國立交通大學

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博士論文

以多指標評比受評單位時之共同權重分析

Common Weight Analysis to Assess Units with Multiple Performance Indices

研 究 生：彭浩軒

指導教授：劉復華 博士

中 華 民 國 九 十 七 年 七 月

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摘　要

一般常見的組織，如銀行、醫院、學校...等等，其管理者經常處理以多指標評比其轄下單位的問題。管理者常會期望決定各指標的權重，以計算各單位的綜合績效，進一步將各單位加以排序，並探討組織內部績效變異的趨勢。而建立共同權重的方式簡單可以分為兩類，事前管理者主觀決定與事後客觀由資料本身來決定，在本研究則是以後者的方式探討組織內部各單位的績效與排序。以搜尋標竿的機制下來決定共同權重以進行排序，透過設定虛擬標竿並最小化所有單位與虛擬標竿的差異來完成，而與虛擬標竿保持無差異之單位即是實質標竿。若以資料包絡分析法的數學模式架構來分析，即是以組織管理者的角度客觀決定一組共同權重，以進行組織內部的各單位排序。另外，為避免績效值上限值造成組織內部各單位績效發展上的阻礙，本研究衍生出第二種數學模式，消除績效值上限為 1 的限制，並以中立妥協的角度產生一組共同權重來進行排序。此一模式類似於回歸分析最小化各單位之間的差異，而其最大不同在於可同時處理多項應變數。最後，我們將此兩種共同權重的分析流程，透過數個範例進行應用上的模擬。

關鍵字：多指標、共同權重、排序、資料包絡分析法、回歸分析

Common Weight Analysis to Assess Units with Multiple Performance Indices

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Abstract

Managers in many fields such as banks, hospitals, and schools frequently assess units under their governance with multiple performance indices. In order to rank the units intuitively with a comparable score, managers always try to determine a common set of weights attached to the indices across all the units. The determination of weights is divided into two ways. One is predetermined subjectively by the manager; the other is determined objectively by the data itself. The methodologies proposed in this research belong to the latter. We propose a procedure to determine the common weights by searching the benchmark unit. One virtual benchmark is defined as units with an efficiency score of 1.0 and all units are asked to approach the virtual benchmark as closely as possible. The units with zero gaps to the virtual benchmark are the real benchmark. In the structure of the data envelopment analysis, the determination of common weights in this research means that the organization manager determines the favorable weight to maximize the organization efficiency. Additionally, in order to avoid the bias in measurement due to the upper bound of efficiency, we develop the second procedure to determine one compromise common set of weights, by eliminating the restriction imposing the upper bound of 1.0 on the efficiency score. The model we propose is similar to the regression analysis model, with the main difference being that the former can handle multiple dependent variables. Finally, the procedures are applied in several numerical cases.

Keywords: multiple performance indices, common set of weights, rank, data envelopment analysis, regression analysis.

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彭浩軒 謹致於

國立交通大學工業工程與管理學系

中華民國九十七年八月十三日

Contents

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Notations

Used in Data Envelopment Analysis (DEA) models

DMU: Decision-making Units

n : the number of DMU

m : the number of input

s : the number of output

i : the index of input i

r : the index of output r

j : the index of DMU j

o : the index of object DMU

v_{io} : the weight assigned to input i of DMU_o

u_{ro} : the weight assigned to output r of DMU_o

v_{ij} : the weight assigned to input i of DMU_j

u_{rj} : the weight assigned to output r of DMU_j

x_{ij} : the input i of DMU_j

y_{rj} : the output r of DMU_j

θ_o^* : the efficiency score of the objective DMU_o in input-oriented optimization model

θ_j^* : the efficiency score of DMU_j in input-oriented optimization model

η_o^* : the efficiency score of the objective DMU_o in output-oriented optimization model

ε : the positive Archimedean infinitesimal constant

E : the set of efficient DMUs

φ_j^* : the efficiency score of DMU_j using common set of weights in DEA structure

Used in Common Weights Analysis (CWA) models

UOA: Unit of Assessment

V_i : the common weight assigned to input i across all UOAs

U_r : the common weight assigned to output r across all UOAs

V_i^* : the common weight assigned to input i across all UOAs in the optimization model

U_r^* : the common weight assigned to output r across all UOAs in the optimization model

V_i' : the arbitrary common weight assigned to input i across all UOAs

U_r' : the arbitrary common weight assigned to output r across all UOAs

X_i : the input i of aggregated DMU

Y_r : the output r of aggregated DMU

Δ_j^O : the total virtual gap of the outputs for UOA_j to the benchmark

Δ_j^I : the total virtual gap of the inputs for UOA_j to the benchmark

Δ_j : the total virtual gap of the inputs and outputs for UOA_j to the benchmark

ζ_j : the CWA efficiency score of UOA_j

ε_r^O : the positive Archimedean infinitesimal constant of output r

ε_i^I : the positive Archimedean infinitesimal constant of input i

π_j : the produced marginal improvement of UOA_j

P_r : the total shortfall of the output r to benchmark for all UOAs

Q_i : the total excess of the input i to benchmark for all UOAs

p_{rj} : the shortfall of the output r to benchmark for UOA_j

q_{ij} : the excess of the input i to benchmark for UOA_j

VWR-CWA : CWA with virtual weights restrictions

ξ_j : the VWR-CWA efficiency score of UOA_j

α_{iw} : the preference of input i to restriction w

β_{rw} : th preference of output r to restriction w

k_w : the intercept of line restriction w

P_{rj}^o : the proportional virtual output r of UOA_j

P_{ij}^I : the proportional virtual input i of UOA_j

a_r : the lower bound of P_{rj}^o for all UOA_j

b_r : the upper bound of P_{rj}^o for all UOA_j

c_i : the lower bound of P_{ij}^I for all UOA_j

d_i : the upper bound of P_{ij}^I for all UOA_j

B_r^{OL} : the lower bound of P_{rj}^o with the parameter δ_r^-

B_r^{OU} : the upper bound of P_{rj}^{o} with the parameter δ_r^{+}

B_i^{IL} : the lower bound of P_{ij}^{I} with the parameter τ_i^{-}

B_i^{IU} : the upper bound of P_{ij}^{I} with the parameter τ_i^{+}

Used in Most Compromise Weights Analysis (MCWA) models

A : the set of UOAs above diagonal line

B : the set of UOAs below diagonal line

σ : the variant slope of DL

A' : the set of UOAs above diagonal line with unknown slope σ

B' : the set of UOAs below diagonal line with unknown slope σ

Δ_j^{A} : the total virtual gap of the inputs and outputs of UOA$_j$ above the diagonal line

Δ_j^{B} : the total virtual gap of the inputs and outputs of UOA$_j$ below the diagonal line

Δ_j^{OA} : the total virtual gap of the outputs of UOA$_j$ above the diagonal line

Δ_j^{IA} : the total virtual gap of the inputs for UOA$_j$ above the diagonal line

Δ_j^{OB} : the total virtual gap of the outputs of UOA$_j$ below the diagonal line

Δ_j^{IB} : the total virtual gap of the inputs of UOA$_j$ below the diagonal line

ε_r^{O} : the positive Archimedean infinitesimal constant of output r

ε_i^{I} : the positive Archimedean infinitesimal constant of input i

η_j : the MCWA efficiency score of UOA$_j$

Đ : the virtual scale unit for the weighted inputs and outputs.

1. Introduction

We will give a brief description about the motivation, background, problem description, assumption, objectives, and organization of the research in this section.

1.1 Motivation and background

Performance measurement of one organization is that of assessing progress toward achieving the predetermined goals. In the process, the variant information derived from performance assessments depends on the aims of the assessment. In general, for the purpose of improving the worst and rewarding the better units, managers often focus the performance management on the generation of objective ranking of the units. However, it is a difficult task to assess the organization units while the general manager considers multiple performance indices simultaneously. This is especially difficult within the organization, for each unit of assessment (UOA), multiple kinds of resources they use, and multiple types of output they produce must be considered. For example, all bank branches would typically use staff and capital assets to generate income activities, such as advancing loans, selling financial products, and carrying out banking transactions for their clients. Generally, the managers predetermine the weights subjectively for each performance index. The traditional determination of weights possesses the advantage that the determination can easily cover the managers' preference in different performance indices, and it is intuitive without redundant numerical analysis. However, the disadvantage is that it is hard to make a decision in determining the weights while the managers have no idea about the relationship among the performance indices. Sometimes, in the above condition, they need some methodologies to assist them in finishing the measurement.

Data Envelopment Analysis (DEA), originated by Charnes et al. [1], is one famous methodology to analyze the relative efficiency of units with multiple performance indices. DEA determines a set of the most favorable weights for each unit against all units. In other words, units employ a different set of weights to create their most favorable efficiency scores individually. DEA successfully separates the units into two subgroups: not-be-dominated and be-dominated units, also known as efficient and inefficient units. The methodology provides preliminary useful and valuable information to the unit of rewarding and improvement. However, ranking of units in each group is still a challenging task. Cooper and Tone [2] proposed that one cannot rank the be-dominated units according to their performance scores, since each of them may have a particular set of not-be-dominated units as their reference set. All the not-be-dominated units are tied numerically with the performance score 1.0 and cannot be ranked. There is a vast amount of literature about ranking the efficient and inefficient units and this will be discussed in the following literature review in section 2.

1.2 Problem description and assumption

The problem is usually expressed as the following: In one organization, n UOAs with the known m inputs and s outputs indices are assessed for the purpose of ranking and performance improvement. For each UOA, say UOA$_j$, the given values of input and output indices are denoted as (x_{1j}, x_{2j}, ..., x_{mj}) and (y_{1j}, y_{2j}, ..., y_{sj}), respectively. The inputs, or resources, are the indices that the lower value is desired in performance measurement. Therefore, the indices possess the characteristic of to-be-minimized, also categorized as "inputs." On the other hand, the outputs, also known as production, are the indices that the higher is the better. Therefore, the indices possess the characteristic of to-be-maximized, which are also categorized as "outputs." The ratio of the weighted sum of outputs (also called virtual output) to the weighted sum of inputs (also called virtual input) is called the efficiency score.

In our solved problem, the indices are assumed that are accumulative to the individual unit to the organization. Hence, the statistics, for instance the mean or variance, are not suitable to be the index candidates. Although any assessment may have uncountable possible performance indices combinations by facing different units even in the same organization, we assume that the performance indices and their corresponding data are given in advance in this research. Hence, we assumed the indices are determined already and that all of them can be quantified as nonnegative real numbers. The data of every UOA in all performance indices has been collected without the consideration of missing data.

1.3 Objectives of the research

Without predetermining the weights of performance indices, in this research, we develop two procedures to determine the common weights, (V_1, V_2, ..., V_m) and (U_1, U_2, ..., U_s), relative to the performance indices (x_{1j}, x_{2j}, ..., x_{mj}) and (y_{1j}, y_{2j}, ..., y_{sj}) across all units of organization. The first one is to determine the common weights by searching the benchmark unit in the organization. One virtual benchmark is defined as units with an efficiency score of 1.0 and all units are asked to approach the virtual benchmark as closely as possible. The units with zero gaps to the virtual benchmark are the real benchmark. In the structure of data envelopment analysis, the determination of common weights in this research means that the organization determines the favorable weight to maximize the organization efficiency. The obtained common weights can assist the organization managers in generating the individual efficiency score for all units and the corresponding ranking problem can be addressed by comparing with the scores.

However, in the first procedure, sometimes there exists some units with the equivalent efficiency score of 1.0, due to the constraint that none of the DMU efficiency scores is allowed to exceed 1.0. This could possibly lead to the obstruction of efficiency development. In order to avoid the bias in measurement due to the upper bound of efficiency, we developed the second procedure to determine one compromise common set of weights by eliminating the restriction with upper bound of 1.0 in the efficiency score. It leads to the more complete ranking, without the repeatable efficiency scores. The proposed model in the second procedure is also similar to the regression analysis model, with the main difference being that the former can handle multiple dependent variables, while the latter only focuses on one dependent variable.

1.4 Organization of the dissertation

In this research, Chapter 2 provides a review of the related literature about DEA, ranking of DEA units, and common weights in the ranking of DEA units. Chapter 3 introduces the ranking procedure, including CWA methodology, in view of the management perspective of benchmark chasing. In addition, in order to enforce the application in reality, the virtual weights restrictions are considered in CWA to satisfy the real conditions in the assessment. Chapter 4 proposes the other methodology, MCWA, to discuss the common weights without considering the upper bound 1.0 of efficiency score. It results in some of the efficiency score exceeding 1.0 with superior ranking. The conclusion and discussion are presented in Chapter 5. The structure of this dissertation is illustrated in Figure 1.

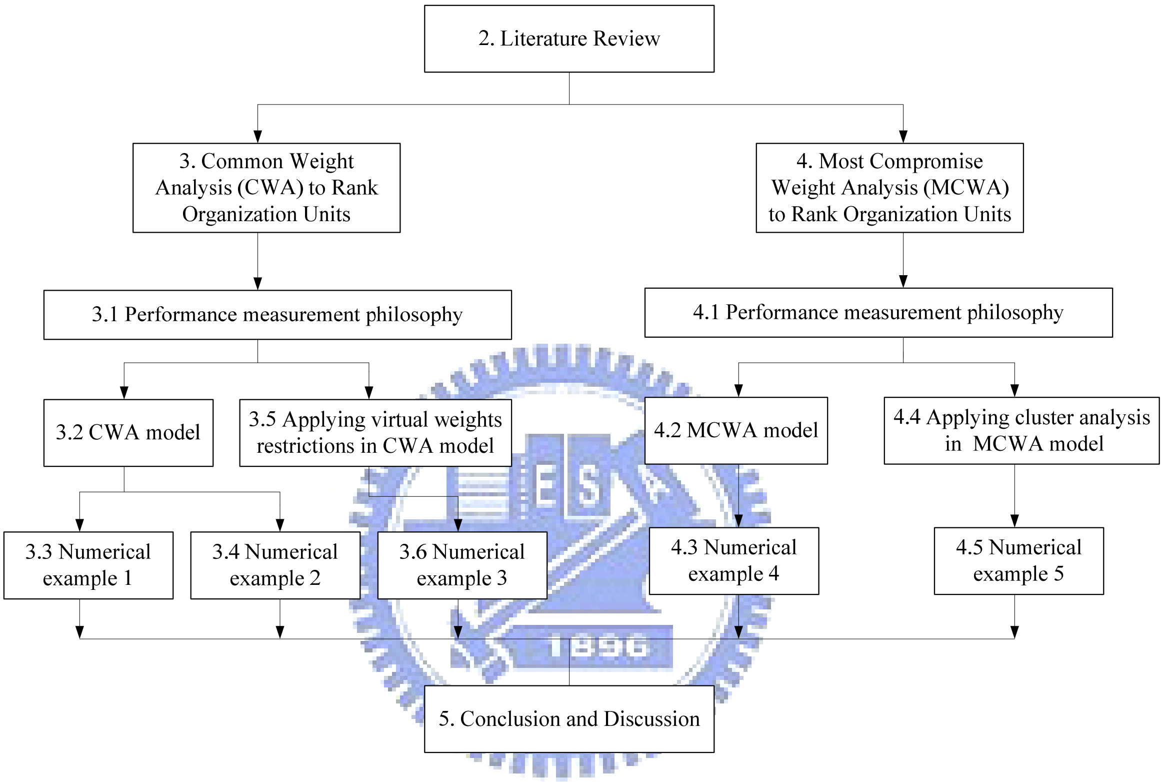

Figure 1. Organization of dissertation

2. Literature Review

We give a series of literature review in DEA, including several famous models and ranking methodologies in DEA structure, especially the common weights used in the ranking methodologies.

2.1 Data envelopment analysis (DEA)

Charnes et al. [1] introduce DEA to assess the relative efficiency of a homogeneous group of decision-making units (DMUs), such as schools, hospitals, or sales outlets. The DMUs usually use a set of resources, referred to as input indices, and transform them into a set of outcomes, referred to as output indices. DEA successfully divides them into two categories: efficient DMUs and inefficient DMUs. The DMUs in the efficient category have identical efficiency score. The initial problem is usually expressed as: n DMUs to be assessed with m inputs and s outputs indices. For each DMU, say DMU$_j$, the given values of indices are denoted as $(x_{1j}, x_{2j}, \ldots, x_{mj})$ and $(y_{1j}, y_{2j}, \ldots, y_{sj})$, respectively. Given the data, DEA measures the best practice comparative efficiency of each DMU once and hence needs n optimizations, one for each DMU$_j$ to be evaluated. Let the DMU$_j$ being evaluated on any trial be designated as DMU$_o$ where o ranges over 1, …, n. We can solve the following multiplier form of fractional programming (2-1) or linear programming (2-2) to obtain objective value (relative efficiency θ_o^*) and one comparative set of weights of inputs (v_{io}^*, i = 1, …, m) and outputs (u_{ro}^*, r = 1, …, s). The symbol ε is a positive Archimedean infinitesimal constant, which is used in order to avoid the appearance of zero weights. This zero case in weights would result in the meaningless of certain indices used in DEA. It is convenient to solve the envelopment form of linear programming (2-3), dual form of (2.2), to obtain objective value (relative efficiency θ_o^*) directly. Besides, linear programming (2-3) can provide the information of output shortfalls (s_r^{+*}) and input excesses (s_i^{-*}) to the frontier.

DEA-CCR-Input Oriented-Multiplier Form-FP

$$\theta_{o}^{*} = \max \quad \frac{\sum_{r=1}^{s} y_{ro} u_{ro}}{\sum_{i=1}^{m} x_{io} v_{io}} \tag{2-1.0}$$

$$s.t. \quad \frac{\sum_{r=1}^{s} y_{rj} u_{ro}}{\sum_{i=1}^{m} x_{ij} v_{io}} \le 1, \quad j = 1, \ldots, n, \tag{2-1.1}$$

$$u_{ro} \ge \varepsilon > 0, \qquad r = 1, \ldots, s, \tag{2-1.2}$$

$$v_{io} \ge \varepsilon > 0, \qquad i = 1, \ldots, m. \tag{2-1.3}$$

DEA-CCR- Input Oriented-Multiplier Form-LP

$$\theta_{o}^{*} = \max \quad \sum_{r=1}^{s} y_{ro} u_{ro} \tag{2-2.0}$$

$$s.t. \quad \sum_{i=1}^{m} x_{io} v_{io} = 1 \tag{2-2.1}$$

$$- \sum_{i=1}^{m} x_{ij} v_{io} + \sum_{r=1}^{s} y_{rj} u_{ro} \le 0, \quad j = 1, \ldots, n, \tag{2-2.2}$$

$$u_{ro} \ge \varepsilon > 0, \qquad r = 1, \ldots, s, \tag{2-2.3}$$

$$v_{io} \ge \varepsilon > 0, \qquad i = 1, \ldots, m. \tag{2-2.4}$$

DEA-CCR- Input Oriented-Envelopment Form

$$\theta_{o}^{*} = \min \ \theta_{o} - \varepsilon \left(\sum_{i=1}^{m} s_{i}^{-} + \sum_{r=1}^{s} s_{r}^{+} \right) \tag{2-3.0}$$

$$s.t \quad \sum_{j=1}^{n} x_{ij} \lambda_{j} + s_{i}^{-} = \theta_{o} x_{io}, \quad i = 1, \ldots, m, \tag{2-3.1}$$

$$\sum_{j=1}^{n} y_{rj} \lambda_{j} - s_{r}^{+} = y_{ro}, \quad r = 1, \ldots, s, \tag{2-3.2}$$

$$\theta_{o} \ge 0 \tag{2-3.3}$$

$$\lambda_{j} \ge 0, \qquad j = 1, \ldots, n, \tag{2-3.4}$$

$$s_{i}^{-} \ge 0, \qquad i = 1, \ldots, m, \tag{2-3.5}$$

$$s_{r}^{+} \ge 0, \qquad r = 1, \ldots, s. \tag{2-3.6}$$

It is claimed that object DMU$_o$ is comparative efficient, with the efficiency θ_o^* = 1.0, also called an efficient DMU. We define $E = \{ j \mid \theta_j^* = 1.0, j = 1, 2, \ldots, n\}$ to represent the set of efficient DMUs. However, it is not appropriate to claim that they have the equivalent performance in actual practice. Managers always face the problem of how to carry out a further comparison among DMUs on the set *E*. In addition, for the category of inefficient DMUs, the efficiency score is derived from comparisons involving performances of different sets of efficient DMUs. Their performances cannot be compared by comparing them with the range of efficiency score generated from the different facets.

DEA-CCR- Output Oriented-Multiplier Form-FP

$$\eta_o^* = \min \quad \frac{\sum_{i=1}^{m} x_{io} v_{io}}{\sum_{r=1}^{s} y_{ro} u_{ro}} \tag{2-4.0}$$

$$s.t. \quad \frac{\sum_{i=1}^{m} x_{ij} v_{io}}{\sum_{r=1}^{s} y_{rj} u_{ro}} \geq 1, \quad j = 1, \ldots, n, \tag{2-4.1}$$

$$u_{ro} \geq \varepsilon > 0, \quad r = 1, \ldots, s, \tag{2-4.2}$$

$$v_{io} \geq \varepsilon > 0, \quad i = 1, \ldots, m. \tag{2-4.3}$$

DEA-CCR- Output Oriented-Multiplier Form-LP

$$\eta_o^* = \min \quad \sum_{i=1}^{m} x_{io} v_{io} \tag{2-5.0}$$

$$s.t. \quad \sum_{r=1}^{s} y_{ro} u_{ro} = 1 \tag{2-5.1}$$

$$\sum_{i=1}^{m} x_{ij} v_{io} - \sum_{r=1}^{s} y_{rj} u_{ro} \geq 0, \quad j = 1, \ldots, n, \tag{2-5.2}$$

$$u_{ro} \geq \varepsilon > 0, \quad r = 1, \ldots, s, \tag{2-5.3}$$

$$v_{io} \geq \varepsilon > 0, \quad i = 1, \ldots, m. \tag{2-5.4}$$

DEA-CCR- Output Oriented-Envelopment Form

$$\eta_o^{\ *} = \max\ \eta_o + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \tag{2-6.0}$$

$$s.t\ \ \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io}\,, \qquad i = 1,\ldots,m, \tag{2-6.1}$$

$$\sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = \eta_o\ y_{ro}\,,\ \ r = 1,\ldots,s, \tag{2-6.2}$$

$$\eta_o \geq 0 \tag{2-6.3}$$

$$\lambda_j \geq 0, \qquad j = 1,\ldots,n, \tag{2-6.4}$$

$$s_i^- \geq 0, \qquad i = 1\ldots,m, \tag{2-6.5}$$

$$s_r^+ \geq 0, \qquad r = 1,\ldots,s. \tag{2-6.6}$$

The type of models (2-1), (2-2), (2-3) is also called *input-oriented* model whose objective is to minimize inputs while producing at least the given output levels. The opposite type of models is called *output-oriented* model that attempt to maximize outputs while using no more than the observed amount of any input, as depicted (2-4), (2-5), (2-6). While combining both orientations in a single model, it is called the *additive* model, as depicted (2-7) and (2-8). The model (2-7) represents that each unit chooses the weight most favorable to obtain its efficiency score. By the dual model (2-8) of multiplier form (2-7), we can observe the input excess (s_i^{-*}) and the output shortfall (s_r^{+*}) of each unit to the frontier. The sum of input excess and the output shortfall stands for the inefficient status of DMU$_o$. When the most inefficient status is that there exist zero slack to the frontier, i.e. the maximum sum of slacks is zero, we call DMU$_o$ is efficient.

DEA-ADD-Multiplier Form

$$\max\ \ \sum_{r=1}^{s} y_{ro}u_{ro} - \sum_{i=1}^{m} x_{io}v_{io} \tag{2-7.0}$$

$$s.t.\ \ \sum_{r=1}^{s} y_{rj}u_{ro} - \sum_{i=1}^{m} x_{ij}v_{io} \leq 0,\ \ j = 1,\ldots,n, \tag{2-7.1}$$

$$u_{ro} \geq \varepsilon > 0, \qquad r = 1,\ldots,s, \tag{2-7.2}$$

$$v_{io} \geq \varepsilon > 0, \qquad i = 1,\ldots,m. \tag{2-7.3}$$

DEA-ADD-Envelopment Form

$$\max \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \tag{2-8.0}$$

$$s.t \quad \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io}, \quad i = 1\ldots, m, \tag{2-8.1}$$

$$\sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{ro}, \quad r = 1,\ldots, s, \tag{2-8.2}$$

$$\lambda_j \geq 0, \qquad j = 1,\ldots, n, \tag{2-8.3}$$

$$s_i^- \geq 0, \qquad i = 1,\ldots, m, \tag{2-8.4}$$

$$s_r^+ \geq 0, \qquad r = 1,\ldots, s. \tag{2-8.5}$$

2.2 Ranking in DEA

Andersen and Petersen [3] evaluate that a DMU's efficiency possibly exceeds the conventional score 1.0, by comparing the DMU being evaluated with a linear combination of other DMUs, while excluding the observations of the DMU being evaluated. They try to discriminate between these efficient DMUs, by using different efficiency scores larger than 1.0. Hashimoto [4] developed a DEA super-efficient model with assurance regions in order to rank the DMUs completely. However, the super-efficient methodology can give specialized DMUs an excessive high ranking. In order to avoid this problem, Sueyoshi [5] introduced specific bounds on the weights in a super-efficient ranking model. The other problem lies with an infeasibility issue, Thrall [6] used the model to identify extreme efficient DMUs and noted that the super-efficiency CCR model may be infeasible. Zhu [7], Dula and Hickman [8] and Seiford and Zhu [9] prove that under some conditions various super-efficient DEA models are infeasible. Mehrabian et al. [10] suggested a modification to the dual formulation in order to ensure the feasibility.

Cook et al. [11] developed prioritization models to rank only the efficient units in DEA. They divide those with equal scores, on the boundary, by imposing the restrictions on the multipliers (weights) in a DEA analysis. Torgersen et al. [12] achieved a complete ranking of efficient DMUs by measuring their importance as a benchmark for inefficient DMUs. Bardhan et al. [13] ranked inefficient DMUs using Measure Inefficiency Dominance (MID) which is based on slack-adjusted DEA models. The measure ranks the inefficient DMUs according to their average proportional inefficiency in all inputs and outputs. Cooper and Tone [2] ranked inefficient units according to scalar Measures Inefficiency Proportion (MIP) in DEA, based on the slack variables. Doyle and Green [14], in their research into the ranking of overall DMUs, developed a ranked scale method utilizing the cross-efficiency matrix, by ranking the average efficiency ratios of each unit.

There is other follow-up research on increasing the DEA's discrimination power, especially the use of weight restrictions – *absolute* weights restrictions and *virtual* weights restrictions – to reduce the number of efficient DMUs. Absolute weights restrictions were first proposed by Thompson et al. [15], imposing acceptable bounds on ratios of weights in DEA that is known as the assurance region method. Dyson and Thanassoulis [16] proposed that meaningful bounds are directly imposed on individual weights. The other famous method, the cone ratio method, proposed and discussed by Charnes et al. [17, 18], is more general than the assurance method. The disadvantage of using absolute weights restrictions is that the bounds setting is dependent on the units of the indices and the orders of magnitude in the indices values. It is not easy for a human to express intuitively their preference for weights restrictions. In order to make it easier for a human to quantify value judgments in terms of percentage values, virtual weights restrictions was first proposed by Wong and Basely [19], setting the lower and/or upper bounds into the ratio of virtual variables. Sarrico and Dyson [20] further brought the concept of assurance regions into virtual weights restrictions. They showed that the use of the assurance region of virtual weights restrictions is more general and preferable to the use of proportional virtual weights restrictions. Bernroider and Stix [21] proposed discussion about the interaction between bound setting in the assurance region method and the validity of ranking outcomes in the assessment of an information system. However, because of the infeasibility problem occurring in the incorporation of lots of weights restrictions, Estellita Lins et al. [22] proposed the existence theorem, which establishes feasibility conditions for DEA with multiple weights restrictions. Review of other several ranking methods was proposed by Alder et al. [23]

2.3 Common weights used in ranking of DEA units

Cook et al. [24] and Roll et al. [25] first introduced the idea of common weights in DEA in the context of applying DEA to evaluate highway maintenance units. Cook and Kress [26, 27] gave a subjective ordinal preference ranking by developing common weights through a series of bounded DEA runs, by closing the gap between the upper and lower limits of the weights. Roll and Golany [28] considered the common weights for all the units, by maximizing the sum of efficiency ratios of all the units, in order to rank each unit. They suggest the potential use of the common weights for ranking DMUs. Sinuany–Stern et al. [29] used linear discriminant analysis in order to find a score function, which ranks DMUs, given the DEA division into efficient and inefficient sets. Friedman and Sinuany–Stern [30] use the CCA method by defining a scaling ratio score as a ratio of linear combinations of inputs and outputs. Then they utilize the common weights for the linear combinations that drawn from the largest eigenvalue of the CCA method. Sinuany–Stern and Friedman [31] developed DR/DEA to provide for given inputs and outputs the best common weights in order to rank all the units on the same scale. Kao and Hung [32] proposed the compromise solution approach to solve the DEA ranking problem with common weights.

The proposed model in this research determines the common weight by maximizing the organization efficiency. It is similar to the analysis by Roll and Golany [28], that proposed maximizing the mean of efficiency ratios of all the units to determine the common weights, as depicted in model (2-9). The major difference between the two is that, to the objective function (2-9.0) of the optimization model, our research used the original data of performance indices, while Roll and Golany [28] used the statistic, mean of efficiency ratio, as the objective function. In fact, in order to describe the organization’s performance, it is not suitable to describe the status of the organization’s operation by the mean of individual efficiency ratio, especially while the organization is able to possess the concrete operation data in performance indices. Hence, we use the accumulated data across all units in each performance index to replace the mean of efficiency ratio in the objective function. In addition, in the DEA structure, each unit in turn chooses the favorable weight to obtain the efficiency score. Following the structure, the organization plays the role of the aggregated unit to choose the favorable weight in combining with the accumulated data of each performance index, for the purpose of obtaining the optimal organization efficiency score. One restriction is noted that the performance indices used in our research should possess the property of accumulation. So the statistics, for instance the mean, is non-meaningful when accumulated across all the units. Therefore, we should avoid the occurrence of statistics in the performance indices.

$$\varphi^* = \max \quad \frac{1}{n}\sum_{j=1}^{n}\varphi_j \tag{2-9.0}$$

$$\text{s.t.} \quad \varphi_j = \frac{\sum_{r=1}^{s} y_{rj}U_r}{\sum_{i=1}^{m} x_{ij}V_i} \le 1, \qquad j = 1,\ldots,n, \tag{2-9.1}$$

$$U_r \ge \varepsilon > 0, \qquad r = 1,\ldots,s, \tag{2-9.2}$$

$$V_i \ge \varepsilon > 0, \qquad i = 1,\ldots,m. \tag{2-9.3}$$

Kao and Hung [32] proposed the compromise solution approach with the following model (2-10). They regarded the DEA efficiency score as the ideal individual benchmark, and minimized the distance between the DEA efficiency score and the score obtained from the common set of weights. The model (2-10) is equivalent to (2-9) with p = 1 and it is shown that the efficiency score obtained by the common set of weights with p = 2 is unique.

$$D_p^* = \min \quad \left[\sum_{j=1}^{n}\left(\theta_j^* - \varphi_j\right)^p\right]^{1/p} \tag{2-10.0}$$

$$\text{s.t.} \quad \varphi_j = \frac{\sum_{r=1}^{s} y_{rj}U_r}{\sum_{i=1}^{m} x_{ij}V_i} \le 1, \qquad j = 1,\ldots,n, \tag{2-10.1}$$

$$U_r \ge \varepsilon > 0, \qquad r = 1,\ldots,s, \tag{2-10.2}$$

$$V_i \ge \varepsilon > 0, \qquad i = 1,\ldots,m. \tag{2-10.3}$$

We give the complete description of our methodologies with several models in the following sections. Simultaneously, the detailed comparison between the sample model (2-9), (2-10) and our models will be introduced in the numerical example 1 in section 3.3. In fact, they show the different results in ranking.

3. Common Weight Analysis (CWA) to Rank Organization Units

In this section, we introduce the CWA methodology from the beginning of performance measurement philosophy and the development including the model transformation and the discussion of alternative optimal solution. Some numerical examples are listed in the final subsections.

3.1 Performance measurement philosophy

In conventional DEA models, each DMU in turn maximizes its efficiency score, under the constraint that none of DMUs' efficiency scores is allowed to exceed 1.0. The general manager always intuitively takes the maximal efficiency score 1.0 as the common *benchmark level* for DMUs. In fact, it also should be the benchmark of organization. In the scenario of organization benchmark chasing, we will take advantage of this benchmark level to help us describe concretely the concept about the generation of common weights here. We introduce one procedure to obtain one common set of weights for ranking the units. First, we focus on the case that all units on the DEA frontier together determine the single most favorable common set of weights in view of maximizing the group's efficiency score. Then, it is expanded to all units including the inefficient ones.

3.2 CWA model

CWA model is generated from the perspective of gaps minimization between the virtual benchmark and real units. By a series of transformation, CWA model also implies the performance measurement philosophy that the organization determines the favorable weights to obtain the maximum organization efficiency in DEA structure.

3.2.1 Development

In Figure 2, the vertical and horizontal axes are set to be the virtual output (weighted sum of s outputs) and virtual input (weighted sum of m inputs), respectively. By the definition of the efficiency score, the common benchmark level is one straight line, with slope 1.0, that passes through the origin in the coordinate. U_r and V_i in the weighted sum denote the decision variables of the common weights for the r-th output and i-th input index, respectively. The notation of a decision variable with superscript symbols " ′ " represents an arbitrary assigned value. For any two UOAs, UOA$_M$ and UOA$_N$, if given one set of weights U'_r and V'_i, then the coordinate of points M' and N' in Figure 2 are $(\sum_{i=1}^{m} x_{iM} V'_i, \sum_{r=1}^{s} y_{rM} U'_r)$ and $(\sum_{i=1}^{m} x_{iN} V'_i, \sum_{r=1}^{s} y_{rN} U'_r)$. The virtual gap between points M' and M'^P is Δ'_M. Similarly, for points N' and N'^P, the virtual gap is Δ'_N. Let the notation of a decision variable with superscript " * " represents the optimal value of the variable. We want to determine an optimal set of weights U^*_r and V^*_i, such that both points M^* and N^* below the benchmark line could approach their projection points, M^{*P} and N^{*P} on the benchmark line, as close as possible. In other words, by adopting the optimal weights, the total virtual gap $\Delta^*_M + \Delta^*_N$ is the shortest from the location of both UOAs to the benchmark line.

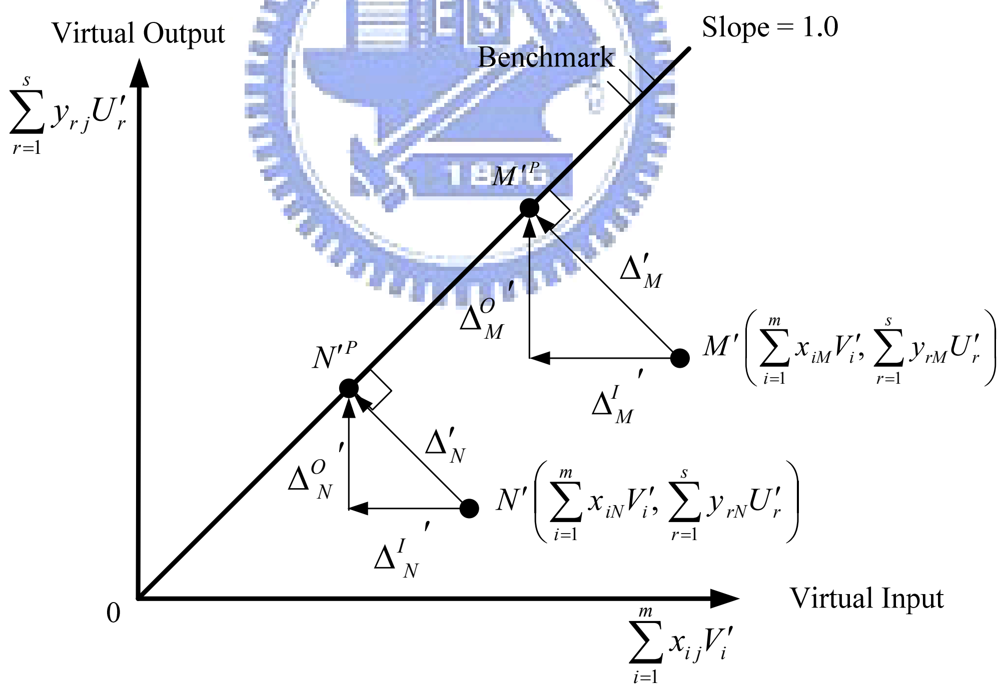

Figure 2. Gap analysis for UOAs below the virtual benchmark line

The following numerical example simulates the above scenario. Table 1 depicts the values of UOA$_A$, UOA$_B$, UOA$_C$, and UOA$_D$ on two input and two output indices. Given an arbitrary set of weights with $U'_r = (U'_1, U'_2) = (1, 2)$ and $V'_i = (V'_1, V'_2) = (25, 1)$, the weighted sum of inputs, weighted sum of outputs, and virtual gap Δ'_j for every UOA are recorded. As Figure 3 depicted, while points A', B', C' and D' are weighted by $U'_r = (U'_1, U'_2) = (1, 2)$ and $V'_i = (V'_1, V'_2) = (25, 1)$, we can obtain the perpendicular virtual gap to their projection points A'^P, B'^P, C'^P and D'^P on the benchmark line. There is a total virtual gap of 106.06 from the four UOAs to the benchmark line. Our methodology, presented in the following subsection, generates one optimal set of weights $U^* = (U_1^*, U_2^*) = (1, 3.33)$ and $V^* = (V_1^*, V_2^*) = (20.33, 1)$ with only the total virtual gap 8.07. As Figure 4 depicted, while the points A^*, B^*, C^* and D^* are weighted by the optimal common set of weights U^* and V^*, we can obtain the minimum perpendicular virtual gap to their projection points A^{*P}, B^{*P}, C^{*P} and D^{*P} on the benchmark line. Obviously, the set of weights is favorable to these UOAs since they are most close to the benchmark line.

Table 1. Simple example to simulate CWA scenario

Index					Assign arbitrary weight $(V'_1, V'_2) = (25, 1)$ $(U'_1, U'_2) = (1, 2)$			Assign optimal weight $(V_1^*, V_2^*) = (20.33, 1)$ $(U_1^*, U_2^*) = (1, 3.33)$		
UOA$_j$	x_1	x_2	y_1	y_2	$25x_1 + x_2$	$y_1 + 2y_2$	Δ'_j	$20.33x_1 + x_2$	$y_1 + 3.33y_2$	Δ_j^*
A	3	5	6	18	80	42	26.87	65.90	65.90	0
B	4	3	5	22	103	49	38.18	84.32	78.26	4.29
C	2	6	14	9	56	32	16.97	46.66	43.97	3.78
D	3	2	13	15	77	43	24.04	62.90	62.90	0
Sum							106.06			8.07

(3-1) expresses the formulation to the original model of our methodology. The objective function (3-1.0) is to minimize the sum of the total virtual gaps of UOAs, in set *E,* to benchmark line. As for the set of constraints (3-1.1), the numerator is the weighted sum of outputs plus the virtual gap Δ_j^O and the denominator is the weighted sum of inputs minus the virtual gap Δ_j^I. They imply that the direction approach to the benchmark line is upwards and leftwards at same time. The ratio of the numerator to the denominator equals to 1.0, which means that the projection point is on the benchmark line. ε_r^O and ε_i^I are the positive Archimedean infinitesimal constant of output *r* and input *i*, respectively. We also avoid a case of zero value of indices obtained by choosing the set of zero weights. In our methodology, we assume the benchmark line is located above all UOAs in set E. The optimal common set of weights U_r^* (r = 1, 2, …, s) and V_i^* (i = 1, 2, …, m) to each efficient UOA would be solved and then each efficient UOA could obtain one absolute efficiency score as the standard for comparison.

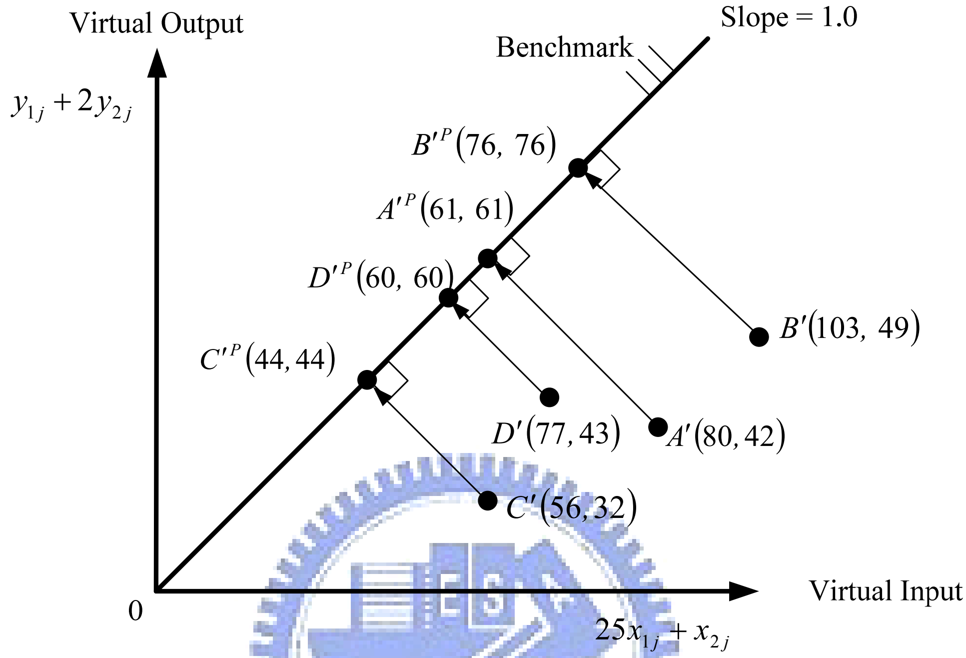

Figure 3. Coordinates of UOAs weighted by arbitrary common set of weights

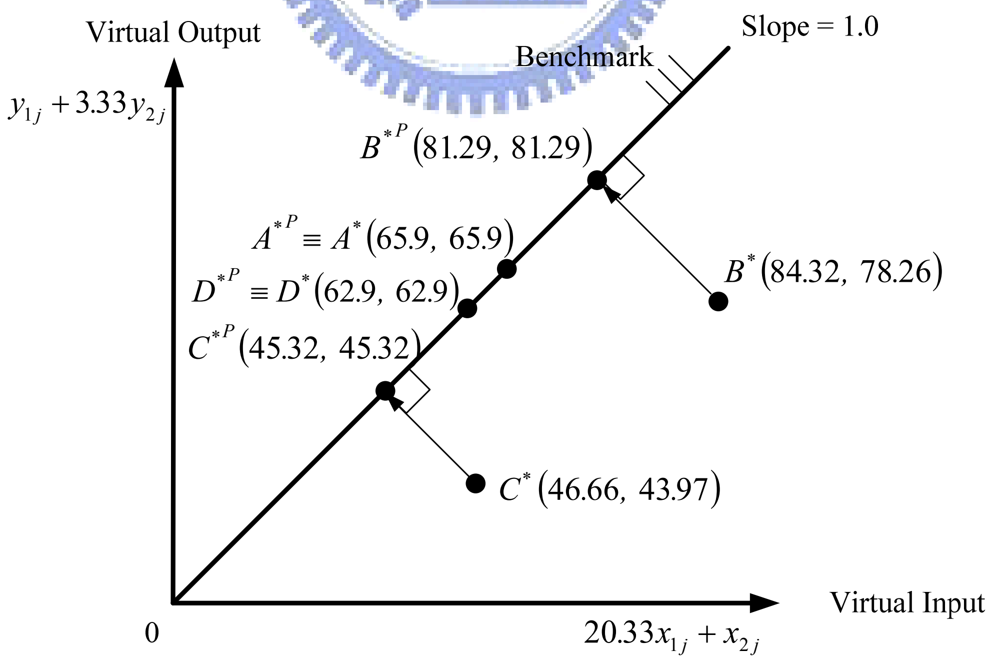

Figure 4. Coordinates of UOAs weighted by optimal common set of weights

CWA-FP

$$\Delta^* = \min \sum_{j \in E} \Delta_j. \tag{3-1.0}$$

$$s.t. \quad \frac{\sum_{r=1}^{s} y_{rj} U_r + \Delta_j^O}{\sum_{i=1}^{m} x_{ij} V_i - \Delta_j^I} = 1, \quad j \in E, \tag{3-1.1}$$

$$\Delta_j^O, \Delta_j^I, \Delta_j \geq 0, \qquad j \in E, \tag{3-1.2}$$

$$U_r \geq \varepsilon_r^O > 0, \qquad r = 1, \dots, s, \tag{3-1.3}$$

$$V_i \geq \varepsilon_i^I > 0, \qquad i = 1, \dots, m. \tag{3-1.4}$$

The ratio form of constraints in (3-1.1) can be rewritten in a linear form, formulated in the constraints (3-2.1). Hence, (3-1) can be transformed into (3-2).

CWA-LP 1

$$\Delta^* = \min \sum_{j \in E} \Delta_j. \tag{3-2.0}$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i + \left(\Delta_j^O + \Delta_j^I\right) = 0, \quad j \in E, \tag{3-2.1}$$

$$\Delta_j^O, \Delta_j^I, \Delta_j \geq 0, \quad j \in E, \tag{3-2.2}$$

$$U_r \geq \varepsilon_r^O > 0, \qquad r = 1, \dots, s, \tag{3-2.3}$$

$$V_i \geq \varepsilon_i^I > 0, \qquad i = 1, \dots, m. \tag{3-2.4}$$

Since the triangular in Figure 2 is one isosceles right triangular, the relationship between the shortest virtual gaps Δ_j, Δ_j^O, and Δ_j^I can be expressed as $\Delta_j = \sqrt{2}\,\Delta_j^O = \sqrt{2}\,\Delta_j^I$. Then, (3-2) is then simplified to the following linear programming (3-3).

CWA-LP 2

$$\Delta^* = \min \sum_{j \in E} \Delta_j \tag{3-3.0}$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i + \sqrt{2}\Delta_j = 0, \quad j \in E, \tag{3-3.1}$$

$$U_r \geq \varepsilon_r^O > 0, \qquad r = 1, \dots, s, \tag{3-3.2}$$

$$V_i \geq \varepsilon_i^I > 0, \qquad i = 1, \dots, m, \tag{3-3.3}$$

$$\Delta_j \geq 0, \qquad j \in E. \tag{3-3.4}$$

(3-3) could be rewritten to the equivalent linear programming (3-4) by taking out the slack variable Δ_j and then aggregating y_{rj} and x_{ij} to be $Y_r = \sum_{j \in E} y_{rj}$ and $X_i = \sum_{j \in E} x_{ij}$, respectively. (3-4) shows that the organization manager determines the common weight by maximizing the organization efficiency in the perspective of aggregated UOA with the status of organization in all performance indices.

CWA-LP 3

$$-\Delta^* = \max \ \sum_{r=1}^{s} Y_r U_r - \sum_{i=1}^{m} X_i V_i \tag{3-4.0}$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i \le 0, \quad j \in E, \tag{3-4.1}$$

$$U_r \ge \varepsilon_r^O > 0, \qquad r = 1, \ldots, s, \tag{3-4.2}$$

$$V_i \ge \varepsilon_i^I > 0, \qquad i = 1, \ldots, m. \tag{3-4.3}$$

In fact, there exists another implicit constraint $\sum_{r=1}^{s} Y_r U_r - \sum_{i=1}^{m} X_i V_i \le 0$ in (3-4). This constraint is redundant since it is a linear combination of the first set of constraints (3-4.1). We regard X_i (i = 1, 2, …, m) and Y_r (r = 1, 2, …, s) as the input and output indices of one aggregated UOA or group. The goal of (3-4.0) is to maximize the efficiency of the aggregated UOA, under the constraints that the efficiency score of each UOA in set E cannot exceed the benchmark level. While the optimal efficiency of the aggregated UOA occurs, one corresponding set of weights is also determined, to be assigned to every UOA in set E. The ranking score that adopts the common set of weights generated from (3-4) makes sense because the organization manager objectively chooses the common weights for the purpose of maximizing group efficiency.

For instance, the general manager of a bank desires to measure the performance of all branches of the bank. A branch would have a higher performance if the required resources could be reduced and the outputs could be increased. The possible resources could be employees, the number of bank service counters, etc., while the outputs could be multiple business items existing in the bank such as deposit business, loan business, credit card business… etc. The general manager desires to have a set of weights for these resources and output indices. However, each branch manager may focus on a different business base, a different strategy, or the limited resources. Therefore, it is difficult for the general manager to set the weight of each business item subjectively for the discrimination requirement of branches. The general manager could take advantage of DEA to distinguish the efficient branches from the inefficient ones. While the detailed ranking of efficient branches is necessary, the general manager could determine one common set of weights for the purpose of maximizing the overall efficient branches' efficiency (group efficiency) under the constraints that every efficient branch's highest efficiency score cannot exceed 1.0. Because of only considering the group of efficient branches, the general manager can take those efficient branches as a *virtual* bank. In other words, the general manager can determine one common set of weights for efficient branches, with the purpose of maximizing the virtual bank's efficiency.

In comparison with additive model (2-7), the difference is that (3-4) chooses the aggregated UOA to be the player of determining the favorable weights. In order to obtain more information, we transform (3-4) to its dual form (3-5). The dual variables associate to (3-4.1), (3-4.2), and (3-4.3) are π_j, P_r, and Q_i, respectively.

CWA-DLP 1

$$\max \left(\sum_{r=1}^{s} \varepsilon_r^o P_r + \sum_{i=1}^{m} \varepsilon_i^I Q_i \right) \qquad (3\text{-}5.0)$$

$$s.t. \quad \sum_{j \in E} y_{rj}\pi_j - P_r = Y_r, \qquad r = 1,\cdots,s, \qquad (3\text{-}5.1)$$

$$\sum_{j \in E} x_{ij}\pi_j + Q_i = X_i, \qquad i = 1,\cdots,m, \qquad (3\text{-}5.2)$$

$$\pi_j \ge 0, \qquad j \in E, \qquad (3\text{-}5.3)$$

$$P_r \ge 0, \qquad r = 1,\cdots,s, \qquad (3\text{-}5.4)$$

$$Q_i \ge 0, \qquad i = 1,\cdots,m. \qquad (3\text{-}5.5)$$

Similarly, (3-5) can be used to compare with the model (2-8) or Phase II extension of a traditional CCR model (3-6) while the parameter θ_o^* is equal to 1.0. The major difference is that P_r and Q_i in (3-5) are respectively the total *shortfalls* and *excesses* of all efficient UOAs relative to the benchmark line, corresponding to the output index r and input index i.

Phase II extension of CCR model

$$\max \quad \varepsilon\left(\sum_{r=1}^{s} s_r^{+} + \sum_{i=1}^{m} s_i^{-}\right) \tag{3-6.0}$$

$$s.t. \quad \sum_{j\in E} y_{rj}\lambda_j - s_r^{+} = y_{ro}, \qquad r = 1,\ldots,s, \tag{3-6.1}$$

$$\sum_{j\in E} x_{ij}\lambda_j + s_i^{-} = x_{io}, \qquad i = 1,\ldots,m, \tag{3-6.2}$$

$$\lambda_j \ge 0, \qquad j \in E, \tag{3-6.3}$$

$$s_r^{+} \ge 0, \qquad r = 1,\ldots,s, \tag{3-6.4}$$

$$s_i^{-} \ge 0, \qquad i = 1,\ldots,m. \tag{3-6.5}$$

The variable value π_j^* in (3-5) is the *shadow price* of UOA_j belongs to set E in the linear programming (3-4). Then, the variations of criterion Eq. (3-7) will result in the variation of constraint Eq. (3-8). That is, if the right-hand side of the j-th constraint increases 1 unit, then the criterion Eq. (3-4.0) will get the variation π_j^* as Eq. (3-8).

$$\sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i \le 0+1 \tag{3-7}$$

$$-\Delta^* = \left(\sum_{r=1}^{s}\left(\sum_{j\in E} y_{rj}\right)U_r - \sum_{i=1}^{m}\left(\sum_{j\in E} x_{ij}\right)V_i\right) + \pi_j^*(0+1) \tag{3-8}$$

π_j^* represents the total virtual gap scale that can be reduced while we release the upper bound of efficiency 1.0 for UOA_j. If there are multiple UOAs on the benchmark line, π_j^* will give valuable information to indicate which one most influences the total virtual gap. It is useful for determining the priority of UOAs on the benchmark line. In the following subsections, we give the following ranking rules for those efficient UOAs.

3.2.2 CWA-efficient and CWA ranking rules

In this section, we will introduce the definition of the CWA-efficient and CWA ranking rules. First, the CWA-efficiency score of UOA$_j$ is defined as Eq. (3-9).

$$\zeta_j^* = \frac{\sum_{r=1}^{s} y_{rj} U_r^*}{\sum_{i=1}^{m} x_{ij} V_i^*}, \quad j \in E \tag{3-9}$$

By the value of CWA-efficiency, we can distinguish the UOAs into two separable classes, UOAs on the benchmark and those below the benchmark.

Definition 1

UOA$_j$ is CWA-efficient (on the benchmark) if $\Delta_j^* = 0$ or $\zeta_j^* = 1.0$. Otherwise, UOA$_j$ is CWA-inefficient (below the benchmark).

The following rules can distinguish the UOAs whether they are on or below the benchmark line.

Rule 1

The performance of UOA$_j$ is better than UOA$_i$ if $\zeta_j^* > \zeta_i^*$.

Rule 2

If $\zeta_j^* = \zeta_i^* = 1$, i.e. they are both CWA-efficient (on benchmark line), then the performance of UOA$_j$ is better than UOA$_i$ if $\pi_j^* > \pi_i^*$.

Each UOA's CWA-efficiency score is limited to no greater than 1.0, so there is no UOA standing above the benchmark line. Furthermore, we can even ensure that there is at least one UOA that joins the assessment located on the benchmark line.

Theorem 1

There is at least one UOA under the assessment located on the benchmark line.

Proof. We will use the proof of contradiction to explain the existence of above theorem. Assume that there is no UOA on benchmark, so we can obtain the optimal criterion and the corresponding optimal value U_r^*, V_i^* and Δ_j^* where $\Delta_j^* > 0$ in (3-3) for all $j \in E$ (formulated in

Eq. (3-10)). That is, each UOA's efficiency is less than 1 (formulated in Eq. (3-11)).

$$\frac{\sum_{r=1}^{s} y_{rj} U_r^* + \sqrt{2}\Delta_j^*}{\sum_{i=1}^{m} x_{ij} V_i^*} = 1, \qquad j \in E \tag{3-10}$$

$$\frac{\sum_{r=1}^{s} y_{rj} U_r^*}{\sum_{i=1}^{m} x_{ij} V_i^*} < 1, \qquad j \in E \tag{3-11}$$

$$\frac{\sum_{r=1}^{s} k_j \left(y_{rj} U_r^*\right)}{\sum_{i=1}^{m} x_{ij} V_i^*} = 1, \qquad j \in E \tag{3-12}$$

We can set the constant k_j ($k_j > 1$) such that the efficiency is equal to 1 for every $j \in E$ (formulated in Eq. (3-12)). Let K be the minimum of set $\{k_j$, for $j \in E\}$, then we can obtain another feasible common set of weights KU_r^* and V_i^* accompanies the smaller Δ_j^* (at least one equals to 0) for all $j \in E$ in (3-3). The case will result in smaller criterion and contradicts the fact that the current criterion has been minimized. Hence, there is at least one UOA locates on benchmark line. ■

3.2.3 Virtual gap analysis

The virtual gaps between virtual input and output indices for each CWA-inefficient UOA could be further decomposed into the real gap of each performance index. We can further analyze this by translating the model (3-4) to the equivalent model (3-13). As model (3-13) showed, P_r and Q_i can be partitioned as $P_r = \sum_{j \in E} p_{rj}$ and $Q_i = \sum_{j \in E} q_{ij}$. And p_{rj} and q_{ij} are the shortfall at the output index r and excess at input index i of UOA$_j$ to the benchmark, respectively. It means that $p_{rj} = P_r\lambda_j$ and $q_{ij} = Q_i\lambda_j$ with convex combinations of multipliers $\lambda_j \geq 0$ and $\sum_{j \in E} \lambda_j$ =1.

CWA-DLP 2

$$\max \quad \sum_{j\in E}\left(\sum_{r=1}^{s}\varepsilon_r^O p_{rj}+\sum_{i=1}^{m}\varepsilon_i^I q_{ij}\right) \tag{3-13.0}$$

$$s.t. \quad \sum_{j\in E} y_{rj}\pi_j=\sum_{j\in E}\left(y_{rj}+p_{rj}\right), \quad r=1,\ldots,s, \tag{3-13.1}$$

$$\sum_{j\in E} x_{ij}\pi_j=\sum_{j\in E}\left(x_{ij}-q_{ij}\right), \quad i=1,\ldots,m, \tag{3-13.2}$$

$$\pi_j \geq 0, \quad j\in E, \tag{3-13.3}$$

$$p_{rj}\geq 0, \quad r=1,\ldots,s, \quad j\in E, \tag{3-13.4}$$

$$q_{ij}\geq 0, \quad i=1,\ldots,m, \quad j\in E. \tag{3-13.5}$$

The shortfall p_{rj}^* and excess q_{ij}^* of (3-13) could be obtained by the following theorem.

Theorem 2

The shortfall p_{rj}^* and excess q_{ij}^* of CWA-inefficient UOA$_j$ to benchmark corresponding to the output index r and input index i are P_r^* (Δ_j^*/Δ^*) and Q_i^* (Δ_j^*/Δ^*).

Proof. Since p_{rj}^* and q_{ij}^* are shortfall and excess of CWA-inefficient UOA$_j$ to the benchmark, we have Eq. (3-14) holds because of Definition 1.

$$\frac{\sum_{r=1}^{s}\left(y_{rj}+p_{rj}^*\right)U_r^*}{\sum_{i=1}^{m}\left(x_{ij}-q_{ij}^*\right)V_i^*}=1 \tag{3-14}$$

$$\frac{\sum_{r=1}^{s}\left(y_{rj}+\frac{P_r^*\Delta_j^*}{\Delta^*}\right)U_r^*}{\sum_{i=1}^{m}\left(x_{ij}-\frac{Q_i^*\Delta_j^*}{\Delta^*}\right)V_i^*}=1 \tag{3-15}$$

To prove Eq. (3-15) is a truth, we first decompose the numerator and denominator to obtain Eq. (3-16) and Eq. (3-17), respectively.

$$\sum_{r=1}^{s} y_{rj}U_r^*+\sum_{r=1}^{s}\frac{P_r^*\Delta_j^*}{\Delta^*}U_r^* \tag{3-16}$$

$$\sum_{i=1}^{m} x_{ij}V_i^*-\sum_{i=1}^{m}\frac{Q_i^*\Delta_j^*}{\Delta^*}V_i^* \tag{3-17}$$

Subtract Eq. (3-17) from Eq. (3-16) resulted Eq. (3-18).

$$\sum_{r=1}^{s} y_{rj}U_r^* - \sum_{i=1}^{m} x_{ij}V_i^* + \frac{\Delta_j^*}{\Delta^*}\left(\sum_{r=1}^{s} P_r^*U_r^* + \sum_{i=1}^{m} Q_i^*V_i^*\right) \tag{3-18}$$

Since the lower bound of U_r^* and V_i^* is ε, according to *Complementary Slackness Theorem*, the following relationship holds.

$$\sum_{r=1}^{s} P_r^*U_r^* + \sum_{i=1}^{m} Q_i^*V_i^* = \left(\sum_{r=1}^{s} \varepsilon_r^O P_r^* + \sum_{i=1}^{m} \varepsilon_i^I Q_i^*\right) = \sqrt{2}\Delta^* \tag{3-19}$$

Therefore, the formula inside the parenthesis in Eq. (3-18) could be substituted by the right-hand-side in Eq. (3-19). Obviously, Eq. (3-18) easily translates to Eq. (3-20).

$$\sum_{r=1}^{s} y_{rj}U_r^* - \sum_{i=1}^{m} x_{ij}V_i^* + \sqrt{2}\Delta_j^* \tag{3-20}$$

Eq. (3-20) is equal to zero by the fulfillment of constraints in (3-3.1). Hence, Eq. (3-15) comes into existence and the theorem is proved. ■

3.2.4. Selection of the alternative optimal common sets of weights

It is worth noting that (3-3) sometimes encounters the existence of alternative weights; moreover, different weights can result in different rankings of efficient UOAs. It is necessary for the general manager to select the applicable one from these efficient UOAs. We propose one approach to assist general managers in dealing with the issue of alternative rankings. While the same weighted sum exists, for the case of one set of output indices combining with different sets of weights, Obata and Ishii [33] propose that it is preferable for output indices to adopt the smaller scale of weight. The choice implies that the current superiority of the weighted sum is originated from the indices value itself, rather than from the weights. Similarly, it is preferable to use the larger scale of weights for input indices in obtaining the weighted sum of inputs. For instance, let $\vec{a} = [\vec{y}, \vec{x}]$ and $\vec{b} = [\vec{U}, -\vec{V}]$, the inner product of $\vec{a}$ and $\vec{b}$ (weighted sum) is equal to $-\sqrt{2}\Delta$, as depicted in (3-32.1). While there exist multiple solutions in $\vec{b}$, we expect the current level of the inner product is generated from the worse or smaller $\vec{b}$ to show the superiority of dataset $\vec{a}$. Therefore, we minimize the scale of $\vec{b}$ by L_1-norm, i.e. the objective function (3-21.0). The following procedure is suggested as a way to search the optimal one of the existing alternative set of weights, using the

L_1-norm.

Stage 1

Solve (3-3) and obtain the optimal value Δ^*.

Stage 2

Solve the following linear programming (3-21) to obtain one optimal common set of weights.

In stage 1, we first have to look for the minimization of the total virtual gap. Then select one appropriate weight in stage 2, under the optimal status of (3-3). Thus, we keep the optimal criteria value in (3-3.0) as one constraint (3-21.2) in the linear programming (3-21) and then take the minimization of the sum of output weights and maximization of the sum of input weights as the criterion.

Optimal weight analysis

$$\min \quad \sum_{r=1}^{s} U_r - \sum_{i=1}^{m} V_i \qquad (3\text{-}21.0)$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i + \sqrt{2}\,\Delta_j = 0, \quad j \in E, \qquad (3\text{-}21.1)$$

$$\sum_{j \in E} \Delta_j = \Delta^* \qquad (3\text{-}21.2)$$

$$U_r \geq \varepsilon_r^o > 0, \qquad r = 1,\ldots,s, \qquad (3\text{-}21.3)$$

$$V_i \geq \varepsilon_i^I > 0, \qquad i = 1,\ldots,m, \qquad (3\text{-}21.4)$$

$$\Delta_j \geq 0, \qquad j \in E. \qquad (3\text{-}21.5)$$

3.3 Numerical example 1

DEA models usually have no more than $n/2$ indices when assessing n UOAs. Otherwise, the number of efficient UOAs becomes unreasonably large. It means that the discriminating power of DEA is reduced. The example uses seven UOAs, with three inputs and three outputs. The last column in Table 2 shows that the seven UOAs are efficient by radial efficiency 1.0 obtained by CCR-Input-oriented model.

Table 2. Example of data with the number of indices is much larger than UOAs

UOA	x_{1j}	x_{2j}	x_{3j}	y_{1j}	y_{2j}	y_{3j}	Efficiency (θ_j^*)
D1	1621	436	205	174	497	22	1
D2	2718	314	221	172	497	22	1
D3	1523	345	215	160	443	22	1
D4	5514	1314	553	487	1925	63	1
D5	1941	507	309	220	521	36	1
D6	1496	321	339	109	699	38	1
D7	932	158	200	37	431	19	1

Table 3 gives the detailed ranking information assessed by adopting CWA. We still find that there are five UOAs still on benchmark line. If we release the upper bound of the efficiency score 1.0 for these UOAs, then π_j^* leads to a reduction in scale in the total virtual gap towards the benchmark line. Obviously, a CWA-efficient UOA with a larger π_j^* is the better one. The total virtual gap can be reduced to a maximum 3.225, compared to the other UOAs on the benchmark line, while we release the upper bound of efficiency score to over 1.0. Therefore, after comparing with π_j^*, we are able to determine the final ranking of CWA-efficient UOAs to be UOA_{D3}, UOA_{D6}, UOA_{D1}, UOA_{D4}, UOA_{D5}, UOA_{D7}, and UOA_{D2}.

Table 3. Corresponding outcomes of Table 2 assessed by CWA

UOA	Δ_j^*	π_j^*	ζ_j^*	Ranking
D3	0	3.225	1.000	1
D6	0	1.772	1.000	2
D1	0	1.118	1.000	3
D4	0	0.922	1.000	4
D5	0	0.028	1.000	5
D7	304.864	0.000	0.847	6
D2	925.362	0.000	0.778	7

Besides, we compare the proposed model CWA with the optimization model (2-9) proposed by Roll and Golany [28] and optimization model (2-10) proposed by Kao and Hung [32] by this dataset. Table 4 and Table 5 show the efficiency score and corresponding ranking. Obviously, we observe that the ranking in Table 3 is different to the ranking in Table 4 and Table 5. Especially the UOA_{D2}, in the perspective of the organization manager, it is the worst one in Table 3. In DEA structure, the common set of weights determined by the organization manager, the aggregated UOA, expresses that the UOA$_{D2}$ is the most inefficient. However, it appears the high ranking in Table 4 and Table 5. On the contrary, UOA$_{D1}$ shows that it is the benchmark in our model while they show the poor ranking in the Table 4 and Table 5. In fact, UOA$_{D1}$ is an efficient unit in DEA from the perspective of organization. In order to accurately describe the organization performance, it is better to represent the status of organization's operation by the concrete operation data in performance indices. The mean of individual efficiency ratio is the indirect information for one organization especially while the value of performance indices across all units can be accumulated as the value of organization.

Table 4. Corresponding outcomes of Table 2 assessed by the model (2-9)

UOA	φ_j^*	Ranking
D2	1.000	1
D3	1.000	1
D4	1.000	1
D5	1.000	1
D6	1.000	1
D1	0.965	6
D7	0.871	7

Table 5. Corresponding outcomes of Table 2 assessed by the model (2-10) with $p = 2$

UOA	φ_j^*	Ranking
D2	1.000	1
D3	1.000	1
D4	1.000	1
D6	1.000	1
D1	0.948	5
D5	0.944	6
D7	0.911	7

3.4 Numerical example 2

In this subsection, we extend the ranking object from UOAs in set E to $E \cup E^C$ in CWA model (3-1) where E^C represents the set of inefficient UOAs. It is unfortunate that a paradoxical case perpaps exists, that some UOAs in E^C are better than UOAs in E by comparing the CWA efficiency score. However, the phenomenon is acceptable and explainable without violating the original concept of DEA. In fact, each UOA in E^C would have a particular reference set that is composed of parts of UOAs in E. One should not declare that one certain UOA in E is better than all UOAs in E^C.

We verify the inference mentioned above by practicing one complete example including UOAs in set $E \cup E^C$. As listed in Table 6, 7 UOAs in set E is extended to 11 UOAs in set $E \cup E^C$. Using DEA model (2-1) and CWA methodology, models (3-3), the results are depicted in Table 7. In view of CWA, we observe UOA_{D2} of set E is ranked 11, and is worse than UOA_{D8}, UOA_{D9}, UOA_{D10}, and UOA_{D11} of set E^C. Although UOA_{D2} belongs to set E, it is not an element of the reference set for UOA_{D8}, UOA_{D9}, UOA_{D10}, and UOA_{D11}. In other words, individual UOA_{D8}, UOA_{D9}, UOA_{D10}, and UOA_{D11} really are not dominated by UOA_{D2}. Therefore, in view of DEA, one should not declare that UOA_{D2} in set E is better than UOA_{D8}, UOA_{D9}, UOA_{D10}, and UOA_{D11} in set E^C. In addition, using DEA model (2-1) to measure the relative efficiency of only these 5 UOAs, at this time one would observe that they belong to the equivalent set E, just depicted in Table 8. Therefore, the CWA ranking seems also to be workable in set $E \cup E^C$ without violating the original concept of DEA. Therefore, CWA ranking reflects two consequences. The first is that it is primarily used in ranking the UOAs in set E. The second is that when it is used in ranking the UOAs in set $E \cup E^C$, one could still obtain a reasonable conclusion without conflicting with the DEA's initial classification.

Table 6. Example of including the UOAs in set $E \cup E^C$

UOA	x_{1j}	x_{2j}	x_{3j}	y_{1j}	y_{2j}	y_{3j}
D1	1621	436	205	174	497	22
D2	2718	314	221	172	497	22
D3	1523	345	215	160	443	22
D4	5514	1314	553	487	1925	63
D5	1941	507	309	220	521	36
D6	1496	321	339	109	699	38
D7	932	158	200	37	431	19
D8	2013	1037	412	198	471	32
D9	1891	976	399	191	491	22
D10	2277	891	418	241	379	28
D11	1995	693	349	167	412	31

Table 7. The reference set, DEA and CWA efficiency score of example in Table 6

UOA	Reference set	DEA Efficiency (θ_j^*)	CWA Efficiency (ζ_j^*)	Rank
D1	D1	1	1	1
D2	D2	1	0.69	11
D3	D3	1	0.99	4
D4	D4	1	0.97	5
D5	D5	1	1	3
D6	D6	1	1	2
D7	D7	1	0.82	6
D8	D1, D5	0.87	0.72	7
D9	D1, D5	0.91	0.73	8
D10	D5	0.93	0.74	9
D11	D5, D6	0.79	0.71	10

Table 8. The DEA efficiency score evaluated only to 5 debatable UOAs in Table 7

UOA	DEA Efficiency (θ_j^*)
D2	1
D8	1
D9	1
D10	1
D11	1

3.5 Applying virtual weights restrictions in CWA model

In order to enforce CWA model in the application of real case and satisfy all kinds of restrictions about the performance indices, we take advantage of the virtual weights restrictions to assist the manager in obtaining a preferable and robust ranking result for units. In order to obtain the preferable ranking, the manager's subjective preference is considered and formulated by the virtual weights restrictions while determining the common weights in the procedure. In addition, in order to obtain a robust ranking, we modify the boundary of the feasible region of virtual weights restrictions in each assessment. The final statistical ranking of all assessments provides the manager with one robust ranking, which is invariant in different feasible regions of virtual weights restrictions in the numerical example.

3.5.1 Review of virtual weights restrictions

Virtual weights restrictions means that the restrictions are imposed on virtual input/output, comprising the product of input/output level and weight for the input/output, rather than on weights directly. It is noted that virtual weights restrictions are developed with reference to the original absolute weights restrictions in DEA formulation. Different to the difficult ascertainment of meaningful bounds in absolute weights restrictions, virtual weights restrictions make it intuitive and easy for a manager to express their subjective preference in the assessment.

The proportional virtual weights restrictions and virtual assurance regions separately provide a different expression in the preference relationships among performance indices. The former represents the importance of one certain input/output attached to the input/output measure, and the latter further expresses the known relationship between any two indices, even among more indices. In this subsection, we give a brief review of virtual assurance regions and proportional virtual weights restrictions.

3.5.1.1 Virtual assurance regions

Sarrico and Dyson [20] proposed that all the virtual weights restrictions can be described by the general set of restrictions expressed by Eq. (3-22).

$$\sum_{i=1}^{m} \alpha_{iw}\, x_{ij} V_i + \sum_{r=1}^{s} \beta_{rw}\, y_{rj} U_r \geq k_w, \quad w = 1,\ldots,W, \quad j = 1,\ldots,n, \tag{3-22}$$

W denotes the number of virtual weights restrictions. α_{iw} denotes the preference of virtual input to restriction w in input i. β_{rw} denotes the preference of virtual output to restriction w in output r. k_w denotes the intercept of line restriction w. While we set $\alpha_{iw} = 0$ (for all i) or $\beta_{rw} = 0$ (for all r) with $k_w = 0$, Eq. (3-22) translates an ordering of preference in inputs and outputs, as expressed in Eq. (3-23) and Eq. (3-24).

$$\sum_{i=1}^{m} \alpha_{iw}\, x_{ij} V_i \geq 0, \quad w = 1,\ldots,W, \quad j = 1,\ldots,n, \tag{3-23}$$

$$\sum_{r=1}^{s} \beta_{rw}\, y_{rj} U_r \geq 0, \quad w = 1,\ldots,W, \quad j = 1,\ldots,n, \tag{3-24}$$

These kinds of restrictions mentioned above in Eq. (3-23) and Eq. (3-24) are useful while managers concentrate the preferences on the indices in the same measure.

Besides, if there is at least one $\alpha_{iw} \neq 0$ (for all i) and one $\beta_{rw} \neq 0$ (for all r) with $k_w = 0$, Eq. (3-22) can be translated as an ordering of preference in input-output, as expressed in Eq. (3-25).

$$\sum_{i=1}^{m} \alpha_{iw}\, x_{ij} V_i + \sum_{r=1}^{s} \beta_{rw}\, y_{rj} U_r \geq 0, \quad w = 1, \ldots, W, \quad j = 1, \ldots, n, \tag{3-25}$$

Eq. (3-25) is used to express a known relationship between a pair of inputs and outputs. For instance, to produce one unit of output, one needs to consume at least a certain level of an input.

3.5.1.2 The proportional virtual weights restrictions

Wong and Basely [19] proposed the use of virtual weights restrictions. In particular, the proportional virtual weights restrictions were intended to make it easier for managers to quantify value judgments in terms of contribution percentage in the same measure, that is, input measure or output measure. Conceptually the proportional virtual output *r* of DMU_j represents the importance attached to the output measure (a similar reasoning can be applied to the virtual input *i*). Let P_{rj}^{O} and P_{ij}^{I} respectively denote the proportional virtual output *r* and input *i* of DMU_j, as follows in Eq. (3-26) and Eq. (3-27). Thus, the manager can intuitively set limits on this proportion to reflect value judgments, as follows in Eq. (3-28) and Eq. (3-29).

$$P_{rj}^{O} = \frac{y_{rj} U_r}{\sum_{r=1}^{s} y_{rj} U_r}, \quad r = 1, \ldots, s, \quad j = 1, \ldots, n, \tag{3-26}$$

$$P_{ij}^{I} = \frac{x_{ij} V_i}{\sum_{i=1}^{m} x_{ij} V_i}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{3-27}$$

$$a_r \leq P_{rj}^{O} \leq b_r, \quad r = 1, \ldots, s, \quad j = 1, \ldots, n, \tag{3-28}$$

$$c_i \leq P_{ij}^{I} \leq d_i, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{3-29}$$

The constant values a_r, b_r, c_i, d_i are the subjective preference limits provided by the manager for output *r* and input *i*. Sarrico and Dyson [20] discussed the possible infeasibility of multiple proportional virtual weights restrictions resulting from the setting of lower and upper bounds, while there exists a large scale range in the index value across all units. They proposed one formulation to determine the feasible lower (upper) bound according to the given upper (lower) bound. Estellita Lins et al. [22] proposed one model to test the feasibility in DEA models with given weight restrictions, including the absolute and virtual weights restrictions, and further modified the bounds using their hyperplane adjusting model while infeasibility occurs.

However, there exists another trap to set the constant value of a_r, b_r, c_i, d_i while the manager has no idea about the implicit restrictions $\sum_{r=1}^{s} P_{rj}^{O} = 1$ and $\sum_{i=1}^{m} P_{ij}^{I} = 1$. For instance, there exit two output indices y_{1j} and y_{2j} in the output measure for all *DMU*$_j$. While the manager sets a_1= 0.2 and b_1= 0.4 with 20%$\leq P_{1j}^{O} \leq$40%, a_2= 0.2 and b_2= 0.4 with 20%$\leq P_{2j}^{O} \leq$40% for intuitive convenience, the setting obviously cannot satisfy the implicit restriction $P_{1j}^{O} + P_{2j}^{O} = 1$. In other words, there exists no such feasible P_{1j}^{O} and P_{2j}^{O} to satisfy these proportional virtual weights restrictions in the output measure. We introduce one method to set initial feasible bounds on the virtual weights to avoid the possible infeasibility in the virtual weights restrictions mentioned above.

3.5.2 The new setting of bounds in proportional virtual weights restrictions

In order to solve the potential infeasibility issue that occurs in the proportional virtual weights restrictions, we propose one systematic setting in the lower and upper bounds of the proportional virtual inputs and outputs to ensure the feasibility of proportional virtual weights restrictions. Besides solving the infeasibility problems, we use the systematic setting to analyze the relationship between ranking and proportional virtual weights restrictions.

Exploring the reasoning of infeasibility in proportional virtual weights restrictions, under the same measure, the sum of the upper bound to all outputs ($\sum_{r=1}^{s} b_r$) cannot reach 1.0 or the sum of the lower bound to all outputs ($\sum_{r=1}^{s} a_r$) exceeds 1.0. In order to avoid this problem, we rewrite Eq. (3-28) and Eq. (3-29) to Eq. (3-30) and Eq. (3-31) respectively by formulating the lower bound and upper bound of input and output with the function of parameters δ_r^-, δ_r^+, τ_i^- and τ_i^+. For the purposes of ensuring the proportion is between 0 and 1.0, we give the following range 0 $\leqq \delta_r^- \leqq$ 1, 0 $\leqq \delta_r^+ \leqq$ *s*-1, 0 $\leqq \tau_i^- \leqq$ 1 and 0 $\leqq \tau_i^+ \leqq$ *m*-1.

$$\frac{1}{s}\left(1-\delta_r^-\right)\le P_{rj}^O \le \frac{1}{s}\left(1+\delta_r^+\right), \quad r=1,\ldots,s, \quad j=1,\ldots,n, \tag{3-30}$$

$$\frac{1}{m}\left(1-\tau_i^-\right)\le P_{ij}^I \le \frac{1}{m}\left(1+\tau_i^+\right), \quad i=1,\ldots,m, \quad j=1,\ldots,n, \tag{3-31}$$

Eq. (3-30) and Eq. (3-31) can then be rewritten as Eq. (3-32) and Eq. (3-33), respectively.

$$B_r^{OL} \le P_{rj}^O \le B_r^{OU}, \quad r=1,\ldots,s, \quad j=1,\ldots,n, \tag{3-32}$$

$$B_i^{IL} \le P_{ij}^I \le B_i^{IU}, \quad i=1,\ldots,m, \quad j=1,\ldots,n, \tag{3-33}$$

P_{rj}^O and P_{ij}^I can only vary within the interval [B_r^{OL}, B_r^{OU}] and [B_i^{IL}, B_i^{IU}], respectively. By combining Eq. (3-30), Eq. (3-31) with Eq. (3-32), Eq. (3-33), we obtain the following range $0 \leqq B_r^{OL} \leqq 1/s$, $1/s \leqq B_r^{OU} \leqq 1$, $0 \leqq B_i^{IL} \leqq 1/m$ and $1/m \leqq B_i^{IU} \leqq 1$. Then, by the setting restrictions, the managers can easily avoid the infeasibility problem generated by the implicit restrictions $\sum_{r=1}^{s} P_{rj}^O = 1$ and $\sum_{i=1}^{m} P_{ij}^I = 1$.

In order to match the virtual assurance region, we have rewritten Eq. (3-32) and Eq. (3-33) as Eq. (3-22) with appropriate values α_{iw} and β_{rw}. For instance, Eq. (3-32) can be divided into two parts, $P_{rj}^O \ge B_r^{OL}$ and $P_{rj}^O \le B_r^{OU}$. The former and the latter can be rewritten as Eq. (3-22) with the setting of parameters α_{iw} and β_{rw} shown in Eq. (3-34) and Eq. (3-35), respectively.

$$\begin{aligned} &\alpha_{iw} = 0, && i=1,\ldots,m, \\ &\beta_{rw} = \begin{cases} -B_z^{OL}, & r \ne z \\ 1-B_z^{OL}, & r = z \end{cases} && r=1,\ldots,s, \quad z=1,\ldots,s, \end{aligned} \tag{3-34}$$

$$\begin{aligned} &\alpha_{iw} = 0, && i=1,\ldots,m, \\ &\beta_{rw} = \begin{cases} B_z^{OU}, & r \ne z \\ B_z^{OU}-1, & r = z \end{cases} && r=1,\ldots,s, \quad z=1,\ldots,s, \end{aligned} \tag{3-35}$$

Similarly, while Eq. (3-33) is divided into two parts, $P_{ij}^I \ge B_i^{IL}$ and $P_{ij}^I \le B_i^{IU}$, they can be rewritten as Eq. (3-22) with the setting of parameters α_{iw} and β_{rw} shown in Eq. (3-36) and Eq. (3-37), respectively.

$$\begin{aligned} &\alpha_{iw} = \begin{cases} -B_z^{IL}, & i \ne z \\ 1-B_z^{IL}, & i = z \end{cases} && i=1,\ldots,m, \quad z=1,\ldots,m, \\ &\beta_{rw} = 0, && r=1,\ldots,s, \end{aligned} \tag{3-36}$$

$$\begin{aligned} &\alpha_{iw} = \begin{cases} B_z^{IU}, & i \ne z \\ B_z^{IU}-1, & i = z \end{cases} && i=1,\ldots,m, \quad z=1,\ldots,m, \\ &\beta_{rw} = 0, && r=1,\ldots,s, \end{aligned} \tag{3-37}$$

As for the amount of restrictions, if there exist *m* inputs and *s* outputs, the proportional virtual weights restrictions, both Eq. (3-32) and Eq. (3-33), can be written as 2*s* + 2*m* restrictions of Eq. (3-22) with $W = 2s + 2m$.

The advantage of bound setting in the proportional virtual weights restrictions is that the manager can systematically choose lower and upper bound to discuss the variation in the ranking of UOAs. For instance, the manager can start the analysis from the unconstrained case with the interval [B_r^{OL}, B_r^{OU}] = [0%, 100%] to P_{rj}^{O} and [B_i^{IL}, B_i^{IU}] = [0%, 100%] to P_{ij}^{I}, and step by step shorten the interval to the extreme cases that each input or output index has equal proportion. They have to note that don't violate the rules $0 \leqq B_r^{OL} \leqq 1/s$, $1/s \leqq B_r^{OU} \leqq 1$, $0 \leqq B_i^{IL} \leqq 1/m$ and $1/m \leqq B_i^{IU} \leqq 1$ to encounter the infeasibility problem.

3.5.3 CWA with virtual weights restrictions (VWR-CWA)

Because the proportional weights restrictions are one case of the virtual assurance regions, we add the general form of virtual weights restriction Eq. (3-22) into the constraints of CWA fractional programming (3-1). Then, (3-1) can be translated into (3-38):

VWR-CWA-FP

$$\Delta^* = \min \sum_{j=1}^{n} \Delta_j \tag{3-38.0}$$

$$s.t. \quad \frac{\sum_{r=1}^{s} y_{rj} U_r + \Delta_j^O}{\sum_{i=1}^{m} x_{ij} V_i - \Delta_j^I} = 1, \quad j = 1, \ldots, n, \tag{3-38.1}$$

$$\sum_{i=1}^{m} \alpha_{iw} x_{ij} V_i + \sum_{r=1}^{s} \beta_{rw} y_{rj} U_r \geq k_w, \quad j = 1, \ldots, n, \quad w = 1, \ldots, W, \tag{3-38.2}$$

$$\Delta_j^O, \Delta_j^I \geq 0, \qquad j = 1, \ldots, n, \tag{3-38.3}$$

$$U_r \geq \varepsilon_r^O > 0, \qquad r = 1, \ldots, s, \tag{3-38.4}$$

$$V_i \geq \varepsilon_i^I > 0, \qquad i = 1, \ldots, m. \tag{3-38.5}$$

Following the transformation of the CWA model, the ratio form (3-38) can be rewritten in a linear form (3-39) and (3-40), step by step:

VWR-CWA-LP1

$$\Delta^* = \min \sum_{j=1}^{n} \Delta_j \tag{3-39.0}$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i + \sqrt{2}\, \Delta_j = 0, \qquad j = 1,\ldots,n, \tag{3-39.1}$$

$$\sum_{i=1}^{m} \alpha_{iw} x_{ij} V_i + \sum_{r=1}^{s} \beta_{rw} y_{rj} U_r \geq k_w, \quad j = 1,\ldots,n, \quad w = 1,\ldots,W, \tag{3-39.2}$$

$$U_r \geq \varepsilon_r^O > 0, \qquad r = 1,\ldots,s, \tag{3-39.3}$$

$$V_i \geq \varepsilon_i^I > 0, \qquad i = 1,\ldots,m, \tag{3-39.4}$$

$$\Delta_j \geq 0, \qquad j = 1,\ldots,n. \tag{3-39.5}$$

VWR-CWA-LP2

$$-\Delta^* = \max \sum_{r=1}^{s} Y_r U_r - \sum_{i=1}^{m} X_i V_i \tag{3-40.0}$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i \leq 0, \qquad j = 1,\ldots,n, \tag{3-40.1}$$

$$\sum_{i=1}^{m} \alpha_{iw} x_{ij} V_i + \sum_{r=1}^{s} \beta_{rw} y_{rj} U_r \geq k_w, \quad j = 1,\ldots,n, \quad w = 1,\ldots,W, \tag{3-40.2}$$

$$Y_r = \sum_{j=1}^{n} y_{rj}, \qquad r = 1,\ldots,s, \tag{3-40.3}$$

$$X_i = \sum_{j=1}^{n} x_{ij}, \qquad i = 1,\ldots,m \tag{3-40.4}$$

$$U_r \geq \varepsilon_r^O > 0, \qquad r = 1,\ldots,s, \tag{3-40.5}$$

$$V_i \geq \varepsilon_i^I > 0, \qquad i = 1,\ldots,m. \tag{3-40.6}$$

Assume that variable value π_j^* is the *shadow price* of the set of constraints (3-40.1). Then, according to the definition of shadow price, the variations of criterion Eq. (3-41) will result in the variation of constraint Eq. (3-42). That is, if the right-hand side of the *j*th constraint increases 1 unit, then the criterion Eq. (3-42) gets the variation π_j^*.

$$\sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i \leq 0 + 1, \tag{3-41}$$

$$-\Delta^* = \left(\sum_{r=1}^{s} \left(\sum_{j=1}^{n} y_{rj} \right) U_r - \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) V_i \right) + \pi_j^* (0+1) \tag{3-42}$$

π_j^* represents the marginal influence on the criteria of linear programming (3-40.0), that is, the marginal influence on the organization's overall performance. It gives another priority reference while UOAs possess equivalent efficiency score. In the following subsections, we analyze further the ranking rules of those UOAs.

3.5.4 VWR-CWA ranking rules

In this subsection, we define the ranking rules by comparing the absolute efficiency score and the shadow price mentioned above with the VWR-CWA efficiency score ξ_j^* of UOA_j, as defined as Eq. (3-43):

$$\xi_j^* = \frac{\sum_{r=1}^{s} y_{rj} U_r^*}{\sum_{i=1}^{m} x_{ij} V_i^*}, \quad j = 1, \ldots, n, \tag{3-43}$$

V_i^* and U_r^* denote the optimal common weights obtained in (3-39) for all UOAs attached to the input index i and output index r. One can easily distinguish the UOAs according the following rules.

Rule 3

The performance of UOA_j is better than UOA_i if $\xi_j^* > \xi_i^*$.

Rule 4

If $\xi_j^* = \xi_i^* = 1.0$, then the performance of UOA_j is better than UOA_i if $\pi_j^* > \pi_i^*$.

3.6 Numerical example 3

In this numerical example, there are two sub-examples to join the discussion. We first give a test example to demonstrate the discrimination power of the proposed approach. The example with the characteristic of a large scale in the values of performance indices across UOAs could appeal to the intuitive ranking of UOAs by merely observing the value in the performance indices. Then, it showed that VWR-CWA obtained the consistent ranking with the intuitive ranking. Secondly, one illustrative example shows how the manager of a retailer could obtain preferable and robust ranking results for all branches.

3.6.1 Test example

Table 9 gives the simulated data set, with two inputs and one output for seven UOAs. The test example possesses the characteristic of a large-scale range in the value across UOAs, such that UOA_1, UOA_2, UOA_3, UOA_5, UOA_6 and UOA_7 are shown to be many times larger than UOA_4. These UOAs are ranked intuitively as UOA_1, UOA_2, UOA_3, UOA_4, UOA_5, UOA_6, UOA_7 by comparing the value of input index x_{1j}. The rankings assessed in CWA, as shown in Table 10, are consistent with intuitive ranking in Table 9. By observing the proportion in input measure, x_{2j} plays a more important role than x_{1j} for all UOAs ($P^I_{1j} < P^I_{2j}$) according to the assessment results of CWA. We try to add the preference of the performance indices to understand whether VWR-CWA works to obtain the consistent ranking with intuitive ranking.

The general form of virtual assurance region Eq. (3-22) can be rewritten as Eq. (3-44) for the test example with two input indices and one output index.

$$\alpha_1 x_{1j} V_1 + \alpha_2 x_{2j} V_2 + \beta_1 y_{1j} U_1 \geq 0, \quad j = 1, \ldots, 7, \tag{3-44}$$

Table 9. Test example with large scale ranges across UOAs

UOA_j	Input index		Output Index	Intuitive ranking
	x_{1j}	x_{2j}	y_{1j}	
UOA_1	470000	700000	200000	1
UOA_2	4800	7000	2000	2
UOA_3	49	70	20	3
UOA_4	5	7	2	4
UOA_5	510	700	200	5
UOA_6	52000	70000	20000	6
UOA_7	530000	700000	200000	7

If we have the preference that the proportion of x_{1j} is larger than twice of x_{2j}, then the parameters (α_1, α_2, β_1, β_2) are substituted by (1, -2, 0, 0). Eq. (3-44) is further rewritten as Eq. (3-45) for all UOA_j:

$$x_{1j} V_1 - 2 x_{2j} V_2 \geq 0, \quad j = 1, \ldots, 7, \tag{3-45}$$

Similar to CWA, as Table 10 depicted, VWR-CWA obtains a consistent ranking in the large scale range in the value of performance indices across UOAs with our preference in input measure. It implies that VWR-CWA provides the available discrimination power in assessing the UOAs.

Table 10. The assessment results of CWA and VWR-CWA in test example

	(1) CWA $(V_1^*, V_2^*, U_1^*) = (1.00, 1.00, 5.85)$					(2) VWR-CWA with Eq. (3-45) $(V_1^*, V_2^*, U_1^*) = (2.98, 1.00, 10.50)$				
UOA_j	ξ_j^*	Ranking	P_{1j}^I	P_{2j}^I	P_{1j}^O	ξ_j^*	Ranking	P_{1j}^I	P_{2j}^I	P_{1j}^O
UOA_1	1.000	1	40%	60%	100%	1.000	1	67%	33%	100%
UOA_2	0.998	2	41%	59%	100%	0.986	2	67%	33%	100%
UOA_3	0.996	3	41%	59%	100%	0.972	3	68%	32%	100%
UOA_4	0.994	4	42%	58%	100%	0.959	4	68%	32%	100%
UOA_5	0.992	5	42%	58%	100%	0.946	5	68%	32%	100%
UOA_6	0.990	6	43%	57%	100%	0.933	6	69%	31%	100%
UOA_7	0.988	7	43%	57%	100%	0.921	7	69%	31%	100%

3.6.2 Illustrative example

A manager of a retail company governs eight branches and periodically assesses them by observing four performance indices: number of Employees, Cost, Turnover, and Profit, as depicted in Table 11. Employees and Cost are treated as input indices, while Turnover and Profit are the output indices. Lower inputs and higher outputs are preferred to generate a higher efficiency score. Different to the first example, the characteristic of a large scale in the value is across indices, not UOAs (branches). In the following subsections, we illustrate how to obtain the preferable ranking and robust ranking for the manager.

Table 11. The indices data in illustrative example

Branch j	Input index		Output index	
	Employee x_{1j}	Cost x_{2j}	Turnover y_{1j}	Profit y_{2j}
A	20	6583	7929	419
B	21	7713	8414	406
C	18	6980	8020	359
D	24	8273	9947	373
E	28	8566	9741	412
F	23	8397	9408	500
G	29	7011	7890	621
H	26	8680	9701	705

3.6.2.1 Preferable ranking

In order to discuss the proportion of each index in different models, we assess these branches by using DEA (CCR input-oriented model), VWR-DEA (CCR input-oriented model with virtual weights restrictions), CWA and VWR-CWA models. The general form of virtual weights restrictions Eq. (3-22) can be rewritten as Eq. (3-46) for the current numerical example, with two input and two output indices:

$$\alpha_1 x_{1j} V_1 + \alpha_2 x_{2j} V_2 + \beta_1 y_{1j} U_1 + \beta_2 y_{2j} U_2 \geq 0, \quad j = A, \ldots, H, \tag{3-46}$$

If the manager has the preference that the proportion of Profit is no less than half of Turnover, then the parameters (α_1, α_2, β_1, β_2) are substituted by (0, 0, -1, 2). Eq. (3-46) is further rewritten as Eq. (3-47) for all branches j:

$$- y_{1j} U_1 + 2 y_{2j} U_2 \geq 0,, \quad j = A, \ldots, H, \tag{3-47}$$

The proportion allocation of each index obtained from the original DEA model, as depicted in column (1) of Table 12, is extremely disproportional in most branches, even though we add the virtual weights restriction Eq. (3-47) in the DEA model (VWR-DEA), as depicted in column (2) of Table 12. For instance, despite the preferable virtual weights restriction Eq. (3-47), branches G and H still choose their favorable weight to create a feasible disproportion in Turnover (0%) and Profit (100%). Besides, comparing DEA with CWA, as depicted in column (1) of Tables 12 and Table 13, the proportion allocation in the DEA model is more unstable than the CWA model, without large variation in all branches. The comparison between VWR-DEA and VWR-CWA, as depicted in column (2) of Tables 12 and 13, would have similar results. These results imply that the proportion allocation obtained, whether in the DEA or VWR-DEA models, cannot reflect the manager's preference altogether.

CWA provided the assessment results in column (1) of Table 13. They show that branch A and B are the best and worst, respectively. Following these common weights (V_1^*, V_2^*, U_1^*, U_2^*) = (1.00, 1.27, 1.00, 1.00) used in CWA, as depicted in column (1) of Table 13, the manager would observe a large difference in relative proportion, whether between the virtual inputs (P_{1j}^I, P_{2j}^I) or outputs (P_{1j}^O, P_{2j}^O); for instance, in the row of branch A, Employee (0.02%) vs. Cost (99.8%) and Turnover (95.0%) vs. Profit (5.0%).

From a managerial scenario, it reveals that the input index Cost and output index Turnover take a considerably large proportion of branch A's rating. The other branches appear to be in a similar situation. This kind of extreme disproportion may not be accepted under specific practical exercises, even though the manager expects quick business development. In fact, in any case, Profit still plays an important role in rating. The virtual assurance region can assist the manager in easily adding his preference in Profit.

Table 12. The proportion results of DEA and VWR-DEA in illustrative example

Branch	(1) DEA				(2) VWR-DEA with Eq. (3-47)			
j	P_{1j}^I	P_{2j}^I	P_{1j}^O	P_{2j}^O	P_{1j}^I	P_{2j}^I	P_{1j}^O	P_{2j}^O
A	35.2%	64.8%	95.1%	4.9%	13.0%	87.0%	58.7%	41.3%
B	45.8%	54.2%	82.6%	17.4%	100.0%	0.0%	60.8%	39.2%
C	65.4%	34.6%	78.1%	21.9%	100.0%	0.0%	62.6%	37.4%
D	34.1%	65.9%	96.5%	3.5%	100.0%	0.0%	66.6%	33.4%
E	0.0%	100.0%	100.0%	0.0%	0.0%	100.0%	63.9%	36.1%
F	46.0%	54.0%	81.2%	18.8%	11.7%	88.3%	58.4%	41.6%
G	10.1%	89.9%	72.4%	27.6%	30.1%	69.9%	0.0%	100.0%
H	7.5%	92.5%	73.9%	26.1%	100.0%	0.0%	0.0%	100.0%

Table 13. The assessment results of CWA and VWR-CWA in illustrative example

Branch	(1) CWA $(V_1^*, V_2^*, U_1^*, U_2^*) = (1.00, 1.27, 1.00, 1.00)$						(2) VWR-CWA with Eq. (3-47) $(V_1^*, V_2^*, U_1^*, U_2^*) = (94.25, 1.92, 1.00, 13.30)$					
j	ζ_j^*	Ranking	P_{1j}^I	P_{2j}^I	P_{1j}^O	P_{2j}^O	ξ_j^*	Ranking	P_{1j}^I	P_{2j}^I	P_{1j}^O	P_{2j}^O
A	1.000	1	0.02%	99.8%	95.0%	5.0%	1.000	1	13.0%	87.0%	58.7%	41.3%
B	0.902	8	0.02%	99.8%	95.4%	4.6%	0.935	6	11.8%	88.2%	60.9%	39.1%
C	0.947	4	0.02%	99.8%	95.7%	4.3%	0.996	3	11.2%	88.8%	62.6%	37.4%
D	0.984	2	0.02%	99.8%	96.4%	3.6%	0.964	5	12.5%	87.5%	66.6%	33.4%
E	0.934	6	0.03%	99.7%	95.9%	4.1%	0.878	7	13.8%	86.2%	63.9%	36.1%
F	0.931	7	0.02%	99.8%	95.0%	5.0%	0.975	4	11.9%	88.1%	58.5%	41.5%
G	0.956	3	0.03%	99.7%	92.7%	7.3%	0.875	8	16.9%	83.1%	48.8%	51.2%
H	0.945	5	0.02%	99.8%	93.2%	7.8%	0.998	2	12.8%	87.2%	50.8%	49.2%

The manager reassesses these branches using the VWR-CWA model. The assessment results of VWR-CWA are arranged in column (2) of Table 13 by using the other common weights $(V_1^*,\ V_2^*,\ U_1^*,\ U_2^*) = (94.25, 1.92, 1.00, 13.3)$. Focusing on the row of branch A in Table 13, the proportion of Turnover (P_{1j}^O) vs. Profit (P_{2j}^O) changes from the CWA disproportion 95.0% vs. 5.0% to the 58.7% vs. 41.3% in VWR-CWA. Similar changes also can be seen in other branches. The rankings of the eight branches under CWA and VWR-CWA are completely different. However, the ranking obtained from VWR-CWA is more preferable and reliable to the manager because its preference is considered.

Obviously, the virtual restriction Eq. (3-47) has an influence on the final ranking of the branches. In the above case, Eq. (3-47) is one of general form Eq. (3-22) with the parameter

$W = 1$. In order to strengthen the preference for the manager, they can add more restrictions to obtain its most preferable ranking for all branches in VWR-CWA. In addition, the different preferable constraints also can be only assigned to certain UOAs to keep the original characteristics in performance indices of each UOA.

3.6.2.2 Robust ranking

Column (2) of Table 13 shows the single preference that the manager assigned. It is common that there exists a situation that the manager has no preference about the relationship among indices. What they concerned is one acceptable and feasible proportion of virtual inputs and virtual outputs in the same measure. The manager can determine the acceptable interval [B_r^{OL} , B_r^{OU}] and [B_i^{IL} , B_i^{IU}] for P_{rj}^O and P_{ij}^I , respectively. For the current numerical example with two inputs ($m = 2$) and two outputs ($s = 2$), all the values of δ_r^- , δ_r^+ , τ_i^- and τ_i^+ are set within 0 and 1 to ensure that P_{rj}^O and P_{ij}^I are between 0 and 1. For the purposes of clearly illustrating our approach, we set the lower bound B_r^{OL} = 0.4 and upper bound B_r^{OU} = 0.6, respectively. In other words, P_{rj}^O would be limited within the interval [40%, 60%]. If a larger interval is allowed, one may set the interval [20%, 80%].

From a managerial scenario, while managers desire to understand the ranking of branches under variant kinds of limitations for P_{rj}^O and P_{ij}^I , Eq. (3-30) and Eq. (3-31) provide one systematic setting of lower bound and upper bound. For the cases where δ_r^- and δ_r^+ are set at five levels 0.2, 0.4, 0.6, 0.8, and 1.0, P_{rj}^O would be limited in the gradually wider intervals [40%, 60%], [30%, 70%], [20%, 80%], [10%, 90%] and [0%, 100%], respectively. With the same setting for τ_i^- and τ_i^+ , P_{ij}^I would have the same limitations as above.

Table 14. The 25 combinations of interval limitation for P_{rj}^O and P_{ij}^I

Combination symbol		[B_i^{IL} , B_i^{IU}]				
		[0%, 100%]	[10%, 90%]	[20%, 80%]	[30%, 70%]	[40%, 60%]
[B_r^{OL} , B_r^{OU}]	[0%, 100%]	C1	C2	C3	C4	C5
	[10%, 90%]	C6	C7	C8	C9	C10
	[20%, 80%]	C11	C12	C13	C14	C15
	[30%, 70%]	C16	C17	C18	C19	C20
	[40%, 60%]	C21	C22	C23	C24	C25

As depicted in Table 14, there are 25 combinations of interval limitation for P_{rj}^{O} and P_{ij}^{I}. Obviously, different interval limitations for P_{rj}^{O} and P_{ij}^{I} may have different assessment results for the ranking. In this numerical example, we can employ the VWR-CWA model in carrying out an assessment for each combination with corresponding intervals [B_i^{IL}, B_i^{IU}] and [B_r^{OL}, B_r^{OU}]. For instance, the results for C12 and C22 are depicted in Table 15. For the combination C12, the general virtual weights restrictions Eq. (3-44) can be rewritten as Eq. (3-48) to Eq. (3-51) by removing four of the same and repeatable restrictions for all branches from the parameters setting in Eq. (3-34) to Eq. (3-37):

$$0.8 y_{1j} U_1 - 0.2 y_{2j} U_2 \geq 0, \qquad j = A, \ldots, H, \tag{3-48}$$

$$-0.2 y_{1j} U_1 + 0.8 y_{2j} U_2 \geq 0, \qquad j = A, \ldots, H, \tag{3-49}$$

$$0.9 x_{1j} V_1 - 0.1 x_{2j} V_2 \geq 0, \qquad j = A, \ldots, H, \tag{3-50}$$

$$-0.1 x_{1j} V_1 + 0.9 x_{2j} V_2 \geq 0, \qquad j = A, \ldots, H, \tag{3-51}$$

As Table 15 depicted, the ranking is inconsistent between the two combinations C12 and C22. For managers, it is expected that more outcomes form all kinds of combinations that can help them make more accurate and robust judgments in the ranking of branches.

Table 15. The assessment results in VWR-CWA of C12 and C22

Branch j	(1) VWR-CWA of C12 (V_1, V_2, U_1, U_2) = (191.30, 1.09, 1.00, 6.67)						(2) VWR-CWA of C22 (V_1, V_2, U_1, U_2) = (121.38, 2.20, 1.00, 17.78)					
	ξ_j^*	Ranking	P_{1j}^I	P_{2j}^I	P_{1j}^O	P_{2j}^O	ξ_j^*	Ranking	P_{1j}^I	P_{2j}^I	P_{1j}^O	P_{2j}^O
A	0.975	2	35%	65%	74%	26%	0.909	3	14%	86%	52%	48%
B	0.895	7	32%	68%	76%	24%	0.801	6	13%	87%	54%	46%
C	0.942	3	31%	69%	77%	23%	0.821	5	12%	88%	56%	44%
D	0.914	5	34%	66%	80%	20%	0.785	7	14%	86%	60%	40%
E	0.850	8	36%	64%	78%	22%	0.767	8	15%	85%	57%	43%
F	0.940	4	32%	68%	74%	26%	0.860	4	13%	87%	51%	49%
G	0.912	6	42%	58%	66%	34%	1.000	1	19%	81%	42%	58%
H	1.000	1	34%	66%	67%	33%	0.998	2	14%	86%	44%	56%

While compiling statistics from 25 combinations, we obtained the percentage of occurrence frequency in each ranking, as depicted in Table 16. It is not hard to observe that except for branch G, the high occurrence frequency centralizes in a few rankings for other branches. For instance, branch H is only ranked 1^{st} and 2^{nd}. For branch E, the ranking of 7^{th} and 8^{th} occurs in all combinations. Undoubtedly, branch H is always better than branch E. If managers choose the highest occurrence frequency as the representative branch of each ranking level, the ranking list for 1^{st} to 8^{th} is H, A, C, F, D, B, E and G.

Table 16. The summary of the 25 ranking results with C1 to C25

Ranking / Branch j	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}	8^{th}
A	10	8	7	0	0	0	0	0
B	0	0	0	0	0	18	7	0
C	0	0	15	1	9	0	0	0
D	0	0	0	0	15	3	7	0
E	0	0	0	0	0	0	10	15
F	0	0	1	24	0	0	0	0
G	2	5	2	0	1	4	1	10
H	13	12	0	0	0	0	0	0
Total	25	25	25	25	25	25	25	25
Robust Ranking	H	A	C	F	D	B	E	G

Table 17. The ranking of branch G in the 25 ranking results with C1 to C25

Ranking of Branch G		$[B_i^{IL}, B_i^{IU}]$				
		[0%, 100%]	[10%, 90%]	[20%, 80%]	[30%, 70%]	[40%, 60%]
$[B_r^{OL}, B_r^{OU}]$	[0%, 100%]	8^{th}	8^{th}	8^{th}	8^{th}	8^{th}
	[10%, 90%]	8^{th}	8^{th}	8^{th}	8^{th}	8^{th}
	[20%, 80%]	6^{th}	6^{th}	6^{th}	6^{th}	7^{th}
	[30%, 70%]	2^{nd}	2^{nd}	2^{nd}	3^{rd}	5^{th}
	[40%, 60%]	1^{st}	1^{st}	2^{nd}	2^{nd}	3^{rd}

Under the above ranking rule, the ranking of branch G is debatable due to its average occurrence in multiple ranking levels. In other words, branch G's ranking varies largely under different combinations. We further observe the ranking status of branch G in all combinations, as depicted in Table 17. While fixing the interval $[B_r^{OL},\ B_r^{OU}]$ with [0%, 100%] or [10%, 90%] for P_{rG}^{O}, branch G is ranked the last of all branches, whatever the interval $[B_i^{IL},\ B_i^{IU}]$ for P_{iG}^{I}. On the contrary, while we shorten the interval $[B_r^{OL},\ B_r^{OU}]$ step by step from [0%, 100%] to [40%, 60%] for P_{rG}^{O}, fixing the interval $[B_i^{IL},\ B_i^{IU}]$ at [0%, 100%] for P_{iG}^{I}, branch G can reach the best one of all branches.

Following the above observation, we understand that the ranking of branch G is deeply affected by the variation of interval $[B_r^{OL},\ B_r^{OU}]$. If the manager is asked to only select some combinations as the reference of assessment, they should concentrate more attention in determining the appropriate interval [B_r^{OL}, B_r^{OU}]. Branch G will obtain a different ranking while the manager determines a different interval [B_r^{OL}, B_r^{OU}]. As for the determination of interval [B_i^{IL}, B_i^{IU}], in this case it is not necessary for the manager to cost more effort because these combinations show the same ranking while the interval [B_i^{IL}, B_i^{IU}] varies.

In order to explore the cause of the above phenomenon, we observe the relationship between the ranking variations and proportion variations of branch G while varying interval [B_i^{IL}, B_i^{IU}] or [B_r^{OL}, B_r^{OU}], as Table 18 and Table 19 depicted. It is obvious that the values of P_{1G}^{I}(54%) and P_{2G}^{I}(46%) obtained in C1 are simultaneously satisfied with a narrower interval [B_i^{IL}, B_i^{IU}] in C2, C3, C4, and C5. Therefore, as depicted in Table 18, while fixing the interval [B_r^{OL}, B_r^{OU}] at [0%, 100%] and shortening the interval [B_i^{IL}, B_i^{IU}], we still obtain the invariant values of proportion and ranking for branch G. However, as depicted in Table 19, P_{1G}^{O}(74%) and P_{2G}^{O}(26%) obtained in C1 are not satisfied with the narrower interval [B_r^{OL}, B_r^{OU}] in C6, C11, C16, and C21. In order to satisfy narrower intervals [B_r^{OL}, B_r^{OU}], the smaller P_{1G}^{O} and P_{2G}^{O} are necessary. Therefore, the above variation in interval [B_r^{OL}, B_r^{OU}] easily results in the variations of P_{1G}^{I}, P_{2G}^{I} and ranking.

Table 18. The proportion variations of indices of branch G while varying [B_i^{IL}, B_i^{IU}]

Combination	Ranking	[B_i^{IL}, B_i^{IU}]	P_{1G}^{I}	P_{2G}^{I}	[B_r^{OL}, B_r^{OU}]	P_{1G}^{O}	P_{2G}^{O}
C1	8th	[0%, 100%]	54%	46%	[0%, 100%]	74%	26%
C2	8th	[10%, 90%]	54%	46%	[0%, 100%]	74%	26%
C3	8th	[20%, 80%]	54%	46%	[0%, 100%]	74%	26%
C4	8th	[30%, 70%]	54%	46%	[0%, 100%]	74%	26%
C5	8th	[40%, 60%]	54%	46%	[0%, 100%]	74%	26%

Table 19. The proportion variations of indices of branch G while varying [B_r^{OL}, B_r^{OU}]

Combination	Ranking	[B_i^{IL}, B_i^{IU}]	P_{1G}^{I}	P_{2G}^{I}	[B_r^{OL}, B_r^{OU}]	P_{1G}^{O}	P_{2G}^{O}
C1	8th	[0%, 100%]	54%	46%	[0%, 100%]	74%	26%
C6	8th	[0%, 100%]	54%	46%	[10%, 90%]	74%	26%
C11	6th	[0%, 100%]	42%	58%	[20%, 80%]	66%	34%
C16	2nd	[0%, 100%]	23%	77%	[30%, 70%]	53%	47%
C21	1st	[0%, 100%]	19%	81%	[40%, 60%]	42%	58%

Following the above discussion, we conclude that given the fixed interval [B_r^{OL} , B_r^{OU}], if the value of P_{1G}^{I} and P_{2G}^{I} obtained in C1 is feasible in the narrowest interval [B_i^{IL} , B_i^{IU}] of C5, then the values of P_{1G}^{I} and P_{2G}^{I} are also feasible in C2, C3, and C4. Most importantly, the ranking is invariant with the same proportion in these combinations. If the manager needs to complete all combinations, it is helpful for them to deduce the times of assessment by omitting C2, C3, and C4 while fixing the interval [B_r^{OL} , B_r^{OU}] at [0%, 100%].

4. Most Compromise Weight Analysis (MCWA) to Rank Organization Units

MCWA model discuss the ranking of organization units by releasing the restrictions that the efficiency score of all units can't exceed the value 1.0 in CWA model. It generates the common weights from the perspective of gap minimization similar to the regression analysis model. Besides, we consider the cluster analysis in the application of MCWA to solve the problem of bias ranking in units because of the different scales.

4.1 Performance measurement philosophy

The CWA methodology in section 3 follows the DEA structure, using the organization manager as the object units to determine the favorable weights. CWA also limits that none of DMUs' efficiency scores is allowed to exceed 1.0. However, the restriction possibly prevents the efficiency from being developed for some excellent units. It easily results in the bias in the assessment results. In order to make the units possessing the freedom in creating the efficiency score, we develop the second procedure to determine one compromise common set of weights by eliminating the restriction with upper bound 1.0 in efficiency score. We expand the MCWA model proposed by Liu et al. [34] in this section to complete the corresponding analysis in ranking. The proposed model in the second procedure is similar to the regression analysis model with the main difference that the proposed model can handle multiple dependent variables while regression model focus on one dependent variable.

4.2 MCWA model

We expand the scenario in Figure 2 to the scenario in Figure 5. Given two UOA_M and UOA_N with one known common weight $\tilde{U}_r$ (r = 1, 2, …, s) and $\tilde{V}_i$ (i = 1, 2, …, m) is assigned, the sum of virtual gap to Diagonal Line (DL) is $\tilde{\Delta}_N^A + \tilde{\Delta}_M^B$, which mean the performance variation of two UOAs. However, we want to choose one optimal common weight to minimize the sum of virtual gap for purpose of expressing the minimal variation between the two UOAs. The minimized sum of virtual gaps is the objective function value of the linear program. For instance, as Figure 5 depicted, the simplified scenario can be modeled to Eq. (4-1). The constraints Eq. (4-1.1) and Eq. (4-1.2) stands for the adjustment of two UOAs to DL with combination of the virtual gaps Δ_N^{OA}, Δ_N^{IA}, Δ_M^{OB} and Δ_M^{IB}.

Assume that there are n UOAs, the model (4-1) can be expanded into model (4-2) with some UOAs belong to the set A (UOAs above DL) and the others belong to the set B (UOAs below DL). The parameters ε_i^I and ε_r^O are the positive Archimedean infinitesimal constant with scale unit "Đ/ unit of the associated i-th input and r-th output indices", respectively. The symbol Đ denotes the virtual scale unit for the weighted input and outputs. For instance, while y_{1j} takes the dollar ($) as its unit, then we obtain the unit of ε_1^O to be the ratio (Đ/$). The products $\varepsilon_i^I x_{ij}$ and $\varepsilon_r^O y_{rj}$ finally obtain values in Đ that are equivalent to the virtual gaps: ($\Delta_j^{IA}, \Delta_j^{IB}$) and ($\Delta_j^{OA}, \Delta_j^{OB}$), respectively.

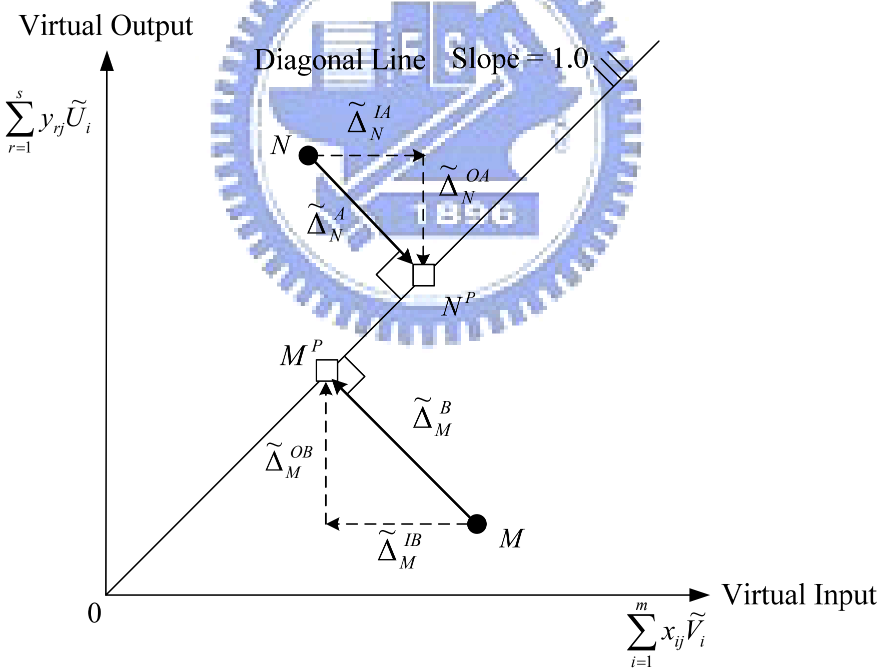

Figure 5. Gap analysis for UOAs locating on the both side of diagonal line

In addition, the fractional model (4-2) can be transformed into the linear model (4-3) for computation. The triangular in Figure 5 is one isosceles right triangular, so it leads to $\Delta_j^{OA} = \Delta_j^{IA} = \frac{1}{\sqrt{2}}\Delta_j^{A}$ and $\Delta_j^{OB} = \Delta_j^{IB} = \frac{1}{\sqrt{2}}\Delta_j^{B}$. Then, (4-3) could be rewritten as (4-4).

MCWA-FP1

$$\min \quad \Delta_M^A + \Delta_N^B \tag{4-1.0}$$

$$s.t. \quad \frac{\sum_{r=1}^{s} y_{rN} U_r - \Delta_N^{OA}}{\sum_{i=1}^{m} x_{iN} V_i + \Delta_N^{IA}} = 1, \tag{4-1.1}$$

$$\frac{\sum_{r=1}^{s} y_{rM} U_r + \Delta_M^{OB}}{\sum_{i=1}^{m} x_{iM} V_i - \Delta_M^{IB}} = 1, \tag{4-1.2}$$

$$U_r \geq \varepsilon_r^O > 0, \qquad r = 1, \ldots, s, \tag{4-1.3}$$

$$V_i \geq \varepsilon_i^I > 0, \qquad i = 1, \ldots, m, \tag{4-1.4}$$

$$\Delta_M^{IA}, \Delta_M^{OA}, \Delta_N^{IA}, \Delta_N^{OA}, \Delta_M^A, \Delta_N^B \geq 0. \tag{4-1.5}$$

MCWA-FP2

$$\min \quad \sum_{j \in A} \Delta_j^A + \sum_{j \in B} \Delta_j^B \tag{4-2.0}$$

$$s.t. \quad \frac{\sum_{r=1}^{s} y_{rj} U_r - \Delta_j^{OA}}{\sum_{i=1}^{m} x_{ij} V_i + \Delta_j^{IA}} = 1, \qquad j \in A, \tag{4-2.1}$$

$$\frac{\sum_{r=1}^{s} y_{rj} U_r + \Delta_j^{OB}}{\sum_{i=1}^{m} x_{ij} V_i - \Delta_j^{IB}} = 1, \qquad j \in B, \tag{4-2.2}$$

$$U_r \geq \varepsilon_r^O > 0, \qquad r = 1, \ldots, s, \tag{4-2.3}$$

$$V_i \geq \varepsilon_i^I > 0, \qquad i = 1, \ldots, m, \tag{4-2.4}$$

$$\Delta_j^{IA}, \Delta_j^{OA}, \Delta_j^A \geq 0, \qquad j \in A, \tag{4-2.5}$$

$$\Delta_j^{IB}, \Delta_j^{OB}, \Delta_j^B \geq 0, \qquad j \in B. \tag{4-2.6}$$

We expand the model (4-4) into the model (4-5) by adding Δ_j^A and Δ_j^B in each constraint for the convenience of computation. Since $\Delta_j^B = 0$ while $j \in A$ and $\Delta_j^A = 0$ while $j \in B$, the variables Δ_j^B and Δ_j^A are separately added into (4-5.0), (4-5.1) and (4-5.2).

MCWA-LP1

$$\min \quad \sum_{j\in A}\Delta_j^A + \sum_{j\in B}\Delta_j^B \qquad (4\text{-}3.0)$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i - \Delta_j^{IA} - \Delta_j^{OA} = 0, \quad j \in A, \qquad (4\text{-}3.1)$$

$$\sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i + \Delta_j^{IB} + \Delta_j^{OB} = 0, \quad j \in B, \qquad (4\text{-}3.2)$$

$$U_r \ge \varepsilon_r^O > 0, \qquad r = 1,\dots,s, \qquad (4\text{-}3.3)$$

$$V_i \ge \varepsilon_i^I > 0, \qquad i = 1,\dots,m, \qquad (4\text{-}3.4)$$

$$\Delta_j^{IA}, \Delta_j^{OA}, \Delta_j^A \ge 0, \qquad j \in A, \qquad (4\text{-}3.5)$$

$$\Delta_j^{IB}, \Delta_j^{OB}, \Delta_j^B \ge 0, \qquad j \in B. \qquad (4\text{-}3.6)$$

MCWA-LP2

$$\min \quad \sum_{j\in A}\Delta_j^A + \sum_{j\in B}\Delta_j^B \qquad (4\text{-}4.0)$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i - \sqrt{2}\Delta_j^A = 0, \quad j \in A, \qquad (4\text{-}4.1)$$

$$\sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i + \sqrt{2}\Delta_j^B = 0, \quad j \in B, \qquad (4\text{-}4.2)$$

$$U_r \ge \varepsilon_r^O > 0, \qquad r = 1,\dots,s, \qquad (4\text{-}4.3)$$

$$V_i \ge \varepsilon_i^I > 0, \qquad i = 1,\dots,m, \qquad (4\text{-}4.4)$$

$$\Delta_j^A \ge 0, \qquad j \in A, \qquad (4\text{-}4.5)$$

$$\Delta_j^B \ge 0, \qquad j \in B. \qquad (4\text{-}4.6)$$

MCWA-LP3

$$\min \quad \sum_{j\in A}\left(\Delta_j^A + \Delta_j^B\right) + \sum_{j\in B}\left(\Delta_j^A + \Delta_j^B\right) \qquad (4\text{-}5.0)$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i - \sqrt{2}\Delta_j^A + \sqrt{2}\Delta_j^B = 0, \quad j \in A, \qquad (4\text{-}5.1)$$

$$\sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i - \sqrt{2}\Delta_j^A + \sqrt{2}\Delta_j^B = 0, \quad j \in B, \qquad (4\text{-}5.2)$$

$$U_r \ge \varepsilon_r^O > 0, \qquad r = 1,\dots,s, \qquad (4\text{-}5.3)$$

$$V_i \ge \varepsilon_i^I > 0, \qquad i = 1,\dots,m, \qquad (4\text{-}5.4)$$

$$\Delta_j^A \ge 0,\ \Delta_j^B = 0, \qquad j \in A, \qquad (4\text{-}5.5)$$

$$\Delta_j^B \ge 0,\ \Delta_j^A = 0, \qquad j \in B. \qquad (4\text{-}5.6)$$

Model (4-5) is equivalent to model (4-6) by combining (4-5.1) with (4-5.2) and combining (4-5.5) with (4-5.6), we obtain the formulation (4-6.1) and (4-6.4).

MCWA-LP4

$$\min \quad \sum_{j=1}^{n}\left(\Delta_j^A + \Delta_j^B\right) \tag{4-6.0}$$

$$s.t. \quad \sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i - \sqrt{2}\Delta_j^A + \sqrt{2}\Delta_j^B = 0, \qquad j = 1,\ldots,n, \tag{4-6.1}$$

$$U_r \ge \varepsilon_r^O > 0, \qquad r = 1,\ldots,s, \tag{4-6.2}$$

$$V_i \ge \varepsilon_i^I > 0, \qquad i = 1,\ldots,m, \tag{4-6.3}$$

$$\Delta_j^A, \Delta_j^B \ge 0, \qquad j = 1,\ldots,n. \tag{4-6.4}$$

Here, we define the MCWA efficiency score η_j^* of UOA_j, as defined as Eq. (4-7).

$$\eta_j^* = \frac{\sum_{r=1}^{s} y_{rj}U_r^*}{\sum_{i=1}^{m} x_{ij}V_i^*}, \qquad j = 1,\ldots,n, \tag{4-7}$$

V_i^* and U_r^* denote the optimal common weights obtained in (4-6) for all UOAs attached to the input index *i* and output index *r*.

Model (4-6) is similar to the multiple regression model while we only consider one dependent variable and multiple independent variables. The equation (4-6.1) relating *m* independent variables (x_{1j}, x_{2j}, …, x_{mj}) to a dependent variable *y* of the form can be depicted in (4-8). Then, (4-8) can be transformed into common form (4-9) of multiple regression model.

$$y_j = \sum_{i=1}^{m} \frac{x_{ij}V_i}{U} + \frac{\sqrt{2}\Delta_j^A}{U} - \frac{\sqrt{2}\Delta_j^B}{U}, \qquad j = 1,\ldots,n, \tag{4-8}$$

$$y_j = \sum_{i=1}^{m} x_{ij}\alpha_i + \frac{\sqrt{2}}{U}\left(\Delta_j^A - \Delta_j^B\right), \qquad j = 1,\ldots,n, \tag{4-9}$$

Generally, we desire to minimize the sum of squares $\frac{\sqrt{2}}{U}\left(\Delta_j^A - \Delta_j^B\right)$ for all UOA_j and generate the following model (4-10). In the other words, we can regard MCWA model as the similar regression model while considering one dependent variable.

$$\min \quad \sum_{j=1}^{n}\left(\Delta_j^A + \Delta_j^B\right) \tag{4-10.0}$$

$$s.t. \quad y_j = \sum_{i=1}^{m} x_{ij}\alpha_i + \frac{\sqrt{2}}{U}\left(\Delta_j^A - \Delta_j^B\right), \qquad j = 1,\ldots,n, \tag{4-10.1}$$

$$\alpha_i \geq \varepsilon > 0, \qquad i = 1,\ldots,m, \tag{4-10.2}$$

$$\Delta_j^A, \Delta_j^B \geq 0, \qquad j = 1,\ldots,n. \tag{4-10.3}$$

4.3 Numerical example 4

In order to describe the methodology readily, we use the same set of hypothetical data as depicted in Table 11 to implement the MCWA model. A manager of a retail company governs eight branches and periodically assesses them by observing four performance indices: number of Employees, Cost, Turnover, and Profit. Employees and Cost are treated as input indices, while Turnover and Profit are the output indices.

Table 20. The efficiency score and ranking of using CWA, MCWA and (4-11) models

Branch	(1) CWA			(2) MCWA			(3) (4-11) with $\sigma^* = 1.175$		
j	ζ_j^*	Ranking	Gap	η_j^*	Ranking	Gap	Score	Ranking	Gap
A	1.000	1	0.00	1.057	1	452.26	1.237	1	358.60
B	0.902	8	958.54	0.957	8	398.87	1.119	8	385.43
C	0.947	4	469.24	1.006	3	49.53	1.175	3	0.00
D	0.984	2	169.98	1.042	2	412.27	1.215	2	285.28
E	0.934	6	711.65	0.986	7	147.32	1.150	7	191.90
F	0.931	7	737.85	0.987	6	130.24	1.155	6	148.25
G	0.956	3	387.45	1.000	4	0.00	1.175	3	0.00
H	0.945	5	600.87	1.000	4	0.00	1.175	3	0.00
	Total		4033.58	Total		1590.49	Total		1369.46

Note1: $(V_1^*, V_2^*, U_1^*, U_2^*, U_3^*) = (1.00, 1.27, 1.00, 1.00)$ in CWA model

Note2: $(V_1^*, V_2^*, U_1^*, U_2^*, U_3^*) = (13.24, 1.16, 1.00, 1.00)$ in MCWA model

Note3: $(V_1^*, V_2^*, U_1^*, U_2^*, U_3^*) = (12.70, 1.00, 1.00, 1.26)$ in model (4-11)

We can use the CWA model (3-3) to search the benchmark and the detailed ranking of all branches. The corresponding reference information is depicted in the column (1) of Table 20 and the benchmark is branch-A. In fact, while we use the model (4-6) by releasing the restriction that none of DMUs' efficiency scores is allowed to exceed 1.0, the benchmark is still the branch-A and its efficiency score increases to 1.057 as depicted in the column (2) of Table 20. There also exists little variation in the accurate ranking of branches and the ranking are similar in the most branches.

In order to make MCWA model more flexible, Liu and Lin [35] proposed to replace the efficiency score 1.0 by the unknown slop σ of DL in MCWA model with some UOAs belong to the set A' (UOAs above DL with slop σ) and the others belong to the set B' (UOAs below DL with slop σ). They transform the model (4-2) into the following nonlinear programming model (4-11). We reassess the dataset by the model (4-11) and obtain the score and ranking in the column (3) of Table 20. We obtain the almost consistent results in ranking between MCWA model and model (4-11) and the latter possesses the minimum gap 1369.46.

$$\min \quad \sum_{j\in A'} \Delta_j^{A'} + \sum_{j\in B'} \Delta_j^{B'} \tag{4-11.0}$$

$$s.t. \quad \frac{\sum_{r=1}^{s} y_{rj} U_r - \Delta_j^{OA'}}{\sum_{i=1}^{m} x_{ij} V_i + \Delta_j^{IA'}} = \sigma, \quad j \in A', \tag{4-11.1}$$

$$\frac{\sum_{r=1}^{s} y_{rj} U_r + \Delta_j^{OB'}}{\sum_{i=1}^{m} x_{ij} V_i - \Delta_j^{IB'}} = \sigma, \quad j \in B', \tag{4-11.2}$$

$$U_r \ge \varepsilon_r^O > 0, \quad r = 1,\ldots,s, \tag{4-11.3}$$

$$V_i \ge \varepsilon_i^I > 0, \quad i = 1,\ldots,m, \tag{4-11.4}$$

$$\Delta_j^{IA'}, \Delta_j^{OA'}, \Delta_j^{A'} \ge 0, \quad j \in A', \tag{4-11.5}$$

$$\Delta_j^{IB'}, \Delta_j^{OB'}, \Delta_j^{B'} \ge 0, \quad j \in B'. \tag{4-11.6}$$

4.4 Applying cluster analysis in MCWA model

Cluster analysis is a term used to describe a family of statistical procedures specifically designed to discover classifications within complex data sets. The objective of cluster analysis is to group units into clusters such that units within one cluster share more in common with one another than they do with the units of other clusters. Thus, the purpose of the analysis is to arrange units into relatively homogenous groups based on multivariate observations.

Minimum variation method proposed by Ward [36] is probably the most widely used in the cluster analysis. The relative proximity of a set of units can be described using the concept of sum of squares, the squared sum of the distances of each unit from the mean value of the cluster. Using Ward's method, the cluster that results in the smallest increase in the sum of squares is formed during each step. Every possible combination of cluster formation is considered at each subsequent step. The minimum variation method provides the division of initial units for the purpose to proceed to further ranking analysis.

4.5 Numerical example 5

In order to describe the procedure readily, we use the data of Table 21 as our example. The input indices include *number of employees* and *cost* and the output indices include the *business of credit*, *deposit,* and *loan*. We propose a loop with three major steps to classify the UOAs.

Table 21. Dataset of 12 bank branches with 5 performance indices

Branch j	Input index		Output index		
	Employee (x_{1j})	Cost (x_{2j})	Credit (y_{1j})	Deposit (y_{2j})	Loan (y_{3j})
	Person	\$	\$	\$	\$
A	23	510	1500	78397	7940
B	26	571	1705	78680	7970
C	36	922	1718	63960	6604
D	42	1012	1812	65426	6682
E	48	1056	1746	66546	6988
F	33	799	631	11671	1551
G	28	544	412	8566	974
H	39	880	750	12312	1389
I	40	968	838	13166	1584
J	29	552	621	7011	789
K	37	824	720	9974	1002
L	35	941	695	10087	1192

Step 0: Initialized the number of subgroup, k=1.

Step 1: Use the computational model (4-6) to obtain the common set of weights U_r^* (r = 1, 2, …, s) and V_i^* (i = 1, 2, …, m) for the group of UOAs and compute their efficiency scores. Then go to Step 2.

Step 2: Proceed with a hypothesis test to the obtained efficiency scores with the null hypothesis that the mean of efficiency scores for each subgroup of UOAs is equal to 1.0. If all the subgroups accept the null hypothesis, stop the procedure here. Otherwise, go to Step 3.

Step 3: Employ Ward's cluster analysis method [36] to categorize the whole UOAs into k+1 subgroups. Then the units belong to each of the k+1 subgroups are further processed by Step 1. Update k as k+1.

We randomly generated a large number of experimental datasets. Generally, the final value of k would not be more than three.

Step 1: Applying MCWA to all branches

Apply the dataset, we called S here, in Table 21 to the MCWA model (4-6), the optimal solutions are depicted in Table 22. The objective function value, total virtual gap 2384.92 (Đ×100) of all branches to DL is the minimized one with the common set of weights $(V_1^*, V_2^*, U_1^*, U_2^*, U_3^*)$ = (94.03 Đ/person, 13.45 Đ/$, 1.00 Đ/$, 1.00 Đ/$, 1.00 Đ/$). Then, each branch's coordinates $(\dot{x}_j^*, \dot{y}_j^*)$ are plotted in Figure 6, with x-axis $94.03\,x_1 + 13.45\,x_2$ and y-axis $1y_1 + 1y_2 + 1y_3$.

For instance, branch I locates at coordinate $(\dot{x}_A^*, \dot{y}_A^*)$ = (167.85, 155.88) and is ranked 9th with the efficiency score 0.929. The virtual gap is 8.46. According to Table 22, branches A and L are identified as the best and the worst branches. Besides, relative to DL, the three subgroups of branches {A, B, C, D, E}, {H, I, J, K, L} and {F, G} are located above, below and on the DL, respectively.

Step 2: Proceeding with a hypothesis test to the obtained efficiency scores

According to the efficiency score η_j^* in Table 22, we reject the null hypothesis with p-value 0.043 by using the t-test on the degree 11 and confidence coefficient 0.05. It implies that there exists significant difference in the level of branches such that the common set of weights cannot efficiently minimize the total virtual gap for the point at all branches in DL. The case possibly leads to the bias of ranking. Therefore, we go to the step 3 to classify the branches into two subgroups.

Step 3: Classifying the branches

The general manager who governs the branches may observe the above ranking as one rough reference while there is a significant difference in the efficiency scores for branches. In Step 2, we analyze the dataset of five original performance indices in Table 21 by Ward's method with computer software SPSS 13.0. Then, the 12 branches of initial group S are classified into two subgroups, C_1= {A, B, C, D, E} and C_2 = {F, G, H, I, J, K, L}. The minimum total sum of squares 2.44E+08 for these two subgroups is composed of the individual subgroup's sum of squares 2.15E+08 for C_1 and 2.89E+07 for C_2, respectively. Then proceed to the MCWA model for the branches of each subgroup.

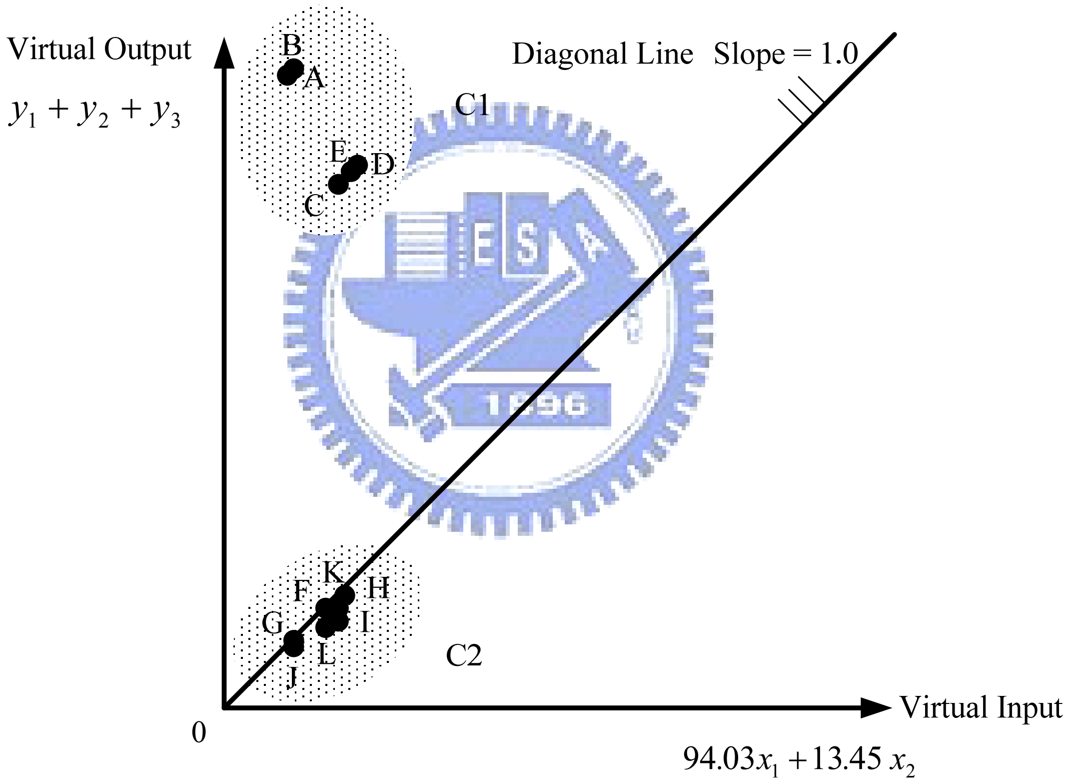

Figure 6. The expression to branches in group S

Back to Step 1: Applying MCWA to each subgroup of branches

The datasets of subgroups C_1= {A, B, C, D, E} and C_2 = {F, G, H, I, J, K, L} are being substituted for the original set S in Table 21 to model (4-6), the results of the two subgroups are arranged in Table 23. Observe Table 23, the ranking of branches in subgroup C_2, F > I > G = H = J > K > L is different from the ranking of branches in Table 22, F = G > H > I > J > K >

L. The branch I obviously changes its order. In terms of scale, the branches in subgroup C_1 are obviously larger than subgroup C_2. It implies that the scale possesses the influence on the common set of weights for the group S. According to our procedure, the branches in subgroup C_2 have similar scales. Eliminating the effect from the branches of subgroup C_1, the common set of weights is fairer in ranking the branches of subgroup C_2.

Table 22. The computation results to 12 branches of group S

Branch j	Score / Rank		Virtual input / output		Virtual gap	
	η_j^*	Rank	$\dot{x}_j^*$ (Đ×100)	$\dot{y}_j^*$ (Đ×100)	Δ_j^{B*} (Đ×100)	Δ_j^{A*} (Đ×100)
A	9.73	1	90.24	878.37	0.00	557.29
B	8.73	2	101.27	883.55	0.00	553.15
C	4.58	3	157.90	722.82	0.00	399.46
D	4.21	4	175.65	739.20	0.00	398.49
E	4.02	5	187.21	752.80	0.00	399.93
F	1.00	6	138.53	138.53	0.00	0.00
G	1.00	6	99.52	99.52	0.00	0.00
H	0.932	8	155.07	144.51	7.47	0.00
I	0.929	9	167.85	155.88	8.46	0.00
J	0.82	10	101.54	84.21	12.25	0.00
K	0.80	11	145.65	116.96	20.29	0.00
L	0.75	12	159.52	119.74	28.13	0.00
Subtotal					76.60	2308.32
Total					2384.92	

Note: $(V_1^*, V_2^*, U_1^*, U_2^*, U_3^*)$ = (94.03 Đ/person, 13.45 Đ/$, 1.00 Đ/$, 1.00 Đ/$, 1.00 Đ/$)

Then, by setting the confidence coefficient as 0.05, the null hypothesis that the mean of the five efficiency scores of subgroup C_1 equals to 1 is accepted with p-value 0.17. Similarly, the seven efficiency scores of subgroups C_2 have also accepted the null hypothesis with p-value 0.31. Figure 7 and Figure 8 depict the locations of the branches of subgroup C_1 and subgroup C_2, respectively.

Table 23. The computation results to branches of 2 disjoint subgroups C_1 and C_2

Branch j		Score / Rank		Virtual input / output		Virtual gap	
		η_j^*	Rank	$\dot{x}_j^*$ (Đ×100)	$\dot{y}_j^*$ (Đ×100)	$\Delta_j^{B^*}$ (Đ×100)	$\Delta_j^{A^*}$ (Đ×100)
C_1	A	2.36	1	372.54	878.37	0.00	357.68
	B	2.12	2	417.10	883.55	0.00	329.83
	C	1.07	3	673.44	722.82	0.00	34.92
	D	1.00	4	739.20	739.20	0.00	0.00
	E	0.98	5	771.38	752.80	13.14	0.00
Total						13.14	722.43
C_2	F	1.04	1	186.85	193.36	0.00	4.60
	G	1.00	3	135.32	135.32	0.00	0.00
	H	1.00	3	209.68	209.68	0.00	0.00
	I	1.01	2	226.40	228.70	0.00	1.62
	J	1.00	3	138.17	138.17	0.00	0.00
	K	0.91	6	197.05	179.53	12.39	0.00
	L	0.84	7	214.40	180.13	24.23	0.00
Total						36.62	6.22

Note1: $(V_1^*, V_2^*, U_1^*, U_2^*, U_3^*)$ = (1.00 Đ/person, 73.00 Đ/\$, 1.00 Đ/\$, 1.00 Đ/\$, 1.00 Đ/\$)

Note2: $(V_1^*, V_2^*, U_1^*, U_2^*, U_3^*)$ = (146.48 Đ/person, 17.34 Đ/\$, 9.69 Đ/\$, 1.00 Đ/\$, 1.00 Đ/\$)

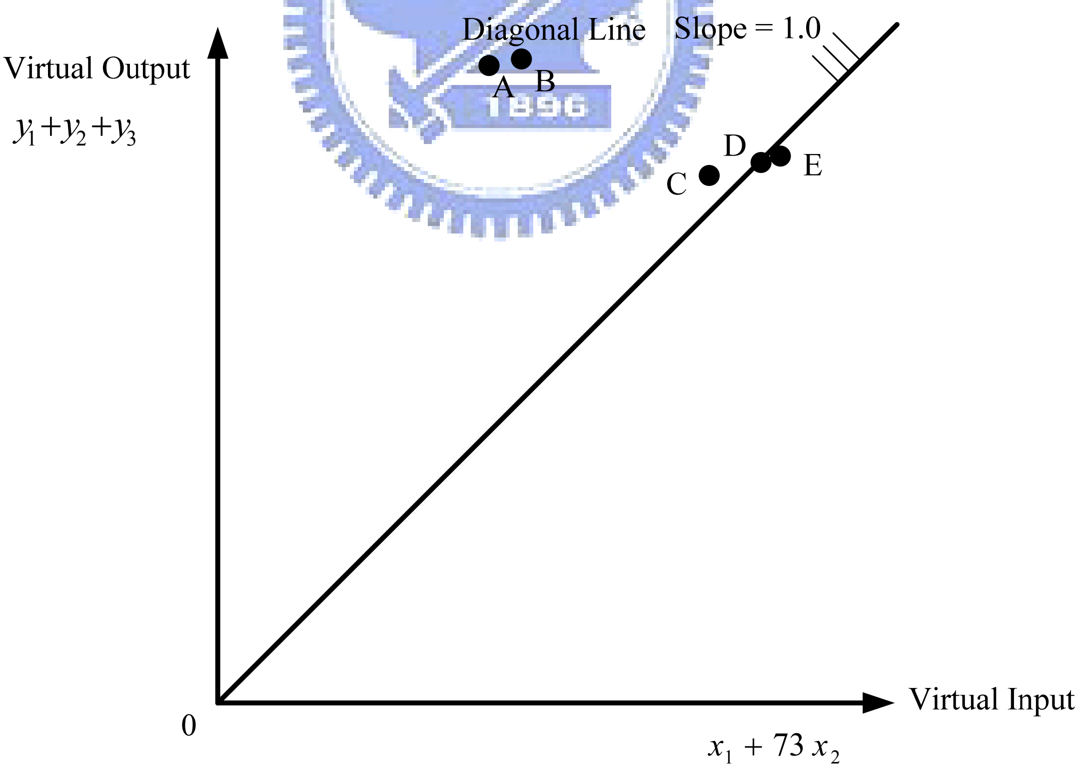

Figure 7. The expression to branches in subgroup C_1

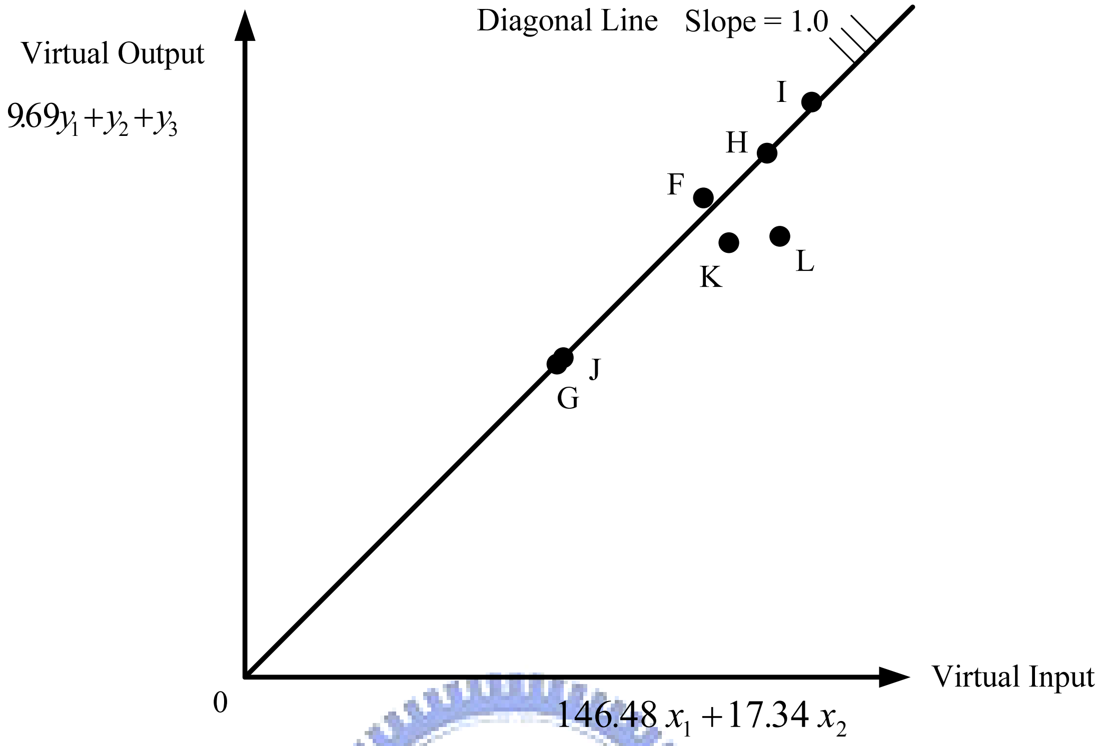

Figure 8. The expression to branches in subgroup C_2

5. Conclusion and Discussion

In this research, we develop two procedures to determine the common weights relative to the performance indices across all units of organization. The first one is to determine the common weights by searching the benchmark unit in the organization. One virtual benchmark is defined as units with efficiency score 1.0 and all units are asked to approach the virtual benchmark as close as possible. The units with zero gaps to the virtual benchmark are the real benchmark. In the structure of data envelopment analysis, the determination of common weights in this research means that the organization determines the favorable weight to maximize the organization efficiency. The obtained common weights can assist the organization managers in generating the individual efficiency score for all units and the corresponding ranking problem can be addressed by comparing with the scores. However, in the first procedure, sometimes there is existing some units with the equivalent efficiency score 1.0 due to the constraint that none of DMUs' efficiency scores is allowed to exceed 1.0. It possibly leads to the obstruction of efficiency development. In order to avoid the bias in measurement due to the upper bound of efficiency, we develop the second procedure to determine one compromise common set of weights by eliminating the restriction with upper bound 1.0 in efficiency score. It leads to the more complete ranking without the repeatable efficiency scores.

Several interesting subjects for the further development of this research are discussed. Besides the scenarios of benchmark chasing and neutral compromise, risk avoidance owns the highest potential for the management. Risk avoidance focuses the prediction of possible and potential UOAs with the worst performance and provides the improvement plan in advance. The excellent risk avoidance always saves a possible significant lost for organizations.

In this research, the common set of weights is applied to all UOAs under different scenarios and the performance indices are assumed given. Some methods for the selection of performance indices will help this research to possess reliable assessment outcomes. Statistical approaches and other methods such as analytic hieratical process (AHP) [37] and analytic network process (ANP) [38] would help to select appropriate combinations.

We used non-negative data for the numerical examples of the procedures proposed. One should examine the applicability of the proposed procedures to the other data types, such as negative data, probabilistic data, fuzzy data, ordinal data, and interval data in determining the common set of weights.

In numerical example 3, Table 14 shows that five interval limitations were set for both the proportional virtual output and proportional virtual input, respectively. In our particular numerical example, we observed that the rankings of the UOAs possess the robustness under the considerable amount of combinations. In fact, how to determine the amount of interval limitations for obtaining the ranking robustness is a critical issue. One would observe the interaction between the setting of boundary intervals and the rankings by observing more and more combinations. Generally, the rule for setting the interval limitations is straightforward; the lower bounds are in increasing order while the upper bounds are in decreasing order.

Section 4 provides the analysis similar to Least Square Method (LSM). While there are existing multiple dependent variables, this model may provide the corresponding analysis. The slope of the diagonal line is also another issue. A non-linear programming model can search for the optimal slope so that the total gap is further minimized.

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