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碩 士 論 文

仿射過程及其應用

Affine Processes and Applications

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仿射過程及其應用

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摘　　　要

仿射過程是種取值於 m 個正實數值及 n 個實數值的馬可夫過程，而仿射過程的特殊性質能廣泛的處理財務上的問題。Duffie、 Filipovic、 Schachermeyer 完整刻劃出仿射過程的主要特徵，再者仿射過程和超擴散過程的關係也已被建立。基於這些觀察我們能建構更多仿射過程來處理財務上的問題。

Affine Processes and Applications

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ABSTRACT

Affine processes is a class of Markov processes taking values in $\mathbb{R}_+^m \times \mathbb{R}^n$. The rich variety of alternative types of random behavior(e.g., mean reversion, stochastic volatility, and jumps) and analytically tractable for affine processes make them ideal models for financial applications. Duffie, Filipovic and Schachermayer[DFS03] characterized all regular affine processes. Connections between regular affine processes and superprocesses with a finite base space were also established. Based on this observation, we construct more general affine processes and investigate sample path properties and financial applications of these processes..

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Contents

1. Intensity Model

There are two reasons why intensity model are important in the study of default risk. First,intensity models clearly seem to be the most elegant way of bridging the gap between credit scoring or default prediction models and the models for pricing default risk. Second,the mathematical machinery of intensity models brings into play the entire machinery of default-free term-structure model.
A process X of state variables with values in $\mathbb{R}^d$ is defined on the probability space (Ω, F, Q). Let λ: $\mathbb{R}^d \to \mathbb{R}$ be nonnegative function . We want to construct a jump process N_t with with the property that $\lambda(X_t)$ is the F_t-intensity of N. Let $F_t = G_t \vee H_t$, where $G_t = \sigma\{X_s : 0 \leq s \leq t\}$ and $H_t = \sigma\{N_s : 0 \leq s \leq t\}$, ie F_t contains the information in both X and jump process. Let J_1 be an exponential random variable with mean 1 , which is independent of $(G_t)_{t\geq 0}$. Define

$$\tau = inf\{t : \int_0^t \lambda(X_s)ds \geq J_1\}.$$

Consider a zero-coupon bond issued by a risky firm at time 0. Suppose the maturity of the bond is T and under the risk-neutral probability measure Q, the default time τ of the issuing firm has an intensity $\lambda(X_t)$, where the setup is precisely as above. Assume that there is a short-rate process $r(X_s)$ such that default free zero-coupon bond prices can be computed as

$$p(t,T) = \mathbb{E}[exp(-\int_t^T r(X_s)ds)|F_t],$$

where T is the maturity date of the bond .On the other hand, the price of the risky bond with zero recovery at time 0 is

$$\begin{aligned} v(0,t) &= \mathbb{E}[exp(-\int_0^T r(X_s)ds)1_{\{\tau>T\}}] \\ &= \mathbb{E}[\mathbb{E}[exp(-\int_0^T r(X_s)ds)1_{\{\tau>T\}}|G_T]] \\ &= \mathbb{E}[exp(-\int_0^T r(X_s)ds)\mathbb{E}[1_{\{\tau>T\}}|G_T]] \\ &= \mathbb{E}[exp(-\int_0^T r(X_s))exp(-\int_0^T \lambda(X_s)ds)] \\ &= \mathbb{E}[exp(-\int_0^T (r+\lambda)(X_s)ds)]. \end{aligned}$$

In general, we want to compute the price S_t of default claim having the expression

$$\mathbb{E}(exp(-\int_t^T r(X_s)ds)f(X_T)1_{\{\tau>T\}}|F_t). \tag{1.1}$$

Lemma 1.1
Let Y be a F_t-measurable random variable , where $F_t = H_t \vee G_t$, then

$$\mathbb{E}(1_{\{\tau>s\}} \cdot Y|F_t) = 1_{\{\tau>t\}}\mathbb{E}(1_{\{\tau>s\}}exp^{\int_0^t ds}Y|G_t), \forall s>t.$$

According to Lemma(1.1) , we obtain

$$\begin{aligned} \mathbb{E} \quad (\quad & exp(-\int_t^T r(X_s)ds)f(X_T)1_{\{\tau>T\}}|F_t) \\ = \quad & 1_{\{\tau>t\}}\mathbb{E}(exp(-\int_t^T r(X_s)ds)exp(\int_0^t \lambda(X_s)ds)f(X_T)1_{\{\tau>T\}}|G_t) \\ = \quad & 1_{\{\tau>t\}}\mathbb{E}(exp(-\int_t^T r(X_s)ds)exp(\int_0^t \lambda(X_s)ds)f(X_T)exp(-\int_0^T \lambda(X_s)ds)|G_t) \\ = \quad & 1_{\{\tau>t\}}\mathbb{E}[exp(-\int_t^T (r+\lambda)(X_s)ds)f(X_T)|G_t]. \end{aligned}$$

2. Affine Processes

A Markov transition function in a measurable space $(E,\mathcal{B})$ is a function $p(r,x;t,B), r < t \in \mathbb{R}, x \in E, B \in \mathcal{B}$ which is $\mathcal{B}$-measurable in x and which is a measure in $\mathcal{B}$ subject to the conditions:

(A) $\int_E p(r,x;t,dy)p(t,y;u,B) = p(r,x;u,B)$ for all $r < t < u, x \in E$ and all $B \in \mathcal{B}$.
(B) $p(r,x;t,E) \leq 1$ for all r,x,t.

To every Markov transition function p there corresponds a family of linear operator T_t^r acting on functions by the formula

$$T_t^r f(x) = \int_E p(r,x;t,dy)f(y).$$

It follows from (B) that $T_s^r T_t^s = T_t^r$ for all $r < s < t \in \mathbb{R}$. We call T the Markov semigroup corresponding to the transition function p.

We assume that $(E, \mathcal{B})$ is a measurable Luzin space. To every Markov transition function p there corresponds a Markov process $\xi = (\xi_t, \mathcal{F}(I), \Pi_{r,x})$ such that

$$\Pi_{r,x}\{\xi_t \in B\} = p(r, x; t, B),$$

$$\Pi_{r,x}\{\xi_{t_1} \in B_1, \cdots, \xi_{t_n} \in B_n\} = \int_{B_1\times\cdots\times B_n} p(r, x; t_1, dy_1)p(t_1, y_1; t_2, dy_2)\cdots p(t_{n-1}, y_{n-1}; t_n, dy_n)$$

for $n \geq 2, t_1 < t_2 < \cdots < t_n$.

If the transition function $p(r, x; t, dy)$ satisfies a condition

$$p(r, x; t, B) = p(r + s, x; t, B)$$

for all r, s, t, B, then $\xi = (\xi_t, \mathcal{F}(I), \Pi_{r,x})$ is time homogeneous. In this case we consider only process $\xi = (\xi_t, \mathcal{F}_t, \Pi_x)$ where $\Pi_x = \Pi_{0,x}, \mathcal{F}_t = \mathcal{F}[0, t]$, and ξ_t is defined for all $t \geq 0$.

From this point on we consider $E = \mathbb{R}^m_+ \times \mathbb{R}^n$ and write $d = m + n$.

Definition 2.1 We say that $\xi = (\xi_t, \mathcal{F}(I), \Pi_{r,x})$ is affine if for every $r < t \in \mathbb{R}$ and $\lambda \in \mathbb{R}^d$, there exists $\varphi(r, t, \lambda) \in \mathbb{C}$ and $\psi(r, t, \lambda) \in \mathbb{C}^d$ such that

$$\Pi_{r,x}[exp\{i < \lambda, \xi_t >\}] = exp\{\varphi(r, t, \lambda)+ < \psi(r, t, \lambda), x >\} \tag{2.1}$$

for all $x \in E$. Here $< x, y >= \sum_{i=1}^d x_i y_i$ for $x = (x_1, x_2, ..., x_d)$ and $y = (y_1, y_2, ..., y_d)$ in $\mathbb{C}^d$. Clearly if ξ is time homogeneous, then we have $\psi(r, t, \lambda) = \psi(t - r, \lambda)$ and $\varphi(r, t, \lambda) = \varphi(t - r, \lambda)$.

Example 1 (Ornstein-Uhlenbeck process)
Consider an Ornstein-Uhlenbeck process

$$d\xi_t = \alpha(l - \xi_t)dt + \sigma dw_t$$

where w is a standard Brownian and α, l and σ are positive constants. Through an application of Ito's formula, we get

$$\xi_t = e^{-\alpha t}[\xi_0 + l(e^{\alpha t} - 1) + \sigma \int_0^t e^{\alpha s} dw_s].$$

Clearly, X_t is normally distributed. Because $e^{\alpha s}$ is continuous in [o,t], $\sum_{s_i\in\Delta^n} e^{\alpha s_i}(W_{s_{i+1}} - W_{s_i})$ converge to $\int_0^t e^{\alpha s} dW_s$ in probability , where Δ^n is subdivisions of [o,t]. Therefore,

$$E[\sum_{s_i\in\Delta^n} e^{\alpha s_i}(W_{s_{i+1}} - W_{s_i})] \; converge \; to \; E[\int_0^t e^{\alpha s} dW s], \tag{2.2}$$

as $s \to t$, $E[\int_0^t e^{\alpha s} dW s] = 0$. Hence $E[\xi_t] = e^{-\alpha t}[x + l(e^{\alpha t} - 1)]$ and $Var\xi_t$ $= E[\xi_t - e^{-\alpha t}[x + l(e^{\alpha t} - 1)]]^2 = \sigma^2 \cdot e^{-2\alpha t} E[\int_0^t e^{\alpha s} dW_s]^2$. By the calculus

$$e^{\alpha t} dW_t \cdot e^{\alpha t} dW_t = e^{2\alpha t} dt, \tag{2.3}$$

we have

$$Var\xi_t = \sigma^2 e^{-2\alpha t} \cdot \int_0^t e^{2\alpha s} ds = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \tag{2.4}$$

Consequently, ξ_t is normally distributed with mean

$$\Pi_x \xi_t = e^{-\alpha t}[x + l(e^{\alpha t} - 1)]$$

and variance

$$Var\xi_t = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

This implies that

$$\Pi_x e^{i\lambda\xi_t} = exp\{v(t,\lambda) + u(t,\lambda)x\}$$

with

$$v(t,\lambda) = i\lambda e^{-\alpha t} l(e^{\alpha t} - 1) - \frac{\lambda^2\alpha^2}{4\alpha}(1 - e^{-2\alpha t})$$

and

$$u(t,\lambda) = i\lambda e^{-\alpha t}.$$

Here, we provide different way.

Consider for $t > 0$

$$dX_s = (a - bX_s)ds + \sigma d\widetilde{W}_s, where a, b, \sigma > 0 \tag{2.5}$$

$$v := E^{s,y}[e^{\lambda X_t}], where v(s,y) : [0,t) \times [0,\infty) \to R \tag{2.6}$$

We want to claim $v(s,y)$ has the form $e^{-\varphi_1(s,t,\lambda)-\psi_2(s,t,\lambda)y}$. Let $v^*(s,y) := e^{-\varphi_1(s,t)-\psi_2(s,t)y}$ where

$$\varphi_1(s,t,\lambda) = -\frac{\lambda a}{b}e^{b(s-t)} + \frac{\lambda^2\sigma^2}{4b}e^{2b(s-t)} + k, \tag{2.7}$$

k is constant

$$\varphi_2(s,t,\lambda) = \lambda e^{b(s-t)} \tag{2.8}$$

Applying the Feynman-Kac Thm to

$$(i) - \frac{\partial v^*}{\partial s}(s,y) = A_s v^*(s,y), A_s v^*(s,y) = e^{b(s-t)}[\frac{\lambda^2\sigma^2}{2}e^{2b(s-t)} - (\lambda a - by)e^{b(s-t)}],$$

$$(ii) v^*(t,y) = e^{\lambda y}, (iii) max_{0\leq s\leq t}|v^*(s,y)| \leq \frac{\lambda a + \lambda^2\sigma^2}{b}$$

satisfying the polynomial growth condition . Hence $v^*(s,y) = E^{s,y}[e^{\lambda X_t}]$

Example 2 (Feller's diffusions)

Feller considered a class of processes that includes the square-root diffusions

$$d\xi_t = \alpha(l - \xi_t)dt + \sigma\sqrt{\xi_t}dw_t$$

where w is a standard Brownian. We consider the case that α, l and σ are positive constants. Based on results of Feller[Fe51], Cox, Ingersoll and Ross[CIR85] noted that the probability density of the interest rate at time s, conditional on its value at the current time t, is given by $f(\xi(t), t; \xi(u), u) = ce^{-w-y}(\frac{y}{w})^{\frac{q}{2}} I_q(2(wy)^{\frac{1}{2}})$ where

$c = \frac{2\alpha}{\sigma^2(1-e^{-\alpha(t-u)})}$

$w = c\xi(u)e^{-\alpha(t-u)}$

$y = c\xi(t)$

$q = \frac{2\alpha l}{\sigma^2} - 1$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q

$$I_q(z) = e^{\frac{-iq\pi}{2}} \cdot J_q(iz) \tag{2.9}$$

where

$$J_q(z) = (\frac{z}{2})^q \cdot \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{z}{2})^{2k}}{k!\Gamma(q+k+1)} \tag{2.10}$$

and $\Gamma(y) = \int_0^\infty e^{-x} \cdot x^{y-1}dx$. Hence, the distribution of ξ_t given ξ_u for some $u < t$, is distributed as $\sigma^2(1-e^{-\alpha(t-u)})/4\alpha$ times a noncentral chi-square distribution $\chi^2_\nu(\lambda)$ with degree of freedom

$$\nu = \frac{4l\alpha}{\sigma^2}$$

and noncentrality

$$\lambda = \frac{4\alpha e^{-\alpha(t-u)}}{\sigma^2(1-e^{-\alpha(t-u)})}\xi_u.$$

Therefore the Laplace transform of ξ_t is given by

$$\Pi_x e^{-\lambda\xi_t} = \frac{1}{(2\lambda c+1)^{2l\alpha/\sigma^2}} exp\{-\frac{\lambda cf}{2\lambda c+1}\}$$

with $c = \sigma^2/4\alpha(1-e^{-\alpha t})$ and $f = 4x\alpha/(\sigma^2(e^{\alpha t}-1))$.
Here, we supply distinct method.
Consider

$$dX_s = (a - bX_s)ds + \sigma\sqrt{X_s}d\widetilde{W}_s \tag{2.11}$$

$$v(s,y) := \widetilde{E}^{s,y}[e^{\lambda X_t}] \tag{2.12}$$

where $v(s,y) : [0,t) \times R^+ \to R^+$
We want to claim $v(s,y)$ has the forme$^{-\varphi(s,t)-\psi(s,t)y}$. Let $v^*(s,y) := e^{-\varphi(s,t)-\psi(s,t)y}$ where $\varphi(s,t), \psi(s,t)$ staisfying

$$\varphi'(s,t) = -a\psi(s,t) \tag{2.13}$$

$$\psi'(s,t) = b\psi(s,t) + \frac{\psi^2(s,t)}{2}\sigma^2 \tag{2.14}$$

where $\varphi(t,t) = 0, \psi(t,t) = \lambda$.

$$\psi(s,t) = \frac{2b\lambda e^{-bt}}{-\sigma^2\lambda e^{-bt} + 2be^{-bs} + e^{-bs}\sigma^2\lambda} \tag{2.15}$$

$$\varphi(s,t) = \frac{2a}{\sigma^2}(bs + log(-\sigma^2\lambda e^{-bt} + (2b+\sigma^2\lambda)e^{-bs})) - \frac{2a}{\sigma^2} \cdot log2b \tag{2.16}$$

Applying to Feynman-Kac Thm to
(2.17)

$$(i) - \frac{\partial v^*}{\partial s}(s,y) = A_s v^*(s,y) \; where \; A_s v^*(s,y) = \frac{y\sigma^2}{2}\varphi_2^2(s,t) - (a-by)\varphi_2(s,t)$$

$$(ii) v^*(t,y) = e^{\lambda y} \tag{2.18}$$

$$(iii) v^*(s,y) \; satisfying \; the \; polynomial \; growth \; condition. \tag{2.19}$$

Hence,we obtain $v^*(s,y) = \widetilde{E}^{s,y}[e^{\lambda X_t}]$

Example 3 (Heston's Model)
An important two-dimensional affine model was used by Heston to model option

prices in settings with stochastic volatility . Here, one supposes that the underlying price process U of an asset satisfies

$$dU_t = U_t(\gamma_0 + \gamma_1 V_t)dt + U_t\sqrt{V_t}dB_{1t},$$

where γ_0 and γ_1 are constants and V is a stochastic -volatility process, which is a Feller diffusion satisfying

$$dV_t = \kappa(\overline{v} - V_t)dt + c\sqrt{V_t}dZ_t \tag{2.20}$$

for constant coefficients $\kappa, \overline{v}$, and c , where $Z = \rho B_1 + \sqrt{1-\rho^2}B_2$ is a Brownian motion that is constructed as a linear combination of independent standard Brownian motions B_1 and B_2.Letting $Y = logU$, a calculation based on Ito's Formula yields

$$dY_t = (\gamma_0 + (\gamma_1 - \frac{1}{2}V_t))dt + \sqrt{V_t}dB_{1t} \tag{2.21}$$

which implies that the two-dimensional process $X = (V, Y)$ is affine, with state space $D = \mathbb{R}_+ \times \mathbb{R}$

Now We have the main characterization results from Duffie et al.(2003a). First, we state an analytic characterization result for regular affine processes.

Definition 2.2 The Markov process $(X, (\mathbb{P}_X)_{X\in D})$, and (P_t) , is called *stochastically continuous* if $p_s(x,\cdot) \to p_t(x,\cdot)$ weakly on D , for $s \to t$, for every $(t, x) \in \mathbb{R} \times D$

Definition 2.3 The Markov process $(X, (\mathbb{P}_X)_{X\in D})$, and (P_t) , is called *regular* if it is stochastically continuous and the right-hand derivative

$$Af_u(x) := \partial_t^+ P_t f_u(x)|_{t=0} \tag{2.22}$$

exists for all $(x, u) \in D \times \mathcal{U}$, and is continuous at $u = 0$ for all $x \in D$.

Theorem 2.1 *Suppose X is regular affine . Then X is a Feller process . Let A be its infinitesimal generator. Then $C_c^\infty(D)$ is a core of A , $C_c^2(D) \subset D(A)$, and there exist admissible parameters $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ such that , for $f \in C_c^2(D)$,*

$$\begin{aligned} Af(x) &= \sum_{k,l=1}^{d}(a_{k,l} + < \alpha_{y,kl}, y >)\frac{\partial^2 f(x)}{\partial x_k \partial x_l} + < b + \beta x, \nabla f(x) > -(c + < \gamma, y >) \\ &+ \int_{D\backslash\{0\}} Gf_0(x,\xi)m(d\xi) \\ &+ \sum_{i=1}^{m}\int_{D\backslash\{0\}} Gf_i(x,\xi)y_i\mu_i(d\xi) \end{aligned} \tag{2.23}$$

where

$$Gf_0(x,\xi) = f(x+\xi) - f(x) - < \nabla_z f(x), \chi_z(\xi) >, \tag{2.24}$$

$$Gf_i(x,\xi) = f(x+\xi) - f(x) - < \nabla_{z(i)} f(x), \chi_{z(i)}(\xi) >, \tag{2.25}$$

Moreover,(2.1) holds for all $(t,u) \in \mathbb{R}_+ \times \mathcal{U}$ where $\varphi(t,u)$ and $\psi(t,u)$ solve the generalized Riccati equations,

$$\varphi(t,u) = \int_0^t F(\psi(s,u))ds \tag{2.26}$$

$$-\partial_t \psi^y(t,u) = R^y(t, \psi^y(t,u), e^{\int_t^T \beta^{\mathcal{Z}}(s)ds} w), \psi^y(T,T,u) = v \tag{2.27}$$

$$\psi^z(t,u) = e^{\beta^{\mathcal{Z}} t} w \tag{2.28}$$

with

$$\begin{aligned} F(u) =& - < au, u > - < b, u > \\ & - \int_{D\backslash 0} (e^{<u,\xi>} - 1 - < u_J, \chi_J(\xi) >) m(d\xi) \\ R_i^y(u) =& - < \alpha_i u, u > - < \beta_i^{\mathcal{Y}}(u) > \\ & - \int_{D\backslash (0)} (e^{<u,\xi>} - 1 - < u_{J(i)}, \chi_{J(i)}(\xi) >) \mu_i(d\xi) \end{aligned}$$

and

$$\begin{aligned} \beta_i^{\mathcal{Y}} :=& (\beta^T)_{i\{1,\dots,d\}} \in \mathbb{R}^d, i \in y \\ \beta^{\mathcal{Z}} :=& (\beta^T)_{JJ} \in \mathbb{R}^{n \times n} \end{aligned}$$

Conversely,let $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ be admissible parameters .Then there exists a unique regular affine semigroup (P_t) with infinitesimal generator (2.23) and (2.1) holds for all $(t,u) \in \mathbb{R} \times U$ where $\varphi(t,u)$ and $\psi(t,u)$ are give by (2.26), (2.27), and (2.28).

3. Transform Analysis And Asset Pricing For Affine Jump-Diffusion

We fix a probability space (Ω, F, P) , and an information filtration(F_t), and suppose that X is a Markov process in some state space $D \subset \mathbb{R}^n$, solving the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dZ_t, \tag{3.1}$$

W is an (F_t)-standard Brownian motion in $\mathbb{R}^n$; $\mu : D \to \mathbb{R}^n$, $\sigma : D \to \mathbb{R}^n$, and Z is a pure jump process whose jumps have a fixed probability distribution ν on $\mathbb{R}^n$ and arrive with intensity $\{\lambda(X_t) : t \geq 0\}$, for some $\lambda : D \to [0, \infty)$. We impose an "affine" structure on $\mu, \sigma\sigma^\top$, and λ, in that all of these functions are assumed to be affine on D. We fix an affine discount-rate function $R : D \to \mathbb{R}$. The affine dependence of $\mu, \sigma\sigma^\top, \lambda$,and R are determined by coefficient (K, H, l, ρ) defined by :

(i)$\mu(x) = K_0 + K_1 x$, for $K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$.
(ii) $(\sigma(x)\sigma(x)^\top)_{ij} = (H_0)_{ij} + (H_1)_{ij} \cdot x$, for $H = (H_0, H_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}$.
(iii)$\lambda(x) = l_0 + l_1 \cdot x$, for $l = (l_0, l_1) \in \mathbb{R} \times \mathbb{R}^n$.
(iv)$R(x) = \rho_0 + \rho_1 \cdot x$, for $\rho = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n$.

Let $\theta(c) = \int_{\mathbb{R}^n} exp(c \cdot z) dv(z)$. The jump transform θ determines the jump-size distribution. The coefficients (K, H, l, θ) of X completely determine its distribution its distribution, given an initial condition $X(0)$. A characteristic $\chi = (K, H, l, \theta, \rho)$ captures both the distribution of X as well as the effects of any discounting, and determines a transform $\psi^{\chi} : \mathbb{C}^n \times D \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ of X_T conditional on F_t when well defined at $t \leq T$, by

$$\psi^{\chi}(u, X_t, t, T) = E^{\chi}[exp(-\int_t^T R(X_s)ds)e^{u \cdot X_T}|F_t], \tag{3.2}$$

where E^{χ} denotes expectation under the distribution of X determined by χ.

Definition 3.1

A characteristic (K, H, l, θ, ρ) is well-behaved at $(u, T) \in \mathbb{C}^n \times [0, \infty)$ if (2.5)-(2.6) are solved uniquely by β and α; and if

$$(1)\ E\int_0^T |\gamma_t| dt < \infty,\ \textit{where}\ \gamma_t = \varphi_t(\theta(\beta(t)) - 1)\lambda(X_t),$$

$$(2)\ E\int_0^T (\eta_t \cdot \eta_t)^{\frac{1}{2}} < \infty\ \textit{where}\ \eta_t = \varphi_t \beta(t)^{\top}\sigma(X_t),$$

and

$$(3)\ E|\varphi_T| < \infty,\ \textit{where}\ \varphi_t = exp(-\int_0^t R(X_s)ds)e^{\alpha(t)+\beta(t)\cdot X(t)}.$$

Proposition 3.1

Suppose (K, H, l, θ, ρ) is well-behaved at (u, T). Then the transform ψ^{χ} of X defined by (3.2) has the form $e^{\alpha(t)+\beta(t)\cdot x}$ where α and β satisfy the complex-valued ODEs.

$$\beta^{'}(t) = \rho_1 - K_1^{\top}\beta(t) - \frac{1}{2}\beta^{\top}(t)H_1\beta(t) - l_1(\theta(\beta(t)) - 1), \tag{3.3}$$

$$\alpha^{'}(t) = \rho_0 - K_0 \cdot \beta(t) - \frac{1}{2}\beta^{\top}H_0\beta(t) - l_0(\theta(\beta(t)) - 1), \tag{3.4}$$

with boundary conditions $\beta(T) = u$ and $\alpha(T) = 0$

4. Connection of Intensity Modeling and affine processes

From now on, we suppose that X is conservative regular affine with with parameters $(a, \alpha, b, \beta, 0, 0, m, \mu)$. Let $l \in \mathbb{R}$, $\lambda = (\lambda^y, \lambda^z) \in \mathbb{R}^m \times \mathbb{R}^n$ and define the affine function $L(x) := l + < \lambda, x >$ on $\mathbb{R}^d$. In many applications $L(x)$ is a model for short rates. The price of a claim of the form $f(X_t)$, where $f \in bD$, is given by the expectation

$$Q_t f(x) := \mathbb{E}_x[exp(-\int_0^t L(X_s)ds)f(X_t)].$$

Suppose that, for fixed $x \in D$ and $t \in \mathbb{R}_+$, we have $\mathbb{E}_x[exp(-\int_0^t L(X_s)ds)] < \infty$. Then $\mathbb{R}^d \to Q_t f_{iq}(x)$ is the characteristic function of X_t with respect to the bounded measure $exp(-\int_0^t L(X_s)ds)\mathbb{P}_x$. Hence, if one knows $Q_t f_u(x)$ for all $u \in \partial U$, the integral can be calculated via Fourier inversion. We use the martingale methods and is more general. But it requires an enlargement of state space and an analytic extension of the exponents $\varphi'(t,.)\psi'(t,.)$ of some (d+1)- dimensional regular affine process X' For $r \in \mathbb{R}$ write

$$R_t^r := r + \int_0^t L(X_s)ds. \tag{4.1}$$

It can be shown that (X, R^r) is a Markov process on $(\Omega, F, (F_t), \mathbb{P}_x)$ for every $x \in D$ and $r \in R$. In fact,we enlarge the state space $D \to D\times\mathbb{R}$ and $U \to U\times iR$, and write accordingly $(x,r) = (y,z,r)$,$(u,q) = (v,w,q) \in U\times i\mathbb{R}$. Let $X' = (Y', Z', R')$ be the regular affine process with state space $D \times \mathbb{R}$ given by the admissible parameters

$$a' = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix},\ \alpha_i' = \begin{bmatrix} \alpha_i & 0 \\ 0 & 0 \end{bmatrix},\ i \in \mathcal{I}$$

$$b' = (b,l),\ \beta_i' = \begin{bmatrix} \beta_i & 0 \\ 0 & 0 \end{bmatrix}$$

$$c' = c = 0,\ \gamma' = \gamma = 0$$

and

$$m'(d\xi, d\rho) = m(d\xi) \times \delta_0(d\rho),$$

$$\mu_i'(d\xi, d\rho) = \mu_i(d\xi) \times \delta_0(d\rho),\ i \in \mathcal{I}.$$

Let X' be defined on the canonical space $(\Omega', F', (F_t'), \mathbb{P}'_{(x,r)\in D\times\mathbb{R}})$. The corresponding mappings F' and $R' = (R'^{\mathcal{Y}}, R'^{\mathcal{Z}}, R'^{\mathcal{R}})$ satisfy

$$F'(u,q) = F(u) + lq,$$

$$R'^{\mathcal{Y}}(u,q) = R^{\mathcal{Y}}(u) + \lambda^y q,$$

$$R'^{\mathcal{Z}}(u,q) = R^{\mathcal{Z}}(u) + \lambda^z q,$$

$$R'^{\mathcal{R}}(u,q) = 0;$$

Let φ', $\psi' = (\psi'^{\mathcal{Y}}, \psi'^{\mathcal{Z}}, \psi'^{\mathcal{R}})$ be the solution of the corresponding generalized Riccati equations ,ie

$$\varphi'(t,u,q) = \int_0^t F(\psi^{\mathcal{Y}}(s,u,q), \psi^{\mathcal{Z}}(s,u,q))ds + tlq, \tag{4.2}$$

$$\psi'^{\mathcal{Y}}(t,u,q) = \int_0^t R^y(\psi^{\mathcal{Y}}(s,u,q), \psi^{\mathcal{Z}}(s,u,q)) + t\lambda^y q. \tag{4.3}$$

$$\psi'^{\mathcal{Z}}(t,u,q) = e^{\beta^z t}w + q\int_0^t e^{\beta^z \lambda^z} ds \tag{4.4}$$

$$\psi'^{\mathcal{R}}(t,u,q) = q. \tag{4.5}$$

with

$$F(u) =< au, u > + < b, u > -c + \int_{D\backslash\{0\}} (e^{<u,\xi>} - 1 - < u_J, \chi_J(\xi) >)m(d\xi),$$

$$R_i^{\mathcal{Y}}(u) =< \alpha_i u, u > + < \beta_i^y, u > -r_i + \int_{D\backslash\{0\}} (e^{<u,\xi>} - 1 - < u_{J(i)}, \chi_{J(i)}(\xi) >)\mu_i(d\xi)$$

for $i \in I$

Proposition 4.1 *Let* $(x,r)\in D \times \mathbb{R}$*. We have* $\mathbb{P}_x \circ (X, R^r)^{-1} = \mathbb{P}'_{(x,r)}$

$$\begin{aligned} \mathbb{E}_x[e^{qR^r_t} f_u(X_t)] &= \mathbb{E}'_{(x,r)}[e^{qR'_t} f_u(Y^{;}_t, Z^{;}_t)] \\ &= e^{\varphi'(t,u,q)+<\psi'^{\mathcal{Y}}(t,u,q),y>+<\psi'^{\mathcal{Z}}(t,u,q),z>+qr} \end{aligned} \tag{4.6}$$

for all $(t,u,q) \in \mathbb{R}_+ \times \mathcal{U} \times i\mathbb{R}$.
The proof can be found in Duffie,D., Filipovic,D., Schachermayer,W.,2003a. We have to ask where has a meaning for $q = -1$.

Proposition 4.2 *Let* X_t *be the conservative regular affine process* $,t \in \mathbb{R}_+$*. Let* U *be an open convex neighborhood of* 0 *in* $\mathbb{C}^d$*. Assume* $\varphi(t,.)$ *and* $\psi(t,.)$ *have an analytic extension on* U*. Then*

$$\mathbb{E}_x[e^{<q,X_t>}] < \infty,$$

$\forall q \in U \cap \mathbb{R}^d$,$\forall x \in D$ *and* (2.2) *holds* $,\forall u \in U$ *with* $Reu \in U \cap \mathbb{R}^d$.

Lemma 4.3 *Let* $V \subset \mathbb{R}^d$ *be open. If*

$$\int_{D\setminus Q(0)} e^{<q,\xi>} m(d\xi) < \infty \tag{4.7}$$

$$\int_{D\setminus Q(0)} e^{<q,\xi>} \mu_i(d\xi) < \infty, \tag{4.8}$$

where $Q(x) := \{\xi \in D : |\xi_k - x_k| \leq 1, 1 \leq k \leq d\}$ and $\forall q \in V$. *Then* F *and* R^y_i *are analytic on the open strip* $S = \{u \in C^d | Reu \in V\}$

Lemma 4.4 *Suppose that* F *and* R^y *are analytic on some open set* U *in* C^d*. Let* $T \leq \infty$ *such that for each* $u \in U$ *, there exists* U*-valued location solution* $\psi(t,u)$ *of (6.1),*$\forall t \in [0,T]$ *Then* φ, ψ *have a unique analytic extension of* $(0,T) \times U$.

Example 4.5
We fix a conservative regular affine process X with semigroup (P_t), and a "discounting" semigroup $(Q_t)_{t\in\mathbb{R}_+}$ based on a short-rate process $L(X)$.

Example 4.5.1 *The term structure of interest rates.*
A central object of study in finance is the term structure $t \to Q_t 1$ of prices of "bonds," assets that pay one unit of account at a given maturity t. In general because $1 = e^{<0,x>}$, the bond price

$$Q_t 1(x) = e^{A(t)+<B(t),x>} \tag{4.9}$$

is easily calculated from the generalized Riccati equations for a broad range of affine processes. Indeed, $A(t) := \varphi'(t,0,-1)$, $B(t) := (\psi'^{\mathcal{Y}}(t,0,-1), \psi'^{\mathcal{Z}}(t,0,-1))$

Example 4.5.2 *Default risk.*
In order to model the timing of default of financial contracts,we assume that N is a nonexplosive counting process that is doubly stochastic driven by X, with intensity

$\{\Lambda(X_{t-}) : t \geq 0\}$, where $x \to \Lambda(x) \geq 0$ is affine. For the doubly stochastic property of N, the survival probability is

$$\mathbb{P}_x(\tau > t) = \mathbb{E}_x[exp(-\int_0^t \Lambda(X_s)ds)].$$

which is of the same form as the bond-price calculation . For a model of the default times $\tau_1, ..., \tau_k$ of $k > 1$ different financial contracts, an approach is to suppose that τ_i is the first jump time of a nonexplosive counting process N_i with intensity $\{\Lambda_i(X_{t-}) : t \geq 0\}$, for affine $x \to \Lambda_i(x) \geq 0$, where $N_1, ..., N_k$ are doubly stochastic driven by X, and moreover are independent conditional on X. Hence, for any increasing sequence of times $t_1, t_2, ..., t_k$ in $\mathbb{R}_+$

$$\mathbb{P}_x(\tau_1 \geq t_1, ..., \tau_k \geq t_k) = \mathbb{E}_x[exp(-\int_0^{t_k} \Lambda(X_s, s)ds)], \tag{4.10}$$

where

$$\Lambda(x, s) = \Sigma_{\{i:s \leq t_i\}} \Lambda_i(x). \tag{4.11}$$

By the law of iterated expectations, the joint distribution of default times is equal to

$$\mathbb{P}_x(\tau_1 \geq t_1, ..., \tau_k \geq t_k) = e^{\varphi_0 + <\psi_0, x>}, \tag{4.12}$$

where φ_i, ψ_i are defined inductively by $\varphi_k = 0$,$\psi_k = 0$, and

$$e^{\varphi_i + <\psi_i, x>} = \mathbb{E}_x[exp(-\int_0^{t_{i+1}-t_i} \Lambda(X_t, t_i + t)dt)e^{\varphi_{i+1} + <\psi_{i+1}, X_{t_{i+1}-t_i}>}],$$

taking $t_0 = 0$. Because the coefficients φ_i and ψ_i are easily calculated recursively from the associated generalized Riccati equations.

Example 4.5.3 *Option pricing.*
Assume that the price of the underlying asset at time t is of the form $f(X_t)$, for some non-negative $f \in C(D)$. The payoff of put option $g(X_t) = max\{K - f(X_t), 0\}$,and the initial price

$$\begin{aligned} Q_t g(x) &= \mathbb{E}_x[exp(-\int_0^t L(X_s)ds)g(X_t)] \\ &= K\mathbb{E}_x[exp(-\int_0^t L(X_s)ds)1_{\{f(X_t) \leq K\}}] && (4.13) \\ &- \mathbb{E}_x[exp(-\int_0^t L(X_s)ds)f(X_t)1_{\{f(X_t) \leq K\}}]. && (4.14) \end{aligned}$$

One can exploit the affine modeling approach computational advantage provided $f(x) = ke^{<b,x>}$,where $k > 0$ in $\mathbb{R}$ and $b \in \mathbb{R}^d$. In this case, both terms in the calculation above of $Q_t g(x)$ are of the form

$$G_{a,b}(q) = \mathbb{E}_x[exp(-\int_0^t L(X_s)ds)e^{<a,X_t>}1_{\{<b,X_t> \leq q\}}]$$

for some $(a, b, q) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. Clearly, $G_{a,b}(.)$ is the distribution function of $< b, X_t >$ with respect to the measure

$$exp(-\int_0^t L(X_s)ds)e^{<a,X_t>}\mathbb{P}_x. \tag{4.15}$$

According to Fourier inversion , we only need to compute the transform

$$g_{a,b}(z) = \int_{-\infty}^{+\infty} e^{izq} G_{a,b}(dq). \tag{4.16}$$

One can see that

$$\begin{aligned} g_{a,b}(z) &= \mathbb{E}_x[exp(-\int_0^t L(X_s)ds)e^{<a,X_t>}e^{iz<b,X_t>}] \\ &= \mathbb{E}_x[exp(-\int_0^t L(X_s)ds)f_u(X_t)], \end{aligned} \tag{4.17}$$

where $u = a + izb$, and the generalized Riccati equations give the solution under nonnegativity of $L(X)$.

5. Solving Riccati Equation

Solving

$$y^{'} = y(\alpha y^2 + \beta y + \gamma) \tag{5.1}$$

, α,β,$\gamma \in \mathbb{R}$
Case 1. $\alpha \neq 0$

$$y^{'} = \alpha y(y^2 + ay + b) \tag{5.2}$$

$a = \frac{\beta}{\alpha}$,$b = \frac{\gamma}{\alpha}$
Subcase 1. $a^2 - 4b < 0$, $y(0) \neq 0$

$$\begin{aligned} \frac{y^{'}}{(y(y^2+ay+b))} &= (\frac{A}{y} + \frac{By+C}{y^2+ay+b})y^{'} \\ &= (\frac{A}{y} + \frac{B(y+\frac{a}{2}) + C - \frac{Ba}{2}}{(y+\frac{a}{2})^2 + b - \frac{a^2}{4}})y^{'} \\ &= (A \cdot lny + \frac{B}{2} \cdot ln(y+\frac{a}{2})^2 + (C - \frac{Ba}{2}) \cdot tan^{-1}(\frac{y+\frac{a}{2}}{\sqrt{b-\frac{a^2}{4}}})) = \alpha \end{aligned}$$

where $A = \frac{1}{b} = \frac{\alpha}{\gamma}$, $B = -\frac{\alpha}{\gamma}$, $C = -\frac{a}{b} = -\frac{\beta}{\gamma}$
Subcase 2. $a^2 - 4b = 0$
$\therefore y^2 + ay + b = (y + \frac{a}{2})^2$ $y(0) \neq 0$

$$\begin{aligned} \frac{y^{'}}{y \cdot (y+\frac{a}{2})^2} &= (\frac{A}{y} + \frac{By+C}{(y+\frac{a}{2})^2})y^{'} \\ &= (\frac{A}{y} + \frac{B(y+\frac{a}{2}) + C - \frac{Ba}{2}}{(y+\frac{a}{2})^2})y^{'} \\ &= (A \cdot lny + \frac{B}{2} \cdot ln(y+\frac{a}{2})^2 + \frac{(C - \frac{Ba}{2})}{y+\frac{a}{2}})^{,} = \alpha \end{aligned}$$

$A = \frac{4}{a^2} = \frac{4\alpha^2}{\beta^2}$, $B = -4\frac{\alpha^2}{\beta^2}$,$C = -Aa = -4\frac{\alpha}{\beta}$

Subcase 3. $a^2-4b>0$ $y^2+ay+b=(y-\theta_1)(y-\theta_2)$

$$\begin{aligned}\frac{y^{'}}{y\cdot(y-\theta_1)(y-\theta_2)} &= (\frac{A}{y}+\frac{B}{y-\theta_1}+\frac{C}{y-\theta_2})y^{'} \\ &= (A\cdot lny+B\cdot ln(y-\theta_1)+C\cdot ln(y-\theta_2))^{'}=\alpha\end{aligned}$$

where $A=\theta_1\theta_2$,$B=\frac{\theta_2^2\theta_1}{\theta_1-\theta_2}$, $C=\frac{\theta_1^2\theta_2}{\theta_2-\theta_1}$
$\theta_1=\frac{-a+\sqrt{a^2-4b}}{2}$ $\theta_2=\frac{-a-\sqrt{a^2-4b}}{2}$.

Case 2. $\alpha=0$

$$y^{'}=\beta\cdot y^2+\gamma y=\beta y(y+a) \tag{5.3}$$

,$a=\frac{\gamma}{\beta}$
Subcase 1. $y\neq -a$,$y(0)\neq 0$

$$\frac{y^{'}}{y(y+a)}=(\frac{A}{y}+\frac{B}{y+a})y^{'}=(A\cdot lny+B\cdot ln(y+a))^{'}=\beta \tag{5.4}$$

where $A=\frac{1}{a}=\frac{\beta}{\gamma}$, $B=-\frac{\beta}{\gamma}$

6. The Basic Three-Firm Model

Let $\mathbf{p}=(p_1,p_2,p_3)$,$\mathbf{q}=(q_1,q_2,q_3)\in\mathbf{I}:=\{0,1\}^3$.We now consider the affine jump-diffusion process $X=(X^0,....,X^6)$ in $\mathbb{R}_+^7$ with generator

$$\begin{aligned}Af &= \sum_{i=0}^{3}\alpha_i x_i\partial_{x_i}^2 f(x)+\sum_{i=0}^{3}(b_i+<\beta_i,x>)\partial_{x_i}f(x) \\ &+ \sum_{\mathbf{p}\in\mathbf{I}}(f(x+p_4e_4+p_5e_5+p_6e_5)-f(x))(l_{\mathbf{p}}+<\lambda_{\mathbf{p}},x>)\end{aligned} \tag{6.1}$$

where $\alpha_i,b_i\geq 0,\beta_i\in\mathbb{R}^7$ with $\beta_{ij}\geq 0,\forall j\neq i$, $l_{\mathbf{p}}\geq 0$, $\lambda_{\mathbf{p}}\in\mathbb{R}_+^7$

X^0 denotes the short rate process. The pair (X^i,X^{3+i}) represents the credit state of firm i, i =1,2,3. We let $X_0^{3+i}=0$ for i= 1,2,3. Then the first jump time τ_i:=inf$\{t|X_t^{3+i}>0\}$ of X^{3+i} models the default time of firm i. The generate implies a rich interdependence structure between the components X^i:

(i) The interest rates ,X^0, influence all credit risk relate variables, $X^1,....X^6$,by β_{i0}(mean-reversion level of X^i) and the respective $\lambda_{\mathbf{p,0}}$(jump intensity of X^{3+i}).

(ii) The credit index of firm i,X^i, i=1,2,3, drives the intensities for (joint) defaults of firms 1,2,and 3 by the respective $\lambda_{\mathbf{p,i}}$.

X^i also influences the mean reversion level for X^j by β_{ji}, j=0,...,3

(iii) The counting process for firm i , X^{3+i},i=1,2,3, influence the intensities for (joint) defaults of firms 1,2 and 3 by the respective $\lambda_{\mathbf{p},3+i}$. Note that this introduces "infectious defaults"and see example 2 below. The following is the main result.

Proposition
For $t \leq T$,$v \in R^7_-$, $\delta \geq 0$ $\mathbf{p} \in \mathbf{I}$ we have

$$E[e^{-\delta\int_t^T X_s^0 ds}e^{<v,X_T>}lim_{k\to\infty}e^{-k(p_4X_T^4+pX_T^5+pX_T^6)}|F_t]$$

$$(6.2) \qquad = e^{\varphi(T-t,v;\delta)+\sum_{i\in 0,\ldots,3\bigcup J_0(\mathbf{p})}\psi_i(T-t,v,\delta;\mathbf{p})X_t^i}\prod_{j\in J_1(p)}1_{\{X_t^j=0\}}$$

where $J_0(\mathbf{p}) := \{4 \leq j \leq 6|p_j = 0\}$,$J_1(\mathbf{p}) := \{4 \leq j \leq 6|p_j = 1\}$ and the R_--valued functions $\varphi = \varphi(t, v, \delta; \mathbf{p})$ and $\psi_i = \psi_i(t, v; \delta; \mathbf{p})$ satisfy

$$(6.3) \qquad \partial_t\varphi = \sum_{k=0}^{3} b_k\psi_k + \sum_{\mathbf{q}\in\mathbf{I}_0(\mathbf{p})} l_{\mathbf{q}}(e^{q_4\psi_4+q_5\psi_5+q_6\psi_6-1}) - \sum_{\mathbf{q}\in\mathbf{I}_1(\mathbf{p})} l_{\mathbf{q}},$$

$$\varphi(0, v; \delta; \mathbf{p}) = 0,$$

$$\begin{aligned}\partial_t\psi_i &= \alpha_i\psi_i^2 + \sum_{k=0}^{3}\beta_{ki}\psi_k + \sum_{\mathbf{q}\in\mathbf{I}_0(\mathbf{p})}\lambda_{\mathbf{q},i}(e^{q_4\psi_4+q_5\psi_5+q_6\psi_6} - 1) \\ &- \sum_{\mathbf{q}\in\mathbf{I}_1(\mathbf{p})}\lambda_{\mathbf{q},i} - \delta 1_{i=0}\end{aligned}$$

$$\psi_i(0, v; \delta; \mathbf{p}) = v_i$$

$$\partial_t\psi_j = \sum_{k=0}^{3}\beta_{kj}\psi_k + + \sum_{\mathbf{q}\in\mathbf{I}_0(\mathbf{p})}\lambda_{\mathbf{q},j}(e^{q_4\psi_4+q_5\psi_5+q_6\psi_6} - 1) - \sum_{\mathbf{q}\in\mathbf{I}_1(\mathbf{p})}\lambda_{\mathbf{q},j}$$

$$\psi_j(0, v; \delta; \mathbf{p}) = v_j$$

for i=0,1,2,3 and $j \in J_0(\mathbf{p})$ where $\mathbf{I}_0(\mathbf{p}):=\{\mathbf{q} \in \mathbf{I}|q_j = 0, \forall j \in J_1(\mathbf{p})\}$ and $I_1(\mathbf{p}) := \mathbf{I}\backslash\mathbf{I}_0(\mathbf{p}) = \{\mathbf{q} \in \mathbf{I}|q_j = 1, j \in J_1(\mathbf{p})\}$

Example 6.1 Let $t \leq T$. TheF_t- condition Laplace transform of X_T with respect to $T-$forward measure condition on $\{T < \tau_1 \wedge \tau_2\}$ is
$\frac{E[e^{-\int_t^T X_s^0 ds}e^{<v,X_T>}1_{\{T<\tau_1\wedge\tau_2\}}|F_t]}{E[e^{-\int_t^T X_s^0 ds}e^{<v,X_T>}|F_t]}$, $v \in R^7_-$ where

$$\begin{aligned}E \quad [\quad & e^{-\int_t^T X_s^0 ds}e^{<v,X_T>}1_{\{T<\tau_1\wedge\tau_2\}}|F_t] \\ = \quad & E[e^{-\int_t^T X_s^0 ds}e^{<v,X_T>}lim_{k\to\infty}e^{-k(X_T^4+X_T^5)}|F_t] \\ = \quad & e^{\varphi(T-t,v;1;1,1,0)+\sum_{i\in\{0,\ldots 3,6\}}\psi_i(T-t,v;1;1,1,0)X_t^i}1_{\{X_t^4=X_t^5=0\}}\end{aligned}$$

With the above discussion , we now discuss the dependence structure of the default time τ_1 and τ_2. Fix $s \geq 0$. For the F_s- condition joint distribution of (τ_1, τ_2) we have

$$\begin{aligned}F(t, T) &= P[\tau_1 \leq t, \tau \leq T|F_s] \\ &= 1 - E[1_{\{t<\tau_1\}}|F_s] - E[1_{\{T<\tau_1\}}|F_s] - E[1_{\{T<\tau_2\}}1_{\{t<\tau_1\}}|F_s]\end{aligned}$$

for $t, T \geq s$The terms involved are

$$\begin{aligned}E[1_{\{t<\tau_1\}}|F_s] &= E[lim_{k\to\infty}e^{-kX_t^4}|F_s] \\ &= e^{\varphi(t-s,0;0;1,0,0)+\sum_{i\in\{0,\ldots,3,5,6\}}\psi_i(t-s,0;0;1,0,0)X_s^i}1_{\{X_s^4=0\}}\end{aligned}$$

$$
\begin{aligned}
E[1_{\{T<\tau_2\}}|F_s] &= E[lim_{k\to\infty}e^{-kX_T^5}|F_s] \\
&= e^{\varphi(T-s,0;0;0,1,0)+\sum_{i\in\{0,\ldots,3,4,6\}}\psi_i(T-s,0;0;0,1,0)X_s^i}1_{\{X_s^5=0\}}
\end{aligned}
$$

and, for $t \le T$,

$$
\begin{aligned}
E[1_{\{T<\tau_2\}}1_{\{t<\tau_1\}}|F_s] &= E[lim_{k\to\infty}e^{-kX_t^4}E[lim_{k\to\infty}e^{-kX_T^5}|F_t]|F_s] \\
&= e^{\varphi(T-t,0;0;0,1,0)}E[lim_{k\to\infty}e^{-k(X_t^4+X_t^5)}e^{\sum_{i\in\{0,\ldots,3,4,6\}}\psi_i(T-t,0;0;0,1,0)X_t^i}|F_s] \\
&= e^{\varphi(T-t,0;0;0,1,0)+\varphi(t-s,\sum_{i\in\{0,\ldots,3,6\}}\psi_i(T-t,0;0;0,1,0)e_i;0;1,1,0)} \\
&\times \; e^{\sum_{j\in\{0,\ldots,3,6\}}\psi_i(t-s,\sum_{i\in\{0,\ldots,3,6\}}\psi_i(T-t,0;0;0,1,0)e_i;0;1,1,0)X_s^j}1_{\{X_s^4=X_s^5=0\}}
\end{aligned}
$$

and similarly for $t \ge T$,

$$
\begin{aligned}
E[1_{\{T<\tau_2\}}1_{\{t<\tau_1\}}|F_s] &= E[lim_{k\to\infty}e^{-kX_t^4}E[lim_{k\to\infty}e^{-kX_T^5}|F_t]|F_s] \\
&= e^{\varphi(t-T,0,0;0,1,0)}E[lim_{k\to\infty}e^{-k(X_t^4+X_t^5)}e^{\sum_{i\in\{0,\ldots,3,4,6\}}\psi_i(t-T,0;0;0,1,0)X_t^i}|F_s] \\
&= e^{\varphi(t-T,0,0;0,1,0)+\varphi(T-s,\sum_{i\in\{0,\ldots,3,6\}}\psi_i(t-T,0;0;0,1,0)e_i;0;1,1,0)} \\
&\times \; e^{\sum_{j\in\{0,\ldots,3,6\}}\psi_i(T-s,\sum_{i\in\{0,\ldots,3,6\}}\psi_i(t-T,0;0;1,0,0)e_i;0;1,1,0)X_s^j}1_{\{X_s^4=X_s^5=0\}}
\end{aligned}
$$

Below we illustrate the three cases where (i) f is only piecewise continuous (Example 2.2),(ii)the density function does not exist (Example 2.3), and (iii) a jointly continuous density function (Example 2.4) .

Example 6.2 Let $l := l_{(1,0,0)} > 0$ and $\lambda := \lambda_{(0,1,0),4} \ge 0$ and all the other parameters be zero .Then the generator is of form
$Af(x) = (f(x+e_4) - f(x))l + (f(x+e_5) - f(x))\lambda x_4$.
This is , firm 1 defaults with a constant intensity l and the default intensity of firm 2 is zero first , jumps to λ at the default time of firm 1(infectious default) and increases by the amount of λ at any further jump time of X^4.Hence ,

$$\partial_t\varphi(t,v;0;1,0,0) = \partial_t\varphi(t,v;0;1,1,0) = -l$$

$$\partial_t\varphi(t,v;0;0,1,0) = l(e^{v_4-\lambda t} - 1)$$

$$\partial_t\psi_4(t,v;0;1,0,0) = \lambda(e^{v_5} - 1)$$

$$\partial_t\psi_4(t,v;0;0,1,0) = \partial_t\psi_4(t,v;0;1,1,0) = -\lambda$$

$$\partial_t\psi_i(t,v;0;\mathbf{p}) \equiv 0$$

for all $i \ne 4$ so that

$$\varphi(t,v;0;1,0,0) = \varphi(t,v;0;1,1,0) = -lt$$

$$\varphi(t,v;0;0,1,0) = l(\frac{e^{v_4}}{\lambda}(1 - e^{-\lambda t}) - t)$$

$$\psi_4(t,v;0;1,0,0) = v_4 + \lambda(e^{v_5} - 1)t$$

$$\psi_4(t,v;0;0,1,0) = \psi_4(t,v;0;1,1,0) = v_4 - \lambda t$$

and $\psi_i(t,v,0,\mathbf{p}) \equiv v_i$ for all $i \ne 4$.
Because $G(t,T) = E[1_{\{t\le\tau_1}1_{\{T\le\tau_2\}}]$,

$$G(t,T) = \begin{cases} e^{\frac{l}{\lambda}(1-e^{-\lambda(T-t)})-lT}, & t \leq T \\ e^{-lt}, & t \geq T. \end{cases}$$

A straightforward calculation show that $\frac{\partial_t \partial_T G(t^-,t)}{G(t,t)} = l\lambda \neq 0$

Example 6.3 Consider the generator $Af(x) = f(x+e_4+e_5) - f(x)$, namely, we let $l_{(1,1,0)} = 1$ and all other parameters are zero.
We get $F(t,T) = 1 - e^{-(t \wedge T)}$.
Clearly,the distribution has no density.

Example 6.4 We now consider an example where τ_1 and τ_2 are conditionally independent given the information $\mathcal{G} = (X_t^0, ..., X_t^3 | t \geq 0)$ generated by $X^0 ... X^3$.Let the generator be of the form

$$\begin{aligned} Af(x) &= \alpha_0 x_0 \partial_{x_0}^2 f(x) + (b_0 + \beta_{00} x_0)\partial_{x_0} f(x) \\ &+ \sum_{i=1}^{2} \alpha_i x_i \partial_{x_i}^2 f(x) + \sum_{i=1}^{2} (b_i + \beta_{i0} x_0 + \beta_{ii} x_i)\partial_{x_i} f(x) \\ &+ (f(x+e_4) - f(x))(\lambda_{(1,0,0),0} x_0 + \lambda_{(1,0,0),1} x_1) + \lambda_{(1,0,0),2} x_2) \\ &+ (f(x+e_5) - f(x))(\lambda_{(0,1,0),0} x_0 + \lambda_{(0,1,0),1} x_1) + \lambda_{(0,1,0),2} x_2) \end{aligned}$$

with the symmetric structure

$$\alpha_1 = \alpha_2, b1 = b_2, \beta_{10} = \beta_{20}, \beta_{11} = \beta_{22},$$

$$\lambda_{(1,0,0),0} = \lambda_{(0,1,0),0}, \lambda_{(1,0,0),1} = \lambda_{(0,1,0),2}, \lambda_{(1,0,0),2} = \lambda_{(0,1,0),1}$$

Since we have $P[\tau_1 \leq t, \tau_2 \leq T|g] = P[\tau_1 \leq t|G] \cdot P[\tau_2 \leq T|G]$ and both of the g-condition distribution functions on the right hand side have a g-measurable continuous density,it is obvious that $F(t,T) = E[P[\tau_1 \leq t, \tau_2 \leq T|G]]$ admits a continuous density.
Although the joint distribution function contain all the information about the dependence of the default times $\tau_1 and \tau_2$,it is interesting to think of the correlation of the events $\{\tau_1 \leq T\}\{\tau_2 \leq T\}$ $\frac{Cov_{12}(T)}{\sqrt{Cov_{11}Cov_{22}(T)}}$

$$\begin{aligned} Cov_{ij} : &= E[1_{\{\tau_i \leq T\}} 1_{\{\tau_j \leq T\}}] - E[1_{\{\tau_i \leq T\}}]E[1_{\{\tau_j \leq T\}}] \\ &= E[1_{\{\tau_i \leq T\}}] - (E[1_{\{\tau_i \leq T\}}])^2, i = j \\ &= F(T,T) - E[1_{\{\tau_i \leq T\}}]E[1_{\{\tau_j \leq T\}}], i \neq j \end{aligned}$$

According to above discussion,we have

$$E[1_{\tau_i \leq T}] = 1 - E[lim_{k \to \infty} e^{-kX_T^{3+i}}] = 1 - e^{\varphi(T,0;0;\mathbf{p}(i)) + \sum_{j=0}^{3} \psi_j(T,0;0;\mathbf{p}(i))X_0^j}$$

where $\mathbf{p}(1) := (1,0,0)$ and $\mathbf{p}(2) := (0,1,0)$.

Example 6.5(Valuing Credit Default Swaps)
Consider the valuation of a plain vanilla credit default swap (CDS) with notional principal 1. The seller (firm3) of a CDS contract provides the buyer(firm2) insurance against the risk of default of a third party called the reference entity (firm1). In return, the buyer makes periodic payments to the seller. T_0 is the start date of CDS and the payment dates by T_1,...,T_n. and $T_k - T_{k-1} \equiv \triangle$ for all $k = 1, ..., n$.

Cash flows take place at dates T_k only , given the events that happened in the preceding periods $(T_{j-1}, T_j]$, j=1,...,k. At time T_k:

(i) if no default has occurred yet($T_k < \tau_1 \wedge \tau_2 \wedge \tau_3$) then the buyer pays to the seller a fixed rate c;

(ii)if the reference entity has defaulted in period $(T_{k-1}, T_k](T_{k-1} < \tau \leq T_k)$and the seller has not defaulted yet $(T_k < \tau_3)$ and the buyer has not defaulted by $T_{k-1}(T_{k-1} < \tau_2)$ then the seller payer pays $1 - G(X_{T_k})$ and the contract terminates , where

$$G(x) = e^{r+<\rho,x>} \leq 1$$

denotes the recovery rate for the bond issued by the reference entity , for some $r \in \mathbb{R}_-$ and $\rho \in \mathbb{R}^7_-$;

(iii)in all other cases there is no payment and the contract terminates.

The value at time $t \leq T_0$ of the buyer's payments accordingly is cB_t, where

$$\begin{aligned} B_t &= \mathbb{E}[\Sigma_{k=1}^n e^{-\int_t^{T_k} X_s^0 ds} \triangle 1_{\{T_k<\tau_1\wedge\tau_2\wedge\tau_3\}}|F_t] \\ &= \triangle\Sigma_{k=1}^n \mathbb{E}[e^{-\int_t^{T_k} X_s^0 ds} lim_{l\to\infty} e^{-l(X^4_{T_k}+X^5_{T_k}+X^6_{T_k})}|F_t] \\ &= \triangle\Sigma_{k=1}^n e^{\Phi(T_k-t,0;1;1,1,1)+\Sigma_{i=0}^3\Psi_i(T_k-t,0;1;1,1,1)X_t^i} 1_{\{X_t^4=X_t^5=X_t^6=0\}} \end{aligned}$$

The value at time $t \leq T_0$ of the seller's payment is

$$\begin{aligned} S_t &= \mathbb{E}[\sum_{k=1}^n e^{-\int_t^{T_k} X_s^0 ds}(1-G(X_{T_k}))1_{\{T_{k-1}<\tau_1\leq T_k\}}1_{\{T_{k-1}<\tau_2\}}1_{\{T_k<\tau_3\}}|F_t] \\ &= \sum_{k=1}^n \mathbb{E}[e^{-\int_t^{T_k} X_s^0 ds}(1-G(X_{T_k})) \times lim_{l,m\to\infty}(e^{-lX^4_{T_{k-1}}} - e^{-mX^4_{T_k}})e^{-lX^5_{T_{k-1}}-mX^6_{T_k}}] \\ &= \sum_{k=1}^n S_t^{1k} - S_t^{2k} - S_t^{3k} + S_t^{4k}, \end{aligned}$$

where,

$$\begin{aligned} S_t^{1k} &= \mathbb{E}[e^{-\int_t^{T_k} X_s^0 ds} lim_{l,m\to\infty} e^{-l(X^4_{T_{k-1}}+X^5_{T_{k-1}})-mX^6_{T_k}}|F_t] \\ &= e^{\Phi(\triangle,0;1;0;0,1)+\Phi(T_{k-1}-t,\Sigma_{i=0}^3\Psi_i(\triangle,0;1;0;0,1)e_i;1;1,1,1)} \\ &\times \ e^{\sum_{j=0}^3 \Psi_j(T_{k-1}-t,\sum_{i=0}^3\Psi_i(\triangle,0;1;0;0,1)e_i;1;1,1,1)X_t^j} 1_{\{X_t^4=X_t^5=X_t^6=0\}}, \end{aligned}$$

$$\begin{aligned} S_t^{2k} &= \mathbb{E}[e^{-\int_t^{T_k} X_s^0 ds} e^{r+<\rho,X_{T_k}>} lim_{l,m\to\infty} e^{-l(X^4_{T_{k-1}}+X^5_{T_{k-1}})-mX^6_{T_k}}|F_t] \\ &= e^{r+\Phi(\triangle,\rho;1;0;0,1)+\Phi(T_{k-1}-t,\Sigma_{i=0}^3\Psi_i(\triangle,\rho;1;0;0,1)e_i;1;1,1,1)} \\ &\times \ e^{\sum_{j=0}^3 \Psi_j(T_{k-1}-t,\sum_{i=0}^3\Psi_i(\triangle,\rho;1;0;0,1)e_i;1;1,1,1)X_t^j} 1_{\{X_t^4=X_t^5=X_t^6=0\}}, \end{aligned}$$

$$\begin{aligned} S_t^{3k} &= \mathbb{E}[e^{-\int_t^{T_k} X_s^0 ds} lim_{l,m\to\infty} e^{-lX^5_{T_{k-1}}-m(X^4_{T_k}+X^6_{T_k})}|F_t] \\ &= e^{\Phi(\triangle,0;1;1;0,1)+\Phi(T_{k-1}-t,\Sigma_{i=0}^3\Psi_i(\triangle,0;1;1;0,1)e_i;1;1,1,1)} \\ &\times \ e^{\sum_{j=0}^3 \Psi_j(T_{k-1}-t,\sum_{i=0}^3\Psi_i(\triangle,0;1;1;0,1)e_i;1;1,1,1)X_t^j} 1_{\{X_t^4=X_t^5=X_t^6=0\}}, \end{aligned}$$

$$\begin{aligned} S_t^{4k} &= \mathbb{E}[e^{-\int_t^{T_k} X_s^0 ds} e^{r+<\rho, X_{T_k}>} lim_{l,m\to\infty} e^{-lX^5_{T_{k-1}} - m(X^4_{T_k} + X^6_{T_k})} | F_t] \\ &= e^{r+\Phi(\triangle,\rho;1;1;0,1)+\Phi(T_{k-1}-t,\Sigma_{j=0}^3 \Psi_i(\triangle,\rho;1;1;0,1)e_i;1;1,1,1)} \\ &\times e^{\sum_{j=0}^3 \Psi_j(T_{k-1}-t,\sum_{i=0}^3 \Psi_i(\triangle,\rho;1;1;0,1)e_i;1;1,1,1)X_t^j} 1_{\{X_t^4=X_t^5=X_t^6=0\}}, \end{aligned}$$

7. Time-Inhomogeneous Affine Processes

Throughout we assume that $f_u(x) := e^{<u,x>}$ for $u \in \mathbb{C}^d$ and $\mathcal{U} := \mathbb{C}^m_- \times i\mathbb{R}^n$, $\partial\mathcal{U} := i\mathbb{R}^d$, $\mathcal{U}^0 := \mathcal{U} \backslash \partial\mathcal{U} = \mathbb{C}^m_{--} \times i\mathbb{R}^n$. Note that $f_u \in \mathbb{C}_b(E)$ if and only if $u \in \mathcal{U}$.

Definition 7.1. *We call $(P_{t,T})$ affine if for every $0 \le t \le T$ and $u \in \partial\mathcal{U}$ there exists $\varphi(t,T,u) \in \mathbb{C}$ and $\psi(t,T,u) = (\psi^{\mathcal{Y}}(t,T,u), \psi^{\mathcal{Z}}(t,T,u)) \in \mathbb{C}^m \times \mathbb{C}^n$ such that*

$$P_{t,T} f_u(x) = e^{\varphi(t,T,u) + <\psi(t,T,u),x>}, \ \forall x \in D. \tag{7.1}$$

Definition 7.2. *We say $p(r,x;t,dy)$ is stochastically continuous if $p(s,x;S,dy) \to p(t,x;T,dy)$ weakly on E for $(s,S) \to (t,T)$, ,for every $0 \le t \le T$ and $x \in E$. Hence P(r,x,;t;dy) is stochastically continuous if and only if $T_t^r f(x)$ is continuous in (r,t) for all $x \in E$ and $f \in C_b(E)$.*

Definition 7.3. *We call $p(r,x;t,dy)$ weakly regular if it is stochastically continuous and the left-hand derivative $\widetilde{A}(t) f_u(x) := -\partial_s^- P_{s,t} f_u(x)|_{s=t}$ exists for all (t,x,u)$\in \mathbb{R}_{++} \times E \times \mathcal{U}$ and is continuous at $u = 0$ for all (t,x)$\in \mathbb{R}_{++} \times E$.*

Example 7.1. *Let $f : \mathbb{R}_+ \to E$ be a measurable function such that $f(r) - f(t) \in E$ for all $0 \le r \le t$. Then*

$$p(r;x;t;dy) := \delta_{x+f(t)-f(r)}(dy)$$

is an affine transition function with

$$\varphi(r,t,\lambda) =< \lambda, f(t) - f(r) >, \psi(r,t,\lambda) = \lambda.$$

Some Notation. For $\alpha, \beta \in C^k$ e write $< \alpha, \beta >:= \alpha_1\beta_1 + ... + \alpha_k\beta_k$.We let$Sem^k$ be the convex cone of symmetric positive semi-definite $k \times k$ matrices.

Definition 7.4. *The t-dependent parameters*

$$(a,\alpha,b,\beta,c,\gamma,m,\mu) = (a(t),\alpha(t),b(t),\beta(t),c(t),\gamma(t),m(t),\mu(t)), t \in R_+$$

are called weakly admissible if for each fixed $t \in R_+$,they are admissible in the sense of that

- *$a(t) \in Sem^d$ with $a_{II} = 0$;*
- *$b(t) \in E$;*
- *$c(t) \in \mathbb{R}_+$;*
- *$\alpha(t) = (\alpha_1(t), \alpha_2(t), ..., \alpha_m(t))$ with $\alpha_i(t) \in Sem^d$ and $(\alpha_i)_{II} = \alpha_{i,ii}(t) Id(i)$ where $Id(i)_{kl} = \delta_{ik}\delta_{kl}$;*
- *$\beta(t) \in \mathbb{R}^{d\times d}$ such that $\beta_{IJ}(t) = 0$ and $\beta_{iI(i)}(t) \in \mathbb{R}_+^{m-1}$ for all $i \in I$;*
- *$\mu(t) = (\mu_1(t), \mu_2(t), ..., \mu_m(t))$ where every $\mu_i(t)$ is a Borel measure on $E\backslash\{0\}$ satisfying*

$$\int_{E\backslash\{0\}} (< \tilde{y}_{I(i)}, 1 > + \|\tilde{y}_{J(i)}\|^2) \mu_i(dy) < \infty;$$

- $\nu(t)$ *is a Borel measure on* $E\backslash\{0\}$ *satisfying*

$$\int_{E\backslash\{0\}} (<\tilde{y}_I, 1> + \|\tilde{y}_J\|^2)\nu(t, dy) < \infty;$$

They are called strongly admissible if in addition they satisfy the following continuous conditions:
(1)$(a(t), \alpha(t), b(t), \beta(t), c(t), \gamma(t), m(t), \mu(t))$are continuous in t$\backslash R_+$

(2)$M(t, d\xi)$and $M_i(t, d\xi)$ are weakly continuous on $D\backslash 0$ int $\in R_+$

Theorem 7.1. *Assume* $p(r, x; t, dy)$ *weakly regular affine. Then there exist some weakly admissible parameters* $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ *such that, for all* $t > 0$,$u = (v, w) \in \mathcal{U}$,$x = (y, z) \in E$,

$$\widetilde{A}(t)f_u(x) = (F(t, u) + < R_1(t, u), y > + < R_2(t, u), z >)f_u(x)$$

with

$$\begin{aligned} F(t, u) &= < a(t)u, u > + < b(t), u > - c(t) \\ &\quad + \int_{D\backslash 0} (e^{<u,\xi>} - 1 - < u_J, \chi_J(\xi) >)m(t, d\xi) \\ R_{1i}(t, u) &= < \alpha_i(t)u, u > + < \beta_i^y(t, u) > - \gamma_i(t) \\ &\quad + \int_{D\backslash(0)} (e^{<u,\xi>} - 1 - < u_{J(i)}, \chi_{J(i)}(\xi) >)\mu_i(t, d\xi) \\ R_2(t, u) &= \beta^z w, \end{aligned}$$

and

$$\begin{aligned} \beta_i^y(t) &:= (\beta^T(t))_{i\{1,\ldots,d\}} \in \mathbb{R} \\ \beta^z(t) &:= (\beta^T(t))_{JJ} \in \mathbb{R}^{n\times n} \end{aligned}$$

Definition 7.5. *We call* $p(r, x; t, dy)$ *strongly regular affine if it is weakly regular affine and the parameters* $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ *from theorem (7.1) are strongly admissible .*

We give an example showing that there are weakly regular affine processes that are not strongly regular affine.

Example 7.2. *Let* $(m, n) = (1, 0)$,$R(t, u) = 0$, *and*

$$F(t, u) = \int_{R\backslash(0)} (e^{uz-1})\frac{1}{z}\delta_{x(t)}(dz) = \frac{e^{ux(t)} - 1}{x(t)},$$

where x *is continuous at* 0 , $x(0) = 0$, $F(0, u) = lim_{t\to 0}F(t, u) = u$.*Hence*

$$b(t) = \begin{cases} 1, & t = 0 \\ 0, & otherwise \end{cases}$$

does not satisfy the continuity conditions.

Theorem 7.2. *Suppose $p(r,x;t,dy)$ is strongly regular affine and $(a,\alpha,b,\beta,c,\gamma,m,\mu)$ the corresponding strongly admissible parameters. Then*
(i) (Θ, X) is a Feller process.

(ii) $C_c^\infty(R_+ \times D)$is a core of $\widetilde{A}$

(iii) for $f \in C_c^{1,2}(R_+ \times D)$,we have

$$\overline{A}f(t,x) = \partial_t f(t,x) + A(t)f(t,x),$$

where $A(t)$is defined on the the function $f(t,\cdot)$ as follows

$$\begin{aligned} A(t)f(t,x): &= \sum_{k,l=1}^{d} (a_{k,l}(t) + < \alpha_{I,kl}(t), y >) \frac{\partial^2 f(t,x)}{\partial x_k \partial x_l} + < b(t) + \beta(t)x, \nabla_x f(t,x) > \\ &- (c(t) + < \gamma(t), y >) f(t,x) \\ &+ \int_{D\setminus\{0\}} (f(t,x+\xi) - f(t,x) - < \nabla_J f(t,x), \chi_J(\xi) >) m(t,d\xi)) \\ &+ \sum_{i=1}^{m} \int_{D\setminus\{0\}} (f(t,x+\xi) - f(t,x) - < \nabla_{J(i)} f(t,x), \chi_{J(i)}(\xi) >) m(t,d\xi)) \end{aligned}$$

(iv)(7.1)holds for all $0 \le t \le T$ and $u \in \mathcal{U}$ where $\varphi(t,T,u)$, and $\psi(t,T,u)$ solve the generalized Riccati equations

$$\varphi(t,T,u) = \int_t^T F(s,\psi(s,T,u))ds$$

$$-\partial_t \psi^y(t,T,u) = R^y(t,\psi^y(t,T,u), e^{\int_t^T \beta^z(s)ds} w), \psi^y(T,T,u) = v$$

$$\psi^z(t,T,u) = e^{\int_t^T \beta^z(s)} w$$

with F,R^y,and β^z are given by Thm 7.1.
Conversely,let$(a,\alpha,b,\beta,c,\gamma,m,\mu)$ be strongly admissible parameters. Then there exists a unique , strongly regular affine Markov process $(P_{t,T})$ whose associated space-time process (Θ, X) has the infinitesimal generator $\overline{A}$ and (7.1) holds for all $0 \le t \le T$ and $u \in \mathcal{U}$ where $\varphi(t,T,u)$ and $\psi(t,T,u)$ are given by above.

8. Characterizations of affine processes

When $m = 1$ and $n = 0$, the affine process ξ takes values in $\mathbb{R}_+$ and is also called a continuous- state process with immigration. It was first studied by Kawazu and Watanabe[KW71] as a continuous limit of Galton-Watson branchinh processes with immigration. Kawazu and Watanabe[KW71] showed that if ξ is a stocastically continuous affine process in $\mathbb{R}_+$, then for every $\lambda > 0, t > 0$ and $x \in \mathbb{R}_+$, we have

$$\Pi_x e^{-\lambda \xi_t} = exp\{-u_t x - \int_0^t \phi(u_s)ds\}$$

where $u_t = u(t,\lambda)$ satisfies

$$\frac{du}{dt} = -\varphi(u) \quad , u(0) = \lambda$$

with

$$\varphi(u) = \alpha u^2 - \beta u - \gamma + \int_{\mathbb{R}_+} [e^{-uy} - 1 + u(1 \wedge y)]\mu(dy)$$

and

$$\phi(u) = c + bu + \int_{\mathbb{R}_+} (1 - e^{-uy})\nu(dy).$$

[Here we assume that

$$\alpha \geq 0, \gamma \geq 0, b \geq 0, c \geq 0, \beta \in \mathcal{R}$$

and μ, ν are two measures on $(0, \infty)$ satisfying

$$\int_0^\infty (1 \wedge y)\mu(dy) < \infty$$

and

$$\int_0^\infty (1 \wedge y)\nu(dy) < \infty.]$$

For general m and n, we write $I = \{1, 2, ..., m\}, J = \{m+1, m+2, ..., m+n\}$. Set $\mathcal{U} = \mathbb{C}_-^m \times i\mathbb{R}^n, I(i) = I\backslash\{i\}$ and $J(i) = J \bigcup \{i\}$ for $1 \leq i \leq m$. $(a, b, c, \alpha, \beta, \gamma, \mu, \nu)$ is called admissible if:

- $a \in Sem^d$ with $a_{II} = 0$;
- $b \in E$;
- $c \in \mathbb{R}_+$;
- $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ with $\alpha_i \in Sem^d$ and $(\alpha_i)_{II} = \alpha_{i,ii} Id(i)$ where $Id(i)_{kl} = \delta_{ik}\delta_{kl}$;
- $\beta \in \mathbb{R}^{d \times d}$ such that $\beta_{IJ} = 0$ and $\beta_{iI(i)} \in \mathbb{R}_+^{m-1}$ for all $i \in I$;
- $\mu = (\mu_1, \mu_2, ..., \mu_m)$ where every μ_i is a Borel measure on $E\backslash\{0\}$ satisfying
$$\int_{E\backslash\{0\}} (< \tilde{y}_{I(i)}, 1 > + \|\tilde{y}_{J(i)}\|^2)\mu_i(dy) < \infty;$$
- ν is a Borel measure on $E\backslash\{0\}$ satisfying
$$\int_{E\backslash\{0\}} (< \tilde{y}_I, 1 > + \|\tilde{y}_J\|^2)\nu(dy) < \infty;$$

Duffie, Filipovic and Schachermayer[DFS03] characterized all regular affine processes. In particular they obtained that if ξ is regular affine, then for all $t, \lambda \in \mathcal{U}$ and $x \in E$, we have

$$\Pi_x e^{<\lambda, x_t>} = exp\{< u(t, \lambda), x > + \int_0^t \phi(u(s, \lambda))ds\}$$

with $u_J(t, \lambda) = e^{\beta^J t}\lambda_J$ and $u(t) = u(t, \lambda)$ satisfies the generalize Riccati equations

$$(\partial_t u)_I = (\Psi(u))_I$$

with $u_I(0) = \lambda_I$ where, for $1 \leq i \leq m$,

$$\Psi(u)(i) =< \alpha_i u, u > + < \beta_i^I, u > -\gamma_i + \int_{E\backslash\{0\}} (e^{<u,y>} - 1 - < u_{J(i)}, \tilde{y}_{J(i)} > \mu_i(dy),$$

and

$$\phi(u) =< au, u > + < b, u > -c + \int_{E\backslash\{0\}} (e^{<u,y>} - 1 - < u_J, \tilde{y}_J >)\nu(dy).$$

In addition, the parameters $(a, b, \alpha, \beta, \mu_i, \nu)$ satisfies for each $f \in C_c^2(D)$

$$\begin{aligned} Af &= \sum_{k,l=1}^{d} (a_{k,l} + < \alpha_{I,kl,}, y >) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + < b + \beta x, \nabla f(x) > \\ &+ \int_{D\setminus\{0\}} Gf_0(x,\xi) m(d\xi) \\ &+ \sum_{i=1}^{m} \int_{D\setminus 0} Gf_i(x,\xi) y_i \mu_i(d\xi) \end{aligned}$$

where

$$Gf_0(x,\xi) = f(x+\xi) - f(x) - < \nabla_J f(x), \chi_J(\xi) >$$

$$Gf_i(x,\xi) = f(x+\xi) - f(x) - < \nabla_{J(i)} f(x), \chi_{J(i)}(\xi) >$$

Conversely,if $(a, b, c, \alpha, \beta, \gamma, \mu, \nu)$ is some admissible parameters, then exists a unique, regular affine $\Pi_{r,x}$ such that (1.1) holds for all $(t, \lambda) \in R^+ \times E$, where $v(t, \lambda), u(t, \lambda)$ are given by

REMARK :

(i) for $n = 0$, we have for every $(t, \lambda) \in \mathbb{R} \times \mathbb{R}_+^m$, there exists $\tilde{u}(t, \lambda) \in \mathbb{R}_+^m$ such that

$$\Pi_x e^{-<\lambda, x_t>} = exp\{- < \tilde{u}(t,\lambda), x > - \int_0^t \tilde{\phi}(\tilde{u}(s,\lambda)) ds\}$$

where $\tilde{u}(t) = \tilde{u}(t, \lambda)$ is a solution of the initial value problem

$$\begin{cases} \partial_t \tilde{u} &= -\tilde{\Psi}(\tilde{u}) \\ \tilde{u}(0) &= \lambda \end{cases}$$

with

$$\tilde{\Psi}(z)(i) = \alpha_{ii} z_i^2 - \sum_j \beta_{ij} z_j - \gamma_i + \int_{\mathbb{R}_+^m \setminus \{0\}} (e^{-<z,y>} - 1 + z_i \tilde{y}_i) \mu_i(dy),$$

and

$$\tilde{\phi}(z) = < b, z > + c + \int_{\mathbb{R}_+^m \setminus \{0\}} (1 - e^{-<z,y>}) \nu(dy).$$

(ii)if $\lambda = (w, 0)$

$$\Pi_x e^{<(w,0),\xi>} = e^{v(t,w,0) + <u(t,w,0),y>}$$

Hence, if ξ_t is regular affine, then (ξ_t^y, Π_x) is a regular affine process with state space R_+^m

It is worth noting that $\tilde{u}$ solves the differential log-Laplace equation corresponding to some superprocess with a finite base space (see, eg, [Dy94]). Based on this observation, we construct more general affine processes through general superprocesses. We also study sample path properties for general affine processes. The rich variety of alternative types of random behavior(e.g., mean reversion, stochastic volatility, and jumps) and analytically tractable for affine processes make them ideal models for financial applications(see, e.g., Duffie, Filipovic and Schachermayer[DFS03] and references therein.)

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