{1}{8}x_3^4x_4 + x_4 - \frac{1}{6}x_4^3 + \frac{1}{120}x_4^5 + \cdots) + k_{13}(y_3 - x_3) \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2(x_3 - \frac{1}{2}x_3x_4^2 + \frac{1}{24}x_3x_4^4 + \frac{1}{2}x_3^3 - \frac{1}{4}x_3^3x_4^2 + \frac{5}{8}x_3^5 + \cdots) + k_{14}(y_4 - x_4) \end{cases} \quad (5\text{-}2\text{-}27)$$

and

$$\begin{cases} \dot{y}_1 = -2a_1(-\frac{1}{2}y_1^2y_2 + \frac{1}{12}y_1^2y_2^3 - \frac{1}{8}y_1^4y_2 + y_2 - \frac{1}{6}y_2^3 + \frac{1}{120}y_2^5 + \cdots) + k_{21}(x_1 - y_1) \\ \dot{y}_2 = -\omega_1(y_1 - y_3) + 2a_1(y_1 - \frac{1}{2}y_1y_2^2 + \frac{1}{24}y_1y_2^4 + \frac{1}{2}y_1^3 - \frac{1}{4}y_1^3y_2^2 + \frac{5}{8}y_1^5 + \cdots) + k_{22}(x_2 - y_2) \\ \dot{y}_3 = -2a_2(-\frac{1}{2}y_3^2y_4 + \frac{1}{12}y_3^2y_4^3 - \frac{1}{8}y_3^4y_4 + y_4 - \frac{1}{6}y_4^3 + \frac{1}{120}y_4^5 + \cdots) + k_{23}(x_3 - y_3) \\ \dot{y}_4 = -\omega_2(y_3 - y_1) + 2a_2(y_3 - \frac{1}{2}y_3y_4^2 + \frac{1}{24}y_3y_4^4 + \frac{1}{2}y_3^3 - \frac{1}{4}y_3^3y_4^2 + \frac{5}{8}y_3^5 + \cdots) + k_{24}(x_4 - y_4) \end{cases} \quad (5\text{-}2\text{-}28)$$

From Eq.(5-2-27), Eq.(5-2-28), and Eq. (5-2-14), the error dynamics is:

$$\begin{aligned} \dot{e}_1 = & -2a_1e_2 + a_1(2x_1y_2 - \frac{1}{6}x_1y_2^3 + \frac{1}{4}x_1y_1y_2(y_1 + 3x_1) + \cdots)e_1 + a_1(x_1^2 + x_2y_2 - \frac{1}{6}x_1(x_1x_2^2 + y_2^3 \\ & + 2x_1y_2^2) + \frac{1}{4}x_1^4 - \frac{1}{12}x_2^2y_2^2 + \cdots)e_2 - a_1(y_2 - \frac{1}{6}y_2^3 + \frac{1}{4}y_1^2y_2 + \cdots)e_1^2 + a_1(-\frac{1}{6}x_1^2y_2 + \cdots)e_2^2 \\ & + a_1(\frac{1}{4}x_1y_2 + \cdots)e_1^3 + a_1(\frac{1}{3} + \cdots)e_2^3 + a_1(-\frac{1}{12}x_2y_2 + \cdots)e_2^3 + a_1(-\frac{1}{60} + \cdots)e_2^5 + \cdots - (k_{11} + k_{21})e_1 \\ \dot{e}_2 = & (2a_1 - \omega_1)e_1 + \omega_1e_3 + a_1(3x_1y_1 - y_2^2 + \frac{1}{12}y_2^4 - \frac{1}{2}y_2^2(y_1^2 + 2x_1^2) + \frac{25}{4}x_1^2y_1^2 + \cdots)e_1 \\ & + a_1(2x_1y_2 + \frac{1}{12}x_1x_2y_2(x_2 + 3y_2) - x_1^3y_2 + \cdots)e_2 + a_1(\frac{1}{2}x_1y_2y_2 + \cdots)e_1^2 - a_1(x_1 + \frac{1}{12}x_1x_2^2 \\ & - \frac{1}{2}x_1^3 + \cdots)e_2^2 + a_1(1 + \frac{25}{4}x_1y_1 + \cdots)e_1^3 + a_1(\frac{1}{12}x_1y_2 + \cdots)e_2^3 + a_1(\frac{5}{4} + \cdots)e_1^5 + \cdots - (k_{12} + k_{22})e_2 \\ \dot{e}_3 = & -2a_2e_4 + a_2(2x_3y_4 - \frac{1}{6}x_3y_4^3 + \frac{1}{4}x_3y_3y_4(y_3 + 3x_3) + \cdots)e_3 + a_2(x_3^2 + x_4y_4 - \frac{1}{6}x_3(x_3x_4^2 + y_4^3 \\ & + 2x_3y_4^2) + \frac{1}{4}x_3^4 - \frac{1}{12}x_4^2y_4^2 + \cdots)e_4 - a_2(y_4 - \frac{1}{6}y_4^3 + \frac{1}{4}y_3^2y_4 + \cdots)e_3^2 + a_2(-\frac{1}{6}x_3^2y_4 + \cdots)e_4^2 \\ & + a_2(\frac{1}{4}x_3y_4 + \cdots)e_3^3 + a_2(\frac{1}{3} + \cdots)e_4^3 + a_2(-\frac{1}{12}x_4y_4 + \cdots)e_4^3 + a_2(-\frac{1}{60} + \cdots)e_4^5 + \cdots - (k_{13} + k_{23})e_3 \end{aligned}$$

$$\dot{e}_4 = (2a_2 - \omega_2)e_3 + \omega_2 e_1 + a_2(3x_3y_3 - y_4^2 + \frac{1}{12}y_4^4 - \frac{1}{2}y_4^2(y_3^2 + 2x_3^2) + \frac{25}{4}x_3^2y_3^2 + \cdots)e_3$$
$$+ a_2(2x_3y_4 + \frac{1}{12}x_3x_4y_4(x_4 + 3y_4) - x_3^3y_4 + \cdots)e_4 + a_2(\frac{1}{2}x_3y_4y_4 + \cdots)e_3^2 - a_2(x_3 + \frac{1}{12}x_3x_4^2$$
$$- \frac{1}{2}x_3^3 + \cdots)e_4^2 + a_2(1 + \frac{25}{4}x_3y_3 + \cdots)e_3^3 + a_2(\frac{1}{12}x_3y_4 + \cdots)e_4^3 + a_2(\frac{5}{4} + \cdots)e_3^5 + \cdots - (k_{14} + k_{24})e_4$$

where $e = (y_1 - x_1,\ \ y_2 - x_2,\ \ y_3 - x_3,\ \ y_4 - x_4)^T$. We can obtain

$$B = \begin{pmatrix} -(k_{11}+k_{21})+B_{11} & -2a_1 + B_{21} & 0 & 0 \\ 2a_1 - \omega_1 + B_{12} & -(k_{12}+k_{22})+B_{22} & \omega_1 & 0 \\ 0 & 0 & -(k_{13}+k_{23})+B_{33} & -2a_2 + B_{43} \\ \omega_2 & 0 & 2a_2 - \omega_2 + B_{34} & -(k_{14}+k_{24})+B_{44} \end{pmatrix}$$

where

$$B_{11} = a_1[2x_1y_2 - \frac{1}{6}x_1y_2^3 + \frac{1}{4}(x_1y_1^2y_2 + 3x_1^2y_1y_2) + \cdots]$$
$$B_{12} = a_1[3x_1y_1 - y_2^2 + \frac{1}{12}y_2^4 - \frac{1}{2}y_2^2(y_1^2 + 2x_1^2) + \frac{25}{4}x_1^2y_1^2 + \cdots]$$
$$B_{21} = a_1[x_1^2 + x_2y_2 - \frac{1}{6}(x_1^2x_2^2 + x_1y_2^3 + 2x_1^2y_2^2) + \frac{1}{4}x_1^4 - \frac{1}{12}x_2^2y_2^2 + \cdots]$$
$$B_{22} = a_1[2x_1y_2 + \frac{1}{12}x_1x_2y_2(x_2 + 3y_2) - x_1^3y_2 + \cdots]$$
$$B_{33} = a_2[2x_3y_4 - \frac{1}{6}x_3y_4^3 + \frac{1}{4}(x_3y_3^2y_4 + 3x_3^2y_3y_4) + \cdots]$$
$$B_{34} = a_2[3x_3y_3 - y_4^2 + \frac{1}{12}y_4^4 - \frac{1}{2}y_4^2(y_3^2 + 2x_3^2) + \frac{25}{4}x_3^2y_3^2 + \cdots]$$
$$B_{43} = a_2[x_3^2 + x_4y_4 - \frac{1}{6}(x_3^2x_4^2 + x_3y_4^3 + 2x_3^2y_4^2) + \frac{1}{4}x_3^4 - \frac{1}{12}x_4^2y_4^2 + \cdots]$$
$$B_{44} = a_2[2x_3y_4 + \frac{1}{12}x_3x_4y_4(x_4 + 3y_4) - x_3^3y_4 + \cdots]$$

Similar to Case I, from Fig. 5-5, $|x_i| < 1,\ \ |y_i| < 1\ \ (i = 1, 2, 3, 4)$, the infinite power series in B_{11}, B_{12}, B_{21}, B_{22}, B_{33}, B_{34}, B_{43} and B_{44} are all convergent and have bounded sums [85-86]. The time histories of B_{11}, B_{12}, B_{21}, B_{22}, B_{33}, B_{34}, B_{43} and B_{44} are shown in Fig. 5-4.

Choose the positive definite symmetric constant matrix $\boldsymbol{P}$ = diag (p_1, p_2, p_3, p_4), $p_i > 0,\ i = 1, 2, 3, 4$ and any constant $\varepsilon > 0$, then $C = B^T P + PB + \varepsilon I$ is negative definite if and only if

$$(-1)^{i+1}\Delta_i < 0, \quad i = 1, 2, 3, 4 \tag{5-2-29}$$

where Δ_i represents the ith order sequential subdeterminant of matrix $\boldsymbol{C}$. Let $\varepsilon = 0.1$, $p_i = 1, i = 1, 2, \cdots, 4$.

$$C = \begin{pmatrix} -2(k_{11}+k_{21}+B_{11})+\varepsilon & -\omega_1+B_{12}+B_{21} & 0 & \omega_2 \\ -\omega_1+B_{12}+B_{21} & -2(k_{12}+k_{22}+B_{22})+\varepsilon & \omega_1 & 0 \\ 0 & \omega_1 & -2(k_{13}+k_{23}+B_{33})+\varepsilon & -\omega_2+B_{34}+B_{43} \\ \omega_2 & 0 & -\omega_2+B_{34}+B_{43} & -2(k_{14}+k_{24}+B_{44})+\varepsilon \end{pmatrix}$$

$$\begin{aligned}
&\Delta_1 = -2(k_{11}+k_{21}-B_{11})+\varepsilon < 0 \\
&\Delta_2 = -2A_1(k_{12}+k_{22}-B_{22})+A_1\varepsilon-(-\omega_1+B_{12}+B_{21})^2 > 0 \\
&\Delta_3 = -2A_2(k_{13}+k_{23}-B_{33})+A_2\varepsilon-\omega_1^2 A_1 < 0 \\
&\Delta_4 = -\omega_2 \begin{pmatrix} -\omega_1+B_{12}+B_{21} & 0 & \omega_2 \\ -2(k_{12}+k_{22})+2B_{22}+\varepsilon & \omega_1 & 0 \\ \omega_1 & -2(k_{13}+k_{23})+2B_{33}+\varepsilon & -\omega_2+B_{34}+B_{43} \end{pmatrix} \\
&\quad -(-\omega_2+B_{34}+B_{43}) \begin{pmatrix} -2(k_{11}+k_{21})+2B_{11} & -\omega_1+B_{12}+B_{21} & \omega_2 \\ -\omega_1+B_{12}+B_{21} & -2(k_{12}+k_{22})+2B_{22}+\varepsilon & 0 \\ 0 & \omega_1 & -\omega_2+B_{34}+B_{43} \end{pmatrix} \\
&\quad -2A_3(k_{14}+k_{24}-B_{44})+A_3\varepsilon \\
&\quad = -\omega_2[\omega_1(-\omega_1+B_{12}+B_{21})(-\omega_2+B_{34}+B_{43})+\omega_2((-2(k_{12}+k_{22}-B_{22})+\varepsilon)(-2(k_{13}+k_{23}) \\
&\quad +2B_{33}+\varepsilon)-\omega_1^2)]-(-\omega_2+B_{34}+B_{43})[\omega_2\omega_1(-\omega_1+B_{12}+B_{21})+(-\omega_2+B_{34}+B_{43})A_2] \\
&\quad -2A_3(k_{14}+k_{24}-B_{44})+2A_3\varepsilon > 0
\end{aligned} \tag{5-2-30}$$

where

$$\begin{cases} A_1 = -2(k_{11}+k_{21}-B_{11})+\varepsilon, \\ A_2 = -2A_1(k_{12}+k_{22}-B_{22})+A_1\varepsilon-(-\omega_1+B_{12}+B_{21})^2, \\ A_3 = -2A_2(k_{13}+k_{23}-B_{33})+A_2\varepsilon-\omega_1^2 A \end{cases} \tag{5-2-31}$$

i.e.

$$(k_{11}+k_{21}) > \max\{B_{11}(t)+\frac{\varepsilon}{2}\} = 8.13 \tag{5-2-32}$$

k_{11} and k_{21} are chosen as 9.2 and 8.4 respectively, by Eq.(5-2-31) A_1=−17.14

$$(k_{12}+k_{22}) > \max\{\frac{A_1(2B_{22}(t)+\varepsilon)-(-\omega_1+B_{12}(t)+B_{21}(t))^2}{2A_1}\} = 7.62 \tag{5-2-33}$$

k_{12} and k_{22} are chosen as 7.6 and 6.7 respectively, by Eq.(5-2-31) A_2=194.61

$$(k_{13}+k_{23}) > \max\{\frac{A_2(2B_{33}(t)+\varepsilon)-\omega_1^2 A_1}{2A_2}\} = 20.11 \quad (5\text{-}2\text{-}34)$$

k_{13} and k_{23} are chosen as 13.2 and 7.6 respectively, by Eq.(5-2-31) A_3=−243.53

$$(k_{14}+k_{24}) > \max\{\frac{A_3(2B_{44}(t)+\varepsilon)-(-\omega_2+B_{34}(t)+B_{43}(t))[\omega_2\omega_1(-\omega_1+B_{12}(t)+B_{21}(t))}{2A_3}$$
$$+\frac{(-\omega_2+B_{34}(t)+B_{43}(t))A_2]-\omega_2[\omega_1(-\omega_1+B_{12}(t)+B_{21}(t))(-\omega_2+B_{34}(t)+B_{43}(t))}{2A_3} \quad (5\text{-}2\text{-}35)$$
$$+\frac{\omega_2((-2(k_{12}+k_{22})+2B_{22}(t)+\varepsilon)(-2(k_{13}+k_{23})+2B_{33}(t)+\varepsilon)-\omega_1^2)]}{2A_3}\} = 569$$

k_{14} and k_{24} are chosen as 284 and 291 respectively.

The Lyapunov asymptotical local stability theorem is satisfied. This means that the synchronization of two Quantum-CNN systems by mutual linear coupling can be achieved. The numerical results are shown in Fig. 5-5.

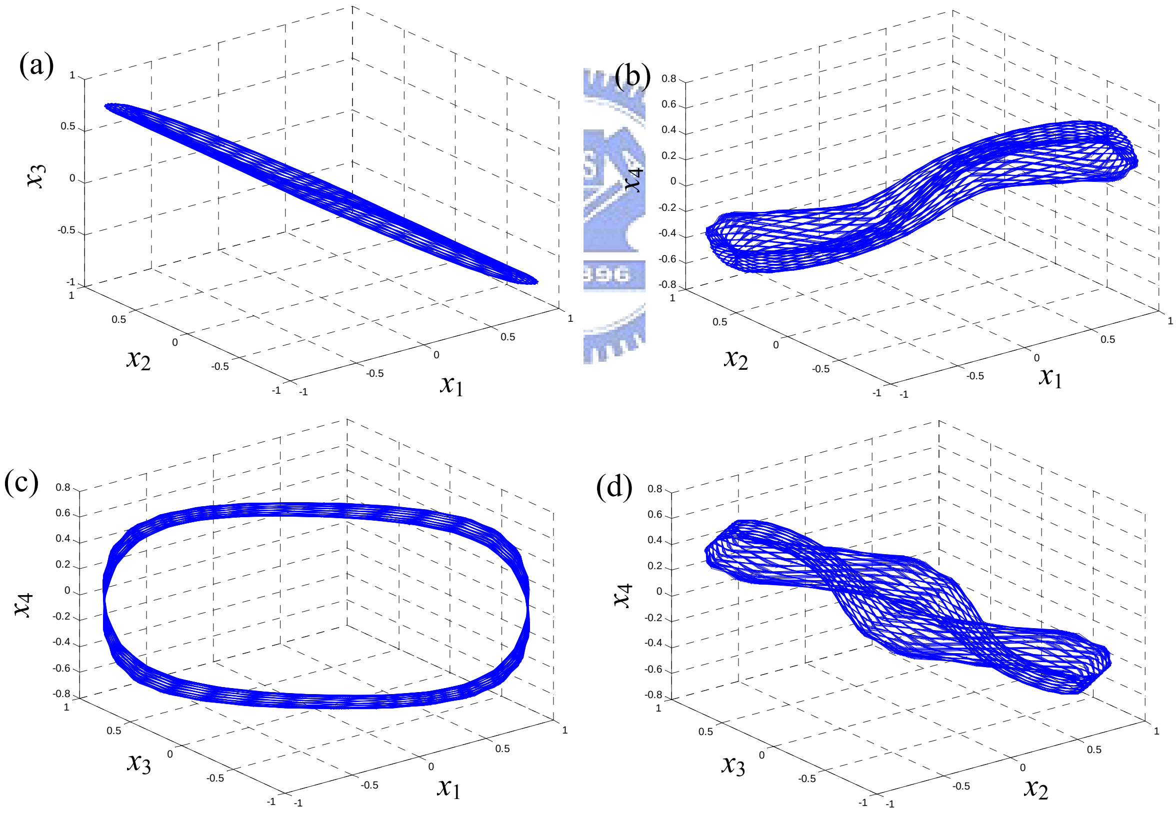


Fig. 5-1. Phase portraits of master system (5-2-10).

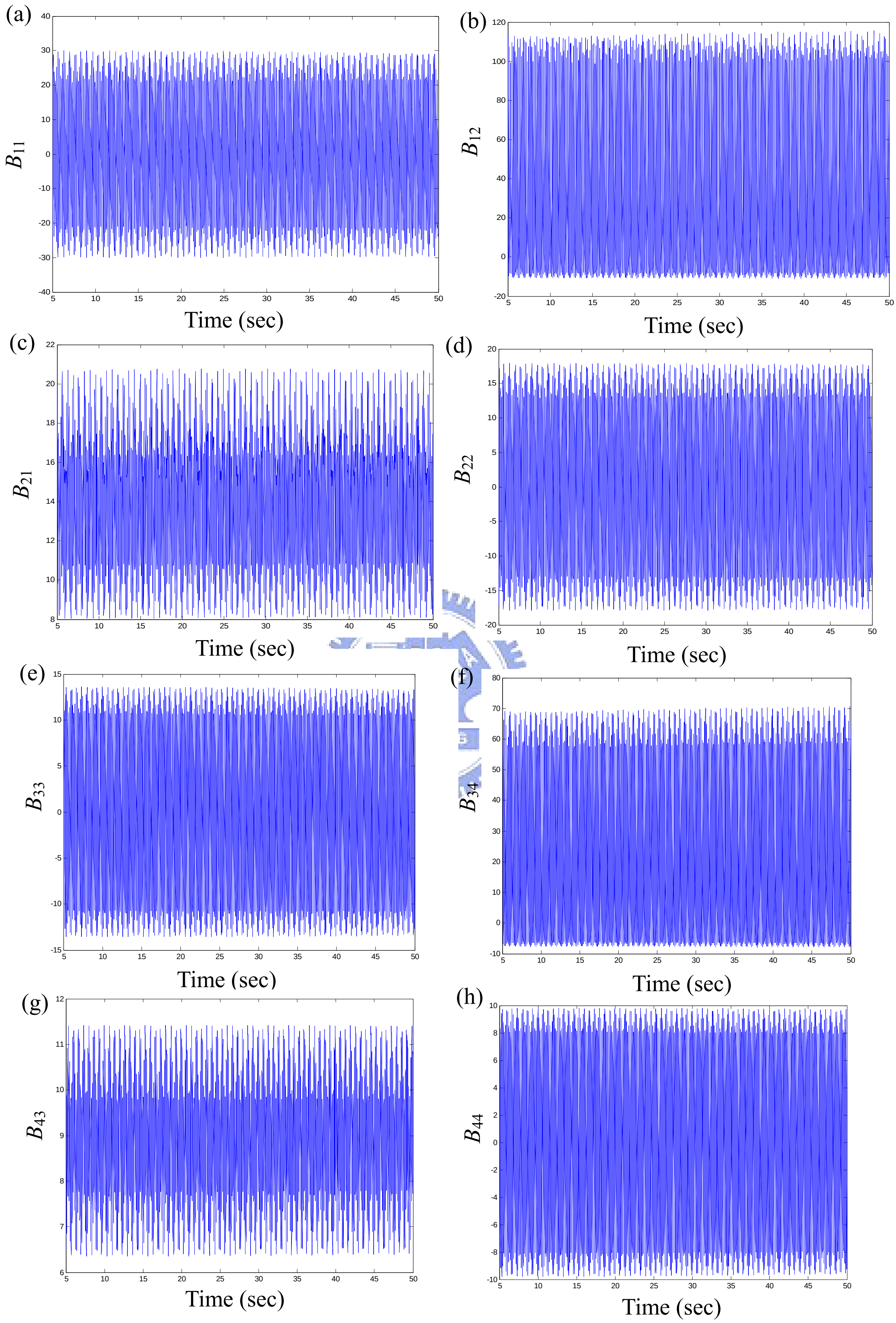


Fig. 5-2. Time histories of B_{11}, B_{12}, B_{21}, B_{22}, B_{33}, B_{34}, B_{43} and B_{44} for Case I.

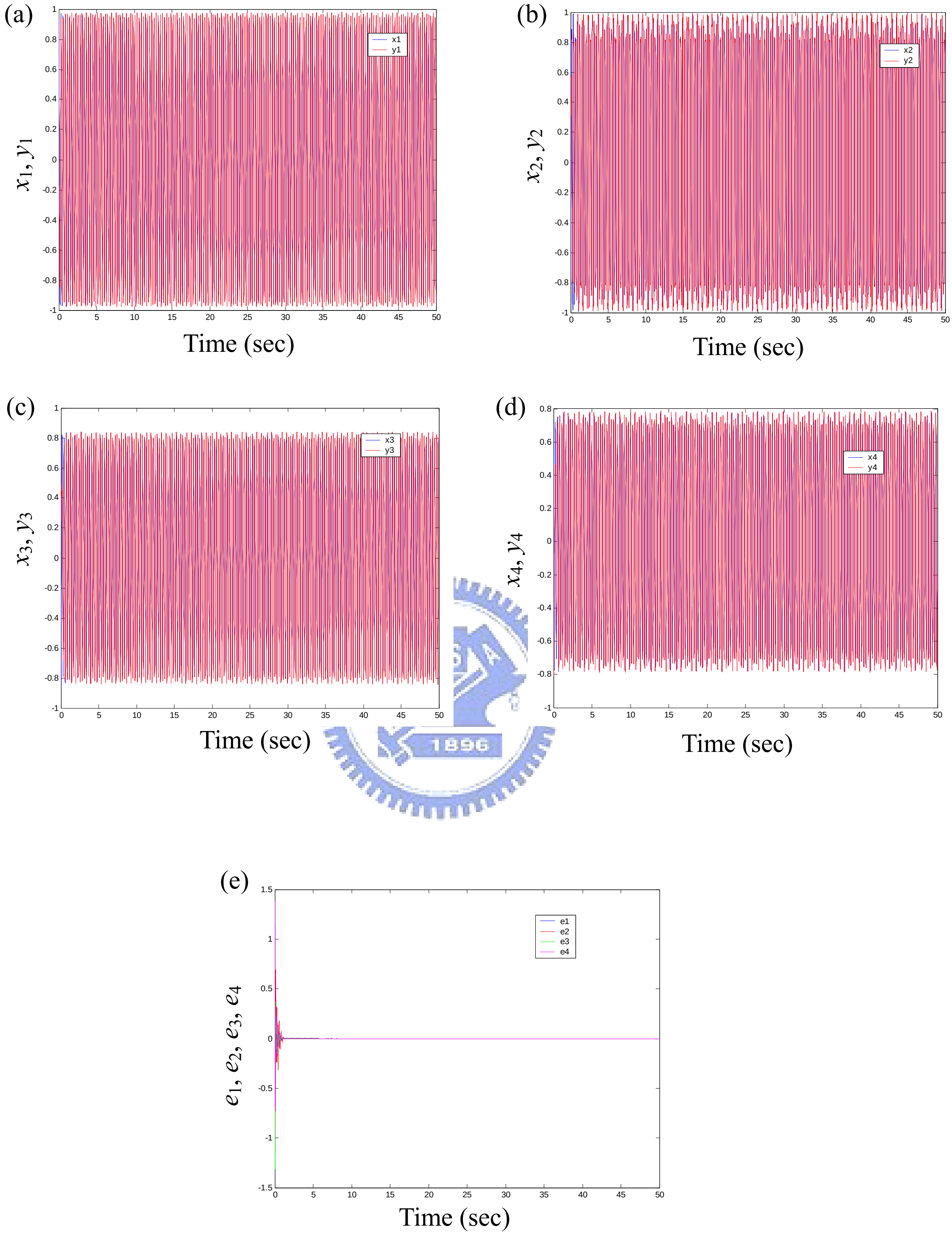


Fig. 5-3. Time histories of states and state errors for Case I.

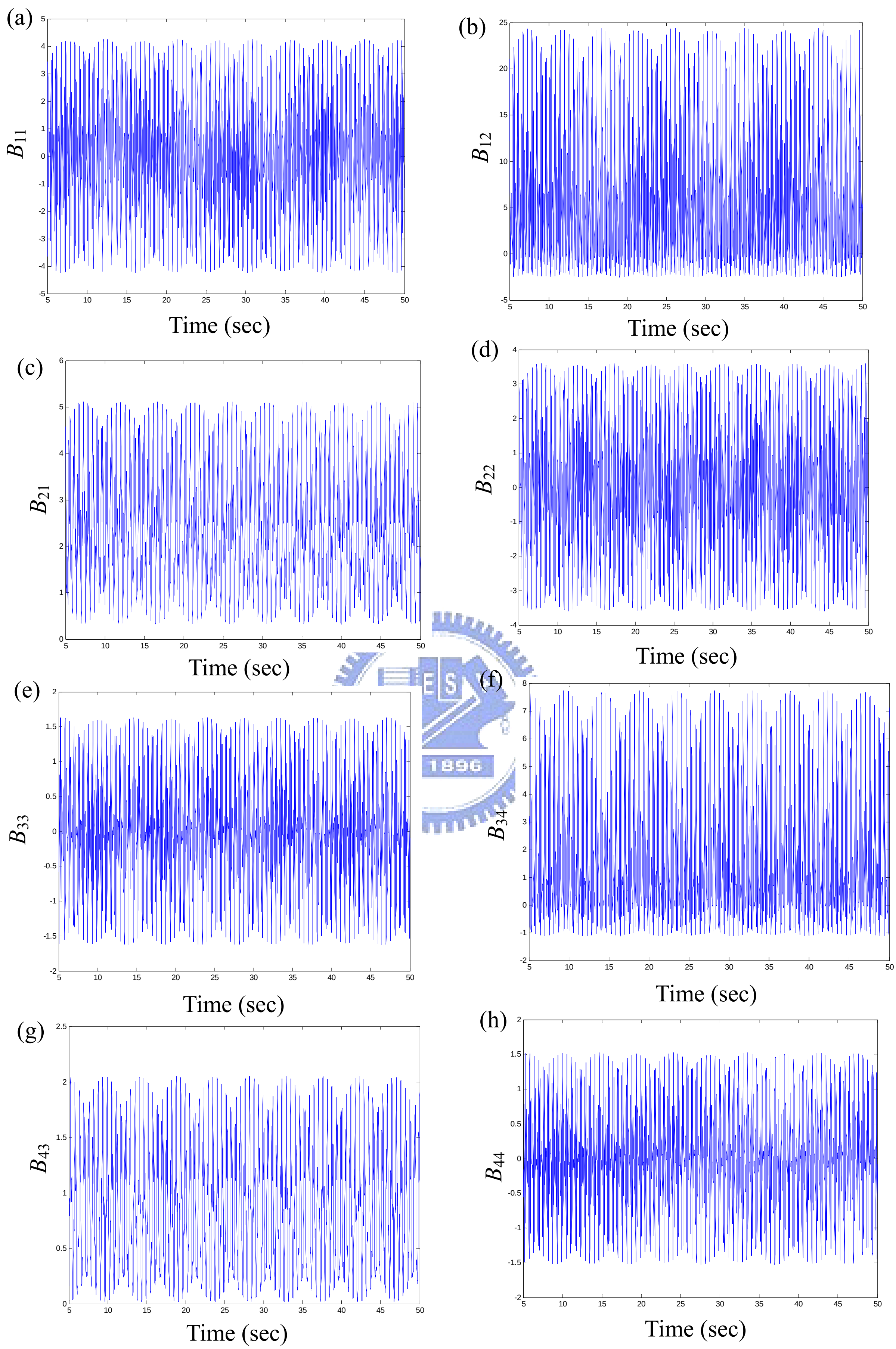


Fig. 5-4. Time histories of B_{11}, B_{12}, B_{21}, B_{22}, B_{33}, B_{34}, B_{43} and B_{44} for Case II.

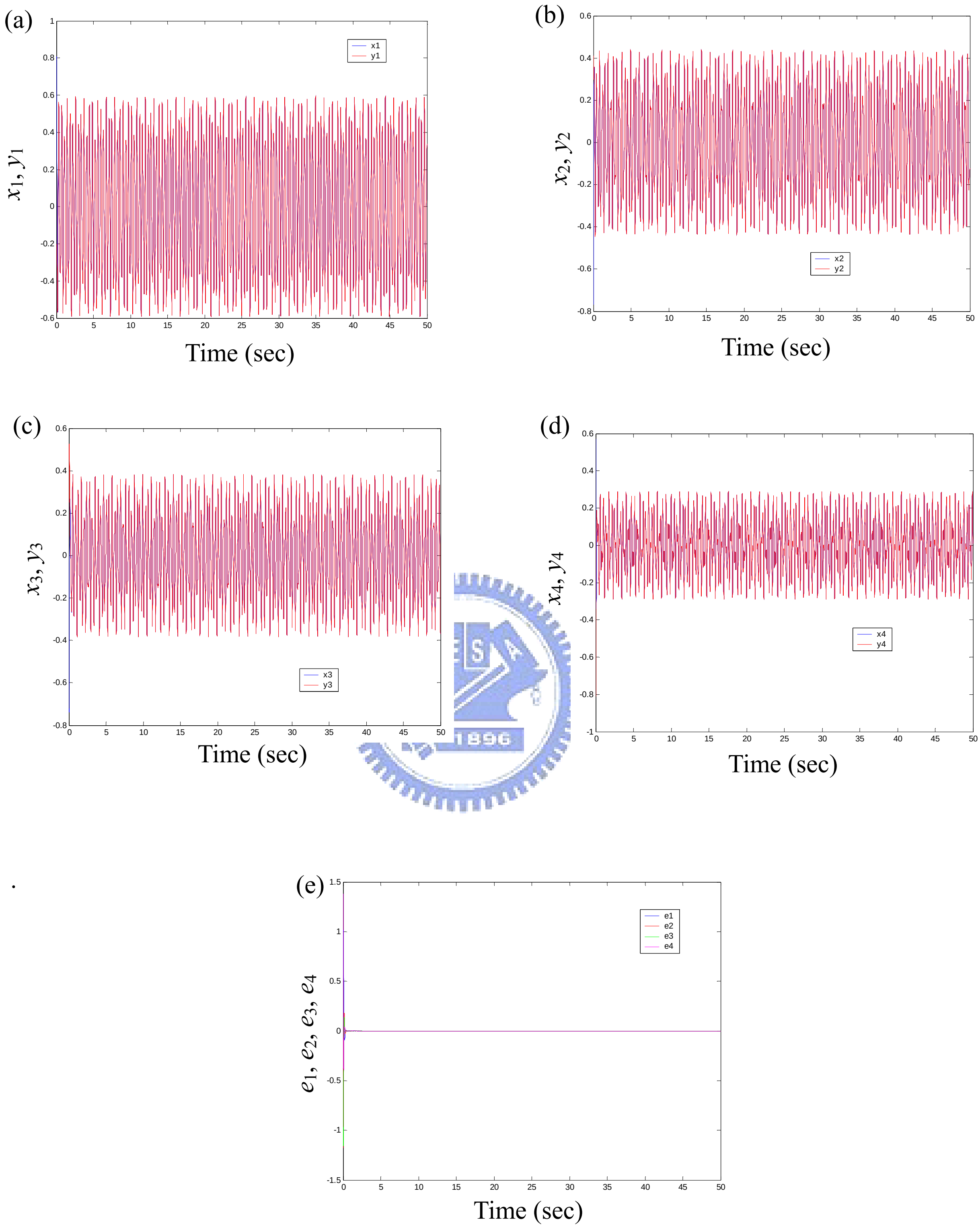


Fig. 5-5. Time histories of states and state errors for Case II.

5-3 Chaos Synchronization of Complex Chaotic Systems in Series Form by Optimal control

5-3-1 Linearly Coupled Chaos Synchronization Scheme by Optimum Control

The optimum control is defined as a method that makes the specified performance index of a system has optimum value when the desired control assignment is fulfilled.

The state equation of the system is

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5-3-1}$$

where $x(t)$ is an n-dimensional state variable of the system, A is an $n \times n$ dimensional constant matrix, and B is an appropriate $n \times r$ dimensional constant matrix. The matrix $[A\ B]$ is controllable entirely and $u(t)$ is an r dimensional control input of the system. Assuming that $u(t)$ has no restriction and $u(0) = 0$, the performance index is

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt\ . \tag{5-3-2}$$

In Eq. (5-3-2), Q is an $n \times n$-dimensional semi-positive definite real symmetric constant matrix; R is an $r \times r$-dimensional positive definite real symmetric constant matrix. The choice of the weighting matrix Q or R is based on eclectic considerations, which can enhance the control performance and reduce the control energy consumption. The aim of the optimum control is to get $u(t)=Kx(t)$ and then make the performance index Eq. (5-3-2) to be minimum, where Kalman gain K is an $r \times n$ dimensional matrix.

So the design of the optimum control system is simplified to get the elements of matrix K. By stability theory, the optimization of the quadratic performance index indicated by Eq. (5-3-2) can be solved.

The feedback gain matrix K of the quadratic optimal regulator is obtained as follows [87]

$$K = R^{-1}B^T S\ . \tag{5-3-3}$$

The matrix S in Eq. (5-3-3) is a positive definite matrix and must meet the following Riccati equation:

$$A^T S + SA - SBR^{-1}B^T S + Q = 0\ . \tag{5-3-4}$$

Then the following nonlinear chaotic system is considered:

$$\dot{x}(t) = Ax(t) + F(t,x) + Bu_1(t) \tag{5-3-5}$$

where A is an $n \times n$ dimensional definite constant matrix, $x = (x_1, x_2, \cdots, x_n) \in R^n$ is the state variable of the system, $F(x) = (F_1, F_2, \cdots, F_n)^T$ is the nonlinear terms of the chaotic system and $u_1(t)=k_a[y(t)-x(t)]$ is a r dimensional control input where k_a is a constant vector. The second chaotic system

$$\dot{y}(t) = Ay(t) + F(t,y) + Bu_2(t)\,. \tag{5-3-6}$$

where B is an appropriate constant matrix, $u_2(t)=k_s[x(t)-y(t)]$ is a r dimensional control input where k_s is also a constant vector.

Define error vector $e = x-y$. From Eqs (5-3-5) and (5-3-6), the error system is

$$\dot{e}(t) = [A - B(k_s + k_a)]e + F(t,x) - F(t,y)\,. \tag{5-3-7}$$

In current scheme of chaos synchronization, maximum values of states must be determined by simulation. They are half analytic method but not pure analytic method. In $F(t, x)-F(t, y)$ nonlinear item is ignored. This is incorrect since there exist linear terms of e in $F(t, x)-F(t, y)$, which can not be ignored. In this thesis, the series expansion form analysis offers a correct method.

The series expansion form is

$$\dot{e} = \left[A + M(x(t), y(t)) - B(k_s + k_a)\right]e + H(x(t), y(t), e) \tag{5-3-8}$$

where $M(x(t), y(t))e + H(x(t), y(t), e) = F(t,x) - F(t,y)$. The elements of $M(x(t), y(t))$ depend on state vectors x, y, and all of them are bounded convergent infinite series of x, y. $H(x(t), y(t), e)$ contains nonlinear terms of e only..

If we choose appropriate k_a and k_s to make $A+M(x(t), y(t))-B(k_s+k_a)$ asymptotically stable, by first approximation theory, the zero solution e=0 of Eq. (5-3-8) is asymptotically stable, i.e. systems (5-3-5) and (5-3-6) are synchronous.

Now we construct an optimal regulator, which is used to synchronize chaotic systems according to the theory of the quadratic optimal regulator, and the aim is to get the feedback gain matrices k_a and k_s of system (5-3-5) and of system (5-3-6), respectively. The steps to get matrix k_a and k_s are: (a) Selecting an $n\times n$ dimensional semi-positive definite real symmetric constant matrices Q, an $r\times r$ dimensional positive definite real symmetric constant matrix R and a constant matrix B, with the definite constant matrix A we can get a Riccati equation as shown in Eq. (5-3-4). Then, we should solve this equation to get matrix S. If the positive definite matrix S exists, then the matrix $A+M(x(t), y(t))-B(k_s+k_a)$ is asymptotically stable and the design of control for the synchronization systems is successful. Otherwise, we should reselect Q, R and B and calculate again. (b) Putting the matrix S in Eq. (5-3-3), we can get the gain matrices k_a and k_s of the regulator. After getting the matrices k_a and k_s according to the above steps, we put k_a and k_s and the matrix B in Eq. (5-3-5) and Eq. (5-3-6), and then we get the synchronized systems.

5-3-2 Numerical Results of the Synchronization of Two Quantum-CNN Oscillator Systems by Unidirectional and by Mutual Linear Coupling

Case I The synchronization by unidirectional linear coupling

For a two-cell Quantum-CNN, the following differential equations are obtained:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1 \dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2 \dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{5-3-9}$$

where x_1 and x_3 are polarizations, x_2 and x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let $a_1 = a_2 = 2.47$, $\omega_1 = 1$ and $\omega_2 = 1$, chaos is obtained for this system.

Two Quantum-CNN chaotic systems using the unidirectional linear coupling can be written as

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2,\ \dot{x}_2 = -\omega_1(x_1-x_3)+2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4,\ \dot{x}_4 = -\omega_2(x_3-x_1)+2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{5-3-10}$$

and

$$\begin{cases} \dot{y}_1 = -2a_1\sqrt{1-y_1^2}\sin y_2 + k_1(x_1-y_1),\ \dot{y}_2 = -\omega_1(y_1-y_3)+2a_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + k_2(x_2-y_2) \\ \dot{y}_3 = -2a_2\sqrt{1-y_3^2}\sin y_4 + k_3(x_3-y_3),\ \dot{y}_4 = -\omega_2(y_3-y_1)+2a_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + k_4(x_4-y_4) \end{cases} \tag{5-3-11}$$

The initial values of linearly coupled Quantum-CNN systems are taken as $x_1(0)=0.8$, $x_2(0)=-0.77$, $x_3(0)=-0.72$, $x_4(0)=0.57$, $y_1(0)=-0.2$, $y_2(0)=0.41$, $y_3(0)=0.25$ and $y_4(0)=-0.81$.

Expand the right hand sides of Eq. (5-3-10) and Eq. (5-3-11) into power series:

$$\begin{cases} \dot{x}_1 = -2a_1(-\frac{1}{2}x_1^2x_2+\frac{1}{12}x_1^2x_2^3-\frac{1}{8}x_1^4x_2+x_2-\frac{1}{6}x_2^3+\frac{1}{120}x_2^5+\cdots) \\ \dot{x}_2 = -\omega_1(x_1-x_3)+2a_1(x_1-\frac{1}{2}x_1x_2^2+\frac{1}{24}x_1x_2^4+\frac{1}{2}x_1^3-\frac{1}{4}x_1^3x_2^2+\frac{5}{8}x_1^5+\cdots) \\ \dot{x}_3 = -2a_2(-\frac{1}{2}x_3^2x_4+\frac{1}{12}x_3^2x_4^3-\frac{1}{8}x_3^4x_4+x_4-\frac{1}{6}x_4^3+\frac{1}{120}x_4^5+\cdots) \\ \dot{x}_4 = -\omega_2(x_3-x_1)+2a_2(x_3-\frac{1}{2}x_3x_4^2+\frac{1}{24}x_3x_4^4+\frac{1}{2}x_3^3-\frac{1}{4}x_3^3x_4^2+\frac{5}{8}x_3^5+\cdots) \end{cases} \tag{5-3-12}$$

and

$$\begin{cases} \dot{y}_1 = -2a_1(-\frac{1}{2}y_1^2y_2+\frac{1}{12}y_1^2y_2^3-\frac{1}{8}y_1^4y_2+y_2-\frac{1}{6}y_2^3+\frac{1}{120}y_2^5+\cdots)+k_1(x_1-y_1) \\ \dot{y}_2 = -\omega_1(y_1-y_3)+2a_1(y_1-\frac{1}{2}y_1y_2^2+\frac{1}{24}y_1y_2^4+\frac{1}{2}y_1^3-\frac{1}{4}y_1^3y_2^2+\frac{5}{8}y_1^5+\cdots)+k_2(x_2-y_2) \\ \dot{y}_3 = -2a_2(-\frac{1}{2}y_3^2y_4+\frac{1}{12}y_3^2y_4^3-\frac{1}{8}y_3^4y_4+y_4-\frac{1}{6}y_4^3+\frac{1}{120}y_4^5+\cdots)+k_3(x_3-y_3) \\ \dot{y}_4 = -\omega_2(y_3-y_1)+2a_2(y_3-\frac{1}{2}y_3y_4^2+\frac{1}{24}y_3y_4^4+\frac{1}{2}y_3^3-\frac{1}{4}y_3^3y_4^2+\frac{5}{8}y_3^5+\cdots)+k_4(x_4-y_4) \end{cases} \tag{5-3-13}$$

From Eq. (5-3-12) and Eq. (5-3-13), the error dynamics is:

$$\dot{e} = [A+M(x(t),y(t)-Bk_s]e+H(x,y,e) \tag{5-3-14}$$

where $e=(y_1-x_1,\ \ y_2-x_2,\ \ y_3-x_3,\ \ y_4-x_4)^T$ and

$$M(x(t), y(t)) = \begin{pmatrix} M_{11} & -2a_1 + M_{21} & 0 & 0 \\ 2a_1 + M_{12} & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & -2a_2 + M_{43} \\ 0 & 0 & 2a_2 + M_{34} & M_{44} \end{pmatrix}$$

in which

$$M_{11} = a_1[2x_1y_2 - \frac{1}{6}x_1y_2^3 + \frac{1}{4}(x_1y_1^2y_2 + 3x_1^2y_1y_2) + \cdots]$$
$$\cdots$$

and $H(x, y, e)$ contains nonlinear terms of e.

The infinite power series of the first element of M, i.e. M_{11} is

$$2x_1y_2 - \frac{1}{6}x_1y_2^3 + \frac{1}{4}(x_1y_1^2y_2 + 3x_1^2y_1y_2) + \cdots \tag{5-3-15}$$

It is well-known [85-86] that a necessary and sufficient condition for the convergence of the infinite series

$$u_1 + u_2 + \cdots + u_n + \cdots$$

is that for any previously assigned positive ε there exists an N such that, for any $n>N$ and for positive p,

$$\left|u_{n+1} + u_{n+2} + \cdots u_{n+p}\right| < \varepsilon \tag{5-3-16}$$

From Fig. 5-6, we know that

$$|x_i| < 1, \quad |y_i| < 1 \quad (i = 1, 2, 3, 4) \tag{5-3-17}$$

Therefore M_{11} and every other element of $M(x(t), y(t))$ are convergent series and each element has a bounded sum.

We can get the optimum gain $k_s=[k_1, k_2, k_3, k_4]^T$ by the method of constructing a quadratic optimal regulator. With

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\omega_1 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & 0 \\ \omega_2 & 0 & -\omega_2 & 0 \end{bmatrix}$$

Choose

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T; \ R = [1]; \ Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}. \tag{5-3-18}$$

After solving the corresponding Riccati equation, we get the gain matrix $k_s=[k_1, k_2, k_3, k_4]^T=[0, 1, 0, 1]^T$.

From the simulation results of Fig. 5-6, it is shown that master system and slave system reach the synchronization state after they are controlled by the quadratic optimal regulator. It is noticed that the synchronization effect is very effective.

Case II The synchronization by mutual linear coupling

Tow Quantum-CNN systems with mutual linear coupling are given:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 + k_{11}(y_1 - x_1), \ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + k_{12}(y_2 - x_2) \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 + k_{13}(y_3 - x_3), \ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + k_{14}(y_4 - x_4) \end{cases} \tag{5-3-19}$$

and

$$\begin{cases} \dot{y}_1 = -2a_1\sqrt{1-y_1^2}\sin y_2 + k_{21}(x_1 - y_1), \ \dot{y}_2 = -\omega_1(y_1 - y_3) + 2a_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + k_{22}(x_2 - y_2) \\ \dot{y}_3 = -2a_2\sqrt{1-y_3^2}\sin y_4 + k_{23}(x_3 - y_3), \ \dot{y}_4 = -\omega_2(y_3 - y_1) + 2a_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + k_{24}(x_4 - y_4) \end{cases} \tag{5-3-20}$$

Expand the right hand sides of Eq. (5-3-19) and Eq. (5-3-20) into power series:

$$\begin{cases} \dot{x}_1 = -2a_1(-\frac{1}{2}x_1^2x_2 + \frac{1}{12}x_1^2x_2^3 - \frac{1}{8}x_1^4x_2 + x_2 - \frac{1}{6}x_2^3 + \frac{1}{120}x_2^5 + \cdots) + k_{11}(y_1 - x_1) \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1(x_1 - \frac{1}{2}x_1x_2^2 + \frac{1}{24}x_1x_2^4 + \frac{1}{2}x_1^3 - \frac{1}{4}x_1^3x_2^2 + \frac{5}{8}x_1^5 + \cdots) + k_{12}(y_2 - x_2) \\ \dot{x}_3 = -2a_2(-\frac{1}{2}x_3^2x_4 + \frac{1}{12}x_3^2x_4^3 - \frac{1}{8}x_3^4x_4 + x_4 - \frac{1}{6}x_4^3 + \frac{1}{120}x_4^5 + \cdots) + k_{13}(y_3 - x_3) \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2(x_3 - \frac{1}{2}x_3x_4^2 + \frac{1}{24}x_3x_4^4 + \frac{1}{2}x_3^3 - \frac{1}{4}x_3^3x_4^2 + \frac{5}{8}x_3^5 + \cdots) + k_{14}(y_4 - x_4) \end{cases} \tag{5-3-21}$$

and

$$
\begin{cases}
\dot{y}_1 = -2a_1(-\frac{1}{2}y_1^2 y_2 + \frac{1}{12}y_1^2 y_2^3 - \frac{1}{8}y_1^4 y_2 + y_2 - \frac{1}{6}y_2^3 + \frac{1}{120}y_2^5 + \cdots) + k_{21}(x_1 - y_1) \\
\dot{y}_2 = -\omega_1(y_1 - y_3) + 2a_1(y_1 - \frac{1}{2}y_1 y_2^2 + \frac{1}{24}y_1 y_2^4 + \frac{1}{2}y_1^3 - \frac{1}{4}y_1^3 y_2^2 + \frac{5}{8}y_1^5 + \cdots) + k_{22}(x_2 - y_2) \\
\dot{y}_3 = -2a_2(-\frac{1}{2}y_3^2 y_4 + \frac{1}{12}y_3^2 y_4^3 - \frac{1}{8}y_3^4 y_4 + y_4 - \frac{1}{6}y_4^3 + \frac{1}{120}y_4^5 + \cdots) + k_{23}(x_3 - y_3) \\
\dot{y}_4 = -\omega_2(y_3 - y_1) + 2a_2(y_3 - \frac{1}{2}y_3 y_4^2 + \frac{1}{24}y_3 y_4^4 + \frac{1}{2}y_3^3 - \frac{1}{4}y_3^3 y_4^2 + \frac{5}{8}y_3^5 + \cdots) + k_{24}(x_4 - y_4)
\end{cases}
\quad (5\text{-}3\text{-}22)
$$

From Eq. (5-3-21) and Eq. (5-3-22), the error dynamics is:

$$
\dot{e} = [A + M(x(t), y(t) - B(k_s + k_a))]e + H(x, y, e) \quad (5\text{-}3\text{-}23)
$$

where $e = (y_1 - x_1,\ \ y_2 - x_2,\ \ y_3 - x_3,\ \ y_4 - x_4)^T$ and

$$
M(x(t), y(t)) = \begin{pmatrix} M_{11} & -2a_1 + M_{21} & 0 & 0 \\ 2a_1 + M_{12} & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & -2a_2 + M_{43} \\ 0 & 0 & 2a_2 + M_{34} & M_{44} \end{pmatrix}
$$

in which

$$
M_{11} = a_1[2x_1 y_2 - \frac{1}{6}x_1 y_2^3 + \frac{1}{4}(x_1 y_1^2 y_2 + 3x_1^2 y_1 y_2) + \cdots]
$$

$$
\cdots
$$

Similar to Case I, from Fig. 5-7, $|x_i| < 1,\ \ |y_i| < 1\ \ (i = 1, 2, 3, 4),$ the infinite power series elements of $M(x(t), y(t))$ are all convergent and have bounded sums [85-86].

The optimum gain $k_a = [k_{11}, k_{12}, k_{13}, k_{14}]^T$ and $k_s = [k_{21}, k_{22}, k_{23}, k_{24}]^T$ can be obtained by the method of constructing a quadratic optimal regulator. With

$$
A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\omega_1 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & 0 \\ \omega_2 & 0 & -\omega_2 & 0 \end{bmatrix}
$$

we choose

$$
B = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T;\ \ R = [1];\ \ Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}. \quad (5\text{-}3\text{-}24)
$$

After solving the corresponding Riccati equation, we then get the gain matrix $k_a=[k_{11}, k_{12}, k_{13}, k_{14}]^T=[0, 0.5, 0, 0.5]^T$ and $k_s=[k_{21}, k_{22}, k_{23}, k_{24}]^T=[0, 0.5, 0, 0.5]^T$.

From the simulation results of Fig. 5-7 it is shown that master system and slave system reach the synchronization state after they are controlled by the quadratic optimal regulator. It is noticed that the synchronization effect is very effective.

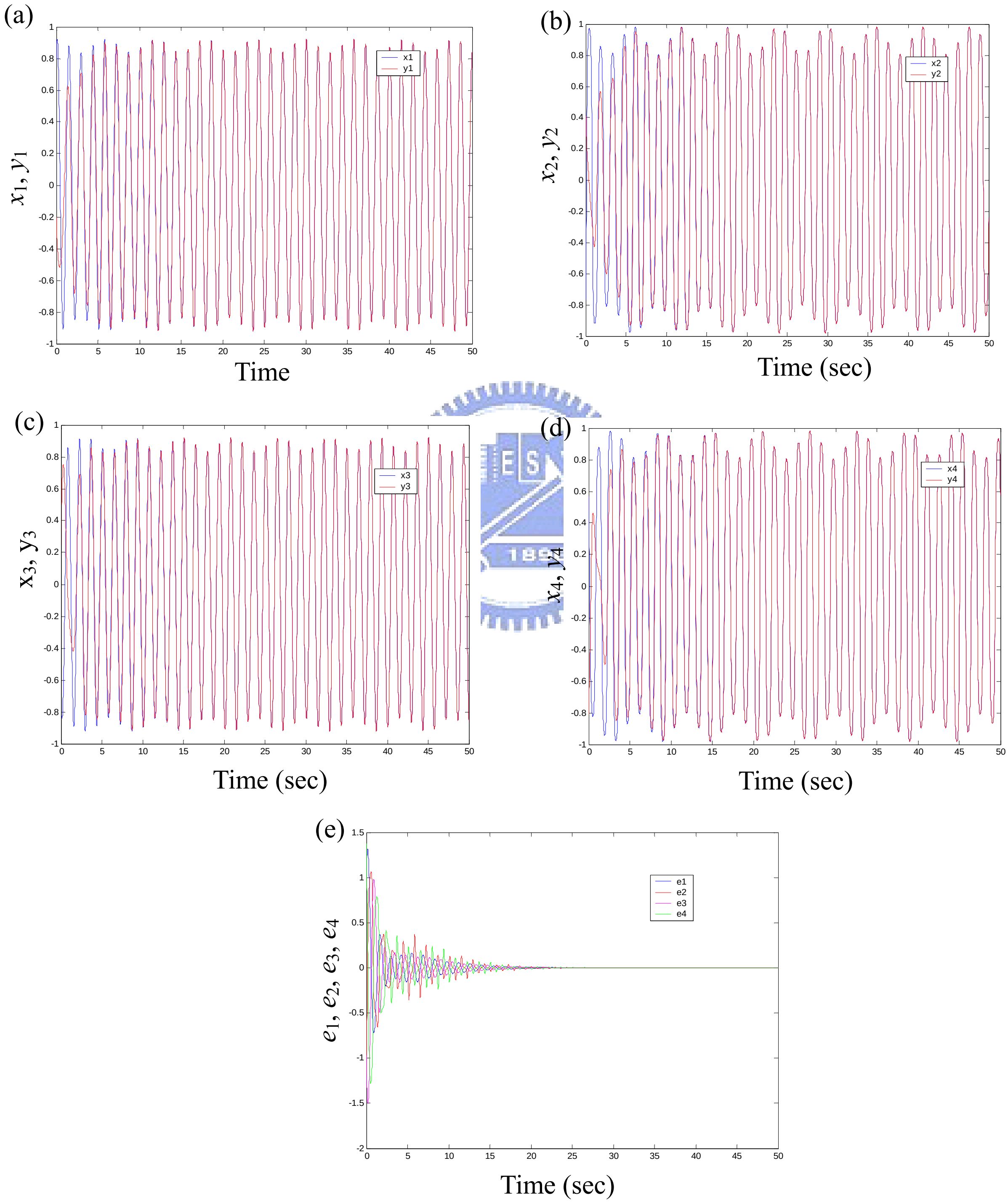


Fig. 5-6. Time histories of states, state errors for uni-direction linear coupling.

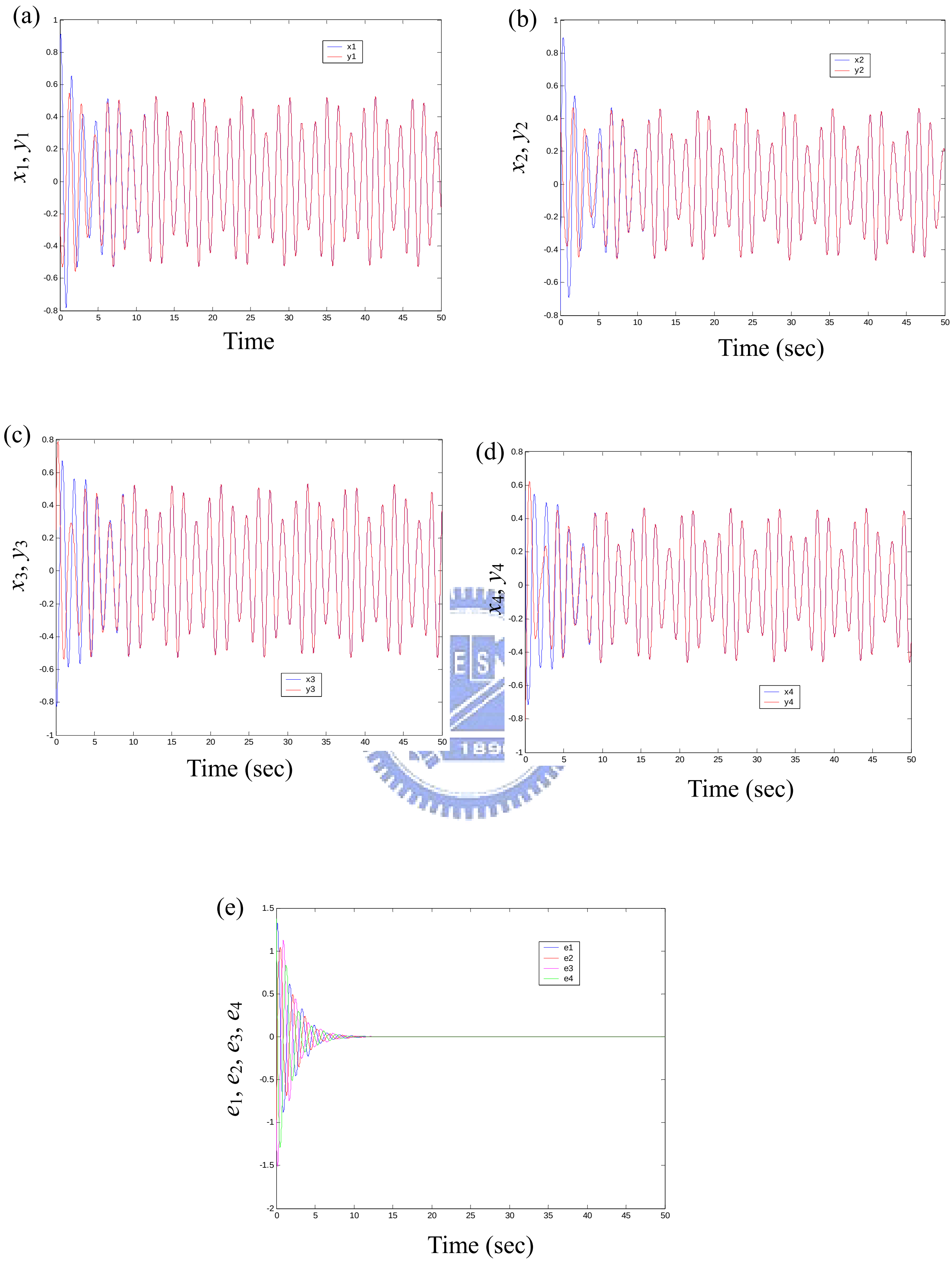


Fig. 5-7. Time histories of states, state errors for mutual linear coupling.

5-4 Chaos Synchronization of Quantum-CNN Chaotic System by Impulse Control

The synchronization by unidirectional/mutual linear coupling with periodic impulse is studied. Two chaos systems using mutual/unidirectional linear coupling with periodic impulse can be written as

$$\dot{x} = Ax + h(x) + u_1(y - x) \tag{5-4-1}$$

and

$$\dot{y} = Ay + h(y) + u_2(x - y) \tag{5-4-2}$$

where $x, y \in R^n$ represent the state vectors of the chaotic systems, $A \in R^{n\times n}$ is a constant matrix, $h \in R^{n\times n}$ is a continuous nonlinear function of x, y, u_1 and u_2 are constant gains of periodic impulse signals which represent the coupled parameters. If u_1 is equal zero then the systems call unidirectional linear coupling synchronization by periodic impulse. If u_1 and u_2 are the systems call mutual linear coupling synchronization by periodic impulse.

As an example, we study unidirectional/mutual linear coupling synchronization by periodic impulse for Quantum-CNN chaotic system. The equations considered are

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 + u_{11}(y_1 - x_1) \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + u_{12}(y_2 - x_2) \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 + u_{13}(y_3 - x_3) \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + u_{14}(y_4 - x_4) \end{cases} \tag{5-4-3}$$

and

$$\begin{cases} \dot{y}_1 = -2a_1\sqrt{1-y_1^2}\sin y_2 + u_{21}(x_1 - y_1) \\ \dot{y}_2 = -\omega_1(y_1 - y_3) + 2a_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + u_{22}(x_2 - y_2) \\ \dot{y}_3 = -2a_2\sqrt{1-y_3^2}\sin y_4 + u_{23}(x_3 - y_3) \\ \dot{y}_4 = -\omega_2(y_3 - y_1) + 2a_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + u_{24}(x_4 - y_4) \end{cases} \tag{5-4-4}$$

The initial values of Quantum-CNN systems are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$, $x_4(0) = 0.57$, $y_1(0) = -0.2$, $y_2(0) = 0.41$, $y_3(0) = 0.25$ and $y_4(0) = -0.81$ respectively.

The result is show in Figs. 5-8 and 5-9 for unidirectional linear coupling and mutual linear coupling, respectively.

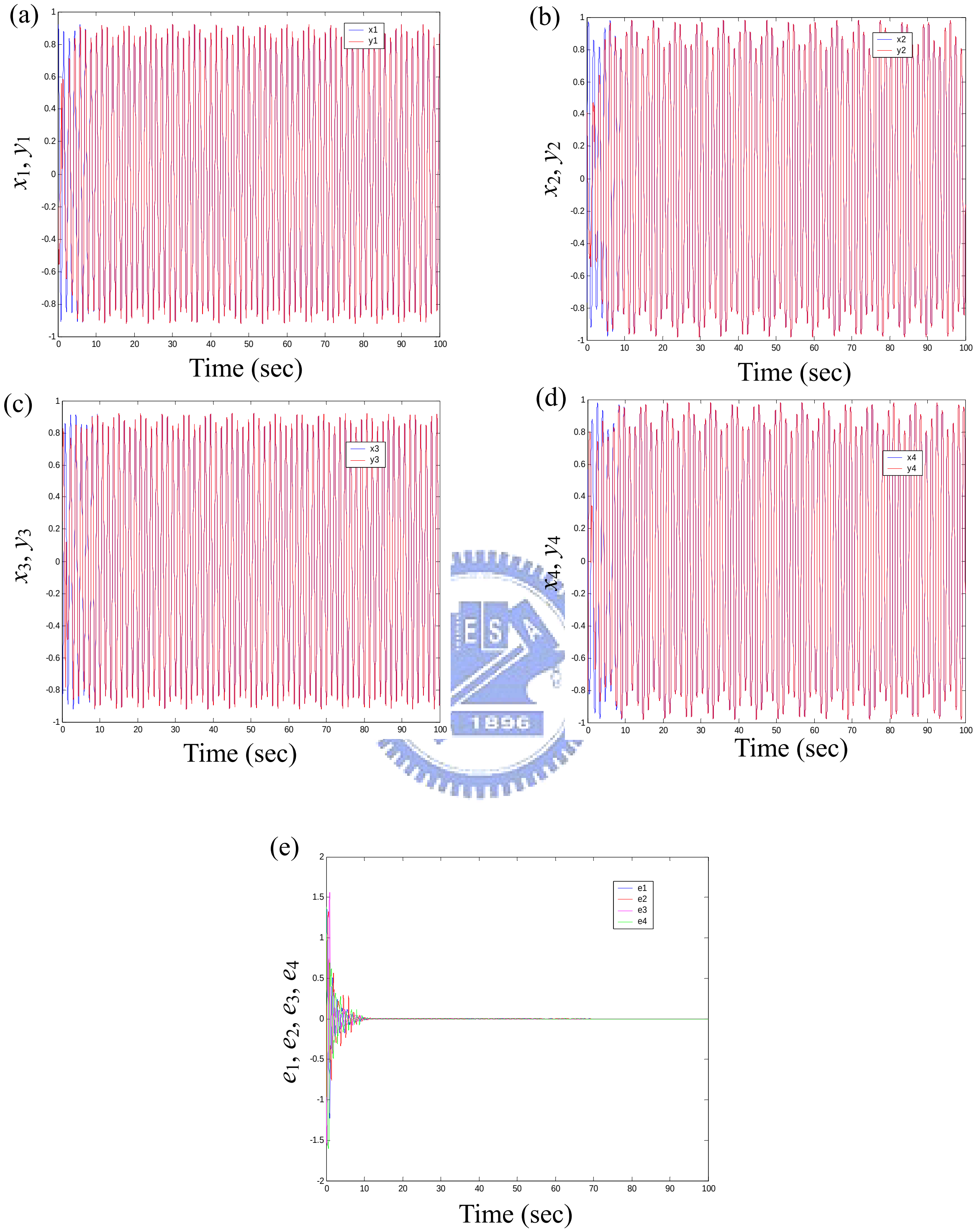


Fig. 5-8. Time histories of states, state errors for uni-direction linear couple.

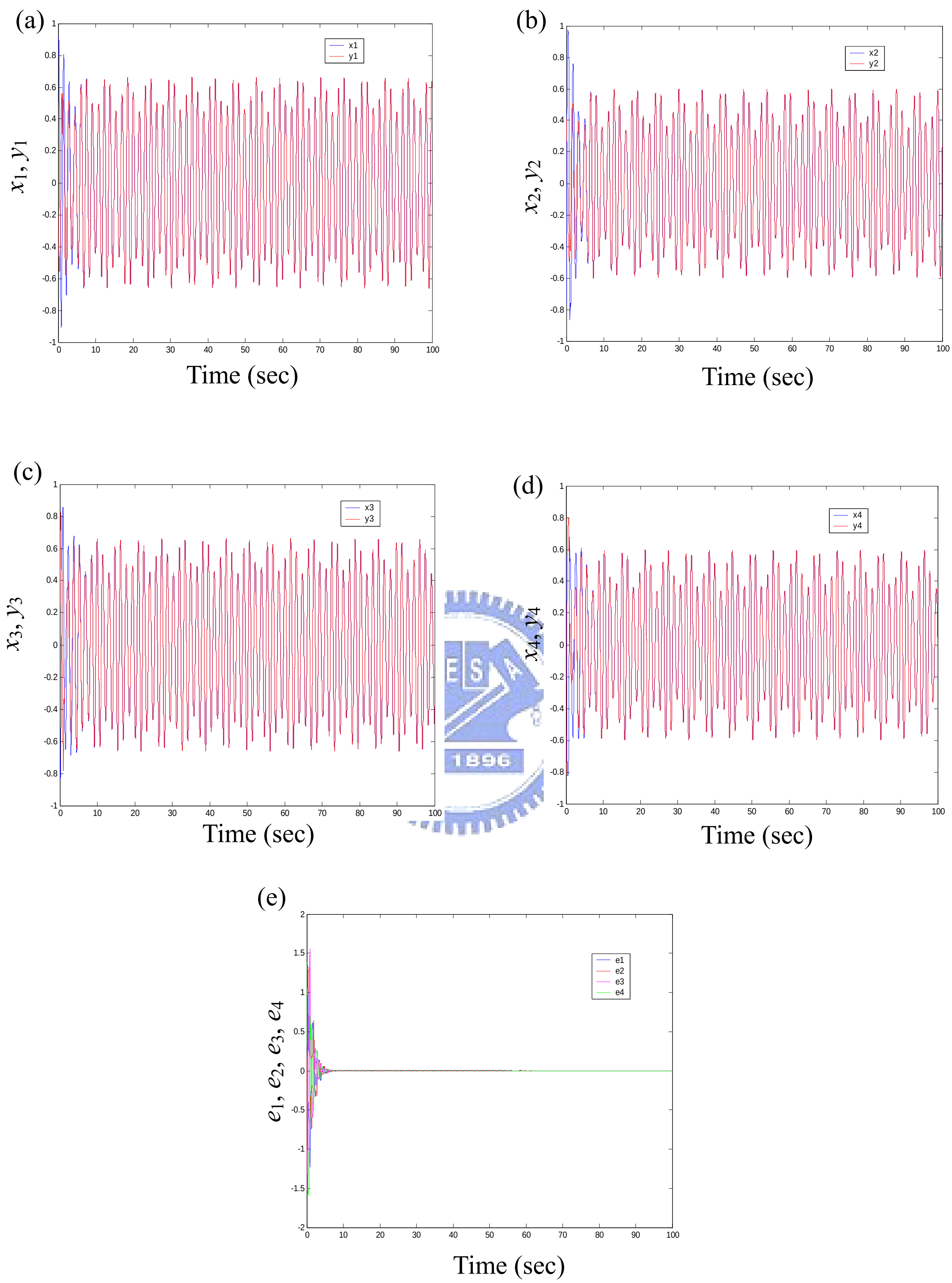


Fig. 5-9. Time histories of states, state errors for mutual linear couple.

Chapter 6

Pragmatical Synchronization and Control

6-1 Preliminary

The synchronization phenomenon has the following feature: the trajectories of the drive and response systems are identical notwithstanding starting from different initial conditions. However, slight errors of initial conditions, for chaotic dynamical systems, will lead to completely different trajectories. Therefore, how to control two chaotic systems to be synchronized is an attractive objective. Many approaches have been presented for the synchronization of chaotic systems such as linear and nonlinear feedback control. Most of them are based on the exact knowledge of the system structure and parameters. But in practice, some or all of the system parameters are uncertain. Moreover, these parameters change from time to time. A lot of works have proceeded to solve this problem by adaptive synchronization. In current scheme of adaptive synchronization, traditional Lyapunov stability theorem and Babalat lemma are used to prove the error vector approaches zero as time approaches infinity. But the question, why the estimated parameters also approach the uncertain values, has still remained without answer. Based on a pragmatical asymptotical stability theorem and an assumption of equal probability for ergodic initial conditions, the question is answered.

Among many kinds of synchronizations, the generalized synchronization is investigated. It means there exists a given functional relationship between the states of the master and that of the slave $y=G(x)$. In this thesis, a special kind of generalized synchronizations

$$y = G(x) = x + F(t) \tag{6-1-1}$$

is studied, where x, y are the state vectors of the master and of the slave, respectively. $F(t)$ is a given vector function of time which may take various forms, either regular or chaotic function of time. When $F(t)=0$, it reduces to a complete synchronization.

Since chaos control problem was firstly considered by Ott et al., it has been investigated extensively by lots of authors. Adaptive control method was also used to control chaotic system. However, traditional adaptive chaos control is limited for the same system. Proposed pragmatical adaptive control method enlarges the function of chaos control. We can control a chaotic system either to a given simple unchaotic system or to a more complex given chaotic system. Based on a pragmatical theorem of asymptotical stability using the concept of probability, an adaptive control law is derived such that it can be proved strictly that the common null solution of error dynamics and of parameter dynamics is asymptotically stable. Numerical results are given for two examples. A chaotic nano resonator system is controlled to a double harmonic system and to a hyperchaotic Quantum-CNN oscillator system.

A main field of synchronization is the coupled identical chaotic system. Due to the simple configuration and easy implementation, the linearly error feedback coupling scheme can be adopted in many real fields. In practice, it is a key problem to determine the appropriate feedback gain or coupling parameters for realizing the synchronization. So far, there have been many specific results about determining the feedback gain or coupling parameters for particular systems with constant uncertain parameters by adaptive control. In this thesis, the synchronization of general chaotic systems that satisfy Lipschitz condition only, with uncertain **variable/chaotic** parameters by linear coupling and pragmatical adaptive tracking is firstly studied.

6-2 Pragmatical Generalized Synchronization of Chaotic Systems with Uncertain Parameters by Adaptive Control

A new kind of generalized synchronization of two chaotic systems with uncertain parameters is proposed. Based on a pragmatical asymptotical stability theorem and an assumption of equal probability for ergodic initial conditions, an adaptive control law is derived so that it can be proved strictly that the common null solution of error dynamics and of parameter dynamics is actually asymptotically stable, i.e. these two identical systems are in generalized synchronization and the estimated parameters approach the uncertain values. It is called pragmatical generalized synchronization.

6-2-1 Pragmatical Generalized Synchronization Scheme by Adaptive Control

There are two identical nonlinear dynamical systems, and the master system controls the slave system. The master system is given by

$$\dot{x} = Ax + f(x, B) \tag{6-2-1}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ denotes a state vector, A is an $n \times n$ uncertain constant coefficient matrix, f is a nonlinear vector function, and B is a vector of uncertain constant coefficients in f.

The slave system is given by

$$\dot{y} = \hat{A}y + f(y, \hat{B}) + u(t) \tag{6-2-2}$$

where $y = [y_1, y_2, \cdots, y_n]^T \in R^n$ denotes a state vector, $\hat{A}$ is an $n \times n$ estimated coefficient matrix, $\hat{B}$ is a vector of estimated coefficients in f, and $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T \in R^n$ is a control input vector.

Our goal is to design a controller $u(t)$ so that the state vector of the slave system (6-2-2) asymptotically approaches the state vector of the master system (6-2-1) plus a given chaotic vector function $F(t) = [F_1(t), F_2(t), \cdots, F_n(t)]^T$. This is a special kind of generalized synchronization, y is a given function of x

$$y = G(x) = x + F(t). \tag{6-2-3}$$

The synchronization can be accomplished when $t \to \infty$, the limit of the error vector $e(t) = [e_1, e_2, \cdots, e_n]^T$ approaches zero:

$$\lim_{t \to \infty} e = 0 \tag{6-2-4}$$

where

$$e = x - y + F(t). \tag{6-2-5}$$

From Eq. (6-2-5) we have

$$\dot{e} = \dot{x} - \dot{y} + \dot{F}(t) \tag{6-2-6}$$

$$\dot{e} = Ax - \hat{A}y + f(x, B) - f(y, \hat{B}) + \dot{F}(t) - u(t)\,. \tag{6-2-7}$$

A Lyapnuov function $V(e, \tilde{A}_c, \tilde{B}_c)$ is chosen as a positive definite function

$$V(e, \tilde{A}_c, \tilde{B}_c) = \frac{1}{2}e^T e + \frac{1}{2}\tilde{A}_c^T \tilde{A}_c + \frac{1}{2}\tilde{B}_c^T \tilde{B}_c \tag{6-2-8}$$

where $\tilde{A} = A - \hat{A}$, $\tilde{B} = B - \hat{B}$, $\tilde{A}_c$ and $\tilde{B}_c$ are two column matrices whose elements are all the elements of matrix $\tilde{A}$ and of matrix $\tilde{B}$, respectively.

Its derivative along any solution of the differential equation system consisting of Eq. (6-2-7) and update parameter differential equations for $\tilde{A}_c$ and $\tilde{B}_c$ is

$$\dot{V}(e, \tilde{A}_c, \tilde{B}_c) = e^T[Ax - \hat{A}y + Bf(x) - \hat{B}f(y) + \dot{F}(t) - u(t)] + \tilde{A}_c\dot{\tilde{A}}_c + \tilde{B}_c\dot{\tilde{B}}_c \tag{6-2-9}$$

where $u(t)$, $\dot{\tilde{A}}_c$, and $\dot{\tilde{B}}_c$ are chosen so that $\dot{V} = e^T Ce$, C is a diagonal negative definite matrix, and $\dot{V}$ is a negative semi-definite function of e and parameter differences $\tilde{A}_c$ and $\tilde{B}_c$. In current scheme of adaptive synchronization [89-93], traditional Lyapunov stability theorem and Babalat lemma are used to prove the error vector approaches zero, as time approaches infinity. But the question, why the estimated parameters also approach to the uncertain parameters, remains no answer. By pragmatical asymptotical stability theorem, the question can be answered strictly.

6-2-2 Numerical Results of Pragmatical Generalized Chaos Synchronization of Two Quantum -CNN Oscillators by Adaptive Control

Case I The chaotic states of a goal system, a double Duffing chaotic system, used as $F(t)$

For a two-cell Quantum-CNN, the following differential equations are used as master system:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\,\sin x_2,\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\,\sin x_4,\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{6-2-10}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let a_1=4.9, a_2=4.9, ω_1=3.03 and ω_2=1.83.

A slave system is described by

$$\begin{cases} \dot{y}_1 = -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2,\ \dot{y}_2 = -\hat{\omega}_1(y_1 - y_3) + 2\hat{a}_1 \dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 \\ \dot{y}_3 = -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4,\ \dot{y}_4 = -\hat{\omega}_2(y_3 - y_1) + 2\hat{a}_2 \dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 \end{cases} \tag{6-2-11}$$

In order to lead (y_1, y_2, y_3, y_4) to ($x_1+F_1(t)$, $x_2+F_2(t)$, $x_3+F_3(t)$, $x_4+F_4(t)$), we add u_1, u_2, u_3 and u_4 to each equation of Eq. (6-2-11), respectively:

$$\begin{cases} \dot{y}_1 = -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 + u_1,\ \dot{y}_2 = -\hat{\omega}_1(y_1 - y_3) + 2\hat{a}_1 \dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + u_2 \\ \dot{y}_3 = -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 + u_3,\ \dot{y}_4 = -\hat{\omega}_2(y_3 - y_1) + 2\hat{a}_2 \dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + u_4 \end{cases} \tag{6-2-12}$$

Subtracting Eq. (6-2-12) from Eq. (6-2-10), we obtain an error dynamics. The initial values of the master system and the slave system are taken as $x_1(0)$=0.8, $x_2(0)$=−0.77, $x_3(0)$=−0.72, $x_4(0)$=0.57, $y_1(0)$=0.1, $y_2(0)$=0.28, $y_3(0)$=0.42 and $y_4(0)$=−0.72, respectively.

The goal system for generalized synchronization is a double Duffing chaotic system

$$\dot{z}_1 = z_2,\ \dot{z}_2 = z_1 - z_1^3 - \delta_1 z_2 + f_1\cos\psi_1 t,\ \dot{z}_3 = z_4,\ \dot{z}_4 = z_3 - z_3^3 - \delta_2 z_4 + f_2 \cos\psi_2 t \tag{6-2-13}$$

where δ_1=13.5, δ_2=12.5, f_1=−24.9, f_2=−33.1, Ψ_1=10.9, Ψ_2=19.9, $z_1(0)$=0.75, $z_2(0)$=−0.3, $z_3(0)$= −0.4 and $z_4(0)$=0.5.

We have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + z_i) = 0, \quad i = 1, 2, 3, 4 \tag{6-2-14}$$

where $\dot{e} = \dot{x} - \dot{y} + \dot{z}$, and

$$
\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - u_1 + \dot{z}_1 \\
\dot{e}_2 &= -\omega_1(x_1-x_3) + \hat{\omega}_1(y_1-y_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - u_2 + \dot{z}_2 \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - u_3 + \dot{z}_3 \\
\dot{e}_4 &= -\omega_2(x_3-x_1) + \hat{\omega}_2(y_3-y_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - u_4 + \dot{z}_4
\end{aligned}
\tag{6-2-15}
$$

where $e_1=x_1-y_1+z_1$, $e_2=x_2-y_2+z_2$, $e_3=x_3-y_3+z_3$ and $e_4=x_4-y_4+z_4$.

Choose a Lyapunov function in the form of a positive definite function:

$$
V(e_1,e_2,e_3,e_4,\tilde{a}_1,\tilde{a}_2,\tilde{\omega}_1,\tilde{\omega}_2) = \frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2+\tilde{a}_1^2+\tilde{a}_2^2+\tilde{\omega}_1^2+\tilde{\omega}_2^2) \tag{6-2-16}
$$

where $\tilde{a}_1 = a_1 - \hat{a}_1$, $\tilde{a}_2 = a_2 - \hat{a}_2$, $\tilde{\omega}_1 = \omega_1 - \hat{\omega}_1$, $\tilde{\omega}_2 = \omega_2 - \hat{\omega}_2$ and $\hat{a}_1$, $\hat{a}_2$, $\hat{\omega}_1$, $\hat{\omega}_2$ are estimates of uncertain parameters a_1, a_2, ω_1 and ω_2 respectively.

Its time derivative is

$$
\begin{aligned}
\dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - u_1 + z_2] + e_2[-\omega_1(x_1-x_3) + \hat{\omega}_1(y_1 \\
&\quad -y_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - u_2 + z_1 - z_1^3 - \delta_1 z_2 + f_1\cos\psi_1 t] \\
&\quad +e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - u_3 + z_4] + e_4[-\omega_2(x_3-x_1) + \hat{\omega}_2(y_3 \\
&\quad -y_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - u_4 + z_3 - z_3^3 - \delta_2 z_4 + f_2\cos\psi_2 t] \\
&\quad +\tilde{a}_1(-\dot{\hat{a}}_1) + \tilde{a}_2(-\dot{\hat{a}}_2) + \tilde{\omega}_1(-\dot{\hat{\omega}}_1) + \tilde{\omega}_2(-\dot{\hat{\omega}}_2)
\end{aligned}
\tag{6-2-17}
$$

Choose

$$
\begin{aligned}
\dot{\tilde{a}}_1 &= -\dot{\hat{a}}_1 = -\frac{e_1 z_2}{a_1} + \tilde{a}_1 e_1 - e_1^2; \quad \dot{\tilde{\omega}}_1 = -\dot{\hat{\omega}}_1 = \frac{\delta_1}{\omega_1}e_2 z_2 + \tilde{\omega}_1 e_2 - e_2^2 \\
\dot{\tilde{a}}_2 &= -\dot{\hat{a}}_2 = -\frac{e_3 z_4}{a_2} + \tilde{a}_2 e_3 - e_3^2; \quad \dot{\tilde{\omega}}_2 = -\dot{\hat{\omega}}_2 = \frac{\delta_2}{\omega_2}e_4 z_4 + \tilde{\omega}_2 e_4 - e_4^2
\end{aligned}
\tag{6-2-18}
$$

$$u_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 + \hat{a}_1 e_1 + \frac{\hat{a}_1 z_2}{a_1} + \tilde{a}_1^2$$

$$u_2 = 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + z_1 - z_1^3 + f_1\cos\psi_1 t + \hat{\omega}_1 e_2 - \frac{\hat{\omega}_1\delta_1}{\omega_1}z_2$$
$$-\omega_1(x_1 - x_3) + \hat{\omega}_1(y_1 - y_3) + \tilde{\omega}_1^2$$

$$u_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 + \hat{a}_2 e_3 + \frac{\hat{a}_2 z_4}{a_2} + \tilde{a}_2^2 \qquad (6\text{-}2\text{-}19)$$

$$u_4 = 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + z_3 - z_3^3 + f_2\cos\psi_2 t + \hat{\omega}_2 e_4 - \frac{\hat{\omega}_2\delta_2}{\omega_2}z_4$$
$$-\omega_2(x_3 - x_1) + \hat{\omega}_2(y_3 - y_1) + \tilde{\omega}_2^2$$

The initial values of estimate for uncertain parameters are $\hat{a}_1(0) = \hat{a}_2(0) = \hat{\omega}_1(0) = \hat{\omega}_2(0) = 0$. Substituting Eq. (6-2-18) and Eq. (6-2-19) into Eq. (6-2-17), we obtain

$$\dot{V} = -a_1 e_1^2 - \omega_1 e_2^2 - a_2 e_3^2 - \omega_2 e_4^2 \le 0 \qquad (6\text{-}2\text{-}20)$$

which is negative semi-definite function of $e_1, e_2, e_3, e_4, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1$ and $\tilde{\omega}_2$ .The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-2-15) and parameter dynamics (6-2-19) is asymptotically stable. Now, D is an 8-manifold, n=8 and the number of error state variables p=4. When e_1=e_2=e_3=e_4=0 and $\tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{\omega}_2$ take arbitrary values, $\dot{V} = 0$, so X is 4-manifold, m=n–p=8−4=4. m+1<n is satisfied. By pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained. The equilibrium point $e_1 = e_2 = e_3 = e_4 = \tilde{a}_1 = \tilde{a}_2 = \tilde{\omega}_1 = \tilde{\omega}_2 = 0$ is pragmatically asymptotically stable. (see Appendix A) Under the assumption of equal probability, it is actually asymptotically stable. The numerical results are shown in Fig. 6-1. After 10 second, the generalized synchronization is accomplished.

Case II The cubics of chaotic states of the goal system, double Duffing chaotic system, used as $F(t)$

We demand

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + z_i^3) = 0, \quad i = 1, 2, 3, 4 \qquad (6\text{-}2\text{-}21)$$

then

$$\dot{e} = \dot{x} - \dot{y} + 3z^2\dot{z}\ . \tag{6-2-22}$$

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - u_1 + 3z_1^2\dot{z}_1 \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) - \hat{\omega}_1(y_1 - y_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - u_2 + 3z_2^2\dot{z}_2 \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - u_3 + 3z_3^2\dot{z}_3 \\
\dot{e}_4 &= -\omega_2(x_3 - x_1) + \hat{\omega}_2(y_3 - y_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - u_4 + 3z_4^2\dot{z}_4
\end{aligned} \tag{6-2-23}$$

where $e_1 = x_1 - y_1 + z_1^3, e_2 = x_2 - y_2 + z_2^3, e_3 = x_3 - y_3 + z_3^3$ and $e_4 = x_4 - y_4 + z_4^3$.

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1, e_2, e_3, e_4, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{\omega}_2) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2) \tag{6-2-24}$$

where $\tilde{a}_1 = a_1 - \hat{a}_1$, $\tilde{a}_2 = a_2 - \hat{a}_2$, $\tilde{\omega}_1 = \omega_1 - \hat{\omega}_1$, $\tilde{\omega}_2 = \omega_2 - \hat{\omega}_2$ and $\hat{a}_1$, $\hat{a}_2$, $\hat{\omega}_1$, $\hat{\omega}_2$ are estimates of uncertain parameters a_1, a_2, ω_1 and ω_2 respectively.

Its time derivative is

$$\begin{aligned}
\dot{V} &= e_1[-2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - u_1 + 3z_1^2 z_2] + e_2[-\omega_1(x_1 - x_3) + \hat{\omega}_1(y_1 \\
&- y_3) + 3z_2^2(z_1 - z_1^3 - \delta_1 z_2 + f_1\cos\psi_1 t) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 \\
&- u_2] + e_3[-2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - u_3 + 3z_3^2 z_4] + e_4[-\omega_2(x_3 \\
&- x_1) + \hat{\omega}_2(y_3 - y_1) + 3z_4^2(z_3 - z_3^3 - \delta_2 z_4 + f_2\cos\psi_2 t) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \\
&- 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - u_4] + \tilde{a}_1(-\dot{\hat{a}}_1) + \tilde{a}_2(-\dot{\hat{a}}_2) + \tilde{\omega}_1(-\dot{\hat{\omega}}_1) + \tilde{\omega}_2(-\dot{\hat{\omega}}_2)
\end{aligned} \tag{6-2-25}$$

Choose

$$\begin{aligned}
\dot{\tilde{a}}_1 &= -\dot{\hat{a}}_1 = -\frac{3e_1z_1^2z_2}{a_1} + \tilde{a}_1e_1 - e_1^2; \quad \dot{\tilde{\omega}}_1 = -\dot{\hat{\omega}}_1 = \frac{3\delta_1}{\omega_1}e_2z_2^3 + \tilde{\omega}_1e_2 - e_2^2 \\
\dot{\tilde{a}}_2 &= -\dot{\hat{a}}_2 = -\frac{3e_3z_3^2z_4}{a_2} + \tilde{a}_2e_3 - e_3^2; \quad \dot{\tilde{\omega}}_2 = -\dot{\hat{\omega}}_2 = \frac{3\delta_2}{\omega_2}e_4z_4^3 + \tilde{\omega}_2e_4 - e_4^2
\end{aligned} \tag{6-2-26}$$

$$u_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 + \hat{a}_1 e_1 + \frac{3\hat{a}_1 z_1^2 z_2}{a_1} + \tilde{a}_1^2$$

$$u_2 = 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + 3z_2^2(z_1 - z_1^3 + f_1\cos\psi_1 t) + \hat{\omega}_1 e_2 - \frac{3\delta_1}{\omega_1}\hat{\omega}_1 z_2^3$$
$$-\omega_1(x_1 - x_3) + \hat{\omega}_1(y_1 - y_3) + \tilde{\omega}_1^2$$

$$u_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 + \hat{a}_2 e_3 + \frac{3\hat{a}_2 z_3^2 z_4}{a_2} + \tilde{a}_2^2 \qquad (6\text{-}2\text{-}27)$$

$$u_4 = 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + 3z_4^2(z_3 - z_3^3 + f_2\cos\psi_2 t) + \hat{\omega}_2 e_4 - \frac{3\delta_2}{\omega_2}\hat{\omega}_2 z_4^3$$
$$-\omega_2(x_3 - x_1) + \hat{\omega}_2(y_3 - y_1) + \tilde{\omega}_2^2$$

The initial values of estimate for uncertain parameters are $\hat{a}_1(0) = \hat{a}_2(0) = \hat{\omega}_1(0) = \hat{\omega}_2(0) = 0$. Substituting Eq. (6-2-26) and Eq. (6-2-27) into Eq. (6-2-25), it can be rewritten as

$$\dot{V} = -a_1 e_1^2 - \omega_1 e_2^2 - a_2 e_3^2 - \omega_2 e_4^2 \le 0 \qquad (6\text{-}2\text{-}28)$$

which is negative semi-definite function of e_1, e_2, e_3, e_4, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{\omega}_1$ and $\tilde{\omega}_2$.The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-2-23) and parameter dynamics (6-2-27) is asymptotically stable. In our case, $\dot{V} = 0$ when e_1=e_2=e_3=e_4=0 and $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{\omega}_1$ and $\tilde{\omega}_2$ take arbitrary values. n=8, m=4, m+1<n is satisfied. By pragmatical asymptotical stability theorem, the equilibrium point $e_1 = e_2 = e_3 = e_4 = \tilde{a}_1 = \tilde{a}_2 = \tilde{\omega}_1 = \tilde{\omega}_2 = 0$ is pragmatically asymptotically stable. (see Appendix A) Under the assumption of equal probability, it is actually asymptotically stable. The error vector e approaches zero and the estimated parameters approach the uncertain parameters. The numerical results are shown in Fig. 6-2. After 10 second, the generalized synchronization is accomplished.

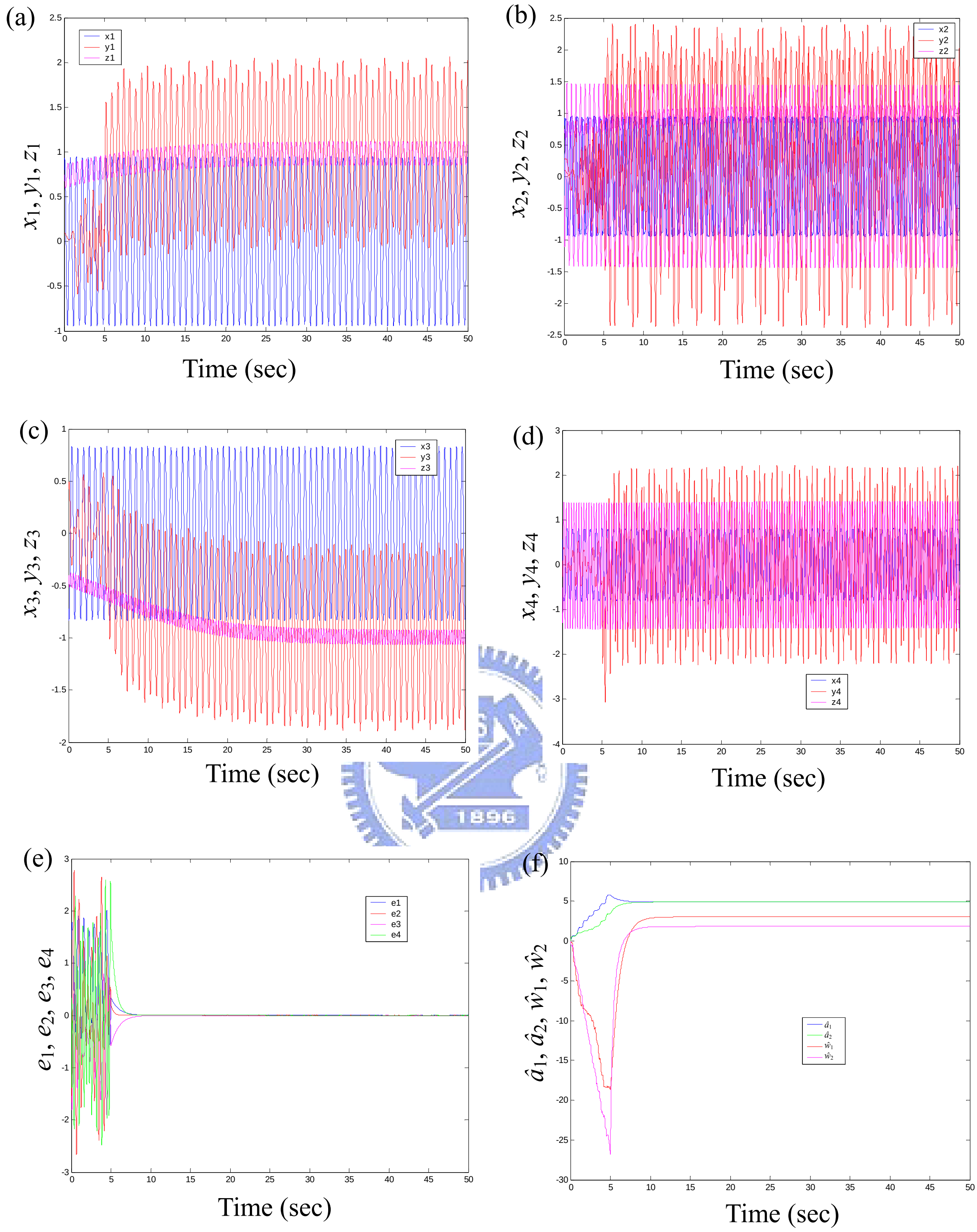


Fig. 6-1. Time histories of states, state errors, z_1, z_2, z_3, z_4, $\hat{a}_1, \hat{a}_2$, $\hat{\omega}_1$ and $\hat{\omega}_2$ for Case I with a_1=4.9, a_2=4.9, ω_1=3.03 and ω_2=1.83.

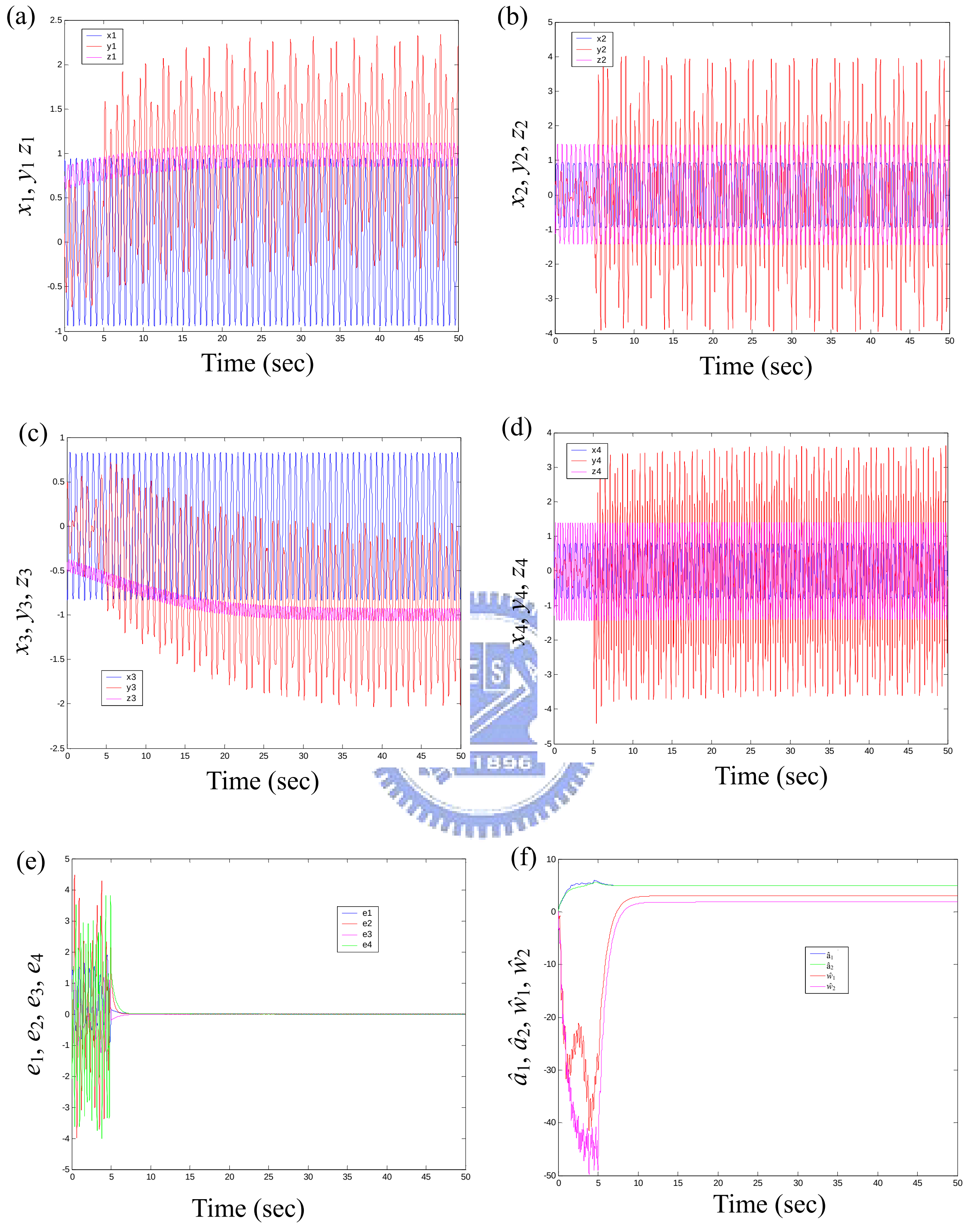


Fig. 6-2. Time histories of states, state errors, z_1, z_2, z_3, z_4, $\hat{a}_1, \hat{a}_2$, $\hat{\omega}_1$ and $\hat{\omega}_2$ for Case II with a_1=4.9, a_2=4.9, ω_1=3.03 and ω_2=1.83.

6-3 Pragmatical Adaptive Control for Different Chaotic Systems

A novel pragmatical adaptive control method for different chaotic systems is proposed. Traditional chaos control is limited for the same system. This method enlarges the function of chaos control. We can control a chaotic system either to a given simple system or to more complex given chaotic system. Based on a pragmatical theorem of asymptotical stability using the concept of probability, an adaptive control law is derived such that it can be proved strictly that the common null solution of error dynamics and of parameter dynamics is asymptotically stable. Numerical results are studied for two examples.

6-3-1 Pragmatical Adaptive Control Scheme

Consider the following chaotic system

$$\dot{x} = f(x, A) + u(t) \tag{6-3-1}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ denotes a state vector, $A = [A_1, A_2, \cdots, A_m]^T \in R^m$ is an original coefficient vector, f is a vector function, and $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T \in R^n$ is a control input vector.

The goal system which can be either chaotic or nonchaotic, is

$$\dot{y} = g(y, \hat{B}) \tag{6-3-2}$$

where $y = [y_1, y_2, \cdots, y_n]^T \in R^n$ denotes a state vector, $\hat{B} = [\hat{B}_1, \hat{B}_2, \cdots, \hat{B}_p]^T \in R^p$ is a goal coefficient vector, and g is a vector function. Our goal is to design an adaptive control method and a controller $u(t)$ so that the state vector of the chaotic system (6-3-1) asymptotically approaches the state vector of the goal system (6-3-2).

The chaos control had accomplished in the sense that the limit of the error vector $e(t) = [e_1, e_2, \cdots, e_n]^T$ approaches zero:

$$\lim_{t \to \infty} e = 0 \tag{6-3-3}$$

where

$$e = y - x\,. \tag{6-3-4}$$

From Eq. (6-3-1) we have

$$\dot{e} = \dot{y} - \dot{x} \tag{6-3-5}$$

$$\dot{e} = g(y, \hat{B}) - f(x, A) - u(t)\,. \tag{6-3-6}$$

A Lyapnuov function $V(e, \tilde{A}, \tilde{B})$ is chosen as a positive definite function

$$V(e, \tilde{A}, \tilde{B}) = \frac{1}{2}e^T e + \frac{1}{2}\tilde{A}^T \tilde{A} + \frac{1}{2}\tilde{B}^T \tilde{B} \tag{6-3-7}$$

where $\tilde{A} = \hat{A} - A$, $\tilde{B} = \hat{B} - B$, $\hat{A}$ and $\hat{B}$ are two column matrices whose elements are all the goal values of corresponding coefficient vector respectively. The A and B are two column matrices whose elements are all the original values of corresponding coefficient vector respectively.

Its derivative along any solution of the differential equation system consisting of Eq. (6-3-6) and update parameter differential equations for $\tilde{A}$ and $\tilde{B}$ is

$$\dot{V}(e) = e^T[g(y, \hat{B}) - f(x, A) - u(t)] + \tilde{A}\dot{\tilde{A}} + \tilde{B}\dot{\tilde{B}} \tag{6-3-8}$$

where $u(t)$, $\dot{\tilde{A}}$, and $\dot{\tilde{B}}$ are chosen so that $\dot{V} = e^T Ce$, C is a diagonal negative definite matrix, and $\dot{V}$ is a negative semi-definite function of e and parameter differences $\tilde{A}$ and $\tilde{B}$. In current scheme of adaptive control of chaotic motion [89-93], traditional Lyapunov stability theorem and Babalat lemma are used to prove the error vector approaches zero, as time approaches infinity. But the question, why the differences between goal coefficients and original coefficients also approach zeros, remains no answer. By GYC pragmatical asymptotical stability theorem, the question can be answered strictly.

6-3-2 Numerical Results of the Chaos Control

Case I Control a chaotic nano resonator system to a double harmonic system

Nano resonator system studied in this thesis is a modified form of nonlinear damped Mathieu system which is obtained when the nano Mathieu oscillator has nonlinear time-dependent spring constant [97]. Then we have the modified nonlinear damped Mathieu system, i.e. the nano resonator system:

$$\begin{cases} \dot{x}_1 = x_2, \dot{x}_2 = -(s_1 + s_2 x_3)x_1 - (s_1 + s_2 x_3)x_1^3 - s_3 x_2 + s_4 x_3 \\ \dot{x}_3 = x_4, \dot{x}_4 = -s_5 x_3 - s_6 x_4^3 \end{cases} \tag{6-3-12}$$

where s_1, s_2, s_3, s_4, s_5 and s_6 are given original constant coefficients of the system with initial conditions $x_1(0)$=0.75, $x_2(0)$=−0.3, $x_3(0)$=−0.4 and $x_4(0)$=0.5. When the given original coefficients are s_1=0.2, s_2=0.2, s_3=0.4, s_4=15, s_5=1 and s_6=0.3, the modified nonlinear damped Mathieu system is chaotic, as shown in Fig. 6-3.

The goal system is a double harmonic system:

$$\dot{y}_1 = y_2,\ \dot{y}_2 = -\hat{\omega}_{n1} y_1, \dot{y}_3 = y_4,\ \dot{y}_4 = -\hat{\omega}_{n2} y_3 \tag{6-3-13}$$

where $\hat{\omega}_{n1}$ and $\hat{\omega}_{n2}$ are constant goal coefficients of the system with initial conditions $y_1(0)$= 0.3, $y_2(0)$=−0.4, $y_3(0)$=−0.7 and $y_4(0)$=0.15.

In order to lead (x_1, x_2, x_3, x_4) to (y_1, y_2, y_3, y_4), we add u_1, u_2, u_3, and u_4 to each equation of Eq. (6-3-12), respectively.

$$\begin{cases} \dot{x}_1 = x_2 + u_1, \dot{x}_2 = -(s_1 + s_2 x_3)x_1 - (s_1 + s_2 x_3)x_1^3 - s_3 x_2 + s_4 x_3 + u_2 \\ \dot{x}_3 = x_4 + u_3, \dot{x}_4 = -s_5 x_3 - s_6 x_4^3 + u_4 \end{cases} \tag{6-3-14}$$

Subtracting Eq. (6-3-14) from the four equations of Eq. (6-3-13), we obtain the error dynamics in which the original coefficients are ω_{n1}=0, ω_{n2}=0, s_1=0.2, s_2=0.2, s_3=0.4, s_4=15, s_5=1 and s_6=0.3.

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(y_i - x_i) = 0, \quad i = 1, 2, 3, 4 \tag{6-3-15}$$

$$\begin{aligned} \dot{e}_1 &= y_2 - x_2 - u_1 \\ \dot{e}_2 &= -\hat{\omega}_{n1} y_1 + (s_1 + s_2 x_3)x_1 + (s_1 + s_2 x_3)x_1^3 + s_3 x_2 - s_4 x_3 - u_2 \\ \dot{e}_3 &= y_4 - x_4 - u_3 \\ \dot{e}_4 &= -\hat{\omega}_{n2} y_3 + s_5 x_3 + s_6 x_4^3 - u_4 \end{aligned} \tag{6-3-16}$$

where $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$ and $e_4 = y_4 - x_4$.

Choose a Lyapunov function in the form of the positive definite function:

$$\begin{aligned} V(e_1, e_2, e_3, e_4, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5, \tilde{s}_6, \tilde{\omega}_{n1}, \tilde{\omega}_{n2}) &= \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{s}_1^2 \\ &+ \tilde{s}_2^2 + \tilde{s}_3^2 + \tilde{s}_4^2 + \tilde{s}_5^2 + \tilde{s}_6^2 + \tilde{\omega}_{n1}^2 + \tilde{\omega}_{n2}^2) \end{aligned} \tag{6-3-17}$$

where $\tilde{s}_1 = \hat{s}_1 - s_1$, $\tilde{s}_2 = \hat{s}_2 - s_2$, $\tilde{s}_3 = \hat{s}_3 - s_3$, $\tilde{s}_4 = \hat{s}_4 - s_4$, $\tilde{s}_5 = \hat{s}_5 - s_5$, $\tilde{s}_6 = \hat{s}_6 - s_6$,

$\tilde{\omega}_{n1} = \hat{\omega}_{n1} - \omega_{n1}$, $\tilde{\omega}_{n2} = \hat{\omega}_{n2} - \omega_{n2}$ and s_1, s_2, s_3, s_4, s_5, s_6, ω_{n1}, ω_{n2} are given original coefficients, $\hat{s}_1 = \hat{s}_2 = \hat{s}_3 = \hat{s}_4 = \hat{s}_5 = \hat{s}_6 = 0$, $\hat{\omega}_{n1} = 1$ and $\hat{\omega}_{n2} = 1$ are goal parameters.

Its time derivative along any solution of Eq. (6-3-16) and parameter dynamics is

$$\begin{aligned}\dot{V} = {} & e_1[y_2 - x_2 - u_1] + e_2[-\hat{\omega}_{n1} y_1 + (s_1 + s_2 x_3) x_1 + (s_1 + s_2 x_3) x_1^3 + s_3 x_2 - s_4 x_3 - u_2] \\ & + e_3[y_4 - x_4 - u_3] + e_4[-\hat{\omega}_{n2} y_3 + s_5 x_3 + s_6 x_4^3 - u_4] + \tilde{s}_1(-\dot{s}_1) + \tilde{s}_2(-\dot{s}_2) + \tilde{s}_3(-\dot{s}_3) \\ & + \tilde{s}_4(-\dot{s}_4) + \tilde{s}_5(-\dot{s}_5) + \tilde{s}_6(-\dot{s}_6) + \tilde{\omega}_{n1}(-\dot{\omega}_{n1}) + \tilde{\omega}_{n2}(-\dot{\omega}_{n2})\end{aligned} \tag{6-3-18}$$

Choose

$$\begin{aligned} u_1 &= \hat{\omega}_{n1} e_1 \\ u_2 &= \hat{\omega}_{n1} e_2 + e_1 + \tilde{s}_1^2 + \tilde{s}_2^3 + \tilde{s}_3^2 + \tilde{s}_4^2 + \tilde{\omega}_{n1}^2 - \omega_{n1} y_1 - \tilde{\omega}_{n1} \omega_{n1} y_1 \\ u_3 &= \hat{\omega}_{n2} e_3 \\ u_4 &= \hat{\omega}_{n2} e_4 + e_3 + \tilde{s}_5^2 + \tilde{s}_6^2 + \tilde{\omega}_{n2}^2 - \omega_{n2} y_3 - \tilde{\omega}_{n2} \omega_{n2} y_3 \end{aligned} \tag{6-3-19}$$

$$\begin{aligned} &\dot{\tilde{s}}_1 = -\dot{s}_1 = -x_1 e_2 - x_1^3 e_2 + \tilde{s}_1 e_2; \quad \dot{\tilde{s}}_2 = -\dot{s}_2 = -x_1 x_3 e_2 - x_3 x_1^3 e_2 + \tilde{s}_2 e_2 \\ &\dot{\tilde{s}}_3 = -\dot{s}_3 = -x_2 e_2 + \tilde{s}_3 e_2; \quad \dot{\tilde{s}}_4 = -\dot{s}_4 = s_4 x_3 e_2 + \tilde{s}_4 e_2 \\ &\dot{\tilde{s}}_5 = -\dot{s}_5 = -x_3 e_4 + \tilde{s}_5 e_4; \quad \dot{\tilde{s}}_6 = -\dot{s}_6 = -x_4^3 e_4 + \tilde{s}_6 e_4 \\ &\dot{\tilde{\omega}}_{n1} = -\dot{\omega}_{n1} = \omega_{n1} y_1 e_2 + \tilde{\omega}_{n1} e_2; \quad \dot{\tilde{\omega}}_{n2} = -\dot{\omega}_{n2} = \omega_{n2} y_3 e_4 + \tilde{\omega}_{n2} e_4 \end{aligned} \tag{6-3-20}$$

Eq.(6-3-20) is the parameter dynamics. Substituting Eq. (6-3-19) and Eq. (6-3-20) into Eq. (6-3-18), we obtain

$$\dot{V} = -\hat{\omega}_{n1} e_1^2 - \hat{\omega}_{n1} e_2^2 - \hat{\omega}_{n2} e_3^2 - \hat{\omega}_{n2} e_4^2 < 0$$

which is negative semi-definite function of e_1, e_2, e_3, e_4, $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5, \tilde{s}_6, \tilde{\omega}_{n1}$ and $\tilde{\omega}_{n2}$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-3-19) and parameter dynamics (6-3-20) is asymptotically stable. Now, D is an 8-manifold, n=12 and the number of error state variables p=4. When $e_1=e_2=e_3=e_4=0$ and $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5, \tilde{s}_6, \tilde{\omega}_{n1}, \tilde{\omega}_{n2}$ take arbitrary values, $\dot{V} = 0$, so X is 4-manifold, $m=n-p=12-4=8$. $m+1<n$ is satisfied. By pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The GYC pragmatical generalized synchronization is obtained. The equilibrium point $e_1 = e_2 = e_3 = e_4 = \tilde{s}_1 = \tilde{s}_2 = \tilde{s}_3 = \tilde{s}_4 = \tilde{s}_5 = \tilde{s}_6 = \tilde{\omega}_{n1} = \tilde{\omega}_{n2} = 0$ is pragmatically asymptotically stable. Under the assumption of

equal probability, it is actually asymptotically stable. (see Appendix A) This means that the chaos control for different systems, from a modified nonlinear damped Mathieu system to a double harmonic system, can be achieved. The simulation results are shown in Fig. 6-4 and Fig. 6-5.

Case II Control a chaotic nano resonator system to a hyperchaotic Quantum-CNN oscillator system with two positive Lyapunov exponents

For a two-cell Quantum-CNN, the following differential equations are obtained:

$$\begin{cases} \dot{z}_1 = -2a_1\sqrt{1-x_1^2}\sin z_2,\ \dot{z}_2 = -\omega_1(z_1 - z_3) + 2a_1\dfrac{z_1}{\sqrt{1-z_1^2}}\cos z_2 \\ \dot{z}_3 = -2a_2\sqrt{1-z_3^2}\sin z_4,\ \dot{z}_4 = -\omega_2(z_3 - z_1) + 2a_2\dfrac{z_3}{\sqrt{1-z_3^2}}\cos z_4 \end{cases} \tag{6-3-21}$$

where z_1, z_3 are polarizations, z_2, z_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 ω_1 and ω_2 are coefficients that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. $\hat{a}_1$, $\hat{a}_2$, $\hat{\omega}_1$ and $\hat{\omega}_2$ are goal coefficients.

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(z_i - x_i) = 0, \quad i = 1, 2, 3, 4 \tag{6-3-22}$$

where $\dot{e} = \dot{z} - \dot{x}$.

$$\begin{aligned} \dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin z_2 - x_2 - u_1 \\ \dot{e}_2 &= -\omega_1(z_1 - z_3) + 2a_1\frac{z_1\cos z_2}{\sqrt{1-z_1^2}} + (s_1 + s_2x_3)x_1 + (s_1 + s_2x_3)x_1^3 + s_3x_2 - s_4x_3 - u_2 \\ \dot{e}_3 &= -2a_2\sqrt{1-z_3^2}\sin z_4 - x_4 - u_3 \\ \dot{e}_4 &= -\omega_2(z_3 - z_1) + 2a_2\frac{z_3\cos z_4}{\sqrt{1-z_3^2}} + s_5x_3 + s_6x_4^3 - u_4 \end{aligned} \tag{6-3-23}$$

where $e_1 = z_1 - x_1$, $e_2 = z_2 - x_2$, $e_3 = z_3 - x_3$ and $e_4 = z_4 - x_4$. When the original parameters $a_1=9\times10^{-5}$, $a_2=9\times10^{-5}$, $\omega_1=10\times10^{-5}$ and $\omega_2=10\times10^{-5}$, the Quantum-CNN system has hyperchaotic motion with two positive Lyapunov exponents.

The given original coefficients in Eq. (6-3-23) are $a_1=0$, $a_2=0$, $\omega_1=0$, $\omega_2=0$, $s_1=0.2$, $s_2=0.2$, $s_3=0.4$, $s_4=50$, $s_5=1$ and $s_6=0.3$.

Choose a Lyapunov function in the form of the positive definite function:

$$V(e_1,e_2,e_3,e_4,\tilde{s}_1,\tilde{s}_2,\tilde{s}_3,\tilde{s}_4,\tilde{s}_5,\tilde{s}_6,\tilde{a}_1,\tilde{a}_2,\tilde{\omega}_1,\tilde{\omega}_2)=\frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2+\tilde{s}_1^2 + \tilde{s}_2^2+\tilde{s}_3^2+\tilde{s}_4^2+\tilde{s}_5^2+\tilde{s}_6^2+\tilde{a}_1^2+\tilde{a}_2^2+\tilde{\omega}_1^2+\tilde{\omega}_2^2) \tag{6-3-24}$$

where $\tilde{s}_1=\hat{s}_1-s_1$, $\tilde{s}_2=\hat{s}_2-s_2$, $\tilde{s}_3=\hat{s}_3-s_3$, $\tilde{s}_4=\hat{s}_4-s_4$, $\tilde{s}_5=\hat{s}_5-s_5$, $\tilde{s}_6=\hat{s}_6-s_6$, $\tilde{a}_1=\hat{a}_1-a_1$, $\tilde{a}_2=\hat{a}_2-a_2$, $\tilde{\omega}_1=\hat{\omega}_1-\omega_1$, $\tilde{\omega}_2=\hat{\omega}_2-\omega_2$ and goal parameters are $\hat{s}_1=\hat{s}_2=\hat{s}_3=\hat{s}_4=\hat{s}_5=\hat{s}_6=0$, $\hat{a}_1=9\times 10^{-5}$, $\hat{a}_2=9\times10^{-5}$, $\hat{\omega}_1=10\times10^{-5}$ and $\hat{\omega}_2=10\times10^{-5}$.

Its time derivative along any solution of Eq. (6-3-23) and parameter dynamics is

$$\begin{aligned}\dot{V}&=e_1[-2a_1\sqrt{1-x_1^2}\sin z_2-x_2-u_1]+e_2[-\omega_1(z_1-z_3)+2a_1\frac{z_1\cos z_2}{\sqrt{1-z_1^2}}+(s_1+s_2x_3)x_1+(s_1+\\&s_2x_3)x_1^3+s_3x_2-s_4x_3-u_2]+e_3[-2a_2\sqrt{1-z_3^2}\sin z_4-x_4-u_3]+e_4[2a_2\frac{z_3\cos z_4}{\sqrt{1-z_3^2}}+s_5x_3\\&-\omega_2(z_3-z_1)+s_6x_4^3-u_4]+\tilde{s}_1(-\dot{s}_1)+\tilde{s}_2(-\dot{s}_2)+\tilde{s}_3(-\dot{s}_3)+\tilde{s}_4(-\dot{s}_4)+\tilde{s}_5(-\dot{s}_5)+\tilde{s}_6(-\dot{s}_6)\\&+\tilde{a}_1(-\dot{a}_1)+\tilde{a}_2(-\dot{a}_2)+\tilde{\omega}_1(-\dot{\omega}_1)+\tilde{\omega}_2(-\dot{\omega}_2)\end{aligned} \tag{6-3-25}$$

Choose

$$\begin{aligned}u_1&=\omega_{n1}e_1+x_2+a_1e_1-2a_1e_1\sqrt{1-x_1^2}\sin z_2+2\tilde{a}_1a_1e_1\sqrt{1-x_1^2}\sin z_2-\tilde{a}_1^2\\u_2&=\omega_{n1}e_2+e_1+\omega_1e_2-(\omega_1(z_1-z_3)-\frac{2a_1z_1\cos z_2}{\sqrt{1-z_1^2}})+\tilde{\omega}_1(\omega_1(z_1-z_3)-\frac{2a_1z_1\cos z_2}{\sqrt{1-z_1^2}})\\&\quad-\tilde{s}_1^2-\tilde{s}_2^2-\tilde{s}_3^2-\tilde{s}_4^2-\tilde{\omega}_1^2\\u_3&=\omega_{n2}e_3+x_4+a_2e_3-2a_2e_3\sqrt{1-z_3^2}\sin z_4+2\tilde{a}_2a_2e_3\sqrt{1-z_3^2}\sin z_4-\tilde{a}_2^2\\u_4&=\omega_{n2}e_4+e_3+\omega_2e_4-(\omega_2(z_3-z_1)-\frac{2a_2z_3\cos z_4}{\sqrt{1-z_3^2}})+\tilde{\omega}_2(\omega_2(z_3-z_1)-\frac{2a_2z_3\cos z_4}{\sqrt{1-z_3^2}})\\&\quad-\tilde{s}_5^2-\tilde{s}_6^2-\tilde{\omega}_2^2\end{aligned} \tag{6-3-26}$$

$$\begin{aligned}&\dot{\tilde{s}}_1=-\dot{s}_1=-(x_1+x_1^3)e_2-\tilde{s}_1e_2;\quad \dot{\tilde{s}}_2=-\dot{s}_2=-(x_1x_3+x_3x_1^3)e_2-\tilde{s}_2e_2\\&\dot{\tilde{s}}_3=-\dot{s}_3=-x_2e_2-\tilde{s}_3e_2;\quad \dot{\tilde{s}}_4=-\dot{s}_4=x_3e_2-\tilde{s}_4e_2\\&\dot{\tilde{s}}_5=-\dot{s}_5=-x_3e_4-\tilde{s}_5e_4;\quad \dot{\tilde{s}}_6=-\dot{s}_6=-x_4^3e_4-\tilde{s}_6e_4\\&\dot{\tilde{a}}_1=-\dot{a}_1=2\hat{a}_1e_1\sqrt{1-x_1^2}\sin z_2-\tilde{a}_1e_1;\quad \dot{\tilde{\omega}}_1=-\dot{\omega}_1=(\omega_1(z_1-z_3)-\frac{2a_1z_1\cos z_2}{\sqrt{1-z_1^2}})e_2-\tilde{\omega}_1e_2\\&\dot{\tilde{a}}_2=-\dot{a}_2=2\hat{a}_2e_3\sqrt{1-z_3^2}\sin z_4-\tilde{a}_2e_3;\quad \dot{\tilde{\omega}}_2=-\omega_2=(\omega_2(z_3-z_1)-\frac{2a_2z_3\cos z_4}{\sqrt{1-z_3^2}})e_4-\tilde{\omega}_2e_4\end{aligned} \tag{6-3-27}$$

Eq. (6-3-27) is the parameter dynamics. Substituting Eq. (6-3-26) and Eq. (6-3-27) into Eq. (6-3-25), it can be rewritten as

$$\dot{V} = -\hat{a}_1 e_1^2 - \hat{\omega}_1 e_2^2 - \hat{a}_2 e_3^2 - \hat{\omega}_2 e_4^2 < 0$$

which is negative semi-definite function of e_1, e_2, e_3, e_4, $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5, \tilde{s}_6, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1$, and $\tilde{\omega}_2$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-3-26) and parameter dynamics (6-3-27) is asymptotically stable. In our case, $\dot{V} = 0$ when $e_1=e_2=e_3=e_4=0$ and $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5, \tilde{s}_6, \tilde{a}_1, \tilde{a}_2\ \tilde{\omega}_1, \tilde{\omega}_2$ take arbitrary values. $n=14$, $m=4$, $m+1<n$ is satisfied. By GYC pragmatical asymptotical stability theorem, the equilibrium point $e_1 = e_2 = e_3 = e_4 = \tilde{s}_1 = \tilde{s}_2 = \tilde{s}_3 = \tilde{s}_4 = \tilde{s}_5 = \tilde{s}_6 = \tilde{a}_1 = \tilde{a}_2 = \tilde{\omega}_1 = \tilde{\omega}_2 = 0$ is pragmatically asymptotically stable. Under the assumption of equal probability, it is actually asymptotically stable. (see Appendix A) The error vector e approaches zero and the given parameters approach the goal parameters. This means that the chaos control for the different systems, from a modified nonlinear damped Mathieu system to a hyperchaotic Quantum-CNN oscillator system with two positive Lyapunov exponents, can be achieved. The simulation results are shown in Fig. 6-6 and Fig. 6-7.

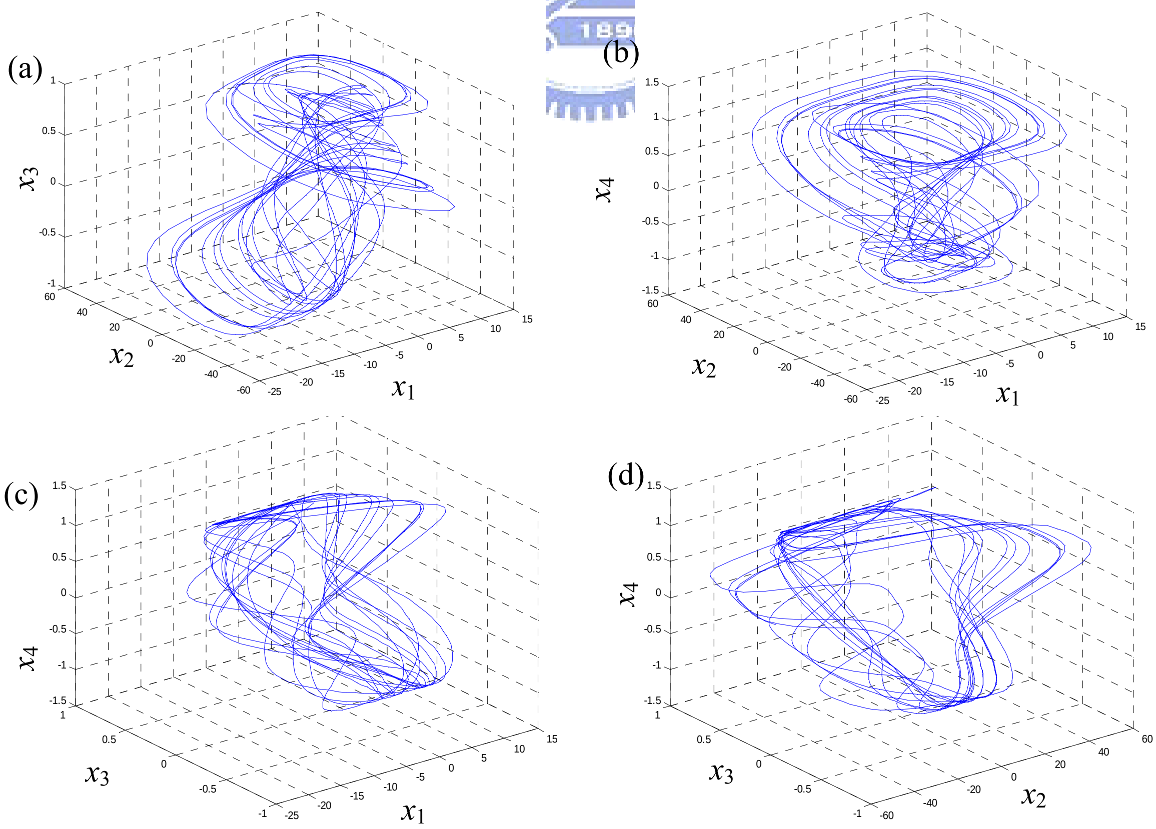


Fig. 6-3. Phase portraits of modified nonlinear damped Mathieu system.

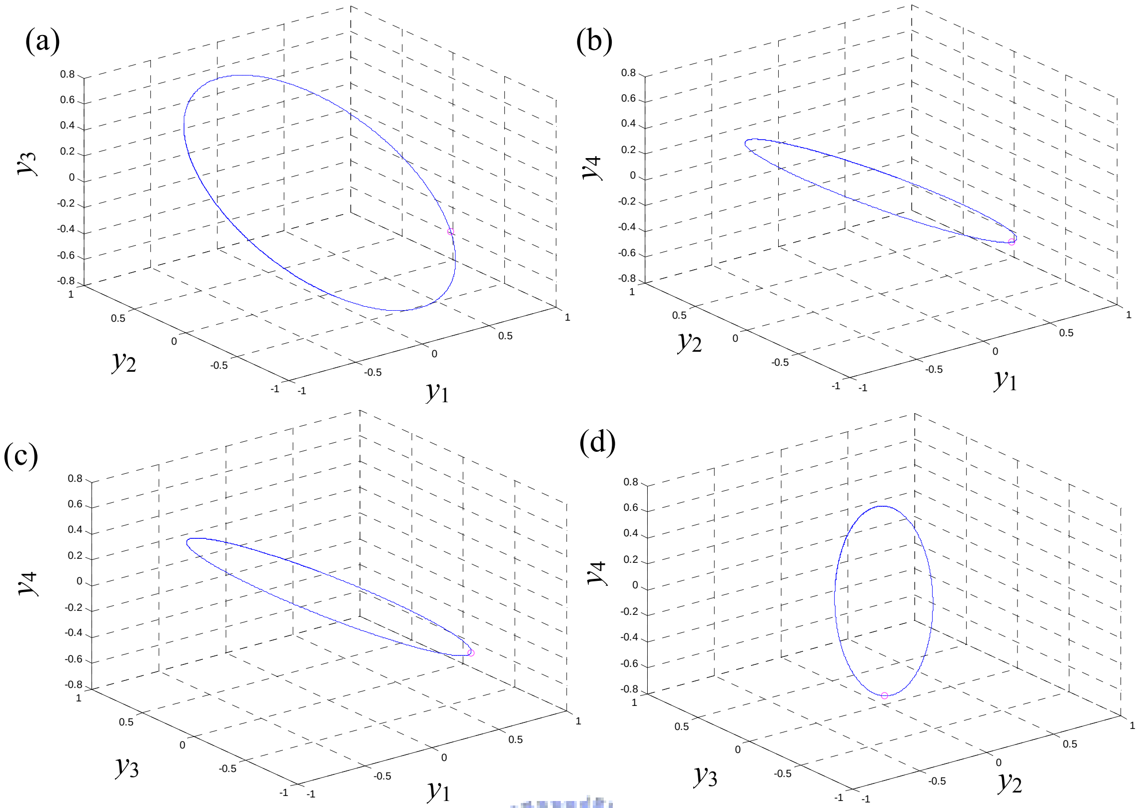


Fig. 6-4. Phase portraits and Poincaré map of a double harmonic system.

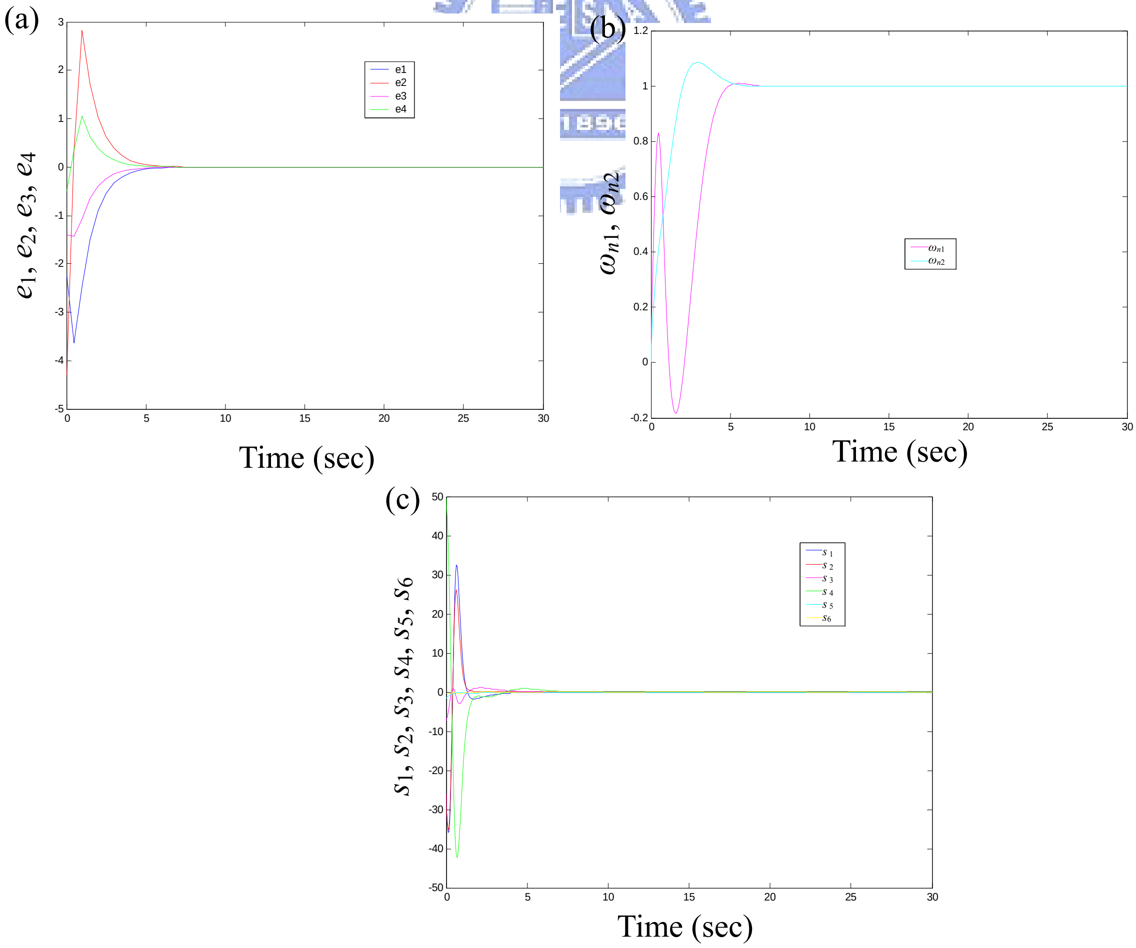


Fig. 6-5.Time histories of state errors, ω_{n1}, ω_{n2}, s_1, s_2, s_3, s_4, s_5 and s_6 for Case I.

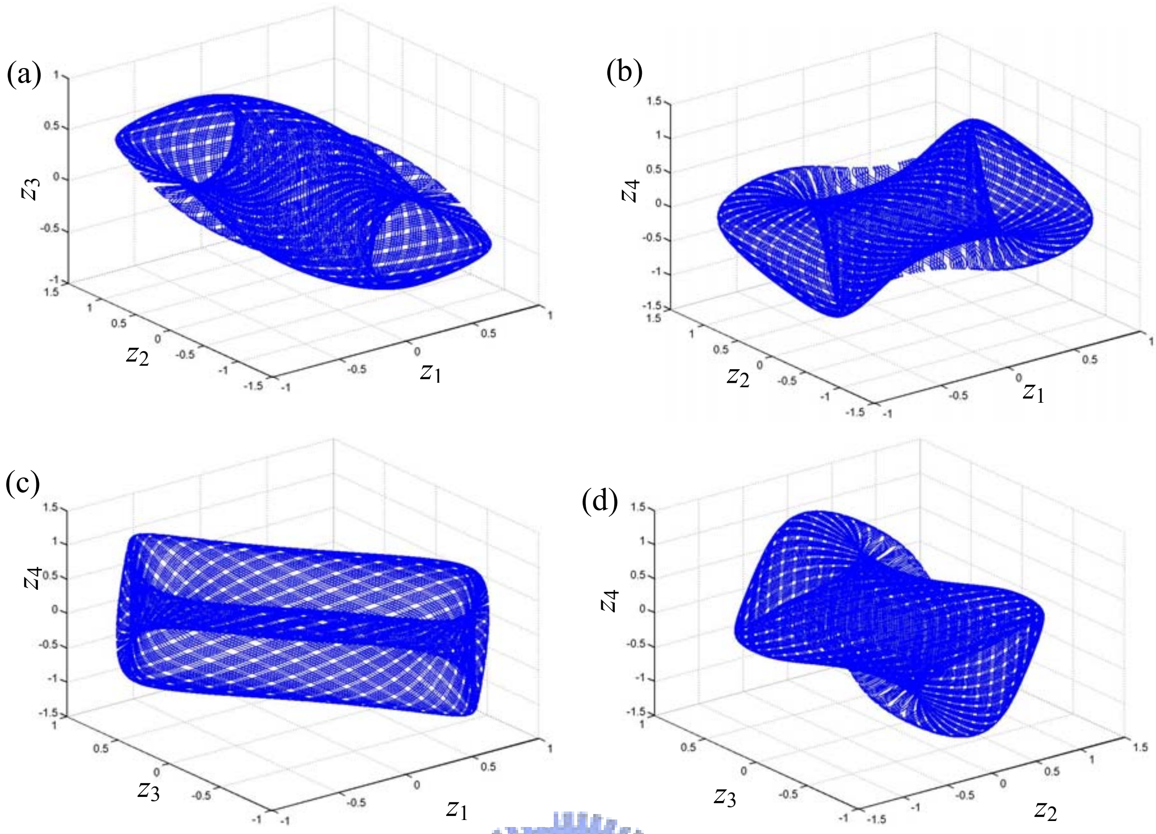


Fig. 6-6. Phase portraits of Quantum-CNN system.

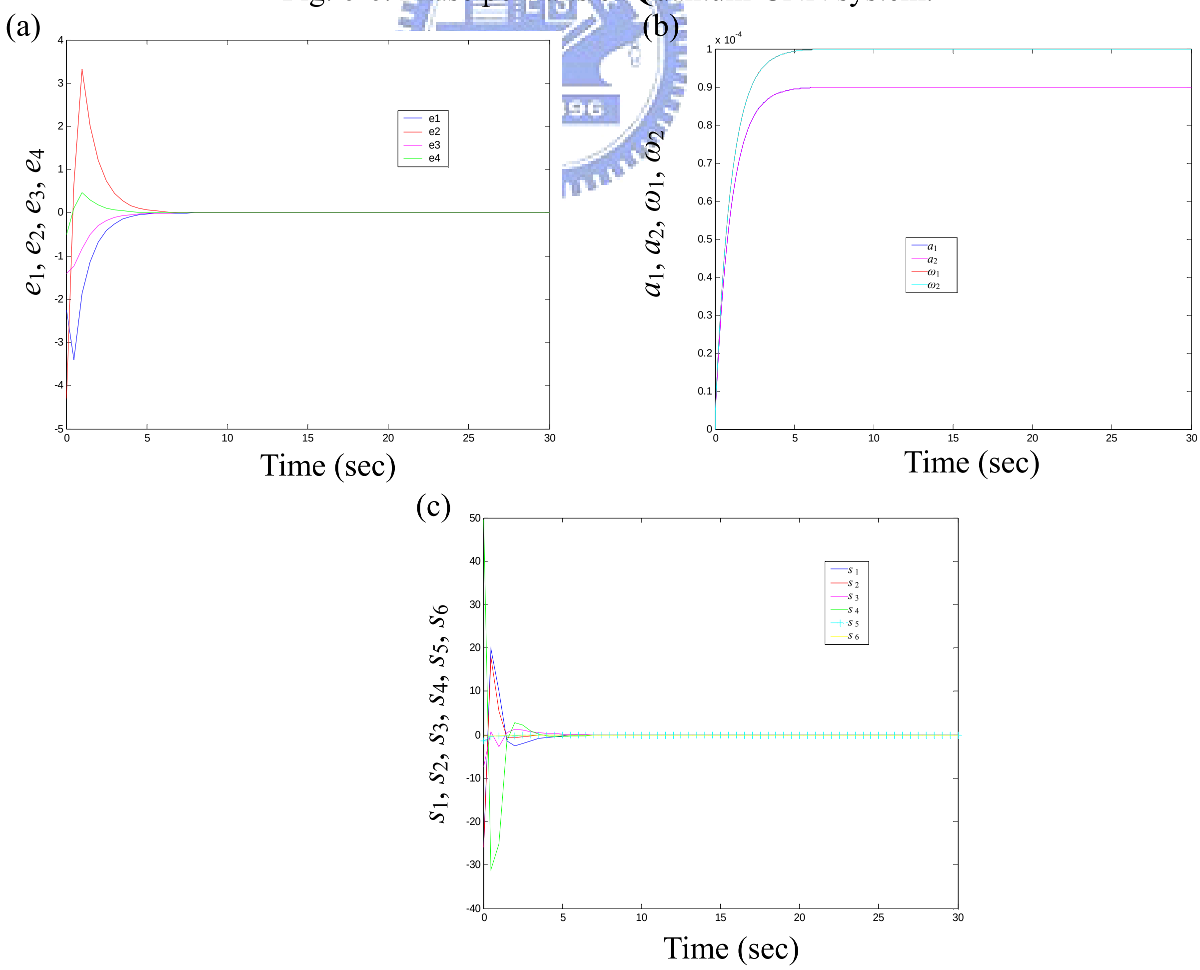


Fig. 6-7.Time histories of state errors, a_1, a_2, ω_1, ω_2, s_1, s_2, s_3, s_4, s_5, and s_6 for Case II.

6-4 Synchronization of Chaotic System with Uncertain Variable/ Chaotic Parameters by Linear Coupling and Pragmatical Adaptive Tracking

This section studies the synchronization of general chaotic systems which satisfy Lipschitz condition only, with uncertain **variable/chaotic** parameters by linear coupling and pragmatical adaptive tracking. The uncertain parameters of a system vary with time due to aging, environment and disturbances. A sufficient condition is given for the asymptotical stability of common zero solution of error dynamics and parameter update dynamics by GYC pragmatical asymptotical stability theorem based on equal probability assumption.

6-4-1 Theoretical Analyses

Consider a nonautonomous system in the form as follows

$$\dot{x} = F(t, x, B(t)) \tag{6-4-1}$$

The slave system is given by

$$\dot{y} = F(t, y, \hat{B}(t)) + \hat{K}(x - y) \tag{6-4-2}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in R^n$, $y = [y_1, y_2, \cdots, y_n]^T \in R^n$, $B = [B_1, B_2, \cdots, B_M]^T \in R^M$ is a vector of uncertain **variable/chaotic** coefficients in *F*, $\hat{B} = \left[\hat{B}_1, \hat{B}_2, \cdots, \hat{B}_M\right]^T \in R^M$ is a vector of estimated coefficients in *F*, $F : \Omega \subset R_+ \times R^n \times R^M \to R^n$ satisfies Lipschitz conditions $\|F(t, x_{\mathrm{I}}, B) - F(t, x_{\mathrm{II}}, B)\| \le G\|x_{\mathrm{I}} - x_{\mathrm{II}}\|$ where x_{I} and x_{II} are two neighbor state vectors, and $\left\|F(t, x, B) - F(t, x, \hat{B})\right\| \le G\left\|B - \hat{B}\right\|$ in Ω_1 with Lipschitz constant *G*. $\hat{K} = diag\left[\hat{K}_1, \cdots, \hat{K}_i, \cdots, \hat{K}_n\right]$ is a constant matrix. $\hat{K}(x - y)$ is the estimated linear coupling term. Ω_1 is domain containing the origin. For given $(t_0, x_0, y_0, B_0) \in \Omega$, the solutions $\left[x^T(t, t_0, x_0, B_0),\ \ y^T(t, t_0, x_0, y_0, B_0)\right]^T$ of Eq. (6-4-1) and Eq. (6-4-2) exist for $t \ge t_0$.

If the synchronization can be accomplished when $t \to \infty$, the limit of the error vector

$e(t)=[e_1, e_2, \cdots, e_n]^T$ must approach zero:

$$\lim_{t\to\infty} e = 0 \tag{6-4-3}$$

where

$$e = x - y\,. \tag{6-4-4}$$

From Eq. (6-4-1), (6-4-2) and (6-4-4) we have

$$\dot{e} = \dot{x} - \dot{y} \tag{6-4-5}$$

$$\dot{e} = F(t,x,B) - F(t,x-e,\hat{B}) - \hat{K}(x-y)\,. \tag{6-4-6}$$

A Lyapnuov function $V(e,\tilde{B},\tilde{G})$ is chosen as a positive definite function

$$V(e,\tilde{B},\tilde{G}) = \frac{1}{2}e^T e + \frac{1}{2}\tilde{B}^T\tilde{B} + \frac{1}{2}\tilde{G}^2 \tag{6-4-7}$$

where $\tilde{G} = G - \hat{G}$; $\hat{G}$ is the estimated Lipschitz constant, $\tilde{B} = B - \hat{B}$.

When $M=n$, the time derivative of V along any solution of the differential equation system consisting of Eq. (6-4-6) and update differential equations for $\tilde{B}$ and $\tilde{G}$ is

$$\begin{aligned}\dot{V}(e,\tilde{B},\tilde{G}) &= e^T[F(t,x,B) - F(t,x-e,B) + F(t,x-e,B) - F(t,x-e,\hat{B}) - \hat{K}e] + \tilde{B}^T\dot{\tilde{B}} + \\ \tilde{G}\dot{\tilde{G}} &= e^T[F(t,x,B) - F(t,x-e,B) - \hat{K}e] + \tilde{G}\dot{\tilde{G}} + e^T[F(t,x-e,B) - F(t,x-e,\hat{B})] + \tilde{B}^T\dot{\tilde{B}}\end{aligned} \tag{6-4-8}$$

By Lipschitz condition,

$$\dot{V}(e,\tilde{B},\tilde{G}) \le G\|e\|^2 - e^T\hat{K}e + \tilde{G}\dot{\tilde{G}} + e^T[F(t,x-e,B) - F(t,x-e,\hat{B})] + \tilde{B}^T\dot{\tilde{B}} \tag{6-4-9}$$

Since

$$\begin{aligned} e^T[F(t,x-e,B) - F(t,x-e,\hat{B})] &\le |e_1|\cdot\left|F_1(t,x-e,B) - F_1(t,x-e,\hat{B})\right| \\ &+ \cdots + |e_n|\cdot\left|F_n(t,x-e,B) - F_n(t,x-e,\hat{B})\right|\end{aligned} \tag{6-4-10}$$

by Schwarz inequality [88] and Lipschitz condition, it is obtained that

$$\begin{aligned}&|e_1|\cdot\left|F_1(t,x-e,B) - F_1(t,x-e,\hat{B})\right| + \cdots + |e_n|\cdot\left|F_n(t,x-e,B) - F_n(t,x-e,\hat{B})\right| \\ &\le \|e\|\cdot\left\|F(t,x-e,B) - F(t,x-e,\hat{B})\right\| \le G\|e\|\cdot\|\tilde{B}\|\end{aligned} \tag{6-4-11}$$

Therefore

$$\dot{V}(e,\tilde{B},\tilde{G}) \le G\|e\|^2 - e^T\hat{K}e + \tilde{G}\dot{\tilde{G}} + G\|e\|\cdot\|\tilde{B}\| + \tilde{B}_1\dot{\tilde{B}}_1 + \cdots + \tilde{B}_n\dot{\tilde{B}}_n \tag{6-4-12}$$

Choose

$$\dot{\hat{G}} = -e^T e\,, \quad \hat{K} = diag[\hat{G} + G] \tag{6-4-13}$$

and

$$\dot{\hat{B}}_1 = -G\tilde{B}_1 \|e\| / \|\tilde{B}\|, \cdots, \dot{\hat{B}}_n = -G\tilde{B}_n \|e\| / \|\tilde{B}\| \tag{6-4-14}$$

we have

$$\tilde{B}^T \dot{\hat{B}} = -G(\tilde{B}_1^2 + \cdots \tilde{B}_N^2)\|e\| / \|\tilde{B}\| = -G\|\tilde{B}\|^2 \cdot \|e\| / \|\tilde{B}\| = -G\|e\| \cdot \|\tilde{B}\|. \tag{6-4-15}$$

Introducing Eqs. (6-4-15), (6-4-13) in (6-4-12), we get

$$\begin{aligned} \dot{V}(e, \tilde{B}, \tilde{G}) &\le G\|e\|^2 - diag[\hat{G} + G]\|e\|^2 - \tilde{G}\|e\|^2 + G\|e\| \cdot \|\tilde{B}\| - G\|e\| \cdot \|\tilde{B}\| \\ &= -G\|e\|^2 = -G(e_1^2 + \cdots + e_n^2) \end{aligned} \tag{6-4-16}$$

$\dot{V}$ is a negative semi-definite of $e, \tilde{B}, \tilde{G}$, by GYC pragmatical asymptotical stability theorem, the solution $e = 0, \tilde{B} = 0, \tilde{G} = 0$ is asymptotically stable, which means that the two coupled systems will synchronize even if different initial conditions are used and the estimation of the parameters is not exact..

When $M \neq n$, on the right-hand side of Eq. (6-4-9), the other terms remain unchanged, we want only to reduce last two terms

$$e^T[F(t, x-e, B) - F(t, x-e, \hat{B})] + \tilde{B}^T \dot{\hat{B}} \tag{6-4-17}$$

When $M > n$, we put

$$e^T = e^T = [e_1, \cdots, e_n, e_{n+1}, \cdots, e_M]^T \tag{6-4-18}$$

where $e_{n+1} = e_{n+2} = \cdots = e_M = 0$. The first term of Eq. (6-4-17) becomes

$$\begin{aligned} &e^T[F(t, x-e, B) - F(t, x-e, \hat{B})] \le |e_1| \cdot \left|F_1(t, x-e, B) - F_1(t, x-e, \hat{B})\right| + \cdots \\ &+ |e_n| \cdot \left|F_n(t, x-e, B) - F_n(t, x-e, \hat{B})\right| + |e_{n+1}| \cdot \left|F_{n+1}(t, x-e, B) - F_{n+1}(t, x-e, \hat{B})\right| \\ &+ \cdots + |e_M| \cdot \left|F_M(t, x-e, B) - F_M(t, x-e, \hat{B})\right| \le G\|e_M\| \cdot \|\tilde{B}\| \end{aligned} \tag{6-4-19}$$

In Eq. (6-4-19), the last term is obtained by Schwarz inequality. Similarly we choose

$$\dot{\hat{B}}_1 = -G\tilde{B}_1 \|e\| / \|\tilde{B}\|, \cdots, \dot{\hat{B}}_M = -G\tilde{B}_M \|e\| / \|\tilde{B}\| \tag{6-4-20}$$

Then

$$\tilde{B}^T \dot{\tilde{B}} = -G(\tilde{B}_1^2 + \cdots \tilde{B}_M^2)\|e\| \Big/ \|\tilde{B}\| = -G\|\tilde{B}\|^2 \|e\| \Big/ \|\tilde{B}\| = -G\|e\| \cdot \|\tilde{B}\|. \qquad (6\text{-}4\text{-}21)$$

Introducing Eqs. (6-4-19), (6-4-21) in Eq. (6-4-9), we can also get lastly

$$\dot{V}(e, \tilde{B}, \tilde{G}) \le -G(e_1^2 + \cdots + e_n^2) \qquad (6\text{-}4\text{-}22)$$

By the same reasoning as when $M=n$, the solution $e=0, \tilde{B}=0, \tilde{G}=0$ is asymptotically stable.

When $M<n$, we put

$$F_i(t, x-e, B) - F_i(t, x-e, \hat{B}) = 0\,, \quad i = M+1, \cdots, n \qquad (6\text{-}4\text{-}23)$$

since $B_{M+1}, \cdots B_N$ do not exist,

$$\tilde{B}_{M+1} = \cdots = \tilde{B}_n = 0 \qquad (6\text{-}4\text{-}24)$$

$$\|\tilde{B}\|^2 = \tilde{B}_1^2 + \cdots + \tilde{B}_M^2 + \tilde{B}_{M+1}^2 + \cdots + \tilde{B}_n^2 \qquad (6\text{-}4\text{-}25)$$

Then by Schwarz inequality,

$$\begin{aligned} & e^T[F(t, x-e, B) - F(t, x-e, \hat{B})] \le |e_1| \cdot \left|F_1(t, x-e, B) - F_1(t, x-e, \hat{B})\right| + \cdots \\ & + |e_M| \cdot \left|F_M(t, x-e, B) - F_M(t, x-e, \hat{B})\right| + |e_{M+1}| \cdot \left|F_{M+1}(t, x-e, B) - F_{M+1}(t, x-e, \hat{B})\right| \\ & + \cdots + |e_n| \cdot \left|F_n(t, x-e, B) - F_n(t, x-e, \hat{B})\right| \le G\|e\| \cdot \|\tilde{B}\| \end{aligned} \qquad (6\text{-}4\text{-}26)$$

Similarly, choose

$$\begin{aligned} & \dot{\tilde{B}}_1 = -G\tilde{B}_1 \|e\| \Big/ \|\tilde{B}\|, \cdots, \dot{\tilde{B}}_M = -G\tilde{B}_M \|e\| \Big/ \|\tilde{B}\|, \dot{\tilde{B}}_{M+1} = -G\tilde{B}_{M+1} \|e\| \Big/ \|\tilde{B}\|, \\ & \qquad \cdots, \dot{\tilde{B}}_n = -G\tilde{B}_n \|e\| \Big/ \|\tilde{B}\| \end{aligned} \qquad (6\text{-}4\text{-}27)$$

$$\tilde{B}^T \dot{\tilde{B}} = -G(\tilde{B}_1^2 + \cdots \tilde{B}_N^2)\|e\| \Big/ \|\tilde{B}\| = -G\|\tilde{B}\|^2 \|e\| \Big/ \|\tilde{B}\| = -G\|e\| \cdot \|\tilde{B}\|. \qquad (6\text{-}4\text{-}28)$$

Introducing Eqs. (6-4-26), (6-4-28) in Eq. (6-4-9), we can also get lastly

$$\dot{V}(e, \tilde{B}, \tilde{G}) \le -G(e_1^2 + \cdots + e_n^2) = -Ge^T e \qquad (6\text{-}4\text{-}29)$$

By the same reasoning as the case $M=n$, the solution $e=0, \tilde{B}=0, \tilde{G}=0$ is asymptotically stable.

Remark. In current scheme of adaptive synchronization [89-93], traditional Lyapunov stability theorem and Babalat lemma are used to prove the error vector approaches zero, as time approaches infinity. But the question, why the estimated parameters also approach to the uncertain parameters, remains no answer. By GYC pragmatical asymptotical stability theorem, the question can be answered strictly. Moreover, the assymtotical stability is global.

6-4-2 Numerical Examples

Case I Periodic parameters for Lorenz system, *M=n*

The master Lorenz system with uncertain variable parameters is

$$\dot{x}_1 = -A_1(t)(x_1 - x_2),\ \dot{x}_2 = A_2(t)x_1 - x_2 - x_1x_3,\ \dot{x}_3 = x_1x_2 - A_3(t)x_3 \tag{6-4-30}$$

where $A_1(t)$, $A_2(t)$ and $A_3(t)$ are uncertain parameters. In simulation, we take

$$A_1(t) = \sigma(1 + d_1 \sin \varpi_1 t),\ A_2(t) = \gamma(1 + d_2 \sin \varpi_2 t),\ A_3(t) = b(1 + d_3 \sin \varpi_3 t) \tag{6-4-31}$$

where σ, γ, b, d_1, d_2, d_3, ϖ_1, ϖ_2 and ϖ_3 are positive constants.

By Eq. (6-4-2), the slave Lorenz system is

$$\begin{cases} \dot{y}_1 = -\hat{A}_1(t)(y_1 - y_2) + (\hat{G} + G)(x_1 - y_1) \\ \dot{y}_2 = \hat{A}_2(t)y_1 - y_2 - y_1y_3 + (\hat{G} + G)(x_2 - y_2) \\ \dot{y}_3 = y_1y_2 - \hat{A}_3(t)y_3 + (\hat{G} + G)(x_3 - y_3) \end{cases} \tag{6-4-32}$$

where $\hat{K} = \hat{G} + G$. $\hat{G}$ is the estimated value of G.

Take $\sigma = 10, \gamma = 28, b = 8/3$, $d_1 = 0.05, d_2 = 0.01, d_3 = 0.1, \varpi_1 = 9, \varpi_2 = 15, \varpi_3 = 18$, and the initial condition be $\begin{bmatrix} x_0^T & y_0^T & \hat{A}_0^T & \hat{G}_0 \end{bmatrix}^T = \begin{bmatrix} 111\,000\,000\,0 \end{bmatrix}^T$.

Subtracting Eq. (6-4-32) from Eq. (6-4-30), we obtain an error dynamics.

$$\begin{aligned} \dot{e}_1 &= -A_1(t)(x_1 - x_2) + \hat{A}_1(t)(y_1 - y_2) - (\hat{G} + G)(x_1 - y_1) \\ \dot{e}_2 &= A_2(t)x_1 - x_2 - x_1x_3 - \hat{A}_2(t)y_1 + y_2 + y_1y_3 - (\hat{G} + G)(x_2 - y_2) \\ \dot{e}_3 &= x_1x_2 - A_3(t)x_3 - y_1y_2 + \hat{A}_3(t)y_3 - (\hat{G} + G)(x_3 - y_3) \end{aligned} \tag{6-4-33}$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2$ and $e_3=x_3-y_3$.

Our aim is

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \quad i = 1, 2, 3 \tag{6-4-34}$$

Let adaptive law be

$$\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e \tag{6-4-35}$$

since G is constant, $\dot{G}=0$. Define

$$\tilde{A}(t)=[\tilde{A}_1(t)\ \ \tilde{A}_2(t)\ \ \tilde{A}_3(t)]^T \tag{6-4-36}$$

$$\tilde{A}_1(t)=A_1(t)-\hat{A}_1(t),\quad \tilde{A}_2(t)=A_2(t)-\hat{A}_2(t),\quad \tilde{A}_3(t)=A_3(t)-\hat{A}_3(t) \tag{6-4-37}$$

then

$$\begin{aligned}&\dot{\tilde{A}}_1(t)=\sigma d_1\varpi_1\cos\varpi_1 t-\dot{\hat{A}}_1(t),\ \ \dot{\tilde{A}}_2(t)=\gamma d_2\varpi_2\cos\varpi_2 t-\dot{\hat{A}}_2(t)\\ &\dot{\tilde{A}}_3(t)=bd_3\varpi_3\cos\varpi_3 t-\dot{\hat{A}}_3(t).\end{aligned} \tag{6-4-38}$$

Choose $\dot{\hat{A}}_1(t)$, $\dot{\hat{A}}_2(t)$ and $\dot{\hat{A}}_3(t)$ as

$$\dot{\hat{A}}_1=-G\tilde{A}_1\|e\|/\|\tilde{A}\|,\ \ \dot{\hat{A}}_2=-G\tilde{A}_2\|e\|/\|\tilde{A}\|,\ \ \dot{\hat{A}}_3=-G\tilde{A}_3\|e\|/\|\tilde{A}\| \tag{6-4-39}$$

Choose a Lyapunov function is given in the form of positive definite function:

$$V(e_1,e_2,e_3,\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,\tilde{G})=\frac{1}{2}(e_1^2+e_2^2+e_3^2+\tilde{A}_1^2+\tilde{A}_2^2+\tilde{A}_3^2+\tilde{G}^2) \tag{6-4-40}$$

Its time derivative along any solution of Eqs. (6-4-33), (6-4-35) and (6-4-39) is

$$\begin{aligned}\dot{V}=&\,e_1[-A_1(t)(x_1-x_2)+\hat{A}_1(t)(y_1-y_2)-(\hat{G}+G)(x_1-y_1)]+e_2[A_2(t)x_1-x_2-x_1x_3\\ &-\hat{A}_2(t)y_1+y_2+y_1y_3-(\hat{G}+G)(x_2-y_2)]+e_3[x_1x_2-A_3(t)x_3-y_1y_2+\hat{A}_3(t)y_3\\ &-(\hat{G}+G)(x_3-y_3)]+\tilde{A}_1\dot{\tilde{A}}_1+\tilde{A}_2\dot{\tilde{A}}_2+\tilde{A}_3\dot{\tilde{A}}_3-\tilde{G}\dot{\hat{G}}\\ \dot{V}=&\,e_1[-A_1(t)(x_1-x_2)+A_1(t)(y_1-y_2)-(\hat{G}+G)(x_1-y_1)]+e_2[A_2(t)x_1-x_2-x_1x_3\\ &-A_2(t)y_1+y_2+y_1y_3-(\hat{G}+G)(x_2-y_2)]+e_3[x_1x_2-A_3(t)x_3-y_1y_2+A_3(t)y_3\\ &-(\hat{G}+G)(x_3-y_3)]+\tilde{A}_1(y_1-y_2)e_1-\tilde{A}_2y_1e_2-\tilde{A}_3y_3e_3-G\|e\|(\tilde{A}_1^2+\tilde{A}_2^2+\tilde{A}_3^2)/\|\tilde{A}\|-\tilde{G}\dot{\hat{G}}\\ \dot{V}\le&\, G\|e\|^2-(\hat{G}+G)\|e\|^2+G\|e\|\|\tilde{A}\|-G\|e\|(\tilde{A}_1^2+\tilde{A}_2^2+\tilde{A}_3^2)/\|\tilde{A}\|-\tilde{G}\dot{\hat{G}}\end{aligned}$$

$\dot{V}$ can be rewritten as

$$\dot{V}\le -G\|e\|^2. \tag{6-4-41}$$

$\dot{V}$ is negative semi-definite function of $e,\tilde{A},\tilde{G}$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-4-33), adaptive laws (6-4-35) and parameter dynamics (6-4-39) is asymptotically stable. Now, D is a 4-manifold, n=7 and the number of error state variables p=3. When $e_i=0$ ($i=1,2,3$) and $\tilde{A}_i,\tilde{G}$ take arbitrary values, $\dot{V}=0$, so X is a 4-manifold, $m=n-p=7-3=4$. $m+1<n$ is satisfied. By

GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained. The equilibrium point $e_i = \tilde{A}_i = \tilde{G} = 0 \quad (i = 1, 2, 3)$ is asymptotically stable. Moreover, the result is global asymptotically stable. (see Appendix A) The numerical results are shown in Fig. 6-8~6-10. The chaos synchronization is accomplished. The coupling strength required is K=2G=38.26.

Case II Exponentially increasing and decreasing parameters for Quantum-CNN system, M=n

For a two-cell Quantum-CNN, the following differential equations are obtained:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2, \ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4, \ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{6-4-42}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. When $a_1 = 6.8$, $a_2 = 4.3$, $\omega_1 = 4.7$ and $\omega_2 = 3.9$, the system is chaotic.

The master Quantum-CNN system with uncertain variable parameters is

$$\begin{cases} \dot{x}_1 = -2A_1(t)\sqrt{1-x_1^2}\sin x_2, \ \dot{x}_2 = -A_3(t)(x_1 - x_3) + 2A_1(t)\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2A_2(t)\sqrt{1-x_3^2}\sin x_4, \ \dot{x}_4 = -A_4(t)(x_3 - x_1) + 2A_2(t)\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{6-4-43}$$

where $A_1(t)$, $A_2(t)$, $A_3(t)$ and $A_4(t)$ are uncertain parameters. In simulation, we take

$$\begin{aligned} &A_1(t) = a_1[1 + c_1(1 - e^{-b_1 t})], \ A_2(t) = a_2[1 + c_2(1 - e^{-b_2 t})] \\ &A_3(t) = \omega_1[1 + c_3(1 - e^{-b_3 t})], \ A_4(t) = \omega_2[1 + c_4(1 - e^{-b_4 t})] \end{aligned} \tag{6-4-44}$$

where b_1, b_2, b_3, b_4, c_1, c_2, c_3 and c_4 are constants. Take b_1=0.05, b_2=0.004, b_3=0.004, b_4=0.005, c_1=−0.25, c_2=0.15, c_3=−0.2 and c_4=0.1.

By Eq. (6-4-2), the slave Quantum-CNN system is

$$
\begin{cases}
\dot{y}_1 = -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 + (\hat{G}+G)(x_1-y_1) \\
\dot{y}_2 = -\hat{\omega}_1(y_1-y_3) + 2\hat{a}_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + (\hat{G}+G)(x_2-y_2) \\
\dot{y}_3 = -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 + (\hat{G}+G)(x_3-y_3) \\
\dot{y}_4 = -\hat{\omega}_2(y_3-y_1) + 2\hat{a}_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + (\hat{G}+G)(x_4-y_4)
\end{cases}
\tag{6-4-45}
$$

where $\hat{K} = \hat{G}+G$. $\hat{G}$ is the estimated value of G. The initial values are taken as $x_1(0)=0.8$, $x_2(0)=-0.77$, $x_3(0)=-0.72$, $x_4(0)=0.57$, $y_1(0)=-0.2$, $y_2(0)=0.41$, $y_3(0)=0.25$, $y_4(0)=-0.81$ and $\begin{bmatrix}\hat{a}_{10} & \hat{a}_{20} & \hat{\omega}_{10} & \hat{\omega}_{20} & \hat{G}_0\end{bmatrix}^T = [00\ 00\ 0]^T$. The error dynamic is

$$
\begin{aligned}
\dot{e}_1 &= -2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1 \\
\dot{e}_2 &= -A_3(t)(x_1-x_3) + 2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + \hat{\omega}_1(y_1-y_3) - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2 \\
\dot{e}_3 &= -2A_2(t)\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3 \\
\dot{e}_4 &= -A_4(t)(x_3-x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + \hat{\omega}_2(y_3-y_1) - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4
\end{aligned}
\tag{6-4-46}
$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2$, $e_3=x_3-y_3$ and $e_4=x_4-y_4$.

Our aim is

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i-y_i) = 0, \quad i=1,2,3,4 \tag{6-4-47}$$

Let adaptive law be

$$\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e \tag{6-4-48}$$

since G is constant, $\dot{G}=0$. Define

$$\tilde{a}_1 = A_1(t)-\hat{a}_1, \quad \tilde{a}_2 = A_2(t)-\hat{a}_2, \quad \tilde{\omega}_1 = A_3(t)-\hat{\omega}_1, \quad \tilde{\omega}_2 = A_4(t)-\hat{\omega}_2 \tag{6-4-49}$$

then

$$\dot{\tilde{a}}_1 = a_1b_1c_1e^{-b_1t} - \dot{\hat{a}}_1,\ \dot{\tilde{a}}_2 = a_2b_2c_2e^{-b_2t} - \dot{\hat{a}}_2,\ \dot{\tilde{\omega}}_1 = \omega_1b_3c_3e^{-b_3t} - \dot{\hat{\omega}}_1,\ \dot{\tilde{\omega}}_2 = \omega_2b_4c_4e^{-b_4t} - \dot{\hat{\omega}}_2 \tag{6-4-50}$$

Let

$$\tilde{A} = [\tilde{a}_1\ \tilde{a}_2\ \tilde{\omega}_1\ \tilde{\omega}_2] \tag{6-4-51}$$

Choose $\dot{\hat{a}}_1$, $\dot{\hat{a}}_2$, $\dot{\hat{\omega}}_1$ and $\dot{\hat{\omega}}_2$ as

$$\dot{\tilde{a}}_1 = -G\tilde{a}_1\|e\|/\|\tilde{A}\|,\ \dot{\tilde{\omega}}_1 = -G\tilde{\omega}_1\|e\|/\|\tilde{A}\|,\ \dot{\tilde{a}}_2 = -G\tilde{a}_2\|e\|/\|\tilde{A}\|\ and\ \dot{\tilde{\omega}}_2 = -G\tilde{\omega}_2\|e\|/\|\tilde{A}\|. \quad (6\text{-}4\text{-}52)$$

A Lyapunov function is given in the form of positive definite function:

$$V(e_1, e_2, e_3, e_4, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{G}) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{G}^2) \quad (6\text{-}4\text{-}53)$$

Its time derivative along any solution of Eqs. (6-4-46), (6-4-48) and (6-4-52) is

$$\begin{aligned}
\dot{V} &= e_1[-2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1] + e_2[-A_3(t)(x_1-x_3) + 2A_1(t) \\
&\times \frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + \hat{\omega}_1(y_1-y_3) - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2] + e_3[-2A_2(t)\sqrt{1-x_3^2}\sin x_4 \\
&+ 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3] + e_4[-A_4(t)(x_3-x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + \hat{\omega}_2(y_3-y_1) \\
&- 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4] + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{\omega}_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\dot{\tilde{\omega}}_2 - \tilde{G}\dot{\hat{G}}
\end{aligned}$$

$$\begin{aligned}
\dot{V} &= e_1[-2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2A_1(t)\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1] + e_2[-A_3(t)(x_1-x_3) + 2A_1(t) \\
&\times \frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + A_3(t)(y_1-y_3) - 2A_1(t)\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2] + e_3[-2A_2(t)\sqrt{1-x_3^2}\sin x_4 \\
&+ 2A_2(t)\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3] + e_4[-A_4(t)(x_3-x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + A_4(t)(y_3-y_1) \\
&- 2A_2(t)\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4] + \tilde{a}_1[2\sqrt{1-y_1^2}\sin y_2 e_1 - \frac{2y_1}{\sqrt{1-y_1^2}}\cos y_2 e_2] + \tilde{\omega}_1[(y_1-y_3)e_2] \\
&+ \tilde{a}_2[2\sqrt{1-y_3^2}\sin y_4 e_3 - \frac{2y_3}{\sqrt{1-y_3^2}}\cos y_4 e_4] + \tilde{\omega}_2[(y_3-y_1)e_4] - G\|e\|(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}
\end{aligned}$$

$$\dot{V} \le G\|e\|^2 - (\hat{G}+G)\|e\|^2 + G\|e\|\|\tilde{A}\| - G\|e\|(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}$$

$\dot{V}$ can be rewritten as

$$\dot{V} \le -G(e_1^2 + e_2^2 + e_3^2 + e_4^2). \quad (6\text{-}4\text{-}54)$$

$\dot{V}$ is a negative semi-definite function of $e_i, \tilde{a}_j, \tilde{\omega}_j, \tilde{G}$ ($i=1,2,3,4$; $j=1,2$). The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-4-46), adaptive laws (6-4-48) and parameter dynamics (6-4-52) is asymptotically stable. Now, D is a 5-manifold, n=9 and the number of error state variables p=4. When $e_i = 0$ ($i=1,2,3,4$) and $\tilde{a}_j, \tilde{\omega}_j, \tilde{G}$ ($j=1,2$) take arbitrary values, $\dot{V}=0$, so X is a 5-manifold,

m=n–p=9−4=5. m+1<n is satisfied. By GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained. The equilibrium point $e_i = \tilde{a}_j = \tilde{\omega}_j = \tilde{G} = 0 \ \ (i = 1,2,3,4\,;\, j = 1,2)$ is asymptotically stable. Moreover, the result is global asymptotically stable. (see Appendix A) The numerical results are shown in Fig. 6-11~6-13. The chaos synchronization is accomplished. The coupling strength required is K=2G=5.54.

Case III Periodically and exponentially increasing and decreasing parameters for Quantum-CNN system, $M < n$

The master Quantum-CNN system with uncertain variable parameters is

$$\begin{cases} \dot{x}_1 = -2A_1(t)\sqrt{1-x_1^2}\sin x_2,\ \ \dot{x}_2 = -A_3(t)(x_1 - x_3) + 2A_1(t)\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2A_2(t)\sqrt{1-x_3^2}\sin x_4,\ \ \dot{x}_4 = -A_4(t)(x_3 - x_1) + 2A_2(t)\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{6-4-55}$$

where $A_1(t)$, $A_2(t)$, $A_3(t)$ and $A_4(t)$ are uncertain parameters. In simulation, we take

$$\begin{aligned} &A_1(t) = a_1[1 + c_1(1 - e^{-b_1 t}\sin \varpi_1 t)],\ \ A_2(t) = a_2[1 + c_2(1 - e^{-b_2 t}\sin \varpi_2 t)], \\ &A_3(t) = \omega_1[1 + c_3(1 - e^{-b_3 t}\sin \varpi_3 t)],\ \ A_4(t) = \omega_2[1 + c_4(1 - e^{-b_4 t}\sin \varpi_4 t)]. \end{aligned} \tag{6-4-56}$$

where b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, ϖ_1, ϖ_2, ϖ_3 and ϖ_4 are constants. Take b_1=0.001, b_2=0.002, b_3=0.004, b_4=0.005, c_1=−0.25, c_2=0.15, c_3=−0.2, c_4=0.1, $\varpi_1 = 5$, $\varpi_2 = 1$, $\varpi_3 = 3$ and $\varpi_4 = 6$. System (6-4-55) is chaotic.

By Eq. (6-4-2), the slave Quantum-CNN system is

$$\begin{cases} \dot{y}_1 = -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)(y_1 - x_1) \\ \dot{y}_2 = -\hat{\omega}_1(y_1 - y_3) + 2\hat{a}_1\dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)(y_2 - x_2) \\ \dot{y}_3 = -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)(y_3 - x_3) \\ \dot{y}_4 = -A_4(t)(y_3 - y_1) + 2\hat{a}_2\dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)(y_4 - x_4) \end{cases} \tag{6-4-57}$$

where $\hat{K} = \hat{G} + G$. $\hat{G}$ is the estimated value of G. The error dynamic is

$$\begin{aligned}
\dot{e}_1 &= -2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1 \\
\dot{e}_2 &= -A_3(t)(x_1-x_3) + 2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + \hat{\omega}_1(y_1-y_3) - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2 \\
\dot{e}_3 &= -2A_2(t)\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3 \\
\dot{e}_4 &= -A_4(t)(x_3-x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + A_4(t)(y_3-y_1) - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4
\end{aligned} \quad (6\text{-}4\text{-}58)$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2$, $e_3=x_3-y_3$ and $e_4=x_4-y_4$.

Our aim is

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \quad i = 1, 2, 3, 4 \quad (6\text{-}4\text{-}59)$$

Let adaptive law be

$$\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e \quad (6\text{-}4\text{-}60)$$

since G is constant, $\dot{G} = 0$. Define

$$\tilde{a}_1 = A_1(t) - \hat{a}_1, \quad \tilde{a}_2 = A_2(t) - \hat{a}_2, \quad \tilde{\omega}_1 = A_3(t) - \hat{\omega}_1, \quad (6\text{-}4\text{-}61)$$

then

$$\begin{aligned}
\dot{\tilde{a}}_1 &= a_1 b_1 c_1 e^{-b_1 t}\sin\varpi_1 t - a_1 c_1 \varpi_1 e^{-b_1 t}\cos\varpi_1 t - \dot{\hat{a}}_1 \\
\dot{\tilde{\omega}}_1 &= \omega_1 b_3 c_3 e^{-b_3 t}\sin\varpi_3 t - \omega_1 c_3 \varpi_3 e^{-b_3 t}\cos\varpi_3 t - \dot{\hat{\omega}}_1 \\
\dot{\tilde{a}}_2 &= a_2 b_2 c_2 e^{-b_2 t}\sin\varpi_2 t - a_2 c_2 \varpi_2 e^{-b_2 t}\cos\varpi_2 t - \dot{\hat{a}}_2
\end{aligned} \quad (6\text{-}4\text{-}62)$$

Let

$$\tilde{A} = [\tilde{a}_1\ \tilde{a}_2\ \tilde{\omega}_1] \quad (6\text{-}4\text{-}63)$$

Choose $\dot{\tilde{a}}_1$, $\dot{\tilde{a}}_2$ and $\dot{\tilde{\omega}}_1$ as

$$\dot{\tilde{a}}_1 = -G\tilde{a}_1 \|e\| / \|\tilde{A}\|, \ \dot{\tilde{\omega}}_1 = -G\tilde{\omega}_1 \|e\| / \|\tilde{A}\| \text{ and } \dot{\tilde{a}}_2 = -G\tilde{a}_2 \|e\| / \|\tilde{A}\|. \quad (6\text{-}4\text{-}64)$$

A Lyapunov function is given in the form of positive definite function:

$$V(e_1, e_2, e_3, e_4, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{G}) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{G}^2) \quad (6\text{-}4\text{-}65)$$

Its time derivative along any solution of Eqs. (6-4-58), (6-4-60) and (6-4-64) is

$$\dot{V} = e_1[-2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1] + e_2[-A_3(t)(x_1-x_3) + 2A_1(t)$$
$$\times\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + \hat{\omega}_1(y_1-y_3) - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2] + e_3[-2A_2(t)\sqrt{1-x_3^2}\sin x_4$$
$$+2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3] + e_4[-A_4(t)(x_3-x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + A_4(t)(y_3-y_1)$$
$$-2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4] + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{\omega}_1\dot{\tilde{\omega}}_1 - \tilde{G}\dot{\hat{G}}$$

$$\dot{V} = e_1[-2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2A_1(t)\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1] + e_2[-A_3(t)(x_1-x_3) + 2A_1(t)$$
$$\times\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + A_3(t)(y_1-y_3) - 2A_1(t)\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2] + e_3[-2A_2(t)\sqrt{1-x_3^2}\sin x_4$$
$$+2A_2(t)\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3] + e_4[-A_4(t)(x_3-x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + A_4(t)(y_3-y_1)$$
$$-2A_2(t)\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4] + \tilde{a}_1[2\sqrt{1-y_1^2}\sin y_2 e_1 - \frac{2y_1}{\sqrt{1-y_1^2}}\cos y_2 e_2] + \tilde{\omega}_1[(y_1-y_3)e_2]$$
$$+\tilde{a}_2[2\sqrt{1-y_3^2}\sin y_4 e_3 - \frac{2y_3}{\sqrt{1-y_3^2}}\cos y_4 e_4] - G\|e\|(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}$$

$$\dot{V} \le G\|e\|^2 - (\hat{G}+G)\|e\|^2 + G\|e\|\|\tilde{A}\| - G\|e\|(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}$$

$\dot{V}$ can be rewritten as

$$\dot{V} \le -G(e_1^2 + e_2^2 + e_3^2 + e_4^2). \tag{6-4-66}$$

$\dot{V}$ is a negative semi-definite function of $e_i, \tilde{a}_j, \tilde{\omega}_1, \tilde{G}$ ($i=1,2,3,4$; $j=1,2$). The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-4-58), adaptive laws (6-4-60) and parameter dynamics (6-4-64) is asymptotically stable. Now, *D* is a 4-manifold, *n*=8 and the number of error state variables *p*=4. When $e_i = 0$ ($i=1,2,3,4$) and $\tilde{a}_j, \tilde{\omega}_1, \tilde{G}$ ($j=1,2$) take arbitrary values, $\dot{V}=0$, so *X* is a 4-manifold, *m*=*n*–*p*=8−4=4. *m*+1<*n* is satisfied. By GYC pragmatical asymptotical stability theorem, error vector *e* approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained. The equilibrium point $e_i = \tilde{a}_j = \tilde{\omega}_1 = \tilde{G} = 0$ ($i=1,2,3,4$; $j=1,2$) is asymptotically stable. Moreover, the result is global asymptotically stable. (see Appendix A) The numerical results are shown in Fig. 6-14~6-15. The chaos synchronization is accomplished. The coupling strength required is K=2G=6.34.

Case IV Chaotic parameters for Lorenz system, M=n

The master Lorenz system with uncertain chaotic parameters is

$$\dot{x}_1 = -A_1(t)(x_1 - x_2),\ \ \dot{x}_2 = A_2(t)x_1 - x_2 - x_1x_3,\ \ \dot{x}_3 = x_1x_2 - A_3(t)x_3 \tag{6-4-67}$$

where $A_1(t)$, $A_2(t)$ and $A_3(t)$ are uncertain parameters. In simulation, we take

$$A_1(t) = \sigma(1 + d_1z_1),\ \ A_2(t) = \gamma(1 + d_2z_2),\ \ A_3(t) = b(1 + d_3z_3) \tag{6-4-68}$$

where σ, γ, b,d_1, d_2 and d_3 are positive constants.

The chaotic signal of system

$$\dot{z}_1 = -\sigma_2(z_1 - z_2),\ \ \dot{z}_2 = \gamma_2z_1 - z_2 - z_1z_3,\ \ \dot{z}_3 = z_1z_2 - b_3z_3 \tag{6-4-69}$$

where σ_2=8, γ_2=27, b_2=3.2 and $\left[z_0^T\right]^T = [222]^T$.

By Eq. (6-4-2), the slave Lorenz system is

$$\begin{cases} \dot{y}_1 = -\hat{A}_1(t)(y_1 - y_2) + (\hat{G} + G)(x_1 - y_1) \\ \dot{y}_2 = \hat{A}_2(t)y_1 - y_2 - y_1y_3 + (\hat{G} + G)(x_2 - y_2) \\ \dot{y}_3 = y_1y_2 - \hat{A}_3(t)y_3 + (\hat{G} + G)(x_3 - y_3) \end{cases} \tag{6-4-70}$$

where $\hat{K} = \hat{G} + G$. $\hat{G}$ is the estimated value of G.

Take $\sigma = 10, \gamma = 28, b = 8/3$, and the initial condition be $\left[x_0^T\ \ y_0^T\ \ \hat{A}_0^T\ \ \hat{G}_0\right]^T$ = $[111\,000\,000\,0]^T$. Take d_1=0.005, d_2=0.001 and d_3=0.009.

Subtracting Eq. (6-4-70) from Eq. (6-4-67), we obtain an error dynamics.

$$\begin{aligned} \dot{e}_1 &= -A_1(t)(x_1 - x_2) + \hat{A}_1(t)(y_1 - y_2) - (\hat{G} + G)(x_1 - y_1) \\ \dot{e}_2 &= A_2(t)x_1 - x_2 - x_1x_3 - \hat{A}_2(t)y_1 + y_2 + y_1y_3 - (\hat{G} + G)(x_2 - y_2) \\ \dot{e}_3 &= x_1x_2 - A_3(t)x_3 - y_1y_2 + \hat{A}_3(t)y_3 - (\hat{G} + G)(x_3 - y_3) \end{aligned} \tag{6-4-71}$$

where e_1=x_1−y_1, e_2=x_2−y_2 and e_3=x_3−y_3.

Our aim is

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \quad i = 1, 2, 3 \tag{6-4-72}$$

Let adaptive law be

$$\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e \tag{6-4-73}$$

since G is constant, $\dot{G} = 0$. Define

$$\tilde{A}(t) = [\tilde{A}_1(t) \;\; \tilde{A}_2(t) \;\; \tilde{A}_3(t)]^T \tag{6-4-74}$$

$$\tilde{A}_1(t) = A_1(t) - \hat{A}_1(t), \quad \tilde{A}_2(t) = A_2(t) - \hat{A}_2(t), \quad \tilde{A}_3(t) = A_3(t) - \hat{A}_3(t) \tag{6-4-75}$$

then

$$\dot{\tilde{A}}_1(t) = \sigma d_1 \dot{z}_1 - \dot{\hat{A}}_1(t), \; \dot{\tilde{A}}_2(t) = \gamma d_2 \dot{z}_2 - \dot{\hat{A}}_2(t), \; \dot{\tilde{A}}_3(t) = b d_3 \dot{z}_3 - \dot{\hat{A}}_3(t) \tag{6-4-76}$$

Choose $\dot{\tilde{A}}_1(t)$, $\dot{\tilde{A}}_2(t)$ and $\dot{\tilde{A}}_3(t)$ as

$$\dot{\tilde{A}}_1 = -G\tilde{A}_1 \|e\| / \|\tilde{A}\|, \; \dot{\tilde{A}}_2 = -G\tilde{A}_2 \|e\| / \|\tilde{A}\|, \; \dot{\tilde{A}}_3 = -G\tilde{A}_3 \|e\| / \|\tilde{A}\| \tag{6-4-77}$$

A Lyapunov function is given in the form of positive definite function:

$$V(e_1, e_2, e_3, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{G}) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + \tilde{A}_1^2 + \tilde{A}_2^2 + \tilde{A}_3^2 + \tilde{G}^2) \tag{6-4-78}$$

Its time derivative along any solution of Eqs. (6-4-71), (6-4-73) and (6-4-77) is

$$\begin{aligned}\dot{V} &= e_1[-A_1(t)(x_1 - x_2) + \hat{A}_1(t)(y_1 - y_2) - (\hat{G} + G)(x_1 - y_1)] + e_2[A_2(t)x_1 - x_2 - x_1x_3 \\ &\quad - \hat{A}_2(t)y_1 + y_2 + y_1y_3 - (\hat{G} + G)(x_2 - y_2)] + e_3[x_1x_2 - A_3(t)x_3 - y_1y_2 + \hat{A}_3(t)y_3 \\ &\quad - (\hat{G} + G)(x_3 - y_3)] + \tilde{A}_1\dot{\tilde{A}}_1 + \tilde{A}_2\dot{\tilde{A}}_2 + \tilde{A}_3\dot{\tilde{A}}_3 - \tilde{G}\dot{\hat{G}}\end{aligned}$$

$$\begin{aligned}\dot{V} &= e_1[-A_1(t)(x_1 - x_2) + A_1(t)(y_1 - y_2) - (\hat{G} + G)(x_1 - y_1)] + e_2[A_2(t)x_1 - x_2 - x_1x_3 \\ &\quad - A_2(t)y_1 + y_2 + y_1y_3 - (\hat{G} + G)(x_2 - y_2)] + e_3[x_1x_2 - A_3(t)x_3 - y_1y_2 + A_3(t)y_3 \\ &\quad - (\hat{G} + G)(x_3 - y_3)] + \tilde{A}_1(y_1 - y_2)e_1 - \tilde{A}_2 y_1 e_2 - \tilde{A}_3 y_3 e_3 - G\|e\|(\tilde{A}_1^2 + \tilde{A}_2^2 + \tilde{A}_3^2)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}\end{aligned}$$

$$\dot{V} \le G\|e\|^2 - (\hat{G} + G)\|e\|^2 + G\|e\|\|\tilde{A}\| - G\|e\|\|\tilde{A}\|^2 / \|\tilde{A}\| - \tilde{G}\dot{\hat{G}}$$

$\dot{V}$ can be rewritten as

$$\dot{V}(e_i) \le -G\|e\|^2 . \tag{6-4-79}$$

$\dot{V}$ is negative semi-definite function of $e, \tilde{A}, \tilde{G}$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-4-71), adaptive laws (6-4-73) and parameter dynamics (6-4-77) is asymptotically stable. Now, D is a 4-manifold, n=7 and the number of error state variables p=3. When $e_i = 0$, $(i = 1,2,3)$ and

$\tilde{A}_i, \tilde{G}$ take arbitrary values, $\dot{V}=0$, so X is a 4-manifold, $m=n-p=7-3=4$. $m+1<n$ is satisfied. By GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained. The equilibrium point $e_i = \tilde{A}_i = \tilde{G} = 0$ $(i = 1,2,3)$ is asymptotically stable. Moreover, the result is global asymptotically stable. (see Appendix A) The numerical results are shown in Fig. 6-16~6-18. The chaos synchronization is accomplished. The coupling strength required is K=2G =39.04.

Case V Chaotic parameters for Lorenz system, $M<n$

The master Lorenz system with uncertain chaotic parameters is

$$\dot{x}_1 = -A_1(t)(x_1 - x_2),\ \dot{x}_2 = A_2(t)x_1 - x_2 - x_1x_3,\ \dot{x}_3 = x_1x_2 - A_3(t)x_3 \tag{6-4-80}$$

where $A_1(t)$, $A_2(t)$ and $A_3(t)$ are uncertain parameters. In simulation, we take

$$A_1(t) = \sigma(1 + d_1z_1),\ A_2(t) = \gamma(1 + d_2z_2),\ A_3(t) = b(1 + d_3z_3) \tag{6-4-81}$$

where d_1, d_2 and d_3 are positive constants.

The chaotic signal of system

$$\dot{z}_1 = -\sigma_2(z_1 - z_2),\ \dot{z}_2 = \gamma_2z_1 - z_2 - z_1z_3,\ \dot{z}_3 = z_1z_2 - b_3z_3 \tag{6-4-82}$$

where σ_2=8, γ_2=27, b_2=3.2 and $\left[z_0^T\right]^T = [222]^T$.

By Eq. (6-4-2), the slave Lorenz system is

$$\begin{cases} \dot{y}_1 = -\hat{A}_1(t)(y_1 - y_2) + (\hat{G} + G)(x_1 - y_1) \\ \dot{y}_2 = \hat{A}_2(t)y_1 - y_2 - y_1y_3 + (\hat{G} + G)(x_2 - y_2) \\ \dot{y}_3 = y_1y_2 - A_3(t)y_3 + (\hat{G} + G)(x_3 - y_3) \end{cases} \tag{6-4-83}$$

The initial condition be $\left[x_0^T\ \ y_0^T\ \ \hat{A}_0^T\ \ \hat{G}_0\right]^T$ = $[111\,000\,00\,0]^T$.

Subtracting Eq. (6-4-83) from Eq. (6-4-80), we obtain an error dynamics.

$$\begin{aligned} \dot{e}_1 &= -A_1(t)(x_1 - x_2) + \hat{A}_1(t)(y_1 - y_2) - (\hat{G} + G)(x_1 - y_1) \\ \dot{e}_2 &= A_2(t)x_1 - x_2 - x_1x_3 - \hat{A}_2(t)y_1 + y_2 + y_1y_3 - (\hat{G} + G)(x_2 - y_2) \\ \dot{e}_3 &= x_1x_2 - A_3(t)x_3 - y_1y_2 + A_3(t)y_3 - (\hat{G} + G)(x_3 - y_3) \end{aligned} \tag{6-4-84}$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2$ and $e_3=x_3-y_3$.

Our aim is

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \quad i = 1, 2, 3 \tag{6-4-85}$$

Let adaptive law be

$$\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e \tag{6-4-86}$$

since G is constant, $\dot{G} = 0$. Define

$$\tilde{A}(t) = [\tilde{A}_1(t) \;\; \tilde{A}_2(t)]^T \tag{6-4-87}$$

$$\tilde{A}_1(t) = A_1(t) - \hat{A}_1(t), \quad \tilde{A}_2(t) = A_2(t) - \hat{A}_2(t) \tag{6-4-88}$$

then

$$\dot{\tilde{A}}_1(t) = \sigma d_1 \dot{z}_1 - \dot{\hat{A}}_1(t), \;\; \dot{\tilde{A}}_2(t) = \gamma d_2 \dot{z}_2 - \dot{\hat{A}}_2(t) \tag{6-4-89}$$

Choose $\dot{\tilde{A}}_1(t)$ and $\dot{\tilde{A}}_2(t)$ as

$$\dot{\tilde{A}}_1 = -G\tilde{A}_1 \|e\| / \|\tilde{A}\|, \;\; \dot{\tilde{A}}_2 = -G\tilde{A}_2 \|e\| / \|\tilde{A}\|. \tag{6-4-90}$$

A Lyapunov function is given in the form of positive definite function:

$$V(e_1, e_2, e_3, \tilde{A}_1, \tilde{A}_2, \tilde{G}) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + \tilde{A}_1^2 + \tilde{A}_2^2 + \tilde{G}^2) \tag{6-4-91}$$

Its time derivative along any solution of Eqs. (6-4-84), (6-4-86) and (6-4-90) is

$$\begin{aligned}\dot{V} = {} & e_1[-A_1(t)(x_1 - x_2) + \hat{A}_1(t)(y_1 - y_2) - (\hat{G} + G)(x_1 - y_1)] + e_2[A_2(t)x_1 - x_2 - x_1 x_3 \\ & - \hat{A}_2(t)y_1 + y_2 + y_1 y_3 - (\hat{G} + G)(x_2 - y_2)] + e_3[x_1 x_2 - A_3(t)x_3 - y_1 y_2 + A_3(t)y_3 \\ & - (\hat{G} + G)(x_3 - y_3)] + \tilde{A}_1\dot{\tilde{A}}_1 + \tilde{A}_2\dot{\tilde{A}}_2 - \tilde{G}\dot{\hat{G}}\end{aligned}$$

$$\begin{aligned}\dot{V} = {} & e_1[-A_1(t)(x_1 - x_2) + A_1(t)(y_1 - y_2) - (\hat{G} + G)(x_1 - y_1)] + e_2[A_2(t)x_1 - x_2 - x_1 x_3 \\ & - A_2(t)y_1 + y_2 + y_1 y_3 - (\hat{G} + G)(x_2 - y_2)] + e_3[x_1 x_2 - A_3(t)x_3 - y_1 y_2 + A_3(t)y_3 \\ & - (\hat{G} + G)(x_3 - y_3)] + \tilde{A}_1(y_1 - y_2)e_1 - \tilde{A}_2 y_1 e_2 - G\|e\|(\tilde{A}_1^2 + \tilde{A}_2^2) / \|\tilde{A}\| - \tilde{G}\dot{\hat{G}}\end{aligned}$$

$$\dot{V} \le G\|e\|^2 - (\hat{G} + G)\|e\|^2 + G\|e\|\|\tilde{A}\| - G\|e\|\|\tilde{A}\|^2 / \|\tilde{A}\| - \tilde{G}\dot{\hat{G}}$$

$\dot{V}$ can be rewritten as

$$\dot{V}(e_i) \le -G\|e\|^2 . \tag{6-4-92}$$

$\dot{V}$ is negative semi-definite function of $e, \tilde{A}, \tilde{G}$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-4-84), adaptive laws (6-4-86) and parameter dynamics (6-4-90) is asymptotically stable. Now, D is a 3-manifold, n=6 and the number of error state variables p=3. When $e_i = 0$, $(i = 1,2,3)$ and $\tilde{A}_j, \tilde{G}$, $(j = 1,2)$ take arbitrary values, $\dot{V} = 0$, so X is a 3-manifold, m=n–p=6−3=3. m+1<n is satisfied. By GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained. The equilibrium point $e_i = \tilde{A}_j = \tilde{G} = 0$ $(i = 1,2,3;\ j = 1,2)$ is asymptotically stable. Moreover, the result is global asymptotically stable. (see Appendix A) The numerical results are shown in Fig. 6-19. The chaos synchronization is accomplished. The coupling strength required is K=2G =39.34.

Case VI Chaotic parameters for Quantum-CNN system, M=n

The master Quantum-CNN system with uncertain chaotic parameters is

$$\begin{cases} \dot{x}_1 = -2A_1(t)\sqrt{1-x_1^2}\sin x_2,\ \dot{x}_2 = -A_3(t)(x_1-x_3) + 2A_1(t)\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2A_2(t)\sqrt{1-x_3^2}\sin x_4,\ \dot{x}_4 = -A_4(t)(x_3-x_1) + 2A_2(t)\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{6-4-93}$$

where $A_1(t)$, $A_2(t)$, $A_3(t)$ and $A_4(t)$ are uncertain parameters. In simulation, we take

$$A_1(t) = a_1(1+d_1z_1),\ A_2(t) = a_2(1+d_2z_2),\ A_3(t) = \omega_1(1+d_3z_3),\ A_4(t) = \omega_2(1+d_4z_4). \tag{6-4-94}$$

where d_1, d_2 and d_3 are positive constants. Take d_1=0.039, d_2=0.043, d_3=0.045 and d_4=0.038.

The chaotic signal of system

$$\begin{cases} \dot{z}_1 = -2a_{21}\sqrt{1-z_1^2}\sin z_2,\ \dot{z}_2 = -\omega_{21}(z_1-z_3) + 2a_{21}\dfrac{z_1}{\sqrt{1-z_1^2}}\cos z_2 \\ \dot{z}_3 = -2a_{22}\sqrt{1-z_3^2}\sin z_4,\ \dot{z}_4 = -\omega_{22}(z_3-z_1) + 2a_{22}\dfrac{z_3}{\sqrt{1-z_3^2}}\cos z_4 \end{cases} \tag{6-4-95}$$

where a_{21}=5.2, a_{22}=4.2, ω_{21}=4.7 and ω_{22}=3.5,

By Eq. (6-4-2), the slave Quantum-CNN system is

$$
\begin{cases}
\dot{y}_1 = -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 + (\hat{G}+G)(x_1 - y_1) \\
\dot{y}_2 = -\hat{\omega}_1(y_1 - y_3) + 2\hat{a}_1 \dfrac{y_1}{\sqrt{1-y_1^2}}\cos y_2 + (\hat{G}+G)(x_2 - y_2) \\
\dot{y}_3 = -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 + (\hat{G}+G)(x_3 - y_3) \\
\dot{y}_4 = -\hat{\omega}_2(y_3 - y_1) + 2\hat{a}_2 \dfrac{y_3}{\sqrt{1-y_3^2}}\cos y_4 + (\hat{G}+G)(x_4 - y_4)
\end{cases} \tag{6-4-96}
$$

Subtracting Eq. (96) from Eq. (93), we obtain an error dynamics. The initial values are taken as $z_1(0)=0.5$, $z_2(0)=-0.3$, $z_3(0)=0.1$, and $z_4(0)=0.2$. The error dynamic is

$$
\begin{aligned}
\dot{e}_1 &= -2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1 \\
\dot{e}_2 &= -A_3(t)(x_1 - x_3) + 2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + \hat{\omega}_1(y_1 - y_3) - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2 \\
\dot{e}_3 &= -2A_2(t)\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3 \\
\dot{e}_4 &= -A_4(t)(x_3 - x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + \hat{\omega}_2(y_3 - y_1) - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4
\end{aligned} \tag{6-4-97}
$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2$, $e_3=x_3-y_3$ and $e_4=x_4-y_4$.

Our aim is

$$
\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \quad i = 1, 2, 3, 4 \tag{6-4-98}
$$

Let adaptive law be

$$
\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e \tag{6-4-99}
$$

since G is constant, $\dot{G} = 0$. Define

$$
\tilde{A}(t) = [\tilde{a}_1(t)\ \ \tilde{a}_2(t)\ \ \tilde{\omega}_1(t)\ \ \tilde{\omega}_2(t)]^T \tag{6-4-100}
$$

$$
\tilde{a}_1 = A_1(t) - \hat{a}_1,\ \tilde{a}_2 = A_2(t) - \hat{a}_2,\ \tilde{\omega}_1 = A_3(t) - \hat{\omega}_1,\ \tilde{\omega}_2 = A_4(t) - \hat{\omega}_2, \tag{6-4-101}
$$

$$
\dot{\tilde{a}}_1 = a_1 d_1 \dot{z}_1 - \dot{\hat{a}}_1,\ \dot{\tilde{a}}_2 = a_2 d_2 \dot{z}_2 - \dot{\hat{a}}_2,\ \dot{\tilde{\omega}}_1 = \omega_1 d_3 \dot{z}_3 - \dot{\hat{\omega}}_1,\ \dot{\tilde{\omega}}_2 = \omega_2 d_4 \dot{z}_4 - \dot{\hat{\omega}}_2. \tag{6-4-102}
$$

Choose $\dot{\hat{a}}_1$, $\dot{\hat{a}}_2$, $\dot{\hat{\omega}}_1$ and $\dot{\hat{\omega}}_2$ as

$$
\dot{\hat{a}}_1 = -G\tilde{a}_1\|e\|/\|\tilde{A}(t)\|,\ \dot{\hat{\omega}}_1 = -G\tilde{\omega}_1\|e\|/\|\tilde{A}(t)\|,\ \dot{\hat{a}}_2 = -G\tilde{a}_2\|e\|/\|\tilde{A}(t)\|,\ \dot{\hat{\omega}}_2 = -G\tilde{\omega}_2\|e\|/\|\tilde{A}(t)\|. \tag{6-4-103}
$$

A Lyapunov function is given in the form of positive definite function:

$$
V(e_1, e_2, e_3, e_4, \tilde{a}_1, \tilde{a}_2, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{G}) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{G}^2) \tag{6-4-104}
$$

Its time derivative along any solution of Eqs. (6-4-97), (6-4-99) and (6-4-103) is

$$\dot{V} = e_1[-2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2\hat{a}_1\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1] + e_2[-A_3(t)(x_1-x_3) + 2A_1(t) \times \frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + \hat{\omega}_1(y_1-y_3) - 2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2] + e_3[-2A_2(t)\sqrt{1-x_3^2}\sin x_4 + 2\hat{a}_2\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3] + e_4[-A_4(t)(x_3-x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + \hat{\omega}_2(y_3-y_1) - 2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4] + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{\omega}_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\dot{\tilde{\omega}}_2 - \tilde{G}\dot{\hat{G}}$$

$$\dot{V} = e_1[-2A_1(t)\sqrt{1-x_1^2}\sin x_2 + 2A_1(t)\sqrt{1-y_1^2}\sin y_2 - (\hat{G}+G)e_1] + e_2[-A_3(t)(x_1-x_3) + 2A_1(t) \times \frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + A_3(t)(y_1-y_3) - 2A_1(t)\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2 - (\hat{G}+G)e_2] + e_3[-2A_2(t)\sqrt{1-x_3^2}\sin x_4 + 2A_2(t)\sqrt{1-y_3^2}\sin y_4 - (\hat{G}+G)e_3] + e_4[-A_4(t)(x_3-x_1) + 2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + A_4(t)(y_3-y_1) - 2A_2(t)\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4 - (\hat{G}+G)e_4] + \tilde{a}_1[2\sqrt{1-y_1^2}\sin y_2 e_1 - \frac{2y_1}{\sqrt{1-y_1^2}}\cos y_2 e_2] + \tilde{\omega}_1[(y_1-y_3)e_2] + \tilde{a}_2[2\sqrt{1-y_3^2}\sin y_4 e_3 - \frac{2y_3}{\sqrt{1-y_3^2}}\cos y_4 e_4] + \tilde{\omega}_2[(y_3-y_1)e_4] - G\|e\|(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{\omega}_1 + \tilde{\omega}_2)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}$$

$$\dot{V} \le G\|e\|^2 - (\hat{G}+G)\|e\|^2 + G\|e\|\|\tilde{A}\| - G\|e\|\|\tilde{A}\|^2/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}$$

$\dot{V}$ can be rewritten as

$$\dot{V} \le -G(e_1^2 + e_2^2 + e_3^2 + e_4^2). \tag{6-4-105}$$

$\dot{V}$ is negative semi-definite function of $e, \tilde{a}, \tilde{\omega}, \tilde{G}$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (6-4-97), adaptive laws (6-4-99) and parameter dynamics (6-4-103) is asymptotically stable. Now, D is a 5-manifold, n=9 and the number of error state variables p=4. When $e_i = 0$, ($i = 1,2,3,4$) and $\tilde{a}_j, \tilde{\omega}_j, \tilde{G}$, ($i = 1,2,3,4$; $j = 1,2$) take arbitrary values, $\dot{V} = 0$, so X is a 5-manifold, $m=n-p=9-4=5$. $m+1<n$ is satisfied. By GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained. The equilibrium point $e_i = \tilde{a}_j = \tilde{\omega}_j = \tilde{G} = 0$ ($i = 1,2,3,4$; $j = 1,2$) is asymptotically stable. Moreover, the result is global

asymptotically stable. (see Appendix A) The numerical results are shown in Fig. 6-20~6-22. The chaos synchronization is accomplished. The coupling strength required is K=2G =5.62.

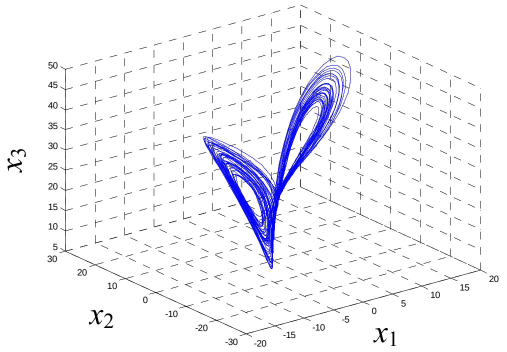


Fig. 6-8. Phase portrait for Lorenz with $\sigma=10, \gamma=28, b=8/3$.

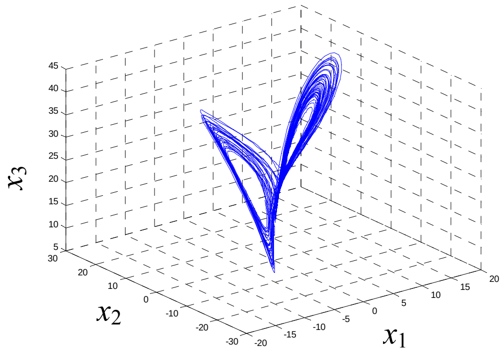


Fig. 6-9. Phase portrait for Eq. (6-4-30) with $A_1(t)=\sigma(1+d_1\sin\varpi_1 t)$, $A_2(t)=\gamma(1+d_2\sin\varpi_2 t)$ and $A_3(t)=b(1+d_3\sin\varpi_3 t)$.

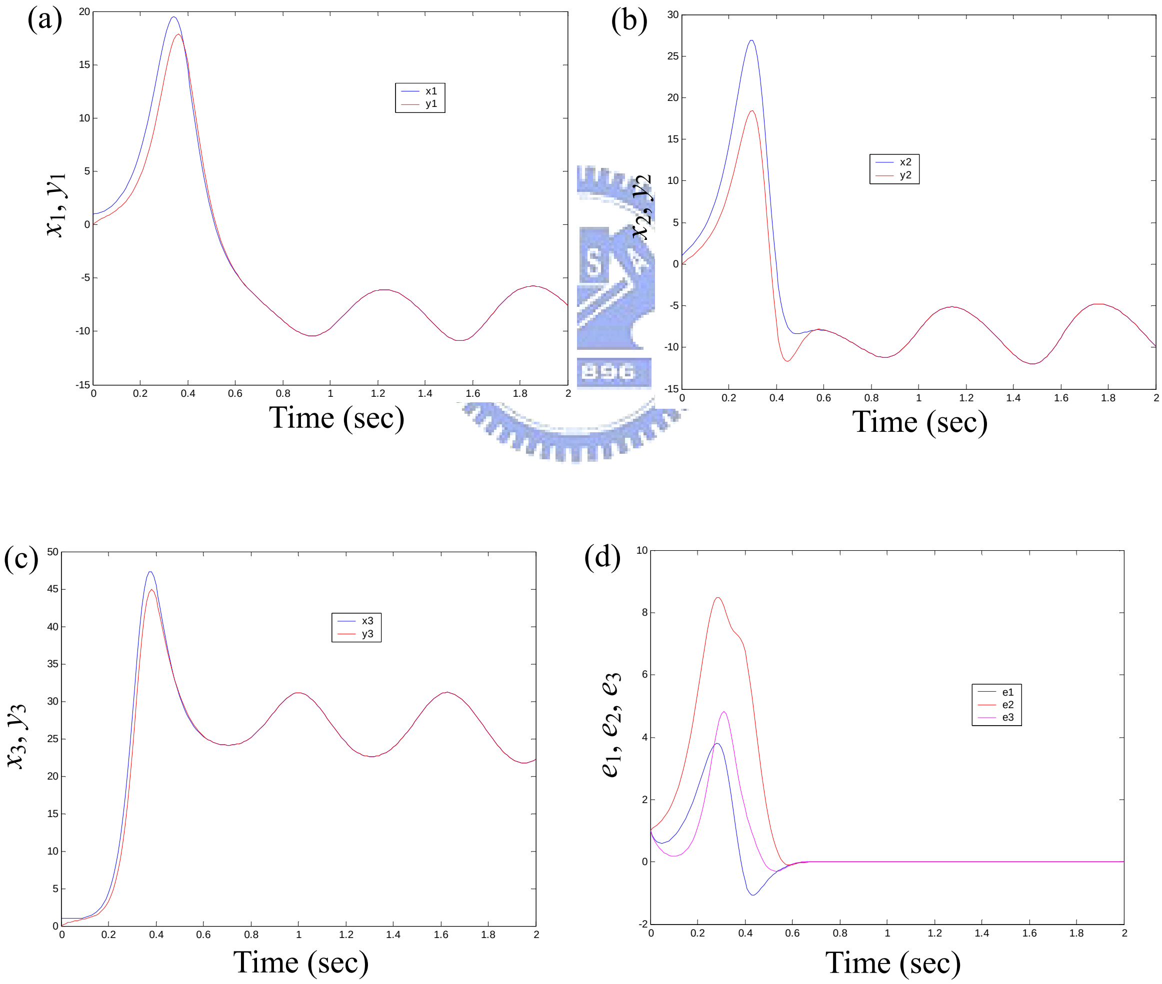
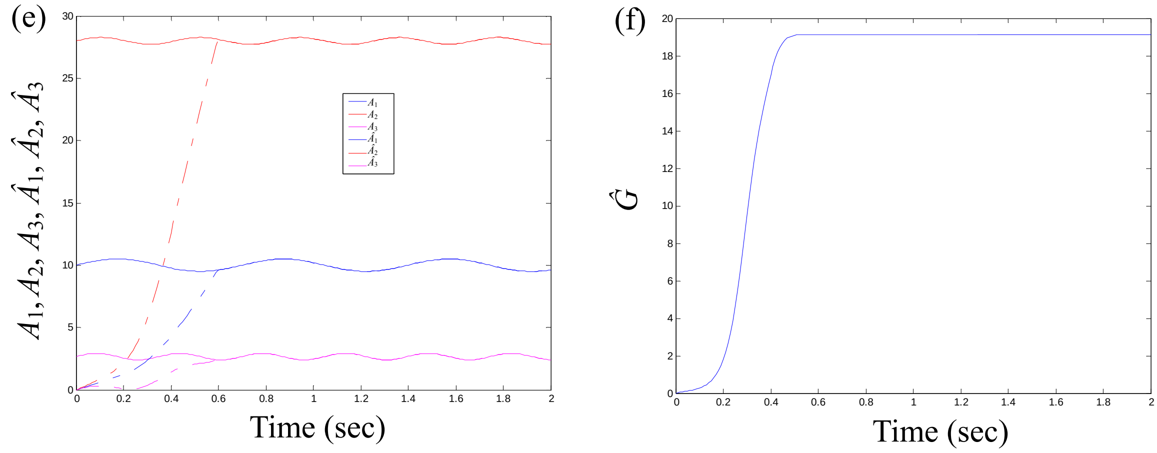


Fig. 6-10. Time histories of states, state errors, $A_1, A_2, A_3, \hat{A}_1, \hat{A}_2, \hat{A}_3$ and estimated Lipschitz constant $\hat{G}$ for Case I.



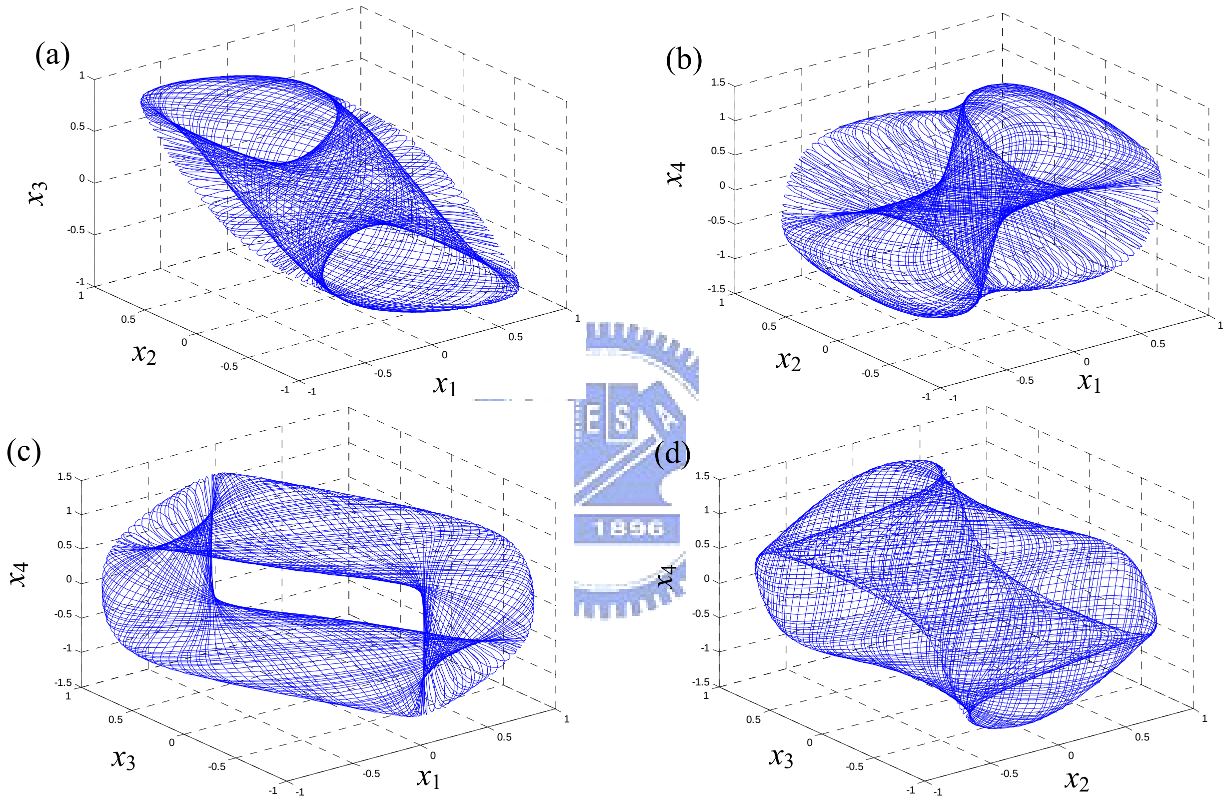


Fig. 6-11. Phase portraits for chaotic system (6-4-42).

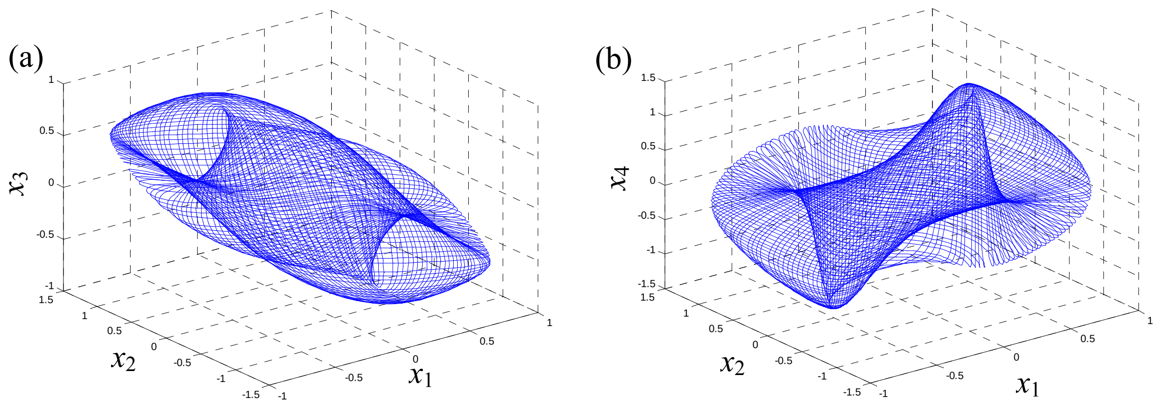


Fig. 6-12. Phase portraits for chaotic system (6-4-43).

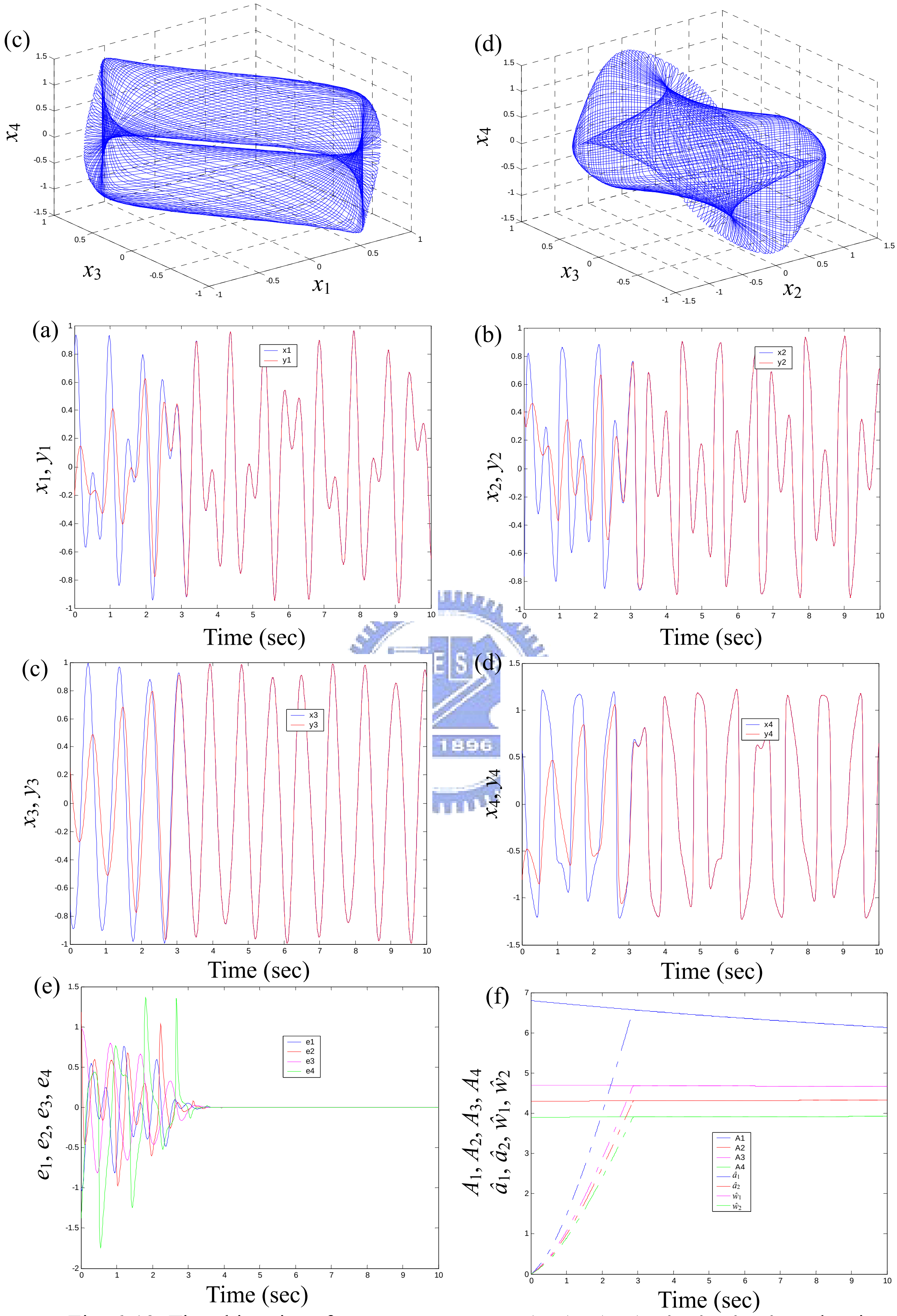
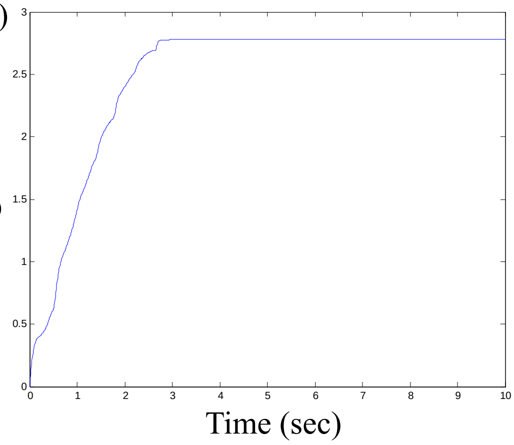


Fig. 6-13. Time histories of states, state errors, A_1, A_2, A_3, A_4, $\hat{a}_1$, $\hat{a}_2$, $\hat{w}_1$, $\hat{w}_2$ and estimated Lipschitz constant $\hat{G}$ for Case II.

(g)



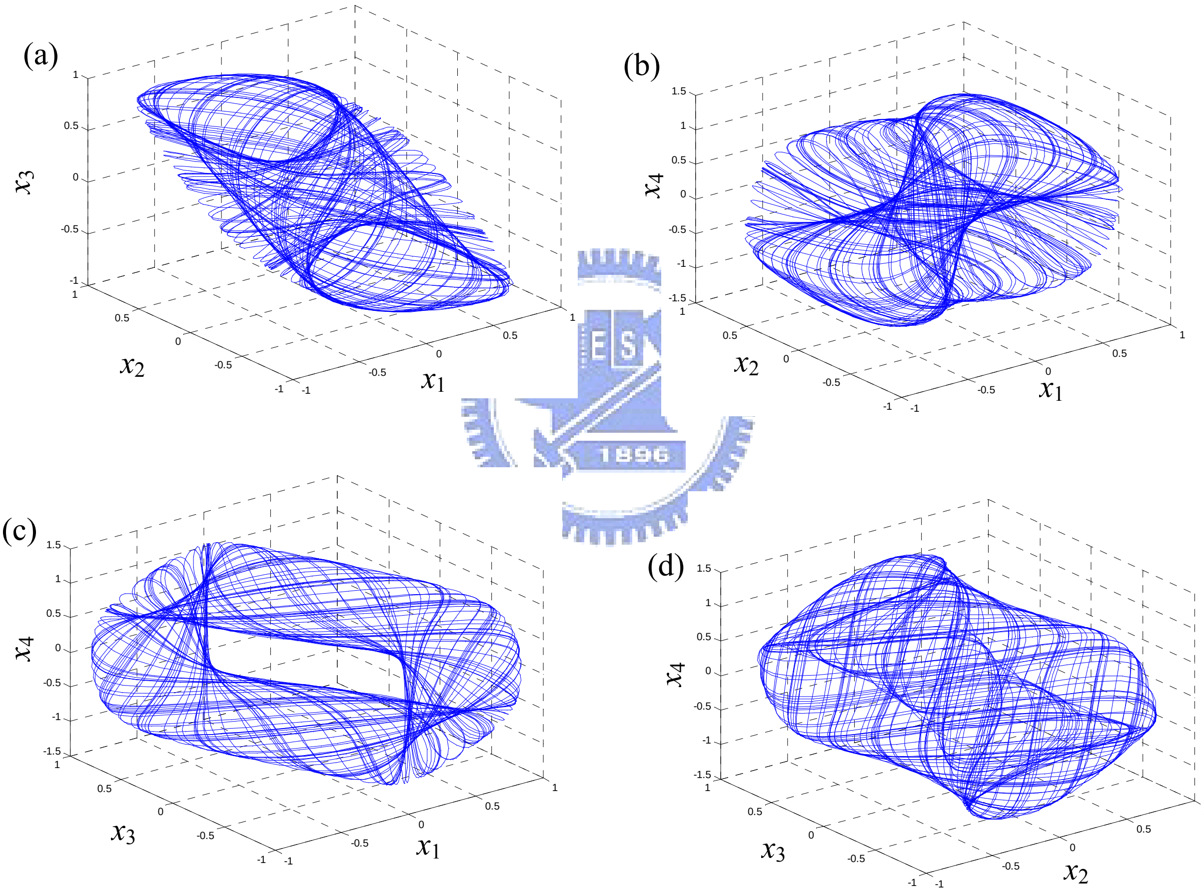


Fig. 6-14. Phase portraits for chaotic system (6-4-54).

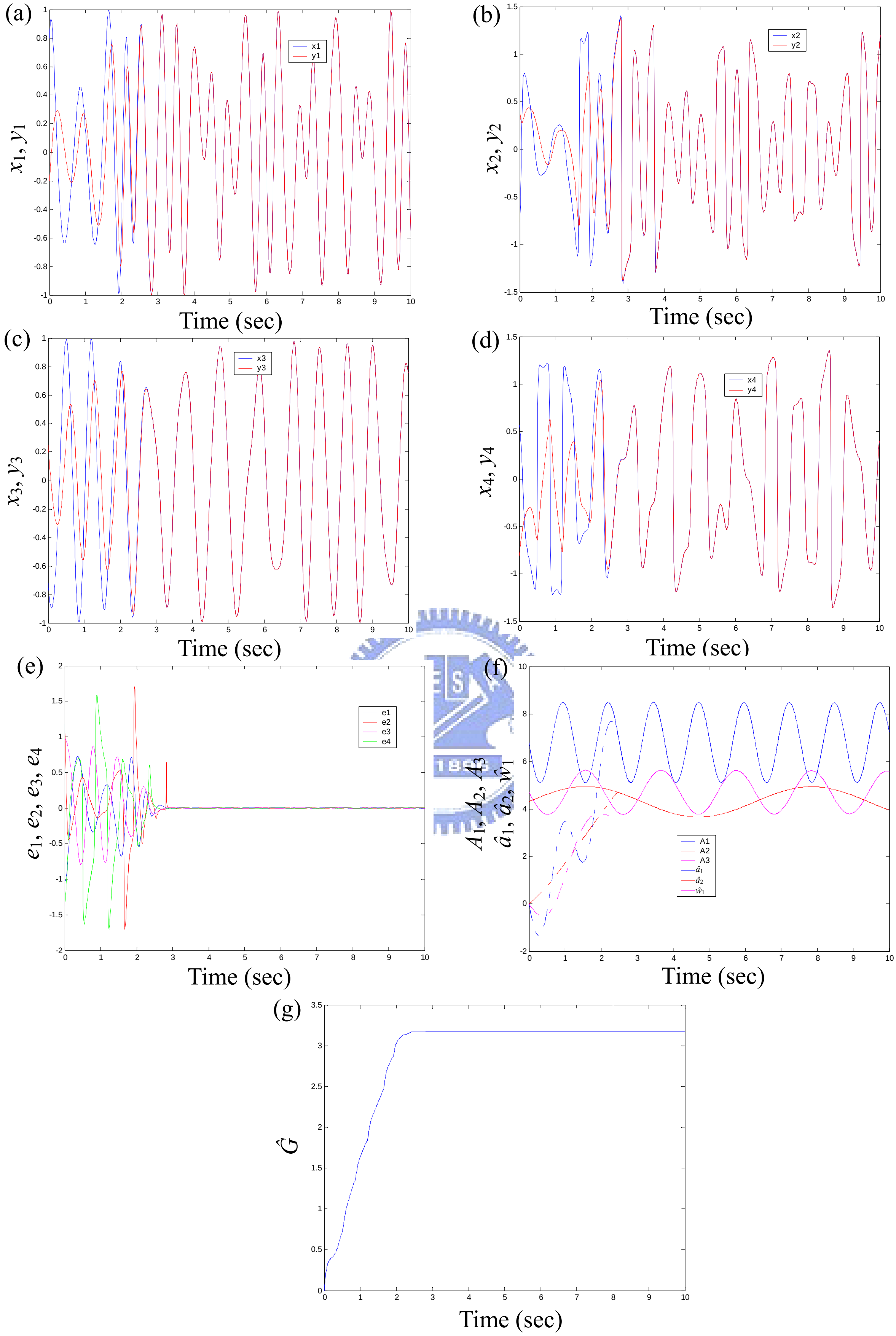


Fig. 6-15. Time histories of states, state errors, A_1, A_2, A_3, $\hat{a}_1$, $\hat{a}_2$, $\hat{w}_1$ and estimated Lipschitz constant $\hat{G}$ for Case III.

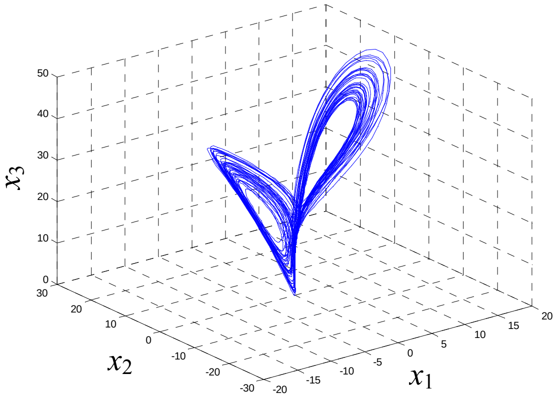


Fig. 6-16. Phase portrait for Lorenz system (6-4-67).

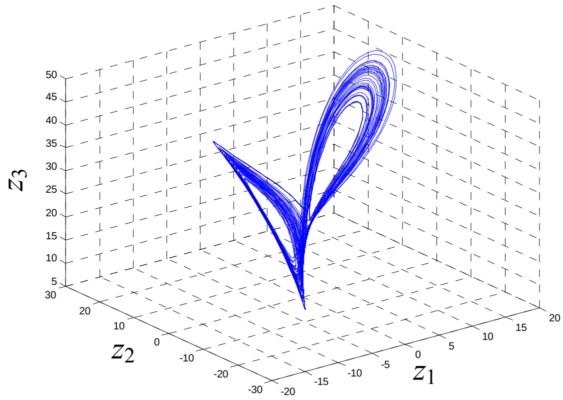


Fig. 6-17. Phase portrait for Lorenz system (6-4-69) with σ=8, γ=27 and b=3.2.

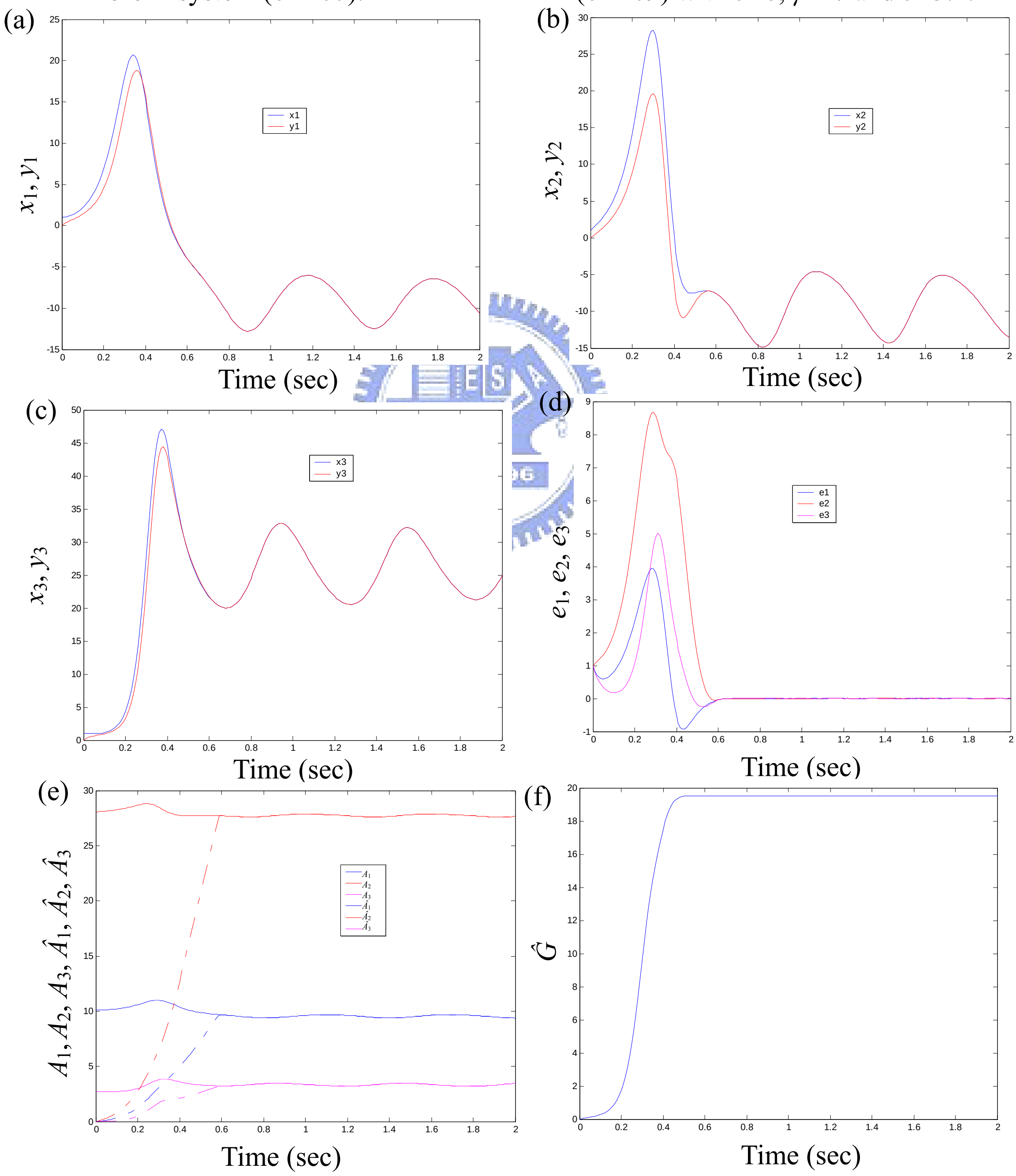


Fig. 6-18. Time histories of states, state errors, $A_1, A_2, A_3, \hat{A}_1, \hat{A}_2, \hat{A}_3$ and estimated Lipschitz constant $\hat{G}$ for Case IV.

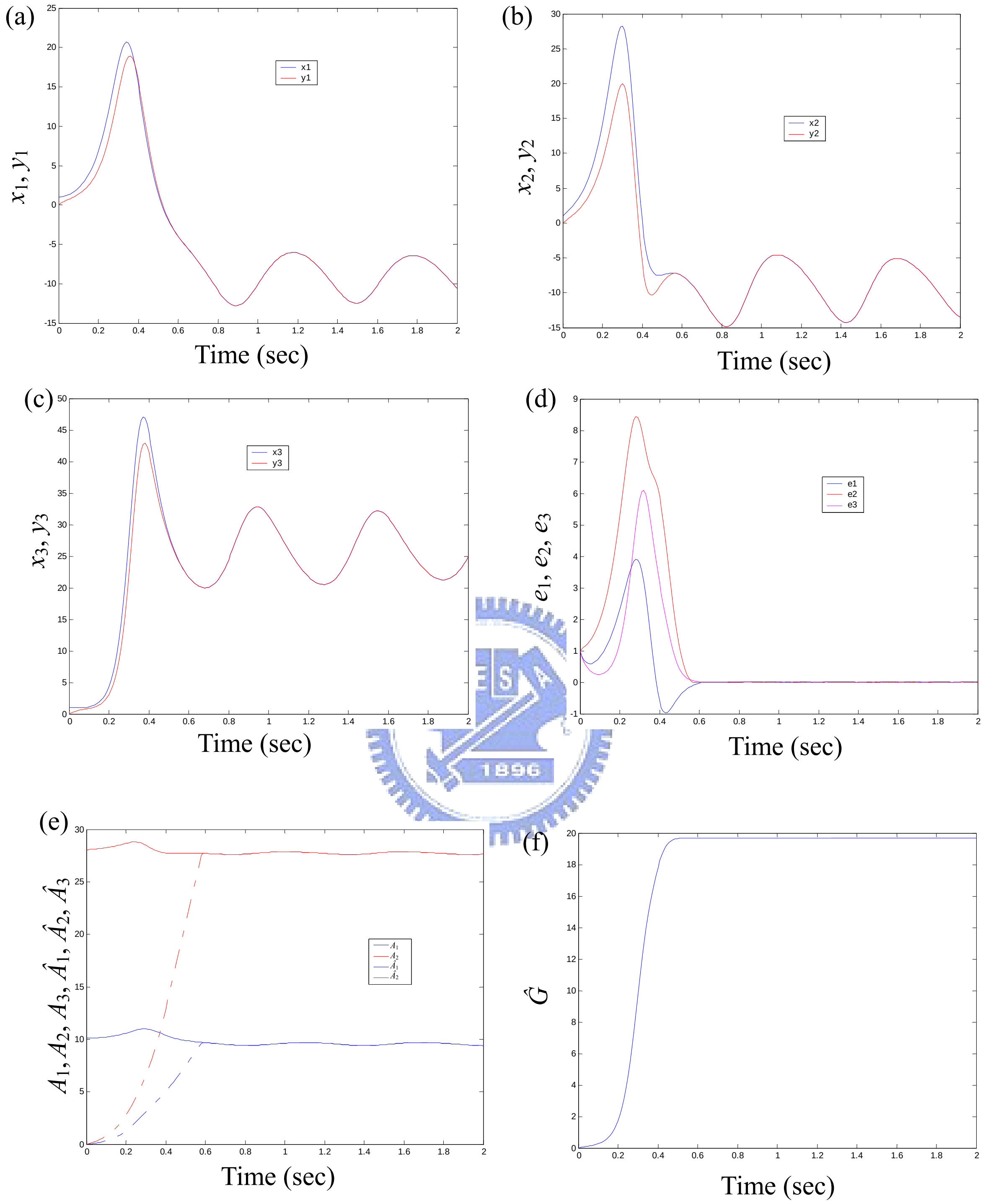


Fig. 6-19. Time histories of states, state errors, $A_1, A_2, \hat{A}_1, \hat{A}_2$ and estimated Lipschitz constant $\hat{G}$ for Case V.

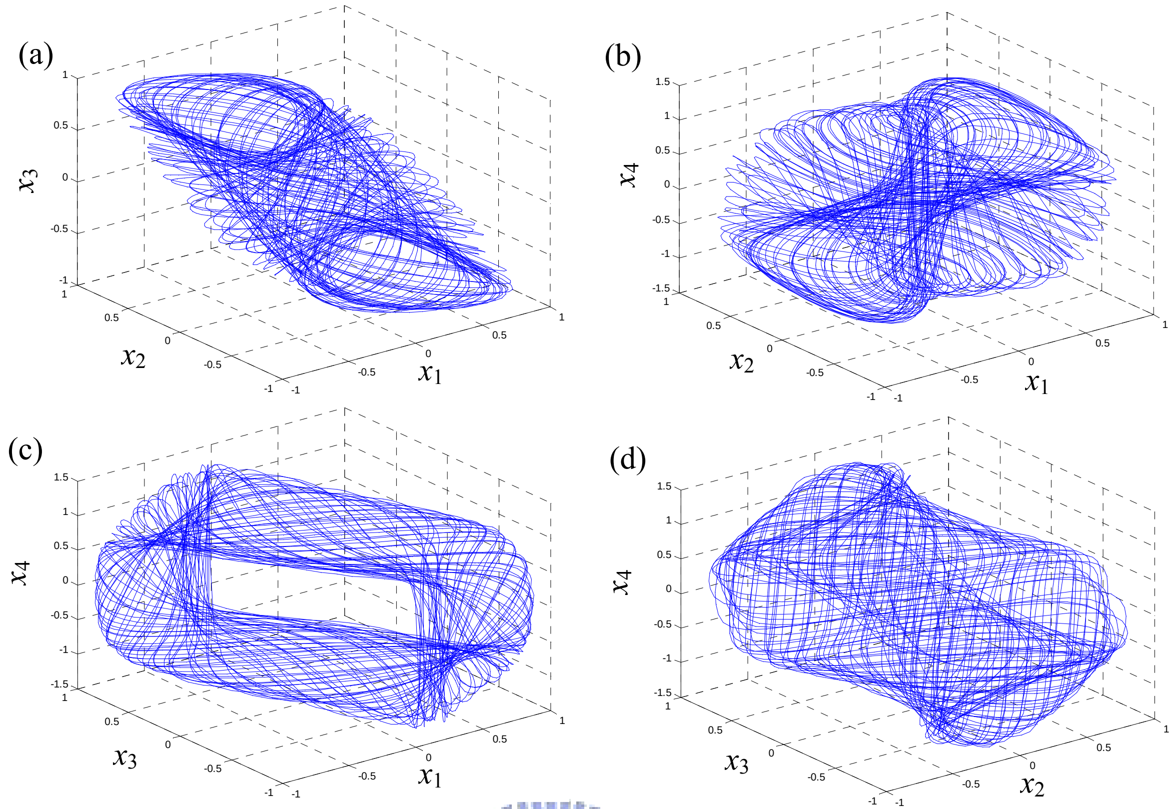


Fig. 6-20. Phase portraits for chaotic system (6-4-93).

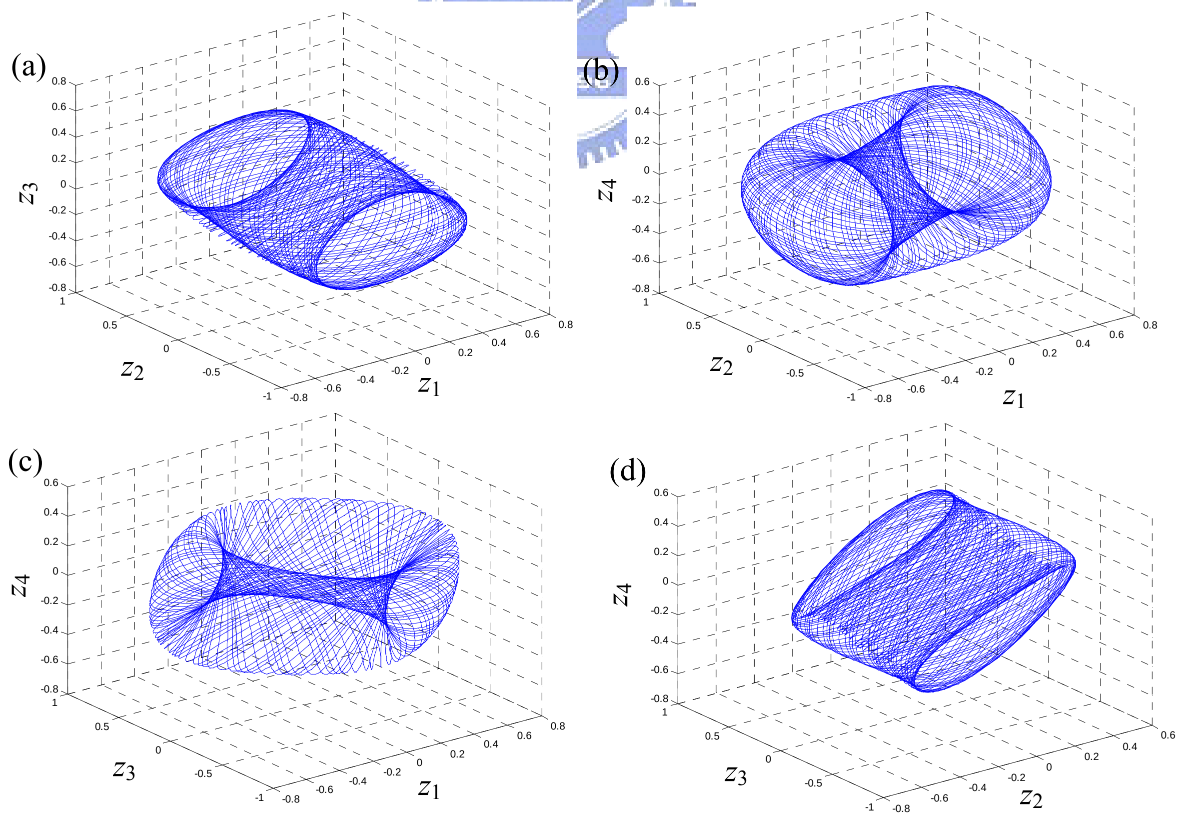


Fig. 6-21. Phase portraits for chaotic system (6-4-95).

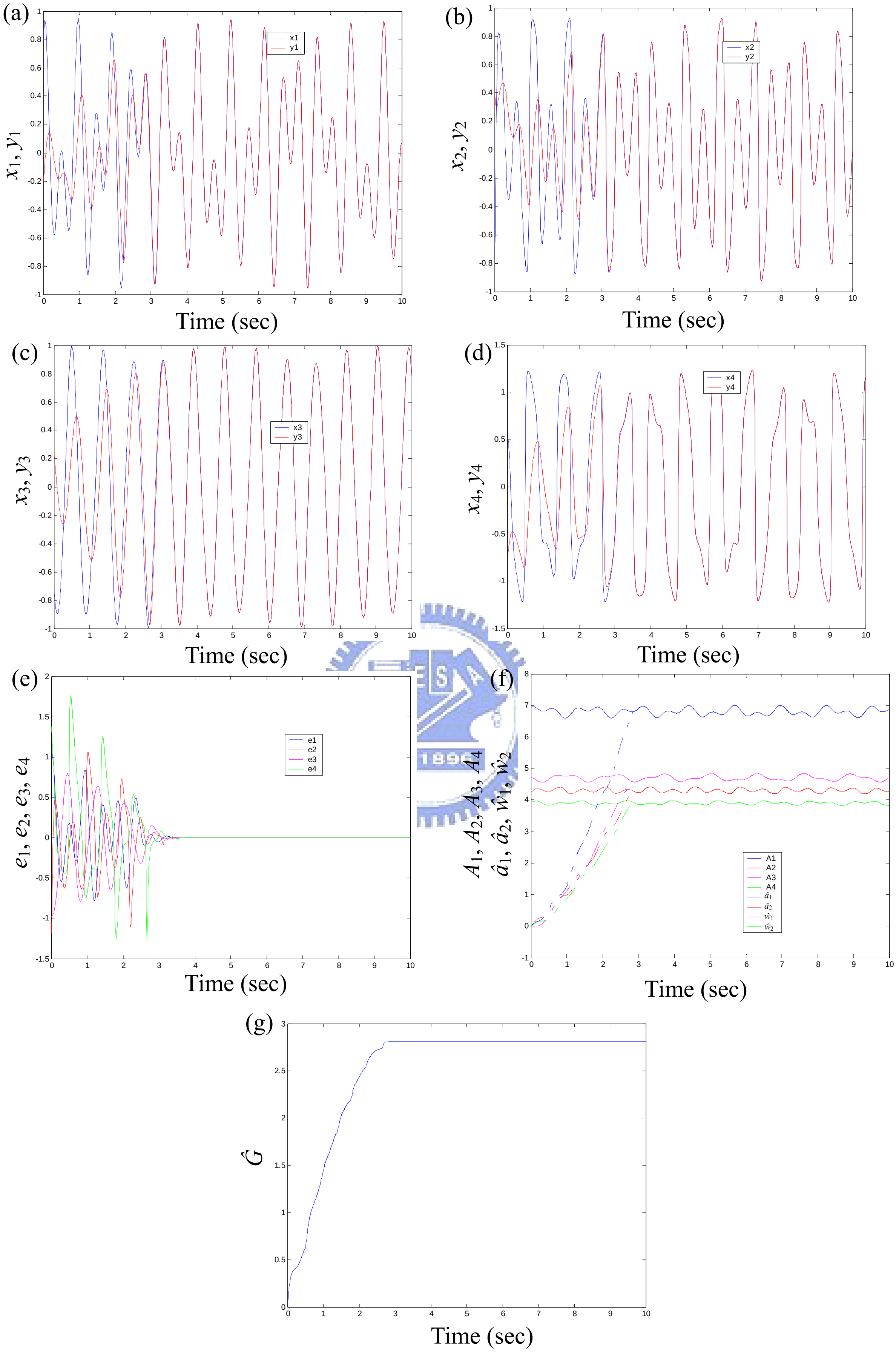


Fig. 6-22. Time histories of states, state errors, A_1, A_2, A_3, A_4, $\hat{a}_1$, $\hat{a}_2$, $\hat{w}_1$, $\hat{w}_2$ and estimated Lipschitz constant $\hat{G}$ for Case VI.

Chapter 7

Chaos Control, Chaotization and Synchronization by GYC Partial Region Stability Theory

7-1 Preliminary

Since chaos control problem was firstly considered by Ott et al., it has been studied extensively. Many linear and nonlinear control methods have been employed to control and anticontrol chaos. There are many control techniques for chaos control, such as linear feedback control, adaptive control, sliding variable method control and active control. However, when tradition Lyapunov stability of zero solution of states is studied, the stability of solutions on the whole neighborhood region of the origin is demanded.

In recent years, synchronization in chaotic dynamic system is a very interesting problem and has been widely studied. Among many kinds of synchronizations, the generalized synchronization is investigated in this thesis. It means that there exists a given functional relationship between the state vector x of the master and the state vector y of the slave $y = G(x,t)$. There are many control techniques to synchronize chaotic systems, such as linear error feedback control, adaptive control and active control

In this chapter, a new scheme to achieve chaos control and chaos generalized synchronization strategy by GYC partial region stability theory is proposed [99, 100]. By applying the GYC partial region stability theory the Lyapunov function used is a simple linear homogeneous function of error states and the controllers are simpler and introduce less simulation error because they are in lower order than that of traditional controllers.

7-2 Chaos Control and Chaotization of Chaotic System to Different Systems

7-2-1 Chaos Control and Chaotization Scheme

Consider the following chaotic systems

$$\dot{x} = f(t, x) \tag{7-2-1}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ is a the state vector, $f : R_+ \times R^n \to R^n$ is a vector function.

The goal system which can be either chaotic or regular, is

$$\dot{y} = g(t, y) \tag{7-2-2}$$

where $y=[y_1, y_2, \cdots, y_n]^T \in R^n$ is a state vector, $g: R_+ \times R^n \to R^n$ is a vector function.

In order to make the chaos state x approaching the goal state y, define $e=x-y$ as the state error. The chaos control is accomplished in the sense that:

$$\lim_{t\to\infty} e = \lim_{t\to\infty}(x-y) = 0 \tag{7-2-3}$$

In this thesis, we will use an example which place the e state in the first quadrant of coordinate system and use GYC partial region stability theory. The Lyapunov function used is a simple linear homogeneous function of states and the controllers are simpler because they are in lower order than that of traditional controllers.

7-2-2 Numerical Results of the Chaos Control

Case I Control a chaotic nano resonator system to a double harmonic system

Nano resonator system studied in this thesis is a modified form of nonlinear damped Mathieu system which is obtained when the nano Mathieu oscillator has nonlinear time-dependent spring constant [97]. The following chaotic system is the nano resonator system of which the old origin is translated to $(x_1, x_2, x_3, x_4) = (100, 100, 100, 100)$ and the chaotic motion always happens in the first quadrant of coordinate system (x_1, x_2, x_3, x_4). Then we have the modified nonlinear damped Mathieu system, i.e. the nano resonator system:

$$\begin{cases} \dot{x}_1 = x_2 - 100 \\ \dot{x}_2 = -(s_1 + s_2(x_3 - 100))(x_1 - 100) - (s_1 + s_2(x_3 - 100))(x_1 - 100)^3 \\ \qquad - s_3(x_2 - 100) + s_4(x_3 - 100) \\ \dot{x}_3 = x_4 - 100 \\ \dot{x}_4 = -s_5(x_3 - 100) - s_6(x_4 - 100)^3 \end{cases} \tag{7-2-4}$$

where s_1, s_2, s_3, s_4, s_5 and s_6 are given original constant coefficients of the system. When the given coefficients are s_1=0.2, s_2=0.2, s_3=0.4, s_4=15, s_5=1 and s_6=0.3, the modified nonlinear damped Mathieu system is chaotic, as shown in Fig. 7-1.

The goal system is a double harmonic system:

$$\dot{y}_1 = y_2,\ \dot{y}_2 = -\omega_{n1} y_1,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -\omega_{n2} y_3 \tag{7-2-5}$$

where ω_{n1} and ω_{n2} are constant coefficients of the system with initial conditions $y_1(0)=0.3$, $y_2(0)=-0.4$, $y_3(0)=-0.7$, and $y_4(0)=0.15$. The given coefficients are $\omega_{n1}=1$ and $\omega_{n2}=1$.

In order to lead (x_1, x_2, x_3, x_4) to (y_1, y_2, y_3, y_4), we add u_1, u_2, u_3 and u_4 to each equation of Eq. (7-2-4), respectively.

$$
\begin{cases}
\dot{x}_1 = x_2 - 100 - u_1 \\
\dot{x}_2 = -(s_1 + s_2(x_3 - 100))(x_1 - 100) - (s_1 + s_2(x_3 - 100))(x_1 - 100)^3 \\
\qquad - s_3(x_2 - 100) + s_4(x_3 - 100) - u_2 \\
\dot{x}_3 = x_4 - 100 - u_3 \\
\dot{x}_4 = -s_5(x_3 - 100) - s_6(x_4 - 100)^3 - u_4
\end{cases}
\tag{7-2-6}
$$

The state error is $e=y-x$, our aim is $\lim_{t\to\infty} e = 0$. We obtain the error dynamics.

$$
\lim_{t\to\infty} e_i = \lim_{t\to\infty}(y_i - x_i) = 0, \quad i = 1, 2, 3, 4 \tag{7-2-7}
$$

$$
\begin{aligned}
\dot{e}_1 &= y_2 - (x_2 - 100) - u_1 \\
\dot{e}_2 &= -\omega_{n1} y_1 + (s_1 + s_2(x_3 - 100))(x_1 - 100) + (s_1 + s_2(x_3 - 100))(x_1 - 100)^3 \\
&\quad + s_3(x_2 - 100) - s_4(x_3 - 100) - u_2 \\
\dot{e}_3 &= y_4 - (x_4 - 100) - u_3 \\
\dot{e}_4 &= -\omega_{n2} y_3 + s_5(x_3 - 100) + s_6(x_4 - 100)^3 - u_4
\end{aligned}
\tag{7-2-8}
$$

where $e_1=y_1-x_1$, $e_2=y_2-x_2$, $e_3=y_3-x_3$ and $e_4=y_4-x_4$.

Choose a Lyapunov function in the form of the positive definite function in the first quadrant

$$
V(e_1, e_2, e_3, e_4) = e_1 + e_2 + e_3 + e_4 \tag{7-2-9}
$$

Its time derivative along any solution of Eq. (7-2-8) and parameter dynamics is

$$
\begin{aligned}
\dot{V} &= [y_2 - (x_2 - 100) - u_1] + [-\omega_{n1} y_1 + (s_1 + s_2(x_3 - 100))(x_1 - 100) \\
&\quad + (s_1 + s_2(x_3 - 100))(x_1 - 100)^3 + s_3(x_2 - 100) - s_4(x_3 - 100) - u_2] \\
&\quad + [y_4 - (x_4 - 100) - u_3] + [-\omega_{n2} y_3 + s_5(x_3 - 100) + s_6(x_4 - 100)^3 - u_4]
\end{aligned}
\tag{7-2-10}
$$

Choose

$$
\begin{aligned}
u_1 &= y_2 - (x_2 - 100) - e_1 \\
u_2 &= -\omega_{n1} y_1 + (s_1 + s_2(x_3 - 100))(x_1 - 100) + (s_1 + s_2(x_3 - 100))(x_1 - 100)^3 \\
&\quad + s_3(x_2 - 100) - s_4(x_3 - 100) - e_2 \\
u_3 &= y_4 - (x_4 - 100) - e_3 \\
u_4 &= -\omega_{n2} y_3 + s_5(x_3 - 100) + s_6(x_4 - 100)^3 - e_4
\end{aligned}
\tag{7-2-11}
$$

Substituting Eq. (7-2-11) into Eq. (7-2-10), we obtain

$$
\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0
$$

which is a negative definite function in first quadrant.

This means that we control the chaotic motion of a modified nonlinear damped Mathieu system to regular motion of a double harmonic system. The simulation results are shown in Fig. 7-1, Fig. 7-2, Fig. 7-3 and Fig. 7-4.

Case II Chaotize the chaotic motion of a chaotic nano resonator system to the hyperchaotic motion of a hyperchaotic Quantum-CNN oscillator system

For a two-cell Quantum-CNN, the following differential equations are obtained [23]:

$$\begin{cases} \dot{z}_1 = -2a_1\sqrt{1-x_1^2}\sin z_2, \ \ \dot{z}_2 = -\omega_1(z_1 - z_3) + 2a_1\dfrac{z_1}{\sqrt{1-z_1^2}}\cos z_2 \\ \dot{z}_3 = -2a_2\sqrt{1-z_3^2}\sin z_4, \ \ \dot{z}_4 = -\omega_2(z_3 - z_1) + 2a_2\dfrac{z_3}{\sqrt{1-z_3^2}}\cos z_4 \end{cases} \tag{7-2-12}$$

where z_1, z_3 are polarizations, z_2, z_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are coefficients that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. $\hat{a}_1$, $\hat{a}_2$, $\hat{\omega}_1$ and $\hat{\omega}_2$ are goal coefficients.

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(z_i - x_i) = 0, \quad i = 1, 2, 3, 4 \tag{7-2-13}$$

where $\dot{e} = \dot{z} - \dot{x}$.

$$\begin{aligned} \dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin z_2 - (x_2 - 100) - u_1 \\ \dot{e}_2 &= -\omega_1(z_1 - z_3) + 2a_1\frac{z_1\cos z_2}{\sqrt{1-z_1^2}} + (s_1 + s_2(x_3 - 100))(x_1 - 100) + (s_1 \\ &\quad + s_2(x_3 - 100))(x_1 - 100)^3 + s_3(x_2 - 100) - s_4(x_3 - 100) - u_2 \\ \dot{e}_3 &= -2a_2\sqrt{1-z_3^2}\sin z_4 - (x_4 - 100) - u_3 \\ \dot{e}_4 &= -\omega_2(z_3 - z_1) + 2a_2\frac{z_3\cos z_4}{\sqrt{1-z_3^2}} + s_5(x_3 - 100) + s_6(x_4 - 100)^3 - u_4 \end{aligned} \tag{7-2-14}$$

where $e_1=z_1-x_1$, $e_2=z_2-x_2$, $e_3=z_3-x_3$ and $e_4=z_4-x_4$.

The parameters are given a_1=0.25, a_2=0.25, ω_1=0.4 and ω_2=0.17.

Choose a Lyapunov function in the form of the positive definite function in the first quadrant

$$V(e_1, e_2, e_3, e_4) = e_1 + e_2 + e_3 + e_4 \tag{7-2-15}$$

Its time derivative along any solution of Eq. (7-2-14) and parameter dynamics is

$$\begin{aligned} \dot{V} &= [-2a_1\sqrt{1-x_1^2}\sin z_2 - (x_2 - 100) - u_1] + [-\omega_1(z_1 - z_3) + 2a_1\frac{z_1\cos z_2}{\sqrt{1-z_1^2}} \\ &+ (s_1 + s_2(x_3 - 100))(x_1 - 100) + (s_1 + s_2(x_3 - 100))(x_1 - 100)^3 + s_3(x_2 - 100) \\ &- s_4(x_3 - 100) - u_2] + [-2a_2\sqrt{1-z_3^2}\sin z_4 - (x_4 - 100) - u_3] \\ &+ [-\omega_2(z_3 - z_1) + 2a_2\frac{z_3\cos z_4}{\sqrt{1-z_3^2}} + s_5(x_3 - 100) + s_6(x_4 - 100)^3 - u_4] \end{aligned} \tag{7-2-16}$$

Choose

$$
\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin z_2 - (x_2-100) + e_1 \\
u_2 &= -\omega_1(z_1 - z_3) + 2a_1\frac{z_1\cos z_2}{\sqrt{1-z_1^2}} + (s_1 + s_2(x_3-100))(x_1-100) \\
&+(s_1 + s_2(x_3-100))(x_1-100)^3 + s_3(x_2-100) - s_4(x_3-100) + e_2 \\
u_3 &= -2a_2\sqrt{1-z_3^2}\sin z_4 - (x_4-100) + e_3 \\
u_4 &= -\omega_2(z_3 - z_1) + 2a_2\frac{z_3\cos z_4}{\sqrt{1-z_3^2}} + s_5(x_3-100) + s_6(x_4-100)^3 + e_4
\end{aligned}
\tag{7-2-17}
$$

Substituting Eq. (7-2-17) into Eq. (7-2-16), it can be rewritten as

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0$$

which is a negative definite function in first quadrant.

This means that we anticontrol the chaotic motion of a modified nonlinear damped Mathieu system to the hyperchaotic motion of a hyperchaotic Quantum-CNN oscillator system with two positive Lyapunov exponents [23]. The simulation results are shown in Fig. 7-5, Fig. 7-6 and Fig. 7-7.

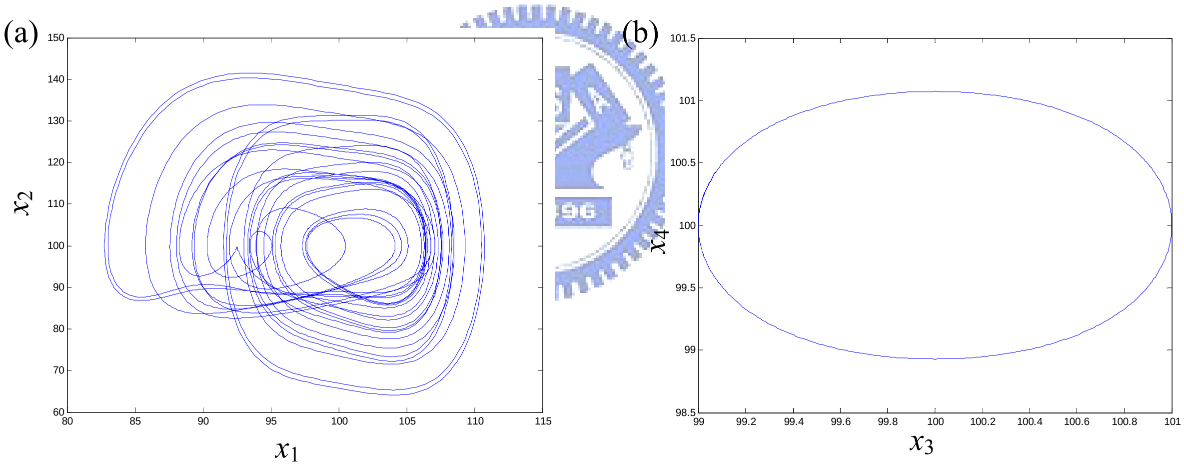


Fig. 7-1. Phase portraits of modified nonlinear damped Mathieu system.

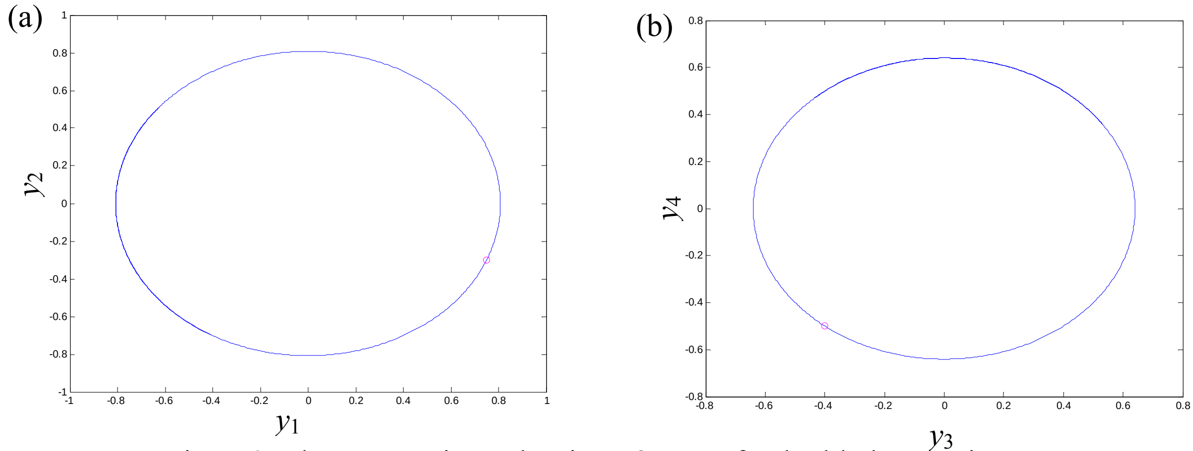


Fig. 7-2. Phase portraits and Poincaré map of a double harmonic system.

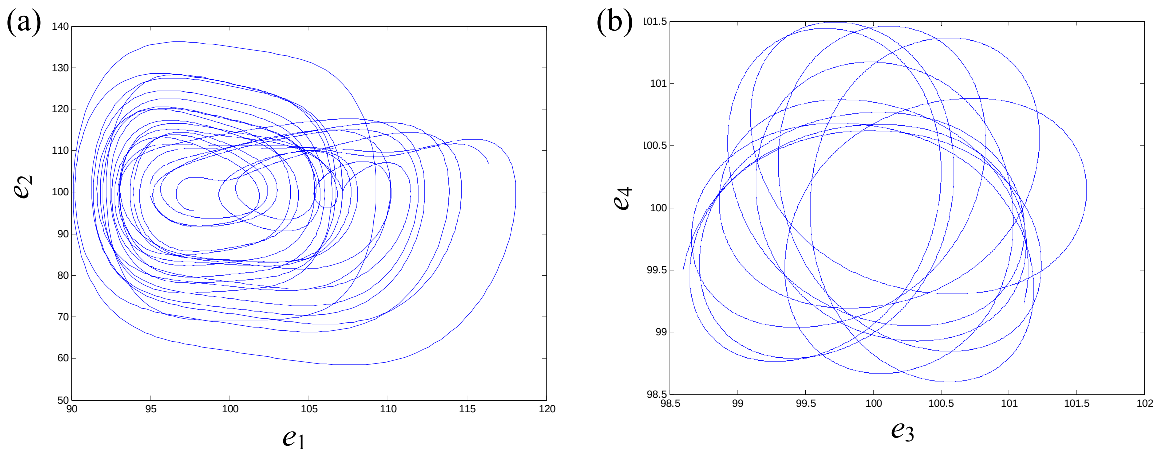


Fig. 7-3. Phase portraits of error dynamic system for Case I.

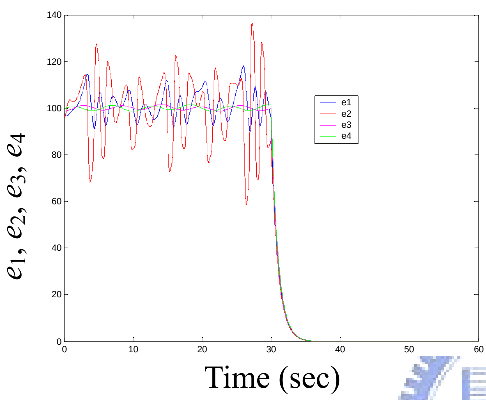


Fig. 7-4.Time histories of state errors for Case I.

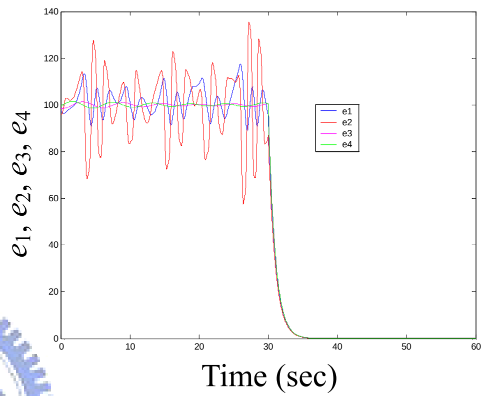


Fig. 7-7.Time histories of state errors for Case II.

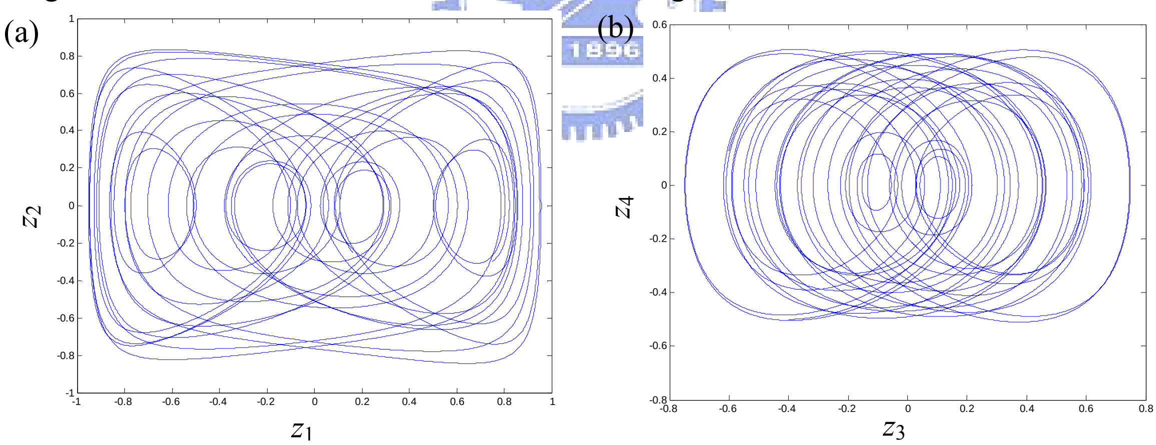


Fig. 7-5. Phase portraits of Quantum-CNN system.

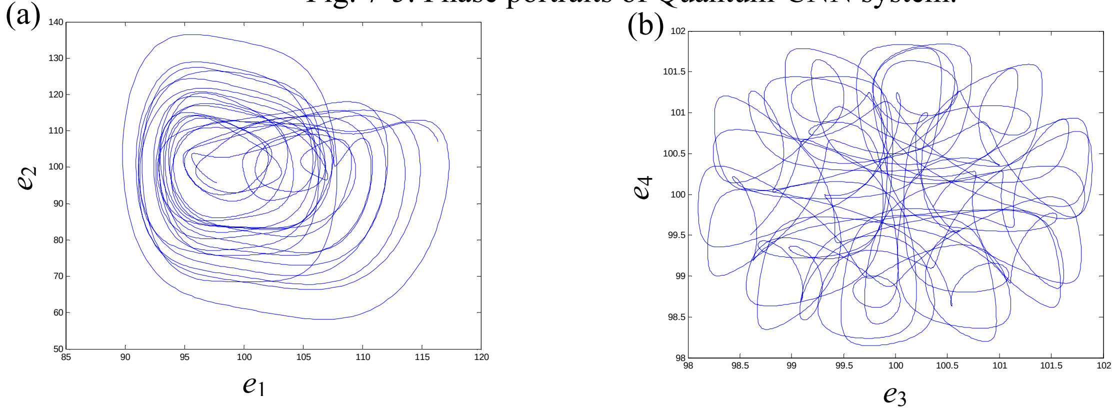


Fig. 7-6. Phase portraits of error dynamic system for Case II.

7-3 The Chaos Generalized Synchronization of a Quantum-CNN Chaotic Oscillator with a Double Duffing Chaotic System

7-3-1 Chaos Generalized Synchronization Strategy

There are two different nonlinear dynamical systems and the master system controls the slave system. The master system is given here

$$\dot{x} = Ax + f(x) \tag{7-3-1}$$

where $x = [x_1, x_2, \cdots, x_{2n}]^T \in R^n$ denotes the state vector, A is a $n \times n$ coefficient matrix, and $f(x)$ is a nonlinear vector function.

The slave system is given here

$$\dot{y} = By + h(y) + u(t) \tag{7-3-2}$$

where $y = [y_1, y_2, \cdots, y_{2n}]^T \in R^n$ denotes a state vector, B is a $n \times n$ coefficient matrix, $h(x)$ is a nonlinear vector function, and $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T \in R^n$ is a control input vector.

Our goal is to design a controller $u(t)$ so that the state vector of the slave system (7-3-2) asymptotically approaches the state vector of the master system (7-3-1) plus a given vector function $F(t) = [F_1(t), F_2(t), \cdots, F_n(t)]^T$ which is either a regular or a chaotic function of time, and finally the synchronization will be accomplished in the sense that the limit of the error vector $e(t) = [e_1, e_2, \cdots, e_n]^T$ approaches zero:

$$\lim_{t \to \infty} e = 0 \tag{7-3-3}$$

where

$$e_i = x_i \mp y_i + F_i(t) + K_i, \quad i = 1, 2, \cdots, n \tag{7-3-4}$$

K_i are positive constants which make the error dynamics always happens in the first quadrant.

From Eq. (7-3-4) we have

$$\dot{e}_i = \dot{x}_i \mp \dot{y}_i + \dot{F}_i(t), \quad i = 1, 2, \cdots, n \tag{7-3-5}$$

Introduce Eq. (7-3-1) and Eq. (7-3-2) in Eq. (7-3-5):

$$\dot{e} = (A - B)[x_i \pm y_i] + f(x) \mp h(y) + \dot{F}(t) \mp u(t), \quad i = 1, 2, \cdots, n \tag{7-3-6}$$

where $[x_i \pm y_i]$ is a $n \times 1$ column matrix.

A Lyapnuov function V(e) is chosen as a positive definite function in the first quadrant.

$$V(e) = e_1 + e_2 + \cdots + e_n \tag{7-3-7}$$

Its derivative along the solution of Eq. (7-3-7) is

$$\dot{V}(e) = (A-B)[x_i \pm y_i] + f(x) \mp h(y) + \dot{F}(t) \mp u(t)\,, \quad i = 1, 2, \cdots, n \tag{7-3-8}$$

where $u(t)$ is chosen so that $\dot{V} = C_1 e_1 + C_2 e_2 + \cdots + C_n e_n$, C_i ($i = 1, 2, \cdots, n$) are negative constants, and $\dot{V}$ is a negative definite function of $e_1, e_2, \cdots, e_n$. When

$$\lim_{t\to\infty} e = 0 \tag{7-3-9}$$

the generalized synchronization is obtained.

By using the GYC partial region stability theory, the Lyapunov function is easier to find, since the terms of first degree can be used to construct the definite Lyapunov function and the controller can be designed in lower order.

7-3-2 Numerical Simulations

Case I Partial generalized synchronization and partial generalized anti-synchronization as the special case of the generalized synchronization

For a two-cell Quantum-CNN, the following differential equations are obtained [23]:

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\,\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1 - x_3) + 2a_1 \dfrac{x_1}{\sqrt{1-x_1^2}} \cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\,\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3 - x_1) + 2a_2 \dfrac{x_3}{\sqrt{1-x_3^2}} \cos x_4 \end{cases} \tag{7-3-10}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. Let $a_1 = 19.4$, $a_2 = 13.1$, $\omega_1 = 9.529$, $\omega_2 = 7.94$.

A double Duffing chaotic system is described by

$$\dot{y}_1 = y_2,\ \dot{y}_2 = y_1 - y_1^3 - \rho_1 y_2 + F_1 \cos\xi_1 t,\ \dot{y}_3 = y_4,\ \dot{y}_4 = y_3 - y_3^3 - \rho_2 y_4 + F_2 \cos\xi_2 t \tag{7-3-11}$$

where $\rho_1 = 0.5, \rho_2 = 1.5, F_1 = 1.9, F_2 = 0.9, \xi_1 = 0.97, \xi_2 = 0.79$.

In order to lead (y_1, y_2, y_3, y_4) to (x_1, x_2, x_3, x_4), we add u_1, u_2, u_3 and u_4 to the first, the second, the third, and the fourth equations of Eq. (7-3-11) respectively.

$$\begin{cases} \dot{y}_1 = y_2 + u_1,\ \dot{y}_2 = y_1 - y_1^3 - \rho_1 y_2 + F_1\cos\xi_1 t + u_2 \\ \dot{y}_3 = y_4 + u_3,\ \dot{y}_4 = y_3 - y_3^3 - \rho_2 y_4 + F_2\cos\xi_2 t + u_4 \end{cases} \tag{7-3-12}$$

Subtracting Eq. (7-3-12) from Eq. (7-3-10), we obtain an error dynamics. The initial values of the Quantum-CNN system and double Duffing systems are taken as $x_1(0)=0.8$, $x_2(0)=-0.77$, $x_3(0)=-0.72$, $x_4(0)=0.57$, $y_1(0)=0.75$, $y_2(0)=-0.3$, $y_3(0)=-0.4$, and $y_4(0)=-0.5$.

When i=1, 2, negative signs before y_i in Eq. (5) are chosen; when i=3, 4, positive signs before y_i in Eq. (7-3-5) are chosen, while $F(t)=0$, $K_1=K_2=K_3=K_4=10$

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + 10) = 0, \quad i=1,2 \quad \text{and} \quad \lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i + y_i + 10) = 0, \quad i=3,4 \tag{7-3-13}$$

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1 \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2 \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + y_4 + u_3 \\
\dot{e}_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + y_3 - y_3^3 - \rho_2 y_4 + F_2\cos\xi_2 t + u_4
\end{aligned} \tag{7-3-14}$$

where $e_1 = x_1 - y_1 + 10$, $e_2 = x_2 - y_2 + 10$, $e_3 = x_3 + y_3 + 10$ and $e_4 = x_4 + y_4 + 10$.

Choose a Lyapunov function in the form of the positive definite function in the first quadrant:

$$V(e_1, e_2, e_3, e_4) = e_1 + e_2 + e_3 + e_4 \tag{7-3-15}$$

Its time derivative along any solution of Eq. (7-3-14) is

$$\begin{aligned}
\dot{V} &= (-2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1) + (-\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\
&\quad - y_1 + y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2) + (-2a_2\sqrt{1-x_3^2}\sin x_4 + y_4 + u_3) \\
&\quad + (-\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + y_3 - y_3^3 - \rho_2 y_4 + F_2\cos\xi_2 t + u_4)
\end{aligned} \tag{7-3-16}$$

Choose

$$\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 + x_1 - y_1 \\
u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 x_2 - F_1\cos\xi_1 t \\
u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - x_3 - y_3 \\
u_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 - \rho_2 x_4 - F_2\cos\xi_2 t
\end{aligned}$$

Eq. (7-3-16) can be rewritten as

$$\dot{V} = -e_1 - \rho_1 e_2 - e_3 - \rho_2 e_4 < 0 \tag{7-3-17}$$

which is negative definite function in the first quadrant. Theorem 2 in Appendix B is satisfied. This means that the partial generalized synchronization and the partial generalized anti-synchronization of Quantum-CNN system with a double Duffing chaotic system can be achieved. The numerical results are shown in Fig. 7-8. After 1 second, our aim is achieved.

Case II Generalized anti-synchronization

When i=1, 2, 3, 4, the positive signs before y_i in Eq. (7-3-5) are chosen, while $F(t)=0$:

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i + y_i + 10) = 0, \quad i=1,2,3,4 \tag{7-3-18}$$

$$\begin{aligned}
\dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 + y_2 + u_1 \\
\dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + y_1 - y_1^3 - \rho_1 y_2 + F_1\cos\xi_1 t + u_2 \\
\dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 + y_4 + u_3 \\
\dot{e}_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + y_3 - y_3^3 - \rho_2 y_4 + F_2\cos\xi_2 t + u_4
\end{aligned} \tag{7-3-19}$$

where $e_1 = x_1 + y_1 + 10$, $e_2 = x_2 + y_2 + 10$, $e_3 = x_3 + y_3 + 10$ and $e_4 = x_4 + y_4 + 10$.

Choose a Lyapunov function in the form of the positive definite function in the first quadrant:

$$V(e_1, e_2, e_3, e_4) = e_1 + e_2 + e_3 + e_4 \tag{7-3-20}$$

Its time derivative along any solution of Eq. (7-3-19) is

$$\begin{aligned}
\dot{V} &= e_1(-2a_1\sqrt{1-x_1^2}\sin x_2 + y_2 + u_1) + e_2(-\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 + y_1 \\
&\quad - y_1^3 - \rho_1 y_2 + F_1\cos\xi_1 t + u_2) + e_3(-2a_2\sqrt{1-x_3^2}\sin x_4 + y_4 + u_3) \\
&\quad + e_4(-\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 + y_3 - y_3^3 - \rho_2 y_4 + F_2\cos\xi_2 t + u_4)
\end{aligned} \tag{7-3-21}$$

Choose

$$\begin{aligned}
u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - x_1 - y_1 \\
u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 - \rho_1 x_2 - F_1\cos\xi_1 t \\
u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - x_3 - y_3 \\
u_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 - \rho_2 x_4 - F_2\cos\xi_2 t
\end{aligned}$$

Eq. (7-3-21) can be rewritten as

$$\dot{V} = -e_1 - \rho_1 e_2 - e_3 - \rho_2 e_4 < 0 \tag{7-3-22}$$

which is negative definite function in the first quadrant. Theorem 2 in Appendix B is satisfied. This means that generalized anti-synchronization of the Quantum-CNN system with a double Duffing chaotic system can be achieved. The numerical results are shown in Fig. 7-9. After 1 second, our aim is achieved.

Case III Generalized synchronization with $F_1(t)=F_3(t)=0$, $F_2(t)=y_1$ and $F_4(t)=y_3$

When $i = 1, 2, 3, 4$, negative signs are chosen before y_i in Eq. (7-3-5). For $i = 1, 3$, $F_1(t) = F_3(t) = 0$ and for $i = 2, 4$, $F_2(t) = y_1(t), F_4(t) = y_3(t)$ are chosen, where $y_1(t)$ and $y_3(t)$ are chaotic functions of time. Then we have

$$\lim_{t\to\infty} e_1 = \lim_{t\to\infty}(x_1 - y_1 + 10) = 0, \quad \lim_{t\to\infty} e_3 = \lim_{t\to\infty}(x_3 - y_3 + 10) = 0 \tag{7-3-23}$$

$$\lim_{t\to\infty} e_2 = \lim_{t\to\infty}(x_2 - y_2 + y_1 + 10) = 0, \quad \lim_{t\to\infty} e_4 = \lim_{t\to\infty}(x_4 - y_4 + y_3 + 10) = 0 \tag{7-3-24}$$

Eq. (7-3-6) becomes

$$\begin{cases} \dot{e}_1 = \dot{x}_1 - \dot{y}_1 \\ \dot{e}_2 = \dot{x}_2 - \dot{y}_2 + \dot{y}_1 \\ \dot{e}_3 = \dot{x}_3 - \dot{y}_3 \\ \dot{e}_4 = \dot{x}_4 - \dot{y}_4 + \dot{y}_3 \end{cases} \tag{7-3-25}$$

Eq. (7-3-7) becomes

$$\begin{aligned} \dot{e}_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1 \\ \dot{e}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2 + y_2 \\ \dot{e}_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - u_3 \\ \dot{e}_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 + \rho_2 y_4 - F_2\cos\xi_2 t - u_4 + y_4 \end{aligned} \tag{7-3-26}$$

where $e_1 = x_1 - y_1 + 10,\ e_2 = x_2 - y_2 + y_1 + 10,\ e_3 = x_3 - y_3 + 10$ and $e_4 = x_4 - y_4 + y_3 + 10$.

Choose a Lyapunov function in the form of the positive definite function in the first quadrant:

$$V(e_1, e_2, e_3, e_4) = e_1 + e_2 + e_3 + e_4 \tag{7-3-27}$$

Its time derivative along any solution of Eq. (7-3-26) is

$$\begin{aligned} \dot{V} &= (-2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 - u_1) + (-\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 \\ &+ y_1^3 + \rho_1 y_2 - F_1\cos\xi_1 t - u_2 + y_2) + (-2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 - u_3) + \\ &(-\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 + \rho_2 y_4 - F_2\cos\xi_2 t - u_4 + y_4) \end{aligned} \tag{7-3-28}$$

Choose

$$\begin{aligned} u_1 &= -2a_1\sqrt{1-x_1^2}\sin x_2 - y_2 + x_1 - y_1 \\ u_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2 - y_1 + y_1^3 + \rho_1 x_2 - F_1\cos\xi_1 t + y_2 + \rho_1 y_1 \\ u_3 &= -2a_2\sqrt{1-x_3^2}\sin x_4 - y_4 + x_3 - y_3 \\ u_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4 - y_3 + y_3^3 + \rho_2 x_4 - F_2\cos\xi_2 t + y_4 + \rho_2 y_3 \end{aligned}$$

Eq. (7-3-28) can be rewritten as

$$\dot{V} = -e_1 - \rho_1 e_2 - e_3 - \rho_2 e_4 < 0 \tag{7-3-29}$$

which is negative definite function in the first quadrant. Theorem 2 in Appendix B is satisfied. The numerical results are shown in Fig. 7-10. After 1 second, our aim is achieved.

Case IV Generalized synchronization with $F_1(t) = G_1\sin\varpi_1 t, F_3(t) = G_3\cos\varpi_3 t$, $F_2(t)=y_1$ and $F_4(t)=y_3$

When $i=1,2,3,4$, negative signs are chosen before y_i in Eq. (7-3-5). For $i=1,3$, $F_1(t)=G_1\sin\varpi_1 t, F_3(t)=G_3\cos\varpi_3 t$, and for $i=2,4$, $F_2(t)=y_1(t), F_4(t)=y_3(t)$ are chosen, where $y_1(t)$ and $y_3(t)$ are chaotic functions of time. We have

$$\begin{aligned}&\lim_{t\to\infty} e_1=\lim_{t\to\infty}(x_1-y_1+G_1\sin\varpi_1 t+10)=0,\quad \lim_{t\to\infty} e_2=\lim_{t\to\infty}(x_2-y_2+y_1+10)=0\\&\lim_{t\to\infty} e_3=\lim_{t\to\infty}(x_3-y_3+G_3\cos\varpi_3 t+10)=0,\quad \lim_{t\to\infty} e_4=\lim_{t\to\infty}(x_4-y_4+y_3+10)=0\end{aligned} \tag{7-3-30}$$

Eq. (7-3-6) becomes

$$\begin{cases}\dot e_1=\dot x_1-\dot y_1+\dot F_1(t), & \dot e_2=\dot x_2-\dot y_2+\dot F_2(t)\\ \dot e_3=\dot x_3-\dot y_3+\dot F_3(t), & \dot e_4=\dot x_4-\dot y_4+\dot F_4(t)\end{cases} \tag{7-3-31}$$

Eq. (7-3-7) becomes

$$\begin{aligned}\dot e_1&=-2a_1\sqrt{1-x_1^2}\sin x_2-y_2-u_1+G_1\varpi_1\cos\varpi_1 t\\ \dot e_2&=-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2-y_1+y_1^3+\rho_1 y_2-F_1\cos\xi_1 t-u_2+y_2\\ \dot e_3&=-2a_2\sqrt{1-x_3^2}\sin x_4-y_4-u_3-G_3\varpi_3\sin\varpi_3 t\\ \dot e_4&=-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4-y_3+y_3^3+\rho_2 y_4-F_2\cos\xi_2 t-u_4+y_4\end{aligned} \tag{7-3-32}$$

where $e_1=x_1-y_1+G_1\sin\varpi_1 t+10,\quad e_2=x_2-y_2+y_1+10,\quad e_3=x_3-y_3+G_3\cos\varpi_3 t+10,$ $e_4=x_4-y_4+y_3+10,\quad G_1=5.37, G_3=5.71, \varpi_1=3.1, \varpi_3=4.9$.

Choose a Lyapunov function in the form of the positive definite function in the first quadrant:

$$V(e_1,e_2,e_3,e_4)=e_1+e_2+e_3+e_4 \tag{7-3-33}$$

Its time derivative along any solution of Eq. (7-3-32) is

$$\begin{aligned}\dot V=&(-2a_1\sqrt{1-x_1^2}\sin x_2-y_2-u_1+G_1\varpi_1\cos\varpi_1 t)+(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\\&-y_1+y_1^3+\rho_1 y_2-F_1\cos\xi_1 t-u_2+y_2)+(-2a_2\sqrt{1-x_3^2}\sin x_4-y_4-u_3-G_3\varpi_3\sin\varpi_3 t)\\&+(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4-y_3+y_3^3+\rho_2 y_4-F_2\cos\xi_2 t-u_4+y_4)\end{aligned} \tag{7-3-34}$$

Choose

$$\begin{aligned}u_1&=-2a_1\sqrt{1-x_1^2}\sin x_2-y_2+x_1-y_1+G_1(\varpi_1\cos\varpi_1 t+\sin\varpi_1 t)\\ u_2&=-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2-y_1+y_1^3+\rho_1 x_2-F_1\cos\xi_1 t+y_2+\rho_1 y_1\\ u_3&=-2a_2\sqrt{1-x_3^2}\sin x_4-y_4+x_3-y_3+G_3(\cos\varpi_3 t-\varpi_3\sin\varpi_3 t)\\ u_4&=-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4-y_3+y_3^3+\rho_2 x_4-F_2\cos\xi_2 t+y_4+\rho_2 y_3\end{aligned}$$

Eq. (7-3-34) can be rewritten as

$$\dot{V} = -e_1 - \delta_1 e_2 - e_3 - \delta_2 e_4 < 0 \tag{7-3-35}$$

which is negative definite function in the first quadrant. Theorem 2 in Appendix B is satisfied. The numerical results are shown in Fig. 7-11. After 1 second, our aim is achieved.

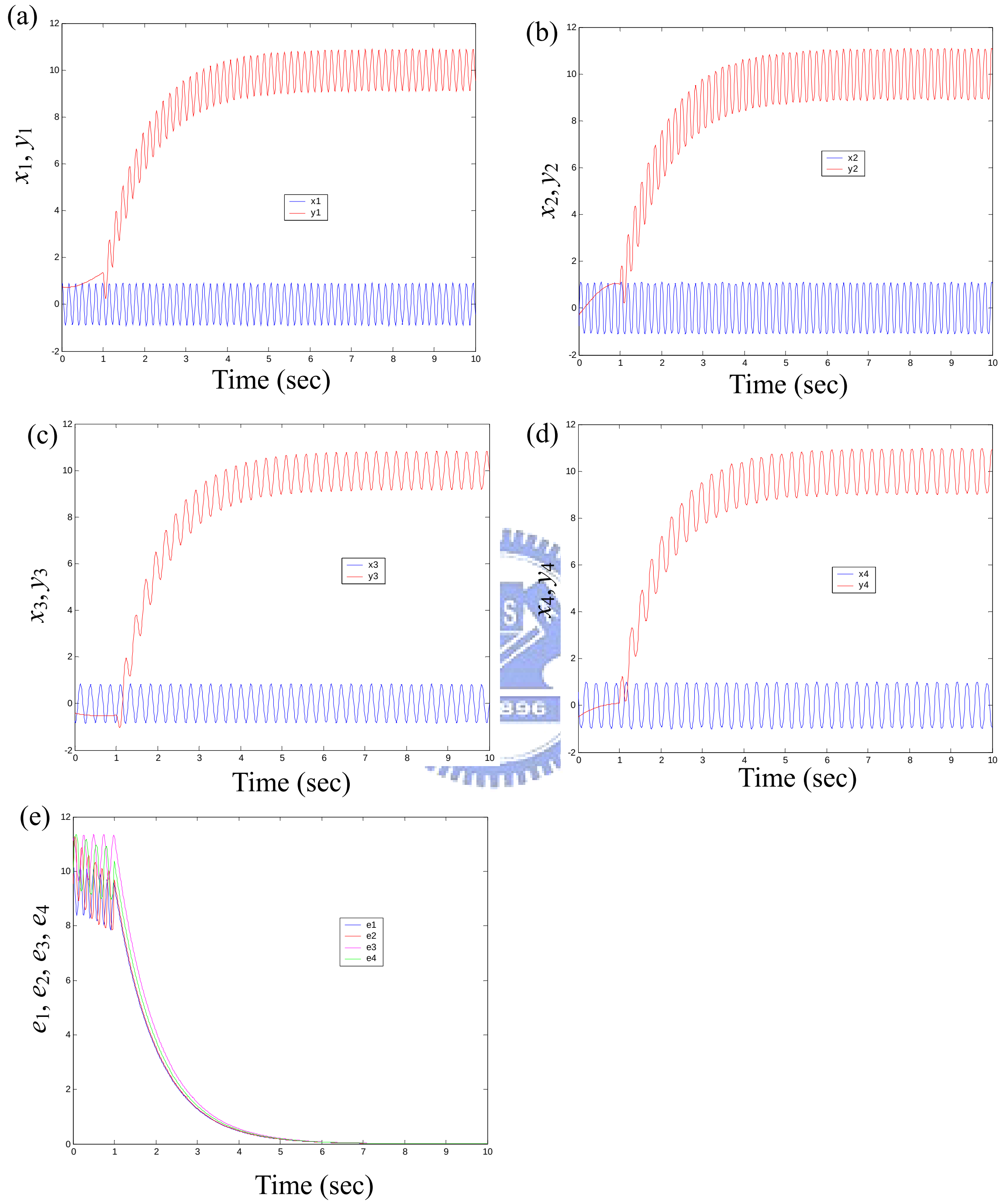


Fig. 7-8. Time histories of the master states, of the slave states and of the errors for Case I.

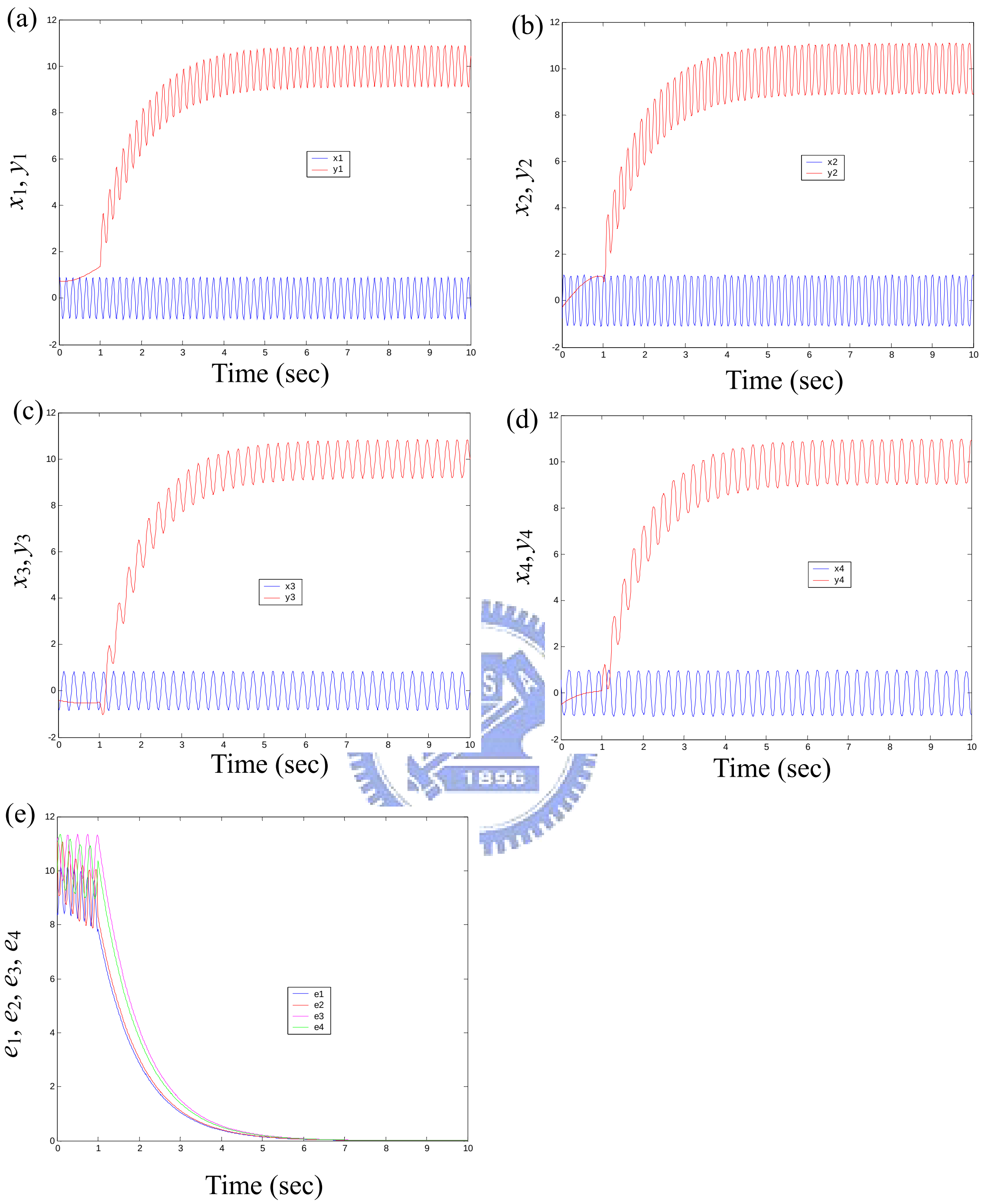


Fig. 7-9. Time histories of the master states, of the slave states and of the errors for Case II.

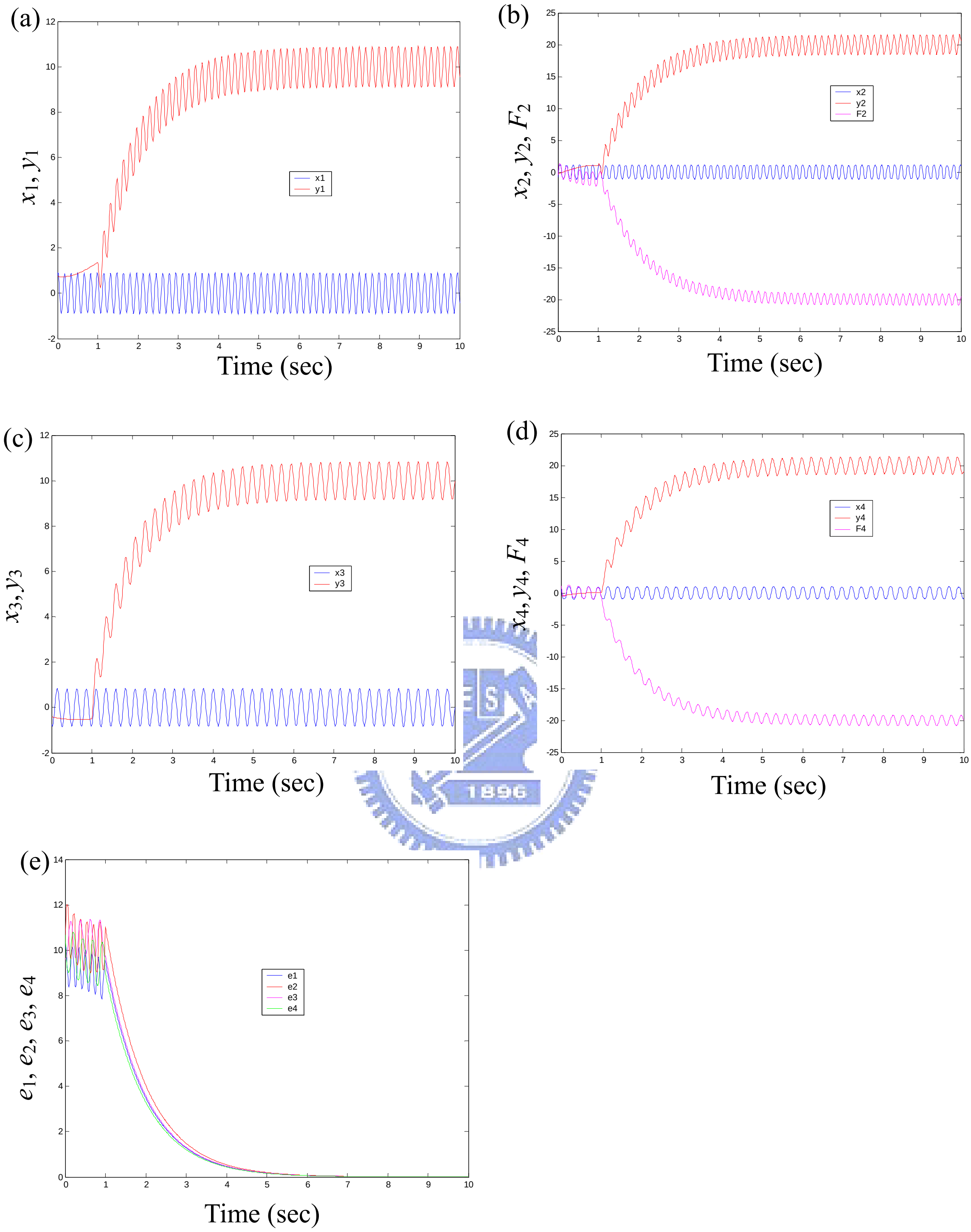


Fig. 7-10. The master state, the slave state, the error, F_2 and F_4 time histories for Case III.

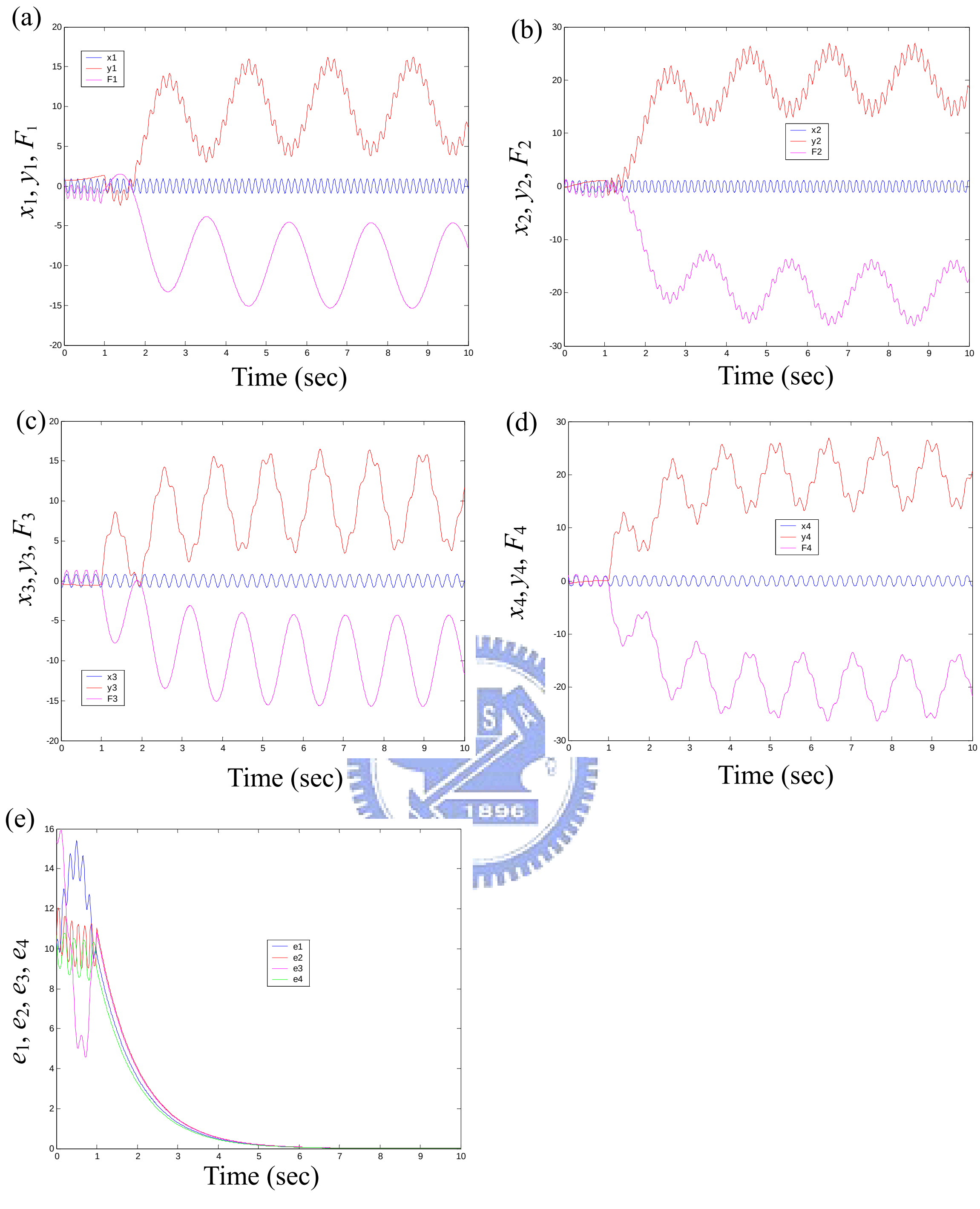


Fig. 7-11. The master state, the slave state, the error, F_1, F_2, F_3 and F_4 time histories for Case IV.

Chapter 8

Conclusions

Hyperchaos, chaos control, chaotization and synchronization for a Quantum Cellular Neural Networks (Quantum-CNN) is studied in this thesis. It is an autonomous four-order nano system. In order to analyze a variety of chaotic phenomena, we employ several numerical techniques such as time history, phase portrait, power spectrum and Lyapunov exponents.

The dynamic characteristics of Quantum-CNN are discussed in Chapter 2. The system model is described, and the numerical results of bounded and of unbounded chaotic phenomenon are presented.

In Chapter 3, the chaos control of the Quantum-CNN system is investigated by using the Routh-Hurwitz theorem and Lyapunov asymptotical stability theorem. The linear, nonlinear sine, and nonlinear state cross cosine nonlinear function feedback controls are used to suppress chaos to unstable equilibrium $O\,(0, 0, 0, 0)$. Numerical simulations are used to verify the effectiveness of the proposed controllers. By the linear function feedback, the states reach equilibrium $O\,(0, 0, 0, 0)$ most quickly and by the nonlinear state cross cosine nonlinear function feedback, the states reach equilibrium $O\,(0, 0, 0, 0)$ most slowly in Fig. 3-6. The chaos controls of a Quantum-CNN system are also studied. The impulse control and variable structure control are used to suppress chaos to fixed point or regulation motion. These chaos control methods can be also used for other chaotic systems. The optimum control is used for chaotization, i.e. to enhance original chaotic state to more complex chaotic state. Numerical simulations are used to verify the effectiveness of the proposed controllers.

In Chapter 4, the generalized chaos synchronization of the different order systems are investigated by using the Lyapunov asymptotical stability theorem. Two different chaotic dynamical systems, the Quantum-CNN system and the Lorenz system, are chosen for the complete synchronization as a special case and for the given regular time function synchronization. The Chen system is chosen for the given chaotic time function synchronizations. The generalized synchronization of chaos system can be used to increase the security of communication. The generalized chaos synchronization of the Quantum-CNN chaotic oscillator with a double Duffing system is investigated by the Lyapunov asymptotical stability theorem. Different chaotic dynamical systems, the Quantum-CNN system and a double Duffing chaotic system, are synchronized in four cases: the partial complete synchronization and the partial anti-synchronization, the anti-synchronization, the partial complete synchronization and the partial slave self-synchronization in which $F_2(t)$ and $F_4(t)$ are given chaotic functions of time and the partial sine/cosine function synchronization and the partial slave self-synchronization in which $F_1(t)$, $F_3(t)$ are given sine/cosine functions of time and $F_2(t)$, $F_4(t)$ are given chaotic functions of time. These generalized synchronizations of chaos systems can be used to increase the security of communication. A new symplectic synchronization of chaos of the Quantum-CNN chaotic oscillator with the Rössler system is implemented by the Lyapunov asymptotical stability theorem. Two different chaotic dynamical systems, the Quantum-CNN system and the Rössler systems, are synchronization in tow cases: the time delay symplectic synchronization and the cubic time delay symplectic synchronization. Symplectic synchronization of chaos systems can be used to increase the security of secure communication.

In Chapter 5, the synchronization of complex chaotic systems in series expansion form are implemented by the Lyapunov asymptotical stability theorem. Two Quantum-CNN systems are synchronized in two cases: unidirectional linear coupling case and mutual linear coupling case. In both cases, by a theorem of convergent series, we prove that all infinite power series in the elements of $B = A + M(x(t), y(t)) - (k_1 + k_2)$ are convergent and have bounded sums. Two Quantum-CNN systems are synchronized in two cases: unidirectional linear coupling by optimum control, mutual linear coupling by optimum control. The number of controllers for optimum control is less than that for synchronization only by linear couplings. This results in lower cost. In chaos synchronization cases, by a theorem of convergent series, we prove that all infinite power series in the elements of $A+M(x(t), y(t))–B(k_s+k_a)$ are convergent and have bounded sums. These synchronizations of chaos systems can be used to increase the security of communication. The chaotic phenomena and chaos controls of a Quantum-CNN system are studied.

In Chapter 6, in this thesis pragmatical generalized synchronization of adaptive control is studied. The pragmatical asymptotical stability theorem fills the vacancy between the actual asymptotical stability and mathematical asymptotical stability, the conditions of the Lyapunov function for pragmatical asymptotical stability are lower than that for traditional asymptotical stability. By using this theorem, with the same conditions for Lyapunov function, $V > 0$, $\dot{V} \le 0$, as that in current scheme of adaptive synchronization, we not only obtain the generalized synchronization of chaotic systems but so prove that the estimated parameters approach the uncertain values. Two Quantum-CNN chaotic systems and a double Duffing chaotic system are used as master system, slave system, and goal system, respectively, in two cases: the chaotic states of a goal system, a double Duffing chaotic system, used as $F(t)$ and the cubics of chaotic states of the same goal system used as $F(t)$. These generalized synchronizations of chaotic systems by adaptive control can be used to increase the security of communication. To control chaotic systems to different systems is study by new pragmatical adaptive control method. Traditional chaos control is limited for the same system. This method enlarges the function of chaos control. We can control a chaotic system either to a given simple system or to more complex given chaotic system. The method also simplifies the controllers and reduces their cost. Using Lipschitz condition, the synchronization of Lorenz chaotic systems and of Quantum-CNN chaotic oscillator systems with uncertain **variable/chaotic** parameters by linear coupling and pragmatical adaptive tracking is implemented by the GYC pragmatical asymptotical stability theorem. Tracking uncertain **variable/chaotic** parameters is firstly studied in this thesis. This is more reasonable, because system parameters always vary due to aging, environment and disturbances. Two Lorenz systems are synchronization in one case: with oscillating parameters. Two Quantum-CNN systems are synchronization in two cases: (1) with exponentially increasing and decreasing parameters (2) with periodically and exponentially increasing and decreasing parameters. The simulation results imply that the present scheme is very satisfactory. Two Lorenz systems are synchronization in two cases: chaotic parameters M=N and chaotic parameters $M < N$. Two Quantum-CNN systems are synchronized for chaotic parameters M=N. The simulation results imply that this scheme is very effective.

In Chapter 7, a new strategy to achieve chaos control by partial region stability is proposed. By using GYC theory of stability on partial region the Lyapunov function is a simple linear homogeneous function of error states and the controller is simpler because they are of lower order. The pragmatical generalized synchronization of adaptive control is studied. The pragmatical asymptotical stability theorem fills the vacancy between the actual asymptotical stability and mathematical asymptotical stability, the conditions of the Lyapunov function for pragmatical asymptotical stability are lower than that for traditional asymptotical stability. By using this theorem, with the same conditions for Lyapunov function, $V > 0$, $\dot{V} \leq 0$, as that in current scheme of adaptive synchronization, we not only obtain the generalized synchronization of chaotic systems but so prove that the estimated parameters approach the uncertain values. Two Quantum-CNN chaotic systems and a double Duffing chaotic system are used as master system, slave system, and goal system, respectively, in two cases: the chaotic states of a goal system, a double Duffing chaotic system, used as $F(t)$ and the cubics of chaotic states of the same goal system used as $F(t)$. These generalized synchronizations of chaotic systems by adaptive control can be used to increase the security of communication.

Appendix A

GYC Pragmatical Asymptotical Stability Theorem

The stability for many problems in real dynamical systems is actual asymptotical stability, although may not be mathematical asymptotical stability. The mathematical asymptotical stability demands that trajectories from all initial states in the neighborhood of zero solution must approach the origin as $t \to \infty$. If there are only a small part or even a few of the initial states from which the trajectories do not approach the origin as $t \to \infty$, the zero solution is not mathematically asymptotically stable. However, when the probability of occurrence of an event is zero, it means the event does not occur actually. If the probability of occurrence of the event that the trajectories from the initial states are that they do not approach zero when $t \to \infty$, i.e. these trajectories are not asymptotical stable for zero solution, is zero, the stability of zero solution is actual asymptotical stability though it is not mathematical asymptotical stability. In order to analyze the asymptotical stability of the equilibrium point of such systems, the pragmatical asymptotical stability theorem is used.

Let X and Y be two manifolds of dimensions m and n ($m<n$), respectively, and φ be a differentiable map from X to Y; then $\varphi(X)$ is subset of Lebesque measure 0 of Y [96]. For an autonomous system

$$\dot{x} = f(x_1, \cdots, x_n) \tag{A-1}$$

where $x = [x_1, \cdots, x_n]^T$ is a state vector, the function $f = [f_1, \cdots, f_n]^T$ is defined on $D \subset R^n$, an n-manifold.

Let x=0 be an equilibrium point for the system (A-1). Then

$$f(0) = 0. \tag{A-2}$$

For a nonautonomous system,

$$\dot{x} = f(x_1, \cdots, x_{n+1}) \tag{A-3}$$

where $x = [x_1, \cdots, x_{n+1}]^T$, the function $f = [f_1, \cdots, f_n]^T$ is define on $D \subset R^n \times R_+$, here

$t = x_{n+1} \subset R_+$. The equilibrium point is

$$f(0, x_{n+1}) = 0. \tag{A-4}$$

Definition The equilibrium point for the system is pragmatically asymptotically stable provided that with initial points on *C* which is a subset of Lebesque measure 0 of *D*, the behaviors of the corresponding trajectories cannot be determined, while with initial points on $D-C$, the corresponding trajectories behave as that agree with traditional asymptotical stability [94-95].

Theorem Let $V = [x_1, x_2, \cdots, x_n]^T$: $D \to R_+$ be positive definite and analytic on *D*, where $x_1, x_2, \cdots, x_n$ are all space coordinates such that the derivative of *V* through Eqs (A-1) or (A-3), $\dot{V}$, is negative semi-definite of $[x_1, x_2, \cdots, x_n]^T$.

For autonomous system, let *X* be the *m*-manifold consisting of point set for which $\forall x \neq 0$, $\dot{V}(x) = 0$ and *D* is an *n*-manifold. If *m*+1<*n*, then the equilibrium point of the system is pragmatically asymptotically stable.

For nonautonomous system, let *X* be the *m*+1-manifold consisting of point set for which $\forall x \neq 0$, $\dot{V}(x_1, x_2, \cdots, x_n) = 0$ and *D* is an *n*+1-manifold. If *m*+1+1<*n*+1, i.e. *m*+1<*n* then the equilibrium point of the system is pragmatically asymptotically stable. Therefore, for both autonomous and nonautonomous system the formula *m*+1<*n* is universal. So the following proof is only for autonomous system. The proof for nonautonomous system is similar.

Proof Since every point of *X* can be passed by a trajectory of Eq. (A-1), which is one-dimensional, the collection of these trajectories, *C*, is a (*m*+1)-manifold [94-95].

If *m*+1 < *n*, then the collection *C* is a subset of Lebesque measure 0 of *D*. By the above definition, the equilibrium point of the system is pragmatically asymptotically stable.

If an initial point is ergodicly chosen in *D*, *the probability of that the initial point falls on the collection C is zero*. *Here, equal probability is assumed for every point chosen as an initial point in the neighborhood of the equilibrium point.* Hence, the event that the initial point is chosen from collection *C does not occur actually*. Therefore, under the equal probability assumption, pragmatical asymptotical stability becomes actual asymptotical stability. When the initial point falls on $D-C$, $\dot{V}(x)<0$, the corresponding trajectories behave as that agree with traditional asymptotical stability because by the existence and uniqueness of the solution of initial-value problem, these trajectories never meet *C*.

For Eq.(6-2-8), Eq.(6-3-7) and Eq.(6-4-7), the Lyapunov function is a positive definite function of *n* variables, i.e. *p* error state variables and *n-p=m* differences between unknown and estimated parameters, while $\dot{V}=e^{T}Ce$ is a negative semi-definite function of *n* variables. Since the number of error state variables is always more than one, *p*>1, *m*+1<*n* is always satisfied, by pragmatical asymptotical stability theorem we have

$$\lim_{t\to\infty} e = 0 \qquad \text{(A-5)}$$

and the estimated parameters approach the uncertain parameters. The pragmatical adaptive control theorem is obtained. Therefore, the equilibrium point of the system is *pragmatically asymptotically stable. Under the equal probability assumption, it is actually asymptotically stable for both error state variables and parameter variables.*

Appendix B

GYC Partial Region Stability Theory

Consider the differential equations of disturbed motion of a nonautonomous system in the normal form

$$\dot{x}_s = X_s(t, x_1, \cdots, x_n), \quad (s = 1, \cdots, n) \tag{B-1}$$

where the function X_s is defined on the intersection of the partial region Ω (shown in Fig. B-1) and

$$\sum_s x_s^2 \leq H \tag{B-2}$$

and $t > t_0$, where t_0 and H are certain positive constants. X_s which vanishes when the variables x_s are all zero, is a real valued function of t, $x_1, \cdots, x_n$. It is assumed that X_s is smooth enough to ensure the existence, uniqueness of the solution of the initial value problem. When X_s does not contain t explicitly, the system is autonomous.

Obviously, $x_s = 0 \quad (s = 1, \cdots, n)$ is a solution of Eq. (B-1). We are interested to the asymptotical stability of this zero solution on partial region Ω (including the boundary) of the neighborhood of the origin which in general may consist of several subregions (Fig. 7-1).

Definition 1:

For any given number $\varepsilon > 0$, if there exists a $\delta > 0$, such that on the closed given partial region Ω when

$$\sum_s x_{s0}^2 \leq \delta, \quad (s = 1, \cdots, n) \tag{B-3}$$

for all $t \geq t_0$, the inequality

$$\sum_s x_s^2 < \varepsilon, \quad (s = 1, \cdots, n) \tag{B-4}$$

is satisfied for the solutions of Eq.(B-1) on Ω, then the disturbed motion $x_s = 0 \quad (s = 1, \cdots, n)$ is stable on the partial region Ω.

Definition 2:

If the undisturbed motion is stable on the partial region Ω, and there exists a $\delta' > 0$, so that on the given partial region Ω when

$$\sum_s x_{s0}^2 \leq \delta', \quad (s = 1, \cdots, n) \tag{B-5}$$

The equality

$$\lim_{t\to\infty}\left(\sum_s x_s^2\right)=0 \tag{B-6}$$

is satisfied for the solutions of Eq. (B-1) on Ω, then the undisturbed motion x_s=0 $(s=1,\cdots,n)$ is asymptotically stable on the partial region Ω.

The intersection of Ω and region defined by Eq. (B-5) is called the region of attraction.

Definition of Functions $V(t,x_1,\cdots,x_n)$ **:**

Let us consider the functions $V(t,x_1,\cdots,x_n)$ given on the intersection Ω_1 of the partial region Ω and the region

$$\sum_s x_s^2 \le h,\ \ (s=1,\cdots,n) \tag{B-7}$$

for $t\ge t_0>0$, where t_0 and h are positive constants. We suppose that the functions are single-valued and have continuous partial derivatives and become zero when $x_1=\cdots=x_n=0$.

Definition 3:

If there exists t_0>0 and a sufficiently small h>0, so that on partial region Ω_1 and $t\ge t_0$, $V\ge 0$ (or ≤ 0), then V is a positive (or negative) semi-definite, in general semi-definite, function on the Ω_1 and $t\ge t_0$.

Definition 4:

If there exists a positive (negative) definitive function $W(x_1,\cdots,x_n)$ on Ω_1, so that on the partial region Ω_1 and $t\ge t_0$

$$V-W\ge 0\ \ (\text{or } -V-W\ge 0) \tag{B-8}$$

then $V(t,x_1,\cdots,x_n)$ is a positive definite function on the partial region Ω_1 and $t\ge t_0$.

Definition 5:

If $V(t,x_1,\cdots,x_n)$ is neither definite nor semi-definite on Ω_1 and $t\ge t_0$, then $V(t,x_1,\cdots,x_n)$ is an indefinite function on partial region Ω_1 and $t\ge t_0$. That is, for any small $h>0$ and any large $t_0>0$, $V(t,x_1,\cdots,x_n)$ can take either positive or negative value on the partial region Ω_1 and $t\ge t_0$.

Definition 6: Bounded function V

If there exist t_0>0, h>0, so that on the partial region Ω_1, we have

$$\left|V(t,x_1,\cdots,x_n)\right|<L \tag{B-9}$$

where L is a positive constant, then V is said to be bounded on Ω_1.

Definition 7: Function with infinitesimal upper bound

If V is bounded, and for any λ>0, there exists μ>0, so that on Ω_1 when $\sum_s x_s^2 \le \mu$, and $t \ge t_0$, we have

$$\left|V(t, x_1, \cdots, x_n)\right| \le \lambda \tag{B-10}$$

then V admits an infinitesimal upper bound on Ω_1.

Theorem of stability and of asymptotical stability on partial region

Theorem 1

If there can be found for the differential equations of the disturbed motion (Eq. (B-1) a definite function $V(t, x_1, \cdots, x_n)$ on the partial region, and for which the derivative with respect to time based on these equations as given by the following:

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} X_s \tag{B-11}$$

is a semi-definite function on the partial region whose sense is opposite to that of V, or if it becomes zero identically, then the undisturbed motion is stable on the partial region.

Proof:

Let us assume for the sake of definiteness that V is a positive definite function. Consequently, there exists a sufficiently large number t_0 and a sufficiently small number $h < H$, such that on the intersection Ω_1 of partial region Ω and

$$\sum_s x_s^2 \le h, \quad (s = 1, \cdots, n) \tag{B-12}$$

and $t \ge t_0$, the following inequality is satisfied

$$V(t, x_1, \cdots, x_n) \ge W(x_1, \cdots, x_n), \tag{B-13}$$

where W is a certain positive definite function which does not depend on t. Besides that, Eq. (B-11) may assume only negative or zero value in this region.

Let ε be an arbitrarily small positive number. We shall suppose that in any case $\varepsilon < h$. Let us consider the aggregation of all possible values of the quantities $x_1, \cdots, x_n$, which are on the intersection ω_2 of Ω_1 and

$$\sum_s x_s^2 = \varepsilon, \tag{B-14}$$

and let us designate by $l > 0$ the precise lower limit of the function W under this condition. By virtue of Eq. (B-8), we shall have

$$V(t, x_1, \cdots, x_n) \geq l \quad \text{for} \quad (x_1, \cdots, x_n) \quad \text{on} \quad \omega_2 . \tag{B-15}$$

We shall now consider the quantities x_s as functions of time, which satisfy the differential equations of disturbed motion. We shall assume that the initial values x_{s0} of these functions for $t=t_0$ lie on the intersection Ω_2 of Ω_1 and the region

$$\sum_s x_s^2 \leq \delta, \tag{B-16}$$

where δ is so small that

$$V(t_0, x_{10}, \cdots, x_{n0}) < l \tag{B-17}$$

By virtue of the fact that $V(t_0, 0, \cdots, 0) = 0$, such a selection of the number δ is obviously possible. We shall suppose that in any case the number δ is smaller than ε. Then the inequality

$$\sum_s x_s^2 < \varepsilon, \tag{B-18}$$

being satisfied at the initial instant will be satisfied, in the very least, for a sufficiently small $t - t_0$, since the functions $x_s(t)$ very continuously with time. We shall show that these inequalities will be satisfied for all values $t > t_0$. Indeed, if these inequalities were not satisfied at some time, there would have to exist such an instant $t=T$ for which this inequality would become an equality. In other words, we would have

$$\sum_s x_s^2(T) = \varepsilon, \tag{B-19}$$

and consequently, on the basis of Eq. (B-15)

$$V(T, x_1(T), \cdots, x_n(T)) \geq l \tag{B-20}$$

On the other hand, since $\varepsilon < h$, the inequality Eq. (B-7) is satisfied in the entire interval of time $[t_0, T]$, and consequently, in this entire time interval $\dot{V} \leq 0$. This yields

$$V(T, x_1(T), \cdots, x_n(T)) \leq V(t_0, x_{10}, \cdots, x_{n0}), \tag{B-21}$$

which contradicts Eq. (B-18) on the basis of Eq. (B-17). Thus, the inequality Eq. (B-4) must be satisfied for all values of $t > t_0$, hence follows that the motion is stable.

Finally, we must point out that from the viewpoint of mathematics, the stability on partial region in general does not be related logically to the stability on whole region. If an undisturbed solution is stable on a partial region, it may be either stable or unstable on the whole region and vice versa. From the viewpoint of dynamics, we were not interesting to the solution starting from Ω_2 and going out of Ω.

Theorem 2

If in satisfying the conditions of theorem 1, the derivative $\dot{V}$ is a definite function on the partial region with opposite sign to that of V and the function V itself permits an infinitesimal upper limit, then the undisturbed motion is asymptotically stable on the partial region.

Proof:

Let us suppose that V is a positive definite function on the partial region and that consequently, $\dot{V}$ is negative definite. Thus on the intersection Ω_1 of Ω and the region defined by Eq. (B-7) and $t \ge t_0$ there will be satisfied not only the inequality Eq. (B-8), but the following inequality as will:

$$\dot{V} \le -W_1(x_1, \cdots, x_n), \tag{B-22}$$

where W_1 is a positive definite function on the partial region independent of t.

Let us consider the quantities x_s as functions of time, which satisfy the differential equations of disturbed motion assuming that the initial values $x_{s0} = x_s(t_0)$ of these quantities satisfy the inequalities Eq. (B-16). Since the undisturbed motion is stable in any case, the magnitude δ may be selected so small that for all values of $t \ge t_0$ the quantities x_s remain within Ω_1. Then, on the basis of Eq. (B-22) the derivative of function $V(t, x_1(t), \cdots, x_n(t))$ will be negative at all times and, consequently, this function will approach a certain limit, as t increases without limit, remaining larger than this limit at all times. We shall show that this limit is equal to some positive quantity different from zero. Then for all values of $t \ge t_0$ the following inequality will be satisfied:

$$V(t, x_1(t), \cdots, x_n(t)) > \alpha \tag{B-23}$$

where $\alpha > 0$.

Since V permits an infinitesimal upper limit, it follows from this inequality that

$$\sum_{s} x_s^2(t) \geq \lambda\,, \quad (s = 1, \cdots, n)\,, \tag{B-24}$$

where λ is a certain sufficiently small positive number. Indeed, if such a number λ did not exist, that is, if the quantity $\sum_{s} x_s(t)$ were smaller than any preassigned number no matter how small, then the magnitude $V(t, x_1(t), \cdots, x_n(t))$, as follows from the definition of an infinitesimal upper limit, would also be arbitrarily small, which contradicts (B-13).

If for all values of $t \geq t_0$ the inequality Eq. (B-24) is satisfied, then Eq. (B-22) shows that the following inequality will be satisfied at all times:

$$\dot{V} \leq -l_1, \tag{B-25}$$

where l_1 is positive number different from zero which constitutes the precise lower limit of the function $W_1(t, x_1(t), \cdots, x_n(t))$ under condition Eq. (B-24). Consequently, for all values of $t \geq t_0$ we shall have:

$$V(t, x_1(t), \cdots, x_n(t)) = V(t_0, x_{10}, \cdots, x_{n0}) + \int_{t_0}^{t} \frac{dV}{dt} dt \leq V(t_0, x_{10}, \cdots, x_{n0}) - l_1(t - t_0), \tag{B-26}$$

which is, obviously, in contradiction with Eq.(B-23). The contradiction thus obtained shows that the function $V(t, x_1(t), \cdots, x_n(t))$ approached zero as t increase without limit. Consequently, the same will be true for the function $W(x_1(t), \cdots, x_n(t))$ as well, from which it follows directly that

$$\lim_{t \to \infty} x_s(t) = 0\,, \quad (s = 1, \cdots, n) \tag{B-27}$$

which proves the theorem.

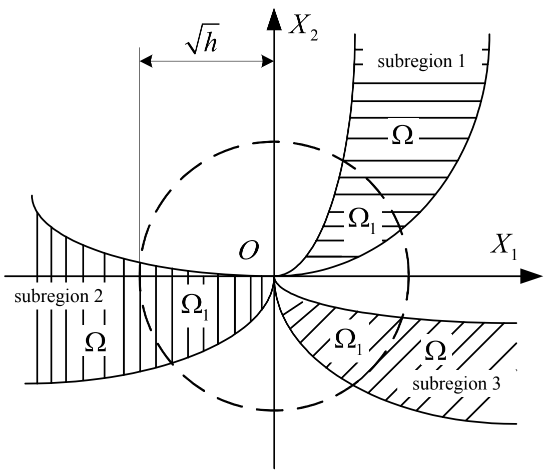


Fig. B-1 Partial Region Ω_1 and Ω_1.

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Chaos, Solitons and Fractals 期刊主編 M. S. El Naschie 教授對 “Generalized Synchronization of Quantum-CNN Chaotic Oscillator with Different Order Systems” 一文的評語

Dipl.-Ing. Prof. Dr. M.S. El Naschie, CPhys FInstP

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Dear prof Ge.

I received your paper
"Generalized Synchronization"

It seems like a very original
and excellent paper.
I will send it just for
refereeing and inform you
as soon as possible

Best Regards

Mohamed Elnaschie