An elliptic problem for single phase flows in random media

Student: You-Xuan Chen **Advisor:**Dr. Li-Ming Yeh

Department of Applied Mathematics

National Chiao Tung University

Hsinchu, Taiwan, R.O.C.

Abstract

Consider the linearized equations of slightly compressible single fluid flows through a highly heterogeneous random porous medium, consisting of two types of material. Due to the high heterogeneity of the two materials, the ratio of their permeability coefficients is of order ε^2, where ε is the characteristic scale of heterogeneities. A homogenized problem is obtained by using the stochastic two scale convergence in the mean and by means of convergence results adapted to a priori estimates and to the random geometry.

Contents

1 Introduction 1

2 Ergodic dynamic system and relative theorems 1

3 ε-problem and main result 4

4 A priori estimate 5

5 Stochastic two-scale convergence 8

6 Auxiliary problem and convergence result 11

Notation

Notation	Definition
(Ω, Ξ, μ)	A probability space
Ω	the sample space
Ξ	a σ-algebra of subsets of Ω
μ	μ is the measure
$\mathcal{T}$	a family of invertible measurable maps $\mathcal{T}(x) : \Omega \to \Omega$, for all $x \in \mathbb{R}^n$
$L^2(\Omega)$	$L^2(\Omega, \Xi, \mu)$
$U(x)$	a n-parameters group of unitary operators on $L^2(\Omega)$
$\mathcal{F}$	a measurable set $\mathcal{F} \in \Xi$ such that $\mu(\mathcal{F}) > 0$ and $\mu(\Omega \setminus \mathcal{F}) > 0$
$F(w)$	$F(w) = \{x \in \mathbb{R}^n : \mathcal{T}(x)w \in \mathcal{F}\}$.
$\mathcal{M}$	$\mathcal{M} = \Omega \setminus \mathcal{F}$
$M(w)$	$M(w) = \mathbb{R}^n \setminus F(w)$
$M_\varepsilon(w)$	$M_\varepsilon(w) = \{x \in \mathbb{R}^n : \varepsilon^{-1}x \in M(w)\}$
Q	Q is a smooth bounded domain in $\mathbb{R}^n$
Q_1^ε	$Q_1^\varepsilon = \{x \in Q : dist(x, \partial Q) \geq \varepsilon\}$
$Q_f^\varepsilon(w)$	$Q_f^\varepsilon(w) = Q \setminus \overline{M_\varepsilon(w) \cap Q_1^\varepsilon}$
$Q_m^\varepsilon(w)$	$Q_m^\varepsilon(w) = Q \setminus \overline{Q_f^\varepsilon(w)}$
$\boldsymbol{L}^2(\Omega)$	$\boldsymbol{L}^2(\Omega) \equiv (L^2(\Omega))^n$
$curl_x \boldsymbol{f}$	$\boldsymbol{f} = \{f_1 ... f_n\}, f_i \in L^2_{loc}(\mathbb{R}^n), i = 1, ..., n. \int_{\mathbb{R}^n} (f_i \frac{\partial \phi}{\partial x_j} - f_j \frac{\partial \phi}{\partial x_i})\, dx = 0, \forall \phi \in C_0^\infty(\mathbb{R}^n)$
$div_x \boldsymbol{f}$	$\boldsymbol{f} = \{f_1 ... f_n\}, f_i \in L^2_{loc}(\mathbb{R}^n), i = 1, ..., n. \int_{\mathbb{R}^n} f_i \frac{\partial \phi}{\partial x_i}\, dx = 0, \forall \phi \in C_0^\infty(\mathbb{R}^n)$
$\boldsymbol{L}^2_{pot}(\Omega)$	$\boldsymbol{L}^2_{pot}(\Omega) = \{f \in \boldsymbol{L}^2(\Omega) : curl_x f(\mathcal{T}(x)w) = 0 \ in \ \mathbb{R}^n\}$
$\boldsymbol{L}^2_{sol}(\Omega)$	$\boldsymbol{L}^2_{sol}(\Omega) = \{f \in \boldsymbol{L}^2(\Omega) : div_x f(\mathcal{T}(x)w) = 0 \ in \ \mathbb{R}^n\}$
$\mathbb{E}\{f\}$	$\mathbb{E}\{f\} \equiv \int_\Omega f(w) d\mu$
$\mathcal{V}^2_{pot}(\Omega)$	$\mathcal{V}^2_{pot}(\Omega) = \{f \in \boldsymbol{L}^2_{pot}(\Omega) : \mathbb{E}\{f\} = 0\}$.
X	the closure of $\mathcal{V}^2_{pot}(\Omega)$ in $\boldsymbol{L}^2(\Omega \setminus \mathcal{M})$
$(D_j f)(w)$	$(D_j f)(w) = \lim_{x_i \to 0, x_j = 0, i \neq j} \dfrac{f(\mathcal{T}(x)w) - f(w)}{x_i}$
$\mathcal{D}_j$	$\mathcal{D}_j = \{ f \in \boldsymbol{L}^2(\Omega) \mid (D_j f)(\mathcal{T}(x)w) = \dfrac{\partial}{\partial x_i} f(\mathcal{T}(x)w), \forall w \in \Omega, \forall x \in \mathbb{R}^n\}$
$\mathcal{D}(\Omega)$	$\mathcal{D}(\Omega) = \bigcap_{j=1}^n \mathcal{D}_j$
Z	$Z = \{z \in \mathcal{D}(\Omega) : z = 0 \text{ on } \Omega \setminus \mathcal{M}\}$

1 Introduction

The main goal of this work is to provide a mathematical justification of the model for a randomly fractured porous medium. Such a mathematical study has already been done mostly for periodically fractured media. For the sake of simplicity and in order to avoid the technical problems associated to the possible loss of ellipticity in the two phase flow model, we consider a weakly compressible single phase flow described by the elliptic equations. The unknown variables in this model will be the density of fluid in the blocks and the density of fluid in the fissures, coupled via the fluxes across the interfaces. In section two, we provide the definition and example of the random structures. The microscopic model describing the exchange between the fractures system and the porous blocks as well as the homogenized macroscopic equation are introduced in the third section. A priori estimates of the density functions in microscopic level are presented in the fourth section. In section five, we employ two-scale convergence in the mean techniques to the microscopic problem. The convergence proof of the homogenized problem is given in the last section.

2 Ergodic dynamic system and relative theorems

Before introducing the elliptic problem, we give the definitions of ergodic dynamic system and state some related theorems in this section (see [2]).

Definition 2.1 *A probability space* (Ω, Ξ, μ) *is a measure space with a measure* μ *that satisfies the probability axioms.* Ω *denotes the sample space.* Ξ *is a* σ*-algebra of subsets of* Ω*.*

Definition 2.2 *Let* (Ω, Ξ, μ) *be a probability space, and assume that a dynamical system* $\mathcal{T}$ *with* n*-dimensional time is given on* Ω *, i.e., there are a family of invertible measurable maps* $\mathcal{T}(x) : \Omega \to \Omega$*, for all* $x \in \mathbb{R}^n$ *such that* $\mathcal{T}(x)$ *and* $\mathcal{T}^{-1}(x)$ *are measurable, and satisfy*

1. $\mathcal{T}(0) = I$ *(I is the identity mappings) on* Ω *and* $\mathcal{T}(x+y) = \mathcal{T}(x) + \mathcal{T}(y)$ *for all* $x, y \in \mathbb{R}^n$*,*

2. *The mappings* $\mathcal{T}(x) : \Omega \to \Omega$ *preserve the measure* μ *on* Ω *i.e., for all* $x \in \mathbb{R}^n$ *and for all* $E \in \Xi$*,* $\mu(\mathcal{T}(x)(E)) = \mu(E)$ *(endomorphism property),*

3. *For all* $E \in \Xi$ *the set* $\{(x, w) \in \mathbb{R}^n \times \Omega : \mathcal{T}(x)w \in E\}$ *is an element of the* σ*-algebra* $\mathcal{L} \times \Xi$ *on* $\mathbb{R}^n \times \Omega$*, where* $\mathcal{L}$ *is the usual Lebesgue* σ*-algebra on* $\mathbb{R}^n$*.*

With the measurable dynamics introduced above, we associate a n-parameters group of unitary operators on $L^2(\Omega) \equiv L^2(\Omega, \Xi, \mu)$ as follows (see [2])

$$(U(x)f)(w) = f(\mathcal{T}(x)w)\,,\; f \in L^2(\Omega).$$

The operator

$$U(x) : L^2(\Omega) \to L^2(\Omega)$$

is unitary for each $x \in \mathbb{R}^n$, and the group $U(x)$ is strongly continuous,

$$\lim_{x \to 0} \|U(x)f - f\|_{L^2(\Omega)} = 0 \text{ for all } f \in L^2(\Omega).$$

Definition 2.3 *Let f(w) be a measurable function on* Ω*. For a fixed* $w \in \Omega$*, the function* $f(\mathcal{T}(x)w)$ *of a argument* $x \in \mathbb{R}^n$ *is said to be a* ***realization*** *of function f.*

Definition 2.4 *A measurable function* f *defined in* Ω *is called* ***invariant*** *if* $f(\mathcal{T}(x)w) = f(w)$ *for any* $x \in \mathbb{R}^n$ *almost everywhere in* Ω *.*

Definition 2.5 *A dynamical system is said to be* ***ergodic****, if every invariant function is constant almost everywhere in* Ω *. In this situation we shall also say that the measure* μ *is* ***ergodic*** *with respect to* $\mathcal{T}(x)$*.*

Remark 2.1 *we now give an example of ergodic dynamic system, which is also a domain with random structure (a quasi-periodic case). Let* $\Omega = \Box$ *be the unit cube in* $\mathbb{R}^n$*, and let* μ *denote the Lebesgue measure on* $\Box$*. For* $x \in \mathbb{R}^n$*, set*

$$\mathcal{T}(x)w \;=\; w + \lambda x\,(mod\,1),$$

*where $\lambda = \lambda_{ij}$ is an $n \times n$ invertible matrix. The mapping $\mathcal{T}(x)w$ satisfies the requirements of **Definition 2.2**:*

1. $\mathcal{T}(0)w = w$ and

$$\mathcal{T}(x+y)w = w + \lambda(x+y) = \mathcal{T}(y) \circ \mathcal{T}(x)w.$$

2. According to the property of Lebesgue measure transition preserving, for every $x \in \mathbb{R}^n$ and every Lebesgue measurable set $\mathcal{F} \subset \Box$,

$$\mathcal{T}(x)\mathcal{F} \text{ is measurable and } \mu(\mathcal{T}(x)\mathcal{F}) = \mu(\mathcal{F}).$$

3. Because the unit cube, $\Box$, is a compact subset of $\mathbb{R}^n$, for any measurable function $f(w)$ on $\Box$ the function $f(\mathcal{T}(x)w)$ is also measurable on the Cartesian product $\Box \times \mathbb{R}^n$.

Then we verify that the dynamic system is ergodic. For any invariant measurable function f (i.e. $f(\mathcal{T}(x)w) = f(w)$ for all $x \in \mathbb{R}^n$) defined in $\Box$, for every $w \in \Box$ there exists $x_w \in \mathbb{R}^n$ such that $\lambda x_w = -w$, i.e., $\forall w \in \Box$, $f(\mathcal{T}(x_w)w) = f(0)$. Thus, every invariant function is constant almost everywhere in $\Box$.

Definition 2.6 *Let $f(x) \in L^1_{loc}(\mathbb{R}^n)$. A number $M\{f\}$ is called the **mean value** of f if*

$$\lim_{\varepsilon \to 0} \int_K f(\varepsilon^{-1}x)\, dx = M\{f\}|K| \tag{2.1}$$

for any Lebesgue measurable bounded set $K \subset \mathbb{R}^n$ ($|K|$ stands for the Lebesgue measure of K). Under additional assumptions on f(x), the definition of the mean value can be expressed in terms of weak convergence. For instance, let the family of functions $f(\varepsilon^{-1}x)$ be bounded in $L^\alpha_{loc}(\mathbb{R}^n)$ for some $\alpha \geq 1$. Since linear combinations of the characteristic functions of the sets K are dense in $L^{\alpha'}_{loc}(\mathbb{R}^n)$, $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$, we can replace (2.1) by

$$f(\varepsilon^{-1}x) \rightharpoonup M\{f\} \text{ in } L^\alpha_{loc}(\mathbb{R}^n). \tag{2.2}$$

Theorem 2.1 ***Birkhoff Ergodic Theorem*** *: Let $f \in L^\alpha(\Omega), \alpha \geq 1$. Then for almost all $w \in \Omega$ the realization $f(\mathcal{T}(x)w)$ possesses a mean value in the sense of*

(2.2). Moreover, the mean value $M\{f(\mathcal{T}(x)w)\}$, considered as a function of $w \in \Omega$, is invariant, and

$$\mathbb{E}\{f\} \equiv \int_{\Omega} f(w)d\mu = \int_{\Omega} M\{f(\mathcal{T}(x)w)\}d\mu.$$

*In particular, if the system $\mathcal{T}(x)$ is **ergodic**, then*

$$M\{f(\mathcal{T}(x)w)\} = \mathbb{E}\{f\} \ for \ almost \ all \ w \in \Omega.$$

3 ε-problem and main result

We now describe the microscopic problem (the ε-problem). The dynamical system $\{\mathcal{T}(x)\}$ will be assumed to be ergodic from now on. Fix a measurable set $\mathcal{F} \in \Xi$ such that $\mu(\mathcal{F}) > 0$ and $\mu(\Omega \setminus \mathcal{F}) > 0$ and define a random fractures system $F(w) \subset \mathbb{R}^n$ for $w \in \Omega$ from $\mathcal{F}$ by setting

$$F(w) = \{x \in \mathbb{R}^n : \mathcal{T}(x)w \in \mathcal{F}\}.$$

In what follows, suppose that $F(w)$ is open and connected a.s. (for almost all $w \in \Omega$). The random matrix blocks set $M(w)$ is constructed in a complementary way by setting

$$\mathcal{M} = \Omega \setminus \mathcal{F}, \quad M(w) = \mathbb{R}^n \setminus F(w).$$

In connection with the random set $M(w)$, define a homothetic structure $M_\varepsilon(w), w \in \Omega$, by

$$M_\varepsilon(w) = \{x \in \mathbb{R}^n : \varepsilon^{-1}x \in M(w)\}.$$

Let Q be a smooth bounded domain in $\mathbb{R}^n$. After having chosen our random structure in $\mathbb{R}^n$, we set

$$Q_1^\varepsilon = \{x \in Q : dist(x, \partial Q) \geq \varepsilon\}.$$

Now introduce the random fracture system in Q by

$$Q_f^\varepsilon(w) = Q \setminus \overline{M_\varepsilon(w) \cap Q_1^\varepsilon}.$$

The random matrix block part of Q is defined as the complement of $Q_f^\varepsilon(w)$ in Q:

$$Q_m^\varepsilon(w) = Q \setminus \overline{Q_f^\varepsilon(w)}.$$

After having defined the random geometry, we introduce the elliptic problem :

$$-\nabla \cdot (K^*(x)\,(\nabla \rho^\varepsilon(x,w) + G_0(x))) = \ f_1(x) \qquad \text{in } Q_f^\varepsilon(w), \tag{3.1}$$

$$-\varepsilon \nabla \cdot (k^\varepsilon(x,w)\,(\varepsilon \nabla \sigma^\varepsilon(x,w) + g_0^\varepsilon(x,w))) = \ f_2(x) \qquad \text{in } Q_m^\varepsilon(w), \tag{3.2}$$

$$K^*(x)\,(\nabla \rho^\varepsilon(x,w) + G_0(x)) \cdot \nu$$
$$= \varepsilon k^\varepsilon(x,w)\,(\varepsilon \nabla \sigma^\varepsilon(x,\varepsilon) + g_0^\varepsilon(x,w)) \cdot \nu \qquad \text{on } \partial Q_m^\varepsilon(w), \tag{3.3}$$

$$\rho^\varepsilon(x,w) = \sigma^\varepsilon(x,w) \qquad \text{on } \partial Q_m^\varepsilon(w), \tag{3.4}$$

$$K^*(x)\,(\nabla \rho^\varepsilon(x,\varepsilon) + G_0(x)) \cdot \nu = 0 \qquad \text{on } \partial Q, \tag{3.5}$$

$$\int_Q \rho^\varepsilon(x,w) + \sigma^\varepsilon(x,w)\,dx = 0. \tag{3.6}$$

In the above equations, $f_i(x), i = 1,2$, represents external source, G_0 and $g_0^\varepsilon(x,w)$ $(= g_0(\mathcal{T}(x/\varepsilon)w))$ are given reference densities. $K^*(x)$ denote the scalar permeability of the fractures set. $k^\varepsilon(x,w) = k(\mathcal{T}(\frac{x}{\varepsilon})w)$ denote the permeability of matrix blocks and it is a symmetric tensor. All above quantities are assumed smooth, uniformly bounded. Besides, $K^*, k^\varepsilon(x,w)$ are positive-definite, i.e. $0 < \underline{C_{K^*}} \le K^*(x) \le \overline{C_{K^*}}$ and $0 < \underline{C_k} \le k^\varepsilon(w) \le \overline{C_k}$.

Main Result :

There is a subsequence of the solutions of (3.1)–(3.6) converging to $\mathcal{D} \in H^1(Q)$ and $\mathcal{D}$ satisfies the following macroscopic equation

$$-\nabla \cdot (\mathcal{A}_N^0 K^*(\nabla_x \mathcal{D} + G_0)) = g \qquad \text{in } Q,$$

where g is the source of the external fracture system and matrix blocks, and $\mathcal{A}_N^0$ is the homogenized tensor depending on the geometry of $\mathcal{F}$.

4 A priori estimate

In this section, we derive a priori estimate for density functions. Define a global density function $\mathcal{D}^\varepsilon$

$$\mathcal{D}^\varepsilon(x,w) = \begin{cases} \rho^\varepsilon(x,w) & \text{in } \overline{Q_f^\varepsilon}(w), \\ \sigma^\varepsilon(x,w) & \text{in } Q_m^\varepsilon(w), \end{cases} \tag{4.1}$$

and global coefficients:

$$\begin{cases} \alpha^\varepsilon(x,w) = G_0 \mathcal{X}_{Q_f^\varepsilon(w)}(x) + \frac{1}{\varepsilon} g_0^\varepsilon(x,w)\mathcal{X}_{Q_m^\varepsilon(w)}(x), \\ \kappa^\varepsilon(x,w) = K^* \mathcal{X}_{Q_f^\varepsilon(w)}(x) + \varepsilon^2 k^\varepsilon(x,w)\mathcal{X}_{Q_m^\varepsilon(w)}(x), \\ F^\epsilon(x,w) = f_1(x)\mathcal{X}_{Q_f^\varepsilon(w)}(x) + f_2(x)\mathcal{X}_{Q_m^\varepsilon(w)}(x). \end{cases} \tag{4.2}$$

Then the variational formulation of (3.1)-(3.6) reads as follows:

Find $\mathcal{D}^\varepsilon \in H^1(Q)$ such that

$$\int_Q \kappa^\varepsilon(x,w)[\nabla \mathcal{D}^\varepsilon(x,w) + \alpha^\varepsilon(x,w)]\nabla\psi(x)\,dx \;=\; \int_Q F^\epsilon(x,w)\psi(x)\,dx \tag{4.3}$$

for any $\psi \in H^1(Q)$.

Proposition 4.1 *Suppose* $F \in L^2(Q)$*, (4.3) is uniquely solvable for all* $\varepsilon > 0$ *almost surely in* w*. Moreover, for all* $\varepsilon > 0$ *we have a.s. in* w*:*

$$(1) \quad \| \mathcal{D}^\varepsilon \|_{L^2(Q)} \le C, \tag{4.4}$$

$$(2) \quad \| \nabla\rho^\varepsilon \|_{L^2(Q_f^\varepsilon(w))^n} \le C, \tag{4.5}$$

$$(3) \quad \| \nabla\sigma^\varepsilon \|_{L^2(Q_m^\varepsilon(w))^n} \le \frac{C}{\varepsilon}. \tag{4.6}$$

where C *is an universal constant independent of* ε *and of* w

Proof: We use Lax-Milgram Lemma to prove the uniqueness and existence of the solution. Let

$$\begin{aligned} a(\mathcal{D}^\varepsilon,\psi) &= \int_{Q_f^\varepsilon} K^*\nabla\rho^\varepsilon \cdot \nabla\psi\,dx + \int_{Q_m^\varepsilon} \varepsilon^2 k^\varepsilon \nabla\sigma^\epsilon \cdot \nabla\psi\,dx \\ &= \int_{Q_f^\varepsilon} f_1\psi - K^* G_0 \cdot \nabla\psi + \int_{Q_m^\varepsilon} f_2\psi - \varepsilon k^\epsilon g_0^\varepsilon \cdot \nabla\psi\,dx. \end{aligned}$$

Claim:

$$1. \quad \exists\, C > 0 \ s.t \ \mid a(\mathcal{D}^\varepsilon,\psi) \mid \le C \parallel \psi \parallel_{H^1(Q)}, \tag{4.7}$$

$$2. \quad \exists\, C^\varepsilon > 0 \ s.t \ a(\mathcal{D}^\varepsilon,\mathcal{D}^\varepsilon) \ge C^\varepsilon \parallel \mathcal{D}^\varepsilon \parallel^2_{H^1(Q)}. \tag{4.8}$$

$Proof\ of\ claim\ 1:$

$$
\begin{aligned}
|\, a(\mathcal{D}^\varepsilon,\psi)\,| &\leq \int_{Q_f^\varepsilon} |\, f_1\,|^2 + |\,\psi\,|^2 + \overline{C_{K*}}\sum_i (|\, G_{0,i}\,|^2 + |\,\frac{\partial \psi}{\partial x_i}\,|^2) \\
&+ \int_{Q_m^\varepsilon} |\, f_2\,|^2 + |\,\psi\,|^2 + \overline{C_k}\sum_i (|\, g_{0,i}^\varepsilon\,|^2 + |\,\frac{\partial \psi}{\partial x_i}\,|^2) \\
&\leq C \parallel \psi \parallel_{H^1(Q)} .
\end{aligned}
$$

$Proof\ of\ claim\ 2:$ Given $\varepsilon > 0$, let $C_1 = min\{\underline{C_{K^*}}, \underline{C_k}\}$

$$
C_1^\varepsilon \parallel \nabla \mathcal{D}^\varepsilon \parallel_{L^2(Q)}^2 = C_1\{\int_{Q_f^\varepsilon} \nabla\rho^\varepsilon \cdot \nabla\rho^\varepsilon\, dx + \int_{Q_m^\varepsilon} \varepsilon^2 \nabla\sigma^\varepsilon \cdot \nabla\sigma^\varepsilon\, dx\} \leq a(\mathcal{D}^\varepsilon, \mathcal{D}^\varepsilon). \quad (4.9)
$$

By Poincare's inequality, there exists $C_\rho > 0$ such that

$$
\parallel \rho^\varepsilon \parallel_{L^2(Q_f^\varepsilon)} \leq C_\rho \parallel \nabla\rho^\varepsilon \parallel_{L^2(Q_f^\varepsilon)} .
$$

Now, extend ρ^ε from Q_f^ε to Q. Let

$$
\pi_\varepsilon : \rho^\varepsilon \longrightarrow \pi_\varepsilon \rho^\varepsilon
$$

where $\pi_\varepsilon\rho^\varepsilon \in H^1(Q)$ and $\pi_\varepsilon\rho^\varepsilon = \rho^\varepsilon$ in Q_f^ε (see [4]). By Poincare's inequality,

$$
\parallel \pi_\varepsilon\rho^\varepsilon \parallel_{L^2(Q)} \leq C_2 \parallel \nabla\pi_\varepsilon\rho^\varepsilon \parallel_{L^2(Q)} \leq C_3 \parallel \nabla\rho^\varepsilon \parallel_{L^2(Q_f^\varepsilon)} .
$$

We know $Q_m^\varepsilon(w) = \bigcup_i \varepsilon\mathcal{M}_i(w)$ where $\mathcal{M}_i(w) \bigcap \mathcal{M}_j(w) = \emptyset$ if $i \neq j$.

$$
\sigma^\varepsilon = (\sigma^\varepsilon - \pi_\varepsilon\rho^\varepsilon) + \pi_\varepsilon\rho^\varepsilon \ \ and\ (\sigma^\varepsilon - \pi_\varepsilon\rho^\varepsilon)\ |_{\partial\varepsilon\mathcal{M}_i} = 0 \ \ for\ \ all\ \ i.
$$

By Poincare's inequality,

$$
\begin{aligned}
\parallel \sigma^\varepsilon - \pi_\varepsilon\rho^\varepsilon \parallel_{L^2(\varepsilon\mathcal{M}_i)} &\leq \varepsilon C_4 \parallel \nabla\sigma^\varepsilon - \nabla\pi_\varepsilon\rho^\varepsilon \parallel_{L^2(\varepsilon\mathcal{M}_i)} \\
&\leq \varepsilon C_4 (\parallel \nabla\sigma^\varepsilon \parallel_{L^2(\varepsilon\mathcal{M}_i)} + \parallel \nabla\pi_\varepsilon\rho^\varepsilon \parallel_{L^2(\varepsilon\mathcal{M}_i)}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\parallel \sigma^\varepsilon \parallel_{L^2(Q_m^\varepsilon)}^2 &= \parallel (\sigma^\varepsilon - \pi_\varepsilon\rho^\varepsilon) + \pi_\varepsilon\rho^\varepsilon \parallel_{L^2(Q_m^\varepsilon)}^2 \\
&\leq c(\parallel (\sigma^\varepsilon - \pi_\varepsilon\rho^\varepsilon) \parallel_{L^2(Q_m^\varepsilon)}^2 + \parallel \pi_\varepsilon\rho^\varepsilon \parallel_{L^2(Q_m^\varepsilon)}^2) \\
&\leq c\left(\sum_i \parallel \sigma^\varepsilon - \pi_\varepsilon\rho^\varepsilon \parallel_{L^2(\varepsilon\mathcal{M}_i)}^2\right) + c \parallel \pi_\varepsilon\rho^\varepsilon \parallel_{L^2(Q)}^2 \\
&\leq c\left(\varepsilon^2 \sum_i (\parallel \nabla\sigma^\varepsilon \parallel_{L^2(\varepsilon\mathcal{M}_i)}^2 + \parallel \nabla\pi_\varepsilon\rho^\varepsilon \parallel_{L^2(\varepsilon\mathcal{M}_i)}^2)\right) + c \parallel \pi_\varepsilon\rho^\varepsilon \parallel_{L^2(Q)}^2 \\
&\leq \varepsilon^2 C_5 \parallel \nabla\sigma^\varepsilon \parallel_{L^2(Q_m^\varepsilon)}^2 + C_6 \parallel \nabla\rho^\varepsilon \parallel_{L^2(Q_f^\varepsilon)}^2 .
\end{aligned}
$$

Therefore,

$$
\parallel \mathcal{D}^\varepsilon \parallel_{L^2(Q)} \leq C_7 \parallel \nabla\mathcal{D}^\varepsilon \parallel_{L^2(Q)} .
$$

5 Stochastic two-scale convergence

Before giving our convergence results, we recall the definition and some properties of the stochastic two-scale convergence in the mean.

Definition 5.1 *(see [1])* $(D_j f)(w) = \lim_{x_i\to 0, x_j=0, i\neq j} \dfrac{f(\mathcal{T}(x)w) - f(w)}{x_i}$.

D_j denotes the infinitesimal generator in $\boldsymbol{L}^2(\Omega)$ of the one-parameter group of translations in x_j, with $\mathcal{D}_j$ its respective domain of definition in $\boldsymbol{L}^2(\Omega)$, i.e., for $f \in \mathcal{D}_j$, for almost all w,

$$(D_j f)(\mathcal{T}(x)w) = \frac{\partial}{\partial x_i} f(\mathcal{T}(x)w), \text{ for almost all } x \in \mathbb{R}^n. \tag{5.1}$$

Then$\{\sqrt{-1}D_j,\ j = 1, ...n\}$ are closed, densely-defined and self-adjoint operators which commute pairwise on $\mathcal{D}(\Omega) = \bigcap_{j=1}^{n} \mathcal{D}_j$. Equipped with the inner product

$$(f, g)_{\mathcal{D}(\Omega)} = (f, g)_{L^2(\Omega)} + \sum_{j=1}^{n} (D_j f, D_j g)_{L^2(\Omega)}. \tag{5.2}$$

$\mathcal{D}(\Omega)$ is a Hilbert space. Now we may define the stochastic gradient $(\nabla_w f)$, divergence $(div_w f)$ and curl $(curl_w f)$, as follows

$$\begin{cases} \nabla_w f = (D_1, ..., D_n f), \\ div_w g = \sum_j D_j g_j, \\ curl_w g = D_i g_j - D_j g_i, \ \ i \neq j. \end{cases} \tag{5.3}$$

Definition 5.2 *(see [2]) A vector field* $f \in (L^2(\Omega))^n \equiv \boldsymbol{L}^2(\Omega)$ *will be called* ***solenoidal*** *if almost all its realizations* $f(\mathcal{T}(x)w)$ *are solenoidal in* $\mathbb{R}^n$*.*
Define

$$\boldsymbol{L}^2_{sol}(\Omega) = \{f \in \boldsymbol{L}^2(\Omega) : \ div_x f(\mathcal{T}(x)w) = 0 \ in \ \mathbb{R}^n\}.$$

Definition 5.3 *(see [2]) A vector field* $f \in (L^2(\Omega))^n \equiv \boldsymbol{L}^2(\Omega)$ *will be called* ***potential*** *if almost all its realizations* $f(\mathcal{T}(x)w)$ *are potential in* $\mathbb{R}^n$*.*
Define

$$\boldsymbol{L}^2_{pot}(\Omega) = \{f \in \boldsymbol{L}^2(\Omega) : \ curl_x f(\mathcal{T}(x)w) = 0 \ in \ \mathbb{R}^n\}.$$

We also use the following spaces:

$$\mathcal{V}^2_{pot}(\Omega) = \{f \in \boldsymbol{L}^2_{pot}(\Omega) : \mathbb{E}\{f\} = 0\},$$
$$\mathcal{V}^2_{sol}(\Omega) = \{f \in \boldsymbol{L}^2_{sol}\Omega : \mathbb{E}\{f\} = 0\}.$$

Proposition 5.1 *(**Weyl's Decomposition**)(see [2]). The following orthogonal decompositions are valid:*

$$\boldsymbol{L}^2(\Omega) = \mathcal{V}^2_{pot}(\Omega) \oplus \boldsymbol{L}^2_{sol}(\Omega) = \mathcal{V}^2_{sol}(\Omega) \oplus \boldsymbol{L}^2_{pot}(\Omega). \tag{5.4}$$

Definition 5.4 *(see [1]) We say that an element ψ of $L^2(Q \times \Omega)$ is **admissible** if the function*

$$\psi_{\mathcal{T}} : (x, w) \to \psi(x, \mathcal{T}(x)w), \ (w, w) \in \ Q \times \Omega,$$

defines an element of $L^2(Q \times \Omega)$.

For example, every linear combination of functions of the form

$$(x, w) \to f(x)g(w), \qquad (w, w) \in Q \times \Omega, f \in L^2(\Omega), g \in L^2(\Omega)$$

is admissible. And as was shown in [2], every element of $L^2(Q, B(\Omega))$ is admissible.

Remark 5.1 *(see[1]) $B(\Omega)$ is of all functions defined everwhere on Ω that are bounded and measurable over Ω. $B(\Omega)$ becomes a Banach space when equipped with the norm $\| \ f \ \|_{B(\Omega)} = sup \mid f(w) \mid$. $L^2(\Omega)$ denote the set of all elements ψ of $L^2(\Omega)$ such that for a.e. $x \in Q, \psi(x, \cdot) \in B(\Omega)$, and*

$$\int_\Omega \| \ \psi(x, \cdot) \ \|^2_{B(\Omega)} \ dx < \infty.$$

Definition 5.5 *(see [1]) A sequence u^ε of functions form $L^2(Q \times \Omega)$ is said to converge **stochastically two-scale in the mean(s.2-s.m.)** towards $u \in L^2(Q \times \Omega)$ if for any admissible $\psi \in L^2(Q \times \Omega)$ we have*

$$\lim_{\varepsilon \to 0} \int_{Q \times \Omega} u^\varepsilon(x, w)\psi(x, \mathcal{T}(\frac{x}{\varepsilon})w) dx d\mu = \int_{Q \times \Omega} u(x, w)\psi(x, w) dx d\mu. \tag{5.5}$$

Definition 5.6 *(see [1]) Definition of $I^2(\Omega)$ and $M^2(\Omega)$.*

Denote $I^2(\Omega)$ the set of all functions in $L^2(\Omega)$ which are invariant for $\mathcal{T}$ relative to μ. $I^2(\Omega) = I^2(\Omega, \Xi, \mu)$ is a norm-closed subspace of $L^2(\Omega)$. $M^2(\Omega)$ denotes the subspace of $L^2(\Omega)$ generated by functions of the form $f(w) = g(w) - g(\mathcal{T}(x)w), g \in L^2(\Omega), x \in \mathbb{R}^n$. Then we have $L^2(\Omega) = I^2(\Omega) \oplus M^2(\Omega)$ and the subspaces $I^2(\Omega)$ and $M^2(\Omega)$ are orthogonal.

Proposition 5.2 *(see [1]) (a) Suppose u^ε is a bounded sequence in $L^2(Q \times \Omega)$, then there exists a subsequence of u^ε which stochastically 2-scale converges in the mean to $u \in L^2(Q \times \Omega)$.*

(b) If u^ε is a sequence in $L^2(Q \times \Omega)$ which stochastically 2-scale mean converges to u, then

$$\int_\Omega u^\varepsilon(\cdot, w)\psi(w)\, d\mu \to \int_\Omega u(\cdot, w)\psi(w)\, d\mu \text{ weakly in } L^2(Q), \forall \psi \in I^2(\Omega).$$

Theorem 5.1 *(see [1])*

1. *Suppose that $\{u^\varepsilon\}$ and $\{\varepsilon \nabla_x u^\varepsilon\}$ are bounded sequences in $L^2(G \times \Omega)$, there exists $u \in L^2(G, H^1(\Omega))$ and a subsequence $\{u^\varepsilon\}$ such that $\{u^\varepsilon\}$ (respectively, $\{\varepsilon \nabla_x u^\varepsilon\}$) stochastically 2-scale converges in the mean to u (respectively, $\nabla_w u$).*

2. *Let X be a norm-closed, convex subset of $H^1(G)$. Suppose $\{u^\varepsilon\}$ is a sequence in $L^2(G \times \Omega)$ which satisfies the following conditions: For $\varepsilon > 0$,*

   (a) $u^\varepsilon(\cdot, w) \in X, for\ \mu\text{-a.e.}\ w \in \Omega$,

   (b) there exists a constant $C > 0$ such that $\int_\Omega \| u^\varepsilon(\cdot, w) \|^2_{H^1(G)}\, d\mu \leq C$,

   then there exist $u \in H^1(G; L^2(\Omega)), v \in (L^2(G \times \Omega))^n$, and a subsequence $\{u^\varepsilon\}$ which satisfy the following conditions :

   (a) for a.e. $x \in G, u(x, \cdot) \in I^2(\Omega)$ and for μ-a.e. $w \in \Omega, u(\cdot, w) \in X$,

   (b) v is contained in the $(L^2(G \times \Omega))^n$-norm closure of $L^2(\Omega) \otimes$ (range of ∇_w), and for a.e. $x \in G, V(x, \cdot) \in (M^2(\Omega))^n$, $curl_w v(x, \cdot) = 0$,

(c) $\{u^\varepsilon\}$ *(respectively,* $\{\nabla_x u^\varepsilon\}$*) stochastically 2-scale converges in the mean to* u *(respectively,* $\nabla_x u + v$*),*

(d) $(\nabla_x u)(x,\cdot) \in (I^2(\Omega))^n$*, for a.e.* $x \in G$*.*

Theorem 5.1 implies the following result.

Theorem 5.2 *(see [3]) Let* X *be the closure of the space* $\mathcal{V}^2_{pot}(\Omega)$ *in* $L^2(\Omega \setminus \mathcal{M})^n$ *and let* $\{u^\varepsilon\} \subset H^1(G)$ *be such a sequence that*

$$\begin{cases} \| u^\varepsilon \|_{L^2(G)} \leq C, \\ \| \nabla u^\varepsilon \|_{L^2(G^\varepsilon_f(w))} \leq C, \\ \| \nabla u^\varepsilon \|_{L^2(G^\varepsilon_m(w))} \leq \frac{C}{\varepsilon}. \end{cases}$$

Then there exist functions $u \in H^1(G), v \in L^2(G;\mathcal{D}(\Omega))$, $v = 0$ *on* $\Omega \setminus \mathcal{M}$*, and* $u_1 \in L^2(G;X), u_1 = 0$ *on* $\mathcal{M}$*, such that, up to a subsequence,*

$$\begin{aligned} u^\varepsilon &\xrightarrow{s.2\text{-}s.m} u(x) + \mathcal{X}_{\mathcal{M}}(w)v(x,w), \\ \mathcal{X}_{G^\varepsilon_f(w)}\nabla u^\varepsilon &\xrightarrow{s.2\text{-}s.m} \mathcal{X}_{\Omega\setminus\mathcal{M}}[\nabla_x u(x) + u_1(x,w)], \\ \varepsilon\mathcal{X}_{G^\varepsilon_m(w)}\nabla u^\varepsilon &\xrightarrow{s.2\text{-}s.m} \mathcal{X}_{\mathcal{M}}(w)\nabla_w v(x,w). \end{aligned}$$

Proposition 5.3 *Let* $\{\mathcal{D}^\varepsilon\}_{\varepsilon>0}$ *satisfy (4.3). Then there exist* $\mathcal{D} \in H^1(Q)$, $v \in \mathcal{D}(\Omega)$, $v = 0$ *on* $\mathcal{F}$ *and* $\mathcal{D}_1 \in X$, $\mathcal{D}_1 = 0$ *on* $\mathcal{M}$*, such that, up to a subsequence,*

$$\mathcal{D}^\varepsilon \xrightarrow{s.2\text{-}s.m} \mathcal{D}(x) + \mathcal{X}_{\mathcal{M}}(w)v(x,w), \tag{5.6}$$

$$\mathcal{X}_{Q^\varepsilon_f(w)}\nabla\mathcal{D}^\varepsilon \xrightarrow{s.2\text{-}s.m} \mathcal{X}_{\Omega\setminus\mathcal{M}}(\nabla_x\mathcal{D}(x) + \mathcal{D}_1(x,w)), \tag{5.7}$$

$$\varepsilon\mathcal{X}_{Q^\varepsilon_m(w)}\nabla\mathcal{D}^\varepsilon \xrightarrow{s.2\text{-}s.m} \mathcal{X}_{\mathcal{M}}\nabla_w v(x,w). \tag{5.8}$$

Proof: This is an immediate consequence of Proposition 4.1 and Theorem 5.2.

6 Auxiliary problem and convergence result

Next we consider an auxiliary problem, which is used to compute the effective permeability. It turns out that the problem is connected with an elliptic problem with

Neumann boundary condition in fracture domain: For any $\eta \in \mathbb{R}^n$, find the minimum value of

$$\inf_{v \in X} \int_{\Omega \setminus \mathcal{M}} |\ \eta + v\ |^2 \ d\mu \tag{6.1}$$

where X is the closure in $\boldsymbol{L}^2(\Omega \setminus \mathcal{M})$ of the space $\mathcal{V}^2_{pot}(\Omega)$. It is easy to show that the problem (6.1) has a unique solution, $v_\eta \in X$, which satisfies the following Euler equation

$$\mathbb{E}\{\mathcal{X}_{\Omega\setminus\mathcal{M}}(\eta + v_\eta)\varphi\} = \int_{\Omega\setminus\mathcal{M}} (\eta + v_\eta(w))\varphi(w) d\mu = 0 \tag{6.2}$$

for all $\varphi \in \mathcal{V}^2_{pot}(\Omega)$.

Proposition 6.1 *(6.2) has a unique solution. Moreover, there is a positive, constant tensor $\mathcal{A}^0_N$ satisfying*

$$\mathcal{A}^0_N \eta = \mathbb{E}\{\mathcal{X}_{\Omega\setminus\mathcal{M}}(\eta + v_\eta)\}, \ \ \forall \eta \in \mathbb{R}^n. \tag{6.3}$$

Proof: The equation (6.2), together with Weyl's decomposition, implies that

$$\mathbf{q} \equiv \mathcal{X}_{\Omega\setminus\mathcal{M}}(\eta + v_\eta) \in \boldsymbol{L}^2(\Omega). \tag{6.4}$$

Let bilinear form $B(\eta, \lambda) = \mathbb{E}\{\mathcal{X}_{\Omega\setminus\mathcal{M}}(\eta + v_\eta)(\lambda + v_\lambda)\}$, where v_η is the solution of equation (6.2) corresponding to η (v_λ respectively λ). Then $B(\eta, \lambda)$ is symmetric and $B(\eta, \eta) = \eta \cdot A^0_N \eta$. (6.2) implies $B(\eta, \lambda) = \mathbb{E}\{\mathbf{q}\} \cdot \lambda$, and therefore (6.3) holds.

Proposition 6.2 *Let $\mathcal{D}$ and v be defined by (5.6)-(5.8). Then $\mathcal{D}$ satisfies the equation*

$$\int_Q \mathcal{A}^0_N K^*(\nabla_x \mathcal{D} + G_0)\nabla_x \psi(x) dx = \int_Q g(x)\psi(x) dx, \quad \forall \psi \in H^1(Q), \tag{6.5}$$

where g is the limit of $F^\epsilon(x, w)$ and $\mathcal{A}^0_N$ is the homogenized tensor depending on the geometry of $\mathcal{F}$.

Proof: Let $\zeta(x) \in C^\infty_0(Q)$ and suppose $\xi \in \mathcal{D}(\Omega)$. Take $\psi = \epsilon\zeta(x)\xi(T(\frac{x}{\varepsilon})w)$ in (4.3) to get

$$\int_Q \mathcal{X}_{Q^\varepsilon_f(w)}\{K^*[\nabla\rho^\varepsilon + G_0]\}\{\varepsilon\nabla_x\zeta\ \xi + \zeta\nabla_w\xi(T(\frac{x}{\varepsilon})w)\}$$
$$+ \int_Q \varepsilon\mathcal{X}_{Q^\varepsilon_m(w)}\{k^\varepsilon[\varepsilon\nabla\sigma^\varepsilon + g^\varepsilon_0]\}\{\varepsilon\nabla_x\zeta\ \xi + \zeta\nabla_w\xi(T(\frac{x}{\varepsilon})w)\} = \int_Q F^\epsilon(x, w)(\varepsilon\zeta\ \xi) dx.$$

By Proposition 5.3, we obtain the equation

$$\int_Q \int_{\Omega\setminus\mathcal{M}} K^*\{\nabla_x \mathcal{D}(x) + \mathcal{D}_1(x,w) + G_0\}\nabla_w \xi(w)\zeta(x)\, d\mu\, dx = 0.$$

Thus,

$$\int_{\Omega\setminus\mathcal{M}} \{\nabla_x \mathcal{D}(x) + \mathcal{D}_1(x,w) + G_0\}\nabla_w \xi(w)\, d\mu = 0.$$

Let

$$\eta = \nabla_x \mathcal{D}(x) + G_0.$$

Proposition 6.1 implies

$$\mathcal{A}_N^0(\nabla\mathcal{D}(x) + G_0) = \mathbb{E}\{\mathcal{X}_{\Omega\setminus\mathcal{M}}(\mathcal{D}_1(x,w) + (\nabla\mathcal{D}(x) + G_0))\}. \tag{6.6}$$

Again we take $\psi \in H^1(Q)$ in equation (4.3), consider the s.2-s.m. convergence, and employ (6.6) to get (6.5).

References

[1] A. Bourgeat, A. Mikelic and S. Wright, On the stochastic two-scale convergence in the mean and applications,J. Reine Angew. Math. (Crelles J.) 456 (1994), 19-51.

[2] V.V. Jikov, S.M. Kozlov and O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer, New York, 1994.

[3] A. Bourgeat, A. Mikelic and A. Piatnitski, On the double porosity model of single phase flow in random media, Asymptotic Analysis, 34 (2003), 311-332.

[4] E, Acerbi V, Chiado Piat, G. Dal Maso, and D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains, Nonlinear Analysis 18(1992) 481-496

[5] Doina Cioranescu, Patrizia Donato, An Introduction to Homogenization, Oxford University Press, 1999.

[6] Lawrence C.Evans, Partial Differential Equations, 1998.