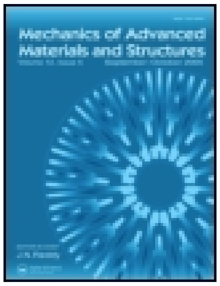


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# Geometrically Induced Stress Singularities of a Thick FGM Plate Based on the Third-Order Shear Deformation Theory

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# Geometrically Induced Stress Singularities of a Thick FGM Plate Based on the Third-Order Shear Deformation Theory

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**Asymptotic solutions for a functionally graded material (FGM) plate are developed to elucidate stress singularities at a plate corner, using a third-order shear deformation theory. The characteristic equations are given explicitly for determining the order of stress singularity at the vertex of a corner with two radial edges having various boundary conditions. The non-homogeneous elasticity properties are present only in the characteristic equations for a corner with one of its two edges simply supported. The effects of material non-homogeneity on the stress singularities are extensively examined. The present results are very useful for developing accurate numerical solutions for an FGM plate under static or dynamic loading when the plate involves stress singularities, such as a V-notch or crack.**

**Keywords** FGM plates, third-order shear deformation theory, stress singularities, asymptotic solutions, eigenfunction expansion

## 1. INTRODUCTION

In functionally graded materials (FGMs), the volume fractions of two or more materials vary continuously as a function of position in a particular dimension(s) to achieve a required functionality. FGMs were first developed in the mid-1980s [1]. Continuous changes in the microstructure of FGMs give mechanical properties better than those of traditional laminated composite materials, which are prone to debonding along layer interfaces because of abrupt changes in material properties across an interface. Gradual changes of material properties in FGMs can be designed for various applications and work environments. Consequently, over the last two decades, FGMs have been extensively explored in various fields including those of electron, chemistry, optics, biomedicine, aeronautical engineering, and mechanical engineering.

Plates in various geometric forms are commonly employed in practical engineering. Since solutions to plate problems based on plate theories are much simpler than those based on three-dimensional elasticity theory, various plate theories have been adopted to study the static or dynamic behaviors of FGM plates [2–8]. Numerous plates of various shapes have re-entrant corners. Stress singularities are typically present at the re-entrant corners, and stress singularity behaviors must be considered for accurate numerical analyses. Corner functions appropriately describing the stress singularity behaviors are used in various famous numerical approaches, such as the Ritz method [9, 10], the traditional finite element method [11, 12] and the mesh-free method [13, 14], for homogenous plates with stress singularities.

Geometrically induced stress singularities are those associated with irregular geometry, such as a notch or an abrupt change in a cross section. Such stress singularities have been comprehensively investigated for homogeneous plates. Williams [15] initially derived characteristic equations for determining singular orders of stress in an isotropic sector plate under bending using classical thin plate theory, while Burton and Sinclair [16], Huang [17–19] and McGee et al. [20] examined stress singularities using various thick plate theories. Different plate theories generally yield different stress singularity behaviors. Based on plane elasticity, Williams [21] pioneered the examination of the stress singularity of an isotropic plate under extension. The eigenfunction expansion technique was applied in these studies [15–21]. Bažant and Estenssoro [22], Keer and Parihar [23], Schmitz et al. [24] and Glushkov et al. [25] employed different numerical solution techniques to elucidate geometrically induced stress singularities at a three-dimensional vertex of a body.

Some analytical studies addressed stress singularities at the interface corner of a plate made of multiple materials. The plates considered in these studies are made of different materials in different regions, each of which is homogenous. For instance, Hein and Erdogan [26], Bogy and Wang [27], Rao [28] and Dempsey and Sinclair [29] utilized different approaches to investigate stress singularities based on plane elasticity theory, while Huang [30, 31] studied thick plates under bending.

These cited studies reveal the need to study geometrically induced stress singularities in FGM plates. Recently, the authors of this study [32] investigated for the first time stress singularities

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of a thin FGM plate using classical thin plate theory. Classical thin plate theory generally yields accurate solutions for plates with thicknesses less than 1/20 of the smallest in-plane dimension. Stress singularities of a thick FGM plate are important and are investigated herein. This work employs eigenfunction expansion to establish asymptotic solutions of the equilibrium equations in terms of displacement functions based on Reddy's plate theory [33], which is a third-order shear deformation theory. Characteristic equations are explicitly presented to determine the stress singularity order at the vertex of a corner with two radial edges having various boundary conditions. Moreover, asymptotic solutions are also explicitly shown for a corner with identical boundary conditions along its two radial edges. The effects of material non-homogeneity on stress singularities are thoroughly examined. This publication is the first to present such results.

## 2. THEORETICAL FORMULATION

The first-order shear deformation plate theory (FSDT), also known as Reissner-Mindlin's plate theory [34, 35], is the simplest plate theory considering shear deformation effect. Since the transverse shear strain is assumed constant through plate thickness, a shear correction coefficient is required. FSDT is considered accurate for analyzing moderately thick plates. The third-order shear deformation plate theory, developed by Reddy [33], is adopted herein to eliminate the need to choose an appropriate shear correction coefficient for FGM plates. Moreover, the displacement field and equilibrium equations for the third-order shear deformation theory are easily converted to those for FSDT, which is shown below. Reddy's plate theory contains a parabolic variation of transverse shear strains through plate thickness and satisfies free transverse shear stress on the top and bottom plate surfaces.

### 2.1. Displacement Field and Strains

The displacement field in Reddy's third-order shear deformation plate theory is assumed to be, in cylindrical coordinates as shown in Figure 1,

$$\bar{u}(r,\theta,z) = u_0(r,\theta) + z[\psi_r(r,\theta) - C_1 z^2(\psi_r(r,\theta) + w_{,r}(r,\theta))], \tag{1a}$$

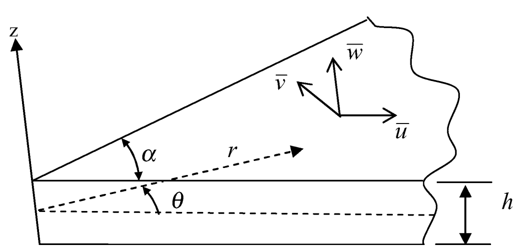



FIG. 1. Coordinate system and positive displacement components for a wedge.

$$\bar{v}(r,\theta,z) = v_0(r,\theta) + z\Big[\psi_\theta(r,\theta) - C_1 z^2\Big(\psi_\theta(r,\theta) + \frac{1}{r}w_{,\theta}(r,\theta)\Big)\Big], \tag{1b}$$

$$\bar{w} = w(r,\theta), \tag{1c}$$

where $\bar{u}$, $\bar{v}$ and $\bar{w}$ are displacement components in the $r$, $\theta$, and $z$ directions, respectively; $u_0$, $v_0$ and $w$ are corresponding displacements on the mid-plane; $\psi_r$ and $\psi_\theta$ are rotations of the mid-plane normal in the radial and circumferential directions, respectively; subscript ,$\beta$ refers to partial differentials with respect to the independent variable $\beta$, and $C_1 = 4/3h^2$. When $C_1$ is set to zero, the displacement field given by Eqs. (1a)–(1c) is identical to that for FSDT.

Linear strain components are expressed in terms of displacement functions as

$$\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \gamma_{r\theta} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{rr}^{(0)} \\ \varepsilon_{\theta\theta}^{(0)} \\ \gamma_{r\theta}^{0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{rr}^{(1)} \\ \varepsilon_{\theta\theta}^{(1)} \\ \gamma_{r\theta}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{rr}^{(3)} \\ \varepsilon_{\theta\theta}^{(3)} \\ \gamma_{r\theta}^{(3)} \end{Bmatrix}$$
$$\begin{Bmatrix} \gamma_{rz} \\ \gamma_{r\theta} \end{Bmatrix} = \begin{Bmatrix} \gamma_{rz}^{(0)} \\ \gamma_{\theta z}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \gamma_{rz}^{(2)} \\ \gamma_{\theta z}^{(2)} \end{Bmatrix} \tag{2}$$

where

$$\varepsilon_{rr}^{(0)} = u_{0,r},\ \varepsilon_{rr}^{(1)} = \psi_{r,r},\ \varepsilon_{rr}^{(3)} = -C_1\ (\psi_{r,r} + w_{,rr}),$$
$$\varepsilon_{\theta\theta}^{(0)} = \frac{1}{r}(u_0 + v_{0,\theta}),\ \varepsilon_{\theta\theta}^{(1)} = \frac{1}{r}\ (\psi_r + \psi_{\theta,\theta}),$$
$$\varepsilon_{\theta\theta}^{(3)} = -C_1\frac{1}{r}\left(\psi_{\theta,\theta} + \frac{1}{r}w_{,\theta\theta} + \psi_r + w_{,r}\right),$$
$$\gamma_{r\theta}^{(0)} = v_{0,r} - \frac{1}{r}v_0 + \frac{1}{r}u_{0,\theta},\ \gamma_{r\theta}^{(1)} = \psi_{\theta,r} + \frac{1}{r}\psi_{r,\theta} - \frac{1}{r}\psi_\theta,$$
$$\gamma_{r\theta}^{(3)} = -C_1\frac{1}{r}\Big[-\psi_\theta - \frac{2}{r}w_{,\theta} + \psi_{r,\theta} + 2\ w_{,r\theta} + r\ \psi_{\theta,r}\Big],$$
$$\gamma_{rz}^{(0)} = \psi_r + w_{,r},\ \gamma_{rz}^{(2)} = -C_2\ (\psi_r + w_{,r}),\ \gamma_{\theta z}^{(0)} = \psi_\theta + \frac{1}{r}w_{,\theta},$$
$$\gamma_{\theta z}^{(2)} = -C_2\left(\psi_\theta + \frac{1}{r}w_{,\theta}\right),\ C_2 = 3C_1. \tag{3}$$

### 2.2. Constitutive Relations

This study considers a thick wedge (or sector plate) (Figure 1), which is made of FGM with material properties varying through the thickness (z direction in Figure 1) only. Poisson's ratio ($\upsilon$) is assumed constant, and elasticity moduli vary according to

$$P(z) = P_b + V(z)\Delta P \tag{4}$$

where $V(z) = (z/h + 1/2)^m$; $P$ denotes a material property such as Young's modulus ($E$) or shear modulus ($G$); $P_b$ denotes $E$ or $G$ at the bottom plate face; $\Delta P$ is the difference between

$P_b$ and the corresponding property at the top plate face; $h$ is plate thickness; $m$ is the parameter that governs the material variation profile through the thickness; $m$ or $\Delta P=0$ represents a homogenous plate.

The linear constitutive relations between stresses and strains are

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \\ \sigma_{rz} \\ \sigma_{\theta z} \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\upsilon^2} & \frac{\upsilon E}{1-\upsilon^2} & 0 & 0 & 0 \\ \frac{\upsilon E}{1-\upsilon^2} & \frac{E}{1-\upsilon^2} & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{\theta z} \end{Bmatrix} \tag{5}$$

The transverse normal stress $\sigma_{zz}$ is assumed zero.

### 2.3. Equilibrium Equations

Using the principle of stationary potential energy and the displacement field given by Eqs. (1a)–(1c), one can obtain the equilibrium equations and boundary conditions. The equilibrium equations without external loading are

$$N_{r,r} + N_{r\theta,\theta}/r + (N_r - N_\theta)/r = 0, \tag{6a}$$

$$N_{r\theta,r} + N_{\theta,\theta}/r + 2N_{r\theta}/r = 0, \tag{6b}$$

$$C_1\left(P_{r,rr} + \frac{2}{r}P_{r,r} + \frac{1}{r^2}P_{\theta,\theta\theta} - \frac{1}{r}P_{\theta,r} + \frac{2}{r}P_{r\theta,r\theta} + \frac{2}{r^2}P_{r\theta,\theta}\right) + \frac{\bar{Q}_r}{r} + \bar{Q}_{r,r} + \frac{1}{r}\bar{Q}_{\theta,\theta} = 0, \tag{6c}$$

$$\bar{M}_{r,r} + \frac{\bar{M}_r}{r} - \frac{\bar{M}_\theta}{r} + \frac{1}{r}\bar{M}_{r\theta,\theta} - \bar{Q}_r = 0, \tag{6d}$$

$$\frac{1}{r}\bar{M}_{\theta,\theta} + \bar{M}_{r\theta,r} + \frac{2\bar{M}_{r\theta}}{r} - \bar{Q}_\theta = 0, \tag{6e}$$

where

$$\bar{M}_{r\theta} = M_{r\theta} - C_1 P_{r\theta},\ \bar{M}_\beta = M_\beta - C_1 P_\beta,\quad \bar{Q}_\beta = Q_\beta - C_2 R_\beta, \tag{7a}$$

$$\begin{Bmatrix} Q_\beta \\ R_\beta \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\beta z} \begin{Bmatrix} 1 \\ z \end{Bmatrix} dz, \tag{7b}$$

$$\begin{Bmatrix} N_\beta \\ M_\beta \\ P_\beta \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\beta\beta} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} dz, \tag{7c}$$

$$\begin{Bmatrix} N_{r\theta} \\ M_{r\theta} \\ P_{r\theta} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{r\theta} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} dz, \tag{7d}$$

and subscript $\beta$ denotes $r$ or $\theta$; $N_r$, $N_\theta$ and $N_{r\theta}$ are in-plane force resultants; $Q_r$ and $Q_\theta$ are shear force resultants; $M_r$, $M_\theta$ and $M_{r\theta}$ are moment resultants; $P_r$, $P_\theta$, $P_{r\theta}$, $R_r$, and $R_\theta$ are the higher-order stress resultants. Notably, Eqs. (6a)–(7a) convert to the equilibrium equations for FSDT if $C_1$ and $C_2$ are set equal to zero. The boundary conditions should be specified as along $\theta = \alpha$,

$$u_0 \text{ or } N_{r\theta},\ v_0 \text{ or } N_\theta,\ \psi_\theta \text{ or } \bar{M}_\theta,\ \psi_r \text{ or } \bar{M}_{r\theta},\ w \text{ or } \bar{Q}_\theta + C_1\left(\frac{2}{r}P_{r\theta} + 2P_{r\theta,r} + \frac{1}{r}P_{\theta,\theta}\right), \text{ and } \frac{w_{,\theta}}{r} \text{ or } P_\theta; \tag{8a}$$

and at $r=R$,

$$u_0 \text{ or } N_r,\ v_0 \text{ or } N_{r\theta},\ \psi_\theta \text{ or } \bar{M}_{r\theta},\ \psi_r \text{ or } \bar{M}_r,\ w \text{ or } \bar{Q}_r + C_1\left(\frac{P_r}{r} + P_{r,r} + \frac{2}{r}P_{r\theta,\theta} - \frac{P_\theta}{r}\right), \text{ and } w_{,r} \text{ or } P_r. \tag{8b}$$

Using Eqs. (2), (3), (5) and (7) yields the relationships (Appendix A) between stress resultants and displacement functions. Substituting those relationships into Eqs. (6a)–(6e) yields the equilibrium equations in terms of displacement functions

$$\bar{E}_0\left(-\frac{u_0}{r^2} + \frac{u_{0,r}}{r} + u_{0,rr} + \frac{1-\upsilon}{2r^2}u_{0,\theta\theta} - \frac{3-\upsilon}{2r^2}v_{0,\theta} + \frac{1+\upsilon}{2r}v_{0,r\theta}\right) + C_1\bar{E}_3\left(\frac{w_{,r}}{r^2} - \frac{w_{,rr}}{r} - w_{,rrr} + \frac{3+\upsilon}{2r^3}w_{,\theta\theta} - \frac{w_{,r\theta\theta}}{r^2}\right) + (\bar{E}_1 - C_1\bar{E}_3)\left(\frac{\psi_r}{r^2} + \frac{\psi_{r,r}}{r} + \psi_{r,rr} + \frac{1-\upsilon}{2r^2}\psi_{r,\theta\theta} - \frac{3-\upsilon}{2r^2}\psi_{\theta,\theta} + \frac{1+\upsilon}{2r}\psi_{\theta,r\theta}\right) = 0, \tag{9a}$$

$$\bar{E}_0\left(\frac{3-\upsilon}{2r^2}u_{0,\theta} + \frac{1+\upsilon}{2r}u_{0,r\theta} - \frac{1-\upsilon}{2r^2}v_0 + \frac{1-\upsilon}{2r}v_{0,r} + \frac{1-\upsilon}{2}v_{0,rr} + \frac{v_{0,\theta\theta}}{r^2}\right) + C_1\bar{E}_3\left(-\frac{w_{,r\theta}}{r^2} - \frac{1+\upsilon}{2r}w_{,rr\theta} - \frac{w_{,\theta\theta\theta}}{r^3}\right) + (\bar{E}_1 - C_1\bar{E}_3)\left(\frac{3-\upsilon}{2r^2}\psi_{r,\theta} + \frac{1+\upsilon}{2r}\psi_{r,r\theta} - \frac{1-\upsilon}{2r^2}\psi_\theta + \frac{1-\upsilon}{2r}\psi_{\theta,r} + \frac{1-\upsilon}{2}\psi_{\theta,rr} + \frac{\psi_{\theta,\theta\theta}}{r^2}\right) = 0, \tag{9b}$$

$$C_1\bar{E}_3\left(\frac{u_0 + v_{0,\theta}}{r^3} - \frac{u_{0,r} + v_{0,r\theta}}{r^2} + \frac{2u_{0,rr} + v_{0,rr\theta}}{r} + u_{0,rrr} + \frac{u_{0,\theta\theta} + v_{0,\theta\theta\theta}}{r^3} + \frac{u_{0,r\theta\theta}}{r^2}\right) + C_1^2\bar{E}_6\left(-\frac{w_{,r}}{r^3} + \frac{w_{,rr}}{r^2} - \frac{4w_{,\theta\theta}}{r^4} + \frac{2w_{,r\theta\theta}}{r^3} - \frac{2w_{,rrr}}{r} - \frac{w_{,\theta\theta\theta\theta}}{r^4} - \frac{2w_{,rr\theta\theta}}{r^2} - w_{,rrrr}\right) + \frac{1-\upsilon}{2}\left(\bar{E}_0 - 2C_2\bar{E}_2 + C_2^2\bar{E}_4\right)\left(\frac{w_{,r}}{r} + w_{,rr} + \frac{\psi_r}{r} + \psi_{r,r} + \frac{\psi_{\theta,\theta}}{r}\right) + \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right)\left(\frac{\psi_r + \psi_{\theta,\theta}}{r^3} - \frac{\psi_{r,r} + \psi_{\theta,r\theta}}{r^2} + \frac{\psi_{r,\theta\theta} + \psi_{\theta,\theta\theta\theta}}{r^3} + \frac{2\psi_{r,rr} + \psi_{\theta,rr\theta}}{r} + \frac{\psi_{r,r\theta\theta}}{r^2} + \psi_{r,rrr}\right) = 0, \tag{9c}$$

$$(\bar{E}_1 - C_1\bar{E}_3)\left(-\frac{u_0}{r^2} + \frac{u_{0,r}}{r} + u_{0,rr} + \frac{1-\upsilon}{2r^2}u_{0,\theta\theta} - \frac{3-\upsilon}{2r^2}v_{0,\theta} + \frac{1+\upsilon}{2r}v_{0,r\theta}\right) + \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right) \times \left(\frac{w_{,r}}{r^2} - \frac{w_{,rr}}{r} + \frac{2w_{,\theta\theta}}{r^3} - w_{,rrr} - \frac{w_{,r\theta\theta}}{r^2}\right) - \frac{1-\upsilon}{2} \times \left(\bar{E}_0 - 2C_2\bar{E}_2 + C_2^2\bar{E}_4\right)(w_{,r} + \psi_r) + \left(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6\right)\left(-\frac{\psi_r}{r^2} + \frac{\psi_{r,r}}{r} + \frac{1-\upsilon}{2r^2}\psi_{r,\theta\theta} + \psi_{r,rr} - \frac{3-\upsilon}{2r^2}\psi_{\theta,\theta} + \frac{1+\upsilon}{2r}\psi_{\theta,r\theta}\right) = 0, \tag{9d}$$

$$(\bar{E}_1 - C_1\bar{E}_3)\left(\frac{3-\upsilon}{2r^2}u_{0,\theta} + \frac{1+\upsilon}{2r}u_{0,r\theta} - \frac{1-\upsilon}{2} \times \left(\frac{v_0}{r^2} - \frac{v_{0,r}}{r} - v_{0,rr}\right) + \frac{v_{0,\theta\theta}}{r^2}\right) - \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right) \times \left(\frac{w_{,r\theta}}{r^2} + \frac{w_{,\theta\theta\theta}}{r^3} + \frac{w_{,rr\theta}}{r}\right) - \frac{1-\upsilon}{2}\left(\bar{E}_0 - 2C_2\bar{E}_2 + C_2^2\bar{E}_4\right)\left(\frac{w_{,\theta}}{r} + \psi_\theta\right) + \left(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6\right) \times \left(\frac{3-\upsilon}{2r^2}\psi_{r,\theta} + \frac{1+\upsilon}{2r}\psi_{r,r\theta} - \frac{1-\upsilon}{2r^2}\psi_\theta + \frac{1-\upsilon}{2r}\psi_{\theta,r} + \frac{\psi_{\theta,\theta\theta}}{r^2} + \frac{1-\upsilon}{2}\psi_{\theta,rr}\right) = 0. \tag{9e}$$

Because this study assumes a constant Poisson's ratio, the following relations are used in deriving Eqs. (9a)–(9e)

$$\bar{D}_i = \upsilon\bar{E}_i \text{ and } \bar{G}_i = \frac{(1-\upsilon)}{2}\bar{E}_i, \tag{10}$$

where $\bar{D}_i$, $\bar{E}_i$ and $\bar{G}_i$ are defined in Appendix A.

## 3. ASYMPTOTIC SOLUTIONS

The eigenfunction expansion method is adopted to solve Eqs. (9a)–(9e). Displacement components are assumed as in the following series:

$$u_0(r,\theta) = \sum_{n=0,1,2}^{\infty} r^{\lambda+n}U_n(\theta,\lambda),\quad v_0(r,\theta) = \sum_{n=0,1,2}^{\infty} r^{\lambda+n}V_n(\theta,\lambda),$$
$$w(r,\theta) = \sum_{n=0,1,2}^{\infty} r^{\lambda+n+1}W_n(\theta,\lambda),$$
$$\psi_r(r,\theta) = \sum_{n=0,1,2}^{\infty} r^{\lambda+n}\Psi_n(\theta,\lambda),\ \psi_\theta(r,\theta) = \sum_{n=0,1,2}^{\infty} r^{\lambda+n}\Phi_n(\theta,\lambda), \tag{11}$$

where the characteristic value $\lambda$ is typically a complex number. The real part of $\lambda$ must exceed zero to satisfy regularity conditions at the sector plate vertex. The regularity conditions require that $u_0$, $v_0$, $\psi_\theta$, $\psi_r$, $w$ and $w_{,r}$ are finite as $r$ approaches zero.

No attempt is made to solve completely Eqs. (9a)–(9e) for all values of $r$. Instead, this work concentrates on the asymptotic solutions as $r$ approaches zero, which specify the singular behaviors of stress resultants when stress singularities are present. Accordingly, inserting Eq. (11) into the equilibrium equations and considering only those terms with the lowest order of $r$ yield,

$$\bar{E}_0[-1+\lambda+\lambda(\lambda-1)]\,U_0 + \frac{1-\upsilon}{2}\bar{E}_0U_{0,\theta\theta} + \left(-\frac{3-\upsilon}{2} + \frac{1+\upsilon}{2}\lambda\right)\bar{E}_0V_{0,\theta} + C_1\bar{E}_3[\lambda+1-\lambda(\lambda+1) - (\lambda+1)\lambda(\lambda-1)]\,W_0 + C_1\bar{E}_3[2-(\lambda+1)]\,W_{0,\theta\theta} + (\bar{E}_1 - C_1\bar{E}_3)\,[\lambda-1+\lambda(\lambda-1)]\,\Psi_0 + (\bar{E}_1 - C_1\bar{E}_3)\frac{1-\upsilon}{2}\Psi_{0,\theta\theta} + (\bar{E}_1 - C_1\bar{E}_3) \times \left(-\frac{3-\upsilon}{2} + \frac{1+\upsilon}{2}\lambda\right)\Phi_{0,\theta} = 0, \tag{12a}$$

$$\bar{E}_0\left(\frac{3-\upsilon}{2} + \frac{1+\upsilon}{2}\lambda\right)U_{0,\theta} + \bar{E}_0\frac{1-\upsilon}{2}[\lambda-1 + \lambda(\lambda-1)]\,V_0 + \bar{E}_0V_{0,\theta\theta} + C_1\bar{E}_3[-(\lambda+1) - \lambda(\lambda+1)]\,W_{0,\theta} - C_1\bar{E}_3W_{0,\theta\theta\theta} + (\bar{E}_1 - C_1\bar{E}_3) \times \left(\frac{3-\upsilon}{2} + \frac{1+\upsilon}{2}\lambda\right)\Psi_{0,\theta} + (\bar{E}_1 - C_1\bar{E}_3) \times \frac{1-\upsilon}{2}[\lambda-1+\lambda(\lambda-1)]\,\Phi_0 + (\bar{E}_1 - C_1\bar{E}_3)\,\Phi_{0,\theta\theta} = 0, \tag{12b}$$

$$C_1\bar{E}_3(\lambda-1)^2\,(\lambda+1)U_0 + C_1\bar{E}_3\,(\lambda+1)\,U_{0,\theta\theta} - C_1^2\bar{E}_6(\lambda+1)^2\,(\lambda-1)^2W_0 + C_1^2\bar{E}_6[-4+2\,(\lambda+1) - 2\lambda\,(\lambda+1)]\,W_{0,\theta\theta} - C_1^2\bar{E}_6W_{0,\theta\theta\theta\theta} + \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right)(\lambda-1)^2\,(\lambda+1)\Psi_0 + \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right)(\lambda+1)\,\Psi_{0,\theta\theta} + \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right) \times (\lambda-1)^2\Phi_{0,\theta} + \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right)\Phi_{0,\theta\theta\theta} = 0, \tag{12c}$$

$$(\bar{E}_1 - C_1\bar{E}_3)\,(\lambda-1)(\lambda+1)U_0 + \frac{1-\upsilon}{2}(\bar{E}_1 - C_1\bar{E}_3)\,U_{0,\theta\theta} + (\bar{E}_1 - C_1\bar{E}_3)\left(-\frac{3-\upsilon}{2} + \frac{1+\upsilon}{2}\lambda\right)V_{0,\theta} - \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right)(\lambda+1)^2(\lambda-1)W_0 + \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right)(1-\lambda) \times W_{0,\theta\theta} + \left(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6\right)(\lambda+1)(\lambda-1)\Psi_0 + \left(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6\right)\left[\frac{1-\upsilon}{2}\Psi_{0,\theta\theta} + \times\left(-\frac{3-\upsilon}{2} + \frac{1+\upsilon}{2}\lambda\right)\Phi_{0,\theta}\right] = 0, \tag{12d}$$

$$(\bar{E}_1 - C_1\bar{E}_3)\left(\frac{3-\upsilon}{2} + \frac{1+\upsilon}{2}\lambda\right)U_{0,\theta} + (\bar{E}_1 - C_1\bar{E}_3)\frac{1-\upsilon}{2}(\lambda+1)(\lambda-1)V_0 + (\bar{E}_1 - C_1\bar{E}_3)\,V_{0,\theta\theta} - \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right)(\lambda+1)^2W_{0,\theta}$$

$$- \left(C_1\bar{E}_4 - C_1^2\bar{E}_6\right) W_{0,\theta\theta\theta} + \left(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6\right)\left(\frac{3-\upsilon}{2} + \frac{1+\upsilon}{2}\lambda\right)\Psi_{0,\theta} + \left(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6\right)\left[\left(-\frac{1-\upsilon}{2} + \frac{1-\upsilon}{2}\lambda^2\right)\Phi_0 + \Phi_{0,\theta\theta}\right] = 0. \quad (12e)$$

Equations (12a)–(12e) are a set of linear differential equations with constant coefficients. The general solutions are easily obtained as

$$U_0(\theta,\lambda) = A_1\cos(\lambda+1)\theta + A_2\sin(\lambda+1)\theta + A_3\cos(\lambda-1)\theta + A_4\sin(\lambda-1)\theta, \quad (13a)$$

$$V_0(\theta,\lambda) = A_2\cos(\lambda+1)\theta - A_1\sin(\lambda+1)\theta + (\kappa_1 A_4 - \kappa_2 B_4)\cos(\lambda-1)\theta + (-\kappa_1 A_3 + \kappa_2 B_3)\sin(\lambda-1)\theta, \quad (13b)$$

$$W_0(\theta,\lambda) = B_1\cos(\lambda+1)\theta + B_2\sin(\lambda+1)\theta + B_3\cos(\lambda-1)\theta + B_4\sin(\lambda-1)\theta, \quad (13c)$$

$$\Psi_0(\theta,\lambda) = D_1\cos(\lambda+1)\theta + D_2\sin(\lambda+1)\theta + D_3\cos(\lambda-1)\theta + D_4\sin(\lambda-1)\theta, \quad (13d)$$

$$\Phi_0(\theta,\lambda) = D_2\cos(\lambda+1)\theta - D_1\sin(\lambda+1)\theta + (\kappa_1 D_4 - \kappa_3 B_4)\cos(\lambda-1)\theta + (-\kappa_1 D_3 + \kappa_3 B_3)\sin(\lambda-1)\theta, \quad (13e)$$

where $A_i$, $B_i$ and $D_i$ ($i$ = 1, 2, 3 and 4) are arbitrary constants, $\kappa_1 = \frac{3+\lambda-\upsilon+\lambda\upsilon}{-3+\lambda+\upsilon+\lambda\upsilon}$,

$$\kappa_2 = -\frac{\dfrac{8C_1\lambda(\bar{E}_2\bar{E}_3 - C_1\bar{E}_3\bar{E}_4 + \bar{E}_1(-\bar{E}_4 + C_1\bar{E}_6))}{\bar{E}_1^2 - 2C_1\bar{E}_1\bar{E}_3 + C_1^2\bar{E}_3^2 - \bar{E}_0\left(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6\right)}}{-3+\lambda+\upsilon+\lambda\upsilon}$$

and

$$\kappa_3 = \frac{\dfrac{8C_1\lambda\left(\bar{E}_1\bar{E}_3 - C_1\bar{E}_3^2 + \bar{E}_0(-\bar{E}_4 + C_1\bar{E}_6)\right)}{\bar{E}_1^2 - 2C_1\bar{E}_1\bar{E}_3 + C_1^2\bar{E}_3^2 - \bar{E}_0\left(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6\right)}}{-3+\lambda+\upsilon+\lambda\upsilon}.$$

The relationships among $A_i$, $B_i$ and $D_i$ ($i$ =1, 2, 3 and 4) and the characteristic value $\lambda$ are determined from boundary conditions along radial edges.

Asymptotic solutions are expressed simply as

$$u_0^{(a)}(r,\theta) = r^\lambda U_0(\theta,\lambda),\ v_0^{(a)}(r,\theta) = r^\lambda V_0(\theta,\lambda),$$
$$\psi_\theta^{(a)}(r,\ \theta) = r^\lambda \Phi_0(\theta,\lambda),\ \psi_r^{(a)}(r,\theta) = r^\lambda \Psi_0(\theta,\lambda)$$
$$\text{and } w_0^{(a)}(r,\theta) = r^{\lambda+1} W_0(\theta,\lambda), \quad (14)$$

where $u_0^{(a)}$, $v_0^{(a)}$, $\psi_\theta^{(a)}$, $\psi_r^{(a)}$ and $w_0^{(a)}$denote asymptotic solutions for $u_0$, $v_0$, $\psi_\theta$, $\psi_r$ and $w$, respectively. These asymptotic solutions are also called corner functions and can be added to regular admissible functions in an energy method to enhance dramatically the accuracy of numerical solutions to dynamic or static problems with stress singularities.

## 4. CHARACTERISTIC EQUATIONS AND CORNER FUNCTIONS

The singularities of stress resultants at $r = 0$ are determined from the real part of $\lambda$ in an asymptotic solution given by Eq. (14). The real part of $\lambda$ less than one leads to singularities of $N_r$, $N_\theta$, $N_{r\theta}$, $M_r$, $M_\theta$, $M_{r\theta}$, $P_r$, $P_\theta$ and $P_{r\theta}$; no singularity for shear forces ($Q_r$ and $Q_\theta$), $R_r$ and $R_\theta$ is produced.

The characteristic value $\lambda$ is affected by boundary conditions along two edges forming a corner. Four types of homogeneous boundary conditions along a radial edge, say $\theta = \alpha$, are considered:

clamped:

$$u_0 = v_0 = w = \psi_r = \psi_\theta = \frac{w_{,\theta}}{r} = 0, \quad (15a)$$

free:

$$N_{r\theta} = N_\theta = \bar{M}_\theta = \bar{M}_{r\theta} = \bar{Q}_\theta + C_1\left(\frac{2}{r}P_{r\theta} + 2P_{r\theta,r} + \frac{1}{r}P_{\theta,\theta}\right) = P_\theta = 0, \quad (15b)$$

type I simply supported:

$$u_0 = v_0 = w = \psi_r = \bar{M}_\theta = P_\theta = 0, \quad (15c)$$

type II simply supported:

$$u_0 = v_0 = w = \bar{M}_\theta = \bar{M}_{r\theta} = P_\theta = 0. \quad (15d)$$

For simplicity, C and F refer to clamped and free boundary conditions, respectively, while S(I) and S(II) denote type I and type II simply supported boundary conditions.

The procedure for deriving the characteristic equation for $\lambda$ and the associated corner function is demonstrated for the case in which both radial edges at $\theta = \pm\alpha/2$ have S(I). Problem symmetry is exploited to divide the asymptotic solutions given by Eq. (14) into symmetric and anti-symmetric parts.

In the symmetric case, $A_i$, $B_i$ and $D_i$ ($i$ = 2 and 4) in Eqs. (13a)–(13e) are zero. Satisfying the boundary conditions yields

$$A_1\cos(\lambda+1)\frac{\alpha}{2} + A_3\cos(\lambda-1)\frac{\alpha}{2} = 0, \quad (16a)$$

$$-A_1\sin(\lambda+1)\frac{\alpha}{2} + (-\kappa_1 A_3 + \kappa_2 B_3)\sin(\lambda-1)\frac{\alpha}{2} = 0, \quad (16b)$$

$$B_1\cos(\lambda+1)\frac{\alpha}{2} + B_3\cos(\lambda-1)\frac{\alpha}{2} = 0, \quad (16c)$$

$$D_1\cos(\lambda+1)\frac{\alpha}{2} + D_3\cos(\lambda-1)\frac{\alpha}{2} = 0, \quad (16d)$$

$$(\lambda+1)\{-A_1\bar{E}_1 + D_1(-\bar{E}_2 + C_1\bar{E}_4) + B_1C_1\bar{E}_4(\lambda+1)\} \times \cos(\lambda+1)\frac{\alpha}{2} + (\lambda-1)\{-A_3\kappa_1\bar{E}_1 + \kappa_1(-\bar{E}_2 + C_1\bar{E}_4) \times D_3 + (\kappa_2\bar{E}_1 + \kappa_3\bar{E}_2 - C_1\bar{E}_4(1+\kappa_3-\lambda))B_3\} \times \cos(\lambda-1)\frac{\alpha}{2} = 0 \quad (16e)$$

$$(\lambda+1)\{-A_1\bar{E}_3 + D_1(-\bar{E}_4 + C_1\bar{E}_6) + B_1C_1\bar{E}_6(\lambda+1)\} \times \cos(\lambda+1)\frac{\alpha}{2} + (\lambda-1)\{-A_3\kappa_1\bar{E}_1 + \kappa_1(-\bar{E}_4 + C_1\bar{E}_6)D_3 + (\kappa_2\bar{E}_3 + \kappa_3\bar{E}_4 - C_1\bar{E}_6(1+\kappa_3-\lambda))B_3\} \times \cos(\lambda-1)\frac{\alpha}{2} = 0 \quad (16f)$$

For a nontrivial solution, the determinant of the coefficients of Eqs. (16a)–(16f) must vanish, leading to

$$\left(\cos(\lambda+1)\frac{\alpha}{2}\right)\left(\cos(\lambda-1)\frac{\alpha}{2}\right) = 0, \quad (17a)$$

or

$$\bar{\gamma}_1\lambda\sin\alpha + \sin\lambda\alpha = 0, \quad (17b)$$

where $\bar{\gamma}_1 = \frac{\bar{\kappa}_1}{\bar{\kappa}_2}$ and the long expressions of $\bar{\kappa}_1$ and $\bar{\kappa}_2$ are elucidated in Appendix B.

When $\lambda$ satisfies $\cos(\lambda+1)\frac{\alpha}{2} = 0$ and $\cos(\lambda-1)\frac{\alpha}{2} \neq 0$, $A_1$, $A_3$, $B_3$ and $B_4$ are zero according to Eqs. (16a)–(16f), and $u_0^{(a)}$and $v_0^{(a)}$are eliminated from Eq. (14). Hence, the corresponding corner functions are

$$w_0^{(a)} = B_1 r^{\lambda+1}\cos(\lambda+1)\theta, \quad \psi_r^{(a)} = D_1 r^{\lambda}\cos(\lambda+1)\theta \quad \text{and} \quad \psi_\theta^{(a)} = -D_1 r^{\lambda}\sin(\lambda+1)\theta. \quad (18)$$

The asymptotic solutions for $u_0$ and $v_0$ are such that the order of $r$ exceeds $\lambda$ and do not produce stress singularities. Similarly, one can find the corner functions corresponding to characteristic equations $\cos(\lambda-1)\frac{\alpha}{2} = 0$ and $\cos(\lambda+1)\frac{\alpha}{2} \neq 0$, $\cos(\lambda+1)\frac{\alpha}{2} = 0$ and $\cos(\lambda-1)\frac{\alpha}{2} = 0$, or $\bar{\gamma}_1\lambda\sin\alpha + \sin\lambda\alpha = 0$ (Table 1). Notably, $\cos(\lambda-1)\frac{\alpha}{2} = 0$ and $\cos(\lambda+1)\frac{\alpha}{2} \neq 0$ yield $A_1 = 0$ and $-\kappa_1 A_3 + \kappa_2 B_3 = 0$, and $v_0^{(a)}$ in Eq. (14) vanishes. Hence, $v_0^{(a)}$ is not included in Table 1 because no stress singularity results from the asymptotic solution of $v_0$.

Similarly, in the anti-symmetric case, when coefficients $A_i$, $B_i$ and $D_i$ ($i$=1 and 3) in Eqs. (13a)–(13e) are set to zero, the characteristic equation for $\lambda$ is

$$(\sin(\lambda+1)\alpha/2)(\sin(\lambda-1)\alpha/2) = 0, \quad (19a)$$

or

$$\sin\lambda\alpha - \bar{\gamma}_1\lambda\sin\alpha = 0. \quad (19b)$$

The corresponding corner functions are also given in Table 1. Notably, $(\sin(\lambda+1)\alpha/2) = 0$ and $(\sin(\lambda-1)\alpha/2) \neq 0$ cause $u_0^{(a)}$and $v_0^{(a)}$ in Eq. (14) to vanish, while $(\sin(\lambda-1)\alpha/2) = 0$ and $(\sin(\lambda+1)\alpha/2) \neq 0$ cause $v_0^{(a)}$ in Eq. (14) to vanish.

Imposing the sets of boundary conditions given by Eqs. (15a)–(15d) and following the procedure described above, characteristic equations for $\lambda$ and the corresponding corner functions can be established for different boundary conditions along the radial edges. Table 2 summarizes the characteristic equations for $\lambda$ corresponding to eight different combinations of boundary conditions. The corner functions that correspond to S(I)_S(I), S(II)_S(II), C_C and F-F can be expressed in a rather compact form, as in Table 1, by taking advantage of the problem symmetry. The corner functions for other boundary conditions can be derived by solving 12 linear equations simultaneously, and their expressions are too lengthy and complex to be included in Table 1.

Notably, Table 2 does not list the characteristic equations for S(I)_F and S(II)_F boundary conditions because they are much more complex and lengthy than those given in Table 2. The characteristic values of $\lambda$ for these boundary conditions can be determined directly by finding the zeros of the 12th-order determinant that is obtained by imposing S(I)_F or S(II)_F boundary conditions.

The characteristic equations for boundary conditions involving S(I) or S(II) may depend on material non-homogeneity along the plate thickness (Table 2). The characteristic equations for boundary conditions not including S(I) or S(II) are identical to those for homogenous plates under extension and bending. For example, the characteristic equation for the symmetrical asymptotic solution with the F-F boundary condition is

$$\lambda(-1+\upsilon)\sin\alpha + (3+\upsilon)\sin\lambda\alpha = 0 \quad (20a)$$

or

$$\lambda\sin\alpha + \sin\lambda\alpha = 0. \quad (20b)$$

Equations (20a) and (20b) were also found by Huang [17] in studying a homogenous plate under bending, while Eq. (20b) was also developed by Williams [21] in investigating a homogenous thin plate under extension. Nevertheless, most corner functions depend on material non-homogeneity as shown in Table 1.

## 5. DEPENDENCE OF STRESS SINGULARITIES ON MATERIAL NON-HOMOGENEITY

Based on the asymptotic solutions in Eq. (14) and the relationships between stress resultants and displacement functions in Appendix A, $N_r$, $N_\theta$, $N_{r\theta}$, $M_r$, $M_\theta$, $M_{r\theta}$, $P_r$, $P_\theta$ and $P_{r\theta}$ should have an $r^{\lambda-1}$ type singularity when $0 < \mathrm{Re}[\lambda]$ (real part of $\lambda$)$<1$. When S(I) or S(II) boundary condition does not apply along one of the two radial edges forming a corner, the value of $\lambda$ does not depend on material non-homogeneity, but rather on the corner angle, as presented in Williams [21] and Huang [17]. This section demonstrates the effects of material non-homogeneity on $\lambda$ for boundary conditions involving S(I) or S(II). The following numerical results were obtained by setting the Poisson's ratio to 0.3.

TABLE 1
Corner functions corresponding to different boundary conditions

| Boundary Conditions | Corner Functions |
|---|---|
| | (1) Symmetric case |
| S(I)-S(I) $(-\frac{\alpha}{2} \leq \theta \leq \frac{\alpha}{2})$ | When $\cos(\lambda-1)\alpha/2 = 0$ & $\cos(\lambda+1)\alpha/2 \neq 0$, |
| | $u_0^{(a)} = A_3 r^{\lambda}\{\cos(\lambda-1)\theta\},\ w_0^{(a)} = A_3 r^{\lambda+1}\{\frac{\kappa_1}{\kappa_2}\cos(\lambda-1)\theta\},$ |
| | $\psi_r^{(a)} = r^{\lambda}\{D_3\cos(\lambda-1)\theta\}, \psi_\theta^{(a)} = r^{\lambda}\{(-\kappa_1 D_3 + \kappa_3 A_3\frac{\kappa_1}{\kappa_2})\sin(\lambda-1)\theta\}.$ |
| | When $\cos(\lambda+1)\alpha/2 = 0$ & $\cos(\lambda-1)\alpha/2 \neq 0$, |
| | $w_0^{(a)} = B_1 r^{\lambda+1}\{\cos(\lambda+1)\theta\},\ \psi_r^{(a)} = D_1 r^{\lambda}\{\cos(\lambda+1)\theta\},$ |
| | $\psi_\theta^{(a)} = D_1 r^{\lambda}\{-\sin(\lambda+1)\theta\}.$ |
| | When $\cos(\lambda-1)\alpha/2 = 0$ & $\cos(\lambda+1)\alpha/2 = 0$, |
| | $u_0^{(a)} = r^{\lambda}\{A_3[(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\kappa_1\cos(\lambda+1)\theta + \cos(\lambda-1)\theta]$ |
| | $+B_3[(\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\kappa_2\cos(\lambda+1)\theta]\},$ |
| | $v_0^{(a)} = r^{\lambda}\{(-\kappa_1 A_3 + \kappa_2 B_3)[(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta]\},$ |
| | $w_0^{(a)} = r^{\lambda+1}\{B_1\cos(\lambda+1)\theta + B_3\cos(\lambda-1)\theta\},$ |
| | $\psi_r^{(a)} = r^{\lambda}\{D_1\cos(\lambda-1)\theta + D_3\cos(\lambda+1)\theta\},$ |
| | $\psi_\theta^{(a)} = r^{\lambda}\{-D_1\sin(\lambda+1)\theta + (-\kappa_1 D_3 + \kappa_3 B_3)\sin(\lambda-1)\theta\}.$ |
| | When $\bar{\gamma}_1\lambda\sin\alpha + \sin\lambda\alpha = 0$, |
| | $u_0^{(a)} = A_3 r^{\lambda}\{(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + \cos(\lambda-1)\theta\},$ |
| | $v_0^{(a)} = A_3 r^{\lambda}\{(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + (-\kappa_1 + \kappa_2\bar{\eta}_2)\sin(\lambda-1)\theta\},$ |
| | $w_0^{(a)} = A_3 r^{\lambda+1}\{\bar{\eta}_2[(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + \cos(\lambda-1)\theta]\},$ |
| | $\psi_r^{(a)} = A_3 r^{\lambda}\{\bar{\eta}_1[(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda-1)\theta + \cos(\lambda+1)\theta]\},$ |
| | $\psi_\theta^{(a)} = A_3 r^{\lambda}\{\bar{\eta}_1(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + (-\kappa_1\bar{\eta}_1 + \kappa_3\bar{\eta}_2)\sin(\lambda-1)\theta\}.$ |
| | (2) Anti-symmetric case |
| | When $\sin(\lambda-1)\alpha/2 = 0$ & $\sin(\lambda+1)\alpha/2 \neq 0$, |
| | $u_0^{(a)} = A_4 r^{\lambda}\{\sin(\lambda-1)\theta\},\ w_0^{(a)} = A_4 r^{\lambda+1}\{\frac{\kappa_1}{\kappa_2}\sin(\lambda-1)\theta\},$ |
| | $\psi_r^{(a)} = r^{\lambda}\{D_4\sin(\lambda-1)\theta\},\ \psi_\theta^{(a)} = r^{\lambda}\{(\kappa_1 D_4 - \kappa_3 A_4\frac{\kappa_1}{\kappa_2})\cos(\lambda-1)\theta\}.$ |
| | When $\sin(\lambda+1)\alpha/2 = 0$ & $\sin(\lambda-1)\alpha/2 \neq 0$, |
| | $w_0^{(a)} = B_2 r^{\lambda+1}\{\sin(\lambda+1)\theta\}, \psi_r^{(a)} = D_2 r^{\lambda}\{\sin(\lambda+1)\theta\},$ |
| | $\psi_\theta^{(a)} = D_2 r^{\lambda}\{\cos(\lambda+1)\theta\}.$ |
| | When $\sin(\lambda-1)\alpha/2 = 0$ & $\sin(\lambda+1)\alpha/2 = 0$, |
| | $u_0^{(a)} = r^{\lambda}\{A_4[-(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\kappa_1\sin(\lambda+1)\theta + \sin(\lambda-1)\theta]$ |
| | $+B_4[(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\kappa_2\sin(\lambda+1)\theta]\},$ |
| | $v_0^{(a)} = r^{\lambda}\{(\kappa_1 A_4 - \kappa_2 B_4)[-(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + \cos(\lambda-1)\theta]\},$ |
| | $w_0^{(a)} = r^{\lambda+1}\{B_2\sin(\lambda+1)\theta + B_4\sin(\lambda-1)\theta\},$ |
| | $\psi_r^{(a)} = r^{\lambda}\{D_2\sin(\lambda-1)\theta + D_4\sin(\lambda+1)\theta\},$ |
| | $\psi_\theta^{(a)} = r^{\lambda}\{D_2\cos(\lambda+1)\theta + (\kappa_1 D_4 - \kappa_3 B_4)\cos(\lambda-1)\theta\}.$ |
| | When $\sin\lambda\alpha - \bar{\gamma}_1\lambda\sin\alpha = 0$, |
| | $u_0^{(a)} = A_4 r^{\lambda}\{(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta\},$ |
| | $v_0^{(a)} = A_4 r^{\lambda}\{(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + (\kappa_1 - \kappa_2\bar{\eta}_3)\cos(\lambda-1)\theta\},$ |
| | $w_0^{(a)} = A_4 r^{\lambda+1}\{\bar{\eta}_3[(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta]\},$ |
| | $\psi_r^{(a)} = A_4 r^{\lambda}\{\bar{\eta}_1[(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta]\},$ |
| | $\psi_\theta^{(a)} = A_4 r^{\lambda}\{\bar{\eta}_1(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + (\kappa_1\bar{\eta}_1 - \kappa_3\bar{\eta}_3)\cos(\lambda-1)\theta\}.$ |
| | (1) Symmetric case |
| S(II)-S(II) $(-\frac{\alpha}{2} \leq \theta \leq \frac{\alpha}{2})$ | When $\cos(\lambda-1)\frac{\alpha}{2} = 0$ and $\cos(\lambda+1)\frac{\alpha}{2} = 0$, |
| | the asymptotic solution is the same with that for S(I)_ S(I). |

*(Continued on next page)*

TABLE 1
Corner functions corresponding to different boundary conditions *(Continued)*

| Boundary Conditions | Corner Functions |
|---|---|
| | When $\bar{\gamma}_1 \lambda \sin\alpha + \sin\lambda\alpha = 0$ or $\lambda\sin\alpha + \sin\lambda\alpha = 0$, $u_0^{(a)} = A_3 r^\lambda\{(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + \cos(\lambda-1)\theta\}$, $v_0^{(a)} = A_3 r^\lambda\{(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + (-\kappa_1+\kappa_2\bar{\eta}_2)\sin(\lambda-1)\theta\}$, $w_0^{(a)} = A_3 r^{\lambda+1}\{\bar{\eta}_2[(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + \cos(\lambda-1)\theta]\}$, $\psi_r^{(a)} = A_3 r^\lambda\{-\frac{1}{\lambda(\upsilon-1)}(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})(\bar{\eta}_4 - (1+\kappa_1-\kappa_1\lambda+\lambda\upsilon)\bar{\eta}_5)$ $\cos(\lambda-1)\theta. - \bar{\eta}_5\cos(\lambda+1)\theta\}$, $\psi_\theta^{(a)} = A_3 r^\lambda\{\frac{1}{\lambda(\upsilon-1)}(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})(\bar{\eta}_4 - (1+\kappa_1-\kappa_1\lambda+\lambda\upsilon)\bar{\eta}_5)$ $\sin(\lambda+1)\theta + (\kappa_1\bar{\eta}_5+\kappa_3\bar{\eta}_2)\sin(\lambda-1)\theta\}$. |
| S(II)-S(II) $(-\frac{\alpha}{2} \le \theta \le \frac{\alpha}{2})$ | (2) Anti-symmetric case<br>When $\sin(\lambda-1)\frac{\alpha}{2} = 0$ and $\sin(\lambda+1)\frac{\alpha}{2} = 0$, the asymptotic solution is the same with that for S(I)- S(I). When $\lambda\sin\alpha - \sin\lambda\alpha = 0$ or $\sin\lambda\alpha - \bar{\gamma}_1\lambda\sin\alpha = 0$, $u_0^{(a)} = A_4 r^\lambda\{(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta\}$, $v_0^{(a)} = A_4 r^\lambda\{(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + (\kappa_1-\kappa_2\bar{\eta}_3)\cos(\lambda-1)\theta\}$, $w_0^{(a)} = A_4 r^{\lambda+1}\{\bar{\eta}_3[(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta]\}$, $\psi_r^{(a)} = A_4 r^\lambda\{-\frac{1}{\lambda(\upsilon-1)}(\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})(\bar{\eta}_6 - (1+\kappa_1-\kappa_1\lambda+\lambda\upsilon)\bar{\eta}_7)\sin(\lambda+1)\theta$ $-\bar{\eta}_7\sin(\lambda-1)\theta\}$, $\psi_\theta^{(a)} = A_4 r^\lambda\{-\frac{1}{\lambda(\upsilon-1)}(\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})(\bar{\eta}_6 - (1+\kappa_1-\kappa_1\lambda+\lambda\upsilon)\bar{\eta}_7)\cos(\lambda+1)\theta$ $+(-\kappa_1\bar{\eta}_7-\kappa_3\bar{\eta}_3)\cos(\lambda-1)\theta\}$. |
| F- F $(-\frac{\alpha}{2} \le \theta \le \frac{\alpha}{2})$ | (1) Symmetric case<br>$u_0^{(a)} = D_3 r^\lambda\{(\frac{\bar{\eta}_{10}\bar{\eta}_8}{\sin(\lambda+1)\alpha/2}(\sin\alpha+\sin\lambda\alpha))\cos(\lambda+1)\theta - \bar{\eta}_8\cos(\lambda-1)\theta\}$, $v_0^{(a)} = D_3 r^\lambda\{-(\frac{\bar{\eta}_{10}\bar{\eta}_8}{\sin(\lambda+1)\alpha/2}(\sin\alpha+\sin\lambda\alpha))\sin(\lambda+1)\theta$ $+(-\kappa_1\bar{\eta}_8 - \kappa_2\frac{\bar{\eta}_9}{C_1(\lambda+1)})\sin(\lambda-1)\theta\}$, $w_0^{(a)} = D_3 r^{\lambda+1}\{\bar{\eta}_9[\frac{\bar{\eta}_{10}}{C_1(\lambda+1)}(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda+1)\theta - \frac{1}{C_1(\lambda+1)}\cos(\lambda-1)\theta]\}$, $\psi_r^{(a)} = D_3 r^\lambda\{-\bar{\eta}_{10}(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda-1)\theta + \cos(\lambda+1)\theta\}$, $\psi_\theta^{(a)} = D_3 r^\lambda\{\bar{\eta}_{10}(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + (-\kappa_1 - \kappa_3\frac{\bar{\eta}_9}{C_1(\lambda+1)})\sin(\lambda-1)\theta\}$.<br>(2) Anti-symmetric case<br>$u_0^{(a)} = D_4 r^\lambda\{-(\frac{\bar{\eta}_{10}\bar{\eta}_8}{\sin(\lambda+1)\alpha/2}(\sin\alpha-\sin\lambda\alpha))\sin(\lambda+1)\theta - \bar{\eta}_8\sin(\lambda-1)\theta\}$, $v_0^{(a)} = D_4 r^\lambda\{-(\frac{\bar{\eta}_{10}\bar{\eta}_8}{\sin(\lambda+1)\alpha/2}(\sin\alpha-\sin\lambda\alpha))\cos(\lambda+1)\theta$ $+(-\kappa_1\bar{\eta}_8 - \kappa_2\frac{\bar{\eta}_9}{C_1(\lambda+1)})\cos(\lambda-1)\theta\}$, $w_0^{(a)} = D_4 r^{\lambda+1}\{\bar{\eta}_9[\frac{\bar{\eta}_{10}}{C_1(\lambda+1)}(\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta - \frac{1}{C_1(\lambda+1)}\sin(\lambda-1)\theta]\}$, $\psi_r^{(a)} = D_4 r^\lambda\{-\bar{\eta}_{10}(\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta\}$, $\psi_\theta^{(a)} = D_4 r^\lambda\{-\bar{\eta}_{10}(\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + (\kappa_1 - \kappa_3\frac{\bar{\eta}_9}{C_1(\lambda+1)})\cos(\lambda-1)\theta\}$. |
| C- C $(-\frac{\alpha}{2} \le \theta \le \frac{\alpha}{2})$ | (1) Symmetric case<br>$u_0^{(a)} = D_3 r^\lambda\{\frac{\kappa_2}{\kappa_3}[(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + \cos(\lambda-1)\theta]\}$, $v_0^{(a)} = D_3 r^\lambda\{\frac{\kappa_2}{\kappa_3}[(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + (-\kappa_1+\kappa_2\bar{\eta}_2)\sin(\lambda-1)\theta]\}$, $w_0^{(a)} = D_3 r^{\lambda+1}\{\frac{\kappa_2}{\kappa_3}\bar{\eta}_2[(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + \cos(\lambda-1)\theta]\}$, $\psi_r^{(a)} = D_3 r^\lambda\{(-\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\cos(\lambda-1)\theta + \cos(\lambda+1)\theta\}$, $\psi_\theta^{(a)} = D_3 r^\lambda\{(\frac{\cos(\lambda-1)\alpha/2}{\cos(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + (-\kappa_1+\kappa_2\bar{\eta}_2)\sin(\lambda-1)\theta\}$. |

*(Continued on next page)*

TABLE 1
Corner functions corresponding to different boundary conditions *(Continued)*

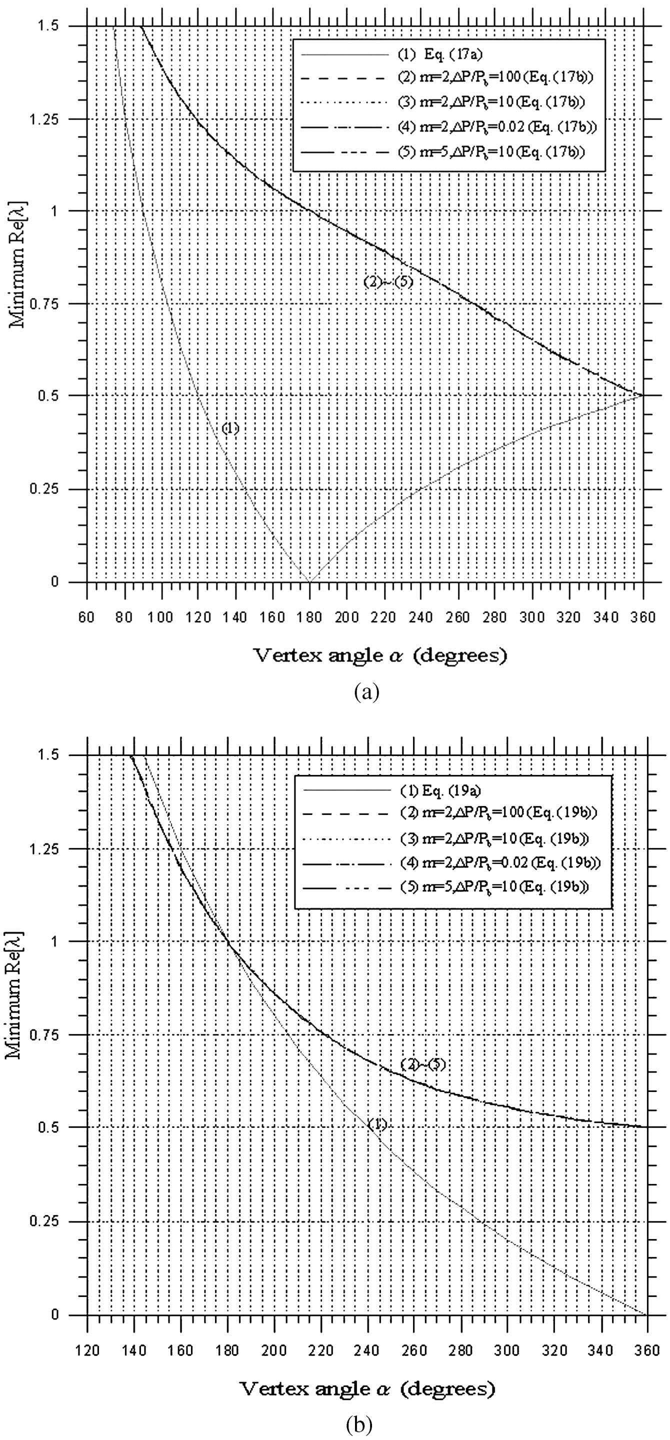
| Boundary Conditions | Corner Functions |
|---|---|
| | (2) Anti-symmetric case<br>$u_0^{(a)} = D_4 r^\lambda\{\frac{\kappa_2}{\kappa_3}[(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta]\}$,<br>$v_0^{(a)} = D_4 r^\lambda\{\frac{\kappa_2}{\kappa_3}[(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + (\kappa_1 - \kappa_2\bar{\eta}_3)\cos(\lambda-1)\theta]\}$,<br>$w_0^{(a)} = D_4 r^{\lambda+1}\{\frac{\kappa_2}{\kappa_3}\bar{\eta}_3[(-\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta]\}$,<br>$\psi_r^{(a)} = D_4 r^\lambda\{-(\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\sin(\lambda+1)\theta + \sin(\lambda-1)\theta\}$,<br>$\psi_\theta^{(a)} = D_4 r^\lambda\{(\frac{\sin(\lambda-1)\alpha/2}{\sin(\lambda+1)\alpha/2})\cos(\lambda+1)\theta + (\kappa_1 - \kappa_2\bar{\eta}_3)\cos(\lambda-1)\theta\}$ |

Note: $\bar{\eta}_1 \sim \bar{\eta}_{10}$ are given in Appendix B.

TABLE 2
Characteristic equations corresponding to various boundary conditions

| Boundary Conditions | Characteristic Equations |
|---|---|
| S(I)- S(I) | Symmetry:<br>$\cos\alpha + \cos\lambda\alpha = 0^*$;  $\bar{\gamma}_1\lambda\sin\alpha + \sin\lambda\alpha = 0$;  where $\bar{\gamma}_1 = \frac{\bar{\kappa}_1}{\bar{\kappa}_2}$.<br>Antisymmetry:<br>$\cos\alpha - \cos\lambda\alpha = 0^*$;  $\sin\lambda\alpha - \bar{\gamma}_1\lambda\sin\alpha = 0$. |
| S(II)-S(II) | Symmetry:<br>$\cos\alpha + \cos\lambda\alpha = 0^*$;  $\lambda\sin\alpha + \sin\lambda\alpha = 0^*$;<br>$\bar{\gamma}_1\lambda\sin\alpha + \sin\lambda\alpha = 0$.<br>Antisymmetry:<br>$\cos\alpha - \cos\lambda\alpha = 0^*$;  $\lambda\sin\alpha - \sin\lambda\alpha = 0^*$;<br>$\sin\lambda\alpha - \bar{\gamma}_1\lambda\sin\alpha = 0$. |
| S(I)- S(II) | $(\cos 2\alpha - \cos 2\lambda\alpha)(\lambda\sin 2\alpha - \sin 2\lambda\alpha) = 0^*$; ....... (T1)<br>$\bar{\gamma}_2\lambda^2\sin^2\alpha + \sin^2\lambda\alpha = 0$; ......(T2)<br>Where $\bar{\gamma}_2 = \frac{\bar{\kappa}_3}{\bar{\kappa}_4}$. |
| F-F | Symmetry:<br>$\lambda(-1+\upsilon)\sin\alpha + (3+\upsilon)\sin\lambda\alpha = 0^*$;  $\lambda\sin\alpha + \sin\lambda\alpha = 0^{*,\#}$.<br>Antisymmetry:<br>$-\lambda(-1+\upsilon)\sin\alpha + (3+\upsilon)\sin\lambda\alpha = 0^*$;  $-\lambda\sin\alpha + \sin\lambda\alpha = 0^{*,\#}$. |
| C-C | Symmetry:<br>$\lambda(1+\upsilon)\sin\alpha + (-3+\upsilon)\sin\lambda\alpha = 0^{*,\#}$;  $\lambda\sin\alpha + \sin\lambda\alpha = 0^*$.<br>Antisymmetry:<br>$\lambda(1+\upsilon)\sin\alpha - (-3+\upsilon)\sin\lambda\alpha = 0^{*,\#}$;  $-\lambda\sin\alpha + \sin\lambda\alpha = 0^*$. |
| C-F | $4 - \lambda^2(1+\upsilon)^2\sin^2\alpha + (-3+\upsilon)(1+\upsilon)\sin^2\lambda\alpha = 0^{*,\#}$;<br>$4 - \lambda^2(1-\upsilon)^2\sin^2\alpha + (3+\upsilon)(-1+\upsilon)\sin^2\lambda\alpha = 0^*$. |
| S(II)- C | $4 - \lambda^2(1+\upsilon)^2\sin^2\alpha + (-3+\upsilon)(1+\upsilon)\sin^2\lambda\alpha = 0^*$; ..... (T3)<br>$\{[2\bar{E}_1(\bar{E}_4 - C_1\bar{E}_6)\frac{\bar{\kappa}_6}{\bar{\kappa}_5}]$<br>$+ \bar{E}_4[4\bar{E}_4(\lambda^2(1+\upsilon)^2\sin^2\alpha - (-3+\upsilon)^2\sin^2\lambda\alpha)(\lambda\sin 2\alpha - \sin 2\lambda\alpha) + 2C_1\bar{E}_3\frac{\bar{\kappa}_6}{\bar{\kappa}_5}]$<br>$- \bar{E}_2[4\bar{E}_6(\lambda^2(1+\upsilon)^2\sin^2\alpha - (-3+\upsilon)^2\sin^2\lambda\alpha)(\lambda\sin 2\alpha - \sin 2\lambda\alpha) + 2\bar{E}_3\frac{\bar{\kappa}_6}{\bar{\kappa}_5}]\} = 0$. ...... (T4) |
| S(I)- C | $-\lambda(1+\upsilon)\sin 2\alpha + (-3+\upsilon)\sin 2\lambda\alpha = 0^*$; ..... (T5)<br>$\{[2\bar{E}_1(\bar{E}_4 - C_1\bar{E}_6)\frac{\bar{\kappa}_6}{\bar{\kappa}_5}]$<br>$+ \bar{E}_4[4\bar{E}_4(\lambda^2(1+\upsilon)^2\sin^2\alpha - (-3+\upsilon)^2\sin^2\lambda\alpha)(\lambda\sin 2\alpha - \sin 2\lambda\alpha) + 2C_1\bar{E}_3\frac{\bar{\kappa}_6}{\bar{\kappa}_5}]$<br>$- \bar{E}_2[4\bar{E}_6(\lambda^2(1+\upsilon)^2\sin^2\alpha - (-3+\upsilon)^2\sin^2\lambda\alpha)(\lambda\sin 2\alpha - \sin 2\lambda\alpha) + 2\bar{E}_3\frac{\bar{\kappa}_6}{\bar{\kappa}_5}]\} = 0$ ...... (T4) |

Note: (1) * denotes that equation can be found for a homogeneous plate under bending [17].
(2) # denotes that equation can be found for a homogeneous plate under extension [21].
(3) $\bar{\kappa}_1 \sim \bar{\kappa}_6$ are given in Appendix B.




(a)

(b)

FIG. 2. Minimum Re[λ] of characteristic equations corresponding to S(I)-S(I) boundary condition: (a)symmetric solution, (b) anti-symmetric solution.

Figure 2 plots the minimum Re[λ] versus α for various $\Delta P/P_b$ (=0.02, 10 or 100) and $m$ (=2 or 5) in Eq. (4) under the boundary condition S(I)–S(I). Symmetric and antisymmetric cases were investigated separately. The minimum Re[λ] of each of Eqs. (17b) and (19b) does not depend significantly on material non-homogeneity. The roots of Eqs. (17a) and (19a) dominate stress singularities, and are independent of material properties. Stress singularities are present at a corner when its angle α exceeds 90°, except at α=180°. When $90^o < \alpha \leq 270°$, the stress singularity order at the corner is determined from the minimum Re[λ] of Eq. (17a), while the minimum Re[λ] of Eq. (19a) determines the singularity order for $270^o \leq \alpha \leq 360°$.

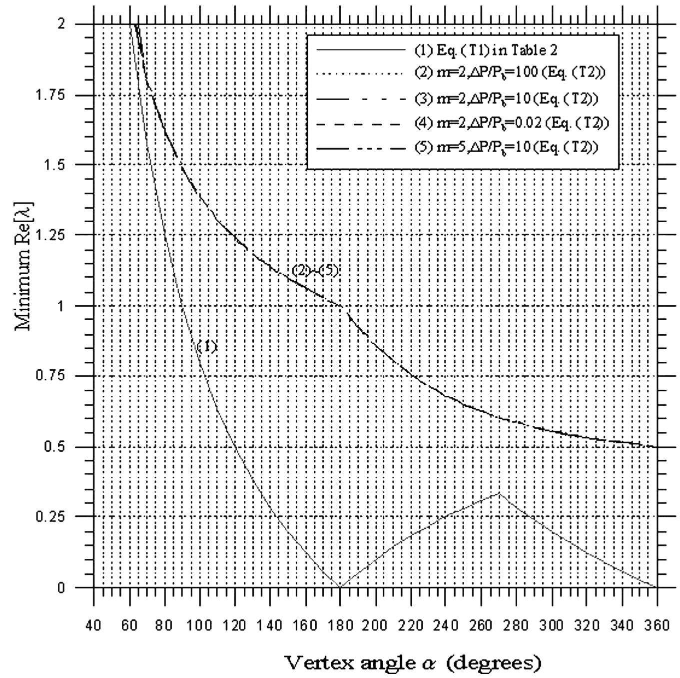



FIG. 3. Minimum Re[λ] of characteristic equations corresponding to S(I)-S(II) boundary condition.

Considering the stress singularity under boundary condition S(II)–S(II) reveals two additional characteristic equations not associated with S(I)–S(I) boundary condition (see Table 2):

$$\lambda \sin \alpha \pm \sin \lambda\alpha = 0. \tag{21}$$

A study of a homogenous plate based on Reddy's plate theory in Huang [17] indicated that the minimum Re[λ] of Eq. (21) exceeds that of Eqs. (17a) and (19a). Consequently, the stress singularity order at the vertex of a sector plate made of FGM and having S(II)-S(II) radial edges is determined by the minimum Re[λ] of Eq. (17a) and Eq. (19a) , and is independent of material properties.

Figure 3 displays the effects of material non-homogeneity on the minimum Re[λ] of the characteristic equations corresponding to the S(I)_S(II) boundary condition. Of the two characteristic equations (Eqs. (T1) and (T2) in Table 2), one equation does not depend on material properties, while the other does. Equation (T1) was also found by Huang [17] in studying a homogenous plate. The material non-homogeneity with $\Delta P/P_b = 0.02$, 10 or 100 and $m = 2$ or 5 slightly affects the minimum Re[λ] of the characteristic equation involving material properties (Eq. (T2)).

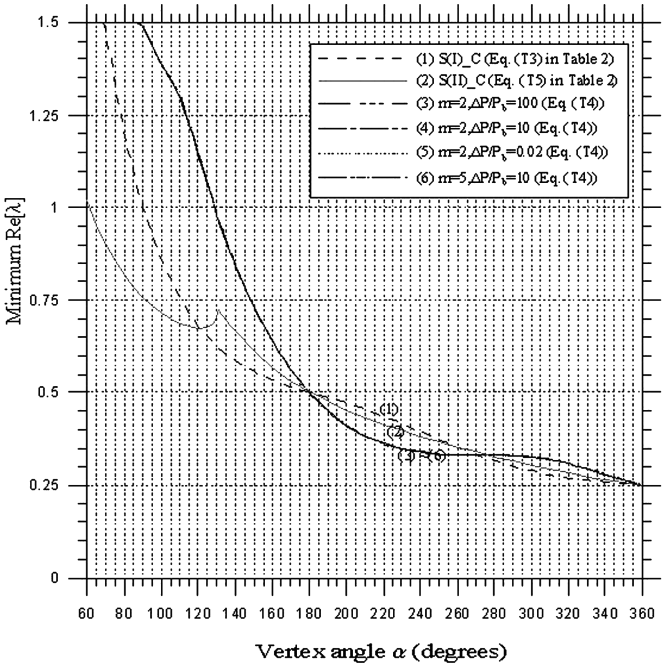



FIG. 4. Minimum Re[λ] of characteristic equations corresponding to S(I)-C and S(II)-C boundary conditions.

The stress singularity order is dominated by the minimum Re[λ] of the characteristic equation independent of material properties (Eq. (T1) in Table 2).

Figure 4 plots the minimum values of Re[λ] of the characteristic equations corresponding to S(I)_C and S(II)_C boundary conditions. Each boundary condition yields two characteristic equations: one involves material non-homogeneity and the other does not. Notably, both boundary conditions have the same characteristic equation involving material non-homogeneity (see Eq. (T4) in Table 2). The characteristic equations not involving material non-homogeneity (Eqs. (T3) and (T5) in Table 2) were also obtained for homogeneous plates under bending [17]. The material non-homogeneity with $\Delta P/P_b = 0.02$, 10 or 100 and $m = 2$ or 5 does not significantly influence the stress singularity order at a corner. The stress singularity order is governed primarily by the roots of the characteristic equations that do not depend on material non-homogeneity when $\alpha$ is less than 180° or larger than 270°.

With reference to S(I)_F and S(II)_F boundary conditions, Figures 5 and 6 demonstrate the relative differences between the minimum values of Re[λ] for non-homogeneous and homogenous plates. The roots of λ were determined from the zeros of the 12th-order determinant based on the boundary conditions under consideration. The relative differences are defined as

$$(\text{minimum Re}[\lambda(m \neq 0)] - \text{minimum Re}[\lambda(m = 0)])/\text{minimum Re}[\lambda(m = 0)],$$

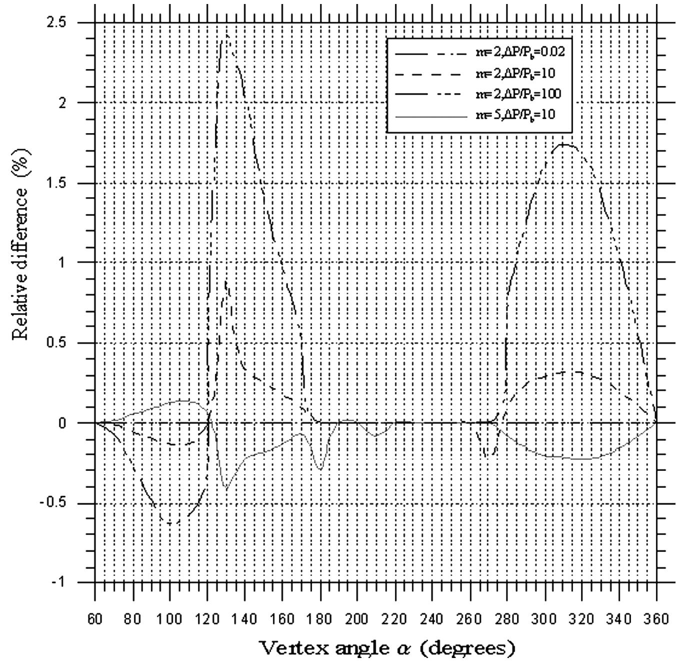



FIG. 5. The effects of material non-homogeneity on minimum Re[λ] of characteristic equations corresponding to S(I)-F boundary condition.

where λ ($m = 0$) is the characteristic value for a homogeneous plate. The stress singularity order for a non-homogeneous plate with $\Delta P/P_b = 0.02$, 10 or 100 and $m = 2$ or 5 differs only slightly from that for a homogeneous plate. The relative differences are less than 3%.

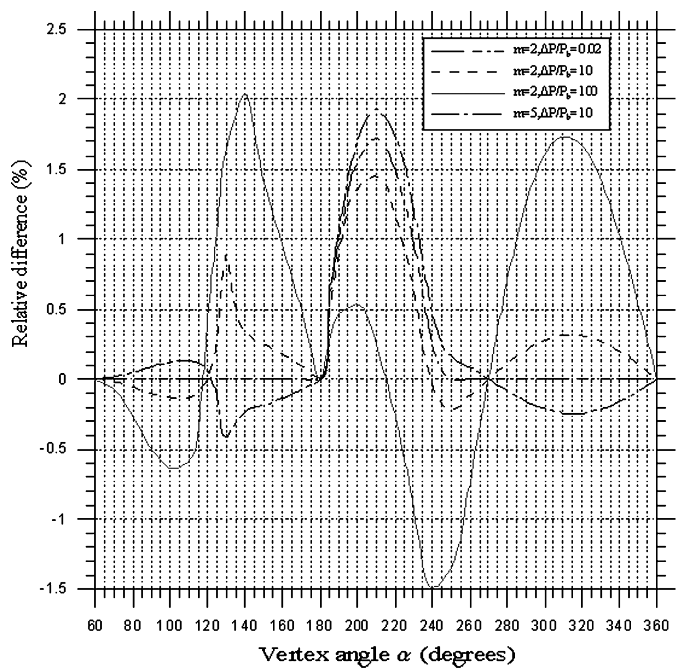



FIG. 6. The effects of material non-homogeneity on minimum Re[λ] of characteristic equations corresponding to S(II)-F boundary condition.

## 6. CONCLUDING REMARKS

The work employs eigenfunction expansion to derive the characteristic equations for eight different combinations of radial edge conditions at the corner of an FGM plate, according to Reddy's plate theory. The corresponding corner functions (or asymptotic solutions) are also explicitly given for two radial edges with identical boundary conditions. When the two radial edges forming a corner do not involve S(I) or S(II) boundary condition, the characteristic equations are identical to those for a homogenous plate. The material non-homogeneity under consideration ($\Delta P/P_b = 0.02$, 10 or 100, and $m = 2$ or 5) does not substantially influence the stress singularity order at a corner that has one of its radial edges with S(I) or S(II) boundary condition. Nevertheless, the asymptotic solutions differ significantly from those for homogenous plates, and most asymptotic solutions for FGM plates show coupling between the in-plane displacement components and out-of-plane displacement component at the mid-plane.

The results shown in the work are the first known in the literature. They will be very useful in obtaining accurate numerical solutions to static or dynamic problems of FGM thick plates involving stress singularities. For instance, the corner functions can be used with additional smooth functions to form admissible functions in the Ritz method to enhance accuracy in determining the natural frequencies of FGM thick plates with a V-notch. The corner functions can also be adopted in a mesh-free method or a finite element approach to determine accurately the stress intensity factors for an FGM thick plate with a V-notch.

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## APPENDIX A. RELATIONSHIPS BETWEEN STRESS RESULTANTS AND DISPLACEMENT FUNCTIONS

$$N_r = \frac{\bar{D}_0}{r}u_0 + \bar{E}_0 u_{0,r} + \frac{\bar{D}_0}{r}v_{0,\theta} - \frac{C_1\bar{D}_3}{r}w_{,r} - C_1\bar{E}_3 w_{,rr} - \frac{C_1\bar{D}_3}{r^2}w_{,\theta\theta} + \frac{1}{r}(\bar{D}_1 - C_1\bar{D}_3)\psi_r + (\bar{E}_1 - C_1\bar{E}_3)\psi_{r,r} + \frac{1}{r}(\bar{D}_1 - C_1\bar{D}_3)\psi_{\theta,\theta}, \tag{A.1}$$

$$N_\theta = \frac{\bar{E}_0}{r}u_0 + \bar{D}_0 u_{0,r} + \frac{\bar{E}_0}{r}v_{0,\theta} - \frac{C_1\bar{E}_3}{r}w_{,r} - C_1\bar{D}_3 w_{,rr} - \frac{C_1\bar{E}_3}{r^2}w_{,\theta\theta} + \frac{1}{r}(\bar{E}_1 - C_1\bar{E}_3)\psi_r + (\bar{D}_1 - C_1\bar{D}_3)\psi_{r,r} + \frac{1}{r}(\bar{E}_1 - C_1\bar{E}_3)\psi_{\theta,\theta}, \tag{A.2}$$

$$N_{r\theta} = \frac{\bar{G}_0}{r}u_{0,\theta} - \frac{\bar{G}_0}{r}v_0 + \bar{G}_0 v_{0,r} + \frac{2C_1\bar{G}_3}{r^2}w_{,\theta} - \frac{2C_1\bar{G}_3}{r}w_{,r\theta} + (\bar{G}_1 - C_1\bar{G}_3)\left(\frac{1}{r}(\psi_{r,\theta} - \psi_\theta) + \psi_{\theta,r}\right), \tag{A.3}$$

$$Q_r = (\bar{G}_0 - C_2\bar{G}_2)(\psi_r + w_{,r}), \tag{A.4}$$

$$Q_\theta = (\bar{G}_0 - C_2\bar{G}_2)\left(\psi_\theta + \frac{1}{r}w_{,\theta}\right), \tag{A.5}$$

$$R_r = (\bar{G}_2 - C_2\bar{G}_4)(\psi_r + w_{,r}), \tag{A.6}$$

$$R_\theta = (\bar{G}_2 - C_2\bar{G}_4)\left(\psi_\theta + \frac{1}{r}w_{,\theta}\right), \tag{A.7}$$

$$M_r = \frac{\bar{D}_1}{r}u_0 + \bar{E}_1 u_{0,r} + \frac{\bar{D}_1}{r}v_{0,\theta} - \frac{C_1\bar{D}_4}{r}w_{,r} - C_1\bar{E}_4 w_{,rr}\frac{C_1\bar{D}_4}{r^2}w_{,\theta\theta} + \frac{1}{r}(\bar{D}_2 - C_1\bar{D}_4)\psi_r + (\bar{E}_2 - C_1\bar{E}_4)\psi_{r,r} + \frac{1}{r}(\bar{D}_2 - C_1\bar{D}_4)\psi_{\theta,\theta}, \tag{A.8}$$

$$M_\theta = \frac{\bar{E}_1}{r}u_0 + \bar{D}_1 u_{0,r} + \frac{\bar{E}_1}{r}v_{0,\theta} - \frac{C_1\bar{E}_4}{r}w_{,r} - C_1\bar{D}_4 w_{,rr} - \frac{C_1\bar{E}_4}{r^2}w_{,\theta\theta} + \frac{1}{r}(\bar{E}_2 - C_1\bar{E}_4)\psi_r + (\bar{D}_2 - C_1\bar{D}_4)\psi_{r,r} + \frac{1}{r}(\bar{E}_2 - C_1\bar{E}_4)\psi_{\theta,\theta}, \tag{A.9}$$

$$M_{r\theta} = \frac{\bar{G}_1}{r}u_{0,\theta} - \frac{\bar{G}_1}{r}v_0 + \bar{G}_1 v_{0,r} + \frac{2C_1\bar{G}_4}{r^2}w_{,\theta} - \frac{2C_1\bar{G}_4}{r}w_{,r\theta} + (\bar{G}_2 - C_1\bar{G}_4)\left(\frac{1}{r}(\psi_{r,\theta} - \psi_\theta) + \psi_{\theta,r}\right), \tag{A.10}$$

$$P_r = \frac{\bar{D}_3}{r}u_0 + \bar{E}_3 u_{0,r} + \frac{\bar{D}_3}{r}v_{0,\theta} - \frac{C_1\bar{D}_6}{r}w_{,r} - C_1\bar{E}_4 w_{,rr}\frac{C_1\bar{D}_6}{r^2}w_{,\theta\theta} + \frac{1}{r}(\bar{D}_4 - C_1\bar{D}_6)\psi_r + (\bar{E}_4 - C_1\bar{E}_6)\psi_{r,r} + \frac{1}{r}(\bar{D}_4 - C_1\bar{D}_6)\psi_{\theta,\theta}, \tag{A.11}$$

$$P_\theta = \frac{\bar{E}_3}{r}u_0 + \bar{D}_3 u_{0,r} + \frac{\bar{E}_3}{r}v_{0,\theta} - \frac{C_1\bar{E}_6}{r}w_{,r} - C_1\bar{D}_6 w_{,rr} - \frac{C_1\bar{E}_6}{r^2}w_{,\theta\theta} + \frac{1}{r}(\bar{E}_4 - C_1\bar{E}_6)\psi_r + (\bar{D}_4 - C_1\bar{D}_6)\psi_{r,r} + \frac{1}{r}(\bar{E}_4 - C_1\bar{E}_6)\psi_{\theta,\theta}, \tag{A.12}$$

$$P_{r\theta} = \frac{\bar{G}_3}{r}u_{0,\theta} - \frac{\bar{G}_3}{r}v_0 + \bar{G}_3 v_{0,r} + \frac{2C_1\bar{G}_6}{r^2}w_{,\theta} - \frac{2C_1\bar{G}_6}{r}w_{,r\theta} + (\bar{G}_4 - C_1\bar{G}_6)\left(\frac{1}{r}(\psi_{r,\theta} - \psi_\theta) + \psi_{\theta,r}\right), \tag{A.13}$$

where

$$\bar{G}_i = \int_{-h/2}^{h/2} G z^i dz,\ \bar{E}_i = \int_{-h/2}^{h/2} \frac{E}{1-\upsilon^2} z^i dz,\ \bar{D}_i = \int_{-h/2}^{h/2} \frac{\upsilon E}{1-\upsilon^2} z^i dz. \quad \text{(A.14)}$$

## APPENDIX B. EXPRESSIONS OF $\bar{\eta}_1 \sim \bar{\eta}_{10}$ AND $\bar{\kappa}_1 \sim \bar{\kappa}_6$

$$\bar{\eta}_1 = \{[\kappa_3(\lambda-1)\bar{E}_2 + C_1(\kappa_3 - 4\lambda - \kappa_3\lambda)\bar{E}_4][\kappa_1 - \cot[(\lambda-1)\alpha/2]\tan[(\lambda+1)\alpha/2]] + \kappa_2\bar{E}_1[1+\lambda-(\lambda-1)\cot[(\lambda-1)\alpha/2]\tan[(\lambda+1)\alpha/2]]]\}/[\kappa_2(-1+\kappa_1(\lambda-1)-\lambda)(\bar{E}_2 - C_1\bar{E}_4)], \quad \text{(B.1)}$$

$$\bar{\eta}_2 = \frac{1}{\kappa_2}\{\kappa_1 - \cot[(\lambda-1)\alpha/2]\tan[(\lambda+1)\alpha/2]\}, \quad \text{(B.2)}$$

$$\bar{\eta}_3 = \frac{1}{\kappa_2}\{\kappa_1 - \cot[(\lambda+1)\alpha/2]\tan[(\lambda-1)\alpha/2]\}, \quad \text{(B.3)}$$

$$\bar{\eta}_4 = -\frac{\kappa_1(-1+\lambda)\bar{E}_1}{\bar{E}_2 - C_1\bar{E}_4} + \frac{(1+\lambda)\bar{E}_1}{\bar{E}_2 - C_1\bar{E}_4} + \frac{C_1(1+\lambda)^2\bar{E}_4\bar{\eta}_2}{(-\bar{E}_2 + C_1\bar{E}_4)} + \frac{\bar{\eta}_2}{(\bar{E}_2 - C_1\bar{E}_4)}(-1+\lambda)[\kappa_2\bar{E}_1 + \kappa_3\bar{E}_2 + C_1(-1-\kappa_3+\lambda)\bar{E}_4], \quad \text{(B.4)}$$

$$\bar{\eta}_5 = \{(-1+\lambda)(-1+\upsilon)(\bar{E}_1 - C_1\bar{E}_3)(\bar{E}_2 - C_1\bar{E}_4)\cos[(1+\lambda)\alpha/2]\sin[(-1+\lambda)\alpha/2] + 2\kappa_1(-1+\lambda)\bar{E}_1[\bar{E}_2 + C_1(-2\bar{E}_4 + C_1\bar{E}_6)]\cos[(-1+\lambda)\alpha/2]\sin[(1+\lambda)\alpha/2] - (1+\lambda)\{C_1(-1+\upsilon)\bar{E}_3(-\bar{E}_2 + C_1\bar{E}_4) + \bar{E}_1[(1+\upsilon)\bar{E}_2 + C_1(-(3+\upsilon)\bar{E}_4 + 2C_1\bar{E}_6)]\} \times \cos[(-1+\lambda)\alpha/2]\sin[(1+\lambda)\alpha/2] - \bar{\eta}_2\{\{-2C_1(1+\lambda)\{(-1+\upsilon)(\bar{E}_2 - C_1\bar{E}_4)(\bar{E}_4 - C_1\bar{E}_6) - (1+\lambda)\bar{E}_4[\bar{E}_2 + C_1(-2\bar{E}_4 + C_1\bar{E}_6)]\}\cos[(-1+\lambda)\alpha/2]\sin[(1+\lambda)\alpha/2] + (-1+\lambda)\{(-1+\upsilon)(\bar{E}_2 - C_1\bar{E}_4)\{-\kappa_3\bar{E}_2 + C_1[2(1+\kappa_3)\bar{E}_4 - C_1(2+\kappa_3)\bar{E}_6]\}\cos[(1+\lambda)\alpha/2]\sin[(-1+\lambda)\alpha/2] - 2[\kappa_2\bar{E}_1 + \kappa_3\bar{E}_2 + C_1(-1+\kappa_3+\lambda)\bar{E}_4][\bar{E}_2 + C_1(-2\bar{E}_4 + C_1\bar{E}_6)]\cos[(-1+\lambda)\alpha/2]\sin[(1+\lambda)\alpha/2]\}\}\}\}/ \{(\bar{E}_2 - C_1\bar{E}_4)(\bar{E}_2 + C_1(-2\bar{E}_4 + C_1\bar{E}_6))[(1+\kappa_1)(-1+\lambda)(-1+\upsilon)\cos[(1+\lambda)\alpha/2] \times \sin[(-1+\lambda)\alpha/2] - 2(1+\kappa_1 - \kappa_1\lambda + \lambda\upsilon)\cos[(-1+\lambda)\alpha/2]\sin[(1+\lambda)\alpha/2]]\}, \quad \text{(B.5)}$$

$$\bar{\eta}_6 = -\frac{\kappa_1(-1+\lambda)\bar{E}_1}{\bar{E}_2 - C_1\bar{E}_4} + \frac{(1+\lambda)\bar{E}_1}{\bar{E}_2 - C_1\bar{E}_4} + \frac{C_1(1+\lambda)^2\bar{E}_4\bar{\eta}_3}{(-\bar{E}_2 + C_1\bar{E}_4)} + \frac{\bar{\eta}_3}{(\bar{E}_2 - C_1\bar{E}_4)}(-1+\lambda)[\kappa_2\bar{E}_1 + \kappa_3\bar{E}_2 + C_1(-1-\kappa_3+\lambda)\bar{E}_4], \quad \text{(B.6)}$$

$$\bar{\eta}_7 = -\{-2\kappa_1(-1+\lambda)\bar{E}_1[\bar{E}_2 + C_1(-2\bar{E}_4 + C_1\bar{E}_6)]\cos[(1+\lambda)\alpha/2]\sin[(-1+\lambda)\alpha/2] + (1+\lambda) \times \{C_1(-1+\upsilon)\bar{E}_3(-\bar{E}_2 + C_1\bar{E}_4) + \bar{E}_1[(1+\upsilon)\bar{E}_2 + C_1(-(3+\upsilon)\bar{E}_4 + 2C_1\bar{E}_6)]\}\cos[(1+\lambda)\alpha/2] \times \sin[(-1+\lambda)\alpha/2] + (-1+\lambda)(-1+\upsilon)(\bar{E}_1 - C_1\bar{E}_3)(-\bar{E}_2 + C_1\bar{E}_4)\cos[(-1+\lambda)\alpha/2]\sin[(1+\lambda) \times \alpha/2] - \bar{\eta}_3\{\{2(-1+\lambda)[\kappa_2\bar{E}_1 + \kappa_3\bar{E}_2 + C_1(-1-\kappa_3+\lambda)\bar{E}_4][\bar{E}_2 + C_1(-2\bar{E}_4 + C_1\bar{E}_6)]\cos[(1+\lambda)\alpha/2] \times \sin[(-1+\lambda)\alpha/2] - 2C_1(1+\lambda)[(-1+\upsilon)(\bar{E}_2 - C_1\bar{E}_4)(-\bar{E}_4 + C_1\bar{E}_6) + (1+\lambda)\bar{E}_4(\bar{E}_2 + C_1(-2\bar{E}_4 + C_1\bar{E}_6))]\cos[(1+\lambda)\alpha/2]\sin[(-1+\lambda)\alpha/2] + (1-\lambda)(-1+\upsilon)(\bar{E}_2 - C_1\bar{E}_4) - \kappa_3\bar{E}_2 + C_1[2(1+\kappa_3)\bar{E}_4 - C_1(2+\kappa_3)\bar{E}_6]\}\cos[(-1+\lambda)\alpha/2]\sin[(1+\lambda)\alpha/2]\}\}/\{(\bar{E}_2 - C_1\bar{E}_4)[\bar{E}_2 + C_1(-2\bar{E}_4 + C_1\bar{E}_6)]\{2(1+\kappa_1 - \kappa_1\lambda + \lambda\upsilon)\cos[(1+\lambda)\alpha/2]\sin[(-1+\lambda)\alpha/2] - (1+\kappa_1)(-1+\lambda)(-1+\upsilon)\cos[(-1+\lambda)\alpha/2]\sin[(1+\lambda)\alpha/2]\}\}, \quad \text{(B.7)}$$

$$\bar{\eta}_8 = \frac{\bar{E}_2\bar{E}_3 - C_1\bar{E}_3\bar{E}_4 + \bar{E}_1(-\bar{E}_4 + C_1\bar{E}_6))}{\bar{E}_1\bar{E}_3 - C_1\bar{E}_3^2 + \bar{E}_0(-\bar{E}_4 + C_1\bar{E}_6))}, \quad \text{(B.8)}$$

$$\bar{\eta}_9 = \frac{\bar{E}_1^2 - 2C_1\bar{E}_1\bar{E}_3 + C_1^2\bar{E}_3^2 - \bar{E}_0(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6))}{(-\bar{E}_1\bar{E}_3 + C_1\bar{E}_3^2 + \bar{E}_0(\bar{E}_4 - C_1\bar{E}_6))},\ \bar{\eta}_{10} = \frac{(3+\lambda(\upsilon-1)+\upsilon)}{(\lambda+1)(\upsilon-1)}, \quad \text{(B.9)}$$

$$\bar{\kappa}_1 = [C_1\bar{E}_4(-\kappa_2\bar{E}_3 + 2(1+\kappa_1)\bar{E}_4) - \kappa_2\bar{E}_1(\bar{E}_4 - C_1\bar{E}_6) - \bar{E}_2(\kappa_2\bar{E}_3 - 2C_1(1+\kappa_1)\bar{E}_6)], \quad \text{(B.10)}$$

$$\bar{\kappa}_2 = [C_1\bar{E}_4(\kappa_2\bar{E}_3 - 2(-1+\kappa_1)\lambda\bar{E}_4) + \kappa_2\bar{E}_1(\bar{E}_4 - C_1\bar{E}_6) - \bar{E}_2(\kappa_2\bar{E}_3 - 2C_1(-1+\kappa_1)\lambda\bar{E}_6)], \quad \text{(B.11)}$$

$$\bar{\kappa}_3 = -\big[2C_1\bar{E}_1\bar{E}_3\bar{E}_4((3+\upsilon)\bar{E}_4 - 2C_1\bar{E}_6) + \bar{E}_2^2\big(2\bar{E}_3^2 - (1+\upsilon)\bar{E}_0\bar{E}_6\big) + \bar{E}_1^2\big(-(\upsilon-1)\bar{E}_4^2 - 4C_1\bar{E}_4\bar{E}_6 + 2C_1^2\bar{E}_6^2\big) + C_1\bar{E}_4^2\big(C_1(1-\upsilon)\bar{E}_3^2 - (1+\upsilon)\bar{E}_0(2\bar{E}_4 - C_1\bar{E}_6)\big) + \bar{E}_2\big((1+\upsilon)\bar{E}_1^2\bar{E}_6 - 2\bar{E}_1\bar{E}_3(2\bar{E}_4 + C_1(-1+\upsilon)\bar{E}_6) + C_1\bar{E}_3^2(-4\bar{E}_4 + C_1(1+\upsilon)\bar{E}_6) + (1+\upsilon)\bar{E}_0\big(\bar{E}_4^2 + 2C_1\bar{E}_4\bar{E}_6 - C_1^2\bar{E}_6^2\big)\big)\big]^2, \quad \text{(B.12)}$$

$$\bar{\kappa}_4 = \Big[ -2C_1\bar{E}_1\bar{E}_3\bar{E}_4((-5+\upsilon)\bar{E}_4 + 2C_1\bar{E}_6) + \bar{E}_2^2\big(2\bar{E}_3^2 + (-3+\upsilon)\bar{E}_0\bar{E}_6\big) + \bar{E}_1^2\big((\upsilon-1)\bar{E}_4^2 - 4C_1\bar{E}_4\bar{E}_6 + 2C_1^2\bar{E}_6^2\big) + C_1\bar{E}_4^2\big(C_1(-1+\upsilon)\bar{E}_3^2 + (-3+\upsilon)\bar{E}_0(2\bar{E}_4 - C_1\bar{E}_6)\big) - \bar{E}_2\big((-3+\upsilon)\bar{E}_1^2\bar{E}_6 + C_1\bar{E}_3^2(4\bar{E}_4 + C_1(-3+\upsilon)\bar{E}_6) + 2\bar{E}_1\bar{E}_3(2\bar{E}_4 - C_1(-1+\upsilon)\bar{E}_6) + (-3+\upsilon)\bar{E}_0\big(\bar{E}_4^2 + 2C_1\bar{E}_4\bar{E}_6 - C_1^2\bar{E}_6^2\big)\big)\Big]^2, \tag{B.13}$$

$$\bar{\kappa}_5 = \bar{E}_1^2 - 2C_1\bar{E}_1\bar{E}_3 + C_1^2\bar{E}_3^2 - \bar{E}_0\big(\bar{E}_2 - 2C_1\bar{E}_4 + C_1^2\bar{E}_6\big), \tag{B.14}$$

$$\bar{\kappa}_6 = [-\bar{E}_2\bar{E}_3 + C_1\bar{E}_3\bar{E}_4 + \bar{E}_1(\bar{E}_4 - C_1\bar{E}_6)][\lambda^3(1+\upsilon)\sin 4\alpha - 2\lambda\sin 2\alpha(\lambda^2(1+\upsilon) + 2(-5+3\upsilon)\sin^2\lambda\alpha) + 4(\lambda^2(-1+3\upsilon)\sin^2\alpha + (-3+\upsilon)\sin^2\lambda\alpha)\sin 2\lambda\alpha]. \tag{B.15}$$