On the Design of Generalized Dartboard

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Abstract

The traditional dartboard design has only one circle. In this paper, we expand one circle to double circles and define the corresponding risk function of the double dartboard. We show how to find an optimal arrangement such that the risk function is maximized for double dartboard problem.

Table of Contents

List of Figures

Chapter 1

Introduction

Darts is a highly popular game. The traditional dartboard has a single circle as shown in figure 1.1. The exciting of darts comes from the difference of adjacent numbers. As the difference is larger, the game of darts is more challenging. Curtis[1] presented a simple greedy algorithm to resolve the dartboard design[6, 7] problem. Consider a multiset of numbers $A = \{a_1, a_2, \ldots, a_n\}$ and arrange these numbers on the dartboard to form a permutation $u = \alpha_1\alpha_2 \ldots \alpha_n$. Then define the single circular risk function as $r = \sum_{i=1}^{n} |\alpha_{i-1} - \alpha_i|^p$, where $\alpha_0 = \alpha_n$, $p \geq 1$, and p is a real number. The dartboard design problem is that given a multiset of numbers A we want to find an optimal permutation such that the single circular risk function has the maximum value.

In this paper, we extend the dartboard design problem from single circle to double circles. For example, see figure 1.2. Given a multiset of numbers $A = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{2n}\}$ divide A into two multisets X and Y such that $|X| = |Y| = n$. We arrange the numbers of X on one circle of the dartboard and arrange the numbers of Y on another circle and form two permutations $u_x = x_1x_2 \ldots x_n$ and $u_y = y_1y_2 \ldots y_n$. We use the following notation to indicate an arrangement on the double dartboard.

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_n \\ y_1 & y_2 & y_3 & y_4 & y_5 & \cdots & y_n \end{matrix}$$

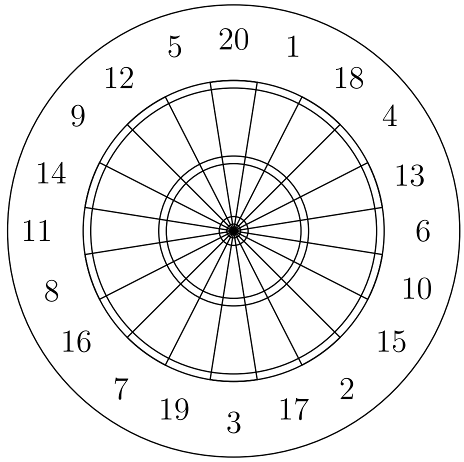

Figure 1.1: A traditional dartboard.

Because the above permutations are arranged on two circles, x_1 is next to x_n, and y_1 is next to y_n. We define the double circular risk function as $r = \sum_{i=1}^{n}\{|x_{i-1} - x_i|^p + |y_{i-1} - y_i|^p + |x_i - y_i|^p\}$, where $x_0 = x_n$, $y_0 = y_n$, $p \geq 1$, and p is a real number. The double circular dartboard design problem is:

> Given $2n$ numbers $a_1, a_2, \ldots, a_{2n}$, divide them into two multisets X and Y such that $|X| = |Y| = n$, and arrange the numbers properly such that the double circular risk function r has the maximum value.

Besides maximizing the risk function, we also consider minimizing the risk function. We call the two cases as the max-dartboard problem and the min-dartboard problem, respectively.

The dartboard design also has been studied by some authors: Eiselt[2] and Laporte found optimal permutations of the traditional dartboard numbers $\{1, 2, \ldots, 20\}$ for $p = 1, 2$, by using a branch-and-bound algorithm[3], and explained that the traditional dartboard permutation is good, but not optimal. Chao[4] and Liang discussed the permutations of n distinct numbers arranged around one circle with $p = 1$ and described a precise characteristic of both arrangements such that the values of the risk function are maximum and minimum. Cohen[5] and Tonkes analyzed optimal permutations for general multisets of numbers by a string reversal algorithm. Recently, Curtis[1] used a greedy algorithm to prove that a permutation of the form $\cdots a_5 a_{n-3} a_3 a_{n-1} a_1 a_n a_2 a_{n-2} a_4 a_{n-4} \cdots$ is an optimal solution of the dartboard design problem where $a_1 \leq a_2 \leq \cdots \leq a_n$.

In the chapter 2, we use a greedy algorithm to resolve the single circular min-dartboard problem and describe the result for the single circular max-dartboard problem. In the chapter 3, first prove that there exists an optimal solution for the double circular problem when all numbers of the multiset X and the multiset Y are fixed, and then resolve the double circular max-dartboard problem.

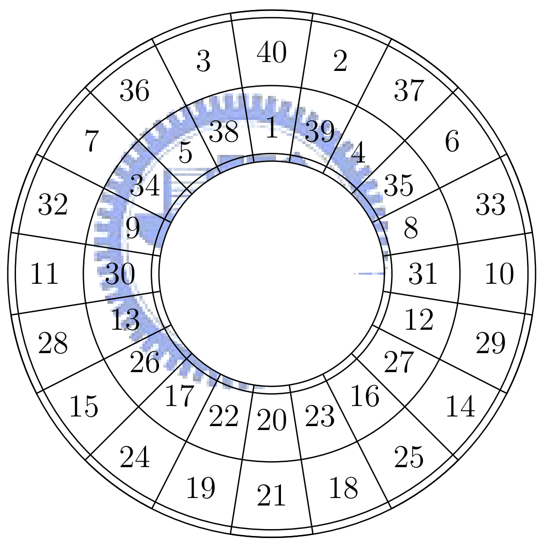

Figure 1.2: A double dartboard

Chapter 2

Arrange numbers on a circle

In this chapter, we consider the single circular dartboard design problem. We first consider the single circular min-dartboard problem. Let A be a multiset of numbers and A_{max} be the maximum number in A and A_{min} be the minimum number in A. The greedy algorithm which we use to resolve the single circular min-dartboard problem is as follows:

Step.1: First, we put the minimum(A_{min}) or maximum(A_{max}) number on the board to form a partial arc.(a permutation of length 1)

Step.2: We choose two numbers x and y in U in a greedy way to make $|x-a|^p+|y-b|^p$ minimum, where a and $b(a \leq b)$ are numbers arranged at two arc ends, and p is a real number such that $p \geq 1$, and U is the multiset of the remaining numbers.
(If there is only one number in the arc, a and b are the same number and they are the left arc end and the right arc end, respectively. If only one number is in U, choose the number directly.)

Step.3: Arrange the number x next to a and arrange the number y next to b to form a new partial arc.

Step.4: Go to the step 2 until all numbers in A are chosen.

Step.5: Return a permutation.

Lemma 1 [1] *Let $l_{min}, l_{max}, r_{min}, r_{max}, p$ be real numbers, with $p \geq 1$. If $l_{min} \leq l_{max}$ and $r_{min} \leq r_{max}$, then $|l_{max} - r_{min}|^p + |l_{min} - r_{max}|^p \geq |l_{max} - r_{max}|^p + |l_{min} - r_{min}|^p$.*

Thus, we see that the number x next to a is less than or equal to the number y next to b by Lemma 1 because $a \leq b$. Note that two numbers selected at each step (after the first number is fixed) are the two smallest numbers or the two largest numbers of the remaining numbers. Next we prove that the single circular min-dartboard problem can be resolved by the greedy algorithm.

Proposition 1 *The single circular min-dartboard problem can be resolved by the greedy algorithm.*

Proof. We will arrange all numbers in the multiset A on the dartboard. We choose the first number that can be the maximum(A_{max}) or minimum (A_{min}) in A and this action does not affect the answer. Then,we claim that the following invariant holds when the greedy algorithm is run.

$$\begin{cases} \text{If the first number is } A_{max}, \text{ then } \forall \alpha \in s: & \alpha \geq U_{max}, \\ \text{If the first number is } A_{min}, \text{ then } \forall \alpha \in s: & \alpha \leq U_{min}, \end{cases}$$

where s is the sequence of numbers arranged on the dartboard so far, and U is the multiset of the remaining numbers. Initially the invariant holds because either there is only A_{max} in s and $U = A - \{A_{max}\}$, or there is only A_{min} in s and $U = A - \{A_{min}\}$.

Then we consider a partial arc sequence s of dartboard during the progress of the algorithm after zero or more greedy steps have been run. Let the greedy step choose a_i, a_j in the multiset U to arrange them to the two ends of sequence s so far. We only discuss the case when the first number is minimum because the situation is symmetric. Without loss of generality we assume that $a_i = U_{min}$ and $a_j = (U - \{U_{min}\})_{min}$ and that the greedy step adds a_i to the front of s and adds a_j to the rear of s, as $a_i s a_j$. Let a_m be the first number in s and a_n be the last number in s. Because a_i is next to a_m and a_j is next to

a_n and $a_i \leq a_j$, we know that $|a_i - a_m|^p + |a_j - a_n|^p \leq |a_i - a_n|^p + |a_j - a_m|^p$ when $a_m \leq a_n$ by Lemma 1, where p is a real number such that $p \geq 1$. For all $a_k \in U$, $a_m \leq a_n \leq a_i \leq a_k$ because $a_i = U_{min}$. Thus, the above invariant holds.

Again, consider any complete optimal permutation of s but not necessarily from the greedy steps. Let a_i and a_j be the two smallest numbers in U. Let a_m be the first number in s and a_n be the last number in s. If a_i and a_j are not arranged next to a_m and a_n, the complete optimal permutation is in the form of figure 2.1. In figure 2.1, a_i, a_j, a_k, a_l, a_p, and a_q are in U. As a_i, a_j are the two smallest numbers in U, $a_i \leq a_k$ and $a_j \leq a_l$. By the above invariant, we know that $a_m \leq a_p$ and $a_n \leq a_q$. We cut in four places of the dartboard: between a_m and a_k, and between a_i and a_p, and between a_n and a_l, and between a_j and a_q. Then we reverse the arc $a_k t a_i$ and the arc $a_l u a_j$ within the dartboard to form a new complete permutation. This reversing action makes the difference at a_m, a_k and a_i, a_p, and a_n, a_l and a_j, a_q changed. Under the new permutation, because $a_i \leq a_k$ and $a_m \leq a_p$, and because $a_j \leq a_l$ and $a_n \leq a_q$, $|a_i - a_m|^b + |a_k - a_p|^b + |a_j - a_n|^b + |a_l - a_q|^b \leq |a_k - a_m|^b + |a_i - a_p|^b + |a_l - a_n|^b + |a_j - a_q|^b$ by Lemma 1, where b is a real number such that $b \geq 1$. So the new risk function value is less than or equal to the old risk function value. Thus, the greedy algorithm can produce an optimal solution for the min-dartboard problem. □

If the number $a_1 \leq a_2 \leq \cdots \leq a_n$ are arranged on the dartboard to form a complete permutation, the following permutation $\cdots a_9 a_7 a_5 a_3 a_1 a_2 a_4 a_6 a_8 \cdots$ is optimal with respect to the risk function for the min-dartboard problem.

In addition, the max-dartboard problem was resolved by Curtis [1]. If the number $a_1 \leq a_2 \leq \cdots \leq a_n$ are placed on the dartboard to form a permutation, the following permutation $\cdots a_5 a_{n-3} a_3 a_{n-1} a_1 a_n a_2 a_{n-2} a_4 a_{n-4} \cdots$ is optimal with respect to the risk function for the max-dartboard problem.

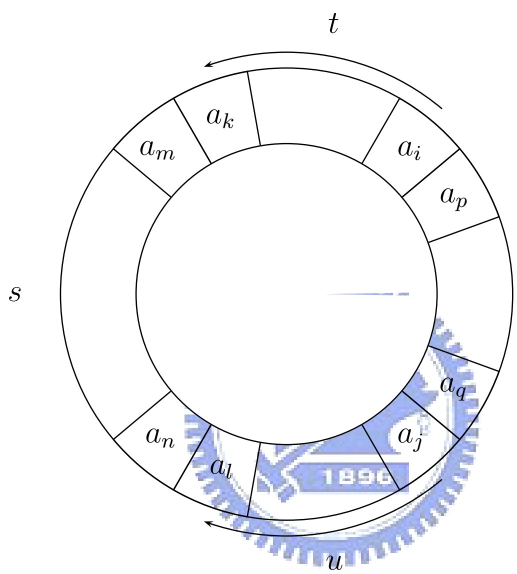

Figure 2.1: An optimal permutation when a_i and a_j are not next to a_m and a_n respectively.

Chapter 3

Arrange numbers on two circles

The difference between single circular dartboard and double circular dartboard is the definition of the risk function. First we define the risk function of double dartboard problem. Let A be a multiset of numbers $\{a_1, a_2, \cdots, a_n, a_{n+1}, \cdots, a_{2n}\}$. Divide A into two multisets X and Y such that $|X| = |Y| = n$ and arrange numbers of X around one circle of the dartboard and arrange numbers of Y around another circle. Then, form two permutations $u_x = x_1 x_2 \cdots x_n$ and $u_y = y_1 y_2 \cdots y_n$. In addition, arrange that x_i is next to y_i, where $i = 1, 2, \ldots, n$. This arrangement is indicated as follows:

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_n \\ y_1 & y_2 & y_3 & y_4 & y_5 & \cdots & y_n \end{array}$$

Definition 1 *Define the double circular risk function $r = \sum_{i=1}^{n}\{|x_{i-1}-x_i|^p + |y_{i-1} - y_i|^p + |x_i - y_i|^p\}$, where $x_0 = x_n$, $y_0 = y_n$, and p is a real number such that $p \geq 1$.*

We view $\sum_{i=1}^{n} |x_{i-1}-x_i|^p$ as the risk of the multiset X and view $\sum_{i=1}^{n} |y_{i-1}-y_i|^p$ as the risk of the multiset Y. $\sum_{i=1}^{n} |x_i - y_i|^p$ is regarded as the risk function between the circle X and the circle Y.

Definition 2 *The double circular dartboard problem is to find one optimal arrangement with respect to the risk function. The case of the risk function* *maximum is called* ***the double max-dartboard problem*** *and the case of the risk function minimum is called* ***the double min-dartboard problem****.*

3.1 Double max-dartboard with X and Y fixed

Lemma 2 *Given two multisets of numbers* $X = \{x_1, x_2, \ldots, x_n\}$ *and* $Y = \{y_1, y_2, \ldots, y_n\}$*. Without loss of generality, assume that* $x_1 \geq x_2 \geq \cdots \geq x_n$*. If* $y_1 \leq y_2 \leq \cdots \leq y_n$*,* $\sum_{i=1}^{n} |x_i - y_i|^p$ *has the maximum value, where* p *is a real number such that* $p \geq 1$*.*

Proof. Assume that $y_1, y_2, \ldots, y_n$ are not sorted in increasing order and $\sum_{i=1}^{n} |x_i - y_i|^p$ is maximum. Thus, there exist i, j and $i < j$ such that $y_i \geq y_j$. As $x_i \geq x_j$, we know that $|x_i - y_i|^p + |x_j - y_j|^p \leq |x_i - y_j|^p + |x_j - y_i|^p$ by Lemma 1. The value of risk function does not decrease after swapping y_i and y_j. By continuing this step, we can rearrange the sequence of $y_i's$ such that there is no inversion. It implies that the sorted order of $y_i's$ can actually achieve the maximum value. □

The above lemma is used to prove the following theorem.

Theorem 1 *If the numbers distributed in the multisets* X *and* Y *are both fixed, the double max-dartboard problem has an explicit optimal solution such that the risk function is maximized.*

Proof. First, we consider the risk function of a single circle. Because the numbers in the multiset X are fixed, we arrange the numbers of X around one circle of the dartboard, then there exists an optimal permutation such that the risk of the multiset X is maximum. Similarly, there exists an optimal permutation such that the risk of the multiset Y is maximum. Assume that $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ with $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. By the above, we see that the permutation $\ldots x_5 x_{n-3} x_3 x_{n-1} x_1 x_n x_2 x_{n-2} x_4 x_{n-4} \ldots$ makes the risk of X maximum and the permutation $\ldots y_5 y_{n-3} y_3 y_{n-1} y_1 y_n y_2 y_{n-2} y_4 y_{n-4} \ldots$ makes the risk of Y maximum.

Then, consider the risk between the circle X and the circle Y. By Lemma 2, $\sum_{i=1}^{n} |x_{n+1-i} - y_i|^p$ is maximum over all the sums of the difference between the numbers of X and Y because $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$.

We claim to arrange the double circular dartboard such that the risk between X and Y and the risk of X and the risk of Y are all maximum. The following arrangement (*) fits our requirement:

$$(*)\begin{cases} \ldots \quad x_5 \quad x_{n-3} \quad x_3 \quad x_{n-1} \quad x_1 \quad x_n \quad x_2 \quad x_{n-2} \quad x_4 \quad x_{n-4} \quad \ldots & (1) \\ \ldots \quad y_{n-4} \quad y_4 \quad y_{n-2} \quad y_2 \quad y_n \quad y_1 \quad y_{n-1} \quad y_3 \quad y_{n-3} \quad y_5 \quad \ldots & (2) \end{cases}$$

Permutation (1) maximizes the risk of X and the permutation (2) maximizes the risk of Y. The risk between X and Y is $\sum_{i=1}^{n} |x_{n+1-i} - y_i|^p$ and is maximum over all the sums of the difference between the numbers of X and Y. Therefore the risk of the arrangement (*) is the sum of the risk of X, the risk of Y and the risk between X and Y, and hence is maximum. $\square$

Corollary 1 *If the numbers distributed in the multiset* $A_i = \{a_{i1}, a_{i2}, \ldots, a_{im}\}$ *are fixed for all* $i = 1, 2, \ldots, n$, *the n-tuple max-dartboard problem has an optimal solution such that the risk function* $r = \sum_{i=1}^{n}\sum_{j=1}^{m} |\alpha_{i(j-1)} - \alpha_{ij}|^p + \sum_{i=1}^{n-1}\sum_{j=1}^{m} + |\alpha_{ij} - \alpha_{(i+1)j}|^p$ *is maximized, where* $\alpha_{i0} = \alpha_{im}$ *for all* $i = 1, 2, \ldots, n$, *and* $\alpha_{i1}\alpha_{i2}\cdots\alpha_{im}$ *is a permutation of* A_i.

3.2 Double min-dartboard with X and Y fixed

Lemma 3 *Given two multisets of numbers* $X = \{x_1, x_2, \ldots, x_n\}$ *and* $Y = \{y_1, y_2, \ldots, y_n\}$. *Without loss of generality, assume that* $x_1 \leq x_2 \leq \cdots \leq x_n$. *If* $y_1 \leq y_2 \leq \cdots \leq y_n$, $\sum_{i=1}^{n} |x_i - y_i|^p$ *has the minimum value, where* p *is a real number such that* $p \geq 1$.

Proof. Assume that $y_i's$ are not sorted in increasing order and $\sum_{i=1}^{n} |x_i - y_i|^p$ is minimum. Thus, there exist i, j and $i < j$ such that $y_i \geq y_j$. As $x_i \leq x_j$, we know that $|x_i - y_i|^p + |x_j - y_j|^p \geq |x_i - y_j|^p + |x_j - y_i|^p$ by Lemma 1. The value of risk function does not increase after swapping y_i and y_j. By continuing this step, we can rearrange the sequence of $y_i's$ such that there is no inversion. It implies that the sorted order of $y_i's$ can actually achieve the minimum value. □

Theorem 2 *If the numbers distributed in the multisets X and Y are both fixed, the double min-dartboard problem has an explicit optimal solution such that the risk function is minimized.*

Proof. First, we consider the risk function of a single circle. As the numbers in the multiset X are fixed, we arrange the numbers of X around one circle of the dartboard, then there exists an optimal permutation such that the risk of the multiset X is minimum. Similarly, there exists an optimal permutation such that the risk of the multiset Y is minimum. Assume that $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ with $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. By the argument in chapter 2, we know that the permutation $\ldots x_9 x_7 x_5 x_3 x_1 x_2 x_4 x_6 x_8 x_{10} \ldots$ minimizes the risk of X and the permutation $\ldots y_9 y_7 y_5 y_3 y_1 y_2 y_4 y_6 y_8 y_{10} \ldots$ minimizes the risk of Y.

Then, consider the risk between X and Y. By Lemma 3, $\sum_{i=1}^{n} |x_i - y_i|^p$ is minimum over all the sums of the difference between the numbers of X and Y because $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$.

We claim to arrange the double circular dartboard such that the risk between X and Y, the risk of X and the risk of Y are all minimum. The following arrangement (#) fits our requirement:

$$(\#)\begin{cases} \ldots \; x_9 \; x_7 \; x_5 \; x_3 \; x_1 \; x_2 \; x_4 \; x_6 \; x_8 \; x_{10} \; \ldots & (3) \\ \ldots \; y_9 \; y_7 \; y_5 \; y_3 \; y_1 \; y_2 \; y_4 \; y_6 \; y_8 \; y_{10} \; \ldots & (4) \end{cases}$$

Permutation (3) minimizes the risk of X and permutation (4) minimizes the risk of Y. The risk between the X and Y is $\sum_{i=1}^{n} |x_i - y_i|^p$ which is minimum over all the possible sums of the difference between numbers of X

and Y. Therefore, the risk of the arrangement(#) is the sum of the risk of X, the risk of Y and the risk between X and Y, and hence is minimum. □

Corollary 2 *If the numbers distributed in the multiset* $A_i = \{a_{i1}, a_{i2}, \ldots, a_{im}\}$ *are fixed for all* $i = 1, 2, \ldots, n$, *the* n*-tuple min-dartboard problem has an optimal solution such that the risk function* $r = \sum_{i=1}^{n}\sum_{j=1}^{m} |\alpha_{i(j-1)} - \alpha_{ij}|^p + \sum_{i=1}^{n-1}\sum_{j=1}^{m} + |\alpha_{ij} - \alpha_{(i+1)j}|^p$ *is minimized, where* $\alpha_{i0} = \alpha_{im}$ *for all* $i = 1, 2, \ldots, n$, *and* $\alpha_{i1}\alpha_{i2}\cdots\alpha_{im}$ *is a permutation of* A_i.

3.3 Double max-dartboard

Theorem 1 and Theorem 2 are both true when the numbers of the multisets X and Y are fixed. Thus, how to divide A into two multisets X and Y such that $|X| = |Y|$ is a key point for the double circular dartboard problem. Without loss of generality, assume that the smallest number is distributed in the multiset X. There exists $\frac{(2n-1)!}{n!(n-1)!}$ possible ways to divide A into two multisets X and Y such that $|X| = |Y|$. To try all possibilities is inefficient. Here we propose an efficient method to resolve the double max-dartboard problem.

Theorem 3 *There is an explicit optimal solution such that the risk function is maximized for the double max-dartboard problem.*

Proof. As the situation is symmetric, without loss of generality assume that $a_1 \le a_2 \le \cdots \le a_{2n}$ and the smallest number a_1 is in the multiset X. When all numbers of the multiset X and Y are fixed and $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$, the following arrangement is optimal:

$$\begin{array}{ccccccccccccc} \cdots & x_5 & x_{n-3} & x_3 & x_{n-1} & x_1 & x_n & x_2 & x_{n-2} & x_4 & x_{n-4} & \cdots \\ \cdots & y_{n-4} & y_4 & y_{n-2} & y_2 & y_n & y_1 & y_{n-1} & y_3 & y_{n-3} & y_5 & \cdots \end{array}$$

Now we want to determine the value of each x_i and y_i. So $x_1 = a_1$. The numbers next to $a_1(x_1)$ are x_n, x_{n-1}, and y_n. Because a_{2n} is at either x_n or

y_n, a_1 can be next to a_{2n}. Now we do not know that a_{2n} is in X or Y. We discuss two possible cases: (1)$y_n = a_{2n}$ and (2)$x_n = a_{2n}$.

Case 1. $y_n = a_{2n}$:

Thus, the following arrangement is optimal:

$$\begin{array}{ccccccccccccc} \dots & x_5 & x_{n-3} & x_3 & x_{n-1} & x_1(a_1) & x_n & x_2 & x_{n-2} & x_4 & x_{n-4} & \dots \\ \dots & y_{n-4} & y_4 & y_{n-2} & y_2 & y_n(a_{2n}) & y_1 & y_{n-1} & y_3 & y_{n-3} & y_5 & \dots \end{array}$$

We know that a_2 is at either x_2 or y_1. x_2 and y_1 are both next to x_n and y_{n-1}. As y_1 is next to y_n and x_2 is next to x_{n-2} and $y_n \geq x_{n-2}$, we know that taking $y_1 = a_2$ leads to a bigger value of the risk by Lemma 1. Thus, $y_1 = a_2$.

We also see that a_{2n-1} is at either x_n or y_{n-1}. x_n and y_{n-1} are both next to x_2 and y_1. As x_n is next to x_1 and y_{n-1} is next to y_3 and $x_1 \leq y_3$, we know that taking $x_n = a_{2n-1}$ leads to a bigger value of the risk by Lemma 1. Thus, $x_n = a_{2n-1}$.

a_3 is at either x_2 or y_2. As y_2 is next to y_n and x_2 is next to x_n and $y_n \geq x_n$ and the positions not arranged yet are symmetric, taking $y_2 = a_3$ makes the value of the risk bigger by Lemma 1. a_{2n-2} is at either x_{n-1} or y_{n-1}. As x_{n-1} is next to x_1 and y_{n-1} is next to y_1 and $x_1 \leq y_1$ and the positions not arranged are symmetric, taking $x_{n-1} = a_{2n-2}$ makes the value of the risk bigger by Lemma 1. Thus, $(x_{n-1}, y_2) = (a_{2n-2}, a_3)$.

Similarly, a_4 is at either x_2 or y_3. As x_2 is next to x_n and y_3 is next to y_{n-3} and $x_n \geq y_{n-3}$, taking $x_2 = a_4$ leads to a bigger value of the risk by Lemma 1. a_{2n-3} is at either x_{n-2} or y_{n-1}. As y_{n-1} is next to y_1 and x_{n-2} is next to x_4 and $y_1 \leq x_4$, taking $y_{n-1} = a_{2n-3}$ leads to a bigger value of the risk by Lemma 1. Thus, $(x_2, y_{n-1}) = (a_4, a_{2n-3})$.

Next we claim to prove by induction that

$$\begin{cases} (x_{2i+1}, y_{n-2i}) = (a_{4i+1}, a_{2n-4i}), \text{ where } i = 0, 1, \dots, \lfloor \frac{n-1}{4} \rfloor. \\ (x_{n-2i}, y_{2i+1}) = (a_{2n-4i-1}, a_{4i+2}), \text{ where } i = 0, 1, \dots, \lfloor \frac{n-2}{4} \rfloor. \\ (x_{n-2i-1}, y_{2i+2}) = (a_{2n-4i-2}, a_{4i+3}), \text{ where } i = 0, 1, \dots, \lfloor \frac{n-3}{4} \rfloor. \\ (x_{2i+2}, y_{n-2i-1}) = (a_{4i+4}, a_{2n-4i-3}), \text{ where } i = 0, 1, \dots, \lfloor \frac{n-4}{4} \rfloor. \end{cases}$$

If for $i = 0, 1, \ldots, k-1$, the above is true. Consider the case when $i = k$. $a_1, a_2, ..., a_{4k-1}, a_{4k}$ have been arranged on the double dartboard and $a_{2n-4k+1}, a_{2n-4k+2}, ..., a_{2n-1}, a_{2n}$ have also been arranged on the double dartboard. Now the partial optimal arrangement is as follows:

$$\begin{array}{ccccccccccc} \cdots & x_{2k+3} & x_{n-2k-1} & x_{2k+1} & a_{2n-4k+2} & \cdots & a_{4k} & x_{n-2k} & x_{2k+2} & x_{n-2k-2} & \cdots \\ \cdots & y_{n-2k-2} & y_{2k+2} & y_{n-2k} & a_{4k-1} & \cdots & a_{2n-4k+1} & y_{2k+1} & y_{n-2k-1} & y_{2k+3} & \cdots \end{array}$$

a_{4k+1} is at either x_{2k+1} or y_{2k+1}. As x_{2k+1} is next to $a_{2n-4k+2}$ and y_{2k+1} is next to $a_{2n-4k+1}$ and the positions not arranged are symmetric, taking $x_{2k+1} = a_{4k+1}$ leads to a bigger value of the risk by Lemma 1. a_{2n-4k} is at either x_{n-2k} or y_{n-2k}. As x_{n-2k} is next to a_{4k} and y_{n-2k} is next to a_{4k-1} and the positions not arranged are symmetric, taking $y_{n-2k} = a_{2n-4k}$ leads to a bigger value of the risk by Lemma 1. Thus, $(x_{2k+1}, y_{n-2k}) = (a_{4k+1}, a_{2n-4k})$.

a_{4k+2} is at either x_{2k+2} or y_{2k+1}. As y_{2k+1} is next to $a_{2n-4k+1}$, taking $y_{2k+1} = a_{4k+2}$ leads to a bigger value of the risk by Lemma 1. $a_{2n-4k-1}$ is at either x_{n-2k} or y_{n-2k-1}. As x_{n-2k} is next to a_{4k}, taking $x_{n-2k} = a_{2n-4k-1}$ leads to a bigger value of the risk by Lemma 1. Thus, $(x_{n-2k}, y_{2k+1}) = (a_{2n-4k-1}, a_{4k+2})$.

a_{4k+3} is at either x_{2k+2} or y_{2k+2}. As x_{2k+2} is next to $a_{2n-4k-1}$ and y_{2k+2} is next to a_{2n-4k} and the positions not arranged are symmetric, taking $y_{2k+2} = a_{4k+3}$ leads to a bigger value of the risk by Lemma 1. $a_{2n-4k-2}$ is at either x_{n-2k-1} or y_{n-2k-1}. As x_{n-2k-1} is next to a_{4k+1} and y_{n-2k-1} is next to a_{4k+2} and the positions not arranged are symmetric, taking $x_{n-2k-1} = a_{2n-4k-2}$ leads to a bigger value of the risk by Lemma 1. Thus, $(x_{n-2k-1}, y_{2k+2}) = (a_{2n-4k-2}, a_{4k+3})$.

a_{4k+4} is at either x_{2k+2} or y_{2k+3}. As x_{2k+2} is next to $a_{2n-4k-1}$, taking $x_{2k+2} = a_{4k+4}$ leads to a bigger value of the risk by Lemma 1. $a_{2n-4k-3}$ is at either x_{n-2k-2} or y_{n-2k-1}. As y_{n-2k-1} is next to a_{4k+2}, taking $y_{n-2k-1} = a_{2n-4k-3}$ leads to a bigger value of the risk by Lemma 1. Thus, $(x_{2k+2}, y_{n-2k-1}) = (a_{4k+4}, a_{2n-4k-3})$.

So the optimal arrangement for the double max-dartboard problem is as follows:

$$\begin{array}{cccccccccccc} \cdots & a_9 & a_{2n-6} & a_5 & a_{2n-2} & a_1 & a_{2n-1} & a_4 & a_{2n-5} & a_8 & a_{2n-9} & \cdots \\ \cdots & a_{2n-8} & a_7 & a_{2n-4} & a_3 & a_{2n} & a_2 & a_{2n-3} & a_6 & a_{2n-7} & a_{10} & \cdots \end{array}$$

Case 2. $x_n = a_{2n}$:

Thus, the following arrangement is optimal:

$$\begin{array}{cccccccccccc} \cdots & x_5 & x_{n-3} & x_3 & x_{n-1} & x_1(a_1) & x_n(a_{2n}) & x_2 & x_{n-2} & x_4 & x_{n-4} & \cdots \\ \cdots & y_{n-4} & y_4 & y_{n-2} & y_2 & y_n & y_1 & y_{n-1} & y_3 & y_{n-3} & y_5 & \cdots \end{array}$$

Because $x_1 = a_1 \leq y_1$, if $x_{n-1} < y_{n-1}$, we can swap the position of x_1 with the position of y_1 to form a new arrangement. In the new arrangement, x_1 is next to y_{n-1}, and y_1 is next to x_{n-1} ,and x_1 and y_1 are both next to x_n and y_n. By Lemma 1, the risk of the new arrangement is no less than the risk of the old arrangement. Because a_1 and a_{2n} in the new arrangement are not on the same circle, it reduces to case 1 if $x_{n-1} < y_{n-1}$. Then, assume that $x_{n-1} \geq y_{n-1}$.

Because $x_n = a_{2n} \geq y_n$, if $x_2 > y_2$, we can swap the position of x_n with the position of y_n to form a new arrangement. In the new arrangement, x_n is next to y_2 and not next to x_2, and y_n is next to x_2 and not next to y_2. By Lemma 1, the risk of the new arrangement is no less than the risk of the old arrangement. Because a_1 and a_{2n} in the new arrangement are not on the same circle, it reduces to case 1. Then, assume that $x_2 \leq y_2$.

Because $x_1 \leq y_1$ and $x_2 \leq y_2$ and $x_{n-1} \geq y_{n-1}$ and $x_n \geq y_n$, if $x_3 > y_3$, we can swap the positions of x_1, x_{n-1} with the positions of y_1, y_{n-1} respectively to form a new arrangement. In the new arrangement, x_{n-1} is next to x_2, y_3 and not next to y_2, x_3, and y_{n-1} is next to y_2, x_3 and not next to x_2, y_3. By Lemma 1, the risk of the new arrangement is no less than the risk of the old arrangement. Because a_1 and a_{2n} in the new arrangement are not on the same circle, it reduces to case 1. Then, assume that $x_3 \leq y_3$.

Similarly, we can use the above to justify that case 2 reduces to case 1 when $x_i > y_i$ or $x_{n+1-i} < y_{n+1-i}$, where $i = 2, \ldots, \lfloor \frac{n}{2} \rfloor$. Thus, we only consider the situation of $x_i \leq y_i$ and $x_{n+1-i} \geq y_{n+1-i}$, for all $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

If n is even, let $n = 2k$. So $\lfloor \frac{n}{2} \rfloor = k$. we swap the positions of $x_1, x_{n-1}, x_3, x_{n-3}, x_5, \ldots$ with the positions of $y_1, y_{n-1}, y_3, y_{n-3}, y_5 \ldots$ respectively to form a new arrangement, as follows:

$$\begin{array}{ccccccccccccc} \cdots & y_5 & y_{n-3} & y_3 & y_{n-1} & y_1 & x_n & x_2 & x_{n-2} & x_4 & x_{n-4} & \cdots \\ \cdots & y_{n-4} & y_4 & y_{n-2} & y_2 & y_n & x_1 & x_{n-1} & x_3 & x_{n-3} & x_5 & \cdots \end{array}$$

In the new arrangement, x_i is next to x_{n+1-i} and not next to y_{n+1-i} , and y_i is next to y_{n+1-i} and not next to x_{n+1-i}, where $i = 2, 3, \ldots, k-1$. By Lemma 1, the risk of the new arrangement is no less than the risk of the old one. Because a_1 and a_{2n} in the new arrangement are not on the same circle, it reduces to case 1.

If n is odd, let $n = 2k+1$. So $\lfloor \frac{n}{2} \rfloor = k$. If $x_{k+1} < y_{k+1}$, we swap the positions of $x_k, x_{n-(k-2)}, x_{k-2}, x_{n-(k-4)}, x_{k-4}, \ldots$ with the positions of $y_k, y_{n-(k-2)}, y_{k-2}, y_{n-(k-4)}, y_{k-4}, \ldots$ to form a new arrangement, as follows:

$$\begin{array}{ccccccccc} \cdots & y_{k-2} & y_{n+1-(k-1)} & y_k & x_{k+1} & x_{n+1-k} & x_{k-1} & x_{n+1-(k-2)} & \cdots \\ \cdots & y_{n+1-(k-2)} & y_{k-1} & y_{n+1-k} & y_{k+1} & x_k & x_{n+1-(k-1)} & x_{k-2} & \cdots \end{array}$$

In the new arrangement, x_i is next to x_{n+1-i} and not next to y_{n+1-i} , and y_i is next to y_{n+1-i} and not next to x_{n+1-i}, where $i = 2, 3, \ldots, k$. In addition, x_k is next to y_{k+1} and not next to x_{k+1}, and y_k is next to x_{k+1} and not next to y_{k+1}. By Lemma 1, the risk of the new arrangement is no less than the risk of the old one. Because a_1 and a_{2n} in the new arrangement are not on the same circle, it reduces to case 1. If $x_{k+1} \geq y_{k+1}$, we swap the positions of $x_{k+1}, x_k, x_{n-(k-2)}, x_{k-2}, x_{n-(k-4)}, x_{k-4}, \ldots$ with the positions of $y_{k+1}, y_k, y_{n-(k-2)}, y_{k-2}, y_{n-(k-4)}, y_{k-4}, \ldots$ to form a new arrangement, as follows:

$$\begin{array}{ccccccccc} \cdots & y_{k-2} & y_{n+1-(k-1)} & y_k & y_{k+1} & x_{n+1-k} & x_{k-1} & x_{n+1-(k-2)} & \cdots \\ \cdots & y_{n+1-(k-2)} & y_{k-1} & y_{n+1-k} & x_{k+1} & x_k & x_{n+1-(k-1)} & x_{k-2} & \cdots \end{array}$$

In the new arrangement, x_i is next to x_{n+1-i} and not next to y_{n+1-i} , and y_i is next to y_{n+1-i} and not next to x_{n+1-i}, where $i = 2, 3, \ldots, k$. In addition, x_{n+1-k} is next to y_{k+1} and not next to x_{k+1}, and y_{n+1-k} is next to x_{k+1} and not next to y_{k+1}. By Lemma 1, the risk of the new arrangement is no less than the old one. Because a_1 and a_{2n} in the new arrangement are not on the same circle, it reduces to case 1.

Thus, we only need to consider the case 1. □

The algorithm for the double max-dartboard problem is as follows:

Step.1: We put the minimum number(A_{min}) on X and the maximum number(A_{max}) on Y and make that A_{min} is next to A_{max}.

Step.2: We choose two numbers U_{min} and U_{max} in U, where U is the multiset of the remaining numbers.

Step.3: Arrange U_{min}, U_{max} next to a, b respectively and arrange U_{min} is next to U_{max}, where one of a and b is at the arc end of X and the other is at the arc end of Y, and a is next to b, and $a \geq b$.

Step.4: Go to the step 2 until all numbers in A are chosen.

Step.5: Return a arrangement.

Thus, the algorithm runs in $O(n \log n)$ time for a multiset of $2n$ numbers: sorting numbers takes $O(n \log n)$ and the greedy algorithm takes $O(n)$.

Chapter 4

Conclusion and remarks

The main results in this paper are: (1) The single circular min-dartboard problem: the permutation $\ldots a_7 a_5 a_3 a_1 a_2 a_4 a_6 a_8 \ldots$ is optimal if $a_1 \leq a_2 \leq \cdots \leq a_n$, and we use a greedy algorithm to resolve this problem. (2) The double circular max-dartboard problem: the following arrangement is optimal if $a_1 \leq a_2 \leq \cdots \leq a_{2n}$.

$$\begin{array}{ccccccccccccc} \ldots & a_9 & a_{2n-6} & a_5 & a_{2n-2} & a_1 & a_{2n-1} & a_4 & a_{2n-5} & a_8 & a_{2n-9} & \ldots \\ \ldots & a_{2n-8} & a_7 & a_{2n-4} & a_3 & a_{2n} & a_2 & a_{2n-3} & a_6 & a_{2n-7} & a_{10} & \ldots \end{array}$$

Open problems: (1) Is there an explicit optimal arrangement for the double circular min-dartboard problem? (2) When we expand from the double circular board to the n-tuple circular board with n($n \geq 3$) levels, is there an explicit optimal arrangement?

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