國立交通大學

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碩 士 論 文

AKNS 系統中兩種 Darboux 變換的等效性

The Equivalence between
Two Kinds of Darboux Transformations
in AKNS Systems

研 究 生：楊川和

指導教授：邵錦昌 教授

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學生：楊川和　　　　　　　　　　　　　　　　　　　　指導教授：邵錦昌

國立交通大學應用數學系（研究所）碩士班

摘　　要

非線性偏微分方程的求解通常是一個重要而困難的問題，但對某些非線性方程（例如，k-dV 方程和 mK-dV 方程…等，）我們已有多種顯式求解的方法了。常見的有反散射變換、Bäcklund 變換和 Darboux 變換。在本文中，我們將集中介紹後兩種方法：Bäcklund 變換和 Darboux 變換；這兩者都是可由已知解出發，利用 AKNS 系統來討論，從而給出新解的方法。我們將寫下這兩種變換的顯式表達式，並以此來證明它們其實是等價的。最後，我們也會舉例來說明如何利用這變換來給出更多我們所考慮的非線性方程的更多解。

The Equivalence between Two Kinds of Darboux Transformations in AKNS Systems

Student : Chuan-Her Yang Advisor : Dr. Jiin-Chang Shaw

Department (Institute) of Applied Mathematics

National Chiao Tung University

Abstract

To find solutions of nonlinear partial differential equations is usually a important and difficult problem, but for some kinds of nonlinear equation (for example, K-dV equation and mK-dV equation etc.), we have haven some method to find out their explicit solutions. These familiar methods which we often use involves the inverse scattering transformation, Bäcklund transformation and Darboux transformation. In this paper, we will focus our discussion on the last two method: explicit Bäcklund transformation and Darboux transformation. These two method are both that starting from a known solution of this nonlinear equation, then applies the AKNS system to derive a new solution. We shall write their explicit expressions down, and then prove that these two transformation are equivalence. Finally, we also illustrate how to use the transformation to find more solutions of the considered nonlinear equation.

誌　　謝

這篇論文的完成，首先要感謝我的指導老師 邵錦昌教授。在我就讀研究所的期間，老師不時指引著我學習及研究的方向，在我遇到困難時，更會適時的點出解決問題的關鍵所在，並給我鼓勵。在老師的指導下，帶領我初窺了物理數學的思考領域中，及一些所需的基礎知識，最重要的是能領悟到了做學問的方法，及找到自己真正的興趣。雖然，愚鈍的自己在做研究這件工作上總不免有些跌跌撞撞的感覺，但在老師的帶領之下，還是可以真正的順利完成這一項對我來說最重要的事。在此，再對邵老師的指導表達學生由衷的感謝。

另外，感謝在應數系上於大學及研究所階段指導過我的各位老師，由於你們的細心教導，讓我從數學領域裡的門外漢，到現在已經能夠完成這篇論文。在口試中，也感謝戈正銘老師、李榮耀老師及張仁煦老師的指導，點出了論文中的一些缺失，以及未來可以繼續探討的課題，讓我對這個研究領域有了更深刻的體會及興趣。

而在這段研究所的生活中，也應該要感謝研究室的各位夥伴，從剛進入研究所時的曾孝捷、陳偉國、涂芳婷和張靖尉，以及現在要畢業前的許倖綺、李雅羚、黃園芳、程千砡及黃碧維。除了課業上的討論問題之外，空暇時也會一起談天說地，讓我能夠增廣見聞，並消除了生活上的緊張及壓力；而就在互相鼓勵、尋找共識之下，一起度過這段充實的歲月，這段美好的回憶我必珍而重之。此外，也要特別感謝一些在這段時間裡幫助過我的學長姊及學弟們，諸如吳恭儉學長、廖康伶學姊、陳冠羽學弟…等等。如果沒有各位的幫忙及指點，我不可能很快地就把學業或論文整理好，並順利考上博士班，感謝各位，未來的時間也還請多多指導。

而回想起來，於剛進研究所時對自己有的的一些期望，在現在來說，有些確實做到了，但仍然有一些事還是沒有能完成，仍是先慢著、留著，而所追求的，依然在水一方。期許在接下來的難走之路上，對於這些舊的期待，及新的目標都能夠一一的落實完成，早日找到自己的天。

最後，我一定要感謝我的家人給我的支持及關心，讓我能沒有後顧之憂的進行我的學業。他們對我的付出，使我能順利地完成碩士班的課業及論文。在接下來的博士班階段，希望自己的學識及能力，繼續不斷的累積及進步，早日有所成就，不再讓他們擔心。

此時，想說的就再套用一句話來說：要感謝的人實在太多了，真要謝的話，就感謝天吧！

謹將這篇論文獻給曾經幫助過我的各位！

目　　　錄

1 Introduction

1.1 The Introduction of Solitary Waves and KdV Equation

Solitons are a special kind of an essential nonlinear wave.

A "soliton" is not precise defined, but is used to describe any solution of a nonlinear equation or system which (i) represents a wave of permanent form; (ii) is localized, decaying or becoming constant at infinity; (iii) may interact strongly with other solitons so that after the interaction it retains its form, almost as if the principle of superposition were valid. The word "soliton" was coined by Zabusky and Kruskal (1965) after "photon", "proton", etc. to emphasize that a soliton is a localized entity which may keep its identity after an interaction (see Fig. 1).

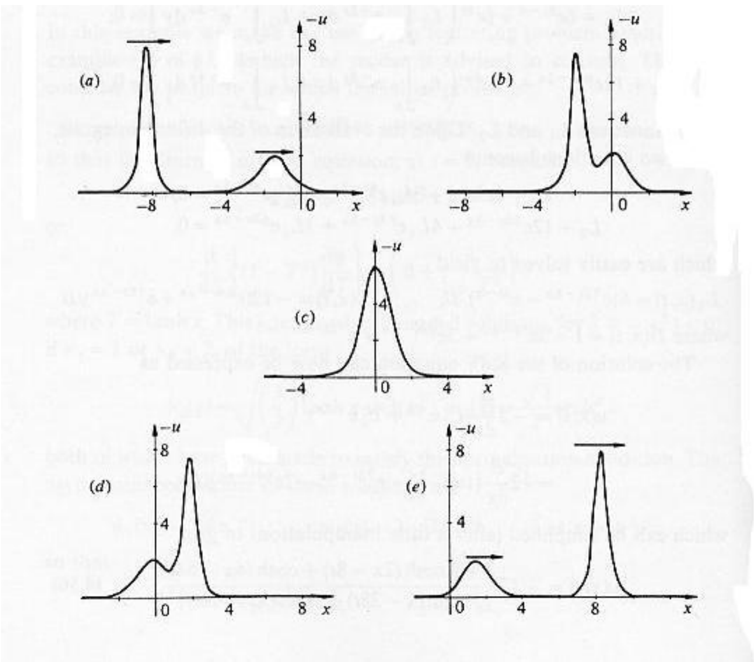

Figure 1: The interaction of two solitary wave

A solitary wave is the first and most celebrated example of a soliton to have been discovered. To realize the definition of a soliton, it is useful to study solitary waves on shallow water. Let us begin at the begging, and relate a little history.

The solitary wave was first observed on the Edinburgh to Glasgow canal in 1834 by J. Scott Russell. He also did some laboratory experiments, generating solitary waves by dropping a weight at one end of a water channel. He deduced empirically that the volume of water in the wave is equal to the volume displaced by the weight and that the steady velocity c of the wave is given by $c^2 = g(h + A)$, where A is the amplitude of the wave and h is the height of the undisturbed water (see Fig. 2). Note that a taller solitary wave travels faster than a smaller one (see Fig. 1).

Boussinesq (1871) and Rayleigh (1876) independently showed essentially that the water wave height ζ about the mean level h is given by

$$\zeta(x,t) = A\ sech^2\ \frac{x - ct}{b},$$

where $b^2 = \dfrac{4h^2(h + A)}{3A}$ for any positive amplitude A.

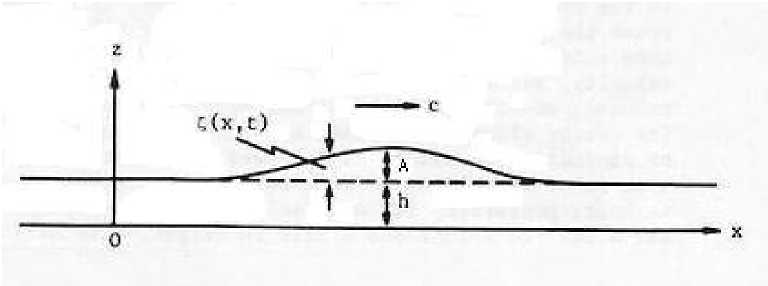

Figure 2: solitary wave

In 1895 Korteweg and de Vries developed this theory, and found an equation governing the one-dimensional motion of nonlinear long waves:

$$\frac{\partial \zeta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}(\zeta\frac{\partial \zeta}{\partial x} + \frac{3}{2}\alpha\frac{\partial \zeta}{\partial x} + \frac{1}{3}\sigma\frac{\partial^3 \zeta}{\partial x^3}),$$

where α is a small constant, $\sigma = \frac{1}{3}h^3 - \frac{Th}{g\rho}$, and T is the surface tension of liquid of density ρ. This is essentially the original form of the Korteweg-dv Vries equation; we shall call it the KdV equation.

Note that by translations and magnifications of the dependent and independent variables,

$$u = k_1\zeta + k_0, \quad X = k_3x + k_2, \quad T = k_4t + k_5,$$

we can write the KdV equation in many equivalent forms by choice of the constants k_0 to k_5. For example:

$$\frac{\partial u}{\partial T} + (1+u)\frac{\partial u}{\partial X} + \frac{\partial^3 u}{\partial X^3} = 0.$$

We can transform the above equation under

$$1 + u \to \alpha u, \quad T \to \beta t, \quad X \to \gamma x,$$

where α, β and γ are real (non-zero) constants, to yield

$$u_t + \frac{\alpha\beta}{\gamma}uu_x + \frac{\beta}{\gamma^3}u_{xxx} = 0.$$

This is a general form of the KdV equation, and a convenient choice, which we shall often use, is

$$u_t - 6uu_x + u_{xxx} = 0. \tag{1.1}$$

We now briefly discuss the solitary-wave solution of the KdV equation. To solve it, first seek wave of permanent shape and size by trying the travelling-wave solutions of this equation such that $u(x,t) = f(\xi)$, where $\xi = x - ct$ for some function f and constant wave velocity c. Thus the equation (1.1) becomes

$$-cf' - 6ff' + f''' = 0,$$

which may be integrated once to yield

$$\frac{d\frac{1}{2}(f')^2}{df} = f'' = 3f^2 + cf + A,$$

where A is a constant of integration. If we use f' as an integrating factor, we may integrate once more to get

$$\frac{1}{2}(f')^2 = f^3 + \frac{1}{2}cf^2 + Af + B,$$

where B is a second constant of integration.

If we want to seek a solitary wave, we may add the boundary conditions f, f', $f'' \to 0$ as $\xi \to \pm\infty$. Thus A and B are both zero,

$$(f')^2 = f^2 \cdot (2f + c),$$

and we can see immediately that a real solution exists only if $(f')^2 \geq 0$ i.e. if $2f + c \geq 0$.

The above equation can be integrated as follows:

$$\pm \int d\xi = \int \frac{df}{f'} = \int \frac{df}{f\sqrt{2f+c}},$$

Then

$$\begin{aligned}
\xi - \xi_0 &= \pm \int \frac{df}{f\sqrt{2f+c}} = \pm \frac{1}{\sqrt{c}} \ln\left|\frac{\sqrt{2f+c} - \sqrt{c}}{\sqrt{2f+c} + \sqrt{c}}\right| \\
&= \mp \frac{2}{\sqrt{c}} \frac{1}{2} \ln\left|\frac{1 + \frac{\sqrt{2f+c}}{\sqrt{c}}}{1 - \frac{\sqrt{2f+c}}{\sqrt{c}}}\right| = \mp \frac{2}{\sqrt{c}} \tanh^{-1} \frac{\sqrt{2f+c}}{\sqrt{c}},
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{2f+c}}{\sqrt{c}} &= \mp \tanh(\frac{\sqrt{c}}{2}(\xi - \xi_1)), \\
2f + c &= c \cdot \tanh^2(\frac{\sqrt{c}}{2}(\xi - \xi_1)), \\
f &= \frac{c}{2}(\tanh^2(\frac{\sqrt{c}}{2}(\xi - \xi_1)) - 1).
\end{aligned}$$

Hence, we shall obtain

$$u(x,t) = f(\xi) = -\frac{1}{2}c \operatorname{sech}^2[\frac{1}{2}\sqrt{c}(x - ct - x_0)], \tag{1.2}$$

where x_0 is an arbitrary constant of integration. The solitary-wave solution (1.2) of equation (1.1) forms a one-parameter family (ignoring x_0), and in fact the solution exists for all $c \geq 0$ no matter how large or small the wave may be.

The most important thing that we usually need to do is the solution of the general initial-value problem for the KdV equation. That is, finding the solution $u(x,t)$ of

$$u_t - 6uu_x + u_{xxx} = 0$$

for all $t > 0$ and $-\infty < x < \infty$, where

$$u(x,0) = g(x)$$

for a given function g.

It can be proved that the method of finding the solution $u(x,t)$ will require a connection to a scattering problem, in fact the classical scattering problem of quantum mechanics. this idea is usually called inverse scattering transform (IST). (ref [1, 2, 6]) The method of inverse scattering or the inverse scattering transform will be used to solve the initial-value problem. (see Fig. 4)

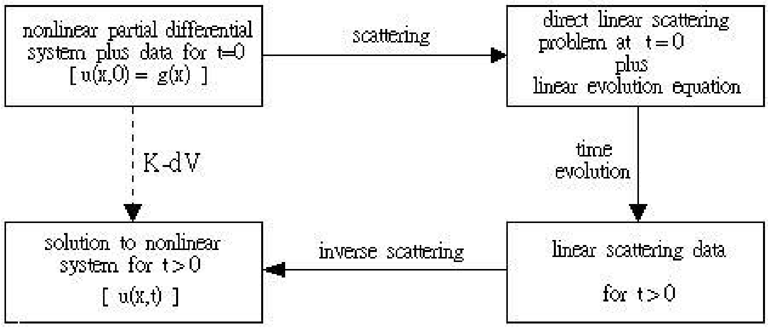

Figure 3: Flow diagram of the method of inverse scattering.

And it will be able to be explained by a deeper and more general argument due to Lax (1968). This abstract argument (the Lex method) will show that the method may be applied to many, through not most, nonlinear initial-value problem. Now, We simply describe the Lax theory as follows.

If the evolution equation

$$u_t = S(u), \tag{1.3}$$

where S is a nonlinear operator which is independent of t, can be expressed as the Lax equation

$$L_t = BL - LB, \tag{1.4}$$

where L and B are some linear operators in x and may depend on $u(x,t)$,(By L_t, we mean the derivative w.r.t. the parameter t as it appears explicitly in the operator; for example, if $L = -\frac{\partial^2}{\partial x^2} + u(x,t)$, then $L_t = u_t$.) and if

$$L\psi = \lambda\psi, \tag{1.5}$$

then $\lambda_t = 0$ and ψ evolves according to

$$\psi_t = B\psi.$$

For example, let us suppose that

$$L = -\frac{\partial^2}{\partial x^2} + u \quad \text{and} \quad B = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x.$$

Therefore, $L_t = BL - LB$ if and only if $u_t = -u_{xxx} + 6uu_x$. Hence, we can find that the K-dV equation

$$u_t + u_{xxx} - 6uu_x = 0 \tag{1.6}$$

is the integrability condition of (1.5).

The integrability condition means: consider the linear equation system

$$\begin{cases} -\psi_{xx} + u\psi = \lambda\psi, \\ \psi_t = -4\psi_{xxx} + 6u\psi_x + 3u_x\psi \end{cases} \tag{1.7}$$

(which is the Lax pair of K-dV equation), where u and ψ are both functions of x and t. For the first equation of (1.7), we derive the $\psi_{xx} = (u-\lambda)\psi$ and then compute the $(\psi_{xx})_t$. And, for the second equation of (1.7), we compute the $(\psi_t)_{xx}$. Then the sufficient and necessary condition of $(\psi_{xx})_t = (\psi_t)_{xx}$ is that u satisfies the K-dV equation (1.6).

It can be seen that we get an eigenvalue problem (1.5) for any nonlinear equation which can be put into the form (1.4) although deriving the form (1.4) from the form (1.3) is not always easy. Hence the scattering and inverse scattering theories appropriate to the eigenvalue problem (1.5) may be used to solve an initial-value problem for the nonlinear system (1.3).

1.2 AKNS System and Its Integrability Condition

In order to generalize the KdV equation to more cases of nonlinear partial equation, V.E. Zakharov, A.B. Shabat, M.J. and M.J. Ablowwitz, D.J. Kaup, A.C. Newell, H. Segur introduce to a more general linear equation pair which in general we call it by AKNS system.

If we want to consider a partial differential equation (system)

$$F(u, u_x, u_t, u_{xx}, \cdots) = 0, \tag{1.8}$$

we shall be able to apply the AKNS form :

$$\begin{aligned} \Phi_x &= U\Phi \;\; [= (-i\lambda J + Q)\Phi] \\ \Phi_t &= V\Phi \;\; [= \sum_{j=0}^{n} V_j \lambda^{n-j}\Phi]. \end{aligned} \tag{1.9}$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \text{ and } V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}.$$

where $\Phi = \Phi(x, t; \lambda)$ is a 2-column vector (or 2×2 matrix) and λ is a complex parameter which we call it by spectrum parameter in general. q and r are λ-independent functions of u and its derivatives (hence also are functions of x, and t).

And the entries of matrix V have the below form

$$\begin{aligned} A &= \sum_{j=0}^{n} a_j(x,t)\lambda^{n-j}, \\ B &= \sum_{j=0}^{n} b_j(x,t)\lambda^{n-j}, \\ \text{and} \quad C &= \sum_{j=0}^{n} c_j(x,t)\lambda^{n-j}, \end{aligned}$$

where a_j, b_j and c_j are some real or complex functions of x, t.

The integrability condition of (1.9) means that when Q is a suitable differential polynomial of u, then (1.8) is exactly the sufficient and necessary condition which the equation $\Phi_{xt} = \Phi_{tx}$ holds for all different powers in λ. It is easy to check that the integrable condition is

$$U_t - V_x + [U, V] = 0, \quad \text{where} \quad [U, V] = UV - VU. \tag{1.10}$$

Thus, we can find that the integrability condition (1.10) of the equation system (1.9) requires

$$\begin{aligned} A_x &= qC - rB, \\ B_x &= q_t - 2i\lambda B - 2qA, \\ \text{and} \quad C_x &= r_t + 2i\lambda C + 2rA \end{aligned} \tag{1.11}$$

which shall become the nonlinear equation (1.8) when we choose the suitable q, r, A, B and C.

Hence, we can discuss the nonlinear partial differential equations of u (equation (1.8)) through the AKNS system (1.9). This class of equations includes the following :

(i) K-dV equation:

$$u_t + u_{xxx} \mp 6uu_x = 0,$$

with

$$q = u, \quad r = \pm 1,$$

$$A = \pm u_x \mp 2i\lambda u - 4i\lambda^3, \quad B = -u_{xx} \pm 2u^2 + 2i\lambda u_x + 4\lambda^2 u, \quad C = 2u \pm 4\lambda^2.$$

(ii) mK-dV equation:

$$u_t + u_{xxx} \mp 6u^2 u_x = 0,$$

with

$$q = u, \quad r = \pm u,$$

$$A = \mp 2i\lambda u^2 - 4i\lambda^3, B = -u_{xx} \pm 2u^3 + 2i\lambda u_x + 4\lambda^2 u, C = \mp u_{xx} + 2u^3 \mp 2i\lambda u_x \pm 4\lambda^2 u.$$

(iii) nonlinear Schrödinger equation:

$$iu_t + u_{xx} \mp 2|u|^2 u = 0,$$

with

$$q = u, \quad r = \pm u^*,$$

$$A = \mp i|u|^2 - 2i\lambda^2, \quad B = iu_x + 2\lambda u, \quad C = \mp iu_x^* \pm 2\lambda u^*.$$

(iv) sine-Gordon equation:

$$u_{xt} = \sin u,$$

with

$$q = -r = \frac{1}{2}u_x, \quad A = \frac{i}{4\lambda}\cos u, \quad B = C = \frac{-i}{4\lambda}\sin u.$$

(v) Liouville equation:

$$u_{xt} = 2e^u,$$

with

$$q = r = \frac{1}{2}u_x, \quad A = -B = C = \frac{i}{2\lambda}e^u.$$

For example, consider the K-dV equation

$$u_t + u_{xxx} \mp 6uu_x = 0$$

and the mK-dV equation

$$u_t + u_{xxx} \mp 6u^2u_x = 0$$

by applying the AKNS system

$$\Phi_x = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}\Phi, \text{ and } \Phi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\Phi.$$

If we choose that A, B and C as follows

$$\begin{aligned} A &= rq_x - qr_x - 2i\lambda qr - 4i\lambda^3 \\ B &= -q_{xx} + 2rq^2 + 2i\lambda q_x + 4\lambda^2 q \\ \text{and} \quad C &= -r_{xx} + 2qr^2 - 2i\lambda r_x + 4\lambda^2 r \end{aligned} \tag{1.12}$$

Now, applying the integrability condition of AKNS system (i.e. (1.11))

$$A_x = qC - rB, \quad B_x = q_t - 2i\lambda B - 2qA, \quad \text{and} \quad C_x = r_t + 2i\lambda C + 2rA,$$

we shall find that the integrable condition for all order of λ is

$$q_t + q_{xxx} - 6rqq_x = 0, \quad \text{and} \quad r_t + r_{xxx} - 6qrr_x = 0. \tag{1.13}$$

Suppose we choose $r = \pm 1$ again, the above equations shall be exactly the K-dV equation

$$q_t + q_{xxx} \mp 6qq_x = 0.$$

Note that if we choose $r = \pm q$, (1.13) becomes the mK-dV equation

$$q_t + q_{xxx} \mp 6q^2q_x = 0.$$

2 Bäcklund Transformations

In this section, we will introduce another method to find a solution of a nonlinear partial differential equation : the Bäcklund transformation. If we have a solution of a nonlinear differential equation, we shall derivative a integrable partial differential equation system. And we can get a new solution from the differential equation system.

The Bäcklund transformations were devised in the 1880s for use in the theories of differential geometry and of differential equations. They arose as a generalization of contact transformations.

2.1 Introductory Ideas

A Bäcklund transformation is essentially defined as a pair of partial differential relations involving two independent variables and their derivatives which together imply that each one of the dependent variables satisfies separately a partial differential equations. Thus, for example, the transformation

$$R_1(u, v, u_x, u_y, \ldots; x, y) = 0 \quad \text{and} \quad R_2(u, v, u_x, u_y, \ldots; x, y) = 0 \tag{2.14}$$

would imply that two functions u and v satisfy partial differential equations of the operational form,

$$P(u) = 0 \quad \text{and} \quad Q(v) = 0 \tag{2.15}$$

where P and Q are two operators which are in general nonlinear. Then $R_i = 0$ is a Bäcklund transformation if it is integrable for v when $P(u) = 0$ and if the resulting v is a solution of $Q(v) = 0$, and vice versa. Of course, this approach to the solution of the equations $P(u) = 0$ and $Q(v) = 0$ is normally only useful if the relations $R_i = 0$ are, in some sense, simpler than the original equations (2.15)

One of the simplest Bäcklund transformations is the pair (written with y rather than t)

$$u_x = v_y \quad , \quad u_y = -v_x,$$

the Cauchy-Riemann relations for Laplace's equation

$$u_{xx} + u_{yy} = 0 \quad ; \quad v_{xx} + v_{yy} = 0.$$

Thus, if $v(x, y) = xy$ (a simple solution of Laplace's equation), then $u(x, y)$ can be determined form

$$u_x = x \quad \text{and} \quad u_y = -y,$$

and so $u(x, y) = \frac{1}{2}(x^2 - y^2)$ is another solution of Laplace's equation.

Another simple example is the Liouville's equation,

$$u_{xt} = e^u. \tag{2.16}$$

First, we introduce a auxiliary variable, v, which satisfies

$$v_{xt} = 0. \tag{2.17}$$

Now, if we consider the pair of first-order equations

$$u_x + v_x = \sqrt{2}e^{(u-v)/2} \quad \text{and} \quad u_t - v_t = \sqrt{2}e^{(u+v)/2}, \tag{2.18}$$

then we can cross-differentiate to obtain

$$u_{xt} + v_{xt} = e^u \quad \text{and} \quad u_{tx} - v_{tx} = e^u. \tag{2.19}$$

It is immediately clear that the two equations (2.19) imply equations (2.16) and (2.17); thus the pair of equations (2.18) constitute a Bäcklund transformation for Liouville's equation and the equation $v_{xt} = 0$. Since this latter equation is easily solved and so, from the Bäcklund transformation (2.18), we shall be able to generate the general solutions of Liouville's equation (cf. [1], p.109-110).

2.2 Bäcklund Transformation for KdV Equation

Next, we shall introduce the Bäcklund transformation for the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.20}$$

in following.

There is a more convenient transformation was developed by Wahlquist and Estabrook (1973), which we shall now describe. We first transform the dependent variable of the KdV equation and then use a Bäcklund transformation. So we define a new dependent variable w by

$$w_x = u \tag{2.21}$$

and the operator Q by $Q(w) = w_t - 3w_x^2 + w_{xxx}$. It follows that

$$[Q(w)]_x = u_t - 6uu_x + u_{xxx}.$$

Hence we find that if u satisfies the KdV equation (2.20) then

$$Q(w) = 0 \ \ \text{(which is called the potential K-dV equation)}. \tag{2.22}$$

Now consider the Bäcklund transformation,

$$w_x^{(0)} + w_x^{(1)} = 2\lambda + \frac{1}{2}(w^{(0)} - w^{(1)})^2 \tag{2.23}$$

and

$$w_t^{(0)} + w_t^{(1)} = -(w^{(0)} - w^{(1)})(w_{xx}^{(0)} - w_{xx}^{(1)}) + 2(w_x^{(0)2} + w_x^{(0)}w_x^{(1)} + w_x^{(1)2}), \tag{2.24}$$

where $w^{(0)}$ and $w^{(1)}$ correspond to $u^{(0)}$ and $u^{(1)}$ respectively, i.e. where $w_x^{(0)} = u^{(0)}$ and $w_x^{(1)} = u^{(1)}$, and λ is a real parameter. On assuming the transformation, we may take $\dfrac{\partial^2(2.23)}{\partial x^2} + (2.24)$ to deduce that

$$Q(w^{(0)}) + Q(w^{(1)}) = 0,$$

and may take $\dfrac{\partial(2.24)}{\partial x} - \dfrac{\partial(2.23)}{\partial t}$ together with $\dfrac{\partial(2.23)}{\partial x}$ to deduce that

$$(w^{(0)} - w^{(1)})[Q(w^{(0)}) - Q(w^{(1)})] = 0.$$

Therefore equations (2.23) and (2.24) shall imply that $Q(w^{(0)}) = 0$ and $Q(w^{(1)}) = 0$. i.e. $w^{(0)}$ and $w^{(1)}$ each satisfy equation (2.22) and thence that $u^{(0)}$ and $u^{(1)}$ each satisfy the KdV equation (2.20).

To give an example of the use of equations (2.23) and (2.24), we shall be able to start from the trivial solution $u^{(0)}(x,t) = 0$ of the KdV equation for all x and t. This corresponds to the solution $w^{(0)}(x,t) = 0$ of equation (2.22). Then the transformation gives

$$w_x^{(1)} = 2\lambda + \frac{1}{2}w^{(1)2}$$

and

$$w_t^{(1)} = -w^{(1)}w_{xx}^{(1)} + 2w_x^{(1)2}.$$

The first of these equations may be integrated directly to yield

$$w^{(1)}(x,t) = -2\kappa \tanh[\kappa x - f(t)] \tag{2.25}$$

where $\lambda = -\kappa^2(< 0)$ and f is an arbitrary function. From the first equation, we can also find $w_{xx}^{(1)} = w^{(1)}w_x^{(1)}$. Therefore the second equation of the transformation gives

$$w_t^{(1)} = -w^{(1)}w_{xx}^{(1)} + 2w_x^{(1)2} = 2w_x^{(1)}(w_x^{(1)} - \frac{1}{2}w^{(1)2}) = 4\lambda w_x^{(1)}.$$

Therefore

$$f'(t) = -4\lambda\kappa = 4\kappa^3,$$

for consistency with equation (2.25), we have

$$f(t) = 4\kappa^3 t + \kappa x_0,$$

where x_0 is an arbitrary constant. Thus the Bäcklund transformation yields the solution

$$w^{(1)}(x,t) = -2\kappa \tanh[\kappa(x - x_0 - 4\kappa^2 t)] \tag{2.26}$$

and so, from equation (2.21), we obtain

$$u^{(1)}(x,t) = -2\kappa^2 \text{sech}^2[\kappa(x - x_0 - 4\kappa^2 t)],$$

the solitary-wave solution of the KdV equation. (Note that solution (2.25) is valid if $|w^{(1)}| < 2\kappa$, but if $|w^{(1)}| > 2\kappa$, then

$$w^{(1)}(x,t) = -2\kappa \coth[\kappa(x - x_0 - 4\kappa^3 t)],$$

a singular solution.)

2.3 Soliton-generating Bäcklund Transformation for Some Nonlinear Equations

When we consider a nonlinear partial differential equation, in general, the Bäcklund transformation pair $R_1(u, v, u_x, u_y, \ldots; x, y)$ and $R_2(u, v, u_x, u_y, \ldots; x, y)$ (in equation system (2.14)) is not easy to find. In this section we will use a more general method to derive a soliton-generating Bäcklund transformation for a class of nonlinear equations associated with the AKNS system which we have introduced in Sec 1.2 :

$$\Phi_x = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix} \Phi, \qquad \Phi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi, \tag{2.27}$$

where $\Phi = \Phi(x,t;\lambda)$ is a 2×2 matrix here. Now, We will use the AKNS system to begin our discussion.

To find a soliton-generating Bäcklund transformation for each of the above nonlinear equations, we will look for a pair of matrix functions $\varphi(x,t;\lambda)$ and $\psi(x,t;\lambda)$ such that $\varphi\psi = I$. We may assume [3]

$$\varphi(x,t;\lambda) = J[I - \frac{\lambda_1-\lambda_2}{\lambda-\lambda_2}P(x,t)], \ \ \psi(x,t;\lambda) = [I + \frac{\lambda_1-\lambda_2}{\lambda-\lambda_1}P(x,t)]J. \tag{2.28}$$

where λ_1 and λ_2 are two distinct complex numbers, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and P is an undtetermined 2×2 projection matrix ($P^2 = P$).

Suppose $u^{(0)}$ and $u^{(1)}$ are two solutions of the nonlinear equations under consideration. Let $\Phi^{(1)}$ is a new solution, and $\Phi^{(0)}$ is a solution of (2.27) which we have known, they satisfy $\Phi^{(0)} = \varphi\Phi^{(1)}$, where $\Phi^{(1)}$ satisfies the equation system $\Phi^{(1)}_x = \begin{pmatrix} -i\lambda & q^{(1)} \\ r^{(1)} & i\lambda \end{pmatrix}\Phi^{(1)}$ and $\Phi^{(1)}_t = \begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & -A^{(1)} \end{pmatrix}\Phi^{(1)}$. Therefore, φ will satisfy

$$\varphi_x = \begin{pmatrix} -i\lambda & q^{(0)} \\ r^{(0)} & i\lambda \end{pmatrix}\varphi - \varphi\begin{pmatrix} -i\lambda & q^{(1)} \\ r^{(1)} & i\lambda \end{pmatrix}, \tag{2.29}$$

$$\varphi_t = \begin{pmatrix} A^{(0)} & B^{(0)} \\ C^{(0)} & -A^{(0)} \end{pmatrix}\varphi - \varphi\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & -A^{(1)} \end{pmatrix}, \tag{2.30}$$

where $q^{(1)}$, $r^{(1)}$, $A^{(1)}$, $B^{(1)}$, $C^{(1)}$ are obtained from $q^{(0)}$, $r^{(0)}$, $A^{(0)}$, $B^{(0)}$, $C^{(0)}$ respectively by replacing $u^{(0)}$ with $u^{(1)}$. Now, we shall use equations (2.28), (2.29) and (2.30) to derive the Bäcklund transformation equations.

First, substituting (2.28) into (2.29), we shall have

$$-J\frac{\lambda_1-\lambda_2}{\lambda-\lambda_2}P_x = \begin{pmatrix} -i\lambda & q^{(0)} \\ r^{(0)} & i\lambda \end{pmatrix} J(I - \frac{\lambda_1-\lambda_2}{\lambda-\lambda_2}P) - J(I - \frac{\lambda_1-\lambda_2}{\lambda-\lambda_2}P)\begin{pmatrix} -i\lambda & q^{(1)} \\ r^{(1)} & i\lambda \end{pmatrix}.$$

Product $(\lambda-\lambda_2)$ on two sides, then

$$\begin{aligned} -J(\lambda_1-\lambda_2)P_x &= \begin{pmatrix} -i\lambda & q^{(0)} \\ r^{(0)} & i\lambda \end{pmatrix} J(\lambda-\lambda_2-(\lambda_1-\lambda_2)P) \\ &\quad -J(\lambda-\lambda_2-(\lambda_1-\lambda_2)P)\begin{pmatrix} -i\lambda & q^{(1)} \\ r^{(1)} & i\lambda \end{pmatrix}. \end{aligned}$$

Let us compare the coefficients of different powers in λ:

(i) For λ^1 :

$$\begin{aligned} 0 &= \begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix} J(-\lambda_2-(\lambda_1-\lambda_2)P) - J(-\lambda_2-(\lambda_1-\lambda_2)P)\begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & q^{(0)} \\ r^{(0)} & 0 \end{pmatrix} J\lambda - J\lambda\begin{pmatrix} 0 & q^{(1)} \\ r^{(1)} & 0 \end{pmatrix}. \end{aligned}$$

Cancel λ on two sides,

$$\begin{aligned} 0 &= i\lambda_2 + i(\lambda_1-\lambda_2)P \\ &\quad -J(i\lambda_2 + i(\lambda_1-\lambda_2)P)J + \begin{pmatrix} 0 & q^{(0)} \\ r^{(0)} & 0 \end{pmatrix} J - J \begin{pmatrix} 0 & q^{(1)} \\ r^{(1)} & 0 \end{pmatrix} \\ 0 &= i(\lambda_1-\lambda_2)(P-JPJ) + \begin{pmatrix} 0 & q^{(0)} \\ r^{(0)} & 0 \end{pmatrix} J - J \begin{pmatrix} 0 & q^{(1)} \\ r^{(1)} & 0 \end{pmatrix}. \end{aligned}$$

Product J on two sides, then we have the equation

$$0 = i(\lambda_1-\lambda_2)[J,P] - \begin{pmatrix} 0 & q^{(0)}+q^{(1)} \\ r^{(0)}+r^{(1)} & 0 \end{pmatrix}. \tag{2.31}$$

(ii) For λ^0:

$$\begin{aligned} J(\lambda_1-\lambda_2)P_x &= \begin{pmatrix} 0 & q^{(0)} \\ r^{(0)} & 0 \end{pmatrix} J(-\lambda_2-(\lambda_1-\lambda_2)P) \\ &\quad -J(-\lambda_2-(\lambda_1-\lambda_2)P) \begin{pmatrix} 0 & q^{(1)} \\ r^{(1)} & 0 \end{pmatrix}. \end{aligned}$$

Product J on two sides and apply $J^2 = I$, then we can get the other equation

$$\begin{aligned} -(\lambda_1-\lambda_2)P_x &= \begin{pmatrix} 0 & q^{(0)} \\ r^{(0)} & 0 \end{pmatrix} (\lambda_2+(\lambda_1-\lambda_2)P) \\ &\quad +(\lambda_2+(\lambda_1-\lambda_2)P) \begin{pmatrix} 0 & q^{(1)} \\ r^{(1)} & 0 \end{pmatrix}. \end{aligned} \tag{2.32}$$

Now we can assume the 2×2 projection matrix to be

$$P = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \quad \text{with} \quad a(1-a) = bc, \tag{2.33}$$

where a, b and c are to be determined in terms of $q^{(0)}$, $r^{(0)}$, $q^{(1)}$ and $r^{(1)}$. Hence the equation (2.31) becomes

$$\begin{pmatrix} 0 & q^{(0)}+q^{(1)} \\ r^{(0)}+r^{(1)} & 0 \end{pmatrix} = 2i(\lambda_1-\lambda_2) \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}. \tag{2.34}$$

Hence we know that

$$b = \frac{q^{(0)}+q^{(1)}}{2i(\lambda_1-\lambda_2)}, \quad c = \frac{r^{(0)}+r^{(1)}}{-2i(\lambda_1-\lambda_2)}, \tag{2.35}$$

and

$$\begin{aligned} a(1-a) &= bc \\ &= \frac{(q^{(0)}+q^{(1)})(r^{(0)}+r^{(1)})}{4(\lambda_1-\lambda_2)^2}. \end{aligned} \tag{2.36}$$

Since we can use equation (2.36) to solve a,

$$a = \frac{1}{2} \pm \frac{\sqrt{(\lambda_1-\lambda_2)^2 - (q^{(0)}+q^{(1)})(r^{(0)}+r^{(1)})}}{2(\lambda_1-\lambda_2)}. \tag{2.37}$$

Substituting (2.33) into (2.32), then we can get the equation:

$$
\begin{aligned}
-(\lambda_1-\lambda_2)\begin{pmatrix} a_x & b_x \\ c_x & -a_x \end{pmatrix} &= \begin{pmatrix} 0 & \lambda_2 q \\ \lambda_2 r & 0 \end{pmatrix} + (\lambda_1-\lambda_2)\begin{pmatrix} cq^{(0)} & q^{(0)}(1-a) \\ ar^{(0)} & br^{(0)} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & \lambda_2 q^{(1)} \\ \lambda_2 r^{(1)} & 0 \end{pmatrix} + (\lambda_1-\lambda_2)\begin{pmatrix} br^{(1)} & aq^{(1)} \\ (1-a)r^{(1)} & cq^{(1)} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \lambda_2(q^{(0)}+q^{(1)}) \\ \lambda_2(r^{(0)}+r^{(1)}) & 0 \end{pmatrix} \\
&\quad +(\lambda_1-\lambda_2)\begin{pmatrix} cq^{(0)}+br^{(1)} & q^{(0)}(1-a)+aq^{(1)} \\ ar^{(0)}+r^{(1)}(1-a) & br^{(0)}+cq^{(1)}) \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\begin{pmatrix} a_x & b_x \\ c_x & -a_x \end{pmatrix} &= \begin{pmatrix} 0 & -\frac{\lambda_2}{\lambda_1-\lambda_2}(q^{(0)}+q^{(1)}) \\ -\frac{\lambda_2}{\lambda_1-\lambda_2}(r^{(0)}+r^{(1)}) & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} -(cq^{(0)}+br^{(1)}) & a(q^{(0)}-q^{(1)})-q^{(0)} \\ -a(r^{(0)}-r^{(1)})-r^{(1)} & -(br^{(0)}+cq^{(1)}) \end{pmatrix} \\
&= \begin{pmatrix} -\frac{r^{(0)}+r^{(1)}}{-2i(\lambda_1-\lambda_2)}q^{(0)} - \frac{q^{(0)}+q^{(1)}}{2i(\lambda_1-\lambda_2)}r^{(1)} & -\frac{1}{2}\frac{2(\lambda_1 q^{(0)}+\lambda_2 q^{(1)})}{\lambda_1-\lambda_2} + a(q^{(0)}-q^{(1)}) \\ -\frac{1}{2}\frac{2(\lambda_2 r^{(0)}+\lambda_1 r^{(1)})}{\lambda_1-\lambda_2} - a(r^{(0)}-r^{(1)}) & -\frac{q^{(0)}+q^{(1)}}{2i(\lambda_1-\lambda_2)}r^{(0)} - \frac{r^{(0)}+r^{(1)}}{-2i(\lambda_1-\lambda_2)}q^{(1)} \end{pmatrix} \\
&= \begin{pmatrix} \frac{r^{(0)}q^{(0)}-q^{(1)}r^{(1)}}{2i(\lambda_1-\lambda_2)} & -i(\lambda_1+\lambda_2)b+(a-\frac{1}{2})(q^{(0)}-q^{(1)}) \\ i(\lambda_1+\lambda_2)c-(a-\frac{1}{2})(r^{(0)}-r^{(1)}) & -\frac{q^{(0)}r^{(0)}-r^{(1)}q^{(1)}}{2i(\lambda_1-\lambda_2)} \end{pmatrix}
\end{aligned}
$$

So, equation (2.32) can be then reduced to

$$
\begin{aligned}
b_x &= -i(\lambda_1+\lambda_2)b+(a-\frac{1}{2})(q^{(0)}-q^{(1)}), && (2.38)\\
c_x &= i(\lambda_1+\lambda_2)c-(a-\frac{1}{2})(r^{(0)}-r^{(1)}), && (2.39)\\
a_x &= \frac{q^{(0)}r^{(0)}-q^{(1)}r^{(1)}}{2i(\lambda_1+\lambda_2)}. && (2.40)
\end{aligned}
$$

Now we need to consider restrictions imposed on λ_1 and λ_2. If we want to get real solutions, we will choose the values of λ_1 and λ_2 as follows:

1. For K-dV, mK-dV, s-G and Liouville, to get soliton-type solutions, we have

$$\lambda_2 = -\lambda_1 \equiv \frac{i}{2}k \quad \text{(imaginary)}; \qquad (2.41)$$

2. For NLS, to get soliton-type solutions, we have

$$\lambda_2 = \overline{\lambda_1} \equiv \frac{1}{2}l + \frac{i}{2}k \quad \text{(complex)}. \qquad (2.42)$$

We now return to the problem of deriving the Bäcklund transformations. Taking the value of λ_1 and λ_2 in (2.41) and (2.42), and the relations r=const. , $r = \pm q$ or $r = \pm q^*$, we can finally reduce (2.38)-(2.40) to a single equation

$$(q^{(0)}+q^{(1)})_x + il(q^{(0)}+q^{(1)}) = \pm(q^{(0)}+q^{(1)})\sqrt{k^2+(q^{(0)}+q^{(1)})(r^{(0)}+r^{(1)})}, \qquad (2.43)$$

where

$$l = 2Re\lambda_2 \quad \text{and} \quad k = 2Im\lambda_2 \quad (l \neq 0 \text{ only for NLS}),$$

and the $\pm$ sign in (2.43) is chosen according to equation (2.37)

$$a - \frac{1}{2} = \pm\frac{1}{2k}\sqrt{k^2 + (q^{(0)} + q^{(1)})(r^{(0)} + r^{(1)})}. \tag{2.44}$$

So, the equation (2.43) shall be one half of the desired Bäcklund transformation equations. And, the other half shall come from the equation (2.30). That is,

$$\varphi_t = \begin{pmatrix} A^{(0)} & B^{(0)} \\ C^{(0)} & -A^{(0)} \end{pmatrix}\varphi - \varphi\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & -A^{(1)} \end{pmatrix}.$$

Let us substitute (2.28) and (2.33) into (2.30), then

$$\begin{aligned}
-\frac{\lambda_1 - \lambda_2}{\lambda - \lambda_2}J\begin{pmatrix} a_t & b_t \\ c_t & -a_t \end{pmatrix} &= \begin{pmatrix} A^{(0)} & B^{(0)} \\ C^{(0)} & -A^{(0)} \end{pmatrix}J[1 - \frac{\lambda_1 - \lambda_2}{\lambda - \lambda_2}\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}] \\
&\quad -J[1 - \frac{\lambda_1 - \lambda_2}{\lambda - \lambda_2}\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}]\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & -A^{(1)} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
-(\lambda_1 - \lambda_2)\begin{pmatrix} a_t & b_t \\ c_t & -a_t \end{pmatrix} &= J\begin{pmatrix} A^{(0)} & B^{(0)} \\ C^{(0)} & -A^{(0)} \end{pmatrix}J[\lambda - \lambda_2 - (\lambda_1 - \lambda_2)\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}] \\
&\quad -[\lambda - \lambda_2 - (\lambda_1 - \lambda_2)\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}]\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & -A^{(1)} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} a_t & b_t \\ c_t & -a_t \end{pmatrix} &= \begin{pmatrix} A^{(0)} & -B^{(0)} \\ -C^{(0)} & -A^{(0)} \end{pmatrix}[-\frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} + \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}] \\
&\quad +[\frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} - \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}]\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & -A^{(1)} \end{pmatrix} \\
&= -\frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2}[\begin{pmatrix} A^{(0)} & -B^{(0)} \\ -C^{(0)} & -A^{(0)} \end{pmatrix} - \begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & -A^{(1)} \end{pmatrix}] \\
&\quad +\begin{pmatrix} A^{(0)} & -B^{(0)} \\ -C^{(0)} & -A^{(0)} \end{pmatrix}\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \\
&\quad -\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & -A^{(1)} \end{pmatrix}.
\end{aligned}$$

Then use (2.41) and (2.42), we shall have

$$\begin{aligned}
b_t &= \frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2}(B^{(0)} + B^{(1)}) + (bA^{(0)} - (1-a)B^{(0)}) - (aB^{(1)} - bA^{(1)}) \\
&= \frac{\lambda - \frac{l}{2} - i\frac{k}{2}}{-ik}(B^{(0)} + B^{(1)}) + b(A^{(0)} + A^{(1)}) - B^{(0)} \\
&\quad +a(B^{(0)} - B^{(1)}) + \frac{1}{2}(B^{(0)} - B^{(1)}) - \frac{1}{2}(B^{(0)} - B^{(1)}) \\
&= i\frac{2\lambda - l}{2k}(B^{(0)} + B^{(1)}) + b(A^{(0)} + A^{(1)}) + (a - \frac{1}{2})(B^{(0)} - B^{(1)}),
\end{aligned} \tag{2.45}$$

$$\begin{aligned}
c_t &= \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}(C^{(0)}-C^{(1)})+(-aC^{(0)}-cA^{(0)})-(cA^{(1)}+(1-a)C^{(1)}) \\
&= \frac{\lambda-\frac{l}{2}-i\frac{k}{2}}{-ik}(C^{(0)}+C^{(1)})-c(A^{(0)}+A^{(1)})-C^{(0)}-a(C^{(0)}-C^{(1)}) \\
&\quad -\frac{1}{2}(C^{(0)}-C^{(1)})+\frac{1}{2}(C^{(0)}-C^{(1)}) \\
&= i\frac{2\lambda-l}{2k}(C^{(0)}+C^{(1)})-c(A^{(0)}+A^{(1)})-(a-\frac{1}{2})(C^{(0)}-C^{(1)}), \qquad (2.46)
\end{aligned}$$

and

$$a_t = -\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}(A^{(0)}-A^{(1)})+(aA^{(0)}-cB^{(0)})-(aA^{(1)}+bC^{(1)}), \qquad (2.47)$$

$$-a_t = -\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}(-A^{(0)}+A^{(1)})-(bC^{(0)}-(1-a)A^{(0)})-(cB^{(1)}-(1-a)A^{(1)}). \qquad (2.48)$$

Let (2.47) + (2.48), then

$$\begin{aligned}
0 &= -b(C^{(0)}+C^{(1)})-c(B^{(0)}+B^{(1)})+2a(A^{(0)}-A^{(1)})-(A^{(0)}-A^{(1)}) \\
&= 2(a-\frac{1}{2})(A^{(0)}-A^{(1)})-c(B^{(0)}+B^{(1)})-b(C^{(0)}+C^{(1)}). \qquad (2.49)
\end{aligned}$$

And, let [(2.47) − (2.48)]/2, we can get

$$\begin{aligned}
a_t &= -\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}(A^{(0)}-A^{(1)})+\frac{1}{2}(A^{(0)}-A^{(1)})-\frac{c}{2}(B^{(0)}-B^{(1)})+\frac{b}{2}(C^{(0)}-C^{(1)}) \\
&= -\frac{\lambda-\frac{l}{2}-i\frac{k}{2}}{-ik}(A^{(0)}-A^{(1)})+\frac{1}{2}(A^{(0)}-A^{(1)})-\frac{c}{2}(B^{(0)}-B^{(1)})+\frac{b}{2}(C^{(0)}-C^{(1)}) \\
&= -i\frac{2\lambda-l}{2k}(A^{(0)}-A^{(1)})-\frac{c}{2}(B^{(0)}-B^{(1)})+\frac{b}{2}(C^{(0)}-C^{(1)}). \qquad (2.50)
\end{aligned}$$

Note that from the above equations (2.45), (2.46), (2.49) and (2.50), a simple equation can be derived, namely,

$$\partial_t[bc-a(1-a)]=0, \qquad (2.51)$$

which is automatically satisfied because of (2.33). Therefore, we can drop one equation (2.50), and we choose the three equations to be (2.45), (2.46) and (2.49).

Now we can simply state the final results of this section: After case by case analysis, we can find that equations (2.46) and (2.49) are already contained in (2.45) and (2.43), and hence can be dropped also. Thus we choose the equations (2.43) and (2.45) to be our final Bäcklund transformation equations.

Notice that the apparent λ-dependence in (2.45) is only spurious, because all λ-dependent terms will cancel out on account of (2.43).

We now illustrate all these by explicitly examining the Bäcklund transformation equations (2.43) and (2.45) for K-dV equation.

The K-dV equation is $u_t + u_{xxx} - 6uu_x = 0$. We shall choose that

$$\begin{aligned}
q &= u \\
r &= 1 \\
A &= u_x - 2i\lambda u - 4i\lambda^3 \\
B &= -u_{xx} + 2u^2 + 2i\lambda u_x + 4\lambda^2 u \\
C &= 2u + 4\lambda^2 \\
l &= 0 \\
b &= \frac{q^{(0)} + q^{(1)}}{2i(\lambda_0 - \mu_0)} = \frac{q^{(0)} + q^{(1)}}{2i(-ik)} = \frac{q^{(0)} + q^{(1)}}{2k} \\
c &= \frac{r^{(0)} + r^{(1)}}{-2i(\lambda_0 - \mu_0)} = \frac{1+1}{-2i(-ik)} = -\frac{1}{k}.
\end{aligned}$$

For equation (2.43),

$$(q^{(0)} + q^{(1)})_x + il(q^{(0)} + q^{(1)}) = \pm(q^{(0)} + q^{(1)})\sqrt{k^2 + (q^{(0)} + q^{(1)})(r^{(0)} + r^{(1)})},$$

we can apply (2.35) and (2.44). Hence we will get

$$\begin{aligned}
0 &= \frac{(q^{(0)} + q^{(1)})_x}{2k} \mp (q^{(0)} - q^{(1)})\frac{1}{2k}\sqrt{k^2 + (q^{(0)} + q^{(1)})(r^{(0)} + r^{(1)})} \\
&= b_x - (q^{(0)} - q^{(1)})(a - \frac{1}{2}). \qquad (2.52)
\end{aligned}$$

For (2.46),

$$\begin{aligned}
c_t &= \frac{i(2\lambda - l)}{2k}(C^{(0)} + C^{(1)}) - c(A^{(0)} + A^{(1)}) - (a - \frac{1}{2})(C^{(0)} - C^{(1)}), \\
(\frac{-2}{2k})_t &= \frac{i(2\lambda - 0)}{2k}(2q^{(0)} + 4\lambda^2 + 2q^{(1)} + 4\lambda^2) \\
&\quad + \frac{2}{2k}(q_x^{(0)} - 2i\lambda^{(0)}q - 4i\lambda^3 + q_x^{(1)} - 2i\lambda q^{(1)} - 4i\lambda^3) \\
&\quad - (a - \frac{1}{2})(2q^{(0)} + 4\lambda^2 - 2q^{(1)} - 4\lambda^2) \\
&= \frac{2i}{2k}\lambda[2(q^{(0)} + q^{(1)}) + 8\lambda^2] \\
&\quad + \frac{2}{2k}[(q^{(0)} + q^{(1)})_x - 2i(q^{(0)} + q^{(1)})\lambda - 8i\lambda^3] \\
&\quad - 2(a - \frac{1}{2})(q^{(0)} - q^{(1)}),
\end{aligned}$$

$$\begin{aligned}
0 &= (\frac{16i}{2k} - \frac{16i}{2k})\lambda^3 \\
&\quad + [\frac{4i}{2k}(q^{(0)} + q^{(1)}) - \frac{4i}{2k}(q^{(0)} + q^{(1)})]\lambda \\
&\quad + [\frac{2}{2k}(q^{(0)} + q^{(1)})_x - 2(a - \frac{1}{2})(q^{(0)} - q^{(1)})] \\
&= 2[\frac{(q^{(0)} + q^{(1)})_x}{2k} - (a - \frac{1}{2})(q^{(0)} - q^{(1)})], \\
0 &= 2[b_x - (a - \frac{1}{2})(q^{(0)} - q^{(1)})].
\end{aligned}$$

For (2.49),

$$
\begin{aligned}
0 &= 2(a-\frac{1}{2})(A^{(0)}-A^{(1)}) - c(B^{(0)}+B^{(1)}) - b(C^{(0)}+C^{(1)}) \\
&= 2(a-\frac{1}{2})(q_x^{(0)} - 2i\lambda q^{(0)} - 4i\lambda^3 - q_x^{(1)} + 2i\lambda q^{(1)} + 4i\lambda^3) \\
&\quad + \frac{2}{2k}(-q_{xx}^{(0)} + 2q^{(0)2} + 2i\lambda q_x^{(0)} + 4\lambda^2 q^{(0)} - q_{xx}^{(1)} + 2q^{(1)2} + 2i\lambda q_x^{(1)} + 4\lambda^2 q^{(1)}) \\
&\quad - \frac{q^{(0)}+q^{(1)}}{2k}(2q^{(0)} + 4\lambda^2 + 2q^{(1)} + 4\lambda^2) \\
&= 2(a-\frac{1}{2})[(q^{(0)}-q^{(1)})_x - 2i(q^{(0)}-q^{(1)})\lambda] \\
&\quad + \frac{2}{2k}[-(q^{(0)}+q^{(1)})_{xx} + 2(q^{(0)2}+q^{(1)2}) + 2i(q^{(0)}+q^{(1)})_x\lambda + 4(q^{(0)}+q^{(1)})\lambda^2] \\
&\quad - \frac{q^{(0)}+q^{(1)}}{2k}[2(q^{(0)}+q^{(1)}) + 8\lambda^2] \\
&= [\frac{8}{2k}(q^{(0)}+q^{(1)}) - \frac{8}{2k}(q^{(0)}+q^{(1)})]\lambda^2 \\
&\quad + [-4i(a-\frac{1}{2})(q^{(0)}-q^{(1)}) + \frac{4i}{2k}(q^{(0)}+q^{(1)})_x]\lambda \\
&\quad + [2(a-\frac{1}{2})(q^{(0)}-q^{(1)})_x - \frac{2}{2k}(q^{(0)}+q^{(1)})_{xx} \\
&\quad + \frac{4}{2k}(q^{(0)2}+q^{(1)2}) - \frac{2}{2k}(q^{(0)}+q^{(1)})^2] \\
&= 4i\lambda[\frac{(q^{(0)}+q^{(1)})_x}{2k} - (a-\frac{1}{2})(q^{(0)}-q^{(1)})] \\
&\quad - 2[\frac{(q^{(0)}+q^{(1)})_{xx}}{2k} - \frac{1}{2k}(q^{(0)}-q^{(1)})^2 - (a-\frac{1}{2})(q^{(0)}-q^{(1)})_x].
\end{aligned}
$$

Since (2.44) and (2.43), we have

$$
\begin{aligned}
[(a-\frac{1}{2})(q^{(0)}-q^{(1)})]_x &= (a-\frac{1}{2})_x(q^{(0)}-q^{(1)}) + (a-\frac{1}{2})(q^{(0)}-q^{(1)})_x \\
&= \pm\frac{1}{2k}\frac{(q^{(0)}+q^{(1)})_x}{[k^2+2(q^{(0)}+q^{(1)})]^{\frac{1}{2}}}(q^{(0)}-q^{(1)}) \\
&\quad + (a-\frac{1}{2})(q^{(0)}-q^{(1)})_x \\
&= \frac{1}{2k}\frac{q^{(0)}-q^{(1)}}{(q^{(0)}+q^{(1)})_x}(q^{(0)}+q^{(1)})_x(q^{(0)}-q^{(1)}) \\
&\quad + (a-\frac{1}{2})(q^{(0)}-q^{(1)})_x \\
&= \frac{1}{2k}(q^{(0)}-q^{(1)})^2 + (a-\frac{1}{2})(q^{(0)}-q^{(1)})_x.
\end{aligned}
$$

Hence

$$
\begin{aligned}
0 &= 4i\lambda[b_x - (a-\frac{1}{2})(q^{(0)}-q^{(1)})] - 2[b_{xx} - [(a-\frac{1}{2})(q^{(0)}-q^{(1)})]_x] \\
&= 4i\lambda[b_x - (a-\frac{1}{2})(q^{(0)}-q^{(1)})] - 2[b_x - (a-\frac{1}{2})(q^{(0)}-q^{(1)})]_x.
\end{aligned}
$$

For (2.45), that is,

$$b_t = i\frac{2\lambda - l}{2k}(B^{(0)} + B^{(1)}) + b(A^{(0)} + A^{(1)}) + (a - \frac{1}{2})(B^{(0)} - B^{(1)}).$$

Substitute the above terms into it, then the equation (2.45) becomes

$$\begin{aligned}
b_t &= i\frac{2\lambda - 0}{2k}(-q^{(0)}_{xx} + 2q^{(0)2} + 2i\lambda q^{(0)}_x + 4\lambda^2 q^{(0)} - q^{(1)}_{xx} + 2q^{(1)2} + 2i\lambda q^{(1)}_x + 4\lambda^2 q^{(1)}) \\
&\quad + \frac{q^{(0)} + q^{(1)}}{2k}(q^{(0)}_x - 2i\lambda q^{(0)} - 4i\lambda^3 + q^{(1)}_x - 2i\lambda q^{(1)} - 4i\lambda^3) \\
&\quad + (a - \frac{1}{2})(-q^{(0)}_{xx} + 2q^{(0)2} + 2i\lambda q^{(0)}_x + 4\lambda^2 q^{(0)} + q^{(1)}_{xx} - 2q^{(1)2} - 2i\lambda q^{(1)}_x - 4\lambda^2 q^{(1)}) \\
&= \frac{2i\lambda}{2k}[-(q^{(0)} + q^{(1)})_{xx} + 2(q^{(0)2} + q^{(0)2}) + 2i(q^{(0)} + q^{(1)})_x\lambda + 4(q^{(0)} + q^{(1)})\lambda^2] \\
&\quad + \frac{q^{(0)} + q^{(1)}}{2k}[(q^{(0)} + q^{(1)})_x - 2i(q^{(0)} + q^{(1)})\lambda - 8i\lambda^3] \\
&\quad + (a - \frac{1}{2})[-(q^{(0)} - q^{(1)})_{xx} + 2(q^{(0)2} - q^{(1)2}) + 2i(q^{(0)} - q^{(1)})_x\lambda + 4(q^{(0)} - q^{(1)})\lambda^2] \\
&= [\frac{8i}{2k}(q^{(0)} + q^{(1)}) - \frac{8i}{2k}(q^{(0)} + q^{(1)})]\lambda^3 \\
&\quad + [-\frac{4}{2k}(q^{(0)} + q^{(1)})_x + 4(a - \frac{1}{2})(q^{(0)} - q^{(1)})]\lambda^2 \\
&\quad + [-\frac{2i}{2k}(q^{(0)} + q^{(1)})_{xx} + \frac{4i}{2k}(q^{(0)2} + q^{(1)2}) \\
&\quad - \frac{2i}{2k}(q^{(0)} + q^{(1)})^2 + 2i(a - \frac{1}{2})(q^{(0)} - q^{(1)})_x]\lambda \\
&\quad + [\frac{q^{(0)} + q^{(1)}}{2k}(q^{(0)} + q^{(1)})_x - (a - \frac{1}{2})(q^{(0)} - q^{(1)})_{xx} + 2(a - \frac{1}{2})(q^{(0)2} - q^{(1)2})] \\
&= -4\lambda^2[b_x - (a - \frac{1}{2})(q^{(0)} - q^{(1)})] \\
&\quad - 2i\lambda[b_{xx} - [\frac{1}{2k}(q^{(0)} - q^{(1)})^2 + (a - \frac{1}{2})(q^{(0)} - q^{(1)})_x]] \\
&\quad - (a - \frac{1}{2})(q^{(0)} - q^{(1)})_{xx} + (q^{(0)} + q^{(1)})[\frac{(q^{(0)} + q^{(1)})_x}{2k} + 2(a - \frac{1}{2})(q^{(0)} - q^{(1)})]
\end{aligned}$$

$$\begin{aligned}
b_t &= -4\lambda^2[b_x - (a - \frac{1}{2})(q^{(0)} - q^{(1)})] - 2i\lambda[b_x - (a - \frac{1}{2})(q^{(0)} - q^{(1)})]_x \\
&\quad - (a - \frac{1}{2})(q^{(0)} - q^{(1)})_{xx} + (q^{(0)} + q^{(1)})[b_x - (a - \frac{1}{2})(q^{(0)} - q^{(1)}) \\
&\quad + 3(a - \frac{1}{2})(q^{(0)} - q^{(1)})],
\end{aligned}$$

which, on account of (2.52), reduces to

$$\begin{aligned}
b_t &= -4\lambda^2 \cdot 0 - 2i\lambda \cdot 0 \\
&\quad - (a - \frac{1}{2})(q^{(0)} - q^{(1)})_{xx} + (q^{(0)} + q^{(1)})[0 + 3(a - \frac{1}{2})(q^{(0)} - q^{(1)})] \\
&= (a - \frac{1}{2})[-(q^{(0)} - q^{(1)})_{xx} + 3(q^{(0)2} - q^{(1)2})], \qquad (2.53)
\end{aligned}$$

which is a λ-independent expression. Therefore, we can write down the equations (2.52) and (2.53) to be our Bäcklund transformation equations for the K-dV equation.

To summarize this subsection, we have derived a general expression for the soliton-generating Bäcklund transformation for the class of nonlinear equations whose linear systems can be written as (2.27). The general expression for the Bäcklund transformation equations is

$$(q^{(0)}+q^{(1)})_x + il(q^{(0)}+q^{(1)}) = (\text{sign})\cdot(q^{(0)}-q^{(1)})\sqrt{k^2+(q^{(0)}+q^{(1)})(r^{(0)}+r^{(1)})}, \quad (2.54)$$

$$\begin{aligned}(q^{(0)}+q^{(1)})_t &= i(2\lambda-l)(B^{(0)}+B^{(1)})+(q^{(0)}+q^{(1)})(A^{(0)}+A^{(1)}) \\ &\quad +(\text{sign})\cdot(B^{(0)}-B^{(1)})\sqrt{k^2+(q^{(0)}+q^{(1)})(r^{(0)}+r^{(1)})}, \end{aligned} \quad (2.55)$$

where (sign) denotes the $\pm$ sign appearing in the expression

$$a-\frac{1}{2} = \pm\frac{1}{2k}\sqrt{k^2+(q^{(0)}+q^{(1)})(r^{(0)}+r^{(1)})}, \quad (2.56)$$

where k and l are two real parameters of Bäcklund transformation ($l \neq 0$ only for the NLS equation).

2.4 Explicit Bäcklund Transformation

In last section, we know that the Bäcklund transformations are useful for generating soliton solutions of some nonlinear equations. However, the Bäcklund transformation equations in their usual form, of which a example is

$$\begin{aligned}(u^{(0)}+u^{(1)})_x &= (u^{(0)}-u^{(1)})\sqrt{k^2\pm 2(u^{(0)}+u^{(1)})}, \\ (u^{(0)}+u^{(1)})_t &= \sqrt{k^2\pm 2(u^{(0)}+u^{(1)})}[-(u^{(0)}-u^{(1)})_{xx}\pm 3(u^{(0)2}-u^{(1)2})],\end{aligned}$$

(i.e. (2.52) and (2.53), for the K-dV equation $u_t+u_{xxx}\mp 6uu_x=0$), are difficult to solve in general, and hence become of limited use in practice for the purpose of constructing a new soliton solutions.

In Section 1.2, we have stated that the class of some nonlinear equations that can be successfully treated by our method includes those which can be represented as the integrability condition of the AKNS linear systems of the following type:

$$\begin{aligned}\Phi_x(x,t;\lambda) &= U(x,t;\lambda)\Phi(x,t;\lambda), \\ \Phi_t(x,t;\lambda) &= V(x,t;\lambda)\Phi(x,t;\lambda), \end{aligned} \quad (2.57)$$

where

$$U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}, \qquad V = \begin{pmatrix} A(x,t;\lambda) & B(x,t;\lambda) \\ C(x,t;\lambda) & -A(x,t;\lambda) \end{pmatrix}. \quad (2.58)$$

In this section, we will state that: for the particular class of systems described in (2.57)-(2.58), there shall be a systematic way to derive an explicit Bäcklund transformation which adds one soliton onto a given solution to make a new solution.

Suppose $u^{(0)}$ is the known solution of the nonlinear equations under consideration and $\Phi^{(0)}$ is a solution of (2.57) which is put $u^{(0)}$ into $q(x,t)$ and $r(x,t)$, that is,

$$\begin{aligned} \Phi_x^{(0)} &= U^{(0)}\Phi^{(0)} \;\; [= \begin{pmatrix} -i\lambda & q^{(0)} \\ r^{(0)} & i\lambda \end{pmatrix} \Phi^{(0)}], \\ \Phi_t^{(0)} &= V^{(0)}\Phi^{(0)} \;\; [= \begin{pmatrix} A^{(0)} & B^{(0)} \\ C^{(0)} & -A^{(0)} \end{pmatrix} \Phi^{(0)}]. \end{aligned} \tag{2.59}$$

Then we now hope to find a 2×2 matrix function $\psi = \psi(x,t;\lambda)$ such that they satisfy the condition

$$\Phi^{(1)} = \psi\Phi^{(0)}, \tag{2.60}$$

where $\Phi^{(1)}$ satisfies the equation system which has the same form with the above equation system:

$$\begin{aligned} \Phi_x^{(1)} &= U^{(1)}\Phi^{(1)} \;\; [= \begin{pmatrix} -i\lambda & q^{(1)} \\ r^{(1)} & i\lambda \end{pmatrix} \Phi^{(1)}], \\ \Phi_t^{(1)} &= V^{(1)}\Phi^{(1)} \;\; [= \begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & -A^{(1)} \end{pmatrix} \Phi^{(1)}]. \end{aligned}$$

We first can observe that

$$\Phi_x^{(1)} = U^{(1)}\Phi^{(1)} = U^{(1)}\psi\Phi^{(0)}.$$

On the other hand, we also have

$$\Phi_x^{(1)} = (\psi\Phi^{(0)})_x = \psi_x\Phi^{(0)} + \psi\Phi_x^{(0)} = \psi_x\Phi^{(0)} + \psi U^{(0)}\Phi^{(0)}.$$

Since the equation should hold for all solution $\Phi^{(0)}$, the following equation must hold:

$$U^{(1)}\psi = \psi_x + \psi U^{(0)}.$$

Hence, we have found that there is a equation as follows:

$$U^{(1)} = \psi U^{(0)}\psi^{-1} + \psi_x\psi^{-1}. \tag{2.61}$$

Similarly, we have the same conclusion for $V^{(1)}$,

$$V^{(1)} = \psi V^{(0)}\psi^{-1} + \psi_x\psi^{-1}. \tag{2.62}$$

To derive such Bäcklund transformations, it is sufficient to ensure that $U^{(1)}$, $V^{(1)}$ have the correct λ structure. As in last section, the transformation function $\psi(x,t;\lambda)$ in (2.60) shall be assumed by

$$\psi(x,t;\lambda) = [I + \frac{\lambda_1 - \lambda_2}{\lambda - \lambda_1}P(x,t)]J, \tag{2.63}$$

where λ_1 and λ_2 are two arbitrary complex numbers, P in an undtetermined 2×2 projection matrix ($P^2 = P$) and

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that ψ^{-1} is given by

$$\psi^{-1}(x,t;\lambda) = J[I - \frac{\lambda_1-\lambda_2}{\lambda-\lambda_2}P(x,t)]. \tag{2.64}$$

It is exactly the matrix φ in last section.

Thus if substitute (2.63) and (2.64) into (2.61) and (2.62), they shall impose the following conditions on P:

$$\begin{aligned} P_x &= (I-P)JU^{(0)}(\lambda_2)JP - PJU^{(0)}(\lambda_1)J(I-P) \\ P_t &= (I-P)JV^{(0)}(\lambda_2)JP - PJV^{(0)}(\lambda_1)J(I-P). \end{aligned} \tag{2.65}$$

Now, if we want to get the expression of $U^{(1)}$, we must find out the explicit form of the matrix P. Fortunately, the projector P in (2.65) can be solved explicitly in terms of calculable matrix function

$$\Phi^{(0)}(x,t;\lambda_j) = \begin{pmatrix} h_{j11} & h_{j12} \\ h_{j21} & h_{j22} \end{pmatrix}_{2\times 2},$$

which is the solution to the linear system (2.59) corresponding to the given solution $q^{(0)}$, $r^{(0)}$ and λ_j, where $j = 1, 2$.

First, let us define a 2×2 matrix $M^{(1)}$ by

$$M^{(1)} \equiv \Phi^{(0)}(x,t;\lambda_2)\begin{pmatrix} m_1 & \frac{1}{n_1} \\ n_1 & \frac{1}{m_1} \end{pmatrix}\Phi^{(0)}(x,t;\lambda_1)^{-1},$$

where m_1 and n_1 are two arbitrary complex constants.

Hence

$$\begin{aligned} M^{(1)} &= \begin{pmatrix} h_{211} & h_{212} \\ h_{221} & h_{222} \end{pmatrix}\begin{pmatrix} m_1 & \frac{1}{n_1} \\ n_1 & \frac{1}{m_1} \end{pmatrix}\Phi^{(0)}(x,t;\lambda_1)^{-1} \\ &= \begin{pmatrix} m_1h_{211}+n_1h_{212} & \frac{1}{n_1}h_{211}+\frac{1}{m_1}h_{212} \\ m_1h_{221}+n_1h_{222} & \frac{1}{n_1}h_{221}+\frac{1}{m_1}h_{222} \end{pmatrix}\begin{pmatrix} h_{122} & -h_{112} \\ -h_{121} & h_{111} \end{pmatrix} \\ &\quad \times\frac{1}{det(\Phi^{(0)}(x,t;\lambda_1))} \\ &= \frac{1}{det(\Phi^{(0)}(x,t;\lambda_1))}\times\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \end{aligned}$$

where

$$\begin{cases} M_{11} = m_1h_{122}h_{211} + n_1h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212} \\ M_{12} = -m_1h_{112}h_{211} - n_1h_{112}h_{212} + \frac{1}{n_1}h_{111}h_{211} + \frac{1}{m_1}h_{111}h_{212} \\ M_{21} = m_1h_{122}h_{221} + n_1h_{122}h_{222} - \frac{1}{n_1}h_{121}h_{221} - \frac{1}{m_1}h_{121}h_{222} \\ M_{22} = -m_1h_{112}h_{221} - n_1h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222} \end{cases},$$

and

$$trM^{(1)} = \frac{1}{det(\Phi^{(0)}(x,t;\lambda_1))}\times(M_{11}+M_{22}).$$

Then (2.65) shall be able to be solved by (ref [4, 7, 8])

$$P = J\tilde{P}J,$$

where

$$\begin{aligned}\tilde{P} &= \frac{M^{(1)}}{trM^{(1)}} \\ &= \frac{1}{M_{11}+M_{22}} \times \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.\end{aligned}$$

Now substitute (2.63) and (2.65) into (2.61) and (2.62), then we shall find

$$\begin{aligned}U^{(1)}(\lambda) = JU^{(0)}(\lambda)J + (\lambda_1-\lambda_2)PJ[\frac{U^{(0)}(\lambda)-U^{(0)}(\lambda_1)}{\lambda-\lambda_1}]J(I-P) \\ -(\lambda_1-\lambda_2)(I-P)J[\frac{U^{(0)}(\lambda)-U^{(0)}(\lambda_2)}{\lambda-\lambda_2}]JP,\end{aligned} \tag{2.66}$$

and a similar equation for $V^{(1)}(\lambda)$.

Now since

$$U^{(0)}(\lambda) = \begin{pmatrix} -i\lambda & q^{(0)} \\ r^{(0)} & i\lambda \end{pmatrix}$$

and $P = J\tilde{P}J$, the above equation (2.66) can be simplified to be

$$U^{(1)}(\lambda) \;=\; \begin{pmatrix} -i\lambda & -q^{(0)} \\ -r^{(0)} & i\lambda \end{pmatrix} - 2i(\lambda_1-\lambda_2)\begin{pmatrix} 0 & \tilde{P}_{12} \\ -\tilde{P}_{21} & 0 \end{pmatrix},$$

manifestly showing that $U^{(1)}(\lambda)$ has the correct λ structure, as desired.

Now, we can first compute $\widetilde{P}_{12}$ and $\widetilde{P}_{21}$ as follows:

$$\begin{aligned}\tilde{P}_{12} &= \frac{M_{12}}{M_{11}+M_{22}} \\ &= (-m_1h_{112}h_{211} - n_1h_{112}h_{212} + \frac{1}{n_1}h_{111}h_{211} + \frac{1}{m_1}h_{111}h_{212}) \\ &\quad /(m_1h_{122}h_{211} + n_1h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212} \\ &\quad -m_1h_{112}h_{221} - n_1h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}), \\ \tilde{P}_{21} &= \frac{M_{21}}{M_{11}+M_{22}} \\ &= (m_1h_{122}h_{221} + n_1h_{122}h_{222} - \frac{1}{n_1}h_{121}h_{221} - \frac{1}{m_1}h_{121}h_{222}) \\ &\quad /(m_1h_{122}h_{211} + n_1h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212} \\ &\quad -m_1h_{112}h_{221} - n_1h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}).\end{aligned}$$

Thus we got the explicit Bäcklunk transformation as follows:

$$\begin{aligned} q^{(1)} &= -q^{(0)} - 2i(\lambda_1 - \lambda_2)\tilde{P}_{12} \\ &= -q^{(0)} - 2i(\lambda_1 - \lambda_2) \\ &\quad \times(-m_1 h_{112}h_{211} - n_1 h_{112}h_{212} + \frac{1}{n_1}h_{111}h_{211} + \frac{1}{m_1}h_{111}h_{212})/ \\ &\quad (m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212} \\ &\quad -m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}), \end{aligned} \tag{2.67}$$

$$\begin{aligned} r^{(1)} &= -r^{(0)} + 2i(\lambda_1 - \lambda_2)\tilde{P}_{21} \\ &= -r^{(0)} - 2i(\lambda_1 - \lambda_2) \\ &\quad \times(-m_1 h_{122}h_{221} - n_1 h_{122}h_{222} + \frac{1}{n_1}h_{121}h_{221} + \frac{1}{m_1}h_{121}h_{222})/ \\ &\quad (m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212} \\ &\quad -m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}). \end{aligned} \tag{2.68}$$

Note that the transformation function ψ is so a important term for the Bäcklunk transformation, we shall also write down the explicit form of ψ:
First, the projection matrix P can be written as

$$\begin{aligned} P &= J\tilde{P}J \\ &= \frac{1}{M_{11}+M_{22}} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{M_{11}+M_{22}} \times \begin{pmatrix} M_{11} & -M_{12} \\ -M_{21} & M_{22} \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \psi &= (I + \frac{\lambda_1 - \lambda_2}{\lambda - \lambda_1}P)J \\ &= \frac{1}{\lambda - \lambda_1}[\begin{pmatrix} \lambda - \lambda_1 & 0 \\ 0 & \lambda - \lambda_1 \end{pmatrix} + \frac{(\lambda_1 - \lambda_2)}{M_{11}+M_{22}} \begin{pmatrix} M_{11} & -M_{12} \\ -M_{21} & M_{22} \end{pmatrix}]J \\ &= \frac{1}{\lambda - \lambda_1} \begin{pmatrix} \lambda - \lambda_1 + (\lambda_1 - \lambda_2)\frac{M_{11}}{M_{11}+M_{22}} & -(\lambda_1 - \lambda_2)\frac{M_{12}}{M_{11}+M_{22}} \\ -(\lambda_1 - \lambda_2)\frac{M_{21}}{M_{11}+M_{22}} & \lambda - \lambda_1 + (\lambda_1 - \lambda_2)\frac{M_{22}}{M_{11}+M_{22}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{\lambda - \lambda_1} \begin{pmatrix} \lambda - \lambda_1 + (\lambda_1 - \lambda_2)\frac{M_{11}}{M_{11}+M_{22}} & (\lambda_1 - \lambda_2)\frac{M_{12}}{M_{11}+M_{22}} \\ -(\lambda_1 - \lambda_2)\frac{M_{21}}{M_{11}+M_{22}} & -\lambda + \lambda_1 - (\lambda_1 - \lambda_2)\frac{M_{22}}{M_{11}+M_{22}} \end{pmatrix} \\ &= \frac{1}{\lambda - \lambda_1} \begin{pmatrix} \psi'_{11} & \psi'_{12} \\ \psi'_{21} & \psi'_{22} \end{pmatrix}, \end{aligned} \tag{2.69}$$

where

$$
\begin{aligned}
\psi'_{11} &= \lambda - \lambda_1 + (\lambda_1 - \lambda_2)\frac{M_{11}}{M_{11}+M_{22}} \\
&= \lambda - \lambda_1 + (\lambda_1 - \lambda_2)\frac{m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212}}{M_{11}+M_{22}} \\
&= \lambda - \lambda_1 + \lambda_1\frac{m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212}}{M_{11}+M_{22}} \\
&\quad -\lambda_2\frac{m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212}}{M_{11}+M_{22}} \\
&= \lambda - \lambda_1\frac{(M_{11}+M_{22}) - (m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212})}{M_{11}+M_{22}} \\
&\quad +\lambda_2\frac{\frac{1}{n_1}h_{121}h_{211} + \frac{1}{m_1}h_{121}h_{212} - m_1 h_{122}h_{211} - n_1 h_{122}h_{212}}{M_{11}+M_{22}} \\
&= \lambda - \frac{\lambda_1}{M_{11}+M_{22}}[(m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212} \\
&\quad -m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}) \\
&\quad -(m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212})] \\
&\quad +\lambda_2\frac{\frac{1}{n_1}h_{121}h_{211} + \frac{1}{m_1}h_{121}h_{212} - m_1 h_{122}h_{211} - n_1 h_{122}h_{212}}{M_{11}+M_{22}} \\
&= \lambda - \lambda_1\frac{\frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222} - m_1 h_{112}h_{221} - n_1 h_{112}h_{222}}{M_{11}+M_{22}} \\
&\quad +\lambda_2\frac{\frac{1}{n_1}h_{121}h_{211} + \frac{1}{m_1}h_{121}h_{212} - m_1 h_{122}h_{211} - n_1 h_{122}h_{212}}{M_{11}+M_{22}} \\
&= \lambda - \frac{1}{M_{11}+M_{22}}[\lambda_1(\frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222} - m_1 h_{112}h_{221} - n_1 h_{112}h_{222}) \\
&\quad +\lambda_2(\frac{1}{n_1}h_{121}h_{211} + \frac{1}{m_1}h_{121}h_{212} - m_1 h_{122}h_{211} - n_1 h_{122}h_{212})],
\end{aligned}
$$

$$
\begin{aligned}
\psi'_{12} &= (\lambda_1 - \lambda_2)\frac{M_{12}}{M_{11}+M_{22}} \\
&= \frac{\lambda_1 - \lambda_2}{M_{11}+M_{22}}(-m_1 h_{112}h_{211} - n_1 h_{112}h_{212} + \frac{1}{n_1}h_{111}h_{211} + \frac{1}{m_1}h_{111}h_{212}),
\end{aligned}
$$

$$
\begin{aligned}
\psi'_{21} &= -(\lambda_1 - \lambda_2)\frac{M_{21}}{M_{11}+M_{22}} \\
&= -\frac{\lambda_1 - \lambda_2}{M_{11}+M_{22}}(m_1 h_{122}h_{221} + n_1 h_{122}h_{222} - \frac{1}{n_1}h_{121}h_{221} - \frac{1}{m_1}h_{121}h_{222}),
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_{22}' &= -\lambda + \lambda_1 - (\lambda_1 - \lambda_2)\frac{M_{22}}{M_{11}+M_{22}} \\
&= -\lambda + \lambda_1 - (\lambda_1 - \lambda_2)\frac{-m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}}{M_{11}+M_{22}} \\
&= -\lambda + \lambda_1 - \lambda_1\frac{-m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}}{M_{11}+M_{22}} \\
&\quad +\lambda_2\frac{-m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}}{M_{11}+M_{22}} \\
&= -\lambda + \lambda_1\frac{(M_{11}+M_{22}) - (-m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222})}{M_{11}+M_{22}} \\
&\quad +\lambda_2\frac{-m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}}{M_{11}+M_{22}} \\
&= -\lambda + \frac{\lambda_1}{M_{11}+M_{22}}[(m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212} \\
&\quad -m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}) \\
&\quad -(-m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222})] \\
&\quad +\lambda_2\frac{-m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}}{M_{11}+M_{22}} \\
&= -\lambda + \lambda_1\frac{m_1 h_{122}h_{211} + n_1 h_{122}h_{212} - \frac{1}{n_1}h_{121}h_{211} - \frac{1}{m_1}h_{121}h_{212}}{M_{11}+M_{22}} \\
&\quad +\lambda_2\frac{-m_1 h_{112}h_{221} - n_1 h_{112}h_{222} + \frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222}}{M_{11}+M_{22}} \\
&= -\lambda + \frac{1}{M_{11}+M_{22}}[-\lambda_1(\frac{1}{n_1}h_{121}h_{211} + \frac{1}{m_1}h_{121}h_{212} - m_1 h_{122}h_{211} - n_1 h_{122}h_{212}) \\
&\quad +\lambda_2(\frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222} - m_1 h_{112}h_{221} - n_1 h_{112}h_{222})].
\end{aligned}
$$

3 Darboux Transformations

In this chapter, we shall introduce the Darboux transformation in matrix form, and how to construct the Darboux matrix. And in final, we will prove that the fact: we shall be able to get the same solution of the nonlinear equation which we consider from a known solution through the Bäcklund transformation and Darboux transformation.

3.1 Introduction of Darboux Transformation

In Sec 1.2, we have introduce the AKNS system. Hence, we can discuss the nonlinear partial differential equations of u

$$F(u, u_x, u_t, u_{xx}, \cdots) = 0 \tag{3.70}$$

through the AKNS system

$$\begin{aligned} \Phi_x &= U\Phi \;\; [= (-i\lambda J + Q)\Phi] \\ \Phi_t &= V\Phi \;\; [= \sum_{j=0}^{n} V_j \lambda^{n-j} \Phi]. \end{aligned} \tag{3.71}$$

when we choose the suitable q, r, A, B and C.

The following topics is that in order to know that how to apply AKNS system to get a new solution of the nonlinear differential equation which we consider form the known solutions, we now introduce the Darboux transformation as follows.

First, we introduce what is the Darboux transformation and the Darboux matrix :

Definition 1. *(Darboux transformation and Darboux matrix)*

For any given matrix $Q^{(0)} = \begin{pmatrix} 0 & q^{(0)} \\ r^{(0)} & 0 \end{pmatrix}$ *and 2-column vector* $\Phi^{(0)} = \begin{pmatrix} \alpha^{(0)} \\ \beta^{(0)} \end{pmatrix}$ *which satisfy (3.71), if the* 2×2 *matrix* $D(x, t; \lambda)$ *and the 2-column vector* $\begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix} = \Phi^{(1)} = D\Phi^{(0)}$ *can also satisfy the linear differential equation system which have the same form with (3.71) :*

$$\begin{aligned} \Phi^{(1)}_x &= U^{(1)}\Phi^{(1)} \;\; [= (-i\lambda J + Q^{(1)})\Phi^{(1)}] \\ \Phi^{(1)}_t &= V^{(1)}\Phi^{(1)} \;\; [= \sum_{j=0}^{n} V^{(1)}_j \lambda^{n-j} \Phi^{(1)}], \end{aligned} \tag{3.72}$$

where $Q^{(1)} = \begin{pmatrix} 0 & q^{(1)} \\ r^{(1)} & 0 \end{pmatrix}$ *is a matrix function with zero diagonal.*

Then we call the transformation $(Q^{(0)}, \Phi^{(0)}) \xrightarrow{D} (Q^{(1)}, \Phi^{(1)})$ *be the Darboux transformation of the AKNS system, and* $D(x, t; \lambda)$ *be a Darboux matrix.* □

3.2 Construct a Darboux Matrix in Explicit Purely Algebraic

From the above definition, we can first find that the explicit Bäcklund transformation is also one kind of Darboux transformation.

Next, the focal point is how to find the matrix D. Before constructing the matrix D, we can first observe that what properties does it have.

Let us consider the linear differential equation system (3.71):

$$\begin{aligned}\Phi_x &= U^{(0)}\Phi^{(0)} \quad [= (-i\lambda J + Q^{(0)})\Phi^{(0)}] \\ \Phi_t &= V^{(0)}\Phi^{(0)} \quad [= \sum_{j=0}^{n} V_j^{(0)}\lambda^{n-j}\Phi^{(0)}]\end{aligned}$$

First, we can suppose that

$$\Phi^{(1)} = D\Phi$$

can satisfies the following equation system (3.72) which has the same form with the above equations:

$$\begin{aligned}\Phi_x^{(1)} &= U^{(1)}\Phi^{(1)} \quad [= (-i\lambda J + Q^{(1)})\Phi^{(1)}] \\ \Phi_t^{(1)} &= V^{(1)}\Phi^{(1)} \quad [= \sum_{j=0}^{n} V_j^{(1)}\lambda^{n-j}\Phi^{(1)}]\end{aligned}$$

As the discussion of the equations (2.61) and (2.62) in Section 2.4, we shall also have the same equations as follows:

$$U^{(1)} = DU^{(0)}D^{-1} + D_xD^{-1} \tag{3.73}$$

and

$$V^{(1)} = DV^{(0)}D^{-1} + D_xD^{-1} \tag{3.74}$$

Now, we are concerned with about the Darboux matrix which λ is of order 1. Without losing of generality, we assume that the Darboux matrix D has the following form

$$D = J(\lambda I - S),$$

where S is a suitable 2×2 matrix, and I is the identity matrix. And we then discuss how to find the Darboux matrix.

3.2.1 Some Properties of The Matrix S

First, we begin our discussion form the first equation of (3.72)

$$\Phi_x^{(1)} = (-i\lambda J + Q^{(1)})\Phi^{(1)}$$

and

$$\Phi^{(1)} = D\Phi^{(0)}.$$

Hance we will have the following formulas :

$$\begin{aligned}\Phi_x^{(1)} &= (D\Phi^{(0)})_x \\ &= [J(\lambda I - S)\Phi^{(0)}]_x \\ &= (\lambda J - JS)_x\Phi^{(0)} + (\lambda J - JS)\Phi_x^{(0)} \\ &= -JS_x\Phi^{(0)} + (\lambda J - JS)(-i\lambda J + Q^{(0)})\Phi^{(0)} \\ &= -JS_x\Phi^{(0)} + (-i\lambda^2 I + \lambda JQ^{(0)} + i\lambda JSJ - JSQ^{(0)})\Phi^{(0)}\end{aligned}$$

and

$$
\begin{aligned}
(-i\lambda J + Q^{(1)})\Phi^{(1)} &= (-i\lambda J + Q^{(1)})(D\Phi^{(0)}) \\
&= (-i\lambda J + Q^{(1)})(J(\lambda I - S))\Phi^{(0)} \\
&= [(-i\lambda J)(J\lambda I) + (-i\lambda J)(-JS) + Q^{(1)}(J\lambda I) + Q^{(1)}(-JS)]\Phi^{(0)} \\
&= (-i\lambda^2 I + i\lambda S + \lambda Q^{(1)}J - Q^{(1)}JS)\Phi^{(0)}.
\end{aligned}
$$

Since the above equation must hold for all solution $\Phi^{(0)}$ of (3.71), we know that

$$
\begin{aligned}
& -iI\lambda^2 + (JQ^{(0)} + iJSJ)\lambda + (-JS_x - JSQ^{(0)}) \\
= \; & -iI\lambda^2 + (iS + Q^{(1)}J)\lambda + (-Q^{(1)}JS). \qquad (3.75)
\end{aligned}
$$

Since the definitions of the symbols

$$
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = J^{-1} \quad \text{and} \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix},
$$

we can find the equation

$$
JQJ = -Q.
$$

Hence, the equation for the coefficient of λ^1 in the equation (3.75) is

$$
\begin{aligned}
JQ^{(0)} + iJSJ &= iS + Q^{(1)}J, \\
JQ^{(0)}J + iJS &= iSJ + Q^{(1)},
\end{aligned}
$$

so we shall find

$$
\begin{aligned}
Q^{(1)} &= JQ^{(0)}J + iJS - iSJ \\
&= -Q^{(0)} + i[J, S]. \qquad (3.76)
\end{aligned}
$$

And, the equation of the coefficient of λ^0 in the equation (3.75) is

$$
\begin{aligned}
-JS_x - JSQ^{(0)} &= -Q^{(1)}JS, \\
S_x + SQ^{(0)} &= JQ^{(1)}JS.
\end{aligned}
$$

Hence, we can also get

$$
\begin{aligned}
0 &= S_x + SQ^{(0)} + (-JQ^{(1)}JS) \\
&= S_x + SQ^{(0)} + Q^{(1)}S \\
&= S_x + SQ^{(0)} + (-Q^{(0)} + i[J, S])S \\
&= S_x + SQ^{(0)} + (-Q^{(0)} + iJS - iSJ)S \\
&= S_x + SQ^{(0)} - Q^{(0)}S + iJSS - iSJS \\
&= S_x + [S, -iJS + Q^{(0)}]. \qquad (3.77)
\end{aligned}
$$

Next, in order to find the equations about S_t, we will proceed to observe the second equation of (3.72):

$$
\Phi_t^{(1)} = (\sum_{j=0}^{n} V_j^{(1)} \lambda^{n-j})\Phi^{(1)}.
$$

Then we shall get the equation :

$$\begin{aligned}
LHS &= (D\Phi^{(0)})_t \\
&= [J(\lambda I - S)\Phi^{(0)}]_t \\
&= (\lambda J - JS)_t\Phi^{(0)} + (\lambda J - JS)\Phi_t^{(0)} \\
&= -JS_t\Phi^{(0)} + (\lambda J - JS)(\sum_{j=0}^{n} V_j^{(0)}\lambda^{n-j})\Phi^{(0)} \\
&= [(\lambda J - JS)V_0^{(0)}\lambda^n + (\lambda J - JS)V_1^{(0)}\lambda^{n-1} + \cdots \\
&\quad +(\lambda J - JS)V_{n-1}^{(0)}\lambda + (\lambda J - JS)V_n^{(0)} - JS_t]\Phi \\
&= [\lambda^{n+1}JV_0^{(0)} + \lambda^n JV_1^{(0)} + \cdots + \lambda^2 JV_{n-1}^{(0)} + \lambda JV_n^{(0)} \\
&\quad -\lambda^n JSV_0^{(0)} - \lambda^{n-1}JSV_1^{(0)} - \cdots - \lambda JSV_{n-1}^{(0)} - JSV_n^{(0)} - JS_t]\Phi \qquad (3.78)
\end{aligned}$$

and

$$\begin{aligned}
RHS &= (\sum_{j=0}^{n} V_j^{(1)}\lambda^{n-j})(J(\lambda I - S)\Phi) \\
&= [V_0^{(1)}\lambda^n(\lambda J - JS) + V_1^{(1)}\lambda^{n-1}(\lambda J - JS) + \cdots + V_{n-1}^{(1)}\lambda(\lambda J - JS) + V_n^{(1)}(\lambda J - JS)]\Phi \\
&= [\lambda^{n+1}V_0^{(1)}J + \lambda^n V_1^{(1)}J + \cdots + \lambda^2 V_{n-1}^{(1)}J + \lambda V_n^{(1)}J \\
&\quad -\lambda^n V_0^{(1)}JS - \lambda^{n-1}V_1^{(1)}JS - \cdots - \lambda V_{n-1}^{(1)}JS - V_n^{(1)}JS]\Phi. \qquad (3.79)
\end{aligned}$$

Similarly, the above formulas must also equal for all solutions $\Phi^{(0)}$ of (3.71). So we know that the coefficient for λ^0 is,

$$\begin{aligned}
-V_n^{(1)}JS &= -JSV_n^{(0)} - JS_t \\
JS_t &= V_n^{(1)}JS - JSV_n^{(0)} \\
S_t &= JV_n^{(1)}JS - SV_n^{(0)}, \qquad (3.80)
\end{aligned}$$

and for the coefficient of λ^{n+1}, we have

$$V_0^{(1)}J = JV_0^{(0)}.$$

And from the equations (3.78) and (3.79), for λ^j, where $j = 1, \cdots, n$,

$$\begin{aligned}
\lambda^n &: V_1^{(1)}J - V_0^{(1)}JS = JV_1^{(0)} - JSV_0^{(0)}, \\
&\vdots \\
\lambda^1 &: V_n^{(1)}J - V_{n-1}^{(1)}JS = JV_n^{(0)} - JSV_{n-1}^{(0)}.
\end{aligned}$$

From the above relations, we can therefore get the equations

$$V_{j+1}^{(1)}J = JV_{j+1}^{(0)} + V_j^{(1)}JS - JSV_j^{(0)}.$$

That is,

$$V_{j+1}^{(1)} = JV_{j+1}^{(0)}J + V_j^{(1)}JSJ - JSV_j^{(0)}J, \qquad (3.81)$$

where $j = 0, \cdots, n-1$.

Now, let us observe the equation (3.80), then we shall get

$$\begin{aligned}
V_0^{(1)} &= JV_0^{(0)}J, \\
V_1^{(1)} &= JV_1^{(0)}J + V_0^{(1)}JSJ - JSV_0^{(0)}J \\
&= JV_1^{(0)}J + JV_0^{(0)}JJSJ - JSV_0^{(0)}J \\
&= JV_1^{(0)}J + JV_0^{(0)}SJ - JSV_0^{(0)}J \\
&= JV_1^{(0)}J + J[V_0^{(0)}, S]J, \\
V_2^{(1)} &= JV_2^{(0)}J + V_1^{(1)}JSJ - JSV_1^{(0)}J \\
&= JV_2^{(0)}J + (JV_1^{(0)}J + JV_0^{(0)}SJ - JSV_0^{(0)}J)JSJ - JSV_1^{(0)}J \\
&= JV_2^{(0)}J + JV_1^{(0)}SJ + JV_0^{(0)}S^2J - JSV_0^{(0)}SJ - JSV_1^{(0)}J \\
&= JV_2^{(0)}J + J[V_1^{(0)}, S]J + J[V_0^{(0)}, S]SJ \\
&\vdots
\end{aligned}$$

Hence we have

$$V_{j+1}^{(1)} = JV_{j+1}^{(0)}J + J(\sum_{k=1}^{j+1}[V_{j+1-k}^{(0)}, S]S^{k-1})J, \tag{3.82}$$

where $j = 0, \cdots, n-1$.

In particular, for the term of order $j = n-1$ of λ, we find that

$$V_n^{(1)} = JV_n^{(0)}J + J(\sum_{k=1}^{n}[V_{n-k}^{(0)}, S]S^{k-1})J.$$

So, if we apply the above equation and equation (3.80), we can get

$$\begin{aligned}
S_t &= JV_n^{(1)}JS - SV_n^{(0)} \\
&= J(JV_n^{(0)}J + J(\sum_{k=1}^{n}[V_{n-k}^{(0)}, S]S^{k-1})J)JS - SV_n^{(0)} \\
&= V_n^{(0)}S - SV_n^{(0)} + \sum_{k=1}^{n}[V_{n-k}^{(0)}, S]S^k \\
&= V_n^{(0)}S - SV_n^{(0)} + (V_{n-1}^{(0)}S^2 - SV_{n-1}^{(0)}S + V_{n-2}^{(0)}S^3 - SV_{n-2}^{(0)}S^2 + \cdots \\
&\quad - \cdots + V_1^{(0)}S^n - SV_1^{(0)}S^{n-1} + V_0^{(0)}S^{n+1} - SV_0^{(0)}S^n) \\
&= [V_n^{(0)} + V_{n-1}^{(0)}S + V_{n-2}^{(0)}S^2 + \cdots + V_1^{(0)}S^{n-1} + V_0^{(0)}S^n, S] \\
&= [\sum_{k=0}^{n} V_{n-k}^{(0)}S^k, S].
\end{aligned}$$

So

$$S_t + [S, \sum_{k=0}^{n} V_{n-k}^{(0)}S^k] = 0. \tag{3.83}$$

From the above discussions, we can conclude that if the matrix $J(\lambda I - S)$ is a Darboux matrix of the equation system (3.71)

$$\begin{aligned}
\Phi_x^{(0)} &= U^{(0)}\Phi^{(0)} \quad [= (-i\lambda J + Q^{(0)})\Phi^{(0)}] \\
\Phi_t^{(0)} &= V^{(0)}\Phi^{(0)} \quad [= \sum_{j=0}^{n} V_j^{(0)}\lambda^{n-j}\Phi^{(0)}],
\end{aligned}$$

then the matrix S must satisfy the following nonlinear partial differential equation system

$$\begin{aligned} S_x + [S, -iJS + Q^{(0)}] &= 0 \\ \text{and} \quad S_t + [S, \sum_{k=0}^{n} V_{n-k}^{(0)} S^k] &= 0. \end{aligned} \tag{3.84}$$

Now, we shall state and prove this fact in the following theorem :

Theorem 1 (1).
The matrix $J(\lambda I - S)$ is a Darboux matrix of the equation system (3.71)

$$\begin{aligned} \Phi_x^{(0)} &= U^{(0)}\Phi^{(0)} \quad [= (-i\lambda J + Q^{(0)})\Phi^{(0)}] \\ \Phi_t^{(0)} &= V^{(0)}\Phi^{(0)} \quad [= \sum_{j=0}^{n} V_j^{(0)} \lambda^{n-j} \Phi^{(0)}], \end{aligned}$$

if and only if the matrix S satisfies the nonlinear partial differential equation system

$$\begin{aligned} S_x + [S, -iJS + Q^{(0)}] &= 0 \\ S_t + [S, \sum_{k=0}^{n} V_{n-k}^{(0)} S^k] &= 0. \end{aligned}$$

Proof.
($\Rightarrow$)
If the matrix $J(\lambda I - S)$ is a Darboux matrix of equation (3.71),
then (3.84) is exactly the equation (3.77) and (3.83) which we got from the above discussion.
($\Leftarrow$)
Suppose that the equations (3.84) (i.e. (3.77) and (3.83)) hold.
Then for any solution Φ of (3.71) , the equations (3.75) and (3.78) = (3.79) shall also hold.
Therefore, The $P^{(1)}$ which be determined by (3.76), and the $\{V_j^{(1)}\}$ which determined by (3.81) will imply that the equation (3.72) holds.
Hence this matrix $D = J(\lambda I - S)$ will satisfy the definition of the Darboux matrix. □

3.2.2 How to Find The Matrix S

The last subsection say that in order to find the Darboux matrix, we must firstly find the solutions S of the nonlinear partial differential equation (3.80). Next, the following theorem shall provide us a method to constructing a Darboux matrix.

Theorem 2 (2).
Let λ_1 and λ_2 be two distinct complex numbers.
First, we define a matrix $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.
Suppose that h_j are the 2-column vector solution of (3.71) with $\lambda = \lambda_j$, where $j = 1, 2$.
Let $H = \begin{pmatrix} h_1 & h_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix}$ *be a 2×2 matrix.*
If $detH \neq 0$, we can define

$$S = H\Lambda H^{-1}$$

and consequently get the matrix

$$D(x,t;\lambda) = J(\lambda I - S) = \lambda J - JH\Lambda H^{-1}. \tag{3.85}$$

Then the matrix D shall be a Darboux matrix of (3.71) .

Proof.
Suppose that h_j are the solutions of (3.71) with $\lambda = \lambda_j$,that is,

$$\begin{aligned} h_{j,x} &= -i\lambda_j J h_j + Q^{(0)} h_j \\ h_{j,t} &= \sum_{k=0}^{n} V_k^{(0)} \lambda^{n-k} h_j, \end{aligned} \tag{3.86}$$

where $j = 1$,2.
From (3.86), we can compute that H_x $[= \begin{pmatrix} h_1 & h_2 \end{pmatrix}_x]$ and H_t.
Hence we have that

$$\begin{aligned} H_x &= -iJH\Lambda + QH^{(0)} \\ H_t &= \sum_{k=0}^{n} V_k^{(0)} H\Lambda^{n-k}. \end{aligned} \tag{3.87}$$

First, we can compute $(H^{-1})_x$ as follows:

$$\begin{aligned} I &= HH^{-1}, \\ 0 &= (HH^{-1})_x \\ &= H_x H^{-1} + H(H^{-1})_x, \\ H(H^{-1})_x &= -H_x H^{-1}, \\ (H^{-1})_x &= -H^{-1} H_x H^{-1}. \end{aligned}$$

Similarly,

$$(H^{-1})_t = -H^{-1} H_t H^{-1}.$$

Since $S \equiv H\Lambda H^{-1}$,

$$\begin{aligned} S_x &= H_x \Lambda H^{-1} + H\Lambda (H^{-1})_x \\ &= H_x \Lambda H^{-1} + H\Lambda(-H^{-1} H_x H^{-1}) \\ &= H_x H^{-1} H\Lambda H^{-1} - H\Lambda H^{-1} H_x H^{-1} \\ &= H_x H^{-1} S - S H_x H^{-1} \\ &= [H_x H^{-1}, S]. \end{aligned}$$

Employing (3.87), then the above equation becomes

$$\begin{aligned} S_x &= [(-iJH\Lambda + Q^{(0)} H) H^{-1}, S] \\ &= [-iJH\Lambda H^{-1} + Q^{(0)}, S] \\ &= [-iJS + Q^{(0)}, S]. \end{aligned}$$

Similarly,

$$\begin{aligned} S_t &= H_t\Lambda H^{-1} + H\Lambda(H^{-1})_t \\ &= H_t\Lambda H^{-1} + H\Lambda(-H^{-1}H_tH^{-1}) \\ &= H_tH^{-1}H\Lambda H^{-1} - H\Lambda H^{-1}H_tH^{-1} \\ &= H_tH^{-1}S - SH_tH^{-1} \\ &= [H_tH^{-1}, S]. \end{aligned}$$

We can further get the following equation by using (3.87) again:

$$\begin{aligned} S_t &= [(\sum_{k=0}^{n} V_k^{(0)} H\Lambda^{n-k})H^{-1}, S] \\ &= [\sum_{k=0}^{n} V_k^{(0)} \underbrace{(H\Lambda H^{-1})\cdots(H\Lambda H^{-1})}_{n-k}, S] \\ &= [\sum_{k=0}^{n} V_{n-k}^{(0)} S^k, S]. \end{aligned}$$

Hence that says that the matrix S which be defined in (3.85) is a solution of (3.84).

Finally applying Theorem(1), we completes the proof that $D = J(\lambda I - S)$ is a Darboux matrix of (3.71).

□

3.3 General Form of The Darboux Matrix which Be Given in theorems

Now, we shall be able to deduce the general form of a Darboux matrix from the statements which be given in the above theorems.

First, we can choose

$$\begin{aligned} \Lambda &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ \text{and } H &= \begin{pmatrix} h_1 & h_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix}, \end{aligned}$$

where λ_1 and λ_2 be two distinct complex numbers, and h_j are the 2-column vector solution $\Phi^{(0)}(x, t; \lambda_j)$ of the equation system (3.71)

$$\begin{aligned} \Phi_x^{(0)} &= U^{(0)}\Phi^{(0)} \quad [= (-i\lambda J + Q^{(0)})\Phi^{(0)}] \\ \Phi_t^{(0)} &= V^{(0)}\Phi^{(0)} \quad [= \sum_{j=0}^{n} V_j^{(0)} \lambda^{n-j}\Phi^{(0)}] \end{aligned}$$

with $\lambda = \lambda_i$, where $j = 1, 2$.

Then the inverse of the matrix H is

$$H^{-1} = \frac{1}{det(H)} \begin{pmatrix} h_{22} & -h_{21} \\ -h_{12} & h_{11} \end{pmatrix}.$$

Hence the matrix S of the Darboux is

$$\begin{aligned}
S &\equiv H\Lambda H^{-1} \\
&= \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} H^{-1} \\
&= \begin{pmatrix} \lambda_1 h_{11} & \lambda_2 h_{21} \\ \lambda_1 h_{12} & \lambda_2 h_{22} \end{pmatrix} \begin{pmatrix} h_{22} & -h_{21} \\ -h_{12} & h_{11} \end{pmatrix} \times \frac{1}{det(H)} \\
&= \frac{1}{det(H)} \times \begin{pmatrix} \lambda_1 h_{11}h_{22} - \lambda_2 h_{12}h_{21} & -\lambda_1 h_{11}h_{21} + \lambda_2 h_{11}h_{21} \\ \lambda_1 h_{12}h_{22} - \lambda_2 h_{12}h_{22} & -\lambda_1 h_{12}h_{21} + \lambda_2 h_{11}h_{22} \end{pmatrix} \\
&= \frac{1}{det(H)} \times \begin{pmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{pmatrix}
\end{aligned}$$

where the entries of the matrix S are

$$\begin{aligned}
S'_{11} &= \lambda_1 h_{11}h_{22} - \lambda_2 h_{12}h_{21}, \\
S'_{12} &= -\lambda_1 h_{11}h_{21} + \lambda_2 h_{11}h_{21}, \\
S'_{21} &= \lambda_1 h_{12}h_{22} - \lambda_2 h_{12}h_{22}, \\
S'_{22} &= -\lambda_1 h_{12}h_{21} + \lambda_2 h_{11}h_{22}.
\end{aligned}$$

Apply the equation (3.76), we will get the new $Q^{(1)}$ as follows:

$$\begin{aligned}
Q^{(1)} &= -Q^{(0)} + i[J, S] \\
&= -Q^{(0)} + i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S - iS \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= -Q^{(0)} + \frac{i}{det(H)} [\begin{pmatrix} S'_{11} & S'_{12} \\ -S'_{21} & -S'_{22} \end{pmatrix} - \begin{pmatrix} S'_{11} & -S'_{12} \\ S'_{21} & -S'_{22} \end{pmatrix}] \\
&= -Q^{(0)} + \frac{2i}{det(H)} \begin{pmatrix} 0 & S'_{12} \\ -S'_{21} & 0 \end{pmatrix}
\end{aligned}$$

Therefore, let us simplify the above equation as follows:

$$\begin{aligned}
\begin{pmatrix} 0 & q^{(1)} \\ r^{(1)} & 0 \end{pmatrix} &= Q^{(1)} \\
&= -Q^{(0)} + \frac{2i}{det(H)} \begin{pmatrix} 0 & -\lambda_1 h_{11}h_{21} + \lambda_2 h_{11}h_{21} \\ -\lambda_1 h_{12}h_{22} + \lambda_2 h_{12}h_{22} & 0 \end{pmatrix} \\
&= -Q^{(0)} - \frac{2i}{det(H)} (\lambda_1 - \lambda_2) \begin{pmatrix} 0 & h_{11}h_{21} \\ h_{12}h_{22} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -q^{(0)} \\ -r^{(0)} & 0 \end{pmatrix} - \frac{2i(\lambda_1 - \lambda_2)}{h_{11}h_{22} - h_{12}h_{21}} \begin{pmatrix} 0 & h_{11}h_{21} \\ h_{12}h_{22} & 0 \end{pmatrix}
\end{aligned}$$

That is,

$$\begin{aligned}
q^{(1)} &= -q^{(0)} - 2i(\lambda_1 - \lambda_2) \frac{h_{11}h_{21}}{h_{11}h_{22} - h_{12}h_{21}} \\
\text{and } r^{(1)} &= -r^{(0)} - 2i(\lambda_1 - \lambda_2) \frac{h_{12}h_{22}}{h_{11}h_{22} - h_{12}h_{21}} \qquad (3.88)
\end{aligned}$$

shall be also a new solution of the partial differential equation $F(u, u_x, u_t, u_{xx}, \cdots) = 0$ which we consider by choosing some suitable q and r.

3.4 The Equivalence of Darboux Transform. and Explicit Bäcklund Transform.

Next, we shall choose the parameters and observe the relation of the Darboux transformation and the explicit Bäcklund transformation which we discuss in last chapter (Sec 2.4). Finally, we shall prove that the equivalence between the two kind of the transformations.

Now, let us suppose that the entries of the matrix H have the following form :

$$\begin{cases} h_{11} = l_1 h_{111} + l_2 h_{112} \\ h_{12} = l_1 h_{121} + l_2 h_{122} \\ h_{21} = l_3 h_{211} + l_4 h_{212} \\ h_{22} = l_3 h_{221} + l_4 h_{222} \end{cases} .$$

where $\begin{pmatrix} h_{j11} & h_{j12} \\ h_{j21} & h_{j22} \end{pmatrix}_{2\times 2} = \Phi^{(0)}(x,t;\lambda_j)$ is the solution of the linear differential equation system (3.71) which given a solution $q(x,t)$, $r(x,t)$ of the nonlinear equation (3.70) and corresponds to λ_j, and l_1, l_2, l_3, l_4 are complex constants.

First, we compute some terms as follows:

$$\begin{aligned} h_{11}h_{21} &= l_1l_3h_{111}h_{211} + l_1l_4h_{111}h_{212} + l_2l_3h_{112}h_{211} + l_2l_4h_{112}h_{212}, \\ h_{12}h_{22} &= l_1l_3h_{121}h_{221} + l_1l_4h_{121}h_{222} + l_2l_3h_{122}h_{221} + l_2l_4h_{122}h_{222} \end{aligned}$$

and

$$\begin{aligned} det(H) &= h_{11}h_{22} - h_{12}h_{21} \\ &= (l_1l_3h_{111}h_{221} + l_1l_4h_{111}h_{222} + l_2l_3h_{112}h_{221} + l_2l_4h_{112}h_{222}) \\ &\quad -(l_1l_3h_{121}h_{211} + l_1l_4h_{121}h_{212} + l_2l_3h_{122}h_{211} + l_2l_4h_{122}h_{212}). \end{aligned}$$

Then the equations (3.88) become

$$\begin{aligned} q^{(1)} &= -q^0 - 2i(\lambda_1 - \lambda_2)\frac{h_{11}h_{21}}{h_{11}h_{22} - h_{12}h_{21}} \\ &= -q^0 - 2i(\lambda_1 - \lambda_2) \times (l_1l_3h_{111}h_{211} + l_1l_4h_{111}h_{212} + l_2l_3h_{112}h_{211} + l_2l_4h_{112}h_{212}) \\ &\quad /[(l_1l_3h_{111}h_{221} + l_1l_4h_{111}h_{222} + l_2l_3h_{112}h_{221} + l_2l_4h_{112}h_{222}) \\ &\quad -(l_1l_3h_{121}h_{211} + l_1l_4h_{121}h_{212} + l_2l_3h_{122}h_{211} + l_2l_4h_{122}h_{212})], \\ r^{(1)} &= -r^0 - 2i(\lambda_1 - \lambda_2)\frac{h_{12}h_{22}}{h_{11}h_{22} - h_{12}h_{21}} \\ &= -r^0 - 2i(\lambda_1 - \lambda_2) \times (l_1l_3h_{121}h_{221} + l_1l_4h_{121}h_{222} + l_2l_3h_{122}h_{221} + l_2l_4h_{122}h_{222}) \\ &\quad /[(l_1l_3h_{111}h_{221} + l_1l_4h_{111}h_{222} + l_2l_3h_{112}h_{221} + l_2l_4h_{112}h_{222}) \\ &\quad -(l_1l_3h_{121}h_{211} + l_1l_4h_{121}h_{212} + l_2l_3h_{122}h_{211} + l_2l_4h_{122}h_{212})]. \end{aligned}$$

In order to compare the above result with the contents in Section 2.4, we found that the following relations must satisfy:

$$\begin{cases} l_1l_3 = \frac{1}{n_1} \\ l_1l_4 = \frac{1}{m_1} \\ l_2l_3 = -m_1 \\ l_2l_4 = -n_1 \end{cases} .$$

Hence, if we choose the coefficients to be

$$\begin{cases} l_1 = 1 \\ l_2 = -m_1 n_1 \\ l_3 = \frac{1}{n_1} \\ l_4 = \frac{1}{m_1} \end{cases},$$

then the above two functions shall become

$$\begin{aligned} q^{(1)} &= -q^0 - 2i(\lambda_1 - \lambda_2) \times (l_1 l_3 h_{111} h_{211} + l_1 l_4 h_{111} h_{212} + l_2 l_3 h_{112} h_{211} + l_2 l_4 h_{112} h_{212}) \\ &\quad /[(l_1 l_3 h_{111} h_{221} + l_1 l_4 h_{111} h_{222} + l_2 l_3 h_{112} h_{221} + l_2 l_4 h_{112} h_{222}) \\ &\quad -(l_1 l_3 h_{121} h_{211} + l_1 l_4 h_{121} h_{212} + l_2 l_3 h_{122} h_{211} + l_2 l_4 h_{122} h_{212})] \\ &= -q^0 - 2i(\lambda_1 - \lambda_2) \times (\frac{1}{n_1} h_{111} h_{211} + \frac{1}{m_1} h_{111} h_{212} - m_1 h_{112} h_{211} - n_1 h_{112} h_{212}) \\ &\quad /[(\frac{1}{n_1} h_{111} h_{221} + \frac{1}{m_1} h_{111} h_{222} - m_1 h_{112} h_{221} - n_1 h_{112} h_{222}) \\ &\quad -(\frac{1}{n_1} h_{121} h_{211} + \frac{1}{m_1} h_{121} h_{212} - m_1 h_{122} h_{211} - n_1 h_{122} h_{212})] \end{aligned} \tag{3.89}$$

and

$$\begin{aligned} r^{(1)} &= -r^0 - 2i(\lambda_1 - \lambda_2) \times (l_1 l_3 h_{121} h_{221} + l_1 l_4 h_{121} h_{222} + l_2 l_3 h_{122} h_{221} + l_2 l_4 h_{122} h_{222}) \\ &\quad /[(l_1 l_3 h_{111} h_{221} + l_1 l_4 h_{111} h_{222} + l_2 l_3 h_{112} h_{221} + l_2 l_4 h_{112} h_{222}) \\ &\quad -(l_1 l_3 h_{121} h_{211} + l_1 l_4 h_{121} h_{212} + l_2 l_3 h_{122} h_{211} + l_2 l_4 h_{122} h_{212})] \\ &= -r^0 - 2i(\lambda_1 - \lambda_2) \times (\frac{1}{n_1} h_{121} h_{221} + \frac{1}{m_1} h_{121} h_{222} - m_1 h_{122} h_{221} - n_1 h_{122} h_{222}) \\ &\quad /[(\frac{1}{n_1} h_{111} h_{221} + \frac{1}{m_1} h_{111} h_{222} - m_1 h_{112} h_{221} - n_1 h_{112} h_{222}) \\ &\quad -(\frac{1}{n_1} h_{121} h_{211} + \frac{1}{m_1} h_{121} h_{212} - m_1 h_{122} h_{211} - n_1 h_{122} h_{212})] \end{aligned} \tag{3.90}$$

So, we can observe the fact that the expressions (equations (3.89) and (3.90)) of $q^{(1)}$ and $r^{(1)}$ which we got from Darboux transformation is the same with the expressions (equations (2.67) and (2.68)) which be derived by Bäcklund transformation.

Next,we will write down the Darboux matrix here, and compare it with the matrix $\psi(x, t; \lambda)$ (i.e. (2.69)) which we have gotten in the Section 2.4.

Before writing down the explicit expression of the Darboux matrix, we first compute the determinant of matrix H:

$$\begin{aligned} det(H) &= h_{11} h_{22} - h_{12} h_{21} \\ &= (h_{111} - m_1 n_1 h_{112})(\frac{1}{n_1} h_{221} + \frac{1}{m_1} h_{222}) - (h_{121} - m_1 n_1 h_{122})(\frac{1}{n_1} h_{211} + \frac{1}{m_1} h_{212}) \\ &= (\frac{1}{n_1} h_{111} h_{221} + \frac{1}{m_1} h_{111} h_{222} - m_1 h_{112} h_{221} - n_1 h_{112} h_{222}) \\ &\quad -(\frac{1}{n_1} h_{121} h_{211} + \frac{1}{m_1} h_{121} h_{212} - m_1 h_{122} h_{211} - n_1 h_{122} h_{212}) \\ &= (m_1 h_{122} h_{211} + n_1 h_{122} h_{212} - \frac{1}{n_1} h_{121} h_{211} - \frac{1}{m_1} h_{121} h_{212}) \\ &\quad +(-m_1 h_{112} h_{221} - n_1 h_{112} h_{222} + \frac{1}{n_1} h_{111} h_{221} + \frac{1}{m_1} h_{111} h_{222}) \\ &= M_{11} + M_{22} \text{ (which is in the Sec 2.4).} \end{aligned}$$

Therefore, the explicit expression of this Darboux matrix shall be :

$$\begin{aligned}
D(x,t;\lambda) &= J(\lambda I - S) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \lambda - \frac{S'_{11}}{det(H)} & -\frac{S'_{12}}{det(H)} \\ -\frac{S'_{21}}{det(H)} & \lambda - \frac{S'_{22}}{det(H)} \end{pmatrix} \\
&= \begin{pmatrix} \lambda - \frac{S'_{11}}{det(H)} & -\frac{S'_{12}}{det(H)} \\ \frac{S'_{21}}{det(H)} & -\lambda + \frac{S'_{22}}{det(H)} \end{pmatrix} \\
&= \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \qquad (3.91)
\end{aligned}$$

where the entries of the matrix are

$$\begin{aligned}
D_{11} &= \lambda - \frac{S'_{11}}{det(H)} \\
&= \lambda - \frac{\lambda_1 h_{11} h_{22} - \lambda_2 h_{12} h_{21}}{det(H)} \\
&= \lambda - \frac{1}{det(H)}[\lambda_1(h_{111} - m_1 n_1 h_{112})(\frac{1}{n_1}h_{221} + \frac{1}{m_1}h_{222}) \\
&\quad -\lambda_2(h_{121} - m_1 n_1 h_{122})(\frac{1}{n_1}h_{211} + \frac{1}{m_1}h_{212})] \\
&= \lambda - \frac{1}{det(H)}[\lambda_1(\frac{1}{n_1}h_{111}h_{221} + \frac{1}{m_1}h_{111}h_{222} - m_1 h_{112}h_{221} - n_1 h_{112}h_{222}) \\
&\quad -\lambda_2(\frac{1}{n_1}h_{121}h_{211} + \frac{1}{m_1}h_{121}h_{212} - m_1 h_{122}h_{211} - n_1 h_{122}h_{212})] \\
&= \psi'_{11} \quad ,
\end{aligned}$$

$$\begin{aligned}
D_{12} &= -\frac{S'_{12}}{det(H)} \\
&= -\frac{-\lambda_1 h_{11} h_{21} + \lambda_2 h_{11} h_{21}}{det(H)} \\
&= \frac{\lambda_1 - \lambda_2}{det(H)} h_{11} h_{21} \\
&= \frac{\lambda_1 - \lambda_2}{det(H)}(h_{111} - m_1 n_1 h_{112})(\frac{1}{n_1}h_{211} + \frac{1}{m_1}h_{212}) \\
&= \frac{\lambda_1 - \lambda_2}{det(H)}(\frac{1}{n_1}h_{111}h_{211} + \frac{1}{m_1}h_{111}h_{212} - m_1 h_{112}h_{211} - n_1 h_{112}h_{212}) \\
&= \psi'_{12} \quad ,
\end{aligned}$$

$$
\begin{aligned}
D_{21} &= \frac{S'_{21}}{det(H)} \\
&= \frac{\lambda_1 h_{12}h_{22} - \lambda_2 h_{12}h_{22}}{det(H)} \\
&= \frac{\lambda_1 - \lambda_2}{det(H)} h_{12}h_{22} \\
&= \frac{\lambda_1 - \lambda_2}{det(H)} (h_{121} - m_1 n_1 h_{122})(\frac{1}{n_1} h_{221} + \frac{1}{m_1} h_{222}) \\
&= \frac{\lambda_1 - \lambda_2}{det(H)} (\frac{1}{n_1} h_{121}h_{221} + \frac{1}{m_1} h_{121}h_{222} - m_1 h_{122}h_{221} - n_1 h_{122}h_{222}) \\
&= \psi'_{21}
\end{aligned}
$$

and

$$
\begin{aligned}
D_{22} &= -\lambda + \frac{S'_{22}}{det(H)} \\
&= -\lambda + \frac{-\lambda_1 h_{12}h_{21} + \lambda_2 h_{11}h_{22}}{det(H)} \\
&= -\lambda + \frac{1}{det(H)} [-\lambda_1 (h_{121} - m_1 n_1 h_{122})(\frac{1}{n_1} h_{211} + \frac{1}{m_1} h_{212}) \\
&\quad +\lambda_2 (h_{111} - m_1 n_1 h_{112})(\frac{1}{n_1} h_{221} + \frac{1}{m_1} h_{222})] \\
&= -\lambda + \frac{1}{det(H)} [-\lambda_1 (\frac{1}{n_1} h_{121}h_{211} + \frac{1}{m_1} h_{121}h_{212} - m_1 h_{122}h_{211} - n_1 h_{122}h_{212}) \\
&\quad +\lambda_2 (\frac{1}{n_1} h_{111}h_{221} + \frac{1}{m_1} h_{111}h_{222} - m_1 h_{112}h_{221} - n_1 h_{112}h_{222})] \\
&= \psi'_{22} \quad .
\end{aligned}
$$

So, the expression of the Darboux matrix is

$$
\begin{aligned}
D(x,t;\lambda) &= \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \\
&= \begin{pmatrix} \psi'_{11} & \psi'_{12} \\ \psi'_{21} & \psi'_{22} \end{pmatrix} \\
&= (\lambda - \lambda_1)[\frac{1}{\lambda - \lambda_1} \begin{pmatrix} \psi'_{11} & \psi'_{12} \\ \psi'_{21} & \psi'_{22} \end{pmatrix}] \\
&= (\lambda - \lambda_1)[(I + \frac{\lambda_1 - \lambda_2}{\lambda - \lambda_1} P) J] \\
&= (\lambda - \lambda_1)\psi(x,t;\lambda)
\end{aligned}
$$

Note that the terms ψ'_{ij}, where $i, j = 1, 2$, exactly are the entries of the matrix ψ in (2.69) which we haven gotten in the Section 2.4.

Therefore, we have proved that the two transformation of the AKNS system will provide us the same result even though they are derived by distinct ways (in complex analysis and in algebraic).

4 Example: Darboux Transformation of mK-dV Equation

In this section, we will illustrate how to use the Darboux transformation to get another solution from the known solution of the nonlinear partial equation which we consider.

Recall that the AKNS system which we have introduced in Sec 1.2 is

$$\begin{aligned} \Phi_x(x,t;\lambda) &= \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix} \Phi(x,t;\lambda), \\ \Phi_t(x,t;\lambda) &= \begin{pmatrix} A(x,t;\lambda) & B(x,t;\lambda) \\ C(x,t;\lambda) & -A(x,t;\lambda) \end{pmatrix} \Phi(x,t;\lambda). \end{aligned} \tag{4.92}$$

For the mK-dV equation

$$u_t + u_{xxx} - 6u^2 u_x = 0,$$

if we take $r = q$, and A, B, C to be

$$\begin{aligned} A &= rq_x - qr_x - 2i\lambda qr - 4i\lambda^3, \\ B &= -q_{xx} + 2rq^2 + 2i\lambda q_x + 4\lambda^2 q, \\ C &= -r_{xx} + 2qr^2 - 2i\lambda r_x + 4\lambda^2 r, \end{aligned} \tag{4.93}$$

then the integrability conditions for all λ will be exactly the mK-dV equation

$$q_t + q_{xxx} - 6q^2 q_x = 0. \tag{4.94}$$

4.1 The Explicit Form of the Darboux Transformation of mK-dV Equation

First, we observe the first equation of equation (4.92) which we choose $q^{(0)} = r^{(0)} = u$, where u is a known solution of the mK-dV equation. Then

$$\Phi_x^{(0)} = U^{(0)}\Phi^{(0)} = \begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix} \Phi^{(0)}. \tag{4.95}$$

Suppose that $\Phi^{(0)} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is a solution of (4.95) w.r.t. $\lambda = \lambda_0$,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_x = \begin{pmatrix} -i\lambda_0 & u \\ u & i\lambda_0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{4.96}$$

Then it is easy to directly check that $\Phi^{(0)} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ shall be a solution of (4.95) w.r.t. $\lambda = -\lambda_0$, that is,

$$\begin{pmatrix} \beta \\ \alpha \end{pmatrix}_x = \begin{pmatrix} i\lambda_0 & u \\ u & -i\lambda_0 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}.$$

Apply Theorem[2], let us choose that

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix}, \quad H = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}.$$

Then we first compute that

$$H^{-1} = \frac{1}{\alpha^2 - \beta^2}\begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix},$$

and therefore

$$\begin{aligned} S &= H\Lambda H^{-1} \\ &= \frac{\lambda_0}{1+(\frac{i\beta}{\alpha})^2}\begin{bmatrix} 1-(\frac{i\beta}{\alpha})^2 & 2i(\frac{i\beta}{\alpha}) \\ -2i(\frac{i\beta}{\alpha}) & -(1-(\frac{i\beta}{\alpha})^2) \end{bmatrix} \\ &= \frac{\lambda_0}{1+\sigma^2}\begin{bmatrix} 1-\sigma^2 & 2i\sigma \\ -2i\sigma & -(1-\sigma^2) \end{bmatrix} \\ &= \lambda_0 \begin{bmatrix} \cos\theta & i\sin\theta \\ -i\sin\theta & -\cos\theta \end{bmatrix} \end{aligned}$$

where

$$\sigma \equiv \frac{i\beta}{\alpha} \quad \text{and} \quad \tan\frac{\theta}{2} \equiv \sigma.$$

Then the Darboux transformation is

$$\begin{aligned} D &= J(\lambda I - S) \\ &= \begin{pmatrix} \lambda - \lambda_0\cos\theta & -i\lambda_0\sin\theta \\ -i\lambda_0\sin\theta & -(\lambda+\lambda_0\cos\theta) \end{pmatrix}, \end{aligned}$$

and the inverse of D is

$$D^{-1} = \frac{1}{-\lambda^2+\lambda_0^2}\begin{pmatrix} -(\lambda+\lambda_0\cos\theta) & i\lambda_0\sin\theta \\ i\lambda_0\sin\theta & \lambda-\lambda_0\cos\theta \end{pmatrix}.$$

Now, we shall apply the equation (3.73), that is,

$$U^{(1)} = DU^{(0)}D^{-1} + D_xD^{-1}$$

to get $U^{(1)}$.

For later convenience, we can first simplify the term $DU^{(0)}D^{-1}$ as follows:

$$\begin{aligned} DU^{(0)}D^{-1} &= \begin{pmatrix} \lambda - \lambda_0\cos\theta & -i\lambda_0\sin\theta \\ -i\lambda_0\sin\theta & -(\lambda+\lambda_0\cos\theta) \end{pmatrix}\begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix} D^{-1} \\ &= \begin{pmatrix} -i\lambda^2 + i\lambda\lambda_0\cos\theta - i\lambda_0 u\sin\theta & \lambda u - \lambda_0 u\cos\theta + \lambda\lambda_0\sin\theta \\ -\lambda\lambda_0\sin\theta - \lambda u - \lambda_0 u\cos\theta & -i\lambda_0 u\sin\theta - i\lambda^2 - i\lambda\lambda_0\cos\theta \end{pmatrix} \\ &\quad \cdot \frac{1}{-\lambda^2+\lambda_0^2}\begin{pmatrix} -(\lambda+\lambda_0\cos\theta) & i\lambda_0\sin\theta \\ i\lambda_0\sin\theta & \lambda-\lambda_0\cos\theta \end{pmatrix} \\ &= \frac{1}{-\lambda^2+\lambda_0^2}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= i\lambda^3 + 2i\lambda\lambda_0 u\sin\theta + i\lambda\lambda_0^2\sin^2\theta - i\lambda\lambda_0^2\cos^2\theta \\ a_{12} &= 2\lambda^2\lambda_0\sin\theta - 2\lambda\lambda_0^2\sin\theta\cos\theta + \lambda_0^2 u + \lambda^2 u - 2\lambda\lambda_0 u\cos\theta \\ a_{21} &= 2\lambda^2\lambda_0\sin\theta + 2\lambda\lambda_0^2\sin\theta\cos\theta + \lambda_0^2 u + \lambda^2 u + 2\lambda\lambda_0 u\cos\theta \\ \text{and} \quad a_{22} &= -i\lambda^3 - 2i\lambda\lambda_0 u\sin\theta - i\lambda\lambda_0^2\sin^2\theta + i\lambda\lambda_0^2\cos^2\theta \end{aligned}$$

In order to get D_x, we can first compute θ_x. Since

$$\begin{aligned}
\sigma_x &= (\frac{i\beta}{\alpha})_x \\
&= i\frac{\alpha\beta_x - \alpha_x\beta}{\alpha^2} \\
&= i\frac{\alpha(u\alpha + i\lambda_0\beta) - (-i\lambda_0\alpha + u\beta)\beta}{\alpha^2} \\
&= \frac{i}{\alpha^2}(u\alpha^2 + i\lambda_0\alpha\beta + i\lambda_0\alpha\beta - u\beta^2) \\
&= i(u + 2\lambda_0(\frac{i\beta}{\alpha}) + u(\frac{i\beta}{\alpha})^2) \\
&= i(u + 2\lambda_0\sigma + u\sigma^2) \\
&= iu(1+\sigma^2) + 2i\lambda_0\sigma,
\end{aligned}$$

and therefore

$$\begin{aligned}
\theta_x &= (2\tan^{-1}\sigma)_x \\
&= 2\frac{\sigma_x}{1+\sigma^2} \\
&= \frac{2}{1+\sigma^2}(iu(1+\sigma^2) + 2i\lambda_0\sigma) \\
&= 2iu + 2i\lambda_0\frac{2\sigma}{1+\sigma^2} \\
&= 2iu + 2i\lambda_0\sin\theta.
\end{aligned}$$

Hence

$$\begin{aligned}
D_x &= \begin{pmatrix} \lambda - \lambda_0\cos\theta & -i\lambda_0\sin\theta \\ -i\lambda_0\sin\theta & -\lambda - \lambda_0\theta_x\cos\theta \end{pmatrix}_x \\
&= \begin{pmatrix} \lambda_0\theta_x\sin\theta & -i\lambda_0\theta_x\cos\theta \\ -i\lambda_0\theta_x\cos\theta & \lambda_0\theta_x\sin\theta \end{pmatrix} \\
&= \lambda_0\theta_x \begin{pmatrix} \sin\theta & -i\cos\theta \\ -i\cos\theta & \sin\theta \end{pmatrix}.
\end{aligned}$$

Then we compute the term D_xD^{-1} as follows:

$$\begin{aligned}
D_xD^{-1} &= \lambda_0(2iu + 2i\lambda_0\sin\theta)\begin{pmatrix} \sin\theta & -i\cos\theta \\ -i\cos\theta & \sin\theta \end{pmatrix} \\
&\quad \cdot\frac{1}{-\lambda^2+\lambda_0^2}\begin{pmatrix} -(\lambda + \lambda_0\cos\theta) & i\lambda_0\sin\theta \\ i\lambda_0\sin\theta & \lambda - \lambda_0\cos\theta \end{pmatrix} \\
&= \frac{1}{-\lambda^2+\lambda_0^2}\begin{pmatrix} 2i\lambda_0u\sin\theta + 2i\lambda_0^2\sin^2\theta & 2\lambda_0u\cos\theta + 2\lambda_0^2\sin\theta\cos\theta \\ 2\lambda_0u\cos\theta + 2\lambda_0^2\sin\theta\cos\theta & 2i\lambda_0u\sin\theta + 2i\lambda_0^2\sin^2\theta \end{pmatrix} \\
&\quad \cdot\begin{pmatrix} -\lambda - \lambda_0\cos\theta & i\lambda_0\sin\theta \\ i\lambda_0\sin\theta & \lambda - \lambda_0\cos\theta \end{pmatrix} \\
&= \frac{1}{-\lambda^2+\lambda_0^2}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},
\end{aligned}$$

where

$$
\begin{aligned}
b_{11} &= -2i\lambda\lambda_0 u \sin\theta - 2i\lambda\lambda_0^2 \sin^2\theta \\
b_{12} &= -2\lambda_0^2 u - 2\lambda_0^3 \sin\theta + 2\lambda\lambda_0^2 \sin\theta\cos\theta + 2\lambda\lambda_0 u \cos\theta \\
b_{21} &= -2\lambda_0^2 u - 2\lambda_0^3 \sin\theta - 2\lambda\lambda_0^2 \sin\theta\cos\theta - 2\lambda\lambda_0 u \cos\theta \\
b_{22} &= 2i\lambda\lambda_0 u \sin\theta + 2i\lambda\lambda_0^2 \sin^2\theta
\end{aligned}
$$

Finally, from the equation (3.73), we shall get

$$
\begin{aligned}
\begin{pmatrix} -i\lambda & u^{(1)} \\ u^{(1)} & i\lambda \end{pmatrix} &= U^{(1)} \\
&= DU^{(0)}D^{-1} + D_x D^{-1} \\
&= \frac{1}{-\lambda^2+\lambda_0^2}\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \\
&= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
c_{11} &= \frac{1}{-\lambda^2+\lambda_0^2}(i\lambda^3 + 2i\lambda\lambda_0 u \sin\theta + i\lambda\lambda_0^2 \sin^2\theta - i\lambda\lambda_0^2\cos^2\theta \\
&\quad -2i\lambda\lambda_0 u \sin\theta - 2i\lambda\lambda_0^2 \sin^2\theta) \\
&= \frac{1}{-\lambda^2+\lambda_0^2}(i\lambda^3 - i\lambda\lambda_0^2) \\
&= \frac{-i\lambda}{-\lambda^2+\lambda_0^2}(-\lambda^2+\lambda_0^2) \\
&= -i\lambda,
\end{aligned}
$$

$$
\begin{aligned}
c_{12} &= \frac{1}{-\lambda^2+\lambda_0^2}(2\lambda^2\lambda_0\sin\theta - 2\lambda\lambda_0^2\sin\theta\cos\theta + \lambda_0^2 u + \lambda^2 u - 2\lambda\lambda_0 u\cos\theta \\
&\quad +2\lambda\lambda_0^2\sin\theta\cos\theta - 2\lambda_0^2 u + 2\lambda\lambda_0 u\cos\theta - 2\lambda_0^3\sin\theta) \\
&= \frac{1}{-\lambda^2+\lambda_0^2}(-u(-\lambda^2+\lambda_0^2) - 2\lambda_0\sin\theta(-\lambda^2+\lambda_0^2)) \\
&= -u - 2\lambda_0\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
c_{21} &= \frac{1}{-\lambda^2+\lambda_0^2}(2\lambda^2\lambda_0\sin\theta + 2\lambda\lambda_0^2\sin\theta\cos\theta + \lambda_0^2 u + \lambda^2 u + 2\lambda\lambda_0 u\cos\theta \\
&\quad -2\lambda\lambda_0^2\sin\theta\cos\theta - 2\lambda_0^2 u - 2\lambda\lambda_0 u\cos\theta - 2\lambda_0^3\sin\theta) \\
&= \frac{1}{-\lambda^2+\lambda_0^2}(-u(-\lambda^2+\lambda_0^2) - 2\lambda_0\sin\theta(-\lambda^2+\lambda_0^2)) \\
&= -u - 2\lambda_0\sin\theta
\end{aligned}
$$

and

$$\begin{aligned} c_{22} &= \frac{1}{-\lambda^2+\lambda_0^2}(-i\lambda^3 - 2i\lambda\lambda_0 u\sin\theta - i\lambda\lambda_0^2\sin^2\theta + i\lambda\lambda_0^2\cos^2\theta \\ &\quad +2i\lambda\lambda_0 u\sin\theta + 2i\lambda\lambda_0^2\sin^2\theta) \\ &= \frac{1}{-\lambda^2+\lambda_0^2}(-i\lambda^3 + i\lambda\lambda_0^2) \\ &= \frac{i\lambda}{-\lambda^2+\lambda_0^2}(-\lambda^2+\lambda_0^2) \\ &= i\lambda. \end{aligned}$$

Thus we shall get the new solution $u^{(1)}$ form the known solution of the mK-dV equation:

$$\begin{aligned} u^{(1)} &= -u - 2\lambda_0\sin\theta \\ &= -u - 2\lambda_0\frac{2\sigma}{1+\sigma^2} \\ &= -u - 2\lambda_0\frac{2i\alpha\beta}{\alpha^2-\beta^2} \\ &= -u - 4i\lambda_0\frac{\alpha\beta}{\alpha^2-\beta^2}. \end{aligned}$$

4.2 Apply This Darboux Transformation to Get A New Solution

Now, we take the known solution $q^{(0)}$, $r^{(0)}$ to be trivial solution

$$q^{(0)} = r^{(0)} = c \neq 0$$

which is a real constant. Then we can find the fact from (4.93) and (4.94) to be

$$V^{(0)}(\lambda) = (2c^2+4\lambda^2)U^{(0)}(\lambda).$$

This shall imply that $\Phi^{(0)}(\lambda)$ to be the following form:

$$\Phi^{(0)}(\lambda) = \Phi^{(0)}[x+(2c^2+4\lambda^2)t;\lambda)]. \tag{4.97}$$

First, we need to solve the equation as follows:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_x = \begin{pmatrix} -i\lambda & c \\ c & i\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

that is, we will solve the equation system

$$\begin{cases} \alpha_x = -i\lambda\alpha + c\beta \\ \\ \beta_x = c\alpha + i\lambda\beta \end{cases}.$$

Therefore, it will have two linear independent solutions, and so

$$\Phi^{(0)}(x;\lambda) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} l_1\cdot e^{k(x-x_0)} & +l_2\cdot e^{-k(x-x_0)} \\ l_1\cdot \frac{i\lambda+k}{c}e^{k(x-x_0)} & +l_2\cdot\frac{i\lambda-k}{c}e^{-k(x-x_0)} \end{pmatrix},$$

where $k \equiv \sqrt{c^2-\lambda^2}$, and x_0 is a constant which in general may depend on λ and c.

Let us choose that $l_1 = 1$ and $l_2 = -i\dfrac{k+i\lambda}{|c|}$. Then we can compute α and β as follows:

$$\begin{aligned}
\alpha(x;\lambda) &= e^{k(x-x_0)} + (-i\frac{k+i\lambda}{|c|})e^{-k(x-x_0)}, \\
\beta(x;\lambda) &= \frac{k+i\lambda}{c}e^{k(x-x_0)} + (-i\frac{k+i\lambda}{|c|})\frac{-k+i\lambda}{c}e^{-k(x-x_0)} \\
&= \frac{k+i\lambda}{c}e^{k(x-x_0)} + i\frac{|c|}{c}e^{-k(x-x_0)}.
\end{aligned}$$

Before computing the new solution, note that the three equation

$$\begin{cases} k^2 = c^2 - \lambda^2 \\ \lambda^2 = c^2 - k^2 \\ k^2 + \lambda^2 = c^2 \end{cases}$$

and first simplify the term which we will use

$$\begin{aligned}
c^2 - (k+i\lambda)^2 &= c^2 - k^2 - 2ik\lambda + \lambda^2 \\
&= \lambda^2 - 2ik\lambda + \lambda^2 \\
&= -2i\lambda(k+i\lambda)
\end{aligned}$$

Now if let $k_1 > 0$ be an arbitrary constant, and so $\lambda_1 = \sqrt{c^2 - k_1^2}$. Hence we have

$$k = \sqrt{c^2 - \lambda_1^2} = k_1 .$$

Thus, we can begin to compute $q^{(1)}$ as follows:

$$
\begin{aligned}
q^{(1)}(x;\lambda_1) &= -q^{(0)}(x) - 4i\lambda_1 \frac{\alpha(x;\lambda_1)\beta(x;\lambda_1)}{\alpha^2(x;\lambda_1)-\beta^2(x;\lambda_1)} \\
&= -c - 4i\lambda_1\Big[\frac{k_1+i\lambda_1}{c}e^{2k_1(x-x_0)} + \Big(-i\frac{k_1+i\lambda_1}{|c|}\Big)\Big(i\frac{|c|}{c}\Big)e^{-2k_1(x-x_0)} \\
&\quad +\Big(i\frac{|c|}{c} - i\frac{k_1+i\lambda_1}{|c|}\frac{k_1+i\lambda_1}{c}\Big)\Big]/\Big[\Big(1-\Big(\frac{k_1+i\lambda_1}{c}\Big)^2\Big)e^{2k_1(x-x_0)} \\
&\quad +\Big(\Big(-i\frac{k_1+i\lambda_1}{|c|}\Big)^2 - \Big(i\frac{|c|}{c}\Big)^2\Big)e^{-2k_1(x-x_0)} + 2\Big(\Big(-i\frac{k_1+i\lambda_1}{|c|}\Big) - \Big(\frac{k_1+i\lambda_1}{c}\Big)\Big(i\frac{|c|}{c}\Big)\Big)\Big] \\
&= -c - 4i\lambda_1 \frac{\frac{k_1+i\lambda_1}{c}e^{2k_1(x-x_0)} + \frac{k_1+i\lambda_1}{c}e^{-2k_1(x-x_0)} + i\frac{c^2-(k_1+i\lambda_1)^2}{|c|c}}{\big(\frac{c^2-(k_1+i\lambda_1)^2}{c^2}\big)e^{2k_1(x-x_0)} + \big(\frac{c^2-(k_1+i\lambda_1)^2}{c^2}\big)e^{-2k_1(x-x_0)} + \big(-4i\frac{k_1+i\lambda_1}{|c|}\big)} \\
&= -c - 4i\lambda_1 \frac{\frac{k_1+i\lambda_1}{c}e^{2k_1(x-x_0)} + \frac{k_1+i\lambda_1}{c}e^{-2k_1(x-x_0)} + i\frac{-2i\lambda_1(k_1+i\lambda_1)}{|c|c}}{\big(\frac{-2i\lambda_1(k_1+i\lambda_1)}{c^2}\big)e^{2k_1(x-x_0)} + \big(\frac{-2i\lambda_1(k_1+i\lambda_1)}{c^2}\big)e^{-2k_1(x-x_0)} + \big(-4i\frac{k_1+i\lambda_1}{|c|}\big)} \\
&= -c - 4i\lambda_1 \frac{\frac{1}{c}\big(e^{2k_1(x-x_0)} + e^{-2k_1(x-x_0)} + 2\frac{\lambda_1}{|c|}\big)}{-2i\frac{1}{c^2}\big(\lambda_1 e^{2k_1(x-x_0)} + \lambda_1 e^{-2k_1(x-x_0)} + 2|c|\big)} \\
&= -c + 2c\frac{\lambda_1 e^{2k_1(x-x_0)} + \lambda_1 e^{-2k_1(x-x_0)} + 2\frac{c^2}{|c|} - 2\frac{k_1^2}{|c|}}{\lambda_1 e^{2k_1(x-x_0)} + \lambda_1 e^{-2k_1(x-x_0)} + 2|c|} \\
&= -c + 2c - 2c\frac{2k_1^2/|c|}{\lambda_1(e^{2k_1(x-x_0)} + e^{-2k_1(x-x_0)}) + 2|c|} \\
&= c - \frac{2k_1^2 c/|c|}{\lambda_1 \cosh 2k_1(x-x_0) + |c|} \\
&= c - \frac{2k_1^2 c/|c|}{|c| + \sqrt{c^2-k_1^2}\cosh 2k_1(x-x_0)}.
\end{aligned}
$$

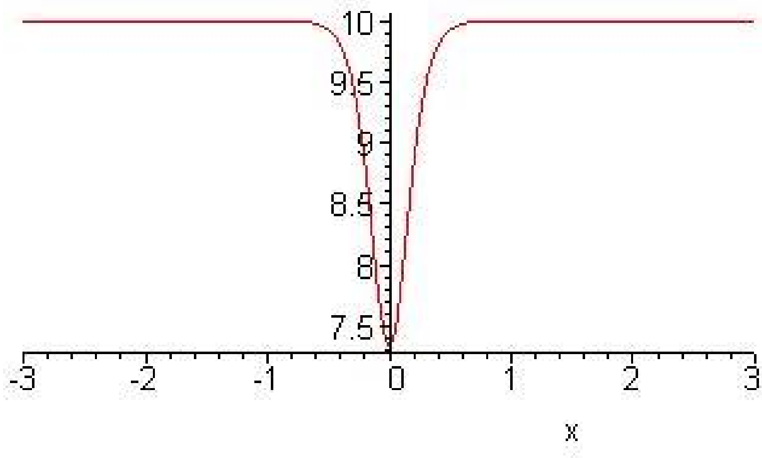

Figure 4: Sketch of the graph of soliton solution $q^{(1)}$ for $c = 10, k_1 = 5, t = 0, x_0 = 0$.

So, for $|c| > k_1$,

$$
q^{(1)}(x,t;\lambda_1) = r^{(1)}(x,t;\lambda_1) = c - \frac{2k_1^2 c/|c|}{|c| + \sqrt{c^2-k_1^2}\cosh 2k_1[x - x_0 + (6c^2 - 4k_1^2)t]}, \quad (4.98)
$$

which shall be a soliton solution again. (see Fig. 4)

However, if we choose $x_0(\lambda, c)$ to be

$$x_0(\lambda, c) = x_1 + \frac{1}{4\sqrt{c^2-\lambda^2}} \ln \frac{|c| + \sqrt{c^2-\lambda^2}}{|c| - \sqrt{c^2-\lambda^2}},$$

where x_1 is a constant, and if we choose λ_1 as before, then the solution (4.98) will take the form

$$\begin{aligned} q^{(1)}(x,t;\lambda_1) = r^{(1)}(x,t;\lambda_1) &= c - (k_1^2 c/|c|)/\{|c| \cosh^2 k_1[x - x_1 + (6c^2 - 4k_1^2)t] \\ &\quad - \frac{k_1}{2} \sinh 2k_1[x - x_1 + (6c^2 - 4k_1^2)t]\}. \end{aligned} \tag{4.99}$$

Now for $\pm c > 0$, we take the limit $k_1 \to \pm c$ again. Hence, (4.99) shall become

$$q^{(1)}(x,t;\lambda_1) = r^{(1)}(x,t;\lambda_1) = \mp c \ \tanh c[x - x_1 + 2c^2 t], \tag{4.100}$$

which is a kink solution. (see Fig. 5)

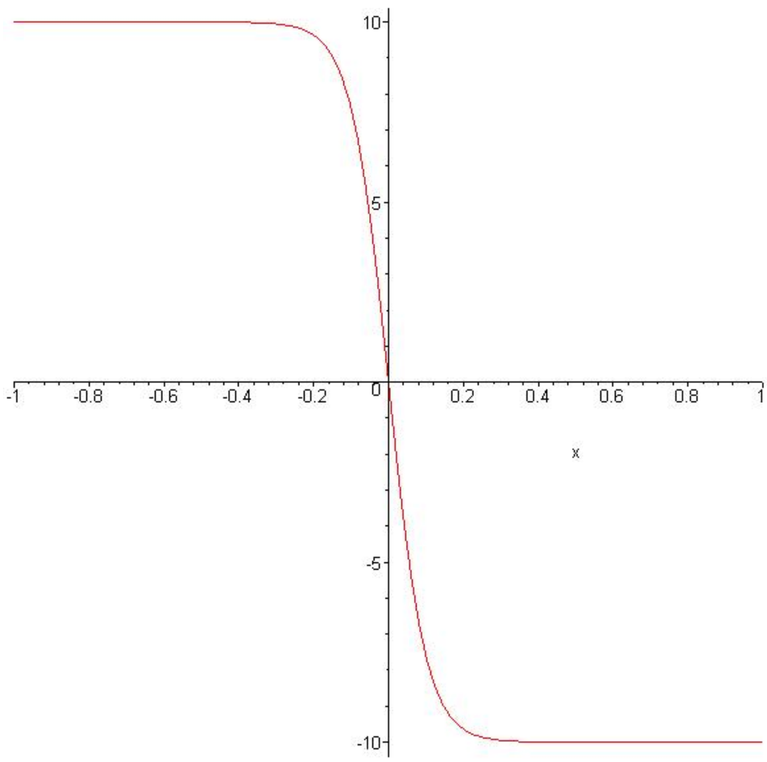

Figure 5: Sketch of the graph of kink solution $q^{(1)}$ for $c = 10, t = 0, x_1 = 0$.

In summary, we have generated a soliton solution (4.98) and a kink solution (4.100) from a trivial solution of the mK-dV equation.

Note that if we observe $q^{(1)}$ and $r^{(1)}$ (i.e. (3.89) and (3.90)), we shall find these two expressions also depends on a pair of parameters (l_1, l_2) (as the pair of parameters (m_1, n_1) in the explicit Bäcklund transformation). On practice of the Darboux transformation (or the explicit Bäcklund transformation), we must take care the choice of this pair of parameters (l_1, l_2). Otherwise, the two functions $q^{(1)}$ and $r^{(1)}$ may not be real functions.

For example, if we choose $l_1 = l_2 = 1$ in above process which we compute the new solution of mK-dV equation, then we have

$$\begin{aligned}\alpha(x;\lambda) &= 1\cdot h_{111} + 1\cdot h_{112}\\ &= e^{k(x-x_0)} + e^{-k(x-x_0)}\ ,\\ \beta(x;\lambda) &= 1\cdot h_{121} + 1\cdot h_{122}\\ &= \frac{k+i\lambda}{c}e^{k(x-x_0)} + \frac{-k+i\lambda}{c}e^{-k(x-x_0)}\ .\end{aligned}$$

Hence,

$$\begin{aligned}q^{(1)} &= -q^{(0)}(x;\lambda) - 4i\lambda_1\frac{\alpha(x;\lambda)\beta(x;\lambda)}{\alpha^2(x;\lambda)-\beta^2(x;\lambda)}\\
&= -c\\
&\quad -4i\lambda_1\frac{\frac{k_1+i\lambda_1}{c}e^{2k_1(x-x_0)} + \frac{-k_1+i\lambda_1}{c}e^{-2k_1(x-x_0)} + (\frac{k_1+i\lambda_1}{c} + \frac{-k_1+i\lambda_1}{c})}{(1-\frac{(k_1+i\lambda_1)^2}{c^2})e^{2k_1(x-x_0)} + (1-\frac{(-k_1+i\lambda_1)^2}{c^2})e^{-2k_1(x-x_0)} + (2-2\frac{(k_1+i\lambda_1)(-k_1+i\lambda_1)}{c^2})}\\
&= -c\\
&\quad -4i\frac{\frac{1}{c}}{\frac{1}{c^2}}\cdot[(k_1\lambda_1 + i(c^2-k_1^2))e^{2k_1(x-x_0)} + (-k_1\lambda_1 + i(c^2-k_1^2))e^{-2k_1(x-x_0)} + 2i(c^2-k_1^2)]\\
&\quad /[(c^2-k_1^2-2ik_1\lambda_1 + (c^2-k_1^2))e^{2k_1(x-x_0)} + (c^2-k_1^2+2ik_1\lambda_1+(c^2-k_1^2))e^{-2k_1(x-x_0)}\\
&\quad +(2c^2+2(k_1^2+c^2-k_1^2))]\\
&= -c + 2c\frac{(c^2-k_1^2-ik_1\lambda_1)e^{2k_1(x-x_0)} + (c^2-k_1^2+ik_1\lambda_1)e^{-2k_1(x-x_0)} + 2(c^2-k_1^2)}{(c^2-k_1^2-ik_1\lambda_1)e^{2k_1(x-x_0)} + (c^2-k_1^2+ik_1\lambda_1)e^{-2k_1(x-x_0)} + 2c^2}\\
&= -c + 2c - 2c\frac{2k_1^2}{(c^2-k_1^2-ik_1\lambda_1)e^{2k_1(x-x_0)} + (c^2-k_1^2+ik_1\lambda_1)e^{-2k_1(x-x_0)} + 2c^2}\\
&= c - \frac{2k_1^2 c}{(c^2-k_1^2)\cosh 2k_1(x-x_0) - ik_1\lambda_1\sinh 2k_1(x-x_0) + c^2}\\
&= c - \frac{2k_1^2 c}{|c|\sqrt{c^2-k_1^2}(\frac{\sqrt{c^2-k_1^2}}{|c|}\cosh 2k_1(x-x_0) + \frac{-ik_1}{|c|}\sinh 2k_1(x-x_0)) + c^2}\ .\end{aligned}$$

For simplifying this expression, we need to define that

$$\begin{cases}\frac{\sqrt{c^2-k_1^2}}{|c|} \equiv \cosh 2k_1x_c\\ \frac{-ik_1}{|c|} \equiv \sinh 2k_1x_c\end{cases}\ .$$

Note that, x_c is not a real number.

Therefore,

$$\begin{aligned}q^{(1)} &= c - \frac{2k_1^2 c}{|c|\sqrt{c^2-k_1^2}(\cosh 2k_1x_c\cosh 2k_1(x-x_0) + \sinh 2k_1x_c\sinh 2k_1(x-x_0)) + c^2}\\ &= c - \frac{2k_1^2c/|c|}{\sqrt{c^2-k_1^2}\cosh 2k_1(x-x_0-x_c) + |c|}\ ,\end{aligned}$$

which is not a real function!!

5 Conclusion

We have introduced the explicit Bäcklund transformation (in Section 2.4) and the Darboux transformation (in Chapter 3). Form the definition of the Darboux transformation stated in Sec 3.1, we can find that the explicit Bäcklund transformation which stated in Sec 2.4 is also one kind of Darboux transformation. And, we have proved that these two kind of Darboux transformations in AKNS system are equivalence in Sec 3.4 even though they are derived by distinct ways (in complex analysis and in algebraic). Hence, we shall be able to use the alternative of these two transformations to get more solutions of the considered equation.

This transformation shall provide us a method to find new solutions explicitly from the known solution of the nonlinear partial differential equation which we consider. Further, we shall be able to make successively the transformations from a given initial solution u to get more soliton-solutions. Once we get $q^{(1)}$, $r^{(1)}$ from the input solution $q^{(0)}$, $r^{(0)}$, we can of course repeat the same procedure to obtain another new solution $q^{(2)}$, $r^{(2)}$, using $q^{(1)}$, $r^{(1)}$ as the input solution this time.

That is, if we want to get a new solution $u^{(1)}(x,t)$ from a given solution $u^{(0)}(x,t)$ of the nonlinear equation which we consider, we only need to input $u^{(0)}(x,t)$ into $Q^{(0)}(x,t)$ in the AKNS system (which is linear) and solve the solution $\Phi^{(0)}(x,t;\lambda)$ out. Then we can choose a parameter λ_1 to get $Q^{(1)}(x,t;\lambda_1)$ which consists $u^{(1)}(x,t) = u^{(1)}(x,t;\lambda_1)$ and construct the Darboux matrix $D^{(1)}(x,t;\lambda_1,\lambda)$. Hence we shall get a new solution $u^{(1)}(x,t)$ of the nonlinear equation.

Since we can directly get $\Phi^{(1)}(x,t;\lambda_1,\lambda) = D^{(1)}(x,t;\lambda_1,\lambda)\Phi^{(0)}(x,t;\lambda)$ which we do not need to solve the AKNS system again. Hence, we can repeat the above procedure to obtain the next solution $u^{(2)}(x,t)$ which consists in $Q^{(2)}(x,t;\lambda_1,\lambda_2,\lambda)$ and construct the Darboux matrix $D^{(2)}(x,t;\lambda_1,\lambda_2,\lambda)$ to get $\Phi^{(2)}(x,t;\lambda_1,\lambda_2,\lambda)$. Therefore, this transformation is really a powerful method of providing us more solutions of some nonlinear equation as follows:

$$(u^{(0)},\Phi^{(0)}) \overset{\lambda_1,D^{(1)}}{\rightarrow} (u^{(1)},\Phi^{(1)}) \overset{\lambda_2,D^{(2)}}{\rightarrow} (u^{(2)},\Phi^{(2)}) \overset{\lambda_3,D^{(3)}}{\rightarrow} \cdots .$$

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