1 Introduction

In this paper, we use the new way to evaluate the entropy of the L-shaped patterns or estimate the lower bounded of entropy. First, the cellular neural network(CNN), as designed by Chua and Yang [1988a, 1988b], is an array of an identical system of cells that are locally coupled. This work investigates the complexity of stable binary patterns of two-dimensional CNN. The state equation of a cell C_{ij} is the set of coupled O.D.E's

$$\dot{x_{ij}}(t) = -x_{ij}(t) + \sum_{|k|,|l| \leq d} a_{kl} y_{i+k,j+l}(t) + z \quad , i, j \in \mathbf{Z}. \tag{1}$$

with output $y_{i,j}(t) = f(x_{i,j}(t))$. Here $y = f(\cdot)$ is a piecewise-linear function expressed as

$$f(x) = \frac{1}{2}(|x+1| - |x-1|) \tag{2}$$

The parameter z is a time-invariant bias and d is a positive integer. The coupling parameters a_{kl}'s are assumed to be space-invariant, which is arranged in a $(2d+1) \times (2d+1)$ matrix A called a *template*. The stationary solution $\mathrm{x} = (x_{ij})$ of (1) are prerequisite for understanding the CNN dynamics. The stationary solutions $\mathrm{x} = (x_{ij})$ of (1) satisfy

$$\frac{dx_{ij}}{dt}(t) = z + \sum_{|k|,|l| \leq d} a_{kl} f(x_{i+k,j+l}), i, j \in \mathbf{Z} \tag{3}$$

A stationary x solution is called a mosaic solution if $|x_{ij}| > 1 \ \forall i, j \in \mathbf{Z}$. The corresponding output $\mathrm{y} = (f(x_{ij}(t)))$ is called a mosaic pattern. Among all stationary solutions, the stable mosaic solutions,which have been studied before [Juang,Lin,1997], are the most fundamental and important applications (finite cells) in image-processing [Chua,1988b,1988]. The complexity of mosaic solutions can be examined according to its entropy. For completeness, the following discussion introduces some definitions and results concerning spatial entropy. Further details can be found in [Chow et al.,1996a]. Denote by $\{-1,1\}^{\mathbb{Z}^2}$ the set of all $\mathrm{y}\colon \mathbb{Z}^2 \to \{-1,1\}$ i.e. the set of all mosaic patterns. Let $\mathcal{U}$ be a translation invariant subset of $\{-1,1\}^{\mathbb{Z}^2}$ and $\mathbb{Z}_{mn} = \{(i,j)\colon 1 \leq i \leq m, 1 \leq j \leq n\}$. The number of distinct patterns observed among the elements of $\mathcal{U}$ when observation is restricted to subset $\mathbb{Z}_{mn}$, is denoted by $\Gamma_{mn}(\mathcal{U})$. The spatial entropy $h(\mathcal{U})$ of $\mathcal{U}$ is defined by

$$h(\mathcal{U}) = \lim_{m,n \to \infty} \frac{\ln \Gamma_{mn}(\mathcal{U})}{mn} \tag{4}$$

$\mathcal{U}$ is called *spatial chaos* if the spatial entropy $h(\mathcal{U})$ is positive. Otherwise, $\mathcal{U}$ is called *pattern formation*.

In this paper, we concentrate the simplest two-dimensional template, L-shaped liked , i.e. A= $\begin{pmatrix} 0 & r & 0 \\ 0 & a & s \\ 0 & 0 & 0 \end{pmatrix}$ The state equation is

$$\frac{dx_{ij}}{dt}(t) = ry_{i,j+1}(t) + sy_{i+1,j}(t) + ay_{ij}(t) + z - 1 \tag{5}$$

We partition the parameter space in the section-2, then we have the eight regions of r-s plane and twenty-five regions of (a-1)-z plane. For the $[\mu, \nu]_d$, we can know the basic set of admissible local L-shaped patterns. In the section-3, we introduce the result of [Ban&Lin], they develop the ordering matrix and use the transition matrix of 2×2 lattice to evaluate the number of $\Gamma_{mn}(\mathcal{U})$. So, we take the L-shaped pattern into a 2×2 lattice, then we also can know the transition matrix of the given $[\mu, \nu]_d$ of L-shaped patterns. And in the section-4, we introduce the result of [Ban, Lin&Lin], they develop the connecting operator to estimate the lower bound of entropy. So, we can use this technique to evaluate the lower bound of entropy of the L-shaped patterns. Finally, we compare the result with the result of [Lin&Yang 2002]. By the previous result of [Lin&Yang 2002], we refer to the building blocks in the process of [Lin&Yang 2002] to know the lowest order of connecting operator. In the new way, we can estimate the entropy better than the previous. Undeniably, the predecessor inspires clues in my study

2 Partitioning the Parameter Space

Then we concentrate the simplest two-dimensional template, L-shaped liked , i.e. A= $\begin{pmatrix} 0 & r & 0 \\ 0 & a & s \\ 0 & 0 & 0 \end{pmatrix}$

The state equation is

$$\frac{dx_{ij}}{dt}(t) = -x_{ij}(t) + ry_{i,j+1}(t) + sy_{i+1,j}(t) + ay_{ij}(t) + z \tag{6}$$

For a mosaic solution $\mathbf{x}$, the output at cell C_{ij} is +1,
i.e., $x_{ij} \geq 1$ ($y_{ij} = 1$), if and only if,

$$(a-1) + z + (ry_{i,j+1} + sy_{i+1,j}) \geq 0 \tag{7}$$

and similarly, the output at cell C_{ij} is -1,
i.e., $x_{ij} \leq -1$ ($y_{ij} = -1$), if and only if,

$$(a-1) - z - (ry_{i,j+1} + sy_{i+1,j}) \geq 0 \tag{8}$$

We discuss (6), it can be rewritten as the following form :

$$(a-1) + z \geq -(ry_{i,j+1} + sy_{i+1,j}) \tag{9}$$

In this inequality, it has four parameter, we can rearrange to three parts:

$\diamondsuit(y_{i,j+1}, y_{i+1,j})$

$\diamondsuit(r, s)$

$\diamondsuit((a-1), z)$

First, $(y_{i,j+1}, y_{i+1,j}) \in \{-1, +1\}^2$, so $(y_{i,j+1}, y_{i+1,j})$ has four cases. Moreover, the right side of (8) also has four cases as the following table :

$(y_{i,j+1}, y_{i+1,j})$	$-(r \cdot y_{i,j+1} + s \cdot y_{i+1,j})$	
$(-1,-1)$	$r+s$	$P_i(-1,-1)$
$(-1,+1)$	$r-s$	$P_i(-1,+1)$
$(+1,-1)$	$-r+s$	$P_i(+1,-1)$
$(+1,+1)$	$-r-s$	$P_i(+1,+1)$

(10)

If $P_i(k,m) = t$, let L_t^+ be the line on (a-1)-z plane with equations:

$$(a-1) + z + kr + ms = 0. \tag{11}$$

According the magnitude relationship between r and s, the sort of $\{P_i(-1,-1), P_i(-1,+1), P_i(+1,-1), P_i(+1,+1)\}$ can be different. For example, if $r > s > 0$, the sort of $\{P_i(-1,-1), P_i(-1,+1), P_i(+1,-1), P_i(+1,+1)\}$ is $P_i(-1,-1) < P_i(-1,+1) < P_i(+1,-1) < P_i(+1,+1)$. In terms of geometry, we can have the following figure :

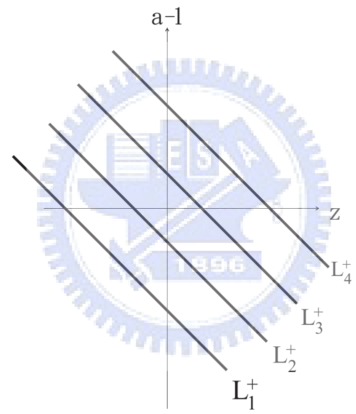


Figure 2-1

Similarly, the inequality (7), it also can be rewritten as the following form :

$$(a-1) - z \geq r y_{i,j+1} + s y_{i+1,j} \tag{12}$$

We also have the following table:

$(y_{i,j+1}, y_{i+1,j})$	$r \cdot y_{i,j+1} + s \cdot y_{i+1,j}$	
$(-1,-1)$	$-r-s$	$N_i(-1,-1)$
$(-1,+1)$	$-r+s$	$N_i(-1,+1)$
$(+1,-1)$	$r-s$	$N_i(+1,-1)$
$(+1,+1)$	$r+s$	$N_i(+1,+1)$

(13)

If $N_i(k,m) = q$, let L_q^- be the line on (a-1)-z plane with equations:

$$z - (a-1) + kr + ms = 0. \tag{14}$$

According the magnitude relationship between r and s, the sort of $\{N_i(-1,-1), N_i(-1,+1), N_i(+1,-1), N_i(+1,+1)\}$ can be different. In terms of geometry, we can have the following figure:

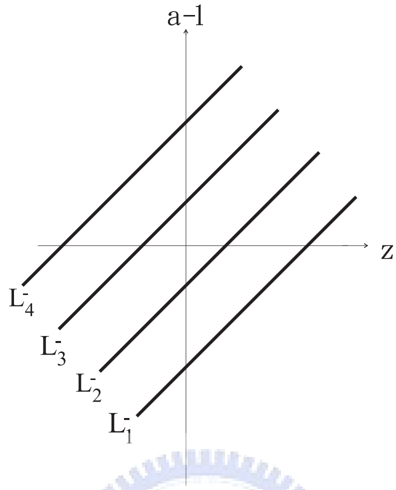


Figure 2-2

We combine the Figure 2-1 and 2-2, we have :

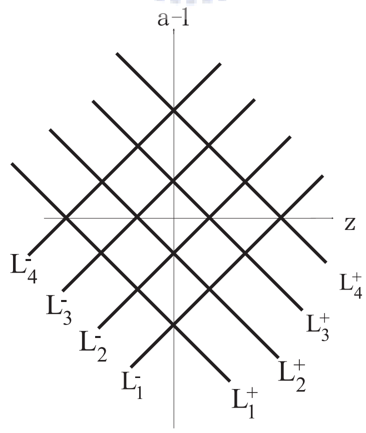


Figure 2-3

Now, we have parted r-s plan into eight open regions and (a-1)-z plane into twenty-five disjoint regions.

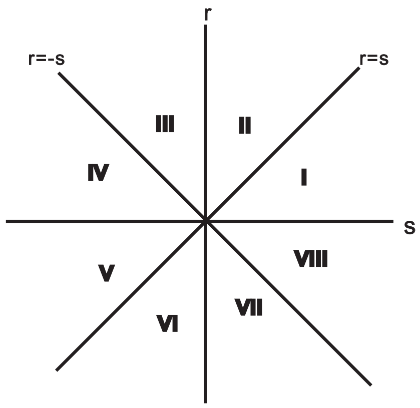


Figure 2-4

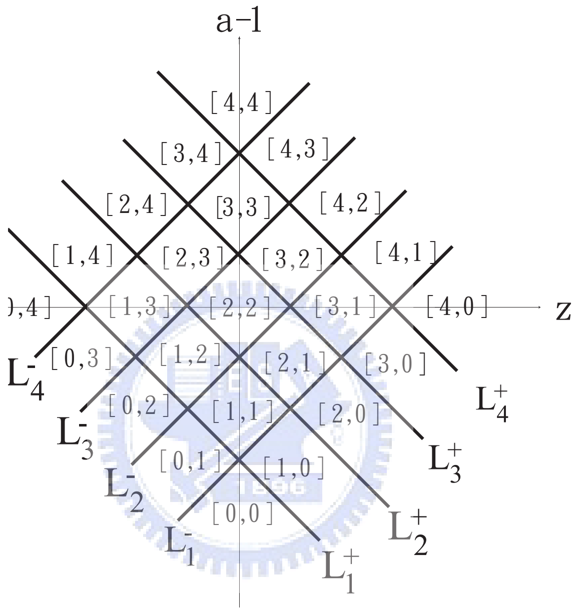


Figure 2-5

The admissible local patterns of $[\mu, \nu]_d$ can be compared with the following figure.

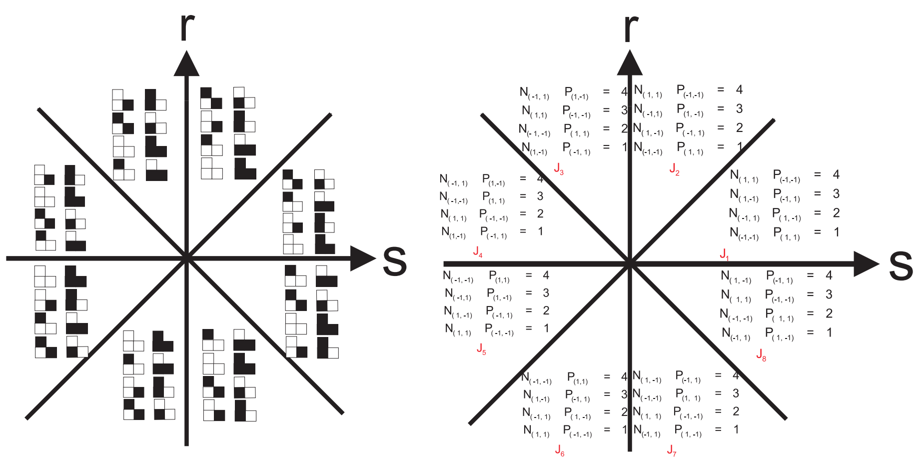


Figure 2-6

Note that the black square means the $y_{i+k,j+l} = +1$ where $k, l \in \{0, 1\}$, also means the square value is +1. The white square means the $y_{i+k,j+l} = -1$ where $k, l \in \{0, 1\}$, also means the square value is 0. After partitioning the parameter space, Proposition2.1 indicates how the admissible local patterns are determined for each region $[\mu, \nu]_d$.

Proposition 2.1 $([Lin\&Yang, 2002])$

For $((a-1), z) \in [\mu, \nu]_d$*, the admissible local patterns are exactly the union of* $\bigcup_{J_i(k,m)\leq\mu} P_{k,m}$ *and* $\bigcup_{J_i(k,m)\leq\nu} N_{k,m}$*,*
where $P_{k,m} = \{c_k : 5 \leq k \leq 8\}$ *and* $N_{k,m} = \{c_k : 1 \leq k \leq 4\}$ *and* $J_i(k,m)$ *are the number of pattern in r-s plane.*

Therefore, for given $[\mu, \nu]_d$, we can find out the admissible local patterns set $\mathcal{B}$, the basic set, $\mathcal{B} = \bigcup_{J_i(k,m)\leq\mu} P_{k,m} \cup \bigcup_{J_i(k,m)\leq\nu} N_{k,m}$.
Note that :

$P(+1,+1)$  $= c_8$ $N(+1,+1)$  $= c_4$
$P(+1,-1)$  $= c_7$ $N(+1,-1)$  $= c_3$ (15)
$P(-1,+1)$  $= c_6$ $N(-1,+1)$  $= c_2$
$P(+1,+1)$  $= c_5$ $N(+1,+1)$  $= c_1$

3 Ordering matrix

3.1 Ordering matrix in 2×2 lattice

According the Ref[Ban&Lin, 2005], we have the following results. On a fixed finite lattice $\mathbb{Z}_{m_1\times m_2}$, the ordering $\chi = \chi_{m_1\times m_2}$ on $\mathbb{Z}_{m_1\times m_2}$ is obtained by

$$\chi((\alpha_1,\alpha_2)) = m_2(\alpha_1 - 1) + \alpha_2 \tag{16}$$

i.e.

m_2	$2m_2$		m_1m_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	m_2+1		$(m_1-1)m_2+1$

(17)

and similarly, the ordering $\chi(U)$ on $\Sigma_{m_1\times m_2}$ is obtained by :
for each $U = (\mu_{\alpha_1,\alpha_2}) \in \Sigma_{m_1\times m_2}$,

$$\chi(U) \equiv \chi_{m_1\times m_2}(U) = 1 + \sum_{\alpha_1=1}^{m_1}\sum_{\alpha_2=1}^{m_2} u_{\alpha_1,\alpha_2}\cdot 2^{m_2(m_1-\alpha_1)+(m_2-\alpha_2)} \tag{18}$$

Obviously, there is an one-to-one correspondence between local patterns in $\Sigma_{m_1\times m_2}$ and positive integer in the set $N_{2^{m_1m_2}} = \{k \in \mathbb{N} : 1 \le k \le 2^{m_1m_2}\}$.
For $1\times n$ pattern $U = (u_k)$, $1 \le k \le n$ in $\Sigma_{1\times n}$, as in (14), there is a number

$$i = \chi(U) = 1 + \sum_{k=1}^{n} u_k \cdot 2^{(n-k)} \tag{19}$$

In particular, when $n = 2$, $i = 1 + 2\mu_1 + \mu_2$ and $x_i = \begin{array}{|c|}\hline u_2 \\ \hline u_1 \\ \hline\end{array}$

A 2×2 pattern $U = (\mu_{\alpha_1,\alpha_2})$ can be obtained by a horizontal or vertical direct sum of two 1×2 or 2×1 patterns,respectively.
i.e.

$$x_{i_1,i_2} \equiv \begin{array}{|c|c|}\hline u_{12} & u_{22} \\ \hline u_{11} & u_{21} \\ \hline\end{array} \tag{20}$$

where

$$i_k = 1 + 2u_{k_1} + u_{k_2}, 1 \le k \le 2 \tag{21}$$

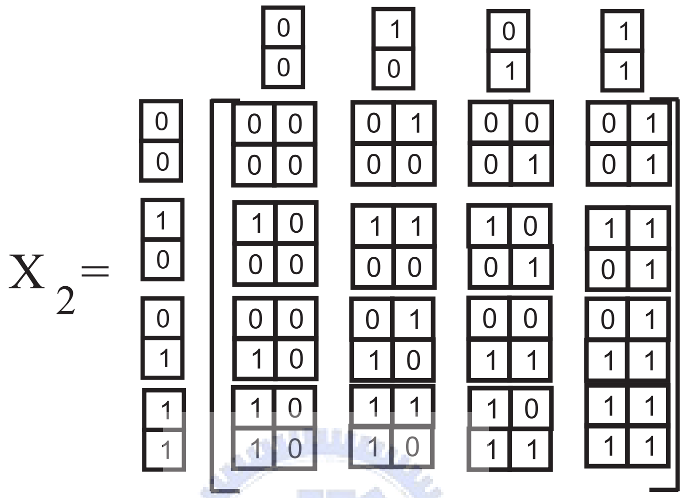
and

$$y_{j_1,j_2} \equiv \begin{array}{|c|c|}\hline u_{12} & u_{22} \\ \hline u_{11} & u_{21} \\ \hline \end{array} \tag{22}$$

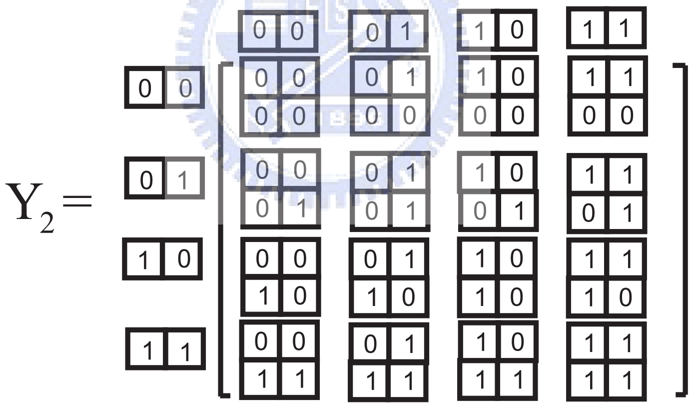
where

$$j_l = 1 + 2u_{1_l} + u_{2_l}, 1 \leq l \leq 2 \tag{23}$$

A 4×4 matrix $\mathbb{X}_2 = [x_{i_1,i_2}]$ and $\mathbb{Y}_2 = [y_{j_1,j_2}]$ can be obtained for $\Sigma_{2\times2}$, i.e.



(24)



(25)

The relationship between $\mathbb{X}_2$ and $\mathbb{Y}_2$ had found.

$$\mathbb{X}_2 = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{2,1} & y_{2,2} \\ y_{1,3} & y_{1,4} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{4,1} & y_{4,2} \\ y_{3,3} & y_{3,4} & y_{4,3} & y_{4,4} \end{bmatrix}, \mathbb{Y}_2 = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{2,1} & x_{2,2} \\ x_{1,3} & x_{1,4} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{4,1} & x_{4,2} \\ x_{3,3} & x_{3,4} & x_{4,3} & x_{4,4} \end{bmatrix}$$

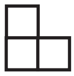
3.1.1 Ordering matrix in L-shaped

For the convenience, we let the color white and black to be 0 and 1,respectively. According Sec3.1, we can put L-pattern into 2×2 lattice by the following method. Let $c_k = \begin{array}{|c|c}\hline l_2 & \\ \hline l_1 & \multicolumn{1}{c|}{l_3} \\ \hline\end{array}$, where $k = 1 + 2^0 l_3 + 2^1 l_2 + 2^2 l_1$, so we have $c_1 \sim c_8$ that means $\begin{array}{|c|c}\hline 0 & \\ \hline 0 & \multicolumn{1}{c|}{0} \\ \hline\end{array} \sim \begin{array}{|c|c}\hline 1 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}$. And, let $\Sigma_L = \{c_k : 1 \leq k \leq 8\}$.

For a L-pattern c_k in a 2×2 lattice $\begin{array}{|c|c|}\hline \alpha_{12} & \alpha_{22} \\ \hline \alpha_{11} & \alpha_{21} \\ \hline\end{array}$.

We can define $\begin{cases} i = 1 + 2^1\alpha_{11} + \alpha_{12} \\ j = 1 + 2^1\alpha_{21} + \alpha_{22} \end{cases}$ $and \begin{cases} l_1 = \alpha_{21} \\ l_2 = \alpha_{12} \\ l_3 = \alpha_{11} \end{cases}$.

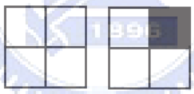
For example,



(26)

$$\begin{cases} i = 1 + 2^1 \cdot 0 + 0 \\ j = 1 + 2^1 0 + \alpha_{22} \end{cases} \Rightarrow \begin{cases} i = 1 \\ j = 1 \ or\ 2\ for\ \alpha_{22} = 0\ or\ 1, resp. \end{cases}$$

i.e.



(27)

3.2 Transition matrix $\mathcal{B} \subset \Sigma_L$

3.2.1 Transition matrix in 2×2 lattice

Now, given a basic set $\mathcal{B} \subset \Sigma_{2\times 2}$, horizontal and vertical transition matrices $\mathbb{H}_2$ and $\mathbb{V}_2$ can be defined by:

Let $\mathbb{H}_2 = [h_{i_1,i_2}]$ and $\mathbb{V}_2 = [v_{j_1,j_2}]$, according to the following rules, we can define the entries of $\mathbb{H}_2$ and $\mathbb{V}_2$ is 0 or 1:

$$\begin{cases} h_{i_1,i_2} = 1 \text{ , if } x_{i_1,i_2} \in \mathcal{B} \\ h_{i_1,i_2} = 0 \text{ , if } x_{i_1,i_2} \in \Sigma_{2\times 2} \backslash \mathcal{B} \end{cases}$$

$$\begin{cases} v_{j_1,j_2} = 1, \ if \ y_{j_1,j_2} \in \mathcal{B} \\ v_{j_1,j_2} = 0, \ if \ y_{j_1,j_2} \in \Sigma_{2\times 2} \backslash \mathcal{B} \end{cases}$$

The transition matrix for $\mathcal{B}$ can be defined by:

$$\mathbb{H}_2 \equiv \mathbb{H}_2(\mathcal{B}) = \begin{bmatrix} v_{11} & v_{12} & v_{21} & v_{22} \\ v_{13} & v_{14} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{41} & v_{42} \\ v_{33} & v_{34} & v_{43} & v_{44} \end{bmatrix}$$

By induction on n, we have Proposition3.1 of the property of $\mathbb{H}_n$

Proposition 3.1 *([Ban&Lin, 2005])*

Let $\mathbb{H}_2$ be a transition matrix for $\mathcal{B}$. Then the higher order transition matrices $\mathbb{H}_n$, $n \geq 3$,we have the following expression. $\mathbb{H}_n$ can be decomposed into a successive 2×2 matrices as follows :

$$\mathbb{H}_n = \begin{bmatrix} \mathbb{H}_{n;1} & \mathbb{H}_{n;2} \\ \mathbb{H}_{n;3} & \mathbb{H}_{n;4} \end{bmatrix}$$

$$\mathrm{H}_{n;j_1\cdots j_k} = \begin{bmatrix} \mathbb{H}_{n;j_1\cdots j_k 1} & \mathbb{H}_{n;j_1\cdots j_k 2} \\ \mathbb{H}_{n;j_1\cdots j_k 3} & \mathbb{H}_{n;j_1\cdots j_k 4} \end{bmatrix} \textit{ for } 1 \leq k \leq n-2 \textit{ and}$$

$$\mathrm{H}_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} v_{j_1\cdots j_{n-1}1} & v_{j_1\cdots j_{n-1}2} \\ v_{j_1\cdots j_{n-1}3} & v_{j_1\cdots j_{n-1}4} \end{bmatrix}.$$

Furthermore, $\mathbb{H}_{n;k} = \begin{bmatrix} v_{k1}\mathbb{H}_{n-1;1} & v_{k2}\mathbb{H}_{n-1;2} \\ v_{k3}\mathbb{H}_{n-1;3} & v_{k4}\mathbb{H}_{n-1;4} \end{bmatrix}.$

3.2.2 Transition matrix in L-pattern

Given a basic set $\mathcal{B}_L \in \Sigma_L$,

if $\begin{array}{|c|c}\hline l_2 & \\ \hline l_1 & \multicolumn{1}{c|}{l_3} \\ \hline \end{array}$ is an admissible local pattern , we can calculate:

$$\begin{cases} i = 1 + 2^0 l_2 + 2^1 l_1 \\ j = 1 + 2^0 \alpha_{22} + 2^1 l_3 \end{cases} \tag{28}$$

then t_{ij}=1 $\forall i, j \in$ (28)

Example 3.2 *if* $\begin{array}{|c|c}\hline 0 & \\ \hline 0 & \multicolumn{1}{c|}{0} \\ \hline \end{array}$ *is admissible* , $\begin{cases} i = 1 + 2^0 0 + 2^1 0 = 1 \\ j = 1 + 2^0 \alpha_{22} + 2^1 0 = \begin{cases} 1, \alpha_{22} = 0 \\ 2, \alpha_{22} = 1 \end{cases} \end{cases}$

the $h_{11} = h_{12} = 1$ *in* $\mathbb{H}_{L;2}$

In the other way,

if $\begin{array}{|c|c}\hline l_2 & \\ \hline l_1 & \multicolumn{1}{c|}{l_3} \\ \hline \end{array}$ is a forbidden local pattern , we can calculate:

$$\begin{cases} i = 1 + 2^0 l_2 + 2^1 l_1 \\ j = 1 + 2^0 \alpha_{22} + 2^1 l_3 \end{cases} \tag{29}$$

then t_{ij}=0 $\forall i, j \in$ (29)

Example 3.3 *if* $\begin{array}{|c|c}\hline 0 & \\ \hline 0 & \multicolumn{1}{c|}{0} \\ \hline \end{array}$ *is forbidden* , $\begin{cases} i = 1 + 2^0 0 + 2^1 0 = 1 \\ j = 1 + 2^0 \alpha_{22} + 2^1 0 = \begin{cases} 1, \alpha_{22} = 0 \\ 2, \alpha_{22} = 1 \end{cases} \end{cases}$

the $h_{11} = h_{12} = 0$ *in* $\mathbb{H}_{L;2}$

In this paper, we use the second way to discuss the entropy of the given $[\mu, \nu]_d$. For given $[\mu, \nu]_d$, we can know the forbidden set, $\mathfrak{F}_{[\mu,\nu]_d}$=$\{c_k\}$. And for every forbidden local pattern c_k, we could find the forbidden set of c_k , namely, $\mathfrak{F}_{c_k} = \{(i, j)\}$. Let $\mathfrak{F}$=$\bigcup_{c_k \in \mathfrak{F}_{[\mu,\nu]_d}} \mathfrak{F}_{c_k}$, the union set of the forbidden set of c_k.

The transition matrix $\mathbb{H}_{L;2}$ could be expressed that $[h_{ij}]$, where $\begin{cases} h_{ij} = 0, if\ (i, j) \in \mathfrak{F} \\ h_{ij} = 1, if\ (i, j) \notin \mathfrak{F} \end{cases}$

In other words, for given $[\mu, \nu]_d$, in Sec.2, we can find the basic set of $[\mu, \nu]_d$, $\mathcal{B}_L$. Then we also can know the forbidden set of $[\mu, \nu]_d$, $\mathfrak{F}_{[\mu,\nu]_d} = \Sigma_L \setminus \mathcal{B}_L$. Now, we have the following theorem.

Theorem 3.4 *([Ban&Lin, 2005])*

Given m,n,d , $0 \leq \mu, \nu \leq 4, 1 \leq d \leq 8,\ \mu,\ \nu,\ d \in \mathbb{N}$,

we have a basic set $\mathcal{B}_L$ *of* $[\mu, \upsilon]_d$ *and a forbidden set* $\mathfrak{F}$ *of* $[\mu, \upsilon]_d$

then the $\mathbb{H}_{L;2} = [h_{ij}]$ *can be written by* 4×4 *matrix where* $\begin{cases} h_{ij} = 0, if\ \ h_{ij} = a_{ij} \in \mathfrak{F} \\ h_{ij} = 1, if\ \ h_{ij} = a_{ij} \notin \mathfrak{F} \end{cases}$

Then the higher order transition matrices $\mathbb{H}_{L;n}, n \geq 3$,*we have the following expression.* $\mathbb{H}_{L;n}$ *can be decomposed into a successive* 2×2 *matrices as follows :*

$$\mathbb{H}_{L;n} = \begin{bmatrix} \mathbb{H}_{L;n;1} & \mathbb{H}_{L;n;2} \\ \mathbb{H}_{L;n;3} & \mathbb{H}_{L;n;4} \end{bmatrix}$$

$$\mathrm{H}_{L;n;j_1\cdots j_k} = \begin{bmatrix} \mathbb{H}_{L;n;j_1\cdots j_k 1} & \mathbb{H}_{L;n;j_1\cdots j_k 2} \\ \mathbb{H}_{L;n;j_1\cdots j_k 3} & \mathbb{H}_{L;n;j_1\cdots j_k 4} \end{bmatrix} \text{ for } 1 \leq k \leq n-2 \text{ and}$$

$$\mathrm{H}_{L;n;j_1\cdots j_{n-1}} = \begin{bmatrix} \upsilon_{j_1\cdots j_{n-1}1} & \upsilon_{j_1\cdots j_{n-1}2} \\ \upsilon_{j_1\cdots j_{n-1}3} & \upsilon_{j_1\cdots j_{n-1}4} \end{bmatrix}.$$

Furthermore, $\mathbb{H}_{L;n;k} = \begin{bmatrix} \upsilon_{k1}\mathbb{H}_{L;n-1;1} & \upsilon_{k2}\mathbb{H}_{L;n-1;2} \\ \upsilon_{k3}\mathbb{H}_{L;n-1;3} & \upsilon_{k4}\mathbb{H}_{L;n-1;4} \end{bmatrix}.$

Proof.

step I:

For given $[\mu, \nu]_d$, we have the transition matrix $\mathbb{H}_{L;2}$ of the basic set of $[\mu, \nu]_d$.

step II:

If the n-th transition matrix is $\mathbb{H}_{L;n} = \begin{bmatrix} \mathbb{H}_{L;n;1} & \mathbb{H}_{L;n;2} \\ \mathbb{H}_{L;n;3} & \mathbb{H}_{L;n;4} \end{bmatrix}$

step III:

By the transition matrix recursive formula in Proposition3.1 $\mathbb{H}_{L;n+1;k} = \begin{bmatrix} \upsilon_{k1} \cdot \mathbb{H}_{L;n;1} & \upsilon_{k2} \cdot \mathbb{H}_{L;n;2} \\ \upsilon_{k3} \cdot \mathbb{H}_{L;n;3} & \upsilon_{k4} \cdot \mathbb{H}_{L;n;4} \end{bmatrix}$,

$1 \leq k \leq n-1$

The proof is complete. □

4 Spatial entropy

The spatial entropy of L-shaped patterns $[\mu, \nu]_d$, we have the following theorem

Theorem 4.1 *([Chow et al., 1996]; [Juang et al., 2000]; [Ban&Lin, 2005])*
Given a basic set $\mathcal{B}_L$ of $[\mu, \nu]_d$, let λ_n be the largest eigenvalue of the associated transition matrix $\mathbb{H}_{L;n}$ which defined in Theorem 2.4.
Then $h(\mathcal{B}_L) = \lim\limits_{n\to\infty} \frac{\log \lambda_n}{n}$

Proof. By the construction of the transition matrix, we know the L-shaped pattern in a 2×2 lattice, the upper right lattice has lost to compare. By the recursive formula, the $m \times n$ lattice paste by L-shaped patterns, we also only lost the upper right lattice. So, the entropy of the L-shaped pattern
$h(\mathcal{B}_L) = \lim\limits_{n,m\to\infty} \frac{\log\ \Gamma_{m\times n}(\mathcal{B}_L)}{m\times n} = \lim\limits_{n,m\to\infty} \frac{\log\ (\Gamma_{m\times n}(\mathcal{B})\div 2)}{m\times n} = \lim\limits_{n,m\to\infty} \frac{\log\ \Gamma_{m\times n}(\mathcal{B})-log2}{m\times n} = \lim\limits_{n,m\to\infty} \frac{\log\ \Gamma_{m\times n}(\mathcal{B})}{m\times n}$
From the construction of transition matrix $\mathbb{H}_n$, we observe that for $m \geq 2$,

$$\Gamma_{m\times n}(\mathcal{B}) = \sum_{1\leq i,j\leq 2^n} (\mathbb{H}_n^{m-1})_{i,j} = \#(\mathbb{H}_n^{m-1}) \tag{30}$$

As in a one dimensional case, we have

$$\lim_{m\to\infty} \frac{\log\ \#(\mathbb{H}_n^{m-1})}{m} = \log \lambda_n \tag{31}$$

e.g. [42].
Therefore, $h(\mathcal{B}_L) = \lim\limits_{n,m\to\infty} \frac{\log\ \Gamma_{m\times n}(\mathcal{B}_L)}{m\times n} = \lim\limits_{n,m\to\infty} \frac{\log\ \Gamma_{m\times n}(\mathcal{B})}{m\times n}$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\lim_{m\to\infty} \frac{\log\ \Gamma_{n\times m}(\mathcal{B})}{m}\right) = \lim_{n\to\infty} \frac{log\lambda_n}{n} \qquad \square$$

But we can not evaluate the exact entropy $\mathbb{H}_n$ for n increasing to infinite. So, the following section will help us estimate the lower bounded of spatial entropy.

4.1 Connecting operator

In Ref[Ban, Lin&Lin, 2006], it found a method to measure the lower bounded of entropy. In this paper, we can use this method to estimate the lower bounded of entropy.

Proposition 4.2 *([Ban, Lin&Lin, 2006])*

For $m \geq 2$ *, define*

$$\mathbb{C}_m = \begin{bmatrix} C_{m;11} & C_{m;12} & C_{m;13} & C_{m;14} \\ C_{m;21} & C_{m;22} & C_{m;23} & C_{m;24} \\ C_{m;31} & C_{m;32} & C_{m;33} & C_{m;34} \\ C_{m;41} & C_{m;42} & C_{m;43} & C_{m;44} \end{bmatrix} = \begin{bmatrix} S_{m;11} & S_{m;12} & S_{m;21} & S_{m;22} \\ S_{m;13} & S_{m;14} & S_{m;23} & S_{m;24} \\ S_{m;31} & S_{m;32} & S_{m;41} & S_{m;42} \\ S_{m;33} & S_{m;34} & S_{m;43} & S_{m;44} \end{bmatrix},$$

where

$$C_{m;\alpha\beta} = (\begin{bmatrix} h_{\alpha 1} & h_{\alpha 2} \\ h_{\alpha 3} & h_{\alpha 4} \end{bmatrix} \circ (\overset{\wedge}{\otimes} \begin{bmatrix} V_{2;1} & V_{2;2} \\ V_{2;3} & V_{2;4} \end{bmatrix}^{m-2})_{2\times 2})_{2^{m-1}\times 2^{m-1}}$$

$$\circ (E_{2^{m-2}\times 2^{m-2}} \otimes (\begin{bmatrix} h_{1\beta} & h_{2\beta} \\ h_{3\beta} & h_{4\beta} \end{bmatrix}))_{2^{m-1}\times 2^{m-1}}$$

Now, $\mathbb{C}_{m+1}$ *can be found from* $\mathbb{C}_m$ *by a recursive formula,*

$$C_{m+1;\alpha\beta} = \begin{bmatrix} h_{\alpha 1} C_{m;1\beta} & h_{\alpha 2} C_{m;2\beta} \\ h_{\alpha 3} C_{m;3\beta} & h_{\alpha 4} C_{m;4\beta} \end{bmatrix},$$

where $\circ$ *is the Hadamard product,* $\otimes$ *is the tensor product and* E_{2^k} *is the* $2^k \times 2^k$ *matrix with 1 as its entries.*

Proposition 4.3 *([Ban, Lin&Lin, 2006])*

For any $m \geq 2$ *and* $n \geq 2$*, let* $S_{m;\alpha\beta}$ *be given as above.*

Then, $X_{m;n;\alpha;\beta} = S_{m;\alpha\beta} X_{m;n-1;\beta}$.

Proposition 4.4 *([Ban, Lin&Lin, 2006])*

Let $\beta_1\beta_2\cdots\beta_K\beta_1$ *be a diagonal cycle.*

Then for any $m \geq 2$, $h(\mathbb{H}_2) \geq \frac{1}{mK} \log\rho(S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K\beta_1})$

In particular, if a diagonal cycle $\beta_1\beta_2\cdots\beta_K\beta_1$ *exists and* $m \geq 2$ *such that*

$\rho(S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K\beta_1}) > 1$

then $h(\mathbb{H}_2) > 0$.

By Nonnegative matrices in the Mathematical Sciences [$Abraham Berman, Robert J. Plemmons$], we have the following proposition.

Proposition 4.5 *If* $0 \leq A \leq B$, *then* $\rho(A) \leq \rho(B)$

4.2 L-shaped

Herein, we first state the result of this phenomenon.

Theorem 4.6 *if* $\mathbb{H}_{L;n}([\mu,\nu]_d) = [h_{i,j}]$ *and* $\mathbb{H}_{L;n}([\nu,\mu]_d) = [h^{'}_{i,j}]$
Then $h^{'}_{i,j} = h_{2^n+1-i,2^n+1-j}$

Proof.
Since the basic set of $[\mu,\nu]_d$ and $[\mu,\nu]_d$ are :

$$\mathcal{B}_{[\mu,\nu]_d} = \bigcup_{J_i(k,m)\le\mu} P_{k,m} \cup \bigcup_{J_i(k,m)\le\nu} N_{k,m}$$

$$\mathcal{B}_{[\nu,\mu]_d} = \bigcup_{J_i(k,m)\le\nu} P_{k,m} \cup \bigcup_{J_i(k,m)\le\mu} N_{k,m}$$

In particular,

$$\bigcup_{J_i(k,m)\le\mu} P_{k,m} \cup \textit{ and } \bigcup_{J_i(k,m)\le\mu} N_{k,m} \textit{ have opposite colors and same number } \mu$$

and

$$\bigcup_{J_i(k,m)\le\nu} P_{k,m} \cup \textit{ and } \bigcup_{J_i(k,m)\le\nu} N_{k,m} \textit{ have opposite colors and same number } \nu$$

Moreover, *every pattern in* $P_{k,m}$, *its* $h'_{i,j} = h_{2^2+1-i,2^2+1-j}$ *in* $N_{k,m}$.
By the recursive formula $\mathbb{H}_2$ to $\mathbb{H}_n$, we can have the following result :
$h'_{i,j} = h_{2^n+1-i,2^n+1-j}$. □

Theorem 4.7 $h([\mu,\nu]_d) = h([\nu,\mu]_d)$

Proof. By Theorem 4.1, we can evaluate spatial entropy by h(H$_L$)=$\lim\limits_{n\to\infty} \ln\lambda_n$. And , we can use the relationship between $\mathrm{H}_n([\mu,\nu]_i) = H$ and $\mathrm{H}_n([\nu,\mu]_i) = H'$ had shown in Theorem4.6. For evaluate the eigenvalue, we need to have the equation det(A-λI)=0, where det(A)=$\sum\limits_{\alpha=1}^{n}(-1)^{1+\alpha}\cdot a_{1,\alpha}\cdot \det(A_{1|\alpha})$,*where* $A_{\alpha|j}$ is the cofactor of $a_{1,\alpha}$. So,we can have the following equality

$$\det(H_n-\lambda I) = \sum_{\alpha=1}^{2^n}(-1)^{1+\alpha}\cdot h_{1,\alpha}\cdot\det((H_n-\lambda I)_{1|\alpha})$$
$$= \sum_{\alpha=1}^{2^n}(-1)^{1+\alpha}\cdot h'_{2^n,\alpha}\cdot\det((H'_n-\lambda I)_{1|\alpha}) = \det(H'_n-\lambda I)$$

Hence, we can know the $\det(H_n-\lambda I) = \det(H'_n-\lambda I)$, and then the maximal eigenvalue of

H_n is equal to the eigenvalue of H'_n. So, the spatial entropy of $[\mu,\nu]_d$ is equal to the spatial entropy of $[\nu,\mu]_d$

□

Since the patterns for $[\mu,\nu]_d$ and $[\nu,\mu]_d$ have same entropy, it is a sufficient to discuss only the cases of $[\nu,\mu]_d$ with $\mu \geq \nu$ and $1 \leq d \leq 8$. By the generation rule in Sec.3 and previous proposition and theorem, we have the main result in this section.

Theorem 4.8

I.exactly solvable

	region	*spatial entropy*	*notation*
[4,4]	R_1 ˜R_8	$\ln 2$	$\mathbb{H}_2 = E \otimes E$
[4,2]	R_4,R_5	$\ln(\frac{1+\sqrt{5}}{2})$	$\mathbb{V}_2 = G' \otimes E$
[4,2]	R_6,R_7	$\ln(\frac{1+\sqrt{5}}{2})$	$\mathbb{H}_2 = G' \otimes E$
[4,2]	R_1,R_8	*0*	$\mathbb{V}_2 = L \otimes E$
[4,2]	R_2,R_3	*0*	$\mathbb{H}_2 = L \otimes E$
[2,2]	R_1 ˜R_8	*0*	$\mathbb{H}_2 = I \otimes E$

II.lower bounded

	region	*Connecting Operator*	*eigenfunction*	ρ^*	*Lower bounded of spatial entropy*
[4,3]	R_1,R_2	$S_{3,11}$	$x^4-4x^3+2x^2$	$2+\sqrt{2}$	$\frac{\ln(2+\sqrt{2})}{3}$
[4,3]	R_3,R_4	$S_{3,44}$	$x^4-4x^3+2x^2$	$2+\sqrt{2}$	$\frac{\ln(2+\sqrt{2})}{3}$
[4,3]	R_5,R_6	$S_{3,44}$	$x^4-3x^3-2x^2$	$\frac{3+\sqrt{17}}{2}$	$\frac{\ln(\frac{3+\sqrt{17}}{2})}{3}$
[4,3]	R_7,R_8	$S_{3,14}S_{3,41}$	$x^4-9x^3+2x^2$	$\frac{9+\sqrt{73}}{2}$	$\frac{\ln(\frac{9+\sqrt{73}}{2})}{6}$
[4,1]	R_5,R_6	$S_{3,44}$	$x^4-x^3-3x^2-x$	$1+\sqrt{2}$	$\frac{\ln(1+\sqrt{2})}{3}$
[3,3]	R_1,R_2	$S_{6,11}S_{6,11}S_{6,14}S_{6,44}S_{6,41}$	$x^2-177x+101$	176.4275	$\frac{\ln(176.4275)}{30}$
[3,3]	R_3,R_4	$S_{4,14}S_{4,41}S_{4,11}$	$x^8-11x^7+10x^6$	10	$\frac{\ln(10)}{12}$
[3,3]	R_5,R_6	$S_{6,14}S_{6,44}S_{6,41}$		120.6018833	$\frac{\ln(120.60188)}{18}$
[3,3]	R_7,R_8	$S_{3,44}S_{3,44}S_{3,41}S_{3,14}S_{3,44}$	x^4-6x^3	6	$\frac{\ln(6)}{15}$
[3,2]	R_4	$S_{3,14}S_{3,44}S_{3,41}$	x^4-5x^3+4x	4	$\frac{\ln(4)}{12}$
[3,2]	R_5	$(S_{4,41}S_{4,14})^2S_{4,44}(S_{4,41}S_{4,14})^2$	$x^8-40x^7+80x^6$	$20+8\sqrt{5}$	$\frac{\ln(20+8\sqrt{5})}{36}$
[3,2]	$R_6(R_5)$	$(S_{4,41}S_{4,14})^2S_{4,44}(S_{4,41}S_{4,14})^2$	$x^8-40x^7+80x^6$	$20+8\sqrt{5}$	$\frac{\ln(20+8\sqrt{5})}{36}$
[3,2]	$R_7(R_4)$	$S_{3,14}S_{3,44}S_{3,41}$	x^4-5x^3+4x	4	$\frac{\ln(4)}{12}$

III.entropy is zero

	region	*upper matrix*
[4,1]	R_1 ~R_4	$\mathbb{H}_2 < L \otimes E$
[4,1]	R_7,R_8	$\mathbb{V}_2 < L \otimes E$
[4,0]	R_1 ~R_8	$\mathbb{H}_2 < L \otimes E$
[3,2]	R_1,R_8	$\mathbb{V}_2 < L \otimes E$
[3,2]	R_2,R_3	$\mathbb{H}_2 < L \otimes E$
[3,1]	R_1 ~R_4	$\mathbb{H}_2 < L \otimes E$
[3,1]	R_5,R_6	*
[3,1]	R_7,R_8	$\mathbb{V}_2 < L \otimes E$
[3,0]	R_1 ~R_8	$\mathbb{H}_2 < L \otimes E$
[2,1]	R_1 ~R_8	$\mathbb{H}_2 < \mathbb{H}_{2,[2,2]}$
[2,0]	R_1 ~R_8	$\mathbb{H}_2 < \mathbb{H}_{2,[2,2]}$
[1,1]	R_1 ~R_4,R_7,R_8	$\mathbb{H}_2 < L \otimes E$
[1,1]	R_5,R_6	$\mathbb{H}_2 < I \otimes E$
[1,0]	R_1 ~R_8	$\mathbb{H}_2 < L \otimes E$

Proof. We have three cases for evaluating spatial entropy. In Part I , we can exact solve spatial entropy of some regions. In Part II , we largely use connecting operator to estimate the lower bound of spatial entropy which is positive. In Part III , we use the character of transition matrix of some regions to evaluate the upper bound of spatial entropy is zero, then we can say the spatial entropy is zero.

Part I :

For $[4,2]_{6,7}$, the transition matrix is :

$$\mathbb{H}_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = G' \otimes E$$

By the recursive formula of the transition matrix, the $\mathbb{H}_3$ can be expressed :

$$\mathbb{H}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = G^{'} \otimes G^{'} \otimes E$$

If $\mathbb{H}_n = G^{'} \otimes G^{'} \otimes \cdots G^{'} \otimes E$ is with (n-1) times of $G^{'}$, also by the recursive formula: we can make sure the $\mathbb{H}_n = G^{'} \otimes G^{'} \otimes \cdots G^{'} \otimes E$ is with n times of $G^{'}$. So induction by n, we can know the $\mathbb{H}_n = G^{'} \otimes G^{'} \otimes \cdots G^{'} \otimes E$ is with (n-1) times of $G^{'}$ $\forall n \in \mathbb{N}$. Then the maximal eigenvalue of $\mathbb{H}_n$ is $\rho(\mathbb{H}_n) = g^{n-1} \cdot 2$, where $g = \frac{1+\sqrt{5}}{2}$. Then,

h($\mathbb{H}_n$)=$\lim\limits_{n\to\infty} \frac{\ln(g^{n-1}\cdot 2)}{n}$=$\lim\limits_{n\to\infty} \frac{(n-1)\ln g + \ln 2}{n} = \ln g = \ln(\frac{1+\sqrt{5}}{2})$.

We can use the same way to evaluate the other regions in Part I.

Part II :

For $[4,3]_{1,2}$, the transition matrix is :

$$\mathbb{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

We can find the trace operator $\mathbb{T}_3$:

$$\mathbb{T}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

By Proposition 4.3, we have find out a diagonal cycle.

$$\mathbf{S}_{3,11} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

which maximal eigenvalue is $2+\sqrt{2}$. By the Proposition 4.4, we can estimate the lower bound of spatial entropy is $\frac{ln(2+\sqrt{2})}{3}$.

We can use the same way to evaluate the other regions in Part II.

Part III :

For $[3,1]_{5,6}$, the transition matrix is :

$$\mathbb{H}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

In particular,

$$\mathbb{T}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We can see that $\mathbb{T}_2$'s every row or column has not more than two 1's.

$$\mathbb{T}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $\mathbb{T}_3$ has the same property. By recursive formula, we can know that $\mathbb{T}_n$ has the same property. Then, by the matrix's property : M=$[m_{i,j}]_{n\times n}$, $\rho(M) \leq \max(\sum_{j=1}^{n} m_{i,j}$, for some

i) or $\max(\sum_{i=1}^{n} m_{i,j}$, for some j) . So the entropy of $[3,1]_{5,6}$ is less than $\lim_{n\to\infty} \frac{\ln 2}{n} = 0$. Therefore, we can see $[3,1]_{5,6}$ is pattern formation.

Otherwise, for $[4,1]_{1,2}$, the transition matrix is :

$$\mathbb{H}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

In particular,

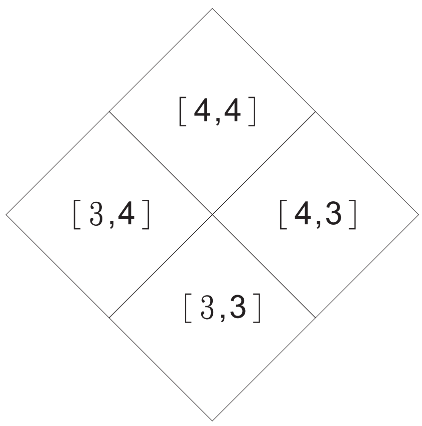
$$\mathbb{H}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

In the Part I , we can know $\mathbb{H}_2 = L \otimes E$ is entropy zero. So we can use Proposition 4.5 to know the upper bound of spatial entropy of $[4,1]_{1,2}$ is zero. So, the spatial entropy of $[4,1]_{1,2}$ is exactly zero.

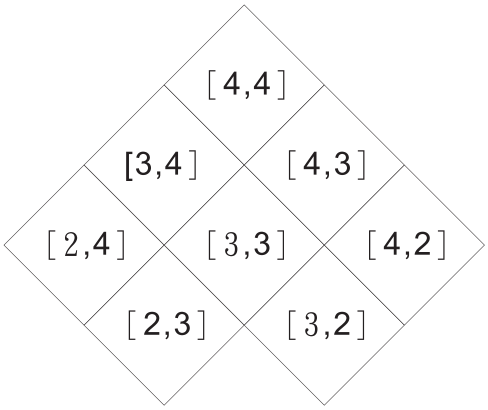
We can use the same way to evaluate the other regions in Part III.

□

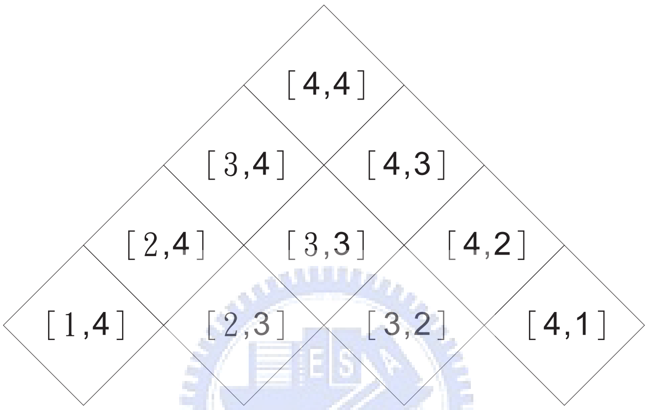
Now, we can conclusion the regions which is spatial chaos in the following figures.



R_1, R_2, R_3, R_8



R_4, R_7



R_5, R_6

Corollary 4.9 *([Lin&Yang, 2000])*

Equation(5) is spatial chaos if and only if r,a,s and z belong to the following regions of $\mathcal{P}^4$ *:*

1. *[4,4]*$_d, 1 \leq d \leq 8$

2.*[4,3]*$_d, 1 \leq d \leq 8$

3.*[4,2]*$_d, d = 4, 5, 6, 7$

4.*[4,1]*$_d, d = 5, 6$

5.*[3,3]*$_d, 1 \leq d \leq 8$

6.*[3,2]*$_d, d = 4, 5, 6, 7$

In the following table,we can see the compare about Prof.Lin and Prof.Yang with my result.

	Prof.Lin & Yang's result(lower bounded)	Huang's result
$[4,4]_{1\sim8}$	$ln2 \approx 0.69315$	$ln2$
$[4,3]_{1,2}$	$\frac{\ln 3}{4} \approx 0.27465$	$\frac{\ln 2+\sqrt{2}}{3} \approx 0.409316$
$[4,3]_{3,4}$	$\frac{\ln 4}{4} \approx 0.347657$	$\frac{\ln 2+\sqrt{2}}{3} \approx 0.409316$
$[4,3]_{5,6}$	$\frac{\ln 5}{4} \approx 0.40236$	$\frac{\ln 3+\sqrt{17}}{3} \approx 0.654448$
$[4,3]_{7,8}$	$\frac{\ln 4}{4} \approx 0.347657$	$\frac{\ln 9+\sqrt{73}}{6} \approx 0.47745$
$[4,2]_{4,5}$	$\frac{\ln 4}{4} \approx 0.347657$	$\ln(\frac{1+\sqrt{5}}{2}) \approx 0.48122$
$[4,2]_{6,7}$	$\frac{\ln 4}{4} \approx 0.347657$	$\ln(\frac{1+\sqrt{5}}{2}) \approx 0.48122$
$[4,1]_{5,6}$	*chaos*	$\frac{\ln 1+\sqrt{2}}{3} \approx 0.293789$
$[3,3]_{1,2}$	$\frac{\ln 20}{20} \approx 0,14979$	0.17234
$[3,3]_{3,4}$	$\frac{\ln 9}{12} \approx 0.1831$	$\frac{\ln 10}{12} \approx 0.191882$
$[3,3]_{5,6}$	$\frac{\ln 9}{9} \approx 0.24414$	$\frac{\ln(120.60188)}{18} \approx 0.26625$
$[3,3]_{7,8}$	$\frac{\ln 9}{20} \approx 0.10986$	$\frac{\ln(6)}{15} \approx 0.11945$
$[3,2]_4$	$\frac{\ln 3}{12} \approx 0.09155$	$\frac{\ln 4}{12} \approx 0.1155245$
$[3,2]_5$	$\frac{\ln 8}{21} \approx 0.099$	$\frac{\ln(20+8\sqrt{5})}{36} \approx 0.10096$
$[3,2]_6$	$\frac{\ln 8}{21} \approx 0.099$	$\frac{\ln(20+8\sqrt{5})}{36} \approx 0.10096$
$[3,2]_7$	$\frac{\ln 3}{15} \approx 0.07324$	$\frac{\ln 4}{12} \approx 0.1155245$

Appendix

	$R_{1\sim2}$	$R_{3\sim4}$	$R_{5\sim6}$	$R_{7\sim8}$
$[4,3]$	$\begin{bmatrix}1&1&1&1\\1&1&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0&0\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1&1\\0&0&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$
$[3,3]$	$\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0&0\\1&1&1&1\\1&1&1&1\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\1&1&1&1\\1&1&1&1\\1&1&0&0\end{bmatrix}$	$\begin{bmatrix}1&1&1&1\\0&0&1&1\\1&1&0&0\\1&1&1&1\end{bmatrix}$
$[4,1]$	$\begin{bmatrix}1&1&0&0\\0&0&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\1&1&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\0&0&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$
$[3,1]$	$\begin{bmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\1&1&0&0\\1&1&1&1\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&1&1\\1&1&1&1\\1&1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\0&0&0&0\\1&1&0&0\\1&1&1&1\end{bmatrix}$
$[1,1]$	$\begin{bmatrix}1&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\1&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&1&1\\1&1&0&0\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\0&0&0&0\\0&0&0&0\\1&1&0&0\end{bmatrix}$
$[1,0]$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&0&0\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&1&0&0\end{bmatrix}$
$[4,0]$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$
$[3,0]$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&1&1\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&1&1\\1&1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&0&0\\1&1&1&1\end{bmatrix}$

	$R_{1\sim 8}$	$R_{2\sim 3}$	$R_{4\sim 5}$	$R_{6\sim 7}$
$[4,2]$	$\begin{bmatrix}1&1&1&1\\0&0&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0&0\\1&1&0&0\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\0&0&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$
$[2,2]$	$\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\1&1&1&1\\1&1&1&1\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\0&0&1&1\\1&1&0&1\\1&1&0&0\end{bmatrix}$
$[2,0]$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&1&1\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&1&0&1\\1&1&0&0\end{bmatrix}$
	R_1	R_2	R_3	R_4
$[3,2]$	$\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0&0\\1&1&0&0\\1&1&1&1\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\1&1&1&1\\1&1&1&1\\0&0&1&1\end{bmatrix}$
	R_5	R_6	R_7	R_8
$[3,2]$	$\begin{bmatrix}0&0&0&0\\1&1&1&1\\1&1&1&1\\1&1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\0&0&1&1\\1&1&1&1\\1&1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1&1\\0&0&0&0\\1&1&0&0\\1&1&1&1\end{bmatrix}$
	R_1	R_2	R_3	R_4
$[2,1]$	$\begin{bmatrix}1&1&0&0\\0&0&0&0\\0&0&0&0\\1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\1&1&0&0\\1&1&1&1\\0&0&0&0\end{bmatrix}$
	R_5	R_6	R_7	R_8
$[2,1]$	$\begin{bmatrix}0&0&0&0\\0&0&1&1\\1&1&1&1\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&1&1\\1&1&0&0\\1&1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\0&0&0&0\\1&1&0&0\\1&1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1&1\\0&0&0&0\\0&0&0&0\\1&1&1&1\end{bmatrix}$

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