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摘要

這篇論文用NIG GARCH 的模型去描述財務市場資產的log return。在這樣的模型假設下，我們可以經由Esscher transform的方法來做資產的定價，而這種方法定出來的價格可以用動態效用函數的架構來說明其合理性。

Normal Inverse Gaussian GARCH Model And Option Pricing

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Abstract

This article uses the NIG GARCH model, the GARCH model with Normal inverse Gaussian innovation, to model the financial asset return. Under this model, we can pricing derivatives via Conditional Esscher transform. The pricing result can be justified by dynamic power utility framework.

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Contents

1 Introduction

GARCH is the widely used model to describe the time changed volatility in financial market. It successfully catch the volatility clustering, but the conditional normal distribution for the innovation still has poor performance to fit the high kurtosis, fat tailed and skewness of the real data. The normal inverse Gaussian , a special subclass of generalized hyperbolic distribution, has been found the out-performance of fitting the financial return. [2] Barndorff-Nielsen(1997) propose the NIG stochastic volatility model, which is the ARCH type time series for the normal inverse Gaussian innovation. [1] Andersson(2001) extended the idea to GARCH type model with more flexible properties. [9] L. Forsberg using different type of parameterization to model the financial data.
For the purpose of pricing derivatives, we must find a equivalent martingale measure and then take the expectaion of payoff function under such probability measure. In the incomplete market we have infinitely many equivalent martingale measure, so [11] Gerber (1994) propose a attractive approach to find a reasonable equivalent martingale measure. Using this method, we can find a martingale measure via Esscher transform under the independent increment model. under Garch or other conditional modeling, we can follow the [16] Tak Kuen Siu's approach via the conditional Esscher transform.

2 Generalized hyperbolic distribution

2.1 Generalized hyperbolic distribution

Distributions that have tails heavier than the normal distribution are ubiquitous in finance. For purposes such as risk management and derivative pricing it is important to use relatively simple models that can capture the heavy tails and other relevant features of financial data. A class of distributions that is very often able to fit the distributions of financial data is the class of generalized hyperbolic distributions. This has been established in numerous investigations. In this paper, the generalized hyperbolic distribution has two different kind of parameter representation. Each representation has good properties in some aspect, and there is a one to one correspondence between two parameterization.

We say $X \sim GH(\lambda, \alpha, \beta, \delta, \mu)$, if it has the density

$$f_X(x) = \frac{(\frac{\gamma}{\delta})^\lambda}{\sqrt{2\pi}K_\lambda(\delta\gamma)} \frac{K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})}{(\sqrt{\delta^2+(x-\mu)^2}/\alpha)^{\frac{1}{2}-\lambda}} e^{\beta(x-\mu)} \tag{2.1}$$

where the parameters satisfies

$$\begin{array}{ll} \delta \geq 0, \alpha \geq 0, \alpha^2 > \beta^2 & \text{if } \lambda > 0. \\ \delta > 0, \alpha > 0, \alpha^2 > \beta^2 & \text{if } \lambda = 0. \\ \delta > 0, \alpha \geq 0, \alpha^2 \geq \beta^2 & \text{if } \lambda < 0. \end{array}$$

Let $\gamma = \sqrt{\alpha^2 - \beta^2}$, $\gamma_z = \sqrt{\alpha^2 - (\beta - z)^2}$
Then we can compute the moment generating function

$$M(z) = e^{\mu z} \frac{\gamma^\lambda K_\lambda(\delta\gamma_z)}{\gamma_z^\lambda K_\lambda(\delta\gamma)}$$

after calculating first and second moments the mean and variance are easily obtained

$$
\begin{aligned}
EX &= \mu + \frac{\delta\beta K_{\lambda+1}(\delta\gamma)}{\gamma K_\lambda(\delta\gamma)} \\
VarX &= \frac{\delta K_{\lambda+1}(\delta\gamma)}{\gamma K_\lambda(\delta\gamma)} + \frac{\beta^2\delta^2}{\gamma^2}\left(\frac{K_{\lambda+2}(\delta\gamma)}{K_\lambda(\delta\gamma)} - \frac{K^2_{\lambda+1}(\delta\gamma)}{K^2_\lambda(\delta\gamma)}\right)
\end{aligned}
$$

We focus on the special subclass of GH for $\lambda = 1/2$ called Normal Inverse Gaussian distribution. A random variable $X \sim NIG(\alpha, \beta, \delta, \mu)$ if the pdf is represented as

$$
f_X(x) = \frac{\alpha}{\pi} e^{(\delta\sqrt{\alpha^2-\beta^2}-\beta\mu)} \frac{K_1(\delta\alpha\sqrt{1+(\frac{x-\mu}{\delta})^2})}{\sqrt{1+(\frac{x-\mu}{\delta})^2}} e^{\beta x} \tag{2.2}
$$

The corresponding moment generating function is:

$$
M(z) = e^{\mu z + \delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+z)^2})} \tag{2.3}
$$

Mean and variance:

$$
EX = \mu + \beta\frac{\delta}{\gamma} \tag{2.4}
$$

$$
VarX = \frac{\delta\alpha^2}{\gamma^3} \tag{2.5}
$$

IF $X_i \sim NIG(\alpha, \beta, \delta_i, \mu_i)$ $i = 1, 2$, the we have $X_1 + X_2 \sim NIG(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2)$.

2.2 Generalized inverse Gaussian distribution

Now, we consider the Inverse Gaussian distribution. $X \sim IG(\delta, \gamma)$, where $\gamma = \sqrt{\alpha^2 - \beta^2}$, if it has the following density:

$$
f_X(x) = \frac{\delta}{\sqrt{2\pi x^3}} e^{-\frac{\gamma^2(x-\frac{\delta}{\gamma})^2}{2x}}
$$

which has mean and variance as:

$$
EX = \frac{\delta}{\gamma} \qquad VarX = \frac{\delta}{\gamma^3}
$$

2.3 Alternative parameterization of NIG

In order to have scale invariant properties for parameters except the scaling and location parameters, we use another parameterization for NIG distribution suggested by [9] Lars

Forsberg (2002) and [10] Lars Forsberg (2002) with

$$\begin{aligned}
\alpha' &= \alpha\delta \\
\beta' &= \beta/\alpha \\
\mu' &= \mu \\
\delta' &= \delta/\alpha \\
\Leftrightarrow & \\
\alpha &= \sqrt{\alpha'/\delta'} \\
\beta &= \sqrt{\alpha'/\delta'}\beta' \\
\mu &= \mu' \\
\delta &= \sqrt{\alpha'\delta'}
\end{aligned}$$

The density of the result parameterization,which we shall denote $X \sim NIG(\alpha', \beta', \mu', \delta')$ can be written with density

$$f_X(x) = \frac{\sqrt{\alpha'}}{\pi\sqrt{\delta'}} e^{\alpha'\sqrt{1-\beta'^2}+\beta'\sqrt{\alpha'/\delta'}(x-\mu')} \frac{K_1\left(\alpha'\sqrt{1+\frac{(x-\mu)^2}{\alpha'\delta'}}\right)}{\sqrt{1+\frac{(x-\mu)^2}{\alpha'\delta'}}} \tag{2.6}$$

where $0 \leq \alpha'$, $\mu \in \mathbb{R}$, $|\beta'| < 1$ and $0 \leq \delta'$. Under this parameterization the moment generating function becomes

$$M(z) = \exp\left(\mu' z + \alpha'\sqrt{1-\beta'^2} - \sqrt{\alpha'^2 - (\alpha'\beta' + \sqrt{\alpha'\delta'}z)^2}\right) \tag{2.7}$$

According to the moment generating function for the new parameterization, we can see that

Lemma 2.1. *The scaling properties of the $NIG(\alpha', \beta', \mu', \delta')$ parameterization are given by the following. Let $Z_1 \sim NIG(\alpha', \beta', \mu', \delta')$, then $cZ_1 + d \sim NIG(\alpha', \beta', c\mu' + d, c^2\delta')$, i.e α' and β' does not change under scaling and shifting.*

The first four central moments can be obtained by the cumulant generating function $\ln(M(u))$

$$\begin{aligned}
\kappa_1 &= EX = \mu' + \frac{\sqrt{\alpha'\delta'}}{\sqrt{1-\beta'^2}}\beta' \\
\kappa_2 &= VarX = \frac{\delta'}{(1-\beta'^2)^{3/2}} \\
\kappa_3 &= \frac{3\delta'^{3/2}\beta'}{\sqrt{\alpha'}(1-\beta'^2)^{5/2}} \\
\kappa_4 &= \frac{3\delta'^2(4\beta'^2+1)}{\alpha'(1-\beta'^2)^{7/2}}
\end{aligned}$$

Here, we have the skewness and kurtosis are given by

$$\begin{aligned}
\text{skewness} &= \frac{\kappa_3}{(\kappa_2)^{3/2}} = \frac{3\beta'}{\sqrt{\alpha'}(1-\beta'^2)^{1/4}} \\
\text{kurtosis} &= \frac{\kappa_4}{\kappa_2^2} = \frac{3(4\beta'^2+1)}{\alpha'\sqrt{1-\beta'^2}}
\end{aligned}$$

2.4 The GARCH NIG model

Assume X_t is the conditional return on its variance Z_t is normally distributed

$$X_t|Z_t \sim N(\mu_t' + \sqrt{\alpha'/\delta}\beta' Z_t, Z_t) \tag{2.8}$$

where $\mu_t' = \mathbb{E}(X_t|Z_t)$ is the conditonal mean, and the variance Z_t is inverse Gaussian distributed given $\mathcal{F}_{t-1}$

$$Z_t|\mathcal{F}_{t-1} \sim IG(\sqrt{\alpha'\delta_t'}, \sqrt{\frac{\alpha'}{\delta_t'}(1-\beta'^2)}) \tag{2.9}$$

with $\mathbb{E}(Z_t|\mathcal{F}_{t-1}) = \frac{\delta_t'}{\sqrt{1-\beta'^2}}$.
Now, the X_t conditionally on $\mathcal{F}_{t-1}$ are normal inverse Gaussian distributed

$$X_t|\mathcal{F}_{t-1} \sim NIG(\alpha', \beta', \mu_t', \delta_t') \tag{2.10}$$

then the conditional variance of X_t is given by $h_t = Var(X_t|\mathcal{F}_{t-1}) = \frac{\delta_t'}{(1-\beta'^2)^{3/2}}$ which follows the GARCH type

$$h_t = \rho_0 + \sum_{i=1}^{q} \rho_i (X_{t-i} - EX_{t-i})^2 + \sum_{j=1}^{p} \pi_j h_{t-j} \tag{2.11}$$

we have the GARCH-NIG(1,1)

$$h_t = \rho_0 + \rho_1 (X_{t-1} - EX_{t-1})^2 + \pi_1 h_{t-1} \tag{2.12}$$

3 Pricing Derivative Under NIG GRACH Model

After constructing the sotck price model. We may be interesting in pricing the derivative. [11] Gerber and Shiu (1994) and [12] Gerber and Shiu(1996) provide a method to pricing options or derivative securities by Esscher transform, which is an efficient tool for pricing many option and contingent claims if the logarithms of the price of the primary securities are certain stochastic process with stationary and independent increment, more generally, the condition of stationarity and independence can be drop, then we can apply this method to GARCH models.

3.1 Conditional Esscher Transform

Our next step is to pricing option under GARCH model, then we need to consider the process without stationarity and independence. Instead of Esscher transform, we need a further tool to obtain the proper equivalent martingale by the so called conditional Esscher transform.
We consider discrete time financial model consisting of one risk-free interest rate r and one risky stock S. For generality, we assume that the innovations process of the underlying stock S is infinitely divisible and that the moment generating function of its distribution exits.
Suppose the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and τ be the time index set $0, 1, 2, \ldots, T$. The stock price process follows

$$S_t = S_{t-1} e^{X_t} \qquad t \in \tau$$

where $X_t|\mathcal{F}_{t-1}$ is normal inverse Gaussian distributed ,we denote it by $X_t|\mathcal{F}_{t-1} \sim NIG(\alpha', 0, \mu', \delta')$ as our second parameterization of NIG we have$\{\delta'_t\}_{t\in\tau}$ a conditional variance process of the underlying stock. We suppose that $\delta'_t \in \mathcal{F}_{t-1}$ $\forall t \in \tau\backslash\{0\}$. Now, assume that $\{X_t\}_{t\in\tau}$ follows a GARCH(1,1) process. So under $\mathbb{P}$

$\forall t \in \tau\backslash\{0\}$

$$\delta'_t = \rho_0 + \rho_1(X_{t-1} - \mu')^2 + \pi_1\delta'_{t-1}. \tag{3.1}$$

where $\rho_0 > 0$, $\alpha_i \geq 0$, and $\pi_1 \geq 0$ $\forall$ $i = 1, 2$

For the covariance stationarity of the GARCH model, we further impose the condition that

$$\rho_1 + \pi_1 < 1 \tag{3.2}$$

[6] Duan (1995) introduced the LRNVR for pricing and assumed that the martingale measure $\mathcal{Q}$ with the LRNVR satisfies some conditions. [16] Tak Kuen Siu (2004) extend the condition for Normal innovation to infinitely divisible distribution. Normal inverse Gaussian is indeed infinitely divisible, and the distribution is invariant under conditional Esscher transform, we also relax the invariant variance condition, we get

1. $\mathbb{Q} \sim \mathbb{P}$
2. $\ln \frac{S_t}{S_{t-1}}$ is normal inverse Gaussian distributed.
3. $\mathbb{E}_{\mathbb{Q}}[\frac{S_t}{S_{t-1}}|\mathcal{F}_{t-1}] = e^r$ a.s.

In the sequel, we construct conditional Esscher transforms for the GARCH process $\{X_t\}_{t\in\tau}$ associated with a sequence of conditional Esscher parameter $\{\theta_t\}_{t\in\tau}$.
Suppose $\{\theta_t\}_{t\in\tau\backslash\{0\}}$ is a stochastic process with $\theta_t \in \mathcal{F}_{t-1}$, for all $t \in \tau\backslash\{0\}$. Let $M_{X_t|\mathcal{F}_{t-1}}(z)$ be the moment generating function of the conditional distribution $X_t|\mathcal{F}_{t-1}$ under $\mathbb{P}$, where $z \in \mathbb{R}$. That is

$$M_{X_t|\mathcal{F}_{t-1}}(z) := \mathbb{E}_{\mathbb{P}}[e^{zX_t}|\mathcal{F}_{t-1}] \tag{3.3}$$

For all $t \in \tau\backslash\{0\}$. Let $M_{X_t|\mathcal{F}_{t-1}}(\theta)$ exists, we define a sequence $\{\Lambda_t\}_{t\in\tau}$ with $\Lambda_0 = 1$ and

$$\Lambda_t = \prod_{k=1}^{t} \frac{e^{\theta_k X_k}}{M_{X_t|\mathcal{F}_{t-1}}(\theta_k)}, \qquad t \in \tau\backslash\{0\} \tag{3.4}$$

[16] Tak K. S. (2004)

Lemma 3.1. $\{\Lambda_t\}_{t\in\tau}$ *is a martingale.*

Here we give a proof that omit in Tak(2004)

Proof.

$$
\begin{aligned}
\mathbb{E}[\Lambda_t|\mathcal{F}_{t-1}] &= \mathbb{E}\left[\prod_{k=1}^{t} \frac{e^{\theta_k X_k}}{M_{X_k|\mathcal{F}_{k-1}}(\theta_k)}\middle| \mathcal{F}_{t-1}\right] \\
&= \prod_{k=1}^{t-1} \frac{e^{\theta_k X_k}}{M_{X_k|\mathcal{F}_{k-1}}(\theta_k)} \mathbb{E}\left[\frac{e^{\theta_t Y_t}}{M_{Y_t|\mathcal{F}_{t-1}}(\theta_t)}\middle| \mathcal{F}_{t-1}\right] \\
&= \Lambda_{t-1}
\end{aligned}
\tag{3.5}
$$

□

Let $\mathbb{P}_t := \mathbb{P}|\mathcal{F}_t \ \forall\ t \in \tau\backslash\{0\}$, and $\mathbb{P}_T = \mathbb{P}$. We define a family of probability measure $\{\mathbb{P}_{t,\Lambda_t}\}_{t\in\tau\backslash\{0\}}$ by the following conditional Esscher transform:

$$
\mathbb{P}_{t,\Lambda_t}(A|\mathcal{F}_{t-1}) = \mathbb{E}_{\mathbb{P}_t}\left(I_A \frac{e^{\theta_t X_t}}{\mathbb{E}_{\mathbb{P}_t}(e^{\theta_t X_t}|\mathcal{F}_{t-1})}\middle| \mathcal{F}_{t-1}\right) \qquad A \in \mathcal{F}_t \tag{3.6}
$$

[16] Tak (2004)

Lemma 3.2. $\mathbb{P}_{t,\Lambda_t} = \mathbb{P}_{t+1,\Lambda_{t+1}}|\mathcal{F}_t \ \forall\ t \in \tau\backslash\{0\}$

Proof. By the martingale property of $\Lambda_{t t\in\tau}$, if$A \in \mathcal{F}_t$

$$
\begin{aligned}
\mathbb{P}_{t+1,\Lambda_{t+1}}[A|\mathcal{F}_t] &= \mathbb{E}_{\mathbb{P}_{t+1}}\left(I_A \frac{e^{\theta_{t+1} X_{t+1}}}{\mathbb{E}_{\mathbb{P}_{t+1}}(e^{\theta_{t+1} X_{t+1}}|\mathcal{F}_t)}\middle| \mathcal{F}_t\right) \\
&= I_A \mathbb{E}_{\mathbb{P}_{t+1}}\left(\frac{e^{\theta_{t+1} X_{t+1}}}{\mathbb{E}_{\mathbb{P}_{t+1}}(e^{\theta_{t+1} X_{t+1}}|\mathcal{F}_t)}\middle| \mathcal{F}_t\right) \\
&= I_A
\end{aligned}
\tag{3.7}
$$

take $\mathbb{E}_{\mathbb{P}_t}$ both side, we get $\mathbb{P}_{t,\Lambda_t}(A) = \mathbb{P}_{t+1,\Lambda_{t+1}}(A)$ □

The associated parameter θ_t is called the conditional Esscher parameter given $\mathcal{F}_{t-1}$. Write $F(x;\theta_t|\mathcal{F}_{t-1})$ for $\mathbb{P}_{t,\Lambda_t}(X_t \le x|\mathcal{F}_{t-1})$ we have

$$
F(x;\theta_t|\mathcal{F}_{t-1}) = \frac{\int_{-\infty}^{y} e^{\theta_t x} dF(x)}{M_{Y_t|\mathcal{F}_{t-1}}(\theta_t)} \tag{3.8}
$$

where $F(x)$ is the cdf of $NIG(\alpha', 0, \mu', \delta')$.

Let $M_{X_t|\mathcal{F}_{t-1}}(z;\theta_t)$ denote the moment generating function of $F(x,\theta_t|\mathcal{F}_{t-1})$, that is

$$
M_{X_t|\mathcal{F}_{t-1}}(z;\theta_t) = \frac{M_{X_t|\mathcal{F}_{t-1}}(z+\theta_t)}{M_{X_t|\mathcal{F}_{t-1}}(\theta_t)} \tag{3.9}
$$

For pricing a derivative V, we construct a martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ by adopting Esscher Transforms.
First, choose a sequence of conditional Esscher parameters $\{\theta_t^q\}_{t\in\tau\backslash\{0\}}$ by solving the following equation

$$
r = \ln[M_{Y_t|\mathcal{F}_{t-1}}(1;\theta_t^q)], \qquad t \in \tau\backslash\{0\} \tag{3.10}
$$

then we can define a family of probability measure $\{\mathbb{P}_{t,\Lambda_t^q}\}_{t\in\tau\setminus\{0\}}$ associated with $\{\theta_t^q\}_{t\in\tau\setminus\{0\}}$. Again, according to above result,

$$\mathbb{P}_{t,\Lambda_t^q} = \mathbb{P}_{s,\Lambda_s^q}|\mathcal{F}_t \qquad s,t\in\tau \quad \text{with } t\leq s \tag{3.11}$$

Lemma 3.3. *Let $\mathbb{Q}=\mathbb{P}_{T,\Lambda_T^q}$ the disounted stock price process $\{e^{-rt}S_t\}_{t\in\tau}$ is a martingale under $\mathbb{Q}$*

Proof.

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[e^{-rt}S_t|\mathcal{F}_{t-1}] &= e^{-rt}\mathbb{E}_{\mathbb{Q}}[S_t|\mathcal{F}_{t-1}]\\
&= e^{-rt}\mathbb{E}_{\mathbb{P}_T}\left[S_{t-1}e^{X_t}\ \frac{e^{\theta_T^q X_T}}{M_{X_T|\mathcal{F}_{T-1}}(\theta_T^q)}\middle|\,\mathcal{F}_{t-1}\right]\\
&= e^{-rt}S_{t-1}\mathbb{E}_{\mathbb{P}_t}\left[e^{X_t}\ \frac{e^{\theta_t^q X_t}}{M_{X_t|\mathcal{F}_{t-1}}(\theta_t^q)}\middle|\,\mathcal{F}_{t-1}\right]\\
&= e^{-rt}S_{t-1}\frac{M_{X_t|\mathcal{F}_{t-1}}(1+\theta_t^q)}{M_{X_t|\mathcal{F}_{t-1}}(\theta_t^q)}\\
&= e^{-rt}S_{t-1}e^{r}\\
&= e^{-r(t-1)}S_{t-1}.
\end{aligned} \tag{3.12}$$

□

Then by risk-neutral pricing formula, the price of the derivative V at time $t\in\tau$ is:

$$V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}V_T\middle|\,\mathcal{F}_t\right] \tag{3.13}$$

we call $\mathbb{Q}$ a conditional risk neutralized Esscher pricing measure.
We can justify the pricing result by solving a dynamic utility maximization problem.
First, consider the sequence of power utility functions $\{u_t\}_{t\in\tau}$ with parameter $\{\gamma_t\}_{t\in\tau}$

$$u_t = \begin{cases} \frac{x^{1-\gamma_t}}{1-\gamma_t} & \text{if } \gamma_t\neq 1\\ \\ \ln x & \text{if}\gamma_t = 1\end{cases} \tag{3.14}$$

We assume that $\widetilde{V_t}$ is the agent's price of the derivative V at time t with $\widetilde{V_T}=V_T$, such that it is optimal for the agent not to buy or sell any unit of derivative V at time t.

Proposition 3.1. *For all $t\in\tau$,$\widetilde{V_t}=V_t$*

Proof. consider one stock S and one risk-free interest rate r. The agent owns m shares of the stock bases his decision on a risk-averse utility function $u_t(x)$. Condider a derivative security V pays V_t at time t. We have

$$\phi(\eta) = \mathbb{E}[u_t(mS_{t+1}+\eta[V_{t+1}-e^{r}\widetilde{V_t}])|\mathcal{F}_t] \tag{3.15}$$

is maximal for $\eta = 0$. From

$$\phi'(0) = 0$$

$$\Rightarrow \phi'(\eta) \quad = \mathbb{E}\left[(V_{t+1} - e^r \widetilde{V}_t) u'_t(mS_{t+1} + \eta[V_{t+1} - e^r \widetilde{V}_t]) \middle| \mathcal{F}_t \right] \tag{3.16}$$

we obtain

$$\widetilde{V}_t = e^{-r} \frac{\mathbb{E}[V_{t+1} u'_t(mS_{t+1})|\mathcal{F}_t]}{\mathbb{E}[u'_t(mS_{t+1})|\mathcal{F}_t]} \tag{3.17}$$

note that

$$\phi''(\eta) = \mathbb{E}[(V_{t+1} - e^r \widetilde{V}_t)^2 u''_t(mS_{t+1} + \eta[V_{t+1} - e^r \widetilde{V}_t])|\mathcal{F}_t] < 0 \tag{3.18}$$

if $u''_t(x) < 0$.

In the particular case of a power utility function with parameter $\gamma_t > 0$, we have $u'_t(x) = x^{-\gamma_t}$. Then

$$\widetilde{V}_t = e^{-r} \frac{\mathbb{E}[V_{t+1}(mS_{t+1})^{-\gamma_t}|\mathcal{F}_t]}{\mathbb{E}[(mS_{t+1})^{-\gamma_t}|\mathcal{F}_t]} = e^{-r} \frac{\mathbb{E}[V_{t+1}(S_{t+1})^{-\gamma_t}|\mathcal{F}_t]}{\mathbb{E}[(S_{t+1})^{-\gamma_t}|\mathcal{F}_t]} \tag{3.19}$$

since $V_{t+1} = S_{t+1}$, we get $\widetilde{V}_t = S_t$, So

$$S_t = \widetilde{V}_t = e^{-r} \frac{\mathbb{E}[(S_{t+1})^{1-\gamma_t}|\mathcal{F}_t]}{\mathbb{E}[(S_{t+1})^{-\gamma_t}|\mathcal{F}_t]} = e^{-r} S_t \frac{\mathbb{E}[e^{Y_{t+1}(1-\gamma_t)}|\mathcal{F}_t]}{\mathbb{E}[e^{Y_{t+1}(-\gamma_t)}|\mathcal{F}_t]} \tag{3.20}$$

this implies

$$e^t = M_{Y_{t+1}|\mathcal{F}_t}(1; -\gamma_t) \tag{3.21}$$

we see that the value of γ_t is $-\theta^q_{t+1}$, and by iteration we have γ_s is $-\theta^q_{s+1}$ $\forall s \geq t$. We have

$$\widetilde{V}_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[V_t \frac{e^{\theta^q_T Y_T}}{M_{Y_T|\mathcal{F}_{T-1}}} \middle| \mathcal{F}_t \right] \tag{3.22}$$

□

3.2 Change of measure for the NIG GARCH(1,1) model

It has been see that from [16] Tak(2004) the conditional Esscher transform perform the same result as [6] Duan (1995).
Now, we apply the conditional Esscher transform to the NIG GARCH(1,1) model. Suppose the log return $\ln(S_t/S_{t-1}) = \{X_t\}_{t\in\mathcal{T}}$ follows a GARCH(1,1) process and $X_t|\mathcal{F}_{t-1} \sim NIG(\alpha', \beta', \mu'_t, \delta'_t)$ under $\mathbb{P}$.more explicitly,

$$\begin{aligned} X_t \quad &= \quad r + \lambda\sqrt{h_t} + \sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4}\sqrt{h_t} + \sqrt{h_t}\varepsilon_t \\ \text{where} \quad & \quad \varepsilon_t \sim NIG(\alpha', \beta', -\sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4}, (1-\beta'^2)^{3/2}) \end{aligned}$$

where h_t follow a GARCH(1,1) process as defined in (2.12), λ is the risk premium and r is the risk free interest rate. we can also claim that the residual ε_t is with zero mean and unit variance

$$
\begin{aligned}
E(\varepsilon_t) &= -\sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4} + \frac{\sqrt{\alpha'}}{\sqrt{1-\beta'^2}}\beta'\sqrt{(1-\beta'^2)^{3/2}} = 0 \\
Var(\varepsilon_t) &= \frac{(1-\beta'^2)^{3/2}}{(1-\beta'^2)^{3/2}} = 1
\end{aligned}
$$

the above specification implies that $X_t \sim NIG(\alpha', \beta', r + \lambda\sqrt{h_t}, (1-\beta'^2)^{3/2}h_t)$, the conditional mean and variance of X_t are

$$
\begin{aligned}
EX_t &= r + (\lambda + \sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4})\sqrt{h_t} \\
VarX_t &= h_t
\end{aligned}
$$

we can see $\lambda + \sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4}$ as total risk premium. Then we want to find the Esscher parameter θ_t^q for this model by solving the following equation

$$
\begin{aligned}
e^r &= \frac{M(\theta_t+1)}{M(\theta_t)} && (3.23)\\
\Rightarrow r &= r + \lambda\sqrt{h_t} + \sqrt{\alpha'^2 - [\alpha'\beta' + \sqrt{\alpha'(1-\beta'^2)^{3/2}h_t}\theta_t]^2} && (3.24)\\
&- \sqrt{\alpha'^2 - [\alpha'\beta' + \sqrt{\alpha'(1-\beta'^2)^{3/2}h_t}(\theta_t+1)]^2} && (3.25)\\
& && (3.26)
\end{aligned}
$$

This equation can be solve explicitly by a quadratic form, the solution must satisfies $|\beta' + \sqrt{\frac{(1-\beta'^2)^{3/2}}{\alpha'}h_t}\theta_t^q| < 1$

Corollary 3.1. *Under our assumption if $X_t|\mathcal{F}_{t-1}$ is $NIG(\alpha', \beta', r + \lambda\sqrt{h_t}, (1-\beta'^2)^{3/2}h_t)$ distribution under physical measure $\mathbb{P}$, then under the risk neutral measure $\mathbb{Q}$ with Esscher parameters θ_t^q, the distribution of $X_t|\mathcal{F}_{t-1}$ becomes $NIG(\alpha', \beta' + \sqrt{\frac{(1-\beta'^2)^{3/2}}{\alpha'}h_t}\theta_t^q, r + \lambda\sqrt{h_t}, (1-\beta'^2)^{3/2}h_t)$*

Proof. the moment generating function $M_{X_t|\mathcal{F}_{t-1}}(z;\theta_t^q)$ is given by

$$
\begin{aligned}
& M_{X_t|\mathcal{F}_{t-1}}(z;\theta_t^q) && (3.27)\\
=& \frac{M_{X_t|\mathcal{F}_{t-1}}(z+\theta_t^q)}{M_{X_t|\mathcal{F}_{t-1}}(\theta_t^q)} && (3.28)\\
=& \exp(r + \lambda\sqrt{h_t} + \sqrt{\alpha'^2 - [\alpha'\beta' + \sqrt{\alpha'(1-\beta'^2)^{3/2}h_t}\theta_t]^2}) && (3.29)\\
\times& \exp(-\sqrt{\alpha'^2 - [\alpha'\beta' + \sqrt{\alpha'(1-\beta'^2)^{3/2}h_t}(\theta_t+z)]^2}) && (3.30)\\
=& \exp(r + \lambda\sqrt{h_t} + \alpha'\sqrt{1 - [\beta' + \sqrt{\frac{(1-\beta'^2)^{3/2}h_t}{\alpha'}}\theta_t]^2}) && (3.31)\\
\times& \exp(-\sqrt{\alpha'^2 - [\alpha'(\beta' + \sqrt{\frac{(1-\beta'^2)^{3/2}h_t}{\alpha'}}\theta_t) + \sqrt{\alpha'(1-\beta'^2)^{3/2}h_t}z]^2}) && (3.32)
\end{aligned}
$$

therefore under $\mathbb{Q}$

$$Y_t|\mathcal{F}_{t-1} \sim NIG(\alpha', \beta' + \sqrt{\frac{(1-\beta')^{3/2}}{\alpha'}} h_t \theta_t^q, r + \lambda\sqrt{h_t}, (1-\beta'^2)^{3/2} h_t) \tag{3.33}$$

□

Corollary 3.2. *We have under original probability measure $\mathbb{P}$ the NIG GARCH model is:*

$$\begin{aligned} X_t &= r + \lambda\sqrt{h_t} + \sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4}\sqrt{h_t} + \eta_t \\ \eta_t &= \sqrt{h_t}\varepsilon_t \sim NIG(\alpha', \beta, -\sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4}\sqrt{h_t}, h_t(1-\beta'^2)^{3/2}) \\ h_t &= \rho_0 + \rho_1\eta_{t-1}^2 + \pi_1 h_{t-1} \end{aligned}$$

After Esscher transform(with Esscher parameters $\{\theta_t^q\}$), under $\mathbb{Q}$ the model becomes:

$$\begin{aligned} X_t &= r + \lambda\sqrt{h_t} + \sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4}\sqrt{h_t} + \bar{\eta}_t \\ \bar{\eta}_t &\sim NIG(\alpha', \beta' + \sqrt{\frac{(1-\beta'^2)^{3/2}}{\alpha'}} h_t \theta_t^q, -\sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4}\sqrt{h_t}, h_t(1-\beta'^2)^{3/2}) \\ h_t &= \rho_0 + \rho_1\bar{\eta}_{t-1}^2 + \pi_1 h_{t-1} \end{aligned}$$

4 Estimation

The parameters $\omega = (\alpha', \beta', \rho_0, \rho_1, \rho_2, \pi_1, \lambda) \in \Theta$, where $\alpha' \geq 0$, $|\beta'| < 1$, $\rho_0 > 0$, $\rho_1, \pi_1 \geq 0$, $\rho_1 + \pi_1 < 1$ and $\lambda \in \mathbb{R}$.

Then we can estimate the parameters by maximizing the log likelihood function

$$L(\alpha', \beta', \rho_0, \rho_1, \pi_1, \lambda) = \sum_{t=1}^{T} L_t(\alpha', \beta', \rho_0, \rho_1, \pi_1, \lambda) = \tag{4.1}$$

$$\frac{n}{2}\ln\alpha' - n\ln\pi - \frac{1}{2}\sum_{t=1}^{T}\ln\delta_t' + n(\alpha'\sqrt{1-\beta'^2}) + \beta'\sum_{t=1}^{T}\sqrt{\alpha'/\delta'}(x_t - \mu_t') \tag{4.2}$$

$$-\frac{1}{2}\sum_{t=1}^{T}\ln\left(1 + \frac{(x_t-\mu_t)^2}{\delta_t'\alpha'}\right) + \sum_{t=1}^{T}\ln K_1\left(\alpha'\sqrt{1 + \frac{(x_t-\mu_t)^2}{\delta_t'\alpha'}}\right) \tag{4.3}$$

where $\mu_t = r + \lambda\sqrt{h_t}$, $\delta_t = h_t(1-\beta'^2)^{3/2}$

$$h_t = \rho_0 + \rho_1(x_{t-1} - m_{t-1})^2 + \pi_1 h_{t-1} \tag{4.4}$$

where $m_t = r + (\lambda + \sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4})\sqrt{h_t}$ All parameters with the following constraint

$$\alpha' \geq 0, \quad |\beta'| < 1, \quad \rho_0 > 0, \quad \rho_1 \text{ and } \pi_1 \geq 0, \quad \rho_0 + \pi_1 < 1, \quad \lambda \in \mathbb{R} \tag{4.5}$$

The gradient is in the appendix.

5 Numerical Examples

We present numerical results of our NIG GARCH pricing model using the colse values of S&P 500 daily index series from Jan 3,2000 to Apr 6,2006, a total of 1578 observation of

log return.
We estimate the parameters of NIG GARCH model via maximum likelihood estimation, the following are the estimated parameters.

α'	β'	λ	ρ_0	ρ_1	π_1
7.4820	-0.1302	0.3832	2×10^{-6}	0.0884	0.8947

After estimating the parameters, we applying the estimated parameters to simulate the option price by Monte Carlo simulation. In order to find the risk neutral Esscher parameter, solving a nonlinear equation is necessary. So the simulation is time consuming, and we use the basic Monte Carlo simulation without any variate reduction technique. For each option price, we produce $100,000$ paths to calculate it by the soft ware MATLAB 7.1. The following tables show the comparison for the option pricing of NIG GARCH model and Black Schole pricing formula.(Assuming the risk free interest rate $r = 0$ and the initial conditional volatility h_0 is equal to the variance of the whole sample)

Maturity	S/K	B-S	NIG GARCH
30	1300/1000	300.0004	300.0725
30	1300/1100	200.1114	200.1037
30	1300/1200	104.0798	104.7615
30	1300/1300	33.2843	32.0355
30	1300/1400	5.3272	4.9065
30	1300/1500	0.4026	0.5229
30	1300/1600	0.0151	0.0707

Maturity	S/K	B-S	NIG GARCH
60	1300/1000	300.0579	300.0500
60	1300/1100	201.3983	202.1829
60	1300/1200	111.7655	111.2655
60	1300/1300	47.0631	44.3521
60	1300/1400	14.2924	12.7812
60	1300/1500	3.1118	3.0055
60	1300/1600	0.4970	0.7438

Maturity	S/K	B-S	NIG GARCH
90	1300/1000	300.0579	300.9501
90	1300/1100	201.3983	204.6041
90	1300/1200	111.7655	117.8821
90	1300/1300	47.0631	53.5759
90	1300/1400	14.2924	19.5506
90	1300/1500	7.2563	6.2298
90	1300/1600	1.9249	2.0444

Comparing with the NIG GARCH option prices, the Black Scholes model always underprices deep out of the money options and it can under price for over price an out of the money options depending on the level of the initial conditional volatility.

6 Conclusion and Further Work

Empirical evidence shows that NIG GARCH model is more flexible to fit the real financial return than the GARCH model with Normal innovation. We can modeling the skewness ,higher Kurtosis for the conditional return. The Esscher transform provide a simple method to find a proper equivalent martingale measure via finding a Esscher parameters θ.
We can find that the Monte Carlo simulation is time consuming, so finding a good method to control variate of the simulation is important task. Since every time step of simulation, we have to solving a nonlinear equation to find the Esscher parameters. Reduce the number of simulation will reduce the simulation time.

7 Appendix

7.1 Modified Bessel Functions

The modified Bessel function of the third kind with order λ, denoted by $K_\lambda(\cdot)$. Here are some properties useful.

- Integral representation

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty y^{\lambda-1}\exp\left(-\frac{x}{2}(y+y^{-1})\right)dy \qquad x > 0 \tag{7.1}$$

- Basic properties

$$K_\lambda(x) = K_{-\lambda}(x) \tag{7.2}$$

$$K_{\lambda+1}(x) = \frac{2\lambda}{x}K_\lambda(x) + K_{\lambda-1}(x) \tag{7.3}$$

- Relation of $K_\lambda(x)$ and $I_\lambda(x)$, asymptotic properties.
  Let $I_\lambda(x)$ be the modified Bessel function of the first kind.

$$K_\lambda(x) = \frac{\pi}{2}\frac{1}{\sin(\pi\lambda)}(I_{-\lambda}(x) - I_\lambda(x)) \tag{7.4}$$

$$K_\lambda(x) \sim \Gamma(\lambda)2^{\lambda-1}x^{-\lambda} \qquad \text{as } x \downarrow 0 \tag{7.5}$$

$$K_0(x) \sim -\ln x \qquad \text{as } x \downarrow 0 \tag{7.6}$$

- Series representation for $\lambda = n + \frac{1}{2}$, $n \in \mathbb{N}$

$$K_{n+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}}x^{-1/2}e^{-x}\left(1 + \sum_{i=1}^{n}\frac{(n+i)!}{(n-i)!i!}(2x)^{-i}\right) \tag{7.7}$$

$$K_{-1/2}(x) = K_{1/2}(x) = \sqrt{\frac{\pi}{2}}x^{-1/2}e^{-x} \tag{7.8}$$

- Derivatives w.r.t x

$$
\begin{aligned}
K_0'(x) &= -K_1(x) &(7.9)\\
K_\lambda'(x) &= -\frac{1}{2}(K_{\lambda+1}(x)+K_{\lambda-1}(x)) = -\frac{\lambda}{x}K_\lambda(x)-K_{\lambda-1}(x) &(7.10)\\
(\ln K_\lambda(x))' &= \frac{\lambda}{x}-R_\lambda(x) &(7.11)\\
(\ln K_\lambda(x))'' &= S_\lambda(x)-\frac{R_\lambda(x)}{x}-\frac{\lambda}{x^2} &(7.12)
\end{aligned}
$$

where

$$R_\lambda(x) := \frac{K_{\lambda+1}(x)}{K_\lambda(x)} \qquad x>0 \tag{7.13}$$

$$S_\lambda(x) := \frac{K_{\lambda+2}(x)K_\lambda(x)-K_{\lambda+1}^2(x)}{K_\lambda^2(x)} \qquad x>0 \tag{7.14}$$

- Properties of R_λ and $\S_\lambda$

$$
\begin{aligned}
R_{-\lambda}(x) &= \frac{1}{K_{\lambda-1}(x)} &(7.15)\\
R_\lambda(x) &= \frac{2\lambda}{x}+R_{-\lambda}(x) &(7.16)\\
R_\lambda'(x) &= \frac{R_\lambda(x)}{x}-S_\lambda(x) &(7.17)\\
R_{-1/2}(x) &= 1, \quad R_{1/2}(x)=1+\frac{1}{x}, \quad R_{-3/2}(x)=\frac{x}{x+1} &(7.18)
\end{aligned}
$$

7.2 Moment structure of Generalized Inverse Gaussian

If we say $Z \sim GIG(\lambda,\delta,\gamma)$. The probability density function of Generalized Inverse Gaussian distribution is denoted by

$$f_Z(z) = \left(\frac{\gamma}{\delta}\right)^\lambda \frac{z^{\lambda-1}}{2K_\lambda(\delta\gamma)} \exp\left(-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 z)\right) \tag{7.19}$$

The moments of Z are given by

$$\mathbb{E}[Z^s] = \left(\frac{\gamma}{\delta}\right)^s \frac{K_{\lambda+s}(\delta\gamma)}{K_\lambda(\delta\gamma)} \tag{7.20}$$

and this formula holds for negative values of s, i.e. for inverse moments, too.

$$\mathbb{E}[\ln Z] = \left.\frac{\partial \mathbb{E}[Z^s]}{\partial s}\right|_{s=0} \tag{7.21}$$

where

$$\frac{\partial \mathbb{E}[Z^s]}{\partial s} = \left(\frac{\gamma}{\delta}\right)^s \ln\left(\frac{\delta}{\gamma}\right)\frac{K_{\lambda+s}(\delta\gamma)}{K_\lambda(\delta\gamma)} + \left(\frac{\gamma}{\delta}\right)^s \frac{1}{K_{\lambda+s}(\delta\gamma)}\frac{\partial}{\partial s}K_{\lambda+s}(\delta\gamma) \tag{7.22}$$

using

$$\frac{\partial}{\partial s}K_{\lambda+s}(\delta\gamma) = \frac{\partial}{\partial(\lambda+s)}K_{\lambda+s}(\delta\gamma)\frac{\partial}{\partial s}(\lambda+s) \tag{7.23}$$

and setting $s=0$, gives

$$\mathbb{E}[\ln Z] = \ln\left(\frac{\delta}{\gamma}\right) + \frac{1}{K_\lambda(\delta\gamma)}\frac{\partial}{\partial\lambda}K_\lambda(\delta\gamma) \tag{7.24}$$

7.3 Gradients of the GARCH NIG models.

In order to find the maximum likelihood estimator, we must derive the gradients of our likelihood function for the numerical method.
Consider the GARCH-NIG models in section 4, the log likelihood for one observation is given by

$$L_t(\alpha', \lambda, \beta', \rho_0, \rho_1, \pi_1) = \tag{7.25}$$

$$\frac{1}{2}\ln\alpha' - \ln\pi - \frac{1}{2}\ln\delta'_t + \alpha'\sqrt{1-\beta'^2} + \beta'\sqrt{\alpha'/\delta'_t}(x_t - \mu'_t) \tag{7.26}$$

$$-\frac{1}{2}\ln\left(1 + \frac{(x_t-\mu_t)^2}{\delta_t\alpha'}\right) + \ln K_1\left(\alpha'\sqrt{1+\frac{(x_t-\mu_t)^2}{\delta_t\alpha'}}\right) \tag{7.27}$$

where $\mu_t = r + \lambda\sqrt{h_t}$, $\delta_t = h_t(1-\beta'^2)^{3/2}$

$$h_t = \rho_0 + \rho_1(x_{t-1} - m_{t-1})^2 + \pi_1 h_{t-1} \tag{7.28}$$

where $m_t = r + (\lambda + \sqrt{\alpha'}\beta'(1-\beta'^2)^{1/4})\sqrt{h_t}$.

$$\nabla L_t(\alpha', \beta', \lambda, \rho_0, \rho_1, \pi_1) = \begin{bmatrix} \frac{\partial L_t(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1)}{\partial\alpha'} \\ \vdots \\ \frac{\partial L_t(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1)}{\partial\pi'_1} \end{bmatrix}_{6\times 1}$$

$$= \begin{bmatrix} \frac{\partial L_t(\alpha',\beta',\mu_t,\delta_t)}{\partial\alpha'} \\ \frac{\partial L_t(\alpha',\beta',\mu_t,\delta_t)}{\partial\beta'} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$+ \left(\begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial\delta_t(\beta',h_t)}{\partial\beta'} \\ \frac{\partial\mu_t(\alpha',\beta',\lambda,h_t)}{\partial\lambda'} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}_{6\times 2} + \begin{bmatrix} \frac{\partial h_t(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1)}{\partial\alpha'} \\ \vdots \\ \frac{\partial h_t(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1)}{\partial\pi'_1} \end{bmatrix}_{6\times 1} \begin{bmatrix} \frac{\partial\mu_t(\alpha',\beta',\lambda,h_t)}{\partial h_t} & \frac{\partial\delta_t(\beta',h_t)}{\partial h_t} \end{bmatrix}_{1\times 2} \right)$$

$$\times \begin{bmatrix} \frac{\partial L_t(\alpha',\beta',\mu_t,\delta_t)}{\partial\mu_t} \\ \frac{\partial L_t(\alpha',\beta',\mu_t,\delta_t)}{\partial\delta_t} \end{bmatrix}_{2\times 1}$$

where

$$\begin{bmatrix} \frac{\partial h_t(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1)}{\partial \alpha'} \\ \vdots \\ \frac{\partial h_t(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1)}{\partial \pi_1'} \end{bmatrix}_{6\times 1} = \begin{bmatrix} \frac{\partial h_t(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1,h_{t-1})}{\partial \alpha'} \\ \vdots \\ \frac{\partial h_t(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1,h_{t-1})}{\partial \pi_1'} \end{bmatrix}_{6\times 1}$$

$$+ \quad \frac{\partial h_t(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1,h_{t-1})}{\partial h_{t-1}} \begin{bmatrix} \frac{\partial h_{t-1}(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1)}{\partial \alpha'} \\ \vdots \\ \frac{\partial h_{t-1}(\alpha',\beta',\lambda,\rho_0,\rho_1,\rho_2,\pi_1)}{\partial \pi_1'} \end{bmatrix}_{6\times 1}$$

Each components can be written down as

$$\begin{aligned}
\frac{\partial L_t(\alpha',\beta',\mu_t,\delta_t)}{\partial \alpha'} &= \frac{1}{2\alpha'} + \sqrt{1-\beta'^2} + \frac{\beta(x_{-}\mu_t)}{2\sqrt{\alpha'\delta_t}} + J_t\Big(\sqrt{1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t}} - \frac{(x_t-\mu_t)^2)}{2\alpha'\delta_t\sqrt{1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t}}}\Big) \\
&+ \frac{(x_t-\mu_t)^2}{2\alpha'^2\delta_t(1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t})} \\
\frac{\partial L_t(\alpha',\beta',\mu_t,\delta_t)}{\partial \beta'} &= -\frac{\alpha'\beta}{\sqrt{1-\beta'^2}} + (x_t-\mu_t)\sqrt{\frac{\alpha'}{\delta_t}} \\
\frac{\partial L_t(\alpha',\beta',\mu_t,\delta_t)}{\partial \mu_t} &= -\beta'\sqrt{\frac{\alpha'}{\delta_t}} - J_t\frac{x_t-\mu_t}{\sqrt{1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t}}\,\delta_t} + \frac{x_t-\mu_t}{\alpha'\delta_t(1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t})} \\
\frac{\partial L_t(\alpha',\beta',\mu_t,\delta_t)}{\partial \delta_t} &= -\frac{1}{2\delta_t} - \frac{\beta'\sqrt{\alpha'}(x_t-\mu_t)}{2\delta_t^{3/2}} - J_t\frac{(x_t-\mu_t)^2}{2\sqrt{1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t}}\,\delta_t^2} + \frac{(x_t-\mu_t)^2}{2\alpha_t'\delta^2(1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t})}
\end{aligned}$$

where

$$J_t = -\frac{K_0(\alpha'\sqrt{1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t}})}{K_1(\alpha'\sqrt{1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t}})} - \frac{1}{\alpha'\sqrt{1+\frac{(x_t-\mu_t)^2}{\alpha'\delta_t}}}$$

$$\begin{aligned}
\frac{\partial \mu_t(\alpha',\beta',\lambda',h_t)}{\partial \lambda'} &= \sqrt{h_t} \\
\frac{\partial \mu_t(\alpha',\beta',\lambda',h_t)}{\partial h_t} &= \frac{\lambda}{2\sqrt{h_t}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \delta_t(\beta',h_t)}{\partial \beta'} &= -3h_t\sqrt{1-\beta'^2}\beta' \\
\frac{\partial \delta_t(\beta',h_t)}{\partial h_t} &= (1-\beta'^2)^{3/2}
\end{aligned}$$

$$\frac{\partial h_t(\alpha', \beta', \lambda, \rho_0, \rho_1, \rho_2, \pi_1, h_{t-1})}{\partial \alpha'} = -\rho_1 \frac{(x_{t-1} - m_{t-1})\beta'(1-\beta'^2)^{1/4}\sqrt{h_{t-1}}}{\sqrt{\alpha'}}$$

$$\frac{\partial h_t(\alpha', \beta', \lambda, \rho_0, \rho_1, \rho_2, \pi_1, h_{t-1})}{\partial \beta'} = -2\rho_1(x_{t-1} - m_{t-1})\frac{\sqrt{\alpha'}(1-2\beta'^2)\sqrt{h_{t-1}}}{2(1-\beta'^2)^{3/4}}$$

$$\frac{\partial h_t(\alpha', \beta', \lambda, \rho_0, \rho_1, \rho_2, \pi_1, h_{t-1})}{\partial \lambda'} = -2\rho_1(x_{t-1} - m_{t-1})\sqrt{h_{t-1}}$$

$$\frac{\partial h_t(\alpha', \beta', \lambda, \rho_0, \rho_1, \rho_2, \pi_1, h_{t-1})}{\partial \rho_0} = 1$$

$$\frac{\partial h_t(\alpha', \beta', \lambda, \rho_0, \rho_1, \rho_2, \pi_1, h_{t-1})}{\partial \rho_1} = (x_{t-1} - m_{t-1})^2$$

$$\frac{\partial h_t(\alpha', \beta', \lambda, \rho_0, \rho_1, \rho_2, \pi_1, h_{t-1})}{\partial \rho_2} = (x_{t-1} - m_{t-1})^2 I_{\{x_{t-1} - m_{t-1} < 0\}}$$

$$\frac{\partial h_t(\alpha', \beta', \lambda, \rho_0, \rho_1, \rho_2, \pi_1, h_{t-1})}{\partial \pi_1} = h_{t-1}$$

$$\frac{\partial h_t(\alpha', \beta', \lambda, \rho_0, \rho_1, \rho_2, \pi_1, h_{t-1})}{\partial h_{t-1}} = -\rho_1 \frac{(x_{t-1} - m_{t-1})(\lambda + \sqrt{\alpha}\beta(1-\beta^2)^{1/4})}{\sqrt{h_{t-1}}} + \pi_1$$

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