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碩 士 論 文

ε 對於 q-KdV 階層系統一孤立子和二孤立子解的修正

The ε correction of one and two soliton solutions for the q-KdV hierarchy

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摘 要

在這篇論文裡透過使用 q-deformed 的 pseudodifferential 算子我們研究 Darboux -Backlund 轉換(DBTs)應用在 q-deformed Korteweg－de Vries 階層系統。算子 T 是由滿足特定線性系統的波函數所構成,有了這個 T 可以帶動 DBTs 的轉換。為了從舊的解去得到新的解 ，我們必須選擇一定特定的算子。反覆疊帶 DBTs 的轉換，我們獲得 one soliton 和 two solitons 的解。另外利用假設 q 趨近於一(這時 q-KdV 會回到原本的 KdV)和 ε 等於 q 減一的假設我們也算出 ε 對於 q-KdV 階層系統一孤立子和二孤立子解的修正 。

The ε correction of one and two soliton solutions for the q-KdV hierarchy

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Abstract

In this thesis we study Darboux -Backlund transformations (DBTs) for the q-deformed Korteweg -de Vries hierarchy by using the q-deformed pseudodifferential operators. The elementary DBTs are triggered by the gauge operators T constructed from the wave functions of the associated linear systems. In order to obtain the new solution from the old one, we have to choose certain gauge operator. Iterating these elementary DBTs, we obtain one and two solitons solutions. In addition we also figure out the ε correction of one and two solitons for the KdV hierarchy by letting $\varepsilon = q - 1$ and q approach 1(which will recovers q-KdV to the ordinary KdV).

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Contents

Chapter 1

Overview

1.1 introduction

There has been remarkable interest in the theory of solitons ever since the discovery of the Inverse Scattering Method for the Korteweg-de Vries (KdV) equation (see, for example, [1]) . Although the ISM has been extended to many nonlinear systems which describe phenomena in many branches of science, the KdV equation still plays an important role in the development of modern soliton theory (see, for example, [2]) . In particular, many concepts were established first for the KdV equation and then generalized to other systems in different ways. A convenient approach to formulate the KdV hierarchy relies on the use of fractional-power pseudo-differential operators associated with the scalar Lax operator $L = \partial^2 + u$ which was further generalized by Gelfand and Dickey [3] to the Nth

KdV hierarchy that has a Lax operator of the form$L_N = \partial^N + u_{N-1}\partial^{N-1} + ... + u_0$. During the past few years, several works have been published concerning the extensions of the KdV hierarchy in Lax formulation, such as Drinfeld-Skolov theory [4] and super-symmetric generalizations , etc. The common features of these extensions show that they preserve the integrable structure of the KdV hierarchy and contain the KdV hierarchy as reductions in some limiting cases. Recently, a new kind of extension called q-deformed KdV(q-Kdv) hierarchy has been aroused much interest in the literature [5-8]. In this extension, two different approaches are commonly used in the study of q-KdV. One of them proposed by Frenke [3] and the other by Khesin, Lyubashenko and Roger (KLR)[4] . Basically, the deformation is performed on the Lax formulation by introducing a parameter q The partial derivative is replaced by the q-deformed differential operator (q-DO) such that the deformed system recovers the ordinary KdV hierarchy as q goes to 1. So far, many integrable structures associated with q-KdV hierarchies have been investigated, such as infinite conservation law [1], bi-Hamiltonian structure [2,3] , Virasoro and W-algebras [2-5] , soliton solutions [5-7] , tau-functions [8] and Backlund - transformations [8] , tau-function , and etc..., here we focus on Darboux-Backlund transformations (DBTs) for this system are still unexplored. It is well known that the DBT is an important property for characterizing the integrability of the hierarchy [9]. Thus, it is worthwhile investigating the DBTs associated with the q-KdV hierarchy. Once this goal can be achieved, it will deepen our understanding of soliton solutions of the hierarchy.

1.2 history

When we first contact with soliton, we may ask what soliton is ? In fact, solitons are nonlinear waves. As a preliminary definition, a soliton is considered as solitary, traveling wave pulse solution of nonlinear partial differential equation (PDE). The nonlinearity will play a significant role. For most dispersive evolution equations these solitary waves would scatter inelastically and lose ’energy’ due to the radiation. Not so for the solitons: after a fully nonlinear interaction, the solitary waves remerge, retaining their identities with same speed and shape (figure.1).

The term ”soliton” was introduced in the 1960’s, but the scientific research of solitons had started in the 19th century, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery, as he described in his ”Report on Waves” [10].

Following this discovery, Scott Russell built a 30’ wave tank in his back garden and made further important observations of the properties of the solitary wave. Throughout his life Russell remained convinced that his solitary wave (the “Wave of Translation”) was of fundamental importance, It was not until the mid 1960’s when applied scientists began to use modern digital computers to study nonlinear wave propagation that the soundness of Russell's early ideas began to be appreciated. He viewed the solitary wave as a self-sufficient dynamic entity, a ”thing” displaying many properties of a particle. From the modern perspective it is used as a constructive element to formulate the complex dynamical behaviour of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, from the elementary particles of matter to the elementary particles of thought. Due to the work of Stokes, Boussinesq, Rayleigh, Korteweg, de Vries, and many others we know that the "great wave of translation" is a special form of a surface water wave. The equation describing the (unidirectional) propagation of waves on the surface of a shallow channel was derived by Korteweg and de Vries in 1895. After performing a Galilean and variety of scaling transformations, the KdV equation can be written in simplified form:

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1}$$

One soliton solution of this nonlinear PDE:

$$u(x,t) = 1/2c sech^2[1/2\sqrt{c}(x - ct + \delta)] \tag{1.2}$$

where c is the speed of the soliton and δ is the phase. This clearly represents the solitary wave observed by John Scott Russell and shows that the peak amplitude is exactely half the speed. Thus larger solitary waves have greater speeds. This suggest a numerical experiment: we start with two solitary wave solutions, with centers well separeted and different amplitude. Nowadays, many model equations of nonlinear phenomena are known to possess soliton solutions such as

$$Sine-Gordonequation: u_{tt} - u_{xx} + sinu = 0 \tag{1.3}$$

$$NonlinearSchrodingerequation: iu_t - u_{xx} + v|u|^2 u = 0 \tag{1.4}$$

$$KadomstevPetviashvili(KP)equation: (u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \tag{1.5}$$

All the equation above also has been remarkable interest in the theory of solitons.

1.3 background

In this thesis, the Darboux-Backlund transformations is investigated for the q-deformed Korteweg-de Vries Hierarchy. Here we adept 2th q-KdV hierarchy that has a Lax operator of the form $L_2 = \partial_q^2 + u_1\partial_q^1 + u_0$. Iterating these elementary DBTs, we obtain one and two soliton solutions [(u_0,u_1) and ($u_0^{(1)}$,$u_1^{(1)}$)]. We also figure out the ϵ correction of one and two solitons for the KdV hierarchy by letting $\epsilon = q - 1$ and $q \to 1$(which will recovers q-KdV to the ordinary KdV).

1.4 Outline of the Thesis

This thesis is organized as follows: In Chapter 2, we recall the basic facts concerning the q-deformed pseudodifferential operators (q-PDO) and define the Nth q-deformed Korteweg-de Vries (q-KdV) hierarchy. In Chapter 3, the concept of Darboux transformation and Lax equation are briefly introduced. We also construct the Darboux-Backlund transformations (DBTs) for the 2th q-KdV hierarchy, which preserve the form of the Lax operator and the hierarchy flows. Iteration of these DBTs generates the q-analogue of soliton solutions of the hierarchy. In Chapter 4, the case for $N = 2$ and $t = 1$(q-KdV hierarchy) is studied in detail to illustrate the q-deformed formulation including one and two solitons solutions and their ϵ correction for the KdV hierarchy also be showed. Concluding remarks are presented in Chapter 5.

Chapter 2

The Conception of q-KdV

2.1 Q-Deformed Pseudodifferential Operator

In this section we introduce some basic conception of q-deformed **KdV** hierarchy. There are much difference between KdV and q-KdV. For example, in q-deformed **KdV**, the partial derivative ∂_x is replaced by the q-deformed differential operator (q-DO) ∂_q such that

$$(\partial_q f(x)) = \frac{f(qx) - f(x)}{x(q-1)} \tag{2.1}$$

but, which will recover the ordinary differentiation $(\partial_x f(x))$ as q goes to 1 , We also let ∂_q^{-1} denote the formal inverse of ∂_q. Following we keep on introducing other useful formulae.

the q- shift operator is

$$\theta(f(x)) = f(qx) \tag{2.2}$$

the shift operation $\theta(f(x)) = f(qx)$ is just a scaling operation, which can be applied correctly to multiplication $\theta(f(x)g(x)) = f(qx)g(qx)$, and linear combination. Then it is easy to show that θ and ∂_q do not commute but satisfy

$$(\partial_q \theta^k(f)) \;=\; q^k \theta^k \,(\partial_q f) \quad k \in \mathbb{Z} \tag{2.3}$$

furthermore we have

$$(\partial_q(fg)) = (\partial_q f)g + \theta(f)(\partial_q g) \tag{2.4}$$

which means

$$\partial_q f = (\partial_q f) + \theta(f)\partial_q \tag{2.5}$$

In general the following q-deformed Leibnitz rule holds:

$$\partial_q^n \circ f = \sum_{k \geqq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q^k f)\partial_q^{n-k}, \; n \in \mathbb{Z} \tag{2.6}$$

where the q-number and the q-binomial are defined by

$$(n)_q = \frac{q^n - 1}{q - 1} \tag{2.7}$$

$$\binom{n}{k}_q = \frac{(n)_q\,(n-1)_q \ldots (n-k+1)_q}{(1)_q\,(2)_q \ldots (k)_q}, \quad \binom{n}{0}_q = 1 \tag{2.8}$$

where $\circ$ means composition of operators, defined by $\partial_q f = (\partial_q f) + \theta(f)\partial_q$. In the coming section, for any function f, will act as $\partial_q f = \partial_q(f)$. For a q-pseudodifferential operator of the form

$$P = \sum_{i=-\infty}^{n} u_i \partial_q^i \tag{2.9}$$

we can separate P into the differential part

$$P_+ = \sum_{i \geqslant 0} u_i \partial_q^i \tag{2.10}$$

and the integral part

$$P_- = \sum_{i \leqslant -1} u_i \partial_q^i \tag{2.11}$$

The conjugate operator '$*$' for P is defined by $P = \sum_{i=-\infty}^{n} (\partial_q^*)^i u_i$ with $\partial_q^* = -\partial_q \theta^{-1} = -\frac{1}{q}\partial_{\frac{1}{q}}, (\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta\partial_q^{-1}$. We can write out the several explicit form for q-derivative ∂_q as

$$\partial_q f = (\partial_q f) + \theta(f)\partial_q, \tag{2.12}$$

$$\partial_q^2 f = (\partial_q^2 f) + (q+1)\theta(\partial_q f)\partial_q + \theta^2(f)\partial_q^2, \tag{2.13}$$

$$\partial_q^3 f = (\partial_q^3 f) + (q^2+q+1)\theta(\partial_q^2 f)\partial_q + (q^2+q+1)\theta^2(\partial_q f)\partial_q^2 + \theta^3(f)\partial_q^3. \tag{2.14}$$

We also denote ∂_q^{-1} as the formal inverse of ∂_q, hence

$$\partial_q^{-1} f = \sum_{k \geqslant 0} (-1)^k q^{-k(k+1)/2} \theta^{-k-1}(\partial_q^k f)\partial_q^{-k-1} \tag{2.15}$$

In particular ∂_q^{-1} havs differnt form

$$\partial_q^{-1} f = \theta^{-1}(f)\partial_q^{-1} + \partial_q^{-1} \circ (\partial_q^* f) \circ \partial_q^{-1} \tag{2.16}$$

For a set of functions $f_1, f_2, ..., f_n$, we define the q-deformd Wronskian determinant

$W_q[f_1, f_2, ..., f_n]$ as

$$W_q[f_1, f_2, ..., f_n] = \begin{vmatrix} f_1 & ... & f_n \\ (\partial_q f_1) & ... & (\partial_q f_1) \\ \vdots & & \vdots \\ (\partial_q f_1) & ... & (\partial_q f_1) \end{vmatrix} \tag{2.17}$$

Finally, the q-exponent $E_q(x)$ is defined following representation:

$$E_q(x) = \exp(\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k) \tag{2.18}$$

the form $E_q(x)$ will play a crucial role in proving the existence of τ function of q-KdV hierarchy.

2.2 The q-KdV in Lax form

In fact, we can define the Nth $q-$KdV hierarchy in Lax form, as

$$\partial_{tn} L = [B_n, L], \quad n = 1, 2, 3, ... \tag{2.19}$$

with

$$L = \partial_q^N + u_{N-1}\partial_q^{N-1} + ... + u_0, \tag{2.20}$$

$$B_n = L_+^{n/N} \tag{2.21}$$

where the coefficients u_i are functions of the variables $(x, t_1, t_2, ...)$ but do not depend on$(t_N, t_{2N}, t_{3N}, ...)$.

In fact, we can rewrite the hierarchy equations (2.19) as:

$$\partial_{tm} B_n - \partial_{tn} B_m - [B_n, B_m] = 0, \tag{2.22}$$

which is called the zero-curvature condition and is equivalent to the whole set of equations of (2.19). If we can find a set of functions $\{u_i, i = 0, 1, ..., N-1\}$ and, hence, a corresponding Lax operator L (or $B_n)$ satisfying (2.19) or , then we have a solution to the Nth q-KdV hierarchy.

For the Lax operator (2.20), we can formally expand $L^{1/N}$ in powers of ∂_q as follows

$$L^{1/N} = \partial_q + s_0 + s_1 \partial_q^{-1} + ... \tag{2.23}$$

such that $(L^{1/N})^N = L$, which gives all the s_i being q-deformed differential polynomials in $\{u_i\}$. Especially, for the coefficient of ∂_q^{N-1}, we have

$$u_{N-1} = s_0 + \theta(s_0) + ... + \theta^{N-1}(s_0). \tag{2.24}$$

The Lax equations (2.19) can be viewed as the compatibility condition of the linear system

$$L\phi = \lambda\phi \tag{2.25}$$

$$\partial_{tn}\, \phi = (B_n \phi), \tag{2.26}$$

where ϕ and λ are called the wave function and eigenvalue of the linear system, respectively. Our purpose is to solve u_0 and u_1 of the $q-KdV$ equation, but it is really difficult to achieve, hence we start doing linear system. Wave function is just assist when we perform DBT . In order to keep the linear system unchanged Lax operator and wave-function have to change at the same time under DBT.

Chapter 3

Elementary Darboux Transformation for the q-KdV

3.1 Darboux Transformations and Linear Equations

In this section we talk about some basic conceptions of **Darboux** transformation[11]. Consider the **Sturm-Liouville** equation

$$-\Psi_{xx} + u\Psi = \lambda\Psi \tag{3.1}$$

we denote the fixed solution of (3.1) taken at the point $\lambda = \lambda_1$ by $\Psi_1, \Psi_1 = \Psi_1(x, \lambda_1)$ and define σ_1 to be the logarithmic derivative of Ψ_1 that means $\sigma_1 = \Psi_{1x}\Psi_1^{-1}$. Now the

Darboux transformation(DT) $\Psi \to \Psi[1]$ of the arbitrary solution of (3.1) is defined by

$$\Psi[1] = (\frac{d}{dx} - \sigma_1)\Psi = \Psi_x - \frac{\Psi_{1x}}{\Psi_1}\Psi = \frac{W(\Psi_1, \Psi)}{\Psi_1}, \tag{3.2}$$

where $W(\Psi_1, \Psi) = \Psi_1\Psi_x - \Psi_{1x}\Psi$ is the usual **Wronskian** determinant. It can be proved that the function $\Psi[1]$ satisfies the differential equation

$$-\Psi_{xx}[1] + u[1]\Psi[1] = \lambda\Psi[1] \tag{3.3}$$

$$u[1] = u - 2\sigma_{1x} = u - 2\frac{d^2}{dx^2}\ln\Psi_1 \tag{3.4}$$

In other words **Darboux**'s theorem declares the **Sturm-Liouville** equation (3.1) is covariant with respect to the action of **Darboux** transformation

$$\Psi \to \Psi[1], \ u \to u[1]$$

The important of the **Darboux** theorem lies in the possibility of obtaining another solvable equation (3.3) starting from the solvable equation (3.1). This will be illustrated by several examples. In fact, by varying Ψ in (3.2) we recover all the solutions of (3.3). This possibility leads to nontrivial resuls even in the simplest case of $u = 0$.

3.2 Darboux Transformation for the q-KdV

In this section, we would like to construct **Darboux** transform for the Nth q-**KdV** hierarchy. To attain this purpose, let us consider the following transformation. Suppose T is pseudo-differential operator, and

$$L \to L^{(1)} = TLT^{-1}, B_n^{(1)} = (L^{(1)})_+^{n/N} \tag{3.5}$$

so that

$$\frac{\partial}{\partial t_n} L^{(1)} = [B_n^{(1)}, L^{(1)}] \tag{3.6}$$

still holds for transformations Lax operator $L^{(1)}$, then T is called a gauge transformation operator of the q-KdV hierarchy.

Theorem 3.1 The operator T is a gauge transformation, if

$$(T \circ B_n \circ T^{-1})_+ = T \circ B_n \circ T^{-1} + \frac{\partial T}{\partial t_n} \circ T^{-1} \tag{3.7}$$

where T is any reasonable q-**PDO** and T^{-1} denotes its inverse. In order to obtain the new solution $(L^{(1)})$ from the old one (L) , the gauge operator T cannot be arbitrarily chosen. It should be constructed in such a way that the transformed Lax operator $L^{(1)}$ preserves the form of L and satisfies the Lax equation (2.19). From the zero curvature condition (2.22) point of view, the operator B_n should be transformed according to

$$B_n \to B_n^{(1)} = TB_nT^{-1} + \partial_{tn}TT^{-1} \tag{3.8}$$

which will, in general, not be a pure q-**DO** although the B_n does. However, if we choose the gauge operator T such that $B_n^{(1)}$, as defined by (3.2), is also a purely q-**DO**, then $B_n^{(1)}$ represents a valid new solution to the Nth q-KdV hierarchy. This is the goal we want to achieve in this thesis. Motivated by the $DBTs$ for the ordinary **KdV** [12] (or Kadomtsev-Petviashvili (KP) [13-14]) hierarchy, we can construct a qualified gauge operator T as follows:

$$T_1 = \theta(\phi_1)\partial_q \phi_1^{-1} = \partial_q - \alpha_1, \tag{3.9}$$

where

$$\alpha_1 = \frac{(\partial_q \phi_1)}{\phi_1}, \tag{3.10}$$

where ϕ_1 is a wave function associated with the linear system (2.24).Under the gauge transformation T_1 , new "eigenfunction" can be got by the way

$$\phi_2^{(1)} = T_1(\phi_2) = (\partial_q \phi_2) - \alpha_1 \phi_2 \tag{3.11}$$

It is not hard to show that the transformed Lax operator $L^{(1)}$is a purely q-**DO** with order N and the Lax equation (2.19) transforms covariantly, i.e. $\partial_{tn} L^{(1)} = [(L^{(1)})_+^{n/N}, L^{(1)}]$. The transformed coefficients $\{u_i^{(1)}\}$ can then be expressed in terms of $\{u_i\}$ and ϕ_1. On the other hand, for a given generic wave function $\phi \neq \phi_1$ its transformed result can be expressed in terms of ϕ_1 and itself.

Theorem 3.2 Under the gauge transformation L$^{(1)} = T(\phi_1) \circ L \circ T^{-1}(\phi_1),$ new "eigenfunction" of the $L^{(1)}$ is

$$\phi \rightarrow \phi^{(1)} = (T(\phi_1) \cdot \phi) = \frac{W_q[\phi_1, \phi]}{\phi_1} \tag{3.12}$$

Then following previous processes we can apply N times iterations of the **DBTs** by using the **DBT**, triggered by the gauge operators T. For example, by iterating the **DBT** triggeredby the gauge operator T, we can express the solution of the Nth q-**KdV** hierarchy through the q-deformed Wronskian representation. This construction starts with n wave functions $\phi_1, \phi_2, ...\phi_n$ of the linear system (2.25) and (2.26). Using ϕ_1, say, to perform the first **DBT**, then all ϕ_i are transformed to $\phi_i^{(1)} = (T_1\phi_i)$. Obviously, we have $\phi_1^{(1)} = 0$. The next step is to perform a subsequent **DBT** triggered by $\phi_2^{(1)}$, which leads to the new wave functions $\phi_i^{(2)}$ with $\phi_2^{(2)} = 0$. Iterating this process such that all the wave functions are exhausted, then an n-step DBT triggered by the gauge operator $T_n = (\partial_q - \alpha_n^{(n-1)})(\partial_q - \alpha_{n-1}^{(n-2)}) \cdots (\partial_q - \alpha_1)$ is obtained, where $\alpha_i^{(j)} \equiv (\partial_q \phi_i^{(j)})/\phi_i^{(j)}$.

It is easy to see that T_n is an nth-order q-**DO** of the form $T_n = \partial_q^n + a_{n-1}\partial_q^{n-1} + \cdots a_0$ with a_i defined by the conditions $(T_n\phi_j) = 0, \quad j = 1, 2, ..., n$. Following Cramer's formula, it turns out that $a_i = -W_q^{(i)}[\phi_1, \phi_2, ..., \phi_n]/W_q[\phi_1, \phi_2, ..., \phi_n]$ where $W_q^{(i)}$ is obtained from W_q with its $i-$th row replaced by $(\partial_q^n\phi_1), ..., (\partial_q^n\phi_n)$.This implies that the n-step transformed wave fuction $\phi^{(n)}(\phi \neq \phi_1)$ is given by

$$\phi^{(n)} = (T_n\phi) = \frac{W_q[\phi_1, \phi_2, ..., \phi_n, \phi]}{W_q[\phi_1, \phi_2, ..., \phi_n]} \tag{3.13}$$

and the n-step gauge operator $T^{(n)}$ can be expression as

$$T^{(n)} = \frac{1}{W_q[\phi_1, \phi_2, ..., \phi_n]} \begin{vmatrix} \phi_1 & ... & \phi_n & 1 \\ (\partial_q \phi_1) & ... & (\partial_q^n \phi_n) & \partial_q \\ \vdots & & \vdots & \vdots \\ (\partial_q^n \phi_1) & ... & (\partial_q^n \phi_n) & \partial_q^n \end{vmatrix} \tag{3.14}$$

where it should be realized that in the expansion of the determinant by the elements of the last column, ∂_i^q have to be written to the right of the minors. Furthermore, $L\phi = \lambda\phi$ holds automatically under the DBT we constructed. In the next section we will construct nontrivial from trivial one.

Chapter 4

The ϵ correction of one of one and two solitons

This chapter is devoted to illustrating the DBTs for the case: q-deformed KdV hierarchy N= 2 and t= 1 . First, construct from the trivial solution (that means $u_1 = u_0 = 0$)we will get one and two soliton solutions of q-KdV equation. Finally, in the end of this chapter we figure out the ϵ correction of one and two solitons.

4.1 One soliton solution of q-KdV

First, for N= 2 and t= 1 we let

$$L = \partial_q^2 + u_1\partial_q + u_0 \tag{4.1}$$

and the Lax equations

$$\partial_{t1} L = [L_+^{n/2}, L], \tag{4.2}$$

define the evolution equations for u_1 and u_0. In particular, for the $t1-$ flow, we have

$$\partial_{t1} u_1 = x(q-1)\partial_{t1} u_0 \tag{4.3}$$

$$\partial_{t1} u_0 = (\partial_q u_0) - (\partial_q^2 s_0) - (\partial_q s_0^2) \tag{4.4}$$

which is nontrivial and recovers the ordinary case as q goes to 1. For higher hierarchy flows, the evolution equations for u_1 and u_0 become more complicated due to the noncommutative nature of the q-deformed formulation.

We now perform the DBT to the Lax operator (4.1)

$$L \to L^{(1)} = T_1 L T_1^{-1} = \partial_q^2 + u_1^{(1)} \partial_q + u_0^{(1)}, \tag{4.5}$$

with

$$T_1 = (\partial_q - \frac{(\partial_q \phi_1)}{\phi_1}) \tag{4.6}$$

then the transformed coeffients become (which prove in the **Appendix A**)

$$u_1^{(1)} = \theta(u_1) - \alpha_1 + \theta^2(\alpha_1) \tag{4.7}$$

$$u_0^{(1)} = \theta(u_0) + (\partial_q u_1) + (q+1)\theta(\partial_q \alpha_1) - \alpha_1 u_1 + \theta(\alpha_1) u_1^{(1)} \tag{4.8}$$

where

$$\alpha_1 = \frac{(\partial_q \phi_1)}{\phi_1} \tag{4.9}$$

This is the new solution of (4.3) and (4.4). Since ϕ_1 is a wave function associated with the Lax operator (4.1), i.e. $L\phi_1 = \lambda_1\phi_1$, one can easily verify that $u_0 + (\partial_q\alpha_1) + \alpha_1 u_1 + \theta(\alpha_1)\alpha_1 = \lambda_1$ and equations (4.7) and (4.8) can be simplified as

$$u_1^{(1)} - u_1 = x(q-1)(u_0^{(1)} - u_0) \tag{4.10}$$

$$u_0^{(1)} - u_0 = \partial_q(u_1 + \alpha_1 + \theta(\alpha_1)) \tag{4.11}$$

Furthermore by the gauge operator T we can obtain $\phi_2^{(1)}$

$$\phi_2^{(1)} = T(\phi_2) = \partial_q\phi_2 - \alpha_1\phi_2 \tag{4.12}$$

also $\phi_2^{(1)}$ satisfies the equation $L^{(1)}\phi_2^{(1)} = \lambda\phi_2^{(1)}$

The soliton solutions can be constructed from the trivial one(that means $u_1 = u_0 = 0$), so we solve $\partial_q^2\phi = p^2\phi$ to obtain the simplest wave-function . Then we can get the following solution

$$\phi = E_q(px)\exp(\sum_{k=0}^{\infty} p^{2k+1}t_{2k+1}) + \gamma_i E_q(-px)\exp(-\sum_{k=0}^{\infty} p^{2k+1}t_{2k+1}) \tag{4.13}$$

with

$$E_q(x) = exp[\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k] \tag{4.14}$$

where γ_i are constants and E_q denotes the q-exponential function which satisfies $\partial_q E_q(px)$=$pE_q(px)$. Equation (4.13) is resembling to the equation satisfied by exponential function in calculus. We can assume that the eigenfunction has the form $exp(\ f(q,x)\ )$ with the series

$f(q,x) = \sum_k a_k(q)x^k$. It is not hard to determine the coefficients $a_k(q)$ by substituting it into $\partial_q^2\phi = p^2\phi$. In fact, the q-deformed exponential function (4.14) is well-known.

Because we start from the trivial one that means $u_0 = u_1 = 0$, equation (4.10) and (4.11) will become

$$
\begin{aligned}
& u_1^{(1)} = \theta^2(\alpha_1) - \alpha_1 = \theta^2(\frac{(\partial_q\phi_1)}{\phi_1}) - \frac{(\partial_q\phi_1)}{\phi_1} \\
= \; & \frac{p_1E_q(q^2p_1x)\exp(p_1t_1) - p_1\gamma_1E_q(-q^2p_1x)\exp(-p_1t_1)}{E_q(q^2p_1x)\exp(p_1t_1) + \gamma_1E_q(-q^2p_1x)\exp(-p_1t_1)} \\
- \; & \frac{p_1E_q(p_1x)\exp(p_1t_1) - p_1\gamma_1E_q(-p_1x)\exp(-p_1t_1)}{E_q(p_1x)\exp(p_1t_1) + \gamma_1E_q(-p_1x)\exp(-p_1t_1)} \\
= \; & \frac{2\gamma_1E_q(q^2p_1x)E_q(-p_1x) - 2\gamma_1E_q(-q^2p_1x)E_q(p_1x)}{E_q(q^2p_1x)[E_q(p_1x)\exp(2p_1t_1) + \gamma_1E_q(-p_1x)] + \gamma_1E_q(-q^2p_1x)[\gamma_1E_q(-p_1x)\exp(-2p_1t_1) + E_q(p_1x)]}
\end{aligned}
$$

and

$$
\begin{aligned}
u_0^{(1)} &= (q+1)\theta(\partial_q\alpha_1) + \theta(\alpha_1)u_1^{(1)} \\
&= (q+1)\theta(\partial_q((\partial_q\phi_1)\cdot\phi_1^{-1})) + \theta(\frac{(\partial_q\phi_1)}{\phi_1})\cdot(\theta^2(\frac{(\partial_q\phi_1)}{\phi_1}) - \frac{(\partial_q\phi_1)}{\phi_1}) \\
&= (q+1)((\frac{p_1^2E_q(p_1x)\exp(p_1t_1) - p_1^2\gamma_1E_q(-p_1x)\exp(-p_1t_1)}{E_q(qp_1x)\exp(p_1t_1) + \gamma_1E_q(-qp_1x)\exp(-p_1t_1)}) \\
&- \frac{p_1E_q(qp_1x)\exp(p_1t_1) - p_1\gamma_1E_q(-qp_1x)\exp(-p_1t_1)}{E_q(qp_1x)\exp(p_1t_1) + \gamma_1E_q(-qp_1x)\exp(-p_1t_1)} \\
&\times (\frac{p_1E_q(p_1x)\exp(p_1t_1) - p_1\gamma_1E_q(-p_1x)\exp(-p_1t_1)}{E_q(p_1x)\exp(p_1t_1) + \gamma_1E_q(-p_1x)\exp(-p_1t_1)} - \frac{u_1^{(1)}}{(q+1)}))
\end{aligned}
$$

with

$$\phi_1 = E_q(p_1x)\exp(p_1t_1) + \gamma_1E_q(-p_1x)\exp(-p_1t_1) \tag{4.15}$$

$$\alpha_1 = \frac{p_1 E_q(p_1 x)\exp(p_1 t_1) - p_1 \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)}{E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)} \tag{4.16}$$

also ϕ_1 satisfies the equation $L\phi_1 = \lambda_1 \phi_1$,

4.2 Two soliton solutions of q-KdV and further discussion

By the same processes as we used before we can get the two soliton solution. Compare to the one soliton, two soliton solution is much complicated when we calculating. First of all, We have to perform the DBT to the Lax operator. Then equation (4.1) will become

$$L^{(1)} \to L^{(2)} = T^{(1)} L^{(1)} (T^{(1)})^{-1} = \partial_q^2 + u_1^{(2)} \partial_q + u_0^{(2)}, \tag{4.17}$$

with

$$T^{(1)} = (\partial_q - \frac{(\partial_q \phi_i^{(1)})}{\phi_i^{(1)}}) \quad \forall i \neq 1 \tag{4.18}$$

see the detail in **Appendix B**. Then the transformation coefficient becomes

$$u_1^{(2)} = \theta(u_1^{(1)}) - \alpha_i^{(1)} + \theta^2(\alpha_i^{(1)}) \tag{4.19}$$

$$u_0^{(2)} = \theta(u_0^{(1)}) + (\partial_q u_1^{(1)}) + (q+1)\theta(\partial_q \alpha_i^{(1)}) - \alpha_i^{(1)} u_1^{(1)} + \theta(\alpha_i^{(1)}) u_1^{(2)} \tag{4.20}$$

with

$$\alpha_i^{(1)} = \frac{(\partial_q \phi_i^{(1)})}{\phi_i^{(1)}}, \;\; \phi_i^{(1)} = T(\phi_i) \quad \forall i \neq 1 \tag{4.21}$$

This is the solution of the equation (4.3) and (4.4), we can also simplify (4.19) and (4.23) as

$$u_1^{(2)} = \theta^3(\alpha_1) - \theta(\alpha_1) - \alpha_i^{(1)} + \theta^2(\alpha_i^{(1)}) \tag{4.22}$$

$$u_0^{(2)} = \partial_q(\theta(\alpha_1) + \theta^2(\alpha_1) + \alpha_i^{(1)} + \theta(\alpha_i^{(1)})) \tag{4.23}$$

and then put (4.14) and (4.15) into (4.22) and (4.23). Then they become

$$u_1^{(2)} = \theta^3(\frac{(\partial_q\phi_1)}{\phi_1}) - \theta(\frac{(\partial_q\phi_1)}{\phi_1}) - \frac{(\partial_q\phi_i^{(1)})}{\phi_i^{(1)}} + \theta^2(\frac{(\partial_q\phi_i^{(1)})}{\phi_i^{(1)}}) \tag{4.24}$$

and

$$u_0^{(2)} = \partial_q(\theta(\frac{(\partial_q\phi_1)}{\phi_1}) + \theta^2(\frac{(\partial_q\phi_1)}{\phi_1}) + \frac{(\partial_q\phi_i^{(1)})}{\phi_i^{(1)}} + \theta(\frac{(\partial_q\phi_i^{(1)})}{\phi_i^{(1)}})) \tag{4.25}$$

with

$$\begin{aligned}\phi_i^{(1)} &= T(\phi_i) = \partial_q\phi_i - \alpha_1\phi_i = \frac{(\partial_q\phi_i)\cdot\phi_1 - (\partial_q\phi_1)\cdot\phi_i}{\phi_1}\\ &= (p_iE_q(p_ix)\exp(p_it_1) - p_i\gamma_iE_q(-p_ix)\exp(-p_it_1)) -\\ &\frac{(p_1E_q(p_1x)\exp(p_1t_1) - p_1\gamma_1E_q(-p_1x)\exp(-p_1t_1))(E_q(p_ix)\exp(p_it_1) + \gamma_iE_q(-p_ix)\exp(-p_it_1))}{(E_q(p_1x)\exp(p_1t_1) + \gamma_1E_q(-p_1x)\exp(-p_1t_1))}\end{aligned}$$

and

$$\alpha_i^{(1)} = \frac{(\partial_q\phi_i^{(1)})}{\phi_i^{(1)}} = \frac{1}{\phi_i^{(1)}}[\partial_q((\partial_q\phi_i) - \alpha_1\phi_i)] = \frac{1}{(\partial_q\phi_i) - \alpha_1\phi_i}[\partial_q((\partial_q\phi_i) - (\partial_q\alpha_1)\phi_i - \theta(\alpha_1)(\partial_q\phi_i)]$$

Also $\phi_i^{(1)}$ satisfies the equation $L^{(1)}\phi_i^{(1)} = \lambda_i\phi_i^{(1)}$, $i \neq 1$. The detail result of two soliton are too complicate so we put it in **Appendix C**.

Following we discuss some other facts about the solution of q-deformed KdV. Using the facts that

$$\partial_{t1}\phi_1 = (L_+^{1/2}\phi_1) = (\ \partial_q\phi_1) + \ s_0\phi_1 \tag{4.26}$$

from the equation (2.24), when $N = 2$ we can get

$$u_1 = \theta(s_0) + s_0 \tag{4.27}$$

we can rewrite (4.11) as

$$u_0^{(1)} = u_0 + \partial_q\partial_{t1}\ln\phi_1\theta(\phi_1). \tag{4.28}$$

Theorem 4.1 Let ϕ_1 be wave functions associated with the Lax operator (4.1). Then, under the DBT, the transformed coefficients are given by

$$u_1^{(1)} - u_1 = x(q-1)(u_0^{(1)} - u_0) \tag{4.29}$$

Equations (4.29) effectively represent the one-step transformations. To obtain the n-step DBT, we just need to iterate the corresponding one-step transformations successively by inserting the triggered wave function (3.13), into the logarithm in Equations (4.28).

Theorem 4.2 Let ϕ_1 wave functions associated with the Lax operator (4.1). Then under the successive DBT of Theorem 4.1, the n-step transformed coefficients are given by

$$u_1^{(n)} - u_1 = x(q-1)(u_0^{(n)} - u_0) \tag{4.30}$$

with

$$u_0^{(n)} = u_0 + \partial_q \partial_{t1} \ln W_q[\phi_1, ..., \phi_1] \theta(W_q[\phi_1, ..., \phi_1]). \tag{4.31}$$

Equations (4.31) provide us with a convenient way to construct new solutions from the old ones. Especially, starting from the trivial solution ($u_1 = u_0 = 0$), we can obtain nontrivial multi-soliton solutions just by putting the wave functions into the formulas .

4.3 The ϵ expansion of soliton solution

In this section we discuss the ϵ expansion of one and two soliton for the KdV hierarchy by letting $\epsilon = q - 1$ and $q \to 1$. By the definition of the q-deformed differential operator and Taylor series expand we have

$$\partial_q f(x) = \frac{f\ (x) + \ f\ '(x)\ x\epsilon + \frac{f\ ''(x)}{2}(x\epsilon)^2 - \ f\ (x)}{x\epsilon} = f'(x) + \frac{f\ ''(x)}{2} x\epsilon \tag{4.32}$$

Let

$$u_1 = u_{10} + \varepsilon u_{11} + ... \tag{4.33}$$

$$s_{01} = s_{00} + \varepsilon s_{01} + ... \tag{4.34}$$

so

$$\theta(s_0) = s_0(qx) = s_{00}(x) + \partial_x s_{00} \cdot x\epsilon + s_{01} \cdot \epsilon + ... \tag{4.35}$$

Then from the equation (4.27) we get

$$u_1 = u_{10} + \epsilon u_{11} = 2s_{00}(x) + \partial_x s_{00} \cdot x\epsilon + s_{01} \cdot 2\epsilon + ... \tag{4.36}$$

because all the s_i are q-deformed differential polynomials in u_i, s_{00} and s_{01} become following representation:

$$s_{00} = \frac{1}{2}u_{10} \tag{4.37}$$

and

$$s_{01} = \frac{1}{2}u_{11} - \frac{1}{4}x \cdot \partial_x u_{10} \tag{4.38}$$

Now the evolution (4.4) becomes

$$\begin{aligned}
\partial_{t1}u_0 &= \partial_{t1}(u_{00} + \epsilon u_{01}) = (\partial_q u_0) - (\partial_q^2 s_0) - (\partial_q s_0^2) \\
&= \partial_x u_{00} - \frac{1}{2}\partial_x^2 u_{10} - \frac{1}{2}u_{10}\partial_x u_{10} \\
&+ \epsilon(\partial_x u_{01} - \frac{1}{2}\partial_x^2 u_{00} - \frac{1}{2}\partial_x^2 u_{11} + \frac{1}{4}\partial_x^2 u_{10} - \frac{1}{4}\partial_x^3 u_{10} - \frac{1}{2}u_{11}\partial_x u_{10} - \frac{1}{2}u_{10}\partial_x u_{11} + \frac{1}{4}u_{10}\partial_x u_{10})
\end{aligned}$$

then we have the equation what u_{00}, and u_{01} are

$$\partial_{t1}u_{00} = \partial_x u_{00} - \frac{1}{2}\partial_x^2 u_{10} - \frac{1}{2}u_{10}\partial_x u_{10} \tag{4.39}$$

and

$$\partial_{t1}u_{01} = \partial_x u_{01} - \frac{1}{2}\partial_x^2 u_{00} - \frac{1}{2}\partial_x^2 u_{11} + \frac{1}{4}\partial_x^2 u_{10} - \frac{1}{4}\partial_x^3 u_{10} - \frac{1}{2}u_{11}\partial_x u_{10} - \frac{1}{2}u_{10}\partial_x u_{11} + \frac{1}{4}u_{10}\partial_x u_{10} \tag{4.40}$$

Also the evolution equation (4.3) becomes

$$\partial_{t1}(u_{10} + \epsilon u_{11}) = x \cdot \epsilon \cdot \partial_{t1}(u_{00} + \epsilon u_{01}) \tag{4.41}$$

then we have

$$\partial_{t1} u_{10} = 0 \tag{4.42}$$

and

$$\partial_{t1} u_{11} = x \cdot (\partial_x u_{00} - \frac{1}{2}\partial_x^2 u_{10} - \frac{1}{2} u_{10}\partial_x u_{10}) \tag{4.43}$$

Now we figure out the ϵ correction of one soliton solution. Let

$$u_1^{(1)} = u_{10}^{(1)} + \epsilon u_{11}^{(1)} + \ldots \tag{4.44}$$

After calculating, we get the result

$$\begin{aligned}
u_{10}^{(1)} &= \frac{p_1 E_q(p_1 x)\exp(p_1 t_1) - p_1\gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)}{E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)} \\
&- \frac{p_1 E_q(p_1 x)\exp(p_1 t_1) - p_1\gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)}{E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)} \\
&= \frac{2\gamma_1 E_q(p_1 x)E_q(-p_1 x) - 2\gamma_1 E_q(-p_1 x)E_q(p_1 x)}{E_q(p_1 x)[E_q(p_1 x)\exp(2p_1 t_1) + \gamma_1 E_q(-p_1 x)] + \gamma_1 E_q(-p_1 x)[\gamma_1 E_q(-p_1 x)\exp(-2p_1 t_1) + E_q(p_1 x)]} \\
&= 0
\end{aligned}$$

and

$$
\begin{aligned}
u_{11}^{(1)} &= \frac{p_1 E_q((1+\varepsilon)^2 p_1 x)\exp(p_1 t_1) - p_1\gamma_1 E_q(-(1+\varepsilon)^2 p_1 x)\exp(-p_1 t_1)}{E_q((1+\varepsilon)^2 p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-(1+\varepsilon)^2 p_1 x)\exp(-p_1 t_1)} \\
&- \frac{p_1 E_q(p_1 x)\exp(p_1 t_1) - p_1\gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)}{E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)} \\
&= \frac{8p_1^2 x\gamma_1}{e^{(2p_1(x+t))} + \ 2\gamma_1 + \ \gamma_1^2 e^{(-2p_1(x+t))}} \\
&= 2xp_1^2 \sec h^2(p_1(x+t) - \frac{\ln\gamma_1}{2})
\end{aligned}
$$

By the same processes we can also get the result about $u_0^{(1)}$. Let

$$
u_0^{(1)} = u_{00}^{(1)} + \epsilon u_{01}^{(1)} + \ldots \tag{4.45}
$$

then we have

$$
\begin{aligned}
u_{00}^{(1)} &= 2(\frac{p_1^2 E_q(p_1 x)\exp(p_1 t_1) - p_1^2\gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)}{E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-qp_1 x)\exp(-p_1 t_1)}) \\
&- \frac{p_1 E_q(qp_1 x)\exp(p_1 t_1) - p_1\gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)}{E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)} \\
&\times (\frac{p_1 E_q(p_1 x)\exp(p_1 t_1) - p_1\gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)}{E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)} - \frac{u_{10}^{(1)}}{(2+\epsilon)})) \\
&= \frac{8p_1^2\gamma_1}{e^{(2p_1(x+t))} + \ 2\gamma_1 + \ \gamma_1^2 e^{(-2p_1(x+t))}}
\end{aligned}
$$

and

$$\begin{aligned}
u_{01}^{(1)} &= (2+\epsilon)\theta(\partial_q\alpha_1) + \theta(\alpha_1)u_1^{(1)} \\
&= (2+\epsilon)\theta(\partial_q((\partial_q\phi_1)\cdot\phi_1^{-1})) + \theta(\frac{(\partial_q\phi_1)}{\phi_1})\cdot(\theta^2(\frac{(\partial_q\phi_1)}{\phi_1}) - \frac{(\partial_q\phi_1)}{\phi_1}) \\
&= (2+\epsilon)((\frac{p_1^2E_q(p_1x)\exp(p_1t_1) - p_1^2\gamma_1E_q(-p_1x)\exp(-p_1t_1)}{E_q((1+\epsilon)p_1x)\exp(p_1t_1) + \gamma_1E_q(-(1+\epsilon)p_1x)\exp(-p_1t_1)}) \\
&- \frac{p_1E_q((1+\epsilon)p_1x)\exp(p_1t_1) - p_1\gamma_1E_q(-(1+\epsilon)p_1x)\exp(-p_1t_1)}{E_q((1+\epsilon)p_1x)\exp(p_1t_1) + \gamma_1E_q(-(1+\epsilon)p_1x)\exp(-p_1t_1)} \\
&\times (\frac{p_1E_q(p_1x)\exp(p_1t_1) - p_1\gamma_1E_q(-p_1x)\exp(-p_1t_1)}{E_q(p_1x)\exp(p_1t_1) + \gamma_1E_q(-p_1x)\exp(-p_1t_1)} - \frac{u_{11}^{(1)}}{(2+\epsilon)})) \\
&= \frac{4p_1^2\gamma_1e^{(2p_1(x+t))}(\ \gamma_1 +\ 4p_1\gamma_1x +\ e^{(2p_1(x+t))} - 4p_1xe^{(2p_1(x+t))})}{e^{(6p_1(x+t))} +\ 3e^{(4p_1(x+t))} +\ 3\gamma_1^2e^{(2p_1(x+t))} +\ \gamma_1^3}
\end{aligned}$$

Note that $u_{10}^{(1)}$,$u_{11}^{(1)}$, $u_{00}^{(1)}$ and $u_{01}^{(1)}$ are the one soliton solution of (4.40), (4.41), (4.43) and (4.44). Equation $u_{00}^{(1)}$ is the one soliton solution of ordinary KdV and $u_{01}^{(1)}$ is the q-KdV correction for the KdV.

Next we can write the ϵ correction of two soliton solution. As the same process above we let

$$u_1^{(2)} = u_{10}^{(2)} + \epsilon u_{01}^{(2)} + \ldots \tag{4.46}$$

then we have

$$\begin{aligned}
u_{10}^{(2)} &= \frac{p_1E_q(p_1x)\exp(p_1t_1) - p_1\gamma_1E_q(-p_1x)\exp(-p_1t_1)}{E_q(p_1x)\exp(p_1t_1) + \gamma_1E_q(-p_1x)\exp(-p_1t_1)} \\
&- \frac{p_1E_q(p_1x)\exp(p_1t_1) - p_1\gamma_1E_q(-p_1x)\exp(-p_1t_1)}{E_q(p_1x)\exp(p_1t_1) + \gamma_1E_q(-p_1x)\exp(-p_1t_1)} \\
&- \alpha_2^{(1)} + \alpha_2^{(1)} \\
&= 0
\end{aligned}$$

and

$$
\begin{aligned}
u_{11}^{(2)} &= \frac{p_1E_q((1+\varepsilon)^3p_1x)\exp(p_1t_1)-p_1\gamma_1E_q(-(1+\varepsilon)^3p_1x)\exp(-p_1t_1)}{E_q((1+\varepsilon)^3p_1x)\exp(p_1t_1)+\gamma_1E_q(-(1+\varepsilon)^3p_1x)\exp(-p_1t_1)} \\
&- \frac{p_1E_q((1+\varepsilon)p_1x)\exp(p_1t_1)-p_1\gamma_1E_q(-(1+\varepsilon)p_1x)\exp(-p_1t_1)}{E_q((1+\varepsilon)p_1x)\exp(p_1t_1)+\gamma_1E_q(-(1+\varepsilon)p_1x)\exp(-p_1t_1)} \\
&- \alpha_2^{(1)}+\theta^2(\alpha_2^{(1)}) \\
&= \frac{8p_1^2x\gamma_1}{e^{(2p_1(x+t))}+\ 2\gamma_1+\ \gamma_1^2e^{(-2p_1(x+t))}} = 2p_1^2\sec h^2(p_1(x+t)-\frac{\ln\gamma_1}{2})
\end{aligned}
$$

with

$$
\begin{aligned}
\alpha_2^{(1)} &= [((p_1^2-p_1^2)(E_q(p_1x)E_q(p_ix)\exp((p_1+p_i)t_1)+\gamma_1E_q(-p_1x)E_q(p_ix)\exp((p_i-p_1)t_1) \\
&+ \gamma_1\gamma_iE_q(-p_1x)E_q(-p_ix)\exp(-(p_1+p_i)t_1)+\gamma_iE_q(-p_ix)E_q(p_1x)\exp((p_1-p_i)t_1))) \\
&+ \frac{p_1(E_q(qp_1x)\exp(p_1t_1)-\gamma_1E_q(-q^2p_1x)\exp(-p_1t_1))}{E_q(q^2p_1x)\exp(p_1t_1)+\gamma_1E_q(-q^2p_1x)\exp(-p_1t_1)} \\
&\times (E_q(p_1x)\exp(p_1t_1)-\gamma_1E_q(-p_1x)\exp(-p_1t_1))] \\
&\div [(p_1+p_i)E_q(p_1x)E_q(p_ix)\exp((p_1+p_i)t_1)+(p_1\gamma_1+p_i\gamma_i)E_q(-p_1x)E_q(p_ix) \\
&\quad \exp((p_i+p_1)t_1)+\gamma_1\gamma_i(p_1-p_i))E_q(-p_1x)E_q(-p_ix)\exp(-(p_1+p_i)t_1) \\
&+ (-\gamma_i(p_i+p_1))E_q(-p_ix)E_q(p_1x)\exp((p_1-p_i)t_1)]
\end{aligned}
$$

and by the same processes we can also get the result of $u_0^{(2)}$. Let

$$u_0^{(2)} = u_{00}^{(2)} + \epsilon u_{01}^{(2)} + \ldots \tag{4.47}$$

the form $u_{00}^{(2)}$ and $u_{01}^{(2)}$ are too complicated , so we put it detail in **Appendix D**. However $u_{10}^{(2)}$,$u_{11}^{(2)}$, $u_{00}^{(2)}$ and $u_{01}^{(2)}$ are the two soliton solution of (4.40), (4.41), (4.43) and (4.44). Equation

$u_{00}^{(2)}$ is the two soliton solution of ordinary KdV and $u_{01}^{(2)}$ is the q-KdV correction for the KdV.

Finally we discuss ϵ vision of wave function ϕ. First we let

$$\phi = \phi_o + \varepsilon\phi_c \tag{4.48}$$

Since all ϕ satisfy the linear system $L\phi = \lambda\phi$ we have

$$[\partial_q^2 + u_1\partial_q + u_0] \circ [\phi_o + \varepsilon\phi_c] = \lambda[\phi_o + \varepsilon\phi_c]$$

that implies

$$[\partial_x^2 + \frac{\varepsilon}{2}\partial_x^2 + \varepsilon x\partial_x^3 + (u_{10} + \varepsilon u_{10})(\partial_x + \frac{\varepsilon}{2}x\partial_x^2) + (u_{00} + \varepsilon u_{01})][\phi_o + \varepsilon\phi_c] = \lambda_1[\phi_o + \varepsilon\phi_c]$$

so we have

$$\lambda_1\phi_o = \partial_x^2\phi_o + u_{10}\partial_x\phi_o + u_{00}\phi_o \tag{4.49}$$

and

$$\lambda_1\phi_c = \frac{1}{2}\partial_x^2\phi_o + x\partial_x^3\phi_o + u_{11}\partial_x\phi_o + \frac{1}{2}xu_{10}\partial_x^2\phi_o + u_{10}\phi_o + \partial_x^2\phi_c + \phi_c u_{00} + u_{10}\partial_x\phi_c \tag{4.50}$$

Also we let the one step transformed wave function $\phi^{(1)}$ as

$$\phi^{(1)} = \phi_o^{(1)} + \varepsilon\phi_c^{(1)} \tag{4.51}$$

which satisfy $\mathrm{L}^{(1)}\phi^{(1)} = \lambda\phi^{(1)}$ that means

$$[\partial_q^2 + u_1^{(1)}\partial_q + u_0^{(1)}] \circ [\phi_o^{(1)} + \varepsilon\phi_c^{(1)}] = \lambda[\phi_o^{(1)} + \varepsilon\phi_c^{(1)}]$$

implies

$$[\partial_x^2+\frac{\varepsilon}{2}\partial_x^2+\varepsilon x\partial_x^3+(u_{10}^{(1)}+\varepsilon u_{10}^{(1)})(\partial_x+\frac{\varepsilon}{2}x\partial_x^2)+(u_{00}^{(1)}+\varepsilon u_{01}^{(1)})][\phi_o^{(1)}+\varepsilon\phi_c^{(1)}]=\lambda[\phi_o^{(1)}+\varepsilon\phi_c^{(1)}]$$

The left hand side equal to

$$\begin{aligned}(&\partial_x^2\phi_o^{(1)}+u_{00}^{(1)}\partial_x\phi_o+u_{00}^{(1)}\phi_o) \\ + &\varepsilon(\frac{1}{2}\partial_x^2\phi_o^{(1)}+x\partial_x^3\phi_o^{(1)}+u_{11}^{(1)}\partial_x\phi_o^{(1)}+\frac{1}{2}xu_{10}^{(1)}\partial_x^2\phi_o^{(1)}+u_{10}^{(1)}\phi_o^{(1)}+\partial_x^2\phi_c^{(1)}+\phi_c^{(1)}u_{00}^{(1)}+u_{10}^{(1)}\partial_x\phi_c^{(1)})\end{aligned}$$

so we have

$$\lambda\phi_o^{(1)}=\partial_x^2\phi_o^{(1)}+u_{00}^{(1)}\partial_x\phi_o+u_{00}^{(1)}\phi_o \tag{4.52}$$

and

$$\lambda\phi_c^{(1)}=\frac{1}{2}\partial_x^2\phi_o^{(1)}+x\partial_x^3\phi_o^{(1)}+u_{11}^{(1)}\partial_x\phi_o^{(1)}+\frac{1}{2}xu_{10}^{(1)}\partial_x^2\phi_o^{(1)}+u_{10}^{(1)}\phi_o^{(1)}+\partial_x^2\phi_c^{(1)}+\phi_c^{(1)}u_{00}^{(1)}+u_{10}^{(1)}\partial_x\phi_c^{(1)} \tag{4.53}$$

Next we put in the trivial solution($u_0=u_1=0$). By applying Binomial theory to the q-exponent it will become following representation:

$$E_q(x)=\exp(\sum_{k=1}^{\infty}\frac{(-1)^k\epsilon^k x^k}{k(-k\epsilon-...-C_m^k\epsilon^{k-1}-...)})=e^x(1-\frac{1}{4}\epsilon x^2) \tag{4.54}$$

so

$E_q(p_1x)=e^x(1-\frac{1}{4}\epsilon(p_1x)^2)$

$E_q(p_ix)=e^x(1-\frac{1}{4}\epsilon(p_ix)^2)$

where p_1^2 is the eigenvalue corresponding to ϕ_1 and p_i^2 is the eigenvalue corresponding to ϕ_1. According to the equation (4.54), equation (4.15) becomes

$$\begin{aligned}
\phi_1 &= \phi_{10} + \varepsilon\phi_{11} \\
&= e^x(1 - \frac{1}{4}\epsilon(p_1x)^2)\exp(p_1t_1) + e^x(1 - \frac{1}{4}\epsilon(p_1x)^2)\exp(-p_1t_1) \\
&= e^x\exp(p_1t_1) + \gamma_1e^x\exp(-p_1t_1) + \varepsilon(-\frac{1}{4}(p_1x)^2e^x\exp(p_1t_1) - \frac{1}{4}(p_1x)^2\gamma_1e^x\exp(-p_1t_1))
\end{aligned}$$

then we have

$$\phi_{10} = e^x\exp(p_1t_1) + \gamma_1e^x\exp(-p_1t_1) \tag{4.55}$$

and

$$\phi_{11} = -\frac{1}{4}(p_1x)^2e^x\exp(p_1t_1) - \frac{1}{4}(p_1x)^2\gamma_1e^x\exp(-p_1t_1) \tag{4.56}$$

Note that ϕ_1 still satisfies the equation $L\phi_1 = p_1^2\phi_1$, so from equation (4.49) and (4.50) we have

$$p_1^2\phi_{10} = \partial_x^2\phi_{10} \tag{4.57}$$

and

$$p_1^2\phi_{11} = \partial_x^2\phi_{11} + \frac{1}{2}\partial_x^2\phi_{10} + x\partial_x^3\phi_{10} \tag{4.58}$$

Next we claim the ϵ expansion of $\phi_i^{(1)}, i \neq 1$. Let

$$
\begin{aligned}
\phi_i^{(1)} &= \phi_{10}^{(1)} + \varepsilon \phi_{11}^{(1)} \\
&= (\partial_x + \frac{x \cdot \epsilon}{2}\partial_x^2) \cdot (E_q(p_i x)\exp(p_i t_1) + \gamma_i E_q(-p_i x)\exp(-p_i t_1)) \\
&- \frac{(\partial_x + \frac{x \cdot \epsilon}{2}\partial_x^2) \cdot (E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1))}{E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1)} \\
&\times (E_q(p_i x)\exp(p_i t_1) + \gamma_i E_q(-p_i x)\exp(-p_i t_1) \\
&= \frac{(-4 + \varepsilon p_i^2 x^2)}{4(e^{p_1(x+t_1)} + \gamma_1 e^{p_1(x+t_1)})} \\
&\times (-p_i e^{((p_i+p_1)(x+t_1))} - \gamma_1 p_i e^{((p_i-p_1)(x+t_1))} + \gamma_i p_i e^{((p_1-p_i)(x+t_1))} + \gamma_1\gamma_i p_i e^{-((p_1+p_i)(x+t_1)} \\
&+ p_1 e^{((p_i+p_1)(x+t_1))} - \gamma_1 p_1 e^{((p_i-p_1)(x+t_1))} + \gamma_i p_1 e^{((p_1-p_i)(x+t_1))} - \gamma_1\gamma_i p_1 e^{-((p_1+p_i)(x+t_1)})
\end{aligned}
$$

then we have

$$
\begin{aligned}
\phi_{10}^{(1)} &= \frac{1}{(e^{p_1(x+t_1)} + \gamma_1 e^{p_1(x+t_1)})} \\
&\times (p_i e^{((p_i+p_1)(x+t_1))} + \gamma_1 p_i e^{((p_i-p_1)(x+t_1))} - \gamma_i p_i e^{((p_1-p_i)(x+t_1))} - \gamma_1\gamma_i p_i e^{-((p_1+p_i)(x+t_1)} \\
&- p_1 e^{((p_i+p_1)(x+t_1))} + \gamma_1 p_1 e^{((p_i-p_1)(x+t_1))} - \gamma_i p_1 e^{((p_1-p_i)(x+t_1))} + \gamma_1\gamma_i p_1 e^{-((p_1+p_i)(x+t_1)})
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_{11}^{(1)} &= \frac{\varepsilon p_i^2 x^2}{4(e^{p_1(x+t_1)} + \gamma_1 e^{p_1(x+t_1)})} \\
&\times (-p_i e^{((p_i+p_1)(x+t_1))} - \gamma_1 p_i e^{((p_i-p_1)(x+t_1))} + \gamma_i p_i e^{((p_1-p_i)(x+t_1))} + \gamma_1\gamma_i p_i e^{-((p_1+p_i)(x+t_1)} \\
&+ p_1 e^{((p_i+p_1)(x+t_1))} - \gamma_1 p_1 e^{((p_i-p_1)(x+t_1))} + \gamma_i p_1 e^{((p_1-p_i)(x+t_1))} - \gamma_1\gamma_i p_1 e^{-((p_1+p_i)(x+t_1)})
\end{aligned}
$$

also $\phi_i^{(1)}$ satisfies the linear system $L^{(1)}\phi_i^{(1)} = p_i^2\phi_i^{(1)}$, $i \neq 1$. So from equation (4.52), (4.53) we get

$$\begin{aligned} p_i^2\phi_{i0}^{(1)} &= \partial_x^2\phi_{i0}^{(1)} + \frac{8p_1^2\gamma_1}{e^{(2p_1(x+t))} + \ 2\gamma_1 + \ \gamma_1^2 e^{(-2p_1(x+t))}}\partial_x\phi_{i0} \\ &+ \frac{8p_1^2\gamma_1}{e^{(2p_1(x+t))} + \ 2\gamma_1 + \ \gamma_1^2 e^{(-2p_1(x+t))}}\phi_{i0} \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} p_i^2\phi_{i1}^{(1)} &= \frac{1}{2}\partial_x^2\phi_{10}^{(1)} + x\partial_x^3\phi_{10}^{(1)} + \frac{8p_1^2 x\gamma_1}{e^{(2p_1(x+t))} + \ 2\gamma_1 + \ \gamma_1^2 e^{(-2p_1(x+t))}}\partial_x\phi_{10}^{(1)} \\ &+ \partial_x^2\phi_{11}^{(1)} + \frac{8p_1^2\gamma_1}{e^{(2p_1(x+t))} + \ 2\gamma_1 + \ \gamma_1^2 e^{(-2p_1(x+t))}}\phi_{11}^{(1)} \end{aligned} \tag{4.50}$$

Chapter 5

Conclusions

5.1 Summary

In this thesis, the elementary Darboux and Backlund transformations for the q-deformed KdV hierarchy, has been performed. Which preserve the form of the Lax operator and are compatible with Lax equations. Iterated application of these elementary **DBTs** produces new soliton solutions (tau-functions) of the q-KdV hierarchy come out of given ones. Most important of all we figure out the ε expansion of one and two soliton solution successfully. When we let $q \to 1$ they will recover to the solution of ordinary KdV. Besides we also let $\varepsilon = q - 1$, this correction term serve as the q-KdV correction for the KdV. It may be useful for some solid state problem.

5.2 Future Works

In the future we will strive to apply our work to real matters such as solid state crystal problem. Althought in this thesis we have done first order ε expansion of one and two solitons, the two or three order ε expansion still under working. Specially the q-KdV is just a reduction of the q-KP by imposing the condition $(L^N)_+ = L^N$, we try to generalized our work to the q-deformed KP We hope we can complete these questions in the future.

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Appendix A

The Transformed coefficient

In this apendix we will show how we get the result of $u_0^{(1)}$and $u_1^{(1)}$ as equation (4.5),(4.6).

Since we have

$$L^{(1)} = TLT^{-1} \Rightarrow L^{(1)}T = TL$$

then R.H.S is equal to

$$(\partial_q - \frac{(\partial_q\phi_1)}{\phi_1})(\partial_q^2 + u_1\partial_q + u_0)$$
$$= \partial_q^3 + (\theta(u_1) - \frac{(\partial_q\phi_1)}{\phi_1})\partial_q^2 + ((\partial_q u_1) + \theta(u_0) - \frac{(\partial_q\phi_1)}{\phi_1}u_1)\partial_q + ((\partial_q u_0) - \frac{(\partial_q\phi_1)}{\phi_1}u_0)$$

and L.H.S is

$$(\partial_q^2 + u_1^{(1)}\partial_q + u_0^{(1)})(\partial_q - \frac{(\partial_q\phi_1)}{\phi_1})$$
$$= \partial_q^3 + (u_1^{(1)} - \theta^2(\frac{(\partial_q\phi_1)}{\phi_1}))\partial_q^2 + (u_0^{(1)} - (q+1)\theta(\partial_q\frac{(\partial_q\phi_1)}{\phi_1}) - u_1^{(1)}\theta(\frac{(\partial_q\phi_1)}{\phi_1}))\partial_q +$$
$$(-\theta(\partial_q^2\frac{(\partial_q\phi_1)}{\phi_1}) - u_1^{(1)}(\partial_q\frac{(\partial_q\phi_1)}{\phi_1}) - u_0^{(1)}\frac{(\partial_q\phi_1)}{\phi_1})$$

from the equation above we can get $u_0^{(1)}$and $u_1^{(1)}$ the coefficient of ∂_q^2 of both side is

right : $\theta(u_1) - \frac{(\partial_q \phi_1)}{\phi_1} = \theta(s_0 + \theta(s_0)) - \alpha_1,$

Left : $u_1^{(1)} - \theta^2(\frac{(\partial_q \phi_1)}{\phi_1})$

so

$$\begin{aligned} u_1^{(1)} &= \theta(u_1) - \frac{(\partial_q \phi_1)}{\phi_1} + \theta^2(\frac{(\partial_q \phi_1)}{\phi_1}) \\ &= \theta(s_0 + \theta(s_0)) - \alpha_1 + \theta^2(\alpha_1) \end{aligned}$$

and the coefficient of ∂_q of both side is

right : $(\partial_q u_1) + \theta(u_0) - \frac{(\partial_q \phi_1)}{\phi_1} u_1$

Left : $u_0^{(1)} - (q+1)\theta(\partial_q \frac{(\partial_q \phi_1)}{\phi_1}) - u_1^{(1)} \theta(\frac{(\partial_q \phi_1)}{\phi_1})$

so we also have

$$u_0^{(1)} = (q+1)\theta(\partial_q \alpha_1) - u_1^{(1)} \theta(\alpha_1) + (\partial_q u_1) + \theta(u_0) - \alpha_1 u_1$$

Appendix B

Darboux transformation for the gauge operator T

If we want to construct the two soliton solution from one soliton solution, we need to apply **Darboux** transformation again.In this process what $\phi_i^{(1)}, i \neq 1$ (if $i = 1$ then $T(\phi_1) = \phi_1^{(1)} = 0$) is needed to be known. Then we can get new gauge operator $T^{(1)}$ by using $\phi_i^{(1)}$. Following we show how to construct the $\phi_i^{(1)}$ and $T^{(1)}$.

as the result of things we mention before, we know

$$T = \partial_q - \alpha_1 \tag{B.1}$$

so we have

$$T^{(1)} = \partial_q - \alpha_i^{(1)} \tag{B.2}$$

here we need to know what $\alpha_i^{(1)}$ is. In fact

$$\alpha_i^{(1)} = \frac{(\partial_q \phi_i^{(1)})}{\phi_i^{(1)}} \tag{B.3}$$

and

$$\phi_i^{(1)} = (T\phi_i) = (\partial_q \phi_i) - \alpha_1 \phi_i, \quad i \neq 1 \tag{B.4}$$

so

$$\alpha_i^{(1)} = \frac{(\partial_q \phi_i^{(1)})}{\phi_i^{(1)}} = \frac{1}{\phi_i^{(1)}}[\partial_q((\partial_q \phi_i) - \alpha_1 \phi_i)] \quad = \frac{1}{(\partial_q \phi_i) - \alpha_1 \phi_i}[\partial_q((\partial_q \phi_i) - (\partial_q \alpha_1)\phi_i - \theta(\alpha_1)(\partial_q \phi_i)] \tag{B.5}$$

in which

$$(\partial_q \alpha_1) = (\partial_q \frac{(\partial_q \phi_1)}{\phi_1}) = (\partial_q(\partial_q \phi_1))\phi_1^{-1} - \theta(\partial_q \phi_1)\theta(\phi_1^{-1})\alpha_1 \tag{B.6}$$

we substitution (B.6) into (B.5) we get the final result of $\alpha_i^{(1)}$, then we can figure out what $T^{(1)}$ is.

Appendix C

The solution of two soliton

In this appendix we will show the result of $u_0^{(2)}$ and $u_1^{(2)}$. We already have equation (4.24),(4.25).

$$u_1^{(2)} = \theta^3(\frac{(\partial_q\phi_1)}{\phi_1}) - \theta(\frac{(\partial_q\phi_1)}{\phi_1}) - \frac{(\partial_q\phi_i^{(1)})}{\phi_i^{(1)}} + \theta^2(\frac{(\partial_q\phi_i^{(1)})}{\phi_i^{(1)}}) \tag{C.1}$$

and

$$u_0^{(2)} = \partial_q(\theta(\frac{(\partial_q\phi_1)}{\phi_1}) + \theta^2(\frac{(\partial_q\phi_1)}{\phi_1}) + \frac{(\partial_q\phi_i^{(1)})}{\phi_i^{(1)}} + \theta(\frac{(\partial_q\phi_i^{(1)})}{\phi_i^{(1)}})) \tag{C.2}$$

when we put ϕ_1 and $\phi_i^{(1)}$ in those equations they will become

$$
\begin{aligned}
u_1^{(2)} &= [2\gamma_1 E_q(q^3 p_1 x)E_q(-p_1 x) - 2\gamma_1 E_q(-q^3 p_1 x)E_q(p_1 x)] \\
&\div [E_q(q^3 p_1 x)E_q(q p_1 x)\exp(2p_1 t_1) + \gamma_1 E_q(-q^3 p_1 x)E_q(q p_1 x) \\
&+ \gamma_1 E_q(q^3 p_1 x)E_q(-q p_1 x) + \gamma_1^2 E_q(-q^3 p_1 x)E_q(-q p_1 x)\exp(-2p_1 t_1)] \\
&- [(p_1^2 - p_i^2)(E_q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-p_1 x)\exp(-p_1 t_1))(E_q(p_i x)\exp(p_i t_1) + \gamma_i E_q(-p_i x)] \\
&\div [(p_1 - p_i)(2E_q(p_1 x)E_q(p_i x)\exp(2p_1 t_1) - 2\gamma_1\gamma_i E_q(-p_1 x)E_q(-p_i x)\exp(-2p_1 t_1)) \\
&+ (p_1 + p_i)(2\gamma_1 E_q(p_i x)E_q(-p_1 x) - 2\gamma_i E_q(p_1 x)E_q(-p_i x))] \\
&+ [(p_1^2 - p_i^2)(E_q(q^2 p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-q^2 p_1 x)\exp(-p_1 t_1))(E_q(q^2 p_i x)\exp(p_i t_1) + \gamma_i E_q(-q^2 p_i x)] \\
&\div [(p_1 - p_i)(2E_q(q^2 p_1 x)E_q(q^2 p_i x)\exp(2p_1 t_1) - 2\gamma_1\gamma_i E_q(-q^2 p_1 x)E_q(-q^2 p_i x) \\
&\quad \exp(-2p_1 t_1)) + (p_1 + p_i)(2\gamma_1 E_q(q^2 p_i x)E_q(-q^2 p_1 x) - 2\gamma_i E_q(q^2 p_1 x)E_q(-q^2 p_i x))] \\
&+ \frac{p_1 E_q(q p_1 x)\exp(p_1 t_1) - p_1\gamma_1 E_q(-q p_1 x)\exp(-p_1 t_1)}{E_q q(p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-q p_1 x)\exp(-p_1 t_1)} \\
&- \frac{p_1 E_q(q^3 p_1 x)\exp(p_1 t_1) - p_1\gamma_1 E_q(-q^3 p_1 x)\exp(-p_1 t_1)}{E_q(q^3 p_1 x)\exp(p_1 t_1) + \gamma_1 E_q(-q^3 p_1 x)\exp(-p_1 t_1)}
\end{aligned}
$$

and

$$
\begin{aligned}
u_0^{(2)} &= \frac{1}{x(q-1)}([2\gamma_1 E_q(q^3p_1x)E_q(-p_1x) - 2\gamma_1 E_q(-q^3p_1x)E_q(p_1x)] \\
&\div [E_q(q^3p_1x)E_q(qp_1x)\exp(2p_1t_1) + \gamma_1 E_q(-q^3p_1x)E_q(qp_1x) \\
&+ \gamma_1 E_q(q^3p_1x)E_q(-qp_1x) + \gamma_1^2 E_q(-q^3p_1x)E_q(-qp_1x)\exp(-2p_1t_1)] \\
&- [(p_1^2-p_i^2)(E_q(p_1x)\exp(p_1t_1) + \gamma_1 E_q(-p_1x)\exp(-p_1t_1))(E_q(p_ix)\exp(p_it_1) + \gamma_i E_q(-p_ix)] \\
&\div [(p_1-p_i)(2E_q(p_1x)E_q(p_ix)\exp(2p_1t_1) - 2\gamma_1\gamma_i E_q(-p_1x)E_q(-p_ix)\exp(-2p_1t_1)) \\
&+ (p_1+p_i)(2\gamma_1 E_q(p_ix)E_q(-p_1x) - 2\gamma_i E_q(p_1x)E_q(-p_ix))] \\
&+ [(p_1^2-p_i^2)(E_q(q^2p_1x)\exp(p_1t_1) + \gamma_1 E_q(-q^2p_1x)\exp(-p_1t_1)) \\
&\quad (E_q(q^2p_ix)\exp(p_it_1) + \gamma_i E_q(-q^2p_ix)]\exp(-p_it_1)] \\
&\div [(p_1-p_i)(2E_q(q^2p_1x)E_q(q^2p_ix)\exp(2p_1t_1) - 2\gamma_1\gamma_i E_q(-q^2p_1x)E_q(-q^2p_ix) \\
&\quad \exp(-2p_1t_1)) + (p_1+p_i)(2\gamma_1 E_q(q^2p_ix)E_q(-q^2p_1x) - 2\gamma_i E_q(q^2p_1x)E_q(-q^2p_ix))] \\
&+ \frac{p_1E_q(qp_1x)\exp(p_1t_1) - p_1\gamma_1 E_q(-qp_1x)\exp(-p_1t_1)}{E_qq(p_1x)\exp(p_1t_1) + \gamma_1 E_q(-qp_1x)\exp(-p_1t_1)} \\
&- \frac{p_1E_q(q^3p_1x)\exp(p_1t_1) - p_1\gamma_1 E_q(-q^3p_1x)\exp(-p_1t_1)}{E_q(q^3p_1x)\exp(p_1t_1) + \gamma_1 E_q(-q^3p_1x)\exp(-p_1t_1)})
\end{aligned}
$$

Appendix D

The ε expansion of u_0^2

In this appendix we will show the ε expansion of $u_0^{(2)}$. We already have equation (4.25), by letting $q \to 1$ and ε = q-1. Then we have

$$
\begin{aligned}
u_{00}^{(2)} &= [4(-p_i^3exp(4p_1x)p_1\gamma_i + p_i^2exp(4p_1x)p_1^2\gamma_i + p_1^3exp(2x(p_i-p_1))\gamma_1^3p_i \\
&- p_1^3exp(2x(p_i+p_1))\gamma_1p_i + p_i^4*xp(4p_1x)p_1\gamma_i + p_i^3exp(4p_1x)p_1^2\gamma_i - p_i^2exp(2x(p_i-p_1))\gamma_1^3p_1^3 \\
&+ p_i^2exp(2x(p_i+p_1))p_1^3\gamma_1 - 6p_i^3exp(2p_ix)p_1^2\gamma_1^2 - 4p_1^4exp(-2p_1x)\gamma_1^3\gamma_i - 2p_i^4exp(-4p_1x)\gamma_1^4\gamma_i \\
&- 8p_i^4exp(2p_1x)\gamma_i\gamma_1 - 8p_i^4exp(-2p_1x)\gamma_1^3\gamma_i + 5p^2exp(2x(p_i+p_1))\gamma_1p_i^2 - 2p_1^4exp(-2x(p_i) \\
&- 2p_1^4exp(-2x(p_i+p_1))\gamma_1^3\gamma_i^2 - 4p_1^4exp(2p_1x)\gamma_1\gamma_i + 5p_1^2exp(2x(p_i-p_1))\gamma_1^3p_i^2 + 16p_1^2\gamma_1^2p_i^2\gamma_i \\
&+ 10p_i^2exp(2p_ix)\gamma_1^2p_1^2 + 4p_i^3p_1^2\gamma_1^2\gamma_i - 4p_1^4exp(-2p_ix)\gamma_1^2\gamma_i^2 - 3p_1^2exp(2x(p_i+p_1))p_i^3\gamma_1 \\
&- 3\gamma_1^3p_1^2exp(2x(p_i-p_1))p_i^3 + p_i^3exp(-4p_1x)\gamma_1^4p_1^2\gamma_i - p_i^4exp(-4p_1x)\gamma_1^4p_1\gamma_i + p_i^3exp(4p_1x)\gamma_1 \\
&+ p_i^2exp(-4p_1x)\gamma_1^4p_1^2\gamma_i + p_1^3exp(-2p_1x)\gamma_1^3\gamma_ip_i - p_i^2exp(-2p_1x)\gamma_1^3p^3\gamma_i + p_i^2exp(2p_1x)p^3\gamma_1\gamma_i \\
&- p_1^3*exp(2p_1x)\gamma_1\gamma_ip_i - 2p_1^4exp(2x(p_i-p_1))\gamma_1^3 - 2p_i^4exp(4p_1x)\gamma_i - 2p_i^4exp(-2p_1x)\gamma_1^3p_1\gamma_i \\
&+ 2p_1^2exp(-2x(p_i-p_1))\gamma_1\gamma_i^2p_i^2 + 9p_i^2exp(-2p_1x)\gamma_1^3p_1^2\gamma_i + 2p_i^3exp(-2p_1x)\gamma_1^3p_1\gamma_i \\
&- 2p_i^3exp(2p_1x)\gamma_1p_1\gamma_i - 12p_i^4\gamma_i\gamma_1^2 - 2p_1^4exp(2x(p_i+p_1))\gamma_1 - 8p_1^4\gamma_1^2\gamma_i - 4p_1^4exp(2p_ix)\gamma_1^2 \\
&+ 2p_i^4exp(2p_1x)p_1\gamma_i\gamma_1 + 4\gamma_i^2p_i^2exp(-2p_ix)\gamma_1^2p_1^2 + 9p_i^2exp(2p_1x)p^2\gamma_1\gamma_i \\
&+ 2p_1^2exp(-2x(p_i+p_1))\gamma_1^3\gamma_i^2p_i^2 + 3\gamma_1^3p^2exp(-2p_1x)p_i^3\gamma_i + 3p_1^2exp(2p_1x)p_i^3\gamma_1\gamma_i)] \\
&\div [(exp(p_1x) + \gamma_1exp(-p_1x))^2/(-p_iexp(x(p_i+p_1)) - p_iexp(x(p_i-p_1))\gamma_1 + \gamma_ip_iexp(x(p_i)) \\
&+ \gamma_ip_iexp(-x(p_i+p_1))\gamma_1 + p_1exp(x(p_i+p_1)) + p_1exp(x(p_i-p_1))\gamma_i - \gamma_1p_1exp(x(p_i-p_1)) \\
&- \gamma_1p_1exp(-x(p_i+p_1)*\gamma_i)^2)]
\end{aligned}
$$

and

$$u_{01}^{(2)} = 1/36(990xp_1^6 exp_1(x(-4p_1+3p_i))\gamma_1^5$$

$$-90p_1^8 exp_1(x(3p_i-2p_1))x^3\gamma_1^4 + 576\gamma_1^4 p_1^5 exp_1(-x(2p_1+p_i))\gamma_i^2$$

$$-1728\gamma_i^2 p_i^5 exp_1(-x(4p_1+p_i))\gamma_1^5 - 288p_i^3 exp_1(x(-4p_1+3p_i))\gamma_1^5 p_1^2$$

$$+1152\gamma_1^4 p_1^5 exp_1(x(-2p_1+p_i))\gamma_i^2 + 288p_i^2 exp_1(x(3p_i+4p_1))p_1^3\gamma_1$$

$$-1728\gamma_i^2 p_i^5 exp_1(-x(p_i-4p_1))\gamma_1 + 45p_1^8 exp_1(x(-4p_1+3p_i))x^3\gamma_1^5$$

$$-5760\gamma_i^2 p_i^5 exp_1(-p_i x)\gamma_1^3 + 1728p_i^5 exp_1(x(4p_1+p_i))\gamma_i\gamma_1 + 9p_1^8 exp_1(x(3p_i+4p_1))x^3\gamma_1$$

$$-1152xp_i^6 exp_1(-x(-6p_1+p_i))\gamma_i^2 + 4446p_1^6 exp_1(x(2p_1+3p_i))x\gamma_1^2$$

$$-1152x\gamma_i p_i^6 exp_1(x(6p_1+p_i)) + 576\gamma_1^4 p_1^5 exp_1(-x(2p_1+p_i))\gamma_i^3$$

$$+144p_1^4 exp_1(x(6p_1+p_i))\gamma_i p_i + 288p_1^3 exp_1(-x(-6p_1+p_i))\gamma_i^3 p_i^2$$

$$+54p_1^6\gamma_1^6 exp_1(3x(-2p_1+p_i))x - 9p_1^8\gamma_1^6 exp_1(3x(-2p_1+p_i))x^3$$

$$+720p_1^7 exp_1(x(3p_i-2p_1))\gamma_1^4 x^2 - 1152\gamma_1^4 p_1^2 exp_1(x(3p_i-2p_1))p_i^3$$

$$+576\gamma_1^4 p_1^5 exp_1(x(3p_i-2p_1))\gamma_i - 1728p_1^2 exp_1(3p_i x)\gamma_1^3 p_i^3$$

$$-576p_1^5 exp_1(-x(-2p_1+p_i))\gamma_i^2\gamma_1^2 + 1728p_1^4 exp_1(3p_i x)\gamma_1^3 p_i$$

$$-288p_i^5 exp_1(-x(6p_1+p_i))\gamma_1^6\gamma_i^2 + 5760p_i^5 exp_1(p_i x)\gamma_1^3\gamma_i$$

$$+288\gamma_1^5 p_1^5 exp_1(-x(4p_1+p_i))\gamma_i^3 + 144p_1^4 exp_1(-x(-6p_1+p_i))\gamma_i^3 p_i$$

$$+144p_1^4 exp_1(-x(-6p_1+p_i))p_i\gamma_i^2 + 144p_1^4 exp_1(x(6p_1+p_i))\gamma_i^2 p_i - 4320\gamma_i^2 p_i^5 exp_1(-x(-2p_1$$

$$+p_i))\gamma_1^2 + 576p_1^5 exp_1(x(p_i-4p_1))\gamma_1^5\gamma_i + 4320p_i^5 exp_1(x(-2p_1+p_i))\gamma_1^4\gamma_i$$

$$+288\gamma_1^5 p_1^5 exp_1(-x(4p_1+p_i))\gamma_i^2 + 7020xp_1^6 exp_1(3p_i x)\gamma_1^3 + 90p_1^8 exp_1(3p_i x)x^3\gamma_1^3$$

$+ 1152p_1^4 exp_1(x(3p_i - 2p_1))\gamma_1^4 p_i + 288p_1^4 exp_1(x(3p_i$

$+ 4p_1))p_i\gamma_1 + 576\gamma_1^5 p_1^5 exp_1(x(p_i - 4p_1))\gamma_i^2$

$+ 576p_1^5 exp_1(x(4p_1 + p_i))\gamma_i^2\gamma_1 + 1728p_i^5 exp_1(x(p_i - 4p_1))\gamma_1^5\gamma_i$

$+ 1152p_1^5 exp_1(x(2p_1 + p_i))\gamma_i^2\gamma_1^2 + 288p_i^5 exp_1(x(-6p_1 + p_i))\gamma_1^6\gamma_i$

$- 45p_1^8 exp_1(x(2p_1 + 3p_i))x^3\gamma_1^2 - 288p_1^3 exp_1(x(-4p_1 + 3p_i))\gamma_1^5 p_i^2$

$+ 288\gamma_1^5 p_1^5 exp_1(x(-4p_1 + 3p_i))\gamma_i - 1152p_i^3 exp_1(x(2p_1 + 3p_i))p_1^2\gamma_1^2 - 144p_1^2 exp_1(x(6p_1 + p_i))\gamma_i^2 p_i^3 - 288p_i^3 exp_1(x(3p_i + 4p_1))p_1^2\gamma_1 - 288\gamma_i^2 p_i^4 exp_1(-x(-6p_1 + p_i))p_1$

$+ 288p_i^2 exp_1(-x(-6p_1 + p_i))p_1^3\gamma_i^2 - 72p_1^7 exp_1(x(3p_i + 4p_1))\gamma_1 x^2 + 4716xp_1^6 exp_1(x(3p_i - 2p_1))\gamma_1^4 + 144p_1^2 exp_1(-x(-6p_1 + p_i))\gamma_i^3 p_i^3 - 288p_1^5 exp_1(x(3p_i + 4p_1))\gamma_i\gamma_1$

$- 864p_1^7 x^2 exp_1(3p_i x)\gamma_1^3 - 144p_i^3 exp_1(x(6p_1 + p_i))p_1^2\gamma_i + 1152p_1^4 exp_1(x(2p_1 + 3p_i))p_i\gamma_1^2 + 4320p_i^5 exp_1(x(2p_1+p_i))\gamma_1^2\gamma_i - 576p_1^5 exp_1(x(4p_1+p_i))\gamma_i\gamma_1 - 576p_1^3 exp_1(x(3p_i-2p_1))\gamma_1^4 p_i^2 - 216p_1^7 exp_1(x(-4p_1 + 3p_i))\gamma_1^5 x^2 + 1152p_1^5 exp_1(x(-2p_1 + p_i))\gamma_1^4\gamma_i + 1206xp_1^6 exp_1(x(3p_i + 4p_1))\gamma_1 - 288p_1^5 exp_1(-x(p_i-4p_1))\gamma_i^3\gamma_1 + 144\gamma_i^2 p_i^3 exp_1(-x(-6p_1+p_i))p_1^2 + 432p_1^7 exp_1(x(2p_1+3p_i))\gamma_1^2 x^2 - 576p_1^5 exp_1(-x(-2p_1+p_i))\gamma_i^3\gamma_1^2 + 288\gamma_1^5 p_1^4 exp_1(x(-4p_1+3p_i))p_i - 576p_1^5 exp_1(x(2p_1+3p_i))\gamma_i\gamma_1^2 + 576xp_i^4 exp_1(x(6p_1+p_i))p_1^2\gamma_i - 240\gamma_1^4 p_1^2 exp_1(x(3p_i-2p_1))\gamma_i p_i^5 x^2 - 184p_1^4 exp_1(x(4p_1+p_i))x^2 p_i^3\gamma_1\gamma_i$

$- 1440\gamma_i^2 p_i^4 exp_1(-x(-2p_1 + p_i))\gamma_1^2 p_1 + 11520xp_1^2 exp_1(p_i x)\gamma_1^3\gamma_i^2 p_i^4 + 648\gamma_1^5 p_1^5 exp_1(x(p_i - 4p_1))x^2 p_i^2\gamma_i + 2142x\gamma_1^5 p_1^5 exp_1(x(p_i - 4p_1))p_i\gamma_i + 216\gamma_1^5 p_1^4 exp_1(-x(3p_i + 4p_1))x^2\gamma_i^3 p_i^3 + 432\gamma_1^2 p_1^5 exp_1(x(2p_1+3p_i))x^2 p_i^2 - 3168\gamma_1^4 p_1^3 exp_1(-x(2p_1+p_i))\gamma_i^2 p_i^2 + 2304xp_1^5 exp_1(-x(2p_1+$

$p_i))\gamma_1^4\gamma_i^3p_i-7236xp_i^2exp_1(3p_ix)\gamma_1^3p_1^4+4608x\gamma_i^2p_i^5exp_1(-x(4p_1+p_i))\gamma_1^5p_1+736p_1^6exp_1(x(-2p_1+$

$p_i))\gamma_1^4\gamma_ip_ix^2 - 1152p_1^2exp_1(x(4p_1 + p_i))p_i^3\gamma_1\gamma_i + 27p_1^6exp_1(-x(-2p_1 + p_i))\gamma_1^2\gamma_i^2p_i^2x^3 -$

$152p_1^7x^2exp_1(-x(p_i-4p_1))\gamma_i^2\gamma_1-992p_1^6exp_1(x(2p_1+p_i))\gamma_1^2p_ix^2\gamma_i+1152\gamma_i^2p_i^4exp_1(-x(4p_1+$

$p_i))\gamma_1^5p_1-864p_1^6exp_1(-x(2p_1+p_i))\gamma_1^4\gamma_i^2p_ix^2-1296\gamma_1^4p_1^4exp_1(-x(2p_1+p_i))\gamma_i^2p_i-864p_1^7x^2exp_1(-3p_ix)\gamma_1^3\gamma_i^3-$

$27p_1^6x^3exp_1(x(4p_1 + p_i))p_i^2\gamma_i\gamma_1 + 64p_1^4exp_1(p_ix)\gamma_1^3\gamma_i^2p_i^3x^2 - 712\gamma_1^5p_1^7x^2exp_1(-x(4p_1 +$

$p_i))\gamma_i^2-2304p_1^4exp_1(-p_ix)\gamma_1^3\gamma_i^3p_i+576x\gamma_1^6p_1^4exp_1(-x(6p_1+p_i))\gamma_i^3p_i^2-4446xp_i^2exp_1(x(2p_1+$

$3p_i))\gamma_1^2p_1^4 - 32\gamma_1^6p_1^3exp_1(3x(-2p_1 + p_i))x^2\gamma_ip_i^4 + 12672xp_1^2exp_1(x(4p_1 + p_i))p_i^4\gamma_1\gamma_i +$

$27p_1^6\gamma_1^5exp_1(x(-4p_1+3p_i))x^3p_i^2+54p_i^3exp_1(-x(2p_1+p_i))\gamma_1^4\gamma_i^2p_1^5x^3+1232\gamma_1^3p_1^4exp_1(p_ix)x^2p_i^3\gamma_i-$

$184p_1^6x^2exp_1(-x(-4p_1+3p_i))\gamma_1\gamma_i^3p_i-16\gamma_1^6p_1^2exp_1(3x(-2p_1+p_i))\gamma_ip_i^5x^2+27p_1^8x^3exp_1(-x(p_i-$

$4p_1))\gamma_i^2\gamma_1 - 1440p_1^3exp_1(-x(2p_1 + p_i))\gamma_1^4\gamma_i^3p_i^2 + 4608xp_1^5exp_1(x(-2p_1 + p_i))\gamma_1^4\gamma_i^2p_i +$

$27p_1^7exp_1(-x(-4p_1 + 3p_i))\gamma_1x^3\gamma_i^3p_i - 23040xp_i^6exp_1(p_ix)\gamma_1^3\gamma_i - 144\gamma_1^4p_1^4exp_1(x(-2p_1 +$

$p_i))p_i\gamma_i+864p_1^5x^2exp_1(3p_ix)\gamma_1^3p_i^2-2466xp_1^5exp_1(-x(-2p_1+p_i))\gamma_1^2\gamma_i^2p_i+1152\gamma_i^3p_i^3exp_1(-x(3p_i-$

$2p_1))\gamma_1^2p_1^2 + 192p_1^6exp_1(3p_ix)\gamma_1^3\gamma_ip_ix^2 - 54p_1^7exp_1(3p_ix)x^3\gamma_1^3p_i - 1314xp_1^5exp_1(x(3p_i +$

$4p_1))p_i\gamma_1+54p_1^6exp_1(-x(-4p_1+3p_i))x\gamma_1\gamma_i^3+2304\gamma_1^5p_1^5exp_1(x(p_i-4p_1))x\gamma_i^2p_i+576xp_i^4exp_1(x(-6p_1+$

$p_i))\gamma_1^6p_1^2\gamma_i - 16\gamma_1^6p_1^4exp_1(3x(-2p_1 + p_i))\gamma_ip_i^3x^2 + 12672xp_1^2exp_1(-x(p_i - 4p_1))\gamma_i^2p_i^4\gamma_1 -$

$40572xp_i^2exp_1(x(-2p_1+p_i))\gamma_1^4p_1^4\gamma_i-3312p_i^3exp_1(x(2p_1+p_i))p_1^2\gamma_1^2\gamma_i+32p_i^4exp_1(x(-6p_1+$

$p_i))\gamma_1^6p_1^3\gamma_ix^2+16p_1^2exp_1(x(6p_1+p_i))\gamma_i^2p_i^5x^2+1152xp_1^3exp_1(-x(-6p_1+p_i))\gamma_i^3p_i^3+240p_1^2exp_1(x(-2p_1+$

$p_i))\gamma_1^4\gamma_i^2p_i^5x^2 + 128p_1^7exp_1(x(-2p_1 + p_i))\gamma_1^4\gamma_i^2x^2 + 1152p_1^6exp_1(-x(4p_1 + p_i))\gamma_1^5x\gamma_i^3 -$

$324xp_i^3exp_1(-p_ix)\gamma_1^3p_1^3\gamma_i^2 + 18p_1^5exp_1(3p_ix)x^3\gamma_1^3p_i^3 + 16p_1^4exp_1(x(2p_1 + 3p_i))\gamma_ip_i^3x^2\gamma_1^2 -$

$$9p_1^5exp_1(x(2p_1+3p_i))x^3\gamma_1^2p_i^3+96\gamma_1^5p_1^2exp_1(x(p_i-4p_1))\gamma_i^2p_i^5x^2-648p_1^6x^2exp_1(x(-4p_1+$$

$$3p_i))\gamma_1^5p_i+9216xp_1^6exp_1(x(2p_1+p_i))\gamma_1^2\gamma_i^2+6912xp_1^6exp_1(3p_ix)\gamma_1^3\gamma_i+3312\gamma_i^2p_i^3exp_1(-x(-2p_1+$$

$$p_i))\gamma_1^2p_1^2-1152xp_i^5exp_1(-x(-6p_1+p_i))\gamma_i^2p_1-32p_1^7x^2exp_1(x(3p_i+4p_1))\gamma_i\gamma_1$$

$$-6912x\gamma_i^2p_i^6exp_1(-x(4p_1+p_i))\gamma_1^5+240p_1^2exp_1(x(2p_1+p_i))\gamma_1^2\gamma_i^2p_i^5x^2-32p_1^4exp_1(x(4p_1+$$

$$p_i))x^2\gamma_i^2p_i^3\gamma_1+3456\gamma_1^5p_1^2exp_1(x(p_i-4p_1))\gamma_i^2p_i^4x+4608x\gamma_1^4p_1^6exp_1(x(3p_i-2p_1))\gamma_i+$$

$$2790xp_i^3exp_1(x(2p_1+p_i))\gamma_1^2p_1^3\gamma_i+128p_1^3exp_1(x(3p_i+4p_1))x^2\gamma_ip_i^4\gamma_1+81p_1^7exp_1(x(-4p_1+$$

$$3p_i))x^3\gamma_1^5p_i-16p_1^4exp_1(3x(2p_1+p_i))\gamma_ip_i^3x^2+680\gamma_1^5p_1^4exp_1(x(p_i-4p_1))x^2p_i^3\gamma_i+288\gamma_i^2p_i^4exp_1(-x(6p_1+$$

$$p_i))\gamma_1^6p_1-4608xp_1^5exp_1(x(2p_1+p_i))\gamma_1^2\gamma_i^2p_i+320p_1^2exp_1(p_ix)\gamma_1^3\gamma_i^2p_i^5x^2+9p_1^5exp_1(-x(3p_i-$$

$$2p_1))x^3\gamma_1^2\gamma_i^3p_i^3+128p_1^6exp_1(x(2p_1+3p_i))\gamma_i\gamma_1^2p_ix^2-16p_i^3exp_1(x(2p_1+p_i))\gamma_1^2p_1^4x^2\gamma_i-$$

$$96\gamma_i^2p_i^5x^2exp_1(-x(p_i-4p_1))p_1^2\gamma_1+2160p_1^2exp_1(-x(-2p_1+p_i))\gamma_1^2\gamma_i^3p_i^3+4608xp_1^6exp_1(-x(2p_1+$$

$$p_i))\gamma_1^4\gamma_i^3+108xp_1^6exp_1(-3p_ix)\gamma_1^3\gamma_i^3+2032p_1^7exp_1(-x(2p_1+p_i))\gamma_1^4\gamma_i^2x^2+64p_1^7exp_1(x(3p_i-$$

$$2p_1))\gamma_1^4\gamma_ix^2-432\gamma_1^4p_1^5exp_1(-x(2p_1+3p_i))\gamma_i^3p_i^2x^2+1424p_1^7exp_1(-x(-2p_1+p_i))\gamma_1^2x^2\gamma_i^2-$$

$$41058xp_i^2exp_1(x(2p_1+p_i))\gamma_1^2p_1^4\gamma_i-27p_1^6exp_1(x(2p_1+3p_i))x^3\gamma_1^2p_i^2-96\gamma_1^5p_1^2exp_1(x(-4p_1+$$

$$3p_i))\gamma_ip_i^5x^2+324xp_1^5exp_1(p_ix)p_i\gamma_1^3\gamma_i-11682xp_1^4exp_1(x(4p_1+p_i))p_i^2\gamma_1\gamma_i-128p_1^3exp_1(x(4p_1+$$

$$p_i))x^2p_i^4\gamma_i\gamma_1-990x\gamma_i^3p_i^2exp_1(-x(-4p_1+3p_i))p_1^4\gamma_1-1440p_i^4exp_1(x(2p_1+p_i))p_1\gamma_1^2\gamma_i-$$

$$64p_1^7exp_1(x(2p_1+3p_i))\gamma_ix^2\gamma_1^2-8064p_1^4exp_1(x(4p_1+p_i))x\gamma_i^2p_i^2\gamma_1+4608xp_1^6exp_1(x(2p_1+$$

$$3p_i))\gamma_i\gamma_1^2+576x\gamma_1^6p_1^2exp_1(x(-6p_1+p_i))\gamma_i^2p_i^4+432p_i^2exp_1(-x(2p_1+p_i))\gamma_1^4p_1^5\gamma_i^2x^2+$$

$$1728\gamma_i^3p_i^3exp_1(-3p_ix)\gamma_1^3p_1^2-27\gamma_1^5p_1^6x^3exp_1(-x(4p_1+p_i))\gamma_i^2p_i^2+8640xp_1^2exp_1(x(-2p_1+$$

$$p_i))\gamma_1^4\gamma_i^2p_i^4-1152xp_1^5exp_1(x(3p_i+4p_1))\gamma_ip_i\gamma_1+4608xp_1^6exp_1(-x(-2p_1+p_i))\gamma_1^2\gamma_i^3+$$

$$2304\gamma_1^5 p_1^6 exp_1(x(p_i-4p_1))x\gamma_i^2-576xp_1^4 exp_1(x(6p_1+p_i))\gamma_i^2 p_i^2-32p_1^6 exp_1(-x(p_i-4p_1))\gamma_i^3 x^2 p_i\gamma_1+$$

$$1152x\gamma_1^6 p_1 exp_1(-x(6p_1+p_i))\gamma_i^2 p_i^5 - 864p_1^5 x^2 exp_1(p_i x)\gamma_1^3 p_i^2\gamma_i + 1152xp_1^6 exp_1(x(3p_i +$$

$$4p_1))\gamma_i\gamma_1-18\gamma_1^3 p_1^5 x^3 exp_1(-3p_i x)\gamma_i^3 p_i^3+2880p_1^2 exp_1(-p_i x)\gamma_i^3 p_i^3\gamma_1^3+576\gamma_i^3 p_i^2 exp_1(-x(3p_i-$$

$$2p_1))p_1^3\gamma_1^2+288\gamma_1^5 p_1^4 exp_1(x(p_i-4p_1))p_i\gamma_i-288\gamma_1^5 p_1^3 exp_1(-x(3p_i+4p_1))\gamma_i^3 p_i^2+1152\gamma_1^5 p_1^2 exp_1(-x(4p_1+$$

$$p_i))\gamma_i^2 p_i^3+990x\gamma_1^5 p_1^3 exp_1(-x(3p_i+4p_1))\gamma_i^3 p_i^3+9540xp_1^6 exp_1(x(-2p_1+p_i))\gamma_1^4\gamma_i+1314p_1^6 exp_1(-x(p_i-$$

$$4p_1))x\gamma_i^2\gamma_1+144\gamma_1^6 p_1^4 exp_1(-x(6p_1+p_i))\gamma_i^2 p_i-27p_1^7 exp_1(x(4p_1+p_i))\gamma_1\gamma_i p_i x^3-216p_1^5 exp_1(x(3p_i+$$

$$4p_1))x^2 p_i^2\gamma_1 - 4932x\gamma_i^3 p_i^2 exp_1(-x(2p_1+3p_i))\gamma_1^4 p_1^4 + 128p_i^4 x^2 exp_1(x(p_i-4p_1))\gamma_1^5 p_1^3\gamma_i -$$

$$648\gamma_i^3 p_i^2 exp_1(-x(3p_i+4p_1))\gamma_1^5 p_1^5 x^2+4284xp_1^5 exp_1(x(-2p_1+p_i))\gamma_1^4\gamma_i p_i-54p_1^6 exp_1(x(-2p_1+$$

$$p_i))\gamma_1^4\gamma_i p_i^2 x^3+45504xp_i^4 exp_1(-x(-2p_1+p_i))\gamma_1^2 p_1^2\gamma_i^2-4608xp_1 exp_1(-x(p_i-4p_1))\gamma_i^2 p_i^5\gamma_1+$$

$$4608xp_i^5 exp_1(x(4p_1+p_i))\gamma_1 p_1\gamma_i+3456p_1^2 exp_1(x(4p_1+p_i))\gamma_i^2 p_i^4 x\gamma_1+864p_1^6 exp_1(-x(-2p_1+$$

$$p_i))\gamma_1^2\gamma_i^2 p_i x^2-2592p_1^7 exp_1(-p_i x)\gamma_1^3 x^2\gamma_i^2-32p_1^3 exp_1(x(6p_1+p_i))x^2 p_i^4\gamma_i+1152p_1^3 exp_1(-x(p_i-$$

$$4p_1))\gamma_i^3\gamma_1 p_i^2-27\gamma_1^6 p_1^7 exp_1(x(-6p_1+p_i))x^3\gamma_i p_i+7236xp_1^6 exp_1(-p_i x)\gamma_1^3\gamma_i^2+27p_1^5 x^3 exp_1(x(4p_1+$$

$$p_i))\gamma_i p_i^3\gamma_1 - 54p_1^6 exp_1(-x(2p_1+p_i))\gamma_1^4\gamma_i^2 p_i^2 x^3 - 17280xp_i^6 exp_1(-x(-2p_1+p_i))\gamma_1^2\gamma_i^2 +$$

$$288\gamma_1^5 p_1^4 exp_1(x(p_i-4p_1))\gamma_i^2 p_i+576xp_1^4 exp_1(-x(-6p_1+p_i))\gamma_i^3 p_i^2-128\gamma_1^5 p_1^3 exp_1(x(-4p_1+$$

$$3p_i))\gamma_i p_i^4 x^2-320p_1^2 exp_1(3p_i x)\gamma_1^3\gamma_i p_i^5 x^2-27p_1^6 x^3 exp_1(-x(p_i-4p_1))\gamma_i^2 p_i^2\gamma_1-288p_1^5 exp_1(x(3p_i+$$

$$4p_1))\gamma_1-576p_1^5 exp_1(x(2p_1+3p_i))\gamma_1^2-162xp_i^3 exp_1(x(-6p_1+p_i))\gamma_1^6 p_1^3\gamma_i-240\gamma_i^2 p_i^5 exp_1(-x(2p_1+$$

$$p_i))x^2\gamma_1^4 p_1^2 + 1152xp_i^3 exp_1(-x(-6p_1 + p_i))\gamma_i^2 p_1^3 + 240p_i^5 exp_1(x(2p_1 + p_i))\gamma_1^2 p_1^2\gamma_i x^2 -$$

$$4770x\gamma_1^5 p_1^4 exp_1(-x(4p_1+p_i))\gamma_i^2 p_i^2+45504xp_i^4 exp_1(-x(2p_1+p_i))\gamma_1^4 p_1^2\gamma_i^2+64p_1^7 exp_1(-x(2p_1+$$

$$p_i))\gamma_1^4\gamma_i^3 x^2+1152p_i^5 exp_1(x(6p_1+p_i))\gamma_i p_1 x-3132xp_i^3 exp_1(-x(2p_1+p_i))\gamma_1^4 p_1^3\gamma_i^2+324p_1^5\gamma_1^4 exp_1(-x(2p_1+$$

$3p_i))x\gamma_i^3p_i+4608\gamma_i^2p_i^3exp_1(-p_ix)\gamma_1^3p_1^2-864p_1^6exp_1(x(2p_1+3p_i))\gamma_1^2x^2p_i-4608xp_i^5exp_1(x(p_i-$

$4p_1))\gamma_1^5p_1\gamma_i-576\gamma_1^3p_1^4exp_1(p_ix)p_i\gamma_i+4932xp_1^6exp_1(-x(2p_1+p_i))\gamma_1^4\gamma_i^2-2304\gamma_1^3p_1^4exp_1(-p_ix)\gamma_i^2p_i-$

$864\gamma_1^5p_1^3exp_1(x(p_i-4p_1))p_i^2\gamma_i+27p_i^3exp_1(x(2p_1+p_i))\gamma_1^2p_1^5x^3\gamma_i+2142xp_i^3exp_1(x(2p_1+$

$3p_i))p_1^3\gamma_1^2+16\gamma_1^4p_1^4exp_1(x(3p_i-2p_1))\gamma_ip_i^3x^2-1638x\gamma_1^5p_1^3exp_1(x(p_i-4p_1))p_i^3\gamma_i-128p_1^6exp_1(-x(-2p_1+$

$p_i))\gamma_1^2\gamma_i^3p_ix^2+288p_1^4exp_1(x(4p_1+p_i))p_i\gamma_1\gamma_i-2970x\gamma_1^5p_1^3exp_1(-x(4p_1+p_i))\gamma_i^2p_i^3+$

$45\gamma_1^5p_1^8exp_1(-x(3p_i+4p_1))x^3\gamma_i^3+432\gamma_i^3p_i^3exp_1(-3p_ix)\gamma_1^3p_1^4x^2+3294xp_1^3exp_1(-x(p_i-$

$4p_1))\gamma_i^2p_i^3\gamma_1-54p_1^7x^3exp_1\gamma_i-736\gamma_1^4p_1^6exp_1(-x(2p_1+3p_i))\gamma_i^3p_ix^2+1152xp_1^6exp_1(-x(p_i-$

$4p_1))\gamma_i^3\gamma_1+32\gamma_1^5p_1^7exp_1(-x(4p_1+p_i))\gamma_i^3x^2+216p_1^5exp_1(-x(p_i-4p_1))x^2\gamma_i^2p_i^2\gamma_1+32\gamma_1^5p_1^7exp_1(x(-4p_1+$

$3p_i))\gamma_ix^2-144p_1^4exp_1(x(-2p_1+p_i))\gamma_1^4\gamma_i^2p_i+32\gamma_1^5p_1^6exp_1(x(-4p_1+3p_i))\gamma_ip_ix^2$

$-324p_1^5\gamma_1^3exp_1(-3p_ix)x\gamma_i^3p_i+54\gamma_1^3p_1^7exp_1(-p_ix)\gamma_i^2x^3p_i-432\gamma_1^3p_1^6exp_1(-p_ix)x^2\gamma_i^2p_i+$

$288\gamma_i^3p_i^2exp_1(-x(-4p_1+3p_i))p_1^3\gamma_1+11520xp_1^2exp_1(-p_ix)\gamma_1^3\gamma_i^3p_i^4-990x\gamma_i^2p_i^3exp_1(-x(6p_1+$

$p_i))\gamma_1^6p_1^3+16p_1^4exp_1(x(-2p_1+p_i))\gamma_1^4\gamma_i^2p_i^3x^2+162xp_iexp_1(3x(-2p_1+p_i))\gamma_1^6p_1^5+288p_1^4exp_1(x(3p_i+$

$4p_1))\gamma_i\gamma_1p_i-162p_1^6exp_1(-x(3p_i+4p_1))x\gamma_1^5\gamma_i^3-1152\gamma_i^2p_i^4exp_1(-x(p_i-4p_1))\gamma_1p_1-$

$320\gamma_i^2p_i^5exp_1(-p_ix)x^2\gamma_1^3p_1^2+162x\gamma_i^3p_iexp_1(-x(-4p_1+3p_i))p_1^5\gamma_1-4608\gamma_1^5p_1^3exp_1(-x(4p_1+$

$p_i))x\gamma_i^3p_i^3+576x\gamma_i^2p_i^4exp_1(-x(6p_1+p_i))\gamma_1^6p_1^2+54p_1^6x^3exp_1(-p_ix)\gamma_1^3\gamma_i^2p_i^2+9\gamma_i^3p_i^3exp_1(-x(3p_i+$

$4p_1))\gamma_1^5p_1^5x^3-648p_1^5x^2exp_1(x(-4p_1+3p_i))\gamma_1^5p_i^2+864\gamma_1^5p_1^2exp_1(-x(4p_1+p_i))\gamma_i^3p_i^3-$

$16p_i^5exp_1(-x(6p_1+p_i))x^2\gamma_1^6p_1^2\gamma_i^2+2160p_1^2exp_1(-x(2p_1+p_i))\gamma_i^3p_i^3\gamma_1^4-1152p_1^4exp_1(-x(p_i-$

$4p_1))x\gamma_i^3p_i^2\gamma_1-36540x\gamma_i^2p_i^2exp_1(-p_ix)\gamma_1^3p_1^4-2160p_1^2exp_1(x(-2p_1+p_i))\gamma_1^4\gamma_i^2p_i^3-2160p_1^2exp_1(x(2p_1+$

$p_i))\gamma_1^2\gamma_i^2p_i^3-576p_1^4exp_1(p_ix)\gamma_1^3\gamma_i^2p_i-432p_i^2exp_1(-x(-2p_1+p_i))\gamma_1^2p_1^5x^2\gamma_i^2+2016p_1^3exp_1(-x(p_i-$

$$4p_1))\gamma_i^2 p_i^2 \gamma_1 + 9p_1^8 x^3 exp_1(-x(-4p_1 + 3p_i))\gamma_1 \gamma_i^3 + 3168 p_1^3 exp_1(-x(-2p_1 + p_i))\gamma_i^2 \gamma_1^2 p_i^2 +$$

$$248p_1^4 exp_1(-x(p_i - 4p_1))x^2 \gamma_i^2 p_i^3 \gamma_1 + 27p_1^5 \gamma_1^6 exp_1(x(-6p_1 + p_i))x^3 p_i^3 \gamma_i - 1360 p_i^3 exp_1(-p_i x)\gamma_1^3 p_1^4 \gamma_i^2 x^2 -$$

$$4608p_i^3 exp_1(p_i x)\gamma_1^3 p_1^2 \gamma_i + 81p_1^7 exp_1(x(2p_1 + 3p_i))x^3 \gamma_1^2 p_i + 27p_1^7 \gamma_1^6 exp_1(-x(6p_1 + p_i))x^3 p_i \gamma_i^2 +$$

$$12672xp_i^4 exp_1(x(p_i - 4p_1))\gamma_1^5 p_1^2 \gamma_i - 9\gamma_1^6 p_1^8 exp_1(-3x(2p_1 + p_i))x^3 \gamma_i^3 - 23292 xp_i^2 exp_1(-x(2p_1 +$$

$$p_i))\gamma_1^4 p_1^4 \gamma_i^2 + 992p_1^6 exp_1(-x(3p_i - 2p_1))\gamma_i^3 \gamma_1^2 p_i x^2 + 14148 xp_1^6 exp_1(p_i x)\gamma_1^3 \gamma_i + 1152\gamma_i^3 p_i^3 exp_1(-x(2p_1 +$$

$$3p_i))\gamma_1^4 p_1^2 + 160\gamma_1^2 p_1^3 exp_1(x(2p_1 + 3p_i))\gamma_i p_i^4 x^2 + 2628xp_1^5 exp_1(-x(2p_1 + p_i))\gamma_1^4 \gamma_i^2 p_i +$$

$$5760xp_i^5 exp_1(x(2p_1 + p_i))\gamma_1^2 p_1 \gamma_i - 16128xp_1^4 exp_1(-p_i x)\gamma_1^3 \gamma_i^3 p_i^2 + 1152\gamma_1^4 p_1^4 exp_1(x(3p_i -$$

$$2p_1))\gamma_i p_i - 135p_1^8 x^3 exp_1(-x(-2p_1 + p_i))\gamma_1^2 \gamma_i^2 - 27072xp_1^4 exp_1(x(2p_1 + p_i))\gamma_1^2 \gamma_i^2 p_i^2 -$$

$$54p_1^6 exp_1(3p_i x)x^3 \gamma_1^3 p_i^2 + 1152p_i^4 exp_1(x(p_i - 4p_1))\gamma_1^5 p_1 \gamma_i + 576p_1^5 exp_1(x(3p_i - 2p_1))\gamma_1^4 +$$

$$1152\gamma_1^5 p_1^6 exp_1(x(-4p_1 + 3p_i))x\gamma_i + 162\gamma_1^6 p_1^6 exp_1(x(-6p_1 + p_i))x\gamma_i - 54p_1^3 \gamma_1^6 exp_1(-3x(2p_1 +$$

$$p_i))x\gamma_i^3 p_i^3 + 8640xp_1^2 exp_1(x(2p_1 + p_i))\gamma_1^2 \gamma_i^2 p_i^4 + 144\gamma_1^6 p_1^4 exp_1(-x(6p_1 + p_i))\gamma_i^3 p_i - 128p_1^6 exp_1(-x(2p_1 +$$

$$p_i))\gamma_1^4 \gamma_i^3 p_i x^2 + 184p_1^6 exp_1(x(4p_1 + p_i))x^2 p_i \gamma_1 \gamma_i - 990xp_i^3 exp_1(x(-4p_1 + 3p_i))\gamma_1^5 p_1^3 +$$

$$27p_1^6 exp_1(x(2p_1 + p_i))\gamma_1^2 \gamma_i p_i^2 x^3 + 54p_1^7 exp_1(-x(2p_1 + p_i))\gamma_1^4 \gamma_i^2 p_i x^3 + 990xp_1^5 exp_1(x(-4p_1 +$$

$$3p_i))\gamma_1^5 p_i + 432p_i^2 exp_1(x(-2p_1 + p_i))\gamma_1^4 p_1^5 x^2 \gamma_i - 90p_1^8 exp_1(-x(2p_1 + 3p_i))\gamma_i^3 \gamma_1^4 x^3 - 9792xp_1^4 exp_1(-x(2p_1 +$$

$$p_i))\gamma_1^4 \gamma_i^3 p_i^2 - 27p_1^5 exp_1(-x(4p_1 + p_i))x^3 \gamma_1^5 p_i^3 \gamma_i^2 - 162x\gamma_1^6 p_1^5 exp_1(-3x(2p_1 + p_i))\gamma_i^3 p_i +$$

$$27p_1^6 exp_1(x(3p_i + 4p_1))x^3 \gamma_1 p_i^2 - 135p_1^8 x^3 exp_1(x(2p_1 + p_i))\gamma_1^2 \gamma_i + 864p_1^2 exp_1(-x(p_i -$$

$$4p_1))\gamma_i^3 p_i^3 \gamma_1 - 17280xp_i^6 exp_1(x(2p_1 + p_i))\gamma_1^2 \gamma_i + 2304x\gamma_1^4 p_1^5 exp_1(x(3p_i - 2p_1))\gamma_i p_i +$$

$$54p_1^6 exp_1(x(3p_i - 2p_1))x^3 \gamma_1^4 p_i^2 - 990p_1^4 exp_1(x(-4p_1 + 3p_i))x\gamma_1^5 p_i^2 - 7236x\gamma_i^3 p_i^2 exp_1(-3p_i x)\gamma_1^3 p_1^4 -$$

$$81p_1^7 exp_1(-x(-2p_1 + p_i))\gamma_1^2 \gamma_i^2 x^3 p_i - 680\gamma_1^5 p_1^6 exp_1(x(p_i - 4p_1))x^2 p_i \gamma_i - 45p_1^8 exp_1(-x(3p_i -$$

$2p_1))\gamma_1^2 x^3\gamma_i^3 + 9216xp_1^6 exp_1(x(-2p_1 + p_i))\gamma_1^4\gamma_i^2 + 288p_1^4 exp_1(x(4p_1 + p_i))\gamma_i^2 p_i\gamma_1$
$+4122xp_1^6 exp_1(-x(-2p_1+p_i))\gamma_1^2\gamma_i^2+72p_1^4 exp_1(x(3p_i+4p_1))p_i^3 x^2\gamma_1-27p_1^7 exp_1(x(3p_i+$
$4p_1))x^3\gamma_1 p_i+6912xp_1^6 exp_1(-p_i x)\gamma_1^3\gamma_i^3-144p_1^4 exp_1(x(2p_1+p_i))\gamma_i^2\gamma_1^2 p_i-2628xp_i^3 exp_1(x(-2p_1+$
$p_i))\gamma_1^4 p_1^3\gamma_i+54\gamma_1^6 p_1^6 exp_1(-3x(2p_1+p_i))x\gamma_i^3+1152p_1^4 exp_1(x(2p_1+3p_i))\gamma_1^2\gamma_i p_i-32\gamma_1^5 p_1^4 exp_1(x(p_i-$
$4p_1))x^2\gamma_i^2 p_i^3 + 16p_i^5 exp_1(x(-6p_1 + p_i))\gamma_1^6 p_1^2 x^2\gamma_i - 27p_1^6\gamma_1^6 exp_1(-3x(2p_1 + p_i))x^3\gamma_i^3 p_i^2 +$
$9p_1^5 exp_1(-x(-4p_1+3p_i))x^3\gamma_1\gamma_i^3 p_i^3-184p_1^7 exp_1(x(4p_1+p_i))\gamma_i x^2\gamma_1+680\gamma_1^5 p_1^6 exp_1(-x(3p_i+$
$4p_1))x^2\gamma_i^3 p_i + 320p_i^5 exp_1(p_i x)x^2\gamma_1^3 p_1^2\gamma_i + 576xp_i^2 exp_1 x)\gamma_1^3 p_1^4\gamma_i + 144p_i^3 exp_1(-x(6p_1 +$
$p_i))\gamma_1^6 p_1^2\gamma_i^2 - 16\gamma_i^2 p_i^5 exp_1(-x(-6p_1 + p_i))p_1^2 x^2 + 414xp_i^2 exp_1(-x(6p_1 + p_i))\gamma_1^6 p_1^4\gamma_i^2 -$
$864\gamma_1^5 p_1^2 exp_1(x(p_i-4p_1))\gamma_i^2 p_i^3-17280x\gamma_i p_i^6 exp_1(x(-2p_1+p_i))\gamma_1^4+45504xp_i^4 exp_1(x(2p_1+$
$p_i))\gamma_1^2 p_1^2\gamma_i - 216p_1^6 exp_1(-x(p_i - 4p_1))x^2\gamma_i^2 p_i\gamma_1 + 54xp_i^3 exp_1(3x(-2p_1 + p_i))\gamma_1^6 p_1^3$
$-4446x\gamma_i^3 p_i^2 exp_1(-x(3p_i-2p_1))p_1^4\gamma_1^2+9\gamma_i^3 p_i^3 exp_1(-3x(2p_1+p_i))\gamma_1^6 p_1^5 x^3+1152\gamma_i^2 p_i^3 exp_1(-x(p_i-$
$4p_1))p_1^2\gamma_1-5760xp_i^5 exp_1(x(-2p_1+p_i))\gamma_1^4 p_1\gamma_i-6912xp_i^6 exp_1(x(4p_1+p_i))\gamma_i\gamma_1+4608p_1^3 exp_1(-x(p_i-$
$4p_1))x\gamma_i^3 p_i^3\gamma_1 - 81\gamma_1^5 p_1^7 x^3 exp_1(-x(4p_1 + p_i))\gamma_i^2 p_i - 576xp_i^2 exp_1(x(6p_1 + p_i))p_1^4\gamma_i$
$-616\gamma_1^5 p_1^4 exp_1(-x(4p_1+p_i))x^2\gamma_i^2 p_i^3+270p_1^8 x^3 exp_1(-p_i x)\gamma_1^3\gamma_i^2-16\gamma_1^6 p_1^4 exp_1(x(-6p_1+$
$p_i))\gamma_i^2 p_i^3 x^2+27\gamma_1^5 p_1^5 x^3 exp_1(x(p_i-4p_1))p_i^3\gamma_i+8730xp_1^6 exp_1(x(2p_1+p_i))\gamma_1^2\gamma_i-27072xp_1^4 exp_1(x(-2p_1+$
$p_i))\gamma_1^4\gamma_i^2 p_i^2 + 1314xp_1^5 exp_1(-x(4p_1 + p_i))\gamma_1^5\gamma_i^2 p_i + 2096p_1^7 exp_1(x(-2p_1 + p_i))\gamma_1^4 x^2\gamma_i +$
$32p_1^6 exp_1(x(3p_i+4p_1))\gamma_1\gamma_i p_i x^2-27\gamma_i p_i^2 exp_1(x(p_i-4p_1))\gamma_1^5 p_1^6 x^3+135\gamma_1^5 p_1^8 x^3 exp_1(-x(4p_1+$
$p_i))\gamma_i^2-4932p_1^4 exp_1(x(3p_i-2p_1))x\gamma_1^4 p_i^2+16p_1^4 exp_1(x(2p_1+p_i))\gamma_1^2\gamma_i^2 p_i^3 x^2-144\gamma_i p_i^3 exp_1(x(-6p_1+$
$p_i))\gamma_1^6 p_1^2 - 162xp_1^5 exp_1(-x(3p_i - 2p_1))\gamma_1^2\gamma_i^3 p_i - 990x\gamma_i^3 p_i^2 exp_1(-x(3p_i + 4p_1))\gamma_1^5 p_1^4 +$

$216p_1^5exp_1(x(4p_1+p_i))x^2p_i^2\gamma_1\gamma_i-990x\gamma_i^2p_iexp_1(-x(p_i-4p_1))p_1^5\gamma_1-81p_1^7x^3exp_1(-x(3p_i-$

$2p_1))\gamma_1^2\gamma_i^3p_i-81p_1^7\gamma_1^5exp_1(-x(3p_i+4p_1))x^3\gamma_i^3p_i-40p_1^7exp_1(-x(-4p_1+3p_i))\gamma_1\gamma_i^3x^2-$

$2304p_1^5exp_1(x(4p_1+p_i))x\gamma_i^2p_i\gamma_1+54p_1^7exp_1(-x(2p_1+3p_i))x^3\gamma_1^4\gamma_i^3p_i-192p_1^6exp_1(-p_ix)\gamma_1^3\gamma_i^3p_ix^2+$

$18p_1^5exp_1(x(3p_i-2p_1))x^3\gamma_1^4p_i^3+2304p_1^6exp_1(x(4p_1+p_i))x\gamma_i^2\gamma_1+864p_1^3exp_1(x(4p_1+$

$p_i))p_i^2\gamma_1\gamma_i+1818xp_1^6exp_1(x(p_i-4p_1))\gamma_1^5\gamma_i+16\gamma_1^6p_1^2exp_1(x(-6p_1+p_i))\gamma_i^2p_i^5x^2+162x\gamma_1^6p_1^5exp_1(x(-6p_1+$

$p_i))p_i\gamma_i-160p_i^4exp_1(x(2p_1+p_i))\gamma_1^2p_1^3\gamma_ix^2-6912xp_i^6exp_1(x(p_i-4p_1))\gamma_1^5\gamma_i-1098p_1^3exp_1(-x(-4p_1+$

$3p_i))x\gamma_1\gamma_i^3p_i^3-144\gamma_1^6p_1^2exp_1(x(-6p_1+p_i))\gamma_i^2p_i^3-27p_1^6exp_1(-x(3p_i-2p_1))\gamma_i^3p_i^2x^3\gamma_1^2-$

$96p_1^2exp_1(x(3p_i+4p_1))\gamma_ip_i^5x^2\gamma_1-1296p_1^4exp_1(-x(-2p_1+p_i))\gamma_i^2\gamma_1^2p_i+108x\gamma_1^4p_1^6exp_1(-x(2p_1+$

$3p_i))\gamma_i^3-576\gamma_1^4p_1^4exp_1(x(3p_i-2p_1))x^2p_i^3-64p_1^7x^2exp_1(x(4p_1+p_i))\gamma_i^2\gamma_1+27p_1^7exp_1(-x(p_i-$

$4p_1))x^3\gamma_i^2p_i\gamma_1-248\gamma_1^5p_1^7exp_1(-x(3p_i+4p_1))\gamma_i^3x^2+135p_1^8exp_1(x(p_i-4p_1))\gamma_1^5x^3\gamma_i-$

$288\gamma_ip_i^4exp_1(x(6p_1+p_i))p_1-1152p_1^5exp_1(x(2p_1+p_i))\gamma_1^2\gamma_i-288p_1^5exp_1(-x(p_i-4p_1))\gamma_i^2\gamma_1-$

$4320\gamma_i^2p_i^5exp_1(-x(2p_1+p_i))\gamma_1^4+576p_i^2exp_1(x(2p_1+3p_i))p_1^3\gamma_1^2-27p_1^7\gamma_1^6exp_1(3x(-2p_1+$

$p_i))x^3p_i+1440p_1^3exp_1(-x(-2p_1+p_i))\gamma_1^2\gamma_i^3p_i^2+27p_1^6exp_1(-x(-4p_1+3p_i))\gamma_1x^3\gamma_i^3p_i^2-$

$1728p_i^2exp_1(x(-2p_1+p_i))\gamma_1^4p_1^3\gamma_i-216\gamma_i^3p_i^2exp_1(-x(-4p_1+3p_i))\gamma_1p_1^5x^2+81\gamma_1^5p_1^7x^3exp_1(x(p_i-$

$4p_1))\gamma_ip_i-1152xp_1^4exp_1(-x(4p_1+p_i))\gamma_1^5\gamma_i^3p_i^2-1152\gamma_1^6p_1^3exp_1(-x(6p_1+p_i))\gamma_i^3p_i^3x-$

$2142p_1^5exp_1(x(2p_1+3p_i))x\gamma_1^2p_i+324xp_1^5exp_1(3p_ix)\gamma_1^3p_i+1152xp_1^5exp_1(-x(4p_1+p_i))\gamma_1^5\gamma_i^3p_i)/(exp_1(p_1x)+$

$\gamma_1exp_1(-p_1x))^3/(-p_iexp_1(x(p_i+p_1))-p_iexp_1(x(p_i-p_1))\gamma_1+\gamma_ip_iexp_1(-x(p_i-p_1))+$

$\gamma_ip_iexp_1(-x(p_i+p_1))\gamma_1+p_1exp_1(x(p_i+p_1))+p_1exp_1(-x(p_i-p_1))\gamma_i-\gamma_1p_1exp_1(x(p_i-$

$p_1))-\gamma_1p_1exp_1(-x(p_i+p_1))\gamma_i)^3)$