國 立 交 通 大 學

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碩 士 論 文

Vogan 圖之距離

The Distance of Vogan Diagram

研 究 生： 程千砡

指導教授： 蔡孟傑 教授

中 華 民 國 九 十 五 年 六 月

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研 究 生：程千砡　　　　　Student：Chien-Yu Chen

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摘　要

Vogan 圖為一個含有 involution 的 Dynkin 圖，並且被固定住的頂點可以是黑色的。若一個 Vogan 圖可以被表示成另一個 Vogan 圖，則他們可以說是等價的。任何一種含有多個黑色頂點的 Vogan 圖都會與某個 Vogan 圖等價，而這個 Vogan 圖只有一個黑色頂點。我們的目的便是要找出含有多個黑色頂點的 Vogan 圖需要多少步驟，才能找到與他等價的單黑色頂點之 Vogan 圖。

這篇論文分為兩個部份，在第一部份(第一節至第五節)，我們會對 Vogan 圖先做一些簡單的介紹以及 Vogan 圖距離的定義，再來便會以 Vogan 圖中幾個典型的圖來證明他們的距離問題；最後的部份(第六節至第九節)則是證明 Vogan 圖中其他特別的圖之距離。

中華民國九十五年六月

The Distance of Vogan Diagrams

Student: Chien-Yu Chen Advisor: Dr. Meng-Kiat Chuah

Department of Applied Mathematics

National Chiao Tung University

Abstract

A Vogan diagram is a Dynkin diagram with an involution, andthe vertices fixed by the involution may be black. If a Vogandiagram can be represented by another Vogan diagrams, then they are equivalent. Any Vogan diagram with many black vertices is equivalent to a diagram with only one black vertex. Our purpose is to find the steps from a Vogan diagram with many black vertices to one black vertex.

This thesis is divided into two parts. In the first part, consisting of Sections 1-5, we give a brief introduction of some fundamental concepts in Vogan diagram and the distance of two Vogan diagrams in classical types with proof. In the last part, Sections 6-9, we prove the distance of two Vogan diagrams in exceptional types.

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兩年的研究生生活，稍縱即逝，很感謝當初蔡孟傑教授願意當我的指導教授，並且悉心教導，讓我能順利完成論文，除此之外，老師也會帶我們去打打羽球，培養感情之餘順便磨練球技。感謝博士班的舉卿學姊，當課業上遇到問題，不論時間早晚，她總是很有耐心教我，她也會告訴我哪裡有好玩好吃的，是位個性開朗、很照顧人的學姐。

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求學路程一路走來，遇到再大的困難，都有家人在背後支撐著我，給我鼓勵，真的很感謝媽媽、爸爸、哥哥、跟妹妹。我是個很戀家的人，即使家住南部仍常每個月都會回家一次，媽媽為了不讓我坐車辛苦，偶爾會北上來看我，還帶很多好吃給我，彼此聊著生活上的大小事，因為母親的相伴，讓我覺得讀書是件很幸福的事。

最後我將本論文的成果獻給我的父母、師長以及所有關心我的朋友，希望一起分享完成論文的這份喜悅。

Contents

1 Introduction

A Dynkin diagram is a certain type of graph. It satisfies the property that two vertices may be connected by 0, 1, 2 or 3 edges, and an orientation (arrow) is assigned to each double or triple-edge. There are altogether seven classes of Dynkin diagrams, labelled as types A, B, C, D, E, F, G ([2, Chapter 11]). A Dynkin diagram in each type is specified by a subscript which indicates the number of vertices in that diagram, for example A_5 is the diagram of type A which has 5 vertices. Then all the Dynkin diagrams are given by A_n, B_n, C_n, D_n for $n \geq 1$, as well as E_6, E_7, E_8, F_4, G_2. The infinite lists of diagrams of types A, B, C, D are called classical, and the finite lists of types E, F, G are called exceptional.

A Vogan diagram is a Dynkin diagram together with extra information. Namely, there is a diagram involution θ, such that the fixed points of θ are colored white or black [3]. Many Dynkin diagrams, for example B_n and C_n, have trivial symmetry. In these cases the involution is the identity.

This thesis studies the following algorithm on the Vogan diagrams. Suppose that v is a Vogan diagram, and p is a black vertex of v. Let F_p be the algorithm which reverses the colors of all the θ-fixed vertices which are adjacent to p (but not p itself), except when the vertex is joint to p with a double-edge with arrow pointing towards p. In this way, $F_p(v)$ is another Vogan diagram. We give some examples as follows.

Example. The vertices are labelled $1, 2, \ldots$ starting from the left.

$v =$ ○—●—●—○

$\Rightarrow F_2(v) =$ ●—●—○—○

$\Rightarrow F_3(v) =$ ○—○—●—●

$u =$ ○—●—○—●⟹●

$\Rightarrow F_4(u) =$ ○—●—●—●⟹○

$\Rightarrow F_5(u) =$ ○—●—○—●⟹●

We say that two Vogan diagrams v and w are equivalent if there is a sequence of algorithms

$$v = v_0 \to v_1 \to \ldots \to v_k = w \tag{1.1}$$

such that each step $v_i \to v_{i+1}$ is given by some F_p. If v and w are equivalent and k in (1.1) is as small as possible, we say that $k = d(v, w)$ is the distance between v and w. Clearly a diagram without black vertex is not equivalent to any other one. Therefore, once and for all, we may consider only Vogan diagrams with black vertices, and denote them by $V(X)$, where X is a Dynkin diagram with trivial diagram involution. Let $V_1(X) \subset V(X)$ denote the diagrams with exactly one black vertex. It is known that every diagram in $V(X)$ is equivalent to a diagram in $V_1(X)$ [1][3]. It allows us to define the distance between $v \in V(X)$ and $V_1(X)$ by

$$d(v, V_1(X)) = \min_{w \in V_1(X)} d(v, w). \tag{1.2}$$

For fixed X, we intend to seek an upper bound for $\{d(v, V_1(X)); v \in V(X)\}$, namely

$$d(V(X)) = \max_{v \in V(X)} d(v, V_1(X)). \tag{1.3}$$

We present the main results of the thesis as follows. The classical Vogan diagrams are denoted by $V(A_n), V(B_n), V(C_n), V(D_n), V(D_n, \theta)$, where θ denotes the nontrivial involution. We need not consider $V(A_n, \theta)$ because it contains only one diagram.

Theorem 1

(a) $d(A_n)$ *is bounded above by* $(n-1)n^2$.

(b) $d(B_n)$ *is bounded above by* $(n-1)n^2$.

(c) $d(C_n)$ *is bounded above by* $(n-1)(n-1)^2$.

(d) $d(D_n)$ *is bounded above by* $(n-1)(n-2)^2$.

(e) $d(D_n, \theta)$ *is bounded above by* $(n-1)(n-2)^2$.

For the proof of Theorem 1, we shall study type A in Section 2, type B in Section 3, type C in Section 4, and type D in Section 5.

For the exceptional diagrams, only E_6 contains nontrivial involution θ. We provide their upper bounds in the following theorem.

Theorem 2

(*a*) *$d(E_6)$ is bounded above by* 130*;*

(*b*) *$d(E_6, \theta)$ is bounded above by* 1*;*

(*c*) *$d(E_7)$ is bounded above by* 228*;*

(*d*) *$d(E_8)$ is bounded above by* 180*;*

(*e*) *$d(F_4)$ is bounded above by* 4*;*

(*f*) *$d(G_2)$ is bounded above by* 1*.*

For the proof of Theorem 2, we shall study E_6 in Section 6, E_7 in Section 7, E_8 in Section 8, and F_4, G_2 in Section 9.

2 Type A

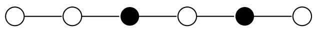
Recall that a Dynkin diagram of type A_n is given by

(2.1)

A Vogan diagram of A_n is labelled as (2.1) with at least one black vertex, denoted by $V(A_n)$. According to the theorem by Borel and de Siebenthal[1, Theorem 6.96], every Vogan diagram is equivalent to a Vogan diagram with 0 or 1 painted vertex. For type A_n, all Vogan diagrams with some black vertices can be equivalent to a painted one. Let $V_k(A_n) \subset V(A_n)$ denote the Vogan diagram of A_n with k black vertices. From (1.2), it also extends to the distance between $V_k(A_n)$ and $V_1(A_n)$ by

$$d(V_k(A_n), V_1(A_n)) = \min\{d(v, w); v \in V_k(A_n), w \in V_1(A_n)\} \qquad (2.2)$$

or just write $d_k(V_1(A_n))$.

For convenience, we write $(i_1, \ldots, i_k) \in V(A_n)$ to denote the diagram with vertices $i_1, \ldots, i_k$ painted. For example $(3, 5) \in V(A_6)$ means 

In this section, we will discuss the distance between $V_k(A_n)$ and $V_1(A_n)$; in other words, we provide an upper bound for it.

$$d(V(A_n)) = \max_{v \in V(A_n)} d(v, V_1(A_n)). \qquad (2.3)$$

Example. For $V(A_4)$, there are 7 cases to discuss.

$(1, 2) \to (1)$

$(1, 3) \to (1, 2, 3) \to (2)$

$(1, 4) \to (1, 2, 4) \to (2, 3, 4) \to (3)$

$(2, 3) \to (1, 2) \to (1)$

$(1, 2, 3) \to (2)$

$(1, 2, 4) \to (2, 3, 4) \to (3)$

$(1, 2, 3, 4) \to (2, 4) \to (2, 3, 4) \to (3)$

By (2.3), $d(V(A_4)) = max\{1, 2, 3, 2, 1, 2, 3\} = 3$.

Proposition 2.1 *Let* $u = (i, j) \in V(A_n)$

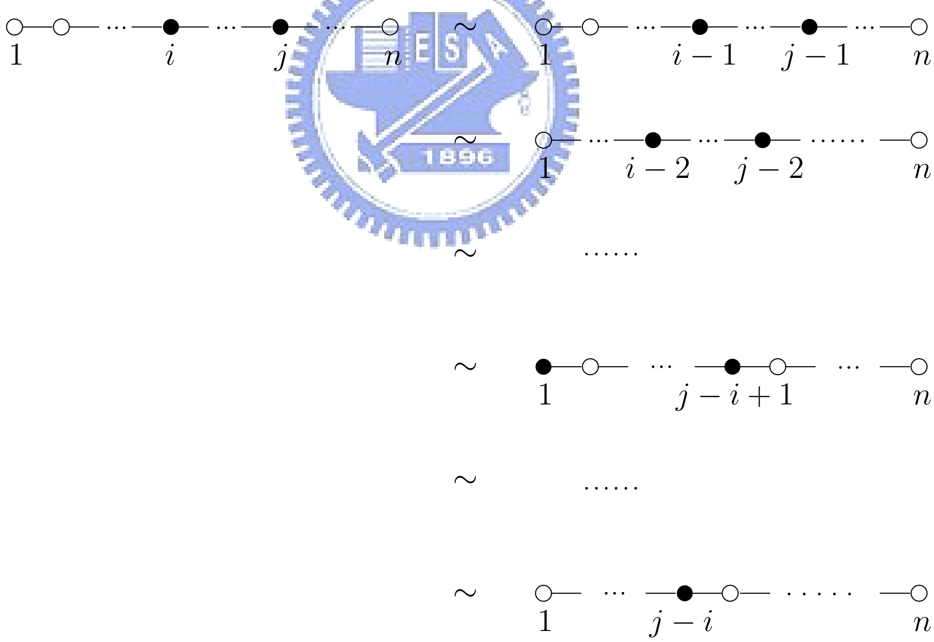
(a) *If* $i - 1 \le n - j$, *then* $d(u, v) \le i(j - i)$ *where* $v = (j - i) \in V(A_n)$.

(b) *If* $i-1 > n-j$, *then* $d(u, w) \le (n-j+1)(j-i)$ *where* $w = (n+i-j+1) \in V(A_n)$.

Proof: u has two black vertices in $V(A_n)$, and it can be equivalent to a diagram in $V_1(A_n)$; in other words, $u \sim s \in V_1(A)$.

There are two ways to make u reduce to a Vogan diagram of A_n with only one painted vertex:

Case 1 Move i and j towards the left:



$u = (i, j) \longrightarrow (i - 1, i, i + 1, j) \longrightarrow (i - 1, i + 1, i + 2, j) \longrightarrow \cdots \longrightarrow (i - 1, j - 2, j - 1, j) \longrightarrow (i - 1, j - 1)$ needs at most $j - i$ steps.

Continuously,

$(i-1, j-1) \longrightarrow (i-2, i-2)$ needs at most $j-i$ steps.

$(i-2, j-2) \longrightarrow (i-3, i-3)$ needs at most $j-i$ steps.

$\vdots$

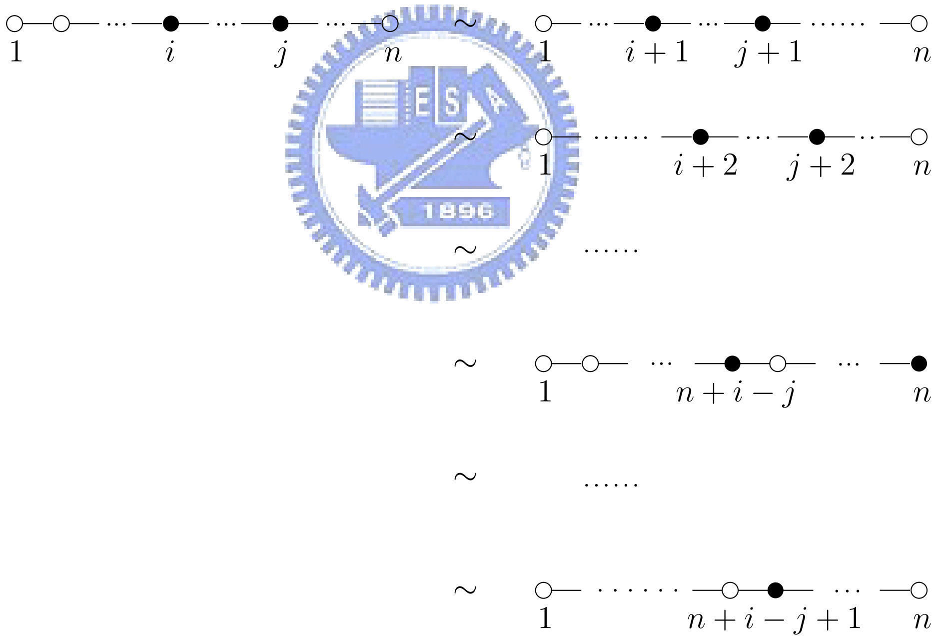
$(2, j-i+2) \longrightarrow (1, j-i+1)$ needs at most $j-i$ steps.

Finally,

$(1, j-i+1) \longrightarrow (1, 2, j-i+1) \longrightarrow (2, 3, j-i+1) \longrightarrow \cdots \longrightarrow (j-i-1, j-i, j-i+1) \longrightarrow (j-i)$ needs at most $j-i$ steps.

Thus, $d(u, (j-i)) \leq (j-i)[(i-1)+1] = i(j-i)$. Let $v = (j-i)$ and $d(u, v) = d_1$.

Case 2 Move i and j towards the right:



$u = (i, j) \longrightarrow (i, j-1, j, j+1) \longrightarrow (i, j-2, j-1, j+1) \longrightarrow \cdots \rightarrow (i, i+1, i+2, j+1) \longrightarrow (i+1, j+1)$ needs at most $j-i$ steps.

Continuously,

$(i+1, j+1) \longrightarrow \cdots \longrightarrow (i+2, j+2)$ needs at most $j-i$ steps.

$(i+2, j+2) \longrightarrow \cdots \longrightarrow (i+3, j+3)$ needs at most $j-i$ steps.

$$\vdots$$

$(n+i-j-1, n-1) \longrightarrow (n+i-j, n)$ needs at most $j-i$ steps.

Finally,

$(n+i-j, n) \longrightarrow (n+i-j, n-1, n) \longrightarrow (n+i-j, n-2, n-1) \longrightarrow \cdots \longrightarrow (n+i-j, n+i-j+1, n+i-j+2) \longrightarrow (n+i-j+1)$ needs at most $j-i$ steps.

Thus,$d(u, (n+i-j+1)) \leq (j-i)[(n+i-j+i)+1] = (n-j+1)(j-i)$. Let $w = (n+i-j+1)$ and $d(u, w) = d_2$.

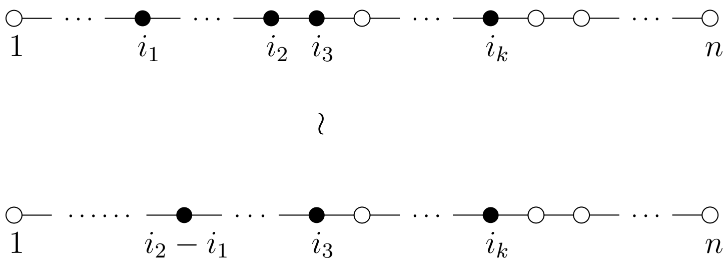
Comparison with Case 1 and Case 2, the proof can be completed as the following:
(a) $i-1 \leq n-j \Rightarrow i < n-j+1 \Rightarrow i(j-i) \leq (n-j+1)(j-i) \Rightarrow d_1 \leq d_2$ with $j-i > 0$. By (1.2), it needs at most $i(j-i)$ steps s.t. $u \to \cdots \to v = (j-i) \in V_1(A_n)$.
(b) $i-1 > n-j \Rightarrow i > n-j+1 \Rightarrow i(j-i) > (n-j+1)(j-i) \Rightarrow d_1 \leq d_2$ with $j-i > 0$. By (1.2), it needs at most $(n-j+1)(j-i)$ steps such that $u \to \cdots \to v = (j-i) \in V_1(A_n)$. $\square$

Therefore, we can conclude that:

Proposition 2.2 *Given $u = (i_1, i_2, ..., i_k) \in V_k(A_n)$, then $d_k(V_1(A_n)) \leq (k-1)n^2$.*

Proof : By Proposition 2.1, we have known that any two painted vertices which can be reduced to be only one painted vertex in $V_k(A_n)$ needs at most d_1 or d_2 steps.

Case 1 If $i_1 - 1 \leq n - i_k$, and just move the leftmost two black vertices (i_1, i_2) in the left direction:



Let $D_m = (i_m - i_{m-1} + i_{m-2} - ... \pm i_1)(i_{m+1} - i_m + i_{m-1} - i_{m-2} + ... \mp i_1)$ where $0 < m < k$. By Proposition 2.1(a), $u \sim (i_2 - i_1, i_3, ..., i_k) \in V_{k-1}$ needs at most D_1 steps.

Continuing the same way,

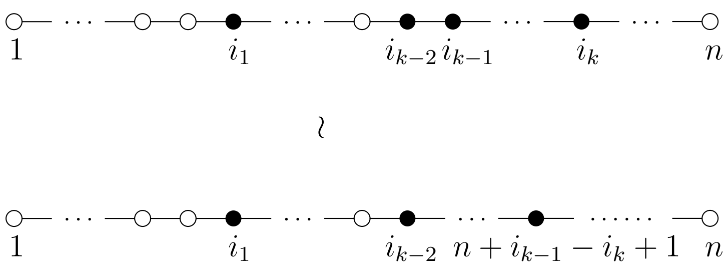
$(i_2 - i_1, i_3, ..., i_k) \sim (i_3 - i_2 + i_1, i_4, ..., i_k) \in V_{k-2}$ needs at most D_2 steps.

$$\vdots$$

$(i_{k-1} - i_{k-2} + ... \pm i_1, i_k) \sim (i_k - i_{k-1} + ... \mp i_1) \in V_1$ needs at most D_{k-1} steps.

Then $D_1 \leq i_2 \cdot i_2 = {i_2}^2 \leq n^2$, $D_2 \leq i_3 \cdot i_3 = {i_3}^2 \leq n^2$, ... , $D_{k-1} \leq i_k \cdot i_k = {i_k}^2 \leq n^2$, and thus $d(u, r) = D_1 + D_2 + ... + D_{k-1} \leq n^2 + n^2 + ... + n^2 = (k-1)n^2$, where $r = (i_k - i_{k-1} + ... \mp i_1) \in V_1$.

Case 2 If $i_1 - 1 > n - i_k$, and just move the rightmost two black vertices (i_{k-1}, i_k) in the right direction:



By Proposition 2.1(b), $u \sim (i_1, ..., i_{k-2}, n + i_{k-1} - i_k + 1) \in V_{k-1}$ needs at most R_1 steps where $R_1 = (n - i_k + 1)(i_k - i_{k-1})$.

Continuously,

$(i_1, ..., i_{k-2}, n + i_{k-1} - i_k + 1) \sim (i_1, ..., i_{k-3}, i_k - i_{k-1} + i_{k-2}) \in V_{k-2}$ needs at most R_2 steps where $R_2 = (i_k - i_{k-1})(n + i_{k-1} - i_k + 1 - i_{k-2})$.

$$\vdots$$

$(i_1, i_k - i_{k-1} + i_{k-2} - ... \pm i_2) \sim (n - i_k + i_{k-1} - ... \mp i_2 + i_1 + 1) \in V_1$ needs at most R_{k-1} steps where $R_{k-1} = (n - i_k + i_{k-1} - ... \mp i_2 + i_1 + 1)(i_k - i_{k-1} + i_{k-2} - ... \pm i_1)$.

Hence $R_1 \leq n \cdot n = n^2$, $R_2 \leq n \cdot n = n^2$, ... , $R_{k-1} \leq n \cdot n = n^2$.

Then $d(u, t) = R_1 + R_2 + ... + R_{k-1} \leq n^2 + n^2 + ... + n^2 = (k-1)n^2$, where $t = (n - i_k + i_{k-1} - ... \mp i_2 + i_1 + 1) \in V_1$. $\square$

Consequently, we can get the result of Theorem 1 (a) immediately.

3 Type B

A Dynkin diagram of type B_n is given by

(3.1)

A Vogan diagram of B_n is labelled as (3.1) with at least one black vertex, denoted by $V(B_n)$. Let $V_k(B_n) \subset V(B_n)$ denote the Vogan diagram of B_n with k black vertices. From (1.2), it also extends to the distance between $V_k(B_n)$ and $V_1(B_n)$ by

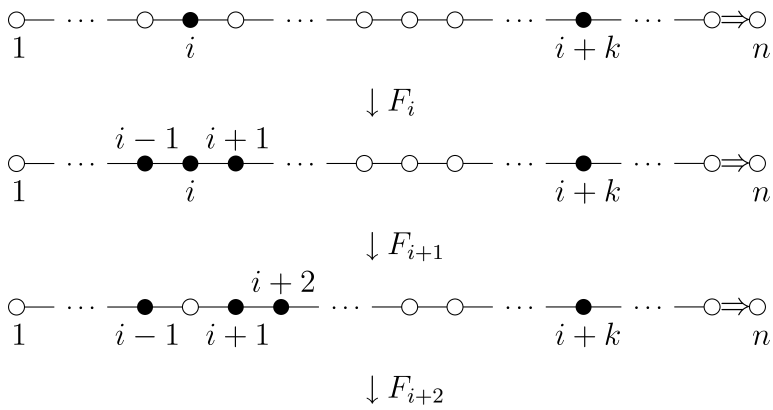
$$d(V_k(B_n), V_1(B_n)) = \min\{d(v, w); v \in V_k(B_n), w \in V_1(B_n)\} \tag{3.2}$$

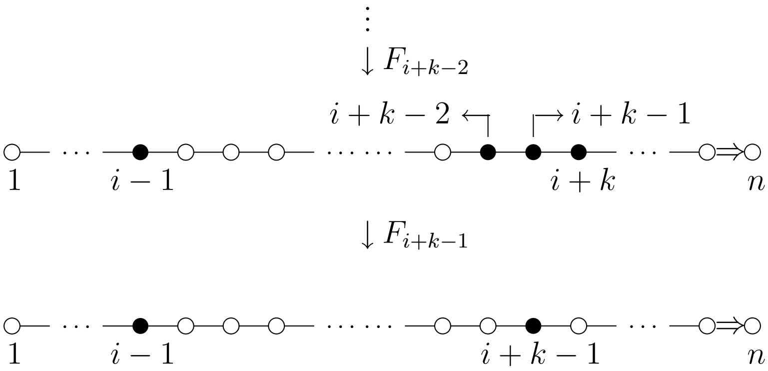
or just write $d_k(V_1(B_n))$. Note that if n is painted, then F_n cannot reverse the color of $n-1$, or we can write $F_n(w) = w$, here $w \in V(B_n)$

In this section, we discuss the distance between $V_k(B_n)$ and $V_1(B_n)$. We will provide an upper bound for it.

Proposition 3.1 *Let $u = (i, i+k) \in V(B_n)$, then $u \sim v$ needs at most ki steps where $v = (k)$.*

Proof : Suppose that we want to obtain a Vogan diagram of B_n with one black vertex from u immediately. Then we should move the painted vertices in the left direction, or the vertex n will be painted and it does not make sense that we mentioned before.





The graph says that $u=(i,i+k) \longrightarrow (i-1,i,i+1,i+k) \longrightarrow (i-1,i+1,i+2,i+k) \longrightarrow \cdots \longrightarrow (i-1,i+k-2,i+k-1,i+k) \longrightarrow (i-1,i+k-1)$ and it needs at most $[(i+k-1)-i]+1=k$ steps.

Continuously, we also can get that

$(i-1,i+k-1) \sim (i-2,i+k-2)$ needs at most k steps.

$(i-2,i+k-2) \sim (i-3,i+k-3)$ needs at most k steps.

$\vdots$

$(2,2+k) \sim (1,1+k)$ needs at most k steps.

Finally, $(1,1+k) \longrightarrow (1,2,1+k) \longrightarrow (2,3,1+k) \longrightarrow \cdots \longrightarrow (k-1,k,k+1) \longrightarrow (k)$ and it needs at least k steps.

Thus, there are at most $k+[(i-1)-1]k+k=ki$ steps to make $(i,i+k) \sim (k)$; in other words, $d(u,v) \le ki$ where $v=(k)$. $\square$

Therefore, we can conclude that:

Proposition 3.2 *Given* $u=(i_1,i_2,...,i_k) \in V_k(B_n)$*, then* $d_k(V_1(B_n)) \le (k-1)n^2$.

Proof : By Proposition 3.1, there are at most $i(j-i)$ steps to make (i,j) reduced to $(j-i)$.

Let $u=(i_1,i_2,...,i_k) \in V_k$. Similar to Case 1 of Proposition 2.2, we move vertices i_1 and i_2 towards the left, so that eventually we get only one black vertex from i_1

and i_2. Repeat this process of moving pairs of leftmost black vertices towards the left. Finally, we prove that $d_k(V_1(B_n)) \leq (k-1)n^2$. Throughout these steps, vertex n remains unpainted. $\square$

Consequently, we can get the result of Theorem 1 (b) immediately.

4 Type C

The Vogan diagram of C_n is similar to B_n except for the direction of arrow on the double-edge, it is indicated by (4.1) and denoted by $V(C_n)$.

$$\begin{array}{ccccccc} \circ & - & \circ & - \ \cdots \ - & \circ & \Leftarrow & \circ \\ 1 & & 2 & & n-1 & & n \end{array} \tag{4.1}$$

By Borel-Siebenthal Theorem, every Vogan diagram is equivalent to one with a black vertex. If a Vogan diagram belongs to $V(C_n)$ and its vertex n is painted, then this diagram is equivalent to (n). For example, $(2,4) \in V_4(C_n)$ and then $(2,4) \to (2,3,4) \to (3,4) \to (4)$.

Before talking about the distance between $V_k(C_n)$ and $V_1(C_n)$, we consider the distance between $V_2(C_n)$ and $V_1(C_n)$ first.

Proposition 4.1 *Let* $u = (i,j) \in V(C_n)$.

(a) *If* $j \neq n$ *(i.e.* n *is unpainted), then*

(i) $d(u,v) \leq i(j-i)$ *if* $i-1 \leq (n-1)-j$, *where* $v = (j-i)$.

(ii) $d(u,w) \leq (n-j)(j-i)$ *if* $i-1 \geq (n-1)-j$, *where* $w = (n-j+i)$.

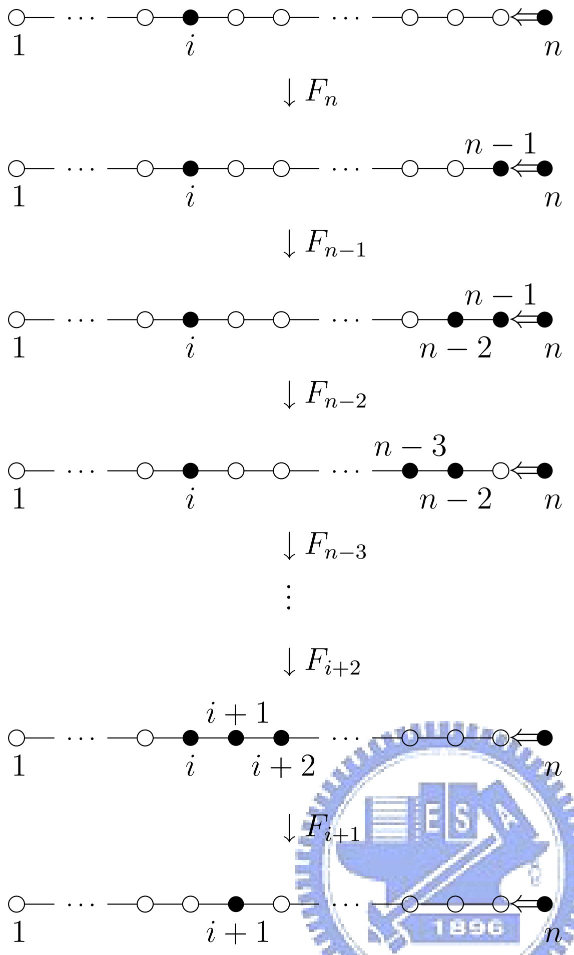
(b) *If* $j = n$ *(i.e.* n *is painted), then* $d(u,s) \leq \frac{(n-i)(n-i+1)}{2}$ *where* $s = (n)$.

Proof :

(a) If n is unpainted and we cannot use any method to make it become painted, then we can ignore the existence of n or just regard this Vogan diagram as $V(A_{n-1})$. Recall the result of Proposition 2.1, then the proof of (i) and (ii) can be completed right away.

(b) If $j = n$, then $u = (i,n)$.

Since n is black, there are no ways to let it change color. Now, we want to move the first black vertex closer to n(in other words, move it in the right direction), or they are too far to get the major distance.



The graph says that $u = (i, n) \longrightarrow (i, n-1, n) \longrightarrow (i, n-2, n-1, n) \longrightarrow (i, n-3, n-2, n) \longrightarrow \cdots \longrightarrow (i, i+1, i+2, n) \longrightarrow (i+1, n)$ and it needs at most $n-(i+1)+1 = n-i$ steps.

By the same way, we can get that

$(i+1, n) \sim (i+2, n)$ needs at most $n-i-1$ steps.

$(i+2, n) \sim (i+3, n)$ needs at most $n-i-2$ steps.

$\vdots$

$(n-2, n) \sim (n-1, n)$ needs at most 2 steps.

$(n-1, n) \longrightarrow (n)$ needs 1 steps.

Let $s = (n)$ and thus $d(u, s) \leq (n-i) + (n-i-1) + (n-i-2) + \cdots + 2 + 1$

$$= \sum_{k=1}^{n-i} k = \frac{(n-i)(n-i-1)}{2}. \qquad \square$$

Therefore, we can conclude that:

Proposition 4.2 *Given* $u = (i_1, i_2, ..., i_k) \in V_k(C_n)$.

(a) *If* $i_k \neq n$, *then* $d_k(V_1(C_n)) \leq (k-1)(n-1)^2$.

(b) *If* $i_k = n$, *then* $d_k(V_1(C_n)) \leq \frac{(k-1)n^2}{2}$.

Proof :

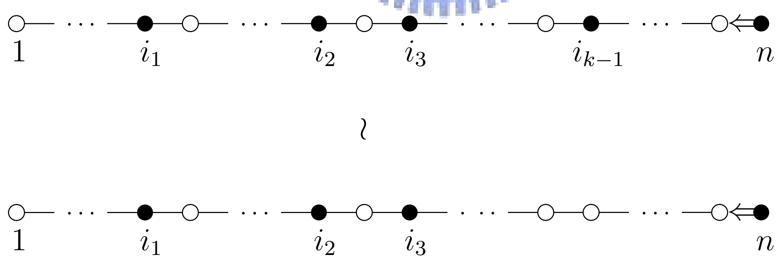
(a) If $i_k \neq n$, then $u \in V(A_{n-1})$. Now, there are two cases to discuss:

$i_1 - 1 \leq (n-1) - i_k$

$i_1 - 1 > (n-1) - i_k$

By Proposition 2.2, the result of (a) can be proved.

(b) First, we want to reduce the number of painted vertices step by step. With the same idea as Proposition 4.1(2), we move the first $k-1$ black vertices closer to n. Therefore, we just move i_{k-1} rightwards and make i_{k-1} and n reduce to one black vertex.



Let $S_p = \frac{1}{2}(n - i_{k-p})(n - i_{k-p} + 1)$ where $0 < p < k$. By Proposition 4.1(b),

$u = (i_1, i_2, \cdots, i_{k-1}, n) \sim (i_1, i_2, i_{k-2}, \cdots, n) \in V_{k-1}(C_n)$ needs at most S_1 steps.

Continuously,

$(i_1, i_2, \cdots, i_{k-2}, n) \sim (i_1, i_2, i_{k-3}, \cdots, n) \in V_{k-2}(C_n)$ needs at most S_2 steps.

$(i_1, i_2, \cdots, i_{k-3}, n) \sim (i_1, i_2, i_{k-4}, \cdots, n) \in V_{k-2}(C_n)$ needs at most S_3 steps.

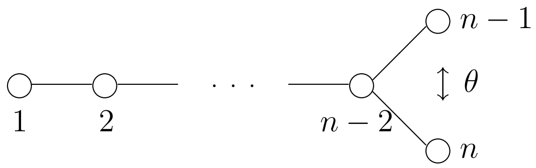
$$\vdots$$

$(i_1, n) \sim (n) \in V_1(C_n)$ needs at most S_{k-1} steps.
Then $S_1 \leq \frac{1}{2}n \cdot n = \frac{n^2}{2}$, $S_2 \leq \frac{n^2}{2}$, $S_3 \leq \frac{n^2}{2}$, ... , $S_{k-1} \leq \frac{n^2}{2}$ and thus $d_k(V_1(C_n)) = S_1 + S_2 + \cdots + S_{k-1} \leq \frac{n^2}{2}(k-1)$. $\square$

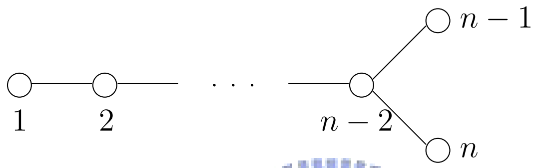
Consequently, we can get the result of Theorem 1 (c) immediately.

5 Type D

A Vogan diagram of D_n is a Dynkin diagram of D_n with a diagram involution θ, denote it by $V(D_n, \theta)$(see (5.1)). Besides, θ can be trivial and we also call it a Vogan diagram of D_n, denote it by $V(D_n)$ (see (5.2)). However the vertices $n-1$ and n fixed by θ can be painted.



(5.1)

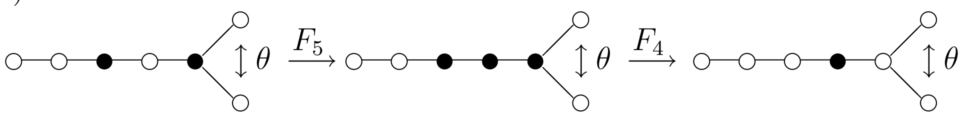


(5.2)

About (5.1) and (5.2), we give two examples to compare the difference between $V(D_n, \theta)$ and $V(D_n)$ if we use the same process.

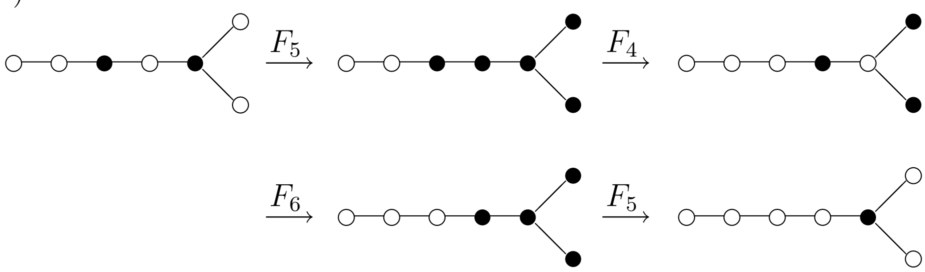
Example :

(1)



(i.e. $(3,5) \in V(D_7, \theta)$ and $(3,5) \sim (4)$ needs 2 steps.)

(2)



(i.e. $(3,5) \in V(D_7)$ and $(3,5) \sim (5)$ needs 4 steps.)

Recall that $V_k(D_n)$ means there are k black vertices in $V_k(D_n)$. From (1.2), it also extends to the distance between $V_k(D_n)$ and $V_1(D_n)$ by

$$d(V_k(D_n), V_1(D_n)) = \min\{d(v,w); v \in V_k(D_n), w \in V_1(D_n)\} \tag{5.3}$$

or just write $d_k(V_1(D_n))$. In this section, we provide an upper bound for the distance between $V_k(D_n)$ and $V_1(D_n)$. Now, we discuss $V(D_n,\theta)$ as follows.

Proposition 5.1 *Let $u = (i,j) \in V(D_n,\theta)$.*

(a) *If $i-1 \leq (n-2)-j$, then $d(u,v) \leq i(j-i)$ where $v = (j-i)$.*

(b) *If $i-1 > (n-2)-j$, then $d(u,w) \leq (n-j-1)(j-i)$ where $w = (n-j+i-1)$.*

Proof : Obviously, we can regard $V(D_n,\theta)$ as $V(A_{n-2})$ since $n-1$ and n have no colors. Then $u = (i,j) \in V(A_{n-2})$ and by Proposition 2.1, the results of (a) and (b) here will be proved. □

Therefore, we can conclude that:

Proposition 5.2 *Given $u = (i_1, i_2, ..., i_k) \in V_k(D_n,\theta)$, then $d_k(V_1(D_n,\theta)) \leq (k-1)\cdot(n-2)^2$.*

Proof : Similarly, $V(D_n,\theta)$ is equivalent to $V(A_{n-2})$. Extending the result of Proposition 2.2, we obtain that $d_k(V_1(D_n,\theta)) \leq (k-1)(n-2)^2$ right away. □

Consequently, we can get the result of Theorem 1 (e) right away. Continuously, we try to find the upper bound for $d(V_k(D_n), V_1(D_n))$ and then discuss $d_k(V_1(D_n))$.

Proposition 5.3 *Let $u = (i,j) \in V(D_n)$.*

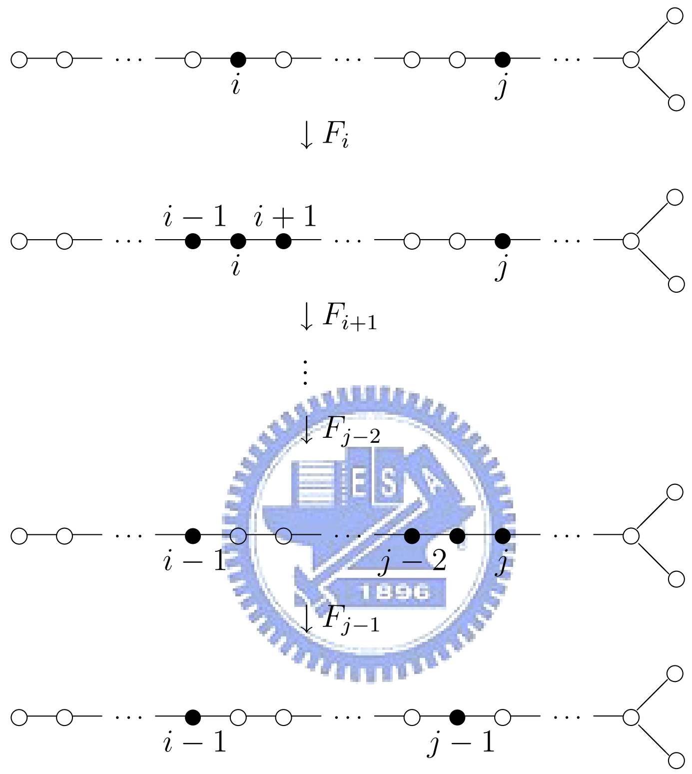
(a) *If $j \notin \{n-1, n\}$, then $d(u,v) \leq i(j-i)$ where $v = (j-i)$.*

(b) *If $i \notin \{n-1,n\}$ and $j \in \{n-1,n\}$, then $d(u,w) \leq \frac{(n-i)(n-i-1)}{2}$ where $w = (n-1)$ or (n).*

(c) *If $u = (n-1,n)$ then $d(u,p) \leq n-1$ where $p = (1)$.*

Proof :

(a) Suppose that $j \notin \{n-1, n\}$. Then $i, j \in \{1, 2, ..., n-2\}$. Now, we want to reduce u to be a Vogan diagram with one painted vertex. If we move these two black vertices toward right, then vertices $n-1$ and n will become painted and the path to a Vogan diagram of $V_1(D_n)$ must be more complicated. Hence we move black vertices toward left.



It means that $u = (i, j) \sim (i-1, j-1)$ needs at most $(j-i)$ steps.

Continuously,

$(i-1, j-1) \sim (i-2, j-2)$ needs at most $(j-i)$ steps.

$(i-2, j-2) \sim (i-3, j-3)$ needs at most $(j-i)$ steps.

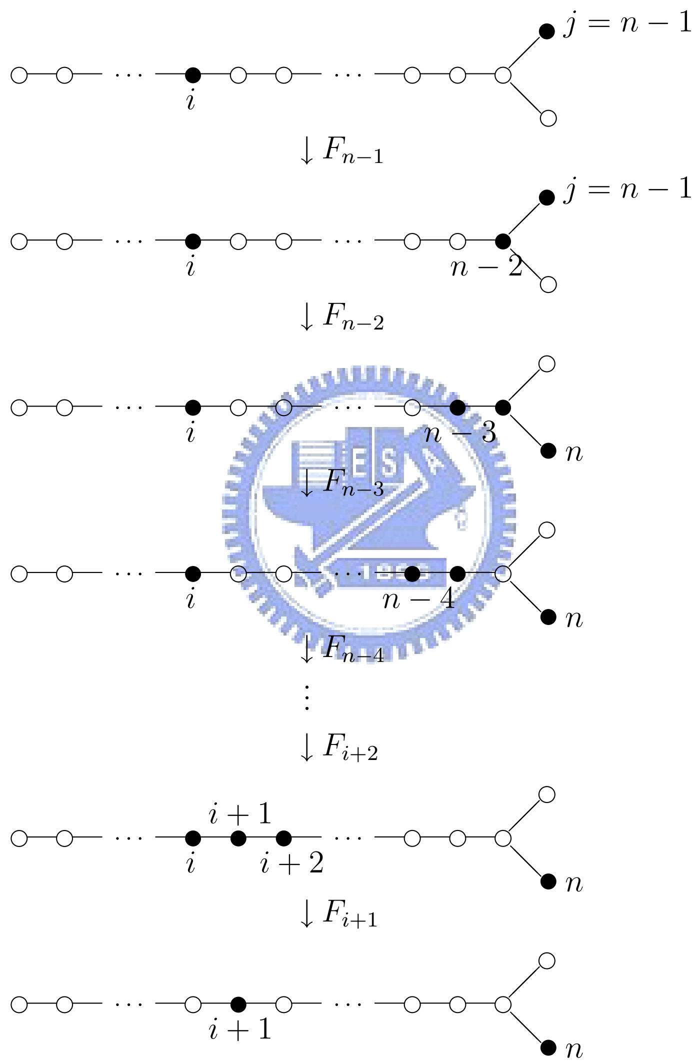
$\vdots$

$(1, j-i+1) \sim (j-i)$ needs at most $(j-i)$ steps.

So, there are at most $i(j-i)$ steps to make $(i, j) \sim (j-i)$; in other words, $d(u, v) \leq i(j-i)$ where $v = (j-i)$.

(b) Suppose that $i \notin \{n-1, n\}$ and $j \notin \{n-1, n\}$. Then $u = (i, n-1)$ or $u = (i, n)$, $0 < i < n-1$. We only talk about $u = (i, n-1)$ because (i, n) is equivalent to $(i, n-1)$, and the conclusions are the same.

Now, we want to move these two black vertices closer and u will be reduced to a Vogan diagram of $V_1(D_n)$ as soon as possible.



We get $(i, n-1) \sim (i+1, n)$ needs at most $(n-1) - (i-1) + 1 = n - i - 1$ steps. By the same way, we also get

$(i+1,n) \sim (i+2,n-1)$ needs at most $n-i-2$ steps.

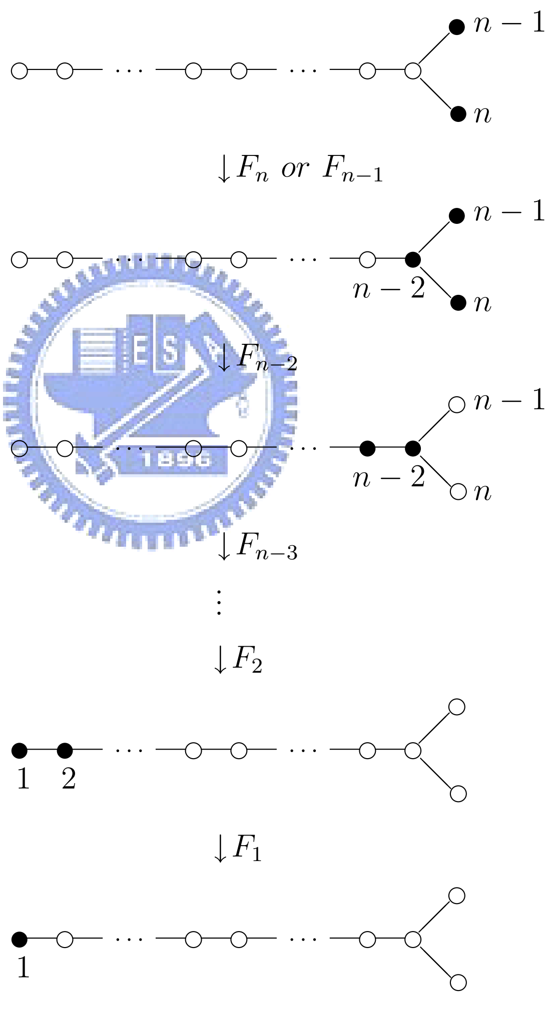
$(i+2,n-1) \sim (i+3,n)$ needs at most $n-i-3$ steps.

$\vdots$

$(n-2,n-1) \sim (n)$ or $(n-2,n) \sim (n-1)$ needs at most 1 steps.

Thus, $d(u,w) \leq 1+2+\cdots+(n-i-1) = \frac{(n-i)(n-i-1)}{2}$ where $w=(n)$ or $(n-1)$.

(c) We have known that vertices $n-1$ and n are painted. The only way to let u be a diagram with one painted vertex is moving them in the left direction.



From the above graph, it tells us that $(n-1,n) \sim (1)$ needs at most $(n-1)-1+1$ steps, i.e. $d(u,p) \leq n-1$ where $p=(1)$. $\square$

Therefore, we can conclude that:

Proposition 5.4 *Given $u = (i_1, i_2, ..., i_k) \in V_k(D_n)$.*

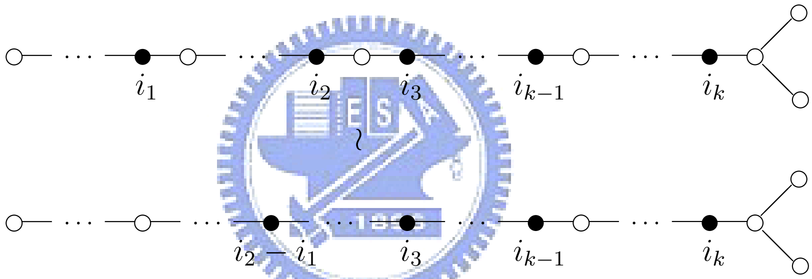
(a) *If $n-1, n \notin u$, then $d_k(V_1(D_n)) \leq (k-1)(n-2)^2$.*

(b) *If $n-1 \in u$ or $n \in u$, then $d_k(V_1(D_n)) \leq \frac{(k-1)(n-1)^2}{2}$.*

(c) *If $n-1, n \in u$, then $d_k(V_1(D_n)) \leq (k-3)(n-2)^2 + (n-2)$.*

Proof :

(a) Suppose that $1 \leq i_1, i_2, \cdots, i_k \leq n-2$ (i.e. vertices $n-1$ and n are unpainted). Looking back the method of Proposition 5.3(a), we just try to move the leftmost two black vertices toward left and then become one black vertex. Step by step, we can finally get only one black vertex in this Vogan diagram.



Let $R_m = (i_m - i_{m-1} + i_{m-2} - ... \pm i_1)(i_{m+1} - i_m + i_{m-1} - i_{m-2} + ... \mp i_1)$ where $0 < m < k$. By Proposition 5.3(a), $u \sim (i_2 - i_1, i_3, ..., i_k) \in V_{k-1}(D_n)$ needs at most R_1 steps. Continuing the same way,

$(i_2 - i_1, i_3, ..., i_k) \sim (i_3 - i_2 + i_1, i_4, ..., i_k) \in V_{k-2}(D_n)$ needs at most R_2 steps.

$\vdots$

$(i_{k-1} - i_{k-2} + ... \pm i_1, i_k) \sim (i_k - i_{k-1} + ... \mp i_1) \in V_1(D_n)$ needs at most R_{k-1} steps.

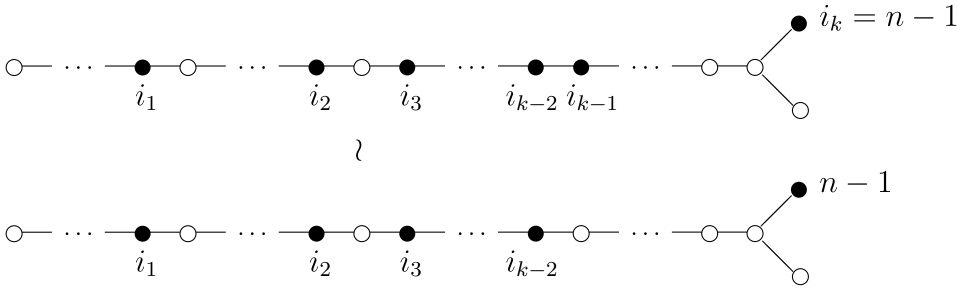
Then $R_1 \leq i_2 \cdot i_2 = {i_2}^2 \leq (n-2)^2$, $R_2 \leq i_3 \cdot i_3 = {i_3}^2 \leq (n-2)^2, \cdots$, $R_{k-1} \leq i_k \cdot i_k = {i_k}^2 \leq (n-2)^2$.

Thus $d(u, r) = R_1 + R_2 + ... + R_{k-1} \leq (n-2)^2 + (n-2)^2 + ... + (n-2)^2 = (k-1)(n-2)^2$, where $r = (i_k - i_{k-1} + ... \mp i_1) \in V_1(D_n)$.

(b) If $n-1 \in u$ or $n \in u$, then one of vertices $n-1$ and $n-1$ is painted (i.e. $i_k = n-1$ or $i_k = n$, but $i_{k-1} \neq n-1$).

Similar to the process of Proposition 5.3(b), we try to move the black vertex i_{k-1} towards the right. Finally, these two vertex will be reduced to a black vertex.

Here we just talk about $i_k = n-1$ because its result is the same as $i_k = n$.



Note that the rightmost black vertex could be $n-1$ or n, but it does not affect the result. Let

$$S_p = \frac{1}{2}(n - i_{k-p})(n - i_{k-p} - 1), \ 1 \leq p \leq k-1$$

. By Proposition 5.3(b),

$u \sim (i_1, i_2, ..., i_{k-2}, n-1)$ needs at most S_1 steps. With the same method,

$(i_1, i_2, ..., i_{k-2}, n-1) \sim (i_1, i_2, ..., i_{k-3}, n-1)$ needs at most S_2 steps.

$(i_1, i_2, ..., i_{k-3}, n-1) \sim (i_1, i_2, ..., i_{k-4}, n-1)$ needs at most S_3 steps.

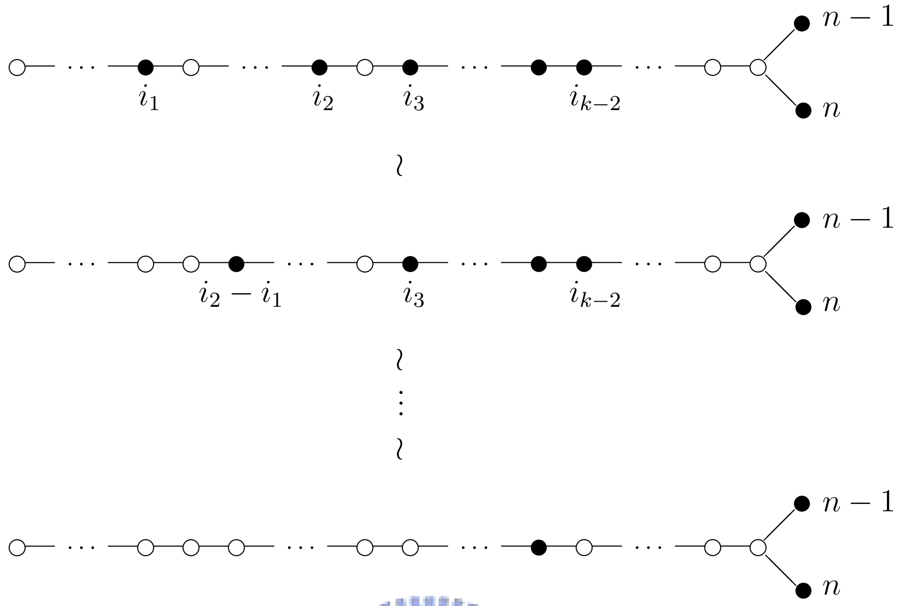
$\vdots$

$(i_1, n-1) \sim (n-1)$ needs at most S_{k-1} steps.

We find that $S_1 \leq \frac{1}{2}(n-2)(n-1)$, $S_2 \leq \frac{1}{2}(n-2)(n-1)$, $\cdots$, $S_{k-1} \leq \frac{1}{2}(n-2)(n-1)$. Thus $d(u, w) = S_1 + S_2 + \cdots + S_{k-1} \leq \frac{1}{2}(k-1)(n-2)(n-1)$, where $w = (n-1)$ or (n) and this proof is completed.

(c) Let $i_{k-1} = n-1$ and $i_k = n$, i.e. $u = (i_1, i_2, ..., i_{k-2}, n-1, n)$. Regardless of vertices $n-1$ and n, we hope to reduce the $k-2$ black vertices in $1, \cdots, n-1$ to one black vertex. Then adding in vertices $n-1$ and n, we finally make these three black vertices become only one black vertex.

Referring to the method of (a), we move the leftmost two painted vertices towards the left.



Let $M_s = (i_s - i_{s-1} + \cdots \pm i_1)(i_{s+1} - i_s + i_{s-1} - \cdots \mp i_1)$ where $1 \leq s \leq k-3$. By Proposition 5.3(a), $u \sim (i_2 - i_1, i_3, \ldots, n-1, n)$ needs at most M_1 steps. Keep on the same way:

$(i_2 - i_1, i_3, \ldots, n-1, n) \sim (i_3 - i_2 + i_1, i_4, \ldots, n-1, n)$ needs at most M_2 steps.

$(i_3 - i_2 + i_1, i_4, \ldots, n-1, n) \sim (i_4 - i_3 + i_2 - i_1, i_5, \ldots, n-1, n)$ needs at most M_3 steps.

$\vdots$

$(i_{k-3} - i_{k-4} + ... \pm i_1, i_{k-2}, n-1, n) \sim (i_{k-2} - i_{k-3} + ... \mp i_1, n-1, n)$ needs at most M_{k-3} steps.

Let $a = i_{k-2} - i_{k-3} + ... \mp i_1$ and the final step is to find vertex i such that $(a, n-1, n) \sim (i)$:

$(a, n-1, n) \longrightarrow (a, n-2, n-1, n) \longrightarrow (a, n-3, n-2) \longrightarrow (a, n-4, n-3) \longrightarrow \cdots \longrightarrow (a, a+1, a+2) \longrightarrow (a+1)$ needs at most $(n-a-1)$ steps and $i = a+1$.

We have known that $1 \leq a \leq n-2$, then $1 \leq n-a-1 \leq n-2$.

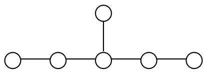
$d_k(V_1(D_n)) \leq M_1 + M_2 + \cdots + M_{k-3} + (n-a-1)$

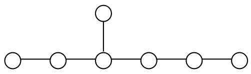
$$
\begin{aligned}
&\leq i_1 \cdot i_2 + i_2 \cdot i_3 + \cdots + i_{k-3} \cdot i_{k-2} + (n-a-1) \\
&\leq (n-2)^2 + (n-2)^2 + \cdots + (n-2) \\
&= (k-3)(n-2)^2 + (n-2). \qquad \square
\end{aligned}
$$

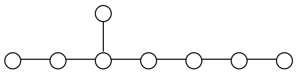
Consequently, we can get the result of Theorem 1 (d) immediately.

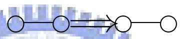
6 Type $\mathbf{E_6}$

After observing $A_n,\ B_n,\ C_n$ and D_n, we talk about other different type: $E_6,\ E_7,\ E_8,\ F_4$ and G_2.

$\mathbf{E_6}$ 

$\mathbf{E_7}$ 

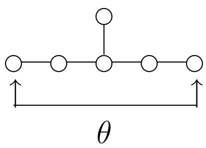
$\mathbf{E_8}$ 

$\mathbf{F_4}$ 

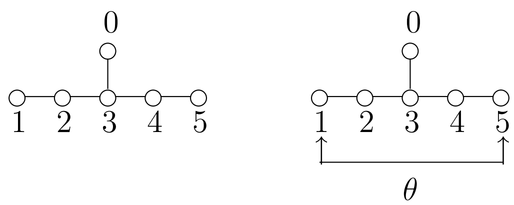
$\mathbf{G_2}$ 

(6.1)

This section will deal with the upper bound for the distance between $V_k(E_6)$ and $V_1(E_6)$. Others are discussed later. Besides (6.1), there is another type of E_6, unfixed by θ, denoted by $V(E_6, \theta)$.



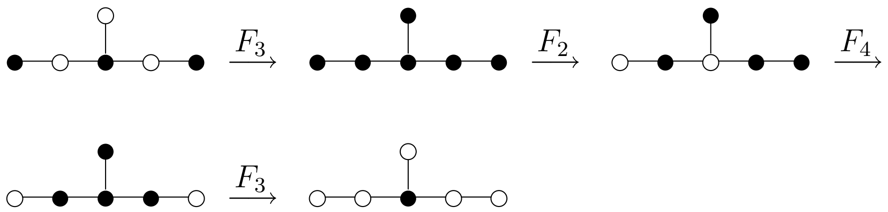
We label the vertices in $V(E_6)$ and $V(E_6, \theta)$ as follows.



We give an example about $V(E_6)$ and $V(E_6, \theta)$ on how to reduce some black vertices to only one black vertex.

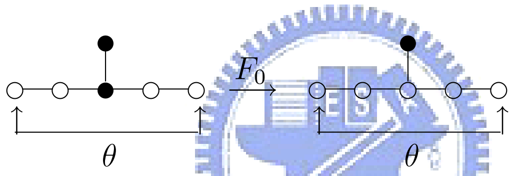
Example :

$V(E_6)$:



i.e. (1,3,5)$\longrightarrow$(0,1,2,3,4,5)$\longrightarrow$(0,2,4,5)$\longrightarrow$(0,2,3,4)$\longrightarrow$(3)

$V(E_6, \theta)$:



i.e. (0,3)$\longrightarrow$(0)

Indeed, there are two vertices in $V(E_6, \theta)$ can be drawn color, vertex 0 and vertex 3, just like the vertices n and $n-1$ in $V(D_n, \theta)$.

Recall that

$$d_k(V_1(E_6)) = d(V_k(E_6), d_1(E_6)) = min\{d(u,v); u \in V_k(E_6), v \in V_1(E_6)\}$$

Proposition 6.1 $d_k(V_1(E_6)) \leq 5(5k-4)$ *where* $1 \leq k \leq 6$.

Proof : Separate E_6 into A_5 and a vertex 0. Regardless of vertex 0, we reduce other black vertices to be one black vertex in $V(A_5)$. There are 4 cases to discuss.

Case 1 Consider vertex 0 is white and all black vertices are in vertex 1 to 3 or vertex 3 to 5. Then vertex 0 will not become painted. By Proposition 2.2, we can get the distance between $V_k(E_6)$ to $V_1(E_6)$ directly. $d_k(V_1(E_6)) = d_k(V_1(A_5)) \leq 5^2(k-1) = 25(k-1)$.

Case 2 Consider vertex 0 is black and all black vertices are in vertex 1 to 3 or vertex 3 to 5. Then vertex 0 will not become unpainted. We arrange the black vertices in $V(A_5)$ to be singly white and finally deal with vertex 0. We have to discuss the following cases.

(i) $u \sim (0,1)$ and $(0,1) \longrightarrow (0,1,2) \longrightarrow (0,2,3) \longrightarrow (3,4) \longrightarrow (4,5) \longrightarrow (5)$. By Proposition 2.2, we have known that $d_k(V_1(A_5)) \leq 5^2(k-1) = 25(k-1)$. Then $d(u,(5)) \leq 25(k-1)+5 = 25k-20$.

(ii) $u \sim (0,2)$ and $(0,2) \longrightarrow (0,2,3) \longrightarrow (3,4) \longrightarrow (4,5) \longrightarrow (5)$.
Then $d(u,(5)) \leq 25(k-1)+4 = 25k-21$.

(iii) $u \sim (0,3)$ and $(0,3) \longrightarrow (0)$. Then $d(u,(5)) \leq 25(k-1)+1 = 25k-24$.

We don't handle with $(0,4)$ and $(0,5)$ because they are equivalent to $(0,2)$ and $(0,1)$. Thus, $d_k(V_1(E_6)) \leq \max\{25k-20, 25k-21, 25k-24\} = 25k-20$ where $1 \leq k \leq 6$.

Case 3 Consider vertex 0 is white. Some black vertices are in vertex 1 to 3 and others are in vertex 3 to 5. Then the process of becoming singly painted will affect the color of vertex 0. We let vertex 0 be black and use the same method as Case 2. Similarly, there are 3 cases to discuss and thus $d_k(V_1(E_6)) \leq 25k-20$ where $1 \leq k \leq 5$.

Case 4 Consider vertex 0 is black. Some black vertices are in vertex 1 to 3 and others are in vertex 3 to 5. The color of vertex 0 will be reversed in the process of reducing to singly painted vertex. We let vertex 0 be white and regard $V(E_7)$ as $V(A_6)$. Thus, $d_k(V_1(E_6)) = d_k(V_1(A_5)) \leq 25(k-1) = 25k-25$ where $1 \leq k \leq 6$.

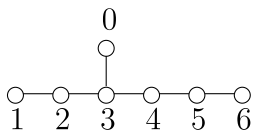
Arrange above 4 cases, we get that $d_k(V_1(E_6)) \leq 25k-20 = 5(5k-4)$ where $1 \leq k \leq 6$. We complete the proof of Theorem 2(a). $\square$

Proposition 6.2 $d_k(V_1(E_6, \theta)) \leq 1$ *where* $k = 2$*.*

Proof : Since vertices 1, 2, 4 and 5 are unfixed by θ, we do not have to paint any color on them. In other words, only vertices 0 and 3 can have color. Hence, the only one case is like the above example, (0,3)$\longrightarrow$(0), needs only one step. Therefore, we get the proof of Theorem 2(b) immediately. $\square$

7 Type $\mathbf{E_7}$

By (6.1), we also label 0,1,...,7 to the Vogan diagram of E_7.



The process of making some black vertices to be singly painted is similar to $V(E_6)$. Now, we discuss how many steps that $V_k(E_7)$ goes to $V_1(E_7)$ at most.

Proposition 7.1 *Given* $u = (i_1, i_2, ..., i_k) \in V_k(E_7)$, $d_k(V_1(E_7)) \leq 12(3k-2)$ *where* $1 \leq k \leq 7$.

Proof : Separate E_7 into A_6 and a vertex 0. We try to discuss the distance between k painted vertices and only one painted vertex by the color of vertex 0.

Case 1 Suppose that vertex 0 is white and all black parts are in vertices 1 to 3 or vertices 3 to 6. Then the process of reducing to only one black vertex cannot affect the color of vertex 0. Thus we just regard $V(E_7)$ as $V(A_6)$ and by Proposition 2.2, $d_k(V_1(E_7)) = d_k(V_1(A_6)) \leq 6^2(k-1) = 36k - 36$.

Case 2 Suppose that vertex 0 is black and other black parts are in vertices 1 to 3 or vertices 3 to 6. Then the process of reducing to only one black vertex cannot affect the color of vertex 0. We make these painted vertices in $V(A_6)$ become one painted and then act on vertex 0. There are 6 cases to be discussed.

(i) $u \sim (0,1)$ and $(0,1) \to (0,1,2) \to (0,2,3) \to (3,4) \to (4,5) \to (5,6) \to (6)$.

By Proposition 2.2, we have known that $d_k(V_1(A_6)) \leq 6^2(k-1) = 36(k-1)$.

Then $d(u,(6)) \leq 36(k-1) + 6 = 36k - 30$.

(ii) $u \sim (0,2)$ and $(0,2) \to (0,2,3) \to (3,4) \to (4,5) \to (5,6) \to (6)$.

Then $d(u,(6)) \leq 36(k-1) + 5 = 36k - 31$.

(iii) $u \sim (0,3)$ and $(0,3) \to (0)$.

Then $d(u,(0)) \leq 36(k-1) + 1 = 36k - 35$.

(iv) $u \sim (0,4)$ and $(0,4) \to (0,3,4) \to (2,3) \to (1,2) \to (1)$.

Then $d(u,(1)) \leq 36(k-1)+4 = 36k-32$.

(v) $u \sim (0,5)$ and $(0,5) \to (0,3,5) \to (2,3,4,5) \to (2,4) \to (1,2,3,4) \to (0,1,3) \to (0,1) \to (0,1,2) \to (0,2,3) \to (3,4) \to (4,5) \to (5,6) \to (6)$.

Then $d(u,(6)) \leq 36(k-1)+12 = 36k-24$.

(vi) $u \sim (0,6)$ and $(0,6) \to (0,5,6) \to (0,4,5) \to (0,3,4) \to (2,3) \to (1,2) \to (1)$.

Then $d(u,(1)) \leq 36(k-1)+6 = 36k-30$.

Thus, $d_k(V_1(E_7)) \leq \max\{36k-30, 36k-31, 36k-35, 36k-32, 36k-24, 36k-30\} = 36k-24$ where $1 \leq k \leq 7$.

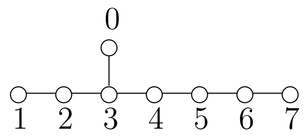
Case 3 Suppose that vertex 0 is white and some black vertices are in vertex 1 to 3 and the other black in vertex 3 to 6. The color of vertex 0 will be reversed in the process of reducing to singly painted vertex. We let vertex 0 be black and use the same method as **Case 2**. Similarly, there are 6 cases to discuss and thus $d_k(V_1(E_7)) \leq 36k-24$ where $1 \leq k \leq 6$.

Case 4 Suppose that vertex 0 is black and some black vertices are in vertex 1 to 3 and others are in vertex 3 to 6. The color of vertex 0 will be reversed in the process of reducing to singly painted vertex. We let vertex 0 be white and look $V(E_7)$ as $V(A_6)$. Thus, $d_k(V_1(E_7)) = d_k(V_1(A_6)) \leq 36(k-1)$.

Combining with **Case 1** to **Case 4**, we can get that $d_k(V_1(E_7)) \leq 36k-24 = 12(3k-2)$ where $1 \leq k \leq 7$. And we also finish the proof of Theorem 2(c). $\square$

8 Type $\mathbf{E_8}$

From the diagram of (6.1), we have known that what the Vogan diagram of E_8 looks like. Similarly, we give 8 numbers on E_8.



This section also talk about what is the maximal distance between $V_k(E_8)$ and $V_1(E_8)$.

Proposition 8.1 *Given* $u = (i_1, i_2, ..., i_k) \in V_k(E_8)$, *then* $d_k(V_1(E_8)) \leq 5(5k-4)$ *where* $1 \leq k \leq 8$.

Proof : Use the same way as Proposition 6.1 and Proposition 7.1, we also have to talk about 4 cases. As above, we just calculate **Case 2** and **Case 3** since they need more steps.

I. Suppose that vertex 0 is black and other black parts are in vertices 1 to 3 or vertices 3 to 7. Then the process of reducing to only one black vertex cannot affect the color of vertex 0. We make these painted vertices in $V(A_7)$ become one painted and then act on vertex 0. There are 7 cases to be discussed.

(i) $u \sim (0,1)$ and $(0,1) \to (0,1,2) \to (0,2,3) \to (3,4) \to (4,5) \to (5,6) \to (6,7) \to (7)$.

By Proposition 2.2, we have known that $d_k(V_1(A_7)) \leq 7^2(k-1) = 49(k-1)$.

Then $d(u,(7)) \leq 49(k-1)+7 = 49k-42$.

(ii) $u \sim (0,2)$ and $(0,2) \to (0,2,3) \to (3,4) \to (4,5) \to (5,6) \to (6,7) \to (7)$.

Then $d(u,(7)) \leq 49(k-1)+6 = 49k-43$.

(iii) $u \sim (0,3)$ and $(0,3) \to (0)$.

Then $d(u,(0)) \leq 49(k-1)+1 = 49k-48$.

(iv) $u \sim (0,4)$ and $(0,4) \to (0,3,4) \to (2,3) \to (1,2) \to (1)$.

Then $d(u,(1)) \leq 49(k-1)+4 = 49k-45$.

(v) $u \sim (0,5)$ and $(0,5) \to (0,3,5) \to (2,3,4,5) \to (2,4) \to (1,2,3,4) \to (0,1,3) \to (0,1) \to (0,1,2) \to (0,2,3) \to (3,4) \to (4,5) \to (5,6) \to (6,7) \to (7)$.

Then $d(u,(7)) \leq 49(k-1)+13 = 49k-36$.

(vi) $u \sim (0,6)$ and $(0,6) \to (0,3,6) \to (2,3,4,6) \to (2,4,5,6) \to (2,5) \to (1,2,3,5) \to (0,1,3,4,5) \to (0,1,4) \to (0,1,3,4) \to (1,2,3) \to (2)$.

Then $d(u,(1)) \leq 49(k-1)+10 = 49k-39$.

(vii) $u \sim (0,7)$ and $(0,7) \to (0,6,7) \to (0,5,6) \to (0,4,5) \to (0,3,4) \to (2,3) \to (1,2) \to (1)$.

Then $d(u,(1)) \leq 49(k-1)+7 = 49k-42$.

Thus, $d_k(V_1(E_8)) \leq \max\{49k-42, 49k-43, 49k-48, 49k-45, 49k-36, 49k-39, 49k-42\} = 49k-36$ where $1 \leq k \leq 8$.

II. Suppose that vertex 0 is white and some black vertices are in vertex 1 to 3 and the other black in vertex 3 to 7. The color of vertex 0 will be reversed in the process of reducing to singly painted vertex. We let vertex 0 be black and use the same method as **I**. Similarly, there are 7 cases to discuss and thus $d_k(V_1(E_8)) \leq 49k-36$ where $1 \leq k \leq 7$.

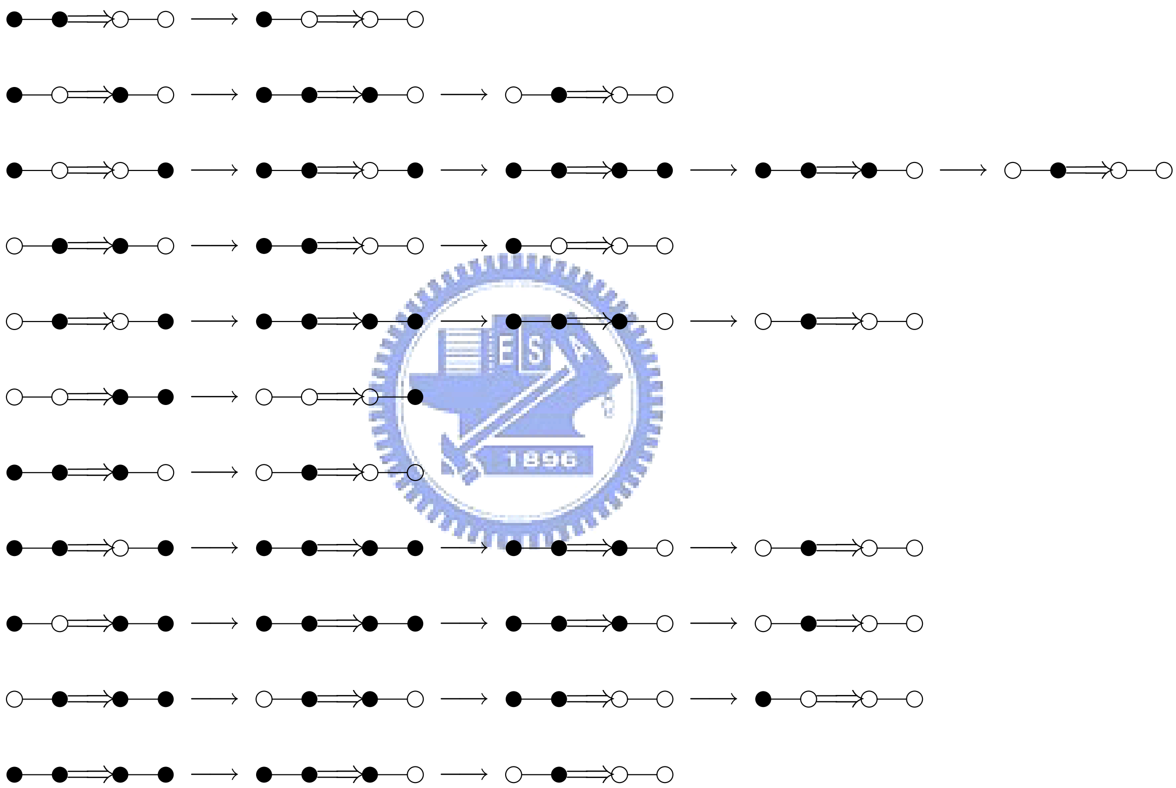
Combining with **I** and **II**, we can get that $d_k(V_1(E_8)) \leq 49k-36$ where $1 \leq k \leq 8$. And we also finish the proof of Theorem 2(d). $\square$

9 Type $\mathbf{F_4}$ and $\mathbf{G_2}$

Finally, we look the remaining Vogan diagram, F_4 and G_2 and observe that the upper bound for $d(V_k(F_4), V_1(F_4))$ $(k = 2, 3, 4)$ and $d(V_k(G_2, V_1(G_2))$.

Proposition 9.1 $d_k(V_1(F_4)) \leq 4$ *where* $k = 2, 3, 4$.

Proof : Since there are only 4 vertices in $V(F_4)$, we just use diagram to find the distance between $V_k(F_4)$ and $V_1(F_4)$, $k = 2, 3, 4$.



From the above process, we can find that it needs at most 4 steps to make a Vogan diagram with some black vertices reduce to only one black vertex. $\square$

Thus, Theorem 2(e) can be completed right away.

Proposition 9.2 $d_k(V_1(G_2)) \leq 1$ *where* $k = 2$.

Proof : $V(G_2)$ is graphed as $\circ\!\Longrightarrow\!\circ$ and there are at most 2 vertices to be painted. So, we just talk about the only one case, $(1, 2)$.

$$\bullet\!\Longrightarrow\!\bullet \longrightarrow \bullet\!\Longrightarrow\!\circ$$

Therefore, $d_2(V_1(G_2)) \leq 1$ and Theorem 2(f) is done. $\square$

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