國 立 交 通 大 學

應用數學系

碩 士 論 文

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Eigencurve Problems For A Class Of Perturbed Block Circulant Matrices

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摘　　要

關心的是，對於β任意固定，擾動區塊循環矩陣$C(\alpha,\beta)$種類的特徵曲線問題。這裡$\alpha>0$是(微波)純量積因子且$\beta\in\mathbb{R}$表示混合邊界常數。$C(\alpha,\beta)$是一個區塊循環矩陣只有在$\beta=1$。對於每個α，$C(\alpha,1)$的特徵值包含它的區塊矩陣作線性組合後的特徵值這件事是已經被知道的。這樣的結果被稱作對於$C(\alpha,1)$的降低特徵值問題。在這篇論文裡，我們得到二個主要結果。首先，對於$C(\alpha,0)$的降低特徵值問題被完全解決了。$C(\alpha,\beta)$的降低特徵值問題也得到一些部分結果。第二，對於$C(\alpha,0)$和$C(\alpha,1)$利用微波方法控制混沌，扮演必要角色的第二大特徵曲線問題將被討論。

Eigencurve Problems For A Class Of Perturbed Block Circulant Matrices

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ABSTRACT

Of concern is the eigencurve problems for a class of "perturbed" block circulant matrices $C(\alpha, \beta)$ with β arbitrary fixed. Here $\alpha > 0$ is a (wavelet) scalar factor and $\beta \in \mathbb{R}$ represents a mixed boundary constant. $C(\alpha, \beta)$ is a block circulant matrix only if $\beta = 1$. It is well-known that for each α the eigenvalues of $C(\alpha, 1)$ consists of eigenvalues of a certain linear combinations of its block matrices. Such results are called the reduced eigenvalue problem for $C(\alpha, 1)$. In this thesis, we obtain two main results. First, the reduced eigenvalue problem for $C(\alpha, 0)$ is completely solved. Some partial results for the reduced eigenvalue problem of $C(\alpha, \beta)$ are also obtained. Second, the second eigencurve problem, which plays essential role for wavelet method for controlling chaos, for $C(\alpha, 0)$ and $C(\alpha, 1)$ are discussed.

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目　錄

1. Introduction

Of concern here is the eigencurve problem for a class of "perturbed" block circulant matrices.

$$C(\alpha,\beta)\mathbf{b} = \lambda(\alpha,\beta)\mathbf{b}. \tag{1.1a}$$

Here $C(\alpha,\beta)$ is an $n \times n$ block matrix of the following form.

$$C(\alpha,\beta) = \begin{pmatrix} C_1(\alpha,\beta) & C_2(\alpha,1) & 0 & \cdots & 0 & C_2^T(\alpha,\beta) \\ C_2^T(\alpha,1) & C_1(\alpha,1) & C_2(\alpha,1) & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & C_2^T(\alpha,1) & C_1(\alpha,1) & C_2(\alpha,1) \\ C_2(\alpha,\beta) & 0 & \cdots & 0 & C_2^T(\alpha,1) & \hat{I}C_1(\alpha,\beta)\hat{I} \end{pmatrix}_{n\times n} \tag{1.1b}$$

Here

$$C_1(\alpha,\beta) = \begin{pmatrix} -1-\beta & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}_{2^j\times 2^j} - \frac{\alpha(1+\beta)}{2^{2j}}ee^T$$

$$=: A_1(\beta, 2^j) - \frac{\alpha(1+\beta)}{2^{2j}}ee^T, \tag{1.1c}$$

where $e = (1,1,...,1)^T$, j is a positive integer, $\alpha > 0$ is a (wavelet) scalar factor and $\beta \in \mathbb{R}$ represents a mixed boundary constant. Moreover,

$$C_2(\alpha,\beta) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ \beta & 0 & \cdots & 0 \end{pmatrix} + \frac{\alpha\beta}{2^{2j}}ee^T$$

$$=: A_2(\beta, 2^j) + \frac{\alpha\beta}{2^{2j}}ee^T \tag{1.1d}$$

$$
\hat{I} = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & & \cdot & \cdot & \cdot & \vdots \\
\vdots & \cdot & \cdot & \cdot & & \vdots \\
0 & 1 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & \cdots & 0 & 0
\end{pmatrix}. \tag{1.1e}
$$

$C(\alpha, \beta)$ is a block circulant matrix (see e.g., [1]) only if $\beta = 1$. It is well-known, see e.g., Theorem 5.6.4 of [1], that for each α the eigenvalues of $C(\alpha, 1)$ consists of eigenvalues of a certain linear combinations of its block matrices. Such results are called the reduced eigenvalue problem for $C(\alpha, 1)$. In this thesis, we obtain two main results. First, the reduced eigenvalue problem for $C(\alpha, 0)$ is completely solved. Some partial results for the reduced eigenvalue problem of $C(\alpha, \beta)$ are also obtained. The second problem in question is to describe the second eigencurve $\lambda_2(\alpha)$, which plays essential role for wavelet method for controlling chaos, of (1.1a) for fixed β. By the second largest eigencurve $\lambda_2(\alpha)$ of $C(\alpha, \beta)$ for fixed β, we mean that for given $\alpha > 0$, $\lambda_2(\alpha, \beta)$ is the second largest eigenvalue of $C(\alpha, \beta)$. We remark that 0 is the largest eigenvalue of $C(\alpha, \beta)$ for any $\alpha > 0$ and $\beta \in \mathbb{R}$. This is to say for fixed β, 0 is the first eigencurve of $C(\alpha, \beta)$. A nontrivial upper bound is conjuctured for the second eigencurve $\lambda_2(\alpha, 0)$ (resp., $\lambda_2(\alpha, 1)$) when α is large is obtained for any j and $n \in \mathbb{N}$. The remainder of this introductory section is devoted to a brief description about how this eigencurve problem arises and its related work.

This problem arises in the wavelet method for chaotic control ([4]). It is found there that the modification of a tiny fraction of wavelet subspaces of a coupling matrix could lead to a dramatic change in chaos synchronizing properties. We begin with describing their work. Let there be N nodes (oscillators). Assume $\mathbf{u}_i$ is the m-dimensional vector of dynamical variables of the ith node. Let the isolated (uncoupling) dynamics be $\dot{\mathbf{u}}_i = f(\mathbf{u}_i)$ for each node. Used in the coupling, $h : \mathbb{R}^m \to \mathbb{R}^m$ is an arbitrary function of each node's variables. Thus, the dynamics of the ith node are

$$
\dot{\mathbf{u}}_i = f(\mathbf{u}_i) + \epsilon \sum_{j=1}^{N} a_{ij} h(\mathbf{u}_j), i = 1, 2, ..., N, \tag{1.2a}
$$

where ϵ is a coupling strength. The sum $\displaystyle\sum_{j=1}^{N} a_{ij} = 0$. Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N)^T$, $F(\mathbf{u}) = (f(\mathbf{u}_1), f(\mathbf{u}_2), ..., f(\mathbf{u}_N))^T$, $H(\mathbf{u}) = (h(\mathbf{u}_1), h(\mathbf{u}_2), ..., h(\mathbf{u}_N))^T$, and $A = (a_{ij})$. We may write (1.1a) as

$$
\dot{\mathbf{u}} = F(\mathbf{u}) + \epsilon A \times H(\mathbf{u}). \tag{1.2b}
$$

Here $\times$ is the direct product of two matrices B and C defined as follows. Let $B = (b_{ij})_{k_1 \times k_2}$ be a $k_1 \times k_2$ matrix and $C = (C_{ij})_{k_2 \times k_3}$ be a $k_2 \times k_3$ block matrix, where each of C_{ij}, $1 \leq i \leq k_2$, $1 \leq j \leq k_3$, is a $k_4 \times k_5$ matrix. Then

$$B \times C = (\sum_{l=1}^{k_2} b_{il} C_{lj})_{k_1 \times k_3}.$$

Many coupling schemes are covered by Equation(1.2b). For example, if the Lorenz system is used and the coupling is through its three components x, y, and z, then the function h is just the matrix

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.3}$$

The choice of A will provide the connectivity of nodes. For instance, the nearest neighbor coupling with periodic, Neumann boundary conditions and mixed boundary conditions are, respectively, given as $A = A_1(1, N) + A_2(1, N) + A_2^T(1, N)$, $A = A_1(0, N) + A_2(1, N)\hat{I}$ and $A = A_1(\beta, N) + A_2(\beta, N) + A_2^T(\beta, N) + (1-\beta)A_2(1, N)\hat{I}$, where those $A's$ are defined in (1.1c,d).
Mathematical speaking ([2]), the second largest eigenvalue λ_2 of A is dominant in controlling the stability of chaotic synchronization, and the critical strength ϵ_c for synchronization can be determined in term of λ_2,

$$\epsilon_c = \frac{L_{max}}{-\lambda_2}. \tag{1.4}$$

The eigenvalues of $A = A_1(1, N)$ are given by $\lambda_i = -4\sin^2 \frac{\pi(i-1)}{N}$, i=1,2,...,N. In general, a larger number of nodes gives a smaller nonzero eigenvalue λ_2 in magnitude and, hence, a larger ϵ_c. In controlling a given system, it is desirable to reduce the critical coupling strength ϵ_c. The wavelet method in [4] would transform A into $C(\alpha, \beta)$. Consequently, it is of great interest to study the second eigencurve of $C(\alpha, \beta)$ for each β. A numerical simulation of a coupled system of $N = 512$ Lorenz oscillators in [4] shows that with $h = I_3$ and $A = A(1, N)$, the critical coupling strength ϵ_c decreases linearly with respect to the increase of α up to a critical value α_c. The smallest ϵ_c is about 6, which is about 10^3 times smaller than the original critical coupling strength, indicating the efficiency of the proposed approach.
The mathematical verification of such phenomena is first achieved by Shieh, Wei, Wang and Lai [3]. Specifically, they solved the second eigencurve problem for $C(\alpha, 1)$ with n being a multiple of 4 and j being any positive integer. Subsequently, in [5], the second eigencurve problem for $C(\alpha, 0)$ and $C(\alpha, 1)$ with n being any positive integer and $j = 1$ is solved.

2. Reduced Eigenvalue problems

Writing the eigenvalue problem $C(\alpha,\beta)\mathbf{b} = \lambda\mathbf{b}$, where $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n)^T$ and $\mathbf{b}_i \in \mathbb{C}^{2^j}$, in block component form, we get

$$C_2^T(\alpha,1)\mathbf{b}_{i-1} + C_1(\alpha,1)\mathbf{b}_i + C_2(\alpha,1)\mathbf{b}_{i+1} = \lambda\mathbf{b}_i, \quad 1 \le i \le n. \tag{2.1a}$$

Mixed boundary conditions would yield that

$$C_2^T(\alpha,1)\mathbf{b}_0 + C_1(\alpha,1)\mathbf{b}_1 + C_2(\alpha,1)\mathbf{b}_2 = \lambda\mathbf{b}_1 = C_1(\alpha,\beta)\mathbf{b}_1 + C_2(\alpha,1)\mathbf{b}_2 + C_2^T(\alpha,\beta)\mathbf{b}_n,$$

and

$$C_2^T(\alpha,1)\mathbf{b}_{n-1} + C_1(\alpha,1)\mathbf{b}_n + C_2(\alpha,1)\mathbf{b}_{n+1} = \lambda\mathbf{b}_n = C_2(\alpha,\beta)\mathbf{b}_1 + C_2^T(\alpha,1)\mathbf{b}_{n-1} + \hat{I}C_1(\alpha,\beta)\hat{I}\mathbf{b}_n,$$

or, equivalently,

$$C_2^T(\alpha,1)\mathbf{b}_0 = (C_1(\alpha,\beta) - C_1(\alpha,1))\mathbf{b}_1 + C_2^T(\alpha,\beta)\mathbf{b}_n$$

$$= [\begin{pmatrix} 1-\beta & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \frac{\alpha(1-\beta)}{2^{2j}} ee^T]\mathbf{b}_1 + [\begin{pmatrix} 0 & \cdots & 0 & \beta \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \frac{\alpha\beta}{2^{2j}} ee^T]\mathbf{b}_n$$

$$= (1-\beta)C_2^T(\alpha,1)\hat{I}\mathbf{b}_1 + \beta C_2^T(\alpha,1)\mathbf{b}_n, \tag{2.1b}$$

and

$$C_2(\alpha,1)\mathbf{b}_{n+1} = (\hat{I}C_1(\alpha,\beta)\hat{I} - C_1(\alpha,1))\mathbf{b}_n + C_2(\alpha,\beta)\mathbf{b}_1$$

$$= (1-\beta)C_2(\alpha,1)\hat{I}\mathbf{b}_n + \beta C_2(\alpha,1)\mathbf{b}_1. \tag{2.1c}$$

To study the block difference equation (2.1), we set

$$\mathbf{b}_j = \delta^j \boldsymbol{v}, \tag{2.2}$$

where $\boldsymbol{v} \in \mathbb{C}^{2^j}$ and $\delta \in \mathbb{C}$.

Substituting (2.2) into (2.1a), we have

$$[C_2^T(\alpha,1)+\delta(C_1(\alpha,1)-\lambda I)+\delta^2 C_2(\alpha,1)]\boldsymbol{v}=0. \tag{2.3}$$

To have a nontrivial solution $\boldsymbol{v}$ satisfying (2.3), we need to have

$$det[C_2^T(\alpha,1)+\delta(C_1(\alpha,1)-\lambda I)+\delta^2 C_2(\alpha,1)]=0. \tag{2.4}$$

Definition 2.1. Equation (2.4) is to be called the characteristic equation of the block difference equation (2.1a). Let $\delta_k=\delta_k(\lambda)\neq 0$ and $\boldsymbol{v}_k=\boldsymbol{v}_k(\lambda)\neq 0$ be complex numbers and vectors, respectively, satisfying (2.3). Here $k=1,2,...,m$ and $m\leq 2^j$. Assume that there exists a $\lambda\in\mathbb{C}$, such that $\mathbf{b}_j=\Sigma_{k=1}^m c_k\delta_k^j(\lambda)\boldsymbol{v}_k(\lambda)$, j=0,1,...,n+1, satisfy equation (2.1b,c), where $c_k\in\mathbb{C}$. If, in addition, $\mathbf{b}_j$, $j=1,2,...,n$, are not all zero vectors. then such $\delta_k(\lambda)$ is called the characteristic value of equation (2.1) or (1.1a) with respect to λ and $\boldsymbol{v}_k(\lambda)$ its corresponding characteristic vector.

Remark 2.1. Clearly, for each α and β, λ in the Definition of 2.1 is an eigenvalue of $C(\alpha,\beta)$.

Should no ambiguity arises, we will write $C_2^T(\alpha,1)=C_2^T$, $C_1(\alpha,1)=C_1$ and $C_2(\alpha,1)=C_2$. Likewise, we will write $A_2(\beta,2^j)=A_2(\beta)$ and $A_1(\beta,2^j)=A_1(\beta)$.

Proposition 2.1. *Let $\rho(\lambda)=\{\delta_i(\lambda):\delta_i(\lambda)$ is a root of equation (2.4)$\}$, and let $\rho'(\lambda)=\{\frac{1}{\delta_i(\lambda)}:\delta_i(\lambda)$ is a root of equation (2.4)$\}$. Then $\rho(\lambda)=\rho'(\lambda)$. Let δ_i and δ_k be in $\rho(\lambda)$. We further assume that δ_i and $\boldsymbol{v}_i=\begin{pmatrix} v_{i1}\\ \vdots \\ v_{i2^j}\end{pmatrix}$ satisfy (2.3). Suppose $\delta_i\cdot\delta_k=1$. Then δ_k and $\boldsymbol{v}_k=\begin{pmatrix} v_{i2^j}\\ v_{i2^j-1}\\ \vdots \\ v_{i2}\\ v_{i1}\end{pmatrix}=:\boldsymbol{v}_i^s$ also satisfy (2.3). Conversely, if $\delta_i\cdot\delta_k\neq 1$, then $\boldsymbol{v}_k\neq\boldsymbol{v}_i^s$.*

Proof. To proof $\rho(\lambda)=\rho'(\lambda)$, we see that

$$det[C_2^T+\delta(C_1-\lambda I)+\delta^2C_2]=\delta^2 det[\frac{1}{\delta^2}C_2^T+\frac{1}{\delta}(C_1-\lambda I)+C_2]$$

$$=\delta^2 det[\frac{1}{\delta^2}C_2^T+\frac{1}{\delta}(C_1-\lambda I)+C_2]^T=\delta^2 det[C_2^T+\frac{1}{\delta}(C_1-\lambda I)+\frac{1}{\delta^2}C_2].$$

Thus, if δ is a root of equation (2.4), then so is $\frac{1}{\delta}$. To see the last assertion of the proposition, we write equation (2.3) with $\delta = \delta_i$ and $\boldsymbol{v} = \boldsymbol{v}_i$ in component form.

$$\sum_{m=1}^{2^j} [(C_2^T)_{lm} v_{im} + \delta_i (\bar{C}_1)_{lm} v_{im} + \delta_i^2 (C_2)_{lm} v_{im}] = 0, l = 1, 2, ..., 2^j. \tag{2.5}$$

Here $\bar{C}_1 = C_1 - \lambda I$. Now the right hand side of (2.5) becomes

$$(\frac{1}{\delta_k})^2 \{ \sum_{m=1}^{2^j} [(C_2)_{l(2^j+1-m)} v_{i(2^j+1-m)} + \delta_k (\bar{C}_1)_{l(2^j+1-m)} v_{i(2^j+1-m)}$$

$$+ \delta_k^2 (C_2^T)_{l(2^j+1-m)} v_{i(2^j+1-m)}] \}$$

$$= (\frac{1}{\delta_k})^2 \{ \sum_{m=1}^{2^j} [(C_2^T)_{(2^j+1-l)m} v_{i(2^j+1-m)} + \delta_k (\bar{C}_1)_{(2^j+1-l)m} v_{i(2^j+1-m)}$$

$$+ \delta_k^2 (C_2)_{(2^j+1-l)m} v_{i(2^j+1-m)}] \}, l = 1, 2, ..., 2^j. \tag{2.6}$$

We have used the fact that

$$(A)_{(2^j+1-l)m} = (A^T)_{l(2^j+1-m)}, \tag{2.7}$$

where $A = C_2^T$ or $\bar{C}_1$ or C_2 to justify the equality in (2.6). However, (2.7) follows from (1.1c) and (1.1d). Letting $v_{i(2^j+1-m)} = v_{km}$, we have that the pair $(\delta_k, \boldsymbol{v}_k)$ satisfies (2.3). Suppose $\boldsymbol{v}_k = \boldsymbol{v}_i^s$, we see, similarly, that the pair $(\frac{1}{\delta_i}, \boldsymbol{v}_k)$ also satisfy (2.3). Thus $\frac{1}{\delta_i} = \delta_k$.

□

Definition 2.2. We shall call $\boldsymbol{v}^s$ and $-\boldsymbol{v}^s$, the symmetric vector and antisymmetric vector of $\boldsymbol{v}$, respectively. A vector $\boldsymbol{v}$ is symmetric (resp., antisymmetric) if $\boldsymbol{v} = \boldsymbol{v}^s$ (resp., $\boldsymbol{v} = -\boldsymbol{v}^s$).

Theorem 2.1. *Let $\delta_k = e^{\frac{\pi k}{n} i}$, k is an integer and $i = \sqrt{-1}$, then δ_{2k}, k=0,1,...,n-1, are characteristic values of equation (2.1) with $\beta = 1$. For each α, if $\lambda \in \mathbb{C}$ satisfies*

$$det[C_2^T + \delta_{2k}(C_1 - \lambda I) + \delta_{2k}^2 C_2] = 0,$$

for some $k \in \mathbb{Z}$, $0 \le k \le n-1$, then λ is an eigenvalue of $C(\alpha, 1)$.

Proof. Let λ be as assumed. Then there exists a $\boldsymbol{v} \in \mathbb{C}^{2^j}$, $\boldsymbol{v} \neq \boldsymbol{0}$ such that

$$[C_2^T + \delta_{2k}(C_1 - \lambda I) + \delta_{2k}^2 C_2] \boldsymbol{v} = \boldsymbol{0}.$$

Let $\mathbf{b}_j = \delta_{2k}^j \boldsymbol{v}$, $0 \le j \le n+1$. Then such $\mathbf{b}_j's$ satisfy (2.1a), (2.1b), and (2.1c). We just proved the assertion of the theorem. □

Corollary 2.1. *Set*

$$\Gamma_k = C_1 + \delta_{2n-k} C_2^T + \delta_k C_2. \tag{2.8}$$

Then the eigenvalues of $C(\alpha, 1)$, for each α, consists of eigenvalues of Γ_k, $k = 0, 2, 4, ..., 2(n-1)$. That is $\rho(C(\alpha,1)) = \bigcup_{k=0}^{n-1} \rho(\Gamma_{2k})$. Here $\rho(A)$ = the spectrum of the matrix A.

Remark 2.2. $C(\alpha, 1)$ is a block circulant matrix. The assertion of Corollary 2.1 is not new (see e.g., Theorem 5.6.4 of [3]). Here we mere gave a different proof.

To study the eigenvalue of $C(\alpha, 0)$ for each α, we begin with considering the eigenvalues and eigenvectors of $C_2^T + C_1 + C_2$ and $C_2^T - C_1 + C_2$.

Proposition 2.2. *Let $T_1(C)$ (resp., $T_2(C)$) be the set of linearly independent eigenvectors of the matrix C that are symmetric (resp., antisymmetric). Then $|T_1(C_2^T + C_1 + C_2)| = |T_2(C_2^T + C_1 + C_2)| = |T_1(C_2^T - C_1 + C_2)| = |T_2(C_2^T - C_1 + C_2)| = 2^{j-1}$. Here $|A|$ denote the cardinality of the set A.*

Proof. We will only illustrate the case for $C_2^T - C_1 + C_2 =: C$. We first observe that $|T_1(C)|$ is less than or equal to 2^{j-1}. So is $|T_2(C)|$. We also remark the cardinality of the set of all linearly independent eigenvectors of C is 2^j. If $0 < |T_1(C)| < 2^{j-1}$, there must exist an eigenvector $\boldsymbol{v}$ for which $\boldsymbol{v} \ne \boldsymbol{v}^s$, $\boldsymbol{v} \ne -\boldsymbol{v}^s$ and $\boldsymbol{v} \notin span\{T_1(C), T_2(C)\}$, the span of the vectors in $T_1(C)$ and $T_2(C)$. It then follows from Proposition 2.1 that $\boldsymbol{v} + \boldsymbol{v}^s$, a symmetric vector, is in the $span\{T_1(C)\}$. Moreover, $\boldsymbol{v} - \boldsymbol{v}^s$ is in $span\{T_2(C)\}$. Hence $\boldsymbol{v} \in span\{T_1(C), T_2(C)\}$, a contradiction. Hence, $|T_1(C)| = 2^{j-1}$. Similarly, we conclude that $|T_2(C)| = 2^{j-1}$. □

Theorem 2.2. *Let $\delta_k = e^{\frac{\pi k}{n} i}$, k is an integer, $i = \sqrt{-1}$. For each α, if $\lambda \in \mathbb{C}$ satisfies*

$$det[C_2^T + \delta_k (C_1 - \lambda I) + \delta_k^2 C_2] = 0,$$

for some $k \in \mathbb{Z}$, $1 \le k \le n-1$, then λ is an eigenvalue of $C(\alpha, 0)$. Let λ be the eigenvalue of $C_2^T + C_1 + C_2$ (resp., $-C_2^T + C_1 - C_2$) for which its associated eigenvector $\boldsymbol{v}$ satisfies $\hat{I}\boldsymbol{v} = \boldsymbol{v}$ (resp., $\hat{I}\boldsymbol{v} = -\boldsymbol{v}$), then λ is also an eigenvalue of $C(\alpha, 0)$.

Proof. For any $1 \le k \le n-1$, let δ_k be as assumed. Let λ_k and $\boldsymbol{v}_k$ be a number and a nonzero vector, respectively, satisfying

$$[C_2^T + \delta_k (C_1 - \lambda_k I) + \delta_k^2 C_2] \boldsymbol{v}_k = \mathbf{0}. \tag{2.9}$$

Using Proposition 2.1, we see that λ_k satisfies

$$det[C_2^T + \delta_{2n-k}(C_1 - \lambda_k I) + \delta_{2n-k}^2 C_2] = 0. \tag{2.10}$$

Let $\boldsymbol{v}_{2n-k}$ be a nonzero vector satisfying $[C_2^T+\delta_{2n-k}(C_1-\lambda_k I)+\delta_{2n-k}^2 C_2]\boldsymbol{v}_{2n-k} = \mathbf{0}$. Letting

$$\mathbf{b}_i = \delta_k^i \boldsymbol{v}_k + \delta_k \delta_{2n-k}^i \boldsymbol{v}_{2n-k}, i = 0, 1, ..., n+1,$$

we conclude, via (2.9) and (2.10), that $\mathbf{b}_i$ satisfy (2.1a) with $\lambda = \lambda_k$. Moreover,

$$\hat{I}\mathbf{b}_1 = \delta_k \hat{I} \boldsymbol{v}_k + \hat{I}\boldsymbol{v}_{2n-k} = \delta_k \boldsymbol{v}_{2n-k} + \boldsymbol{v}_k = \mathbf{b}_0.$$

We have used Proposition 2.1 to justify the second equality above. Similarly, $\mathbf{b}_{n+1} = \hat{I}\mathbf{b}_n$. To see $\lambda = \lambda_k$, $1 \leq k \leq n-1$, is indeed an eigenvalue of $C(\alpha, 0)$ for each α, it remains to show that $\mathbf{b}_i \neq \mathbf{0}$ for some i. Using Proposition 2.1, we have that there exists an m, $1 \leq m \leq 2^j$ such that $v_{km} = v_{(2n-k)(2^j-m+1)} \neq 0$. We first show that $\mathbf{b}_0 \neq \mathbf{0}$. Let m be the index for which $v_{km} \neq 0$. Suppose $\mathbf{b}_0 = \mathbf{0}$. Then

$$v_{km} + \delta_k v_{(2n-k)m} = 0$$

and

$$v_{k(2^j-m+1)} + \delta_k v_{(2n-k)(2^j-m+1)} = v_{(2n-k)m} + \delta_k v_{km} = 0.$$

And so, $v_{km} = \delta_k^2 v_{km}$, a contradiction. Let λ and $\boldsymbol{v}$ be as assumed in the last assertion of theorem. Letting $\mathbf{b}_i = \boldsymbol{v}$ (resp., $\mathbf{b}_i = (-1)^i \boldsymbol{v}$), we conclude that λ is an eigenvalue of $C(\alpha, 0)$ with corresponding eigenvector $\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix}$.

Thus, λ_k is an eigenvalue of $C(\alpha, 0)$ for each α. □

Corollary 2.2. *Let $\delta_k = e^{\frac{\pi k}{n} i}$, k is an integer, $i = \sqrt{-1}$. Then, for each α, $\rho(C(\alpha,0)) = \bigcup_{k=1}^{n-1} \rho(\Gamma_k) \bigcup \rho^S(\Gamma_0) \bigcup \rho^{AS}(\Gamma_n)$, where $\rho^S(A)$ (resp., $\rho^{AS}(A)$) the set of eigenvalues of A for which their corresponding eigenvectors are symmetric (resp., antisymmetric).*

We next consider the eigenvalues of $C(\alpha, \beta)$.

Theorem 2.3. *Let $\delta_k = e^{\frac{\pi k}{n} i}$, k is an integer, $i = \sqrt{-1}$. Then, for each α,*

$$\rho(C(\alpha,\beta)) \supset \begin{cases} \bigcup_{k=1}^{[\frac{n}{2}]} \rho(\Gamma_{2k}) \bigcup \rho^S(\Gamma_0), & n \ \text{is odd}, \\ \bigcup_{k=1}^{\frac{n}{2}-1} \rho(\Gamma_{2k}) \bigcup \rho^S(\Gamma_0) \bigcup \rho^{AS}(\Gamma_n), & n \ \text{is even}. \end{cases}$$

Here $[\frac{n}{2}]$ is the greatest integer that is less than or equal to $\frac{n}{2}$.

Proof. We illustrate only the case that n is even. Let k be such that $1 \le k \le \frac{n}{2} - 1$. Let $\mathbf{b}_i = \delta_{2k}^i \boldsymbol{v}_{2k} + \delta_{2k}\delta_{2n-2k}^i \boldsymbol{v}_{2n-2k}$, we see clearly that such $\mathbf{b}_i$, $i = 0, 1, n, n+1$, satisfy both Neumann and periodic boundary conditions, respectively. And so

$$\mathbf{b}_0 = (1-\beta)\mathbf{b}_0 + \beta\mathbf{b}_0 = (1-\beta)\hat{I}\mathbf{b}_1 + \beta\mathbf{b}_n,$$

and

$$\mathbf{b}_{n+1} = (1-\beta)\mathbf{b}_{n+1} + \beta\mathbf{b}_{n+1} = (1-\beta)\hat{I}\mathbf{b}_n + \beta\mathbf{b}_1.$$

Here, δ_{2k}, $1 \le k \le \frac{n}{2} - 1$, are characteristic values of equation of (2.1). Thus, if $\lambda \in \rho(\Gamma_{2k})$, then λ is an eigenvalue of $C(\alpha,\beta)$. The assertions for Γ_0 and Γ_n can be done similarly. □

Remark 2.3. If n is an even number, for each α and β, half of the eigenvalues of $C(\alpha,\beta)$ are independent of the choice of β. The other characteristic values of (1) seem to depend on β. It is of interest to find them.

3. The Second Eigencurve of $C(\alpha, 0)$ and $C(\alpha, 1)$

We begin with considering the eigencurves of Γ_k, as given in (2.8). Clearly,

$$\Gamma_k = \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & \delta_{2n-k} \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ \delta_k & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}_{m\times m} - \frac{\alpha(2 - 2\cos\frac{\pi k}{n})}{m} ee^T$$

$$=: D_1(k) - \alpha(k)ee^T, \tag{3.1}$$

where $m = 2^j$. We next find a unitary matrix to diagonalize $D_1(k)$.

Remark 3.1. Let $(\lambda(k), \boldsymbol{v}(k))$ be the eigenpair of $D_1(k)$. If $e^T\boldsymbol{v}(k) = 0$, then $\lambda(k)$ is also an eigenvalue of Γ_k.

Proposition 3.1. *Let*

$$\theta_{l,k} = \frac{2l\pi}{m} + \frac{k\pi}{nm}, l = 0, 1, ..., m-1, \tag{3.2a}$$

$$\boldsymbol{p}_l(k) = \begin{pmatrix} e^{i\theta_{l,k}} \\ e^{i2\theta_{l,k}} \\ \vdots \\ e^{im\theta_{l,k}} \end{pmatrix} \tag{3.2b}$$

and

$$P(k) = \begin{pmatrix} \frac{\boldsymbol{p}_0(k)}{\sqrt{m}} & \cdots & \frac{\boldsymbol{p}_{m-1}(k)}{\sqrt{m}} \end{pmatrix}. \tag{3.2c}$$

(i) Then $P(k)$ is a unitary matrix and $P^H(k)D_1(k)P(k) = Diag(\lambda_{0,k} \cdots \lambda_{m-1,k})$, where P^H is the conjugate transpose of P, and

$$\lambda_{l,k} = 2\cos\theta_{l,k} - 2, l = 0, 1, .., m-1. \tag{3.2d}$$

(ii) Moreover, for $0 \le k \le 2n$, the eigenvalues of $D_1(k)$ are distinct if and only if $k \ne 0, n$ or $2n$.

Proof. Let $\mathbf{b} = (b_1, ..., b_m)^T$. Writing the eigenvalue problem $D_1(k)\mathbf{b} = \lambda\mathbf{b}$ in component form, we get

$$b_{j-1} - (2+\lambda)b_j + b_{j+1} = 0, j = 2, 3, ..., m-1, \tag{3.3a}$$

$$-(2+\lambda)b_1 + b_2 + \delta_{2n-k}b_m = 0, \tag{3.3b}$$

$$\delta_k b_1 + b_{m-1} - (2+\lambda)b_m = 0. \tag{3.3c}$$

Set $b_j = \delta^j$, where δ satisfies the characteristic equation $1-(2+\lambda)\delta+\delta^2 = 0$ of the system $D_1(k)\mathbf{b} = \lambda\mathbf{b}$. Then the boundary conditions (3.3b) and (3.3c) are reduced to

$$\delta^m = \delta_k. \tag{3.4}$$

Thus, the solutions $e^{i\theta_{l,k}}$, $l = 0, 1, ..., m-1$, of (3.4) are the candidates for the characteristic values of (3.3). Substituting $e^{i\theta_{l,k}}$ into (3.3a) and solving for λ, we see that $\lambda = \lambda_{l,k}$ are the candidates for the eigenvalues of $D_1(k)$. Clearly, $(\lambda, \mathbf{b}) = (\lambda_{l,k}, \boldsymbol{p}_l(k))$ satisfies $D_1(k)\mathbf{b} = \lambda\mathbf{b}$ and $\mathbf{b} = \boldsymbol{p}_l(k) \ne 0$. To complete the proof of the proposition, it suffices to show that $P(k)$ is unitary. To this end, we have that

$$\boldsymbol{p}_l^H(k) \cdot \boldsymbol{p}_{l'}(k) = \begin{cases} m & l = l' \\ 0 & l \neq l' \end{cases},$$

Clearly, $\boldsymbol{p}_l^H(k) \cdot \boldsymbol{p}_l(k) = m$. Now, let $l \neq l'$, we have that

$$\boldsymbol{p}_l^H(k) \cdot \boldsymbol{p}_{l'}(k) = \sum_{j=1}^{m} e^{ij(\theta_{l,k} - \theta_{l',k})} = \sum_{j=1}^{m} e^{ij(\frac{2(l-l')}{m}\pi)} = \frac{r(1-r^m)}{1-r} = 0,$$

where $r = e^{i(\frac{2(l-l')}{m}\pi)}$. The last assertion of the proposition is obvious. □

Remark 3.2. The eigenvalues and eigenvectors of $D_1(k)$ were first given in the Proposition 2.3 of [3]. For completeness, we give an easier proof above.

To prove the main results in this section, we also need the following proposition.

Proposition 3.2. *Suppose $D = diag(d_1, ..., d_m) \in \mathbb{R}^{m \times m}$ and that the diagonal entries satisfy $d_1 > \cdots > d_m$. Let $\gamma \neq 0$ and $\boldsymbol{z} = (z_1, ..., z_m)^T \in \mathbb{R}^n$. Assume that $(\lambda_i(\gamma), \boldsymbol{v}_i(\gamma))$ are the eigenpairs of $D + \gamma \boldsymbol{z}\boldsymbol{z}^T$ with $\lambda_1(\gamma) \geq \lambda(\gamma) \geq ... \geq \lambda_m(\gamma)$. (i) Let $A = \{k : 1 \leq k \leq m, z_k = 0\}$, $A^c = \{1, ..., m\} - A$. If $k \in A$, then $d_k = \lambda_k$. (ii) Assume $\gamma > 0$. Then the following interlacing relations hold $\lambda_1(\gamma) \geq d_1 \geq \lambda_2(\gamma) \geq d_2 \geq ... \geq \lambda_m(\gamma) \geq d_m$. Moreover, the strict inequality holds for these indexes $i \in A^c$. (iii) Let $i \in A^c$, $\lambda_i(\gamma)$ are strictly increasing in γ and $\lim_{\gamma \to \infty} \lambda_i(\gamma) = \bar{\lambda}_i$ for all i, where $\bar{\lambda}_i$ are the roots of $g(\lambda) = \sum_{k \in A^c} \frac{z_i^2}{d_k - \lambda}$ with $\bar{\lambda}_i \in (d_i, d_{i-1})$. In case that $1 \in A^c$, $d_0 = \infty$.*

Proof. The proof of interlacing relations in (ii) and the assertion in (i) can be found in Theorem 8.6.2 of [4]. We only prove the remaining assertions of the proposition. Rearranging $\mathbf{z}$ so that $\mathbf{z}^T = (0, 0, ..., 0, z_{i_1}, ..., z_{i_k}) =: (0, ..., 0, \bar{\mathbf{z}}^T)$, where $i_1 < i_2 < ... < i_k$ and $i_j \in A^c$, $j = 1, ..., k$. The diagonal matrix D is rearranged accordingly. Let $D = diag(D_1, D_2)$, where $D_2 = diag(d_{i_1}, ..., d_{i_k})$. Following Theorem 8.6.2 of [4], we see that $\lambda_{i_j}(\gamma)$ are the roots of the scalar equation $f_\gamma(\lambda)$, where

$$f_\gamma(\lambda_{i_j}(\gamma)) = 1 + \gamma \sum_{j=1}^{k} \frac{z_{i_j}^2}{d_{i_j} - \lambda_{i_j}(\gamma)} = 0. \tag{3.5}$$

Differentiate the equation above with respect to γ, we get

$$\sum_{j=1}^{k} \frac{z_{i_j}^2}{d_{i_j} - \lambda_{i_j}(\gamma)} + (\gamma \sum_{j=1}^{k} \frac{z_{i_j}^2}{(d_{i_j} - \lambda_{i_k}(\gamma))^2}) \frac{d\lambda_{i_j}(\gamma)}{d\gamma} = 0.$$

Thus,

$$\frac{d\lambda_{i_j}(\gamma)}{d\gamma} = \frac{1}{\gamma^2}\sum_{j=1}^{k}\frac{z_{i_j}^2}{(d_{i_j}-\lambda_{i_j}(\gamma))^2} > 0.$$

Clearly, the limit of $\lambda_{i_j}(\gamma)$ as $\gamma \to \infty$ exists, say $\bar{\lambda}_{i_j}$. Since, for $d_{i_j} < \lambda_{i_j} < d_{i_j-1}$,

$$\sum_{j=1}^{k}\frac{z_{i_j}^2}{d_{i_j}-\lambda_{i_j}(\gamma)} = -\frac{1}{\gamma}.$$

Taking the limit as $\gamma \to \infty$ on both side of the equation above, we get

$$\sum_{j=1}^{k}\frac{z_{i_j}^2}{d_{i_j}-\bar{\lambda}_{i_j}} = 0 \tag{3.6}$$

as desired. □

We are now in the position to state the following theorems.

Theorem 3.1. *For each k and α, denote by $\lambda_{l,k}(\alpha)$,$l = 0, 1, ..., 2^j - 1 =: m-1$, the eigenvalues of Γ_k. For $k = 1, 2, ..., n-1$, let $(\lambda_{l,k}, u_{l,k})$ be the eigenpairs of $D_1(k)$, as defined in (3.1), then there exist $\lambda^*_{l,k}$ such that $\lim_{\alpha\to\infty}\lambda_{l.k}(\alpha) = \lambda^*_{l,k}$. Moreover, $g_k(\lambda^*_{l,k}) = 0$, where*

$$g_k(\lambda) = \sum_{l=1}^{m}\frac{1}{(\lambda_{l-1,k})(\lambda_{l-1,k}+\lambda)}. \tag{3.7}$$

Proof. Let k be as assumed. Set, for $l = 0, 1, ..., m-1$,

$$z_{l+1} = \boldsymbol{p}_l^H(k)e = \sum_{j=1}^{m} e^{ij\theta_{l,k}} = \frac{e^{-i\theta_{l,k}}(1-e^{-im\theta_{l,k}})}{1-e^{-i\theta_{l,k}}} = \frac{e^{-i\theta_{l,k}}(1-e^{-ik\frac{\pi}{n}})}{1-e^{-i\theta_{l,k}}}.$$

Then

$$\bar{\boldsymbol{z}}_{l+1}\boldsymbol{z}_{l+1} = \frac{2-2\cos m\theta_{l,k}}{2-2\cos\theta_{l,k}} = \frac{2\cos\frac{k\pi}{n}-2}{\lambda_{l,k}} \neq 0. \tag{3.8}$$

Let $P(k)$ be as given in (3.2c). Then

$$-P^H(k)\cdot\Gamma_k\cdot P(k) = Diag(-\lambda_{0,k}, ..., -\lambda_{m-1,k}) + \alpha(k)P_l^H(k)e(P_l^H(k)e)^H.$$

Note that if k is as assumed, it follows from Proposition 3.1-(ii) that $\lambda_{l,k}$, $l = 0, ..., m-1$, are distinct. Thus, we are in the position to apply Proposition 3.2.

Specifically, by noting $A^c = \phi$, we see that $\lambda^*_{l,k}$ satisfies $g_k(\lambda) = 0$, where

$$g_k(\lambda) = \sum_{l=1}^{m} \frac{1}{(\lambda_{l-1,k})(\lambda_{l-1,k} + \lambda)}.$$

□

Conjecture 3.1. We conjecture that

$$\lambda^*_{0,k} \leq \lambda_{0,n}, \qquad k = 1, 2, ..., n-1. \tag{3.9}$$

The calculation via computer seems to confirm that (3.9) holds. Note that if $g_k(-\lambda_{0,n}) < 0$, then (3.9) holds true.

Remark 3.3. For $m = 2$, we have that

$$\lambda^*_{0,k} = \frac{\lambda^2_{0,k} + \lambda^2_{1,k}}{\lambda_{0,k} + \lambda_{1,k}}. \tag{3.10}$$

Treating k as a real parameter, and differentiating (3.9) with respect to k, we get

$$(\lambda_{0,k}+\lambda_{1,k})^2 \frac{d\lambda^*_{0,k}}{dk} = 2(\lambda_{0,k}\frac{\lambda_{0,k}}{dk}+\lambda_{1,k}\frac{\lambda_{1,k}}{dk})(\lambda_{0,k}+\lambda_{1,k}) - (\lambda^2_{0,k}+\lambda^2_{1,k})(\frac{\lambda_{0,k}}{dk}+\frac{\lambda_{1,k}}{dk})$$

$$= \lambda^2_{0,k}\frac{\lambda_{0,k}}{dk} + \lambda^2_{1,k}\frac{\lambda_{1,k}}{dk} + (\lambda_{0,k}\lambda_{1,k})(\lambda_{0,k}\frac{\lambda_{0,k}}{dk} + \lambda_{1,k}\frac{\lambda_{1,k}}{dk}) < 0.$$

Consequently $\lambda^*_{0,k} \leq \lambda^*_{0,1}$. A direct computation would yield that $\lambda^*_{0,1} \leq \lambda_{0,n}$.

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