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根系統和Weyl群的軌跡

Root System and Orbits of the Weyl Group

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指導老師: 蔡 孟 傑 教 授

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摘要

令$\mathfrak{g}$為一個有限維的複半單李代數，而$\mathfrak{h}$是$\mathfrak{g}$的Cartan子李代數。則$\mathfrak{g}$會導出一個包含許多根的根系統。每個根沿著它自己的超平面又可導出一個根反射。這些根反射所生成的群叫做Weyl群，這個群在$\mathfrak{h}^*$有群作用。現在給定任意兩個$\mathfrak{h}^*$的向量，我們的目標是藉由觀察Weyl群的結構，找出一個有系統的方法去判斷這兩個向量是否在同一個Weyl群的軌跡裡。對於A_n, B_n, C_n, D_n, G_2型態的李代數，我們觀察Weyl群作用在歐氏空間的行為。對於F_4型態的李代數，觀察F_4的根系統的自同構與D_4的根系統的自同構之間的關係，並藉此用D_4的Weyl群去描述F_4的Weyl群。

中 華 民 國 九 十 五 年 六 月

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Department of Applied Mathematics

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Abstract

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra with the Cartan subalgebra $\mathfrak{h}$. $\mathfrak{g}$ induces a root system containing roots. Each root gives a reflection with respect to its hyperplane. These reflections generate a group W called Weyl group acting on on $\mathfrak{h}^*$. Given two vectors, our purpose is to find a systematic method to judge if they are in the sane W-orbit by observing the structure of W. For type A_n, B_n, C_n, D_n, G_2, we study the W-action on Euclidean space. For type F_4, observe the relation between the automorphism of the root system of F_4 and it of D_4. Then describe the Weyl group of F_4 by the Weyl group of D_4.

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此論文的完成，首先要感謝指導教授蔡孟傑老師，很早就明確地讓我接觸論文題目，也常常會給我在一些學業上規畫的建議，在課業或研究有疑問時，他也很高興看到我們去請教他。也要謝謝胡舉卿學姊，他在研究和體育方面都很優秀，在課業上不但是我和同學的最佳助教，在我參加系排那段時間，也受過他熱心地指導練球，讓我對排球有全新的認識。

還有我的同學千砡，他待人幽默、辦事明快，總是發自內心地關心、照顧著身邊的人，很幸運可以和他互相鼓勵直到完成這篇論文。謝謝我的好朋友玫樺，我們不常聯絡，但他對我的困難常常可以提出有建設性的建議。謝謝羅經凱學長，跟這個電腦軟體高手在同一間研究室，讓我的疑難雜症往往立即解決，不但省了我不少麻煩，還常常可以分享好東西。謝謝從大學時期就一直很照顧我的王立中老師，我跟他學到的只有一點點卻受用至今。謝謝在研一時給我很多幫助的黃大原老師，每當有問題去找他，他一直都很樂意幫忙。

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目 錄

1 Introduction

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\Delta \subset \mathfrak{h}^*$ a choice of simple roots. It corresponds to a diagram $D = Dyn(\mathfrak{g})$, whose vertices are the elements of Δ, known as the Dynkin diagram of $\mathfrak{g}$. The Dynkin diagram is independent of the choice of $\mathfrak{h}$ and Δ. Let $\Phi \subset \mathfrak{h}^*$ be all the roots. Each $\alpha \in \Phi$ defines a reflection which preserves Φ, and these reflections generate a subgroup W of $\mathrm{Aut}(\Phi)$, known as the Weyl group. Let X be the set of all assignments of complex numbers to the vertices of D. By Φ, we can identify $\mathfrak{h}^*$ with X. Namely, the element $\sum_\Delta c_\alpha \alpha \in \mathfrak{h}^*$ can be represented by the assignment of the numbers $\{c_\alpha\}$ on the vertices $\{\alpha\}$ of D. Since W acts on Φ as well as on $\mathfrak{h}^*$, it also acts on X. In this thesis, we study the orbits of the W-action on X.

This thesis is divided into the following sections. In Section 2, we recall the definitions of Cartan subalgebras, root system, simple roots, Dynkin diagram and Weyl group. In Section 3, we introduce some standard actions on $\mathbb{R}^n$ by S_n (symmetric group) and $\mathbb{Z}_2^n$ (n-fold product of $\mathbb{Z}_2$) as well as their semi-direct product, so that we can use them to describe the W-action on $\mathfrak{h}^*$. In Section 4, we present the main result of this thesis, which is the study of the W-orbits on X for the classical Lie algebras.

2 Mathematical Background

In this section, we start from the definition of Cartan subalgebra. Every complex semisimple Lie algebra $\mathfrak{g}$ gives a root system by choosing Cartan subalgebra. A root system of a vector space V induces a Weyl group and a simple system so that we can make use of them to define an equivalence relation on V and begin to observe it type by type. Finally, recall the list of all types of complex semisimple Lie algebras.

Definition 2.1 Let $\mathfrak{g}$ be a Lie algebra, the *adjoint representation* $ad : \mathfrak{g} \to End\mathfrak{g}$ sending X to ad_X is given by $ad_x(Y) = [X, Y]$, for all $X, Y \in \mathfrak{g}$.

Definition 2.2 Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* if
(a)$\mathfrak{h}$ is maximal abelian.
(b) $ad_{\mathfrak{h}}$ is simultaneously diagonalizable.(i.e. there exists basis $\{v_i\}$ of $\mathfrak{g}$ such that each v_i is an eigenvector of ad_X for all $X \in \mathfrak{h}$.)

From (b), we can write $\mathfrak{g}$ as a simultaneous eigenspace decomposition. That is,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \text{ where } \mathfrak{g}_\alpha = \{Y \in \mathfrak{g} | \ ad_X Y = \alpha(X)Y, \forall X \in \mathfrak{h}\}, \mathfrak{h} = \mathfrak{g}_0.$$

Here $\Phi = \{\alpha \in \mathfrak{h}^* | \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}$ and α is a function translating X to the eigenvalue of ad_X with respect to eigenvector Y. α is linear because of the bilinearity of the Lie bracket. Therefore, $\alpha \in \Phi \subseteq \mathfrak{h}^*$. Φ is called the *root system* of $\mathfrak{g}$ and the elements of Φ are called *roots.*

Proposition 2.3 *([4], Proposition2.17, Corollary2.38) Let $B(\ , \ )$ be Killing form on a Lie algebra $\mathfrak{g}$, $\mathfrak{h} \subseteq \mathfrak{g}$ is the Cartan subalgebra. Let α a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then*
(a)There exists $H_\alpha \in \mathfrak{h}$ such that $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{h}$.
(b)If $\mathfrak{h}_0 = span_{\mathbb{R}}\{H_\alpha | \alpha \in \Phi\}$, then $\mathfrak{h}_0$ is a real form of $\mathfrak{h}$ such that $\alpha|_{\mathfrak{h}_0}$ is real on $\mathfrak{h}_0$ for all $\alpha \in \Phi$, hence Φ can be considered as in $\mathfrak{h}_0^*$.

Proposition 2.4 *([4], Corollary2.38) Let $\mathfrak{h}_0 \subseteq \mathfrak{h} \subseteq \mathfrak{g}$ be defined as above. Then $\mathfrak{h}_0^*$ is an inner product space over $\mathbb{R}$.*

Note that the inner product in $\mathfrak{h}_0^*$ is given by $(\alpha, \beta) = B(H_\alpha, H_\beta)$ for all $\alpha, \beta \in \mathfrak{h}_0^*$, where H_α, H_β is the same as Proposition 2.3(a).

Next, we recall the definition of the root system of general real vector space E. By setting $E = \mathfrak{h}_0^*$, $\mathfrak{g}$ determines a root system through its Cartan subalgebra $\mathfrak{h}$. Now let E be an real inner product (,). Any nonzero vector α gives a reflection σ_α by $\sigma_\alpha(\beta) := \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$, for all $\beta \in E$. For convenient, denote the number $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ by $< \beta, \alpha >$. Hence we have $\sigma_\alpha(\beta) = \beta - < \beta, \alpha > \alpha$.

Definition 2.5 A subset Φ of the euclidean space E is called a *(reduced)root system* in E if

(a) Φ is finite, spans E, and does not contain 0.

(b) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(c) If $\alpha \in \Phi$, the reflection σ_a leaves Φ invariant.

(d) If $\alpha, \beta \in \Phi$, then $< \beta, \alpha > \in \mathbb{Z}$

Φ is called *irreducible* if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

In what follows, we see how a root system induces a Weyl group.

Definition 2.6 Let Φ be a root system in E. Define the *Weyl group* of Φ by

$$W = W(\Phi) = \{\sigma_\alpha | \alpha \in \Phi\}.$$

The main question that we want to discuss in this thesis is to study the orbits of Weyl groups. In Section 4, we introduce the root system of every $\mathfrak{g}$ of each type of complex semisimple Lie algebras. Then

Definition 2.7 Let E be a vector space. A subset Δ of Φ is called a *simple system* if

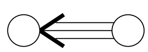
(a) Δ is a basis of E.

(b) each root β can be written as $\beta = \Sigma_{\alpha \in \Delta} c_\alpha \alpha$ with integral coefficients c_α all nonnegative or all nonpositive.

The roots in Δ are called *simple roots*.

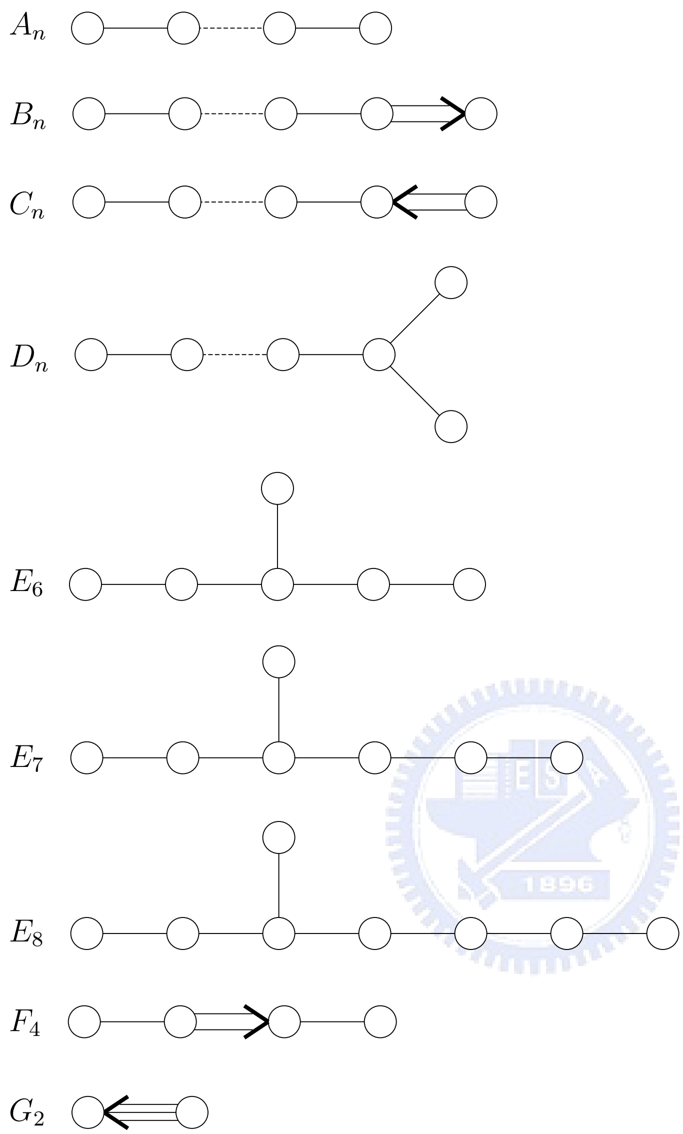
The simple system is a particular basis. We can use the simple system of a root system to draw Dynkin diagram and write numbers on each vertex to represent the elements of $\mathfrak{h}^*$.

Definition 2.8 Let Φ be a root system of rank n, W its Weyl group, $\Delta = \{\alpha_1, ..., \alpha_n\}$ a simple system of Φ. Define the *Dynkin diagram* of Φ to be a graph having n vertices where the ith vertex denotes the simple root α_i. Then set $< \alpha_i, \alpha_j >< \alpha_j, \alpha_i >$ edges between the ith and the jth vertices for all $i \neq j$. Finally, if there exists any edge between two vertices with different length, add an arrow from the longer to the shorter of the two roots.

Example 2.9 G_2 

Recall that a complex semisimple Lie algebra induces a root system, hence a Dynkin diagram. We can classify all complex semisimple Lie algebras through their Dynkin diagram. The following theorem shows that they can be exactly classified in several types. Hence we can study the equivalence relation with respect to each type of complex semisimple Lie algebras.

Theorem 2.10 *If Φ is an irreducible root system of rank n, its Dynkin diagram is one of the following:*


A_n
B_n
C_n
D_n
E_6
E_7
E_8
F_4
G_2

3 Group Action

To understand the orbit of a Weyl group, we observe the structures of those Weyl groups and described them (on $\mathbb{R}^n$) by other groups that we are more familiar in experience. Therefore, we need some definition for those group action on $\mathbb{R}^n$ which describes the Weyl group action on $\mathfrak{h}^*$.

In this article, the elements in Euclidean space are represented by column vector in order to separate from the elements in symmetric groups.

Definition 3.1 Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, we define the *group action* $G \times \mathbb{R}^n \to \mathbb{R}^n$ with respect to the following types of G :

(a) $G = S_n$ (S_n is the symmetric group of degree n) acts on $\mathbb{R}^n$

Define $\sigma.x = \sigma.\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{pmatrix}$, for $\sigma \in G$.

(b) $G = \mathbb{Z}_2^n$ acts on $\mathbb{R}^n$

Define $b.x = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} . \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} (-1)^{b_1} x_1 \\ \vdots \\ (-1)^{b_n} x_n \end{pmatrix}$, for $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in G$.

(c) $G = \mathbb{Z}_2^{n-1}$ acts on $\mathbb{R}^n$

Define $b.x = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} . \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} (-1)^{b_1} x_1 \\ \vdots \\ (-1)^{b_{n-1}} x_{n-1} \\ (-1)^{\Sigma_1^{n-1} b_i} x_n \end{pmatrix}$, for $b = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} \in G$.

(d) $G = \mathbb{Z}_2^n \rtimes S_n$ acts on $\mathbb{R}^n$

Note that the group action in G is defined by $(b_1, \sigma_1)(b_2, \sigma_2) = (b_1 + \sigma_1.b_2, \sigma_1\sigma_2)$, for all $(b_1, \sigma_1), (b_2, \sigma_2) \in G$. Define $(b, \sigma).x = b.(\sigma.x)$, for $b \in \mathbb{Z}_2^n, \sigma \in S_n$, where $(\sigma.x)$ is defined in case (a) previously.

(e) $G = \mathbb{Z}_2 \times S_3$ acts on $\mathbb{R}^3$

Define $(b, \sigma).x = (-1)^b(\sigma.x)$, for $b \in \mathbb{Z}_2, \sigma \in S_3$, where $(\sigma.x)$ is defined in case (a) previously.

Note that the definition in (a) and (b) are special cases of that in (d). They can be obtained by setting identity of the first and the second group of $\mathbb{Z}_2^n \rtimes S_n$ respectively.

Example 3.2 We give some examples for the above definition of group actions. In what follows, the examples (a),(b),(c),(d),(e) correspond respectively to the group actions in Definition 3.1 (a),(b),(c),(d),(e).

(a) Let $n = 3$, then $(1\ 2\ 3).\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}$.

(b) Let $n = 3$, then $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ -7 \end{pmatrix}$.

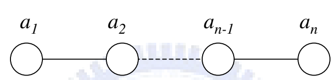
(c) Let $n = 4$, then $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ -7 \\ 8 \end{pmatrix}$.

(d) Let $n = 3$, then $\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, (1\ 2)\right) . \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 6 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \\ -7 \end{pmatrix}$.

(e) $(1,(2\ 1\ 3)) \cdot \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} -6 \\ -7 \\ -5 \end{pmatrix}$.

Definition 3.3 Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Let $Dyn(\mathfrak{g})$ be the Dynkin diagram of $\mathfrak{g}$ and fix a simple system $\{\alpha_i\}_{i=1}^n$ of $\mathfrak{h}_0^*$, we can denote elements of $\mathfrak{h}_0^*$ by writing numbers on the vertices of $Dyn(\mathfrak{g})$ corresponding to the coefficients of the linear combination of Δ.

Example 3.4 Fix a simple system $\{\alpha_i\}_{i=1}^n$ of A_n, then



denotes $\Sigma_{i=1}^n a_i\alpha_i$.

Definition 3.5 Let $\Delta = \{\alpha_i\}_{i=1}^n$ be a simple system of a vector space V, $W = (\{\sigma_{\alpha_i} | i = 1, ..., n\})$ be the Weyl group. Let $a = \Sigma a_i\alpha_i$, $b = \Sigma b_i\alpha_i \in V$ we say a is equivalent to b with respect to W if there exists $\sigma \in W$ such that $\sigma a = b$, and denote it by $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \sim \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ or $a \sim b$ if we denote a, b by Dynkin diagram.

Example 3.6 Let $V = \mathbb{R}^3$ with simple system

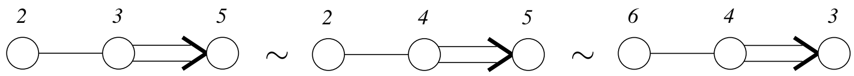
$$\Delta = \Delta(B_3) = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3\}$$

Consider a vector $v = 2\alpha_1 + 3\alpha_2 + 5\alpha_3$. Then

$$\sigma_{\alpha_2}(v) = 2(\alpha_1 + \alpha_2) + 3(-\alpha_2) + 5(\alpha_2 + \alpha_3) = 2\alpha_1 + 4\alpha_2 + 5\alpha_3$$

$$\sigma_{\alpha_3}\sigma_{\alpha_2}(v) = 2\alpha_1 + 4(\alpha_2 + 2\alpha_3) + 5(-\alpha_3) = 6\alpha_1 + 4\alpha_2 + 3\alpha_3$$

and



The main question that we are curious is if there exists some convenient method to check whether two vectors are equivalent or not. Next, to solve the problem, we are going to observe the equivalence relation on a different basis through some groups isomorphic to the Weyl groups.

4 Equivalence Relation:Classical cases

In Section 4 and 5, we introduce some construction of root systems of complex semisimple Lie algebras where Section 4 is for the classical cases and Section 5 is for the exceptional cases. After observing the action of their Weyl groups on another basis of $\mathfrak{h}^*$, we will find that the behaviors of those actions are very straight forward.

To observe the relation between bases and Weyl group action later, it is useful to define some notation to represent a vector with respect to a basis for the discussion.

Definition 4.1 Let V be an n-dimensional vector space over a field F, $\beta = \{v_1, \ldots, v_n\}, \gamma = \{u_1, \ldots, u_n\}$ be two bases of V.

(a) For $x \in V$, let $a_1, ...a_n$ be the unique scalars such that $x = \Sigma_{i=1}^n a_i v_i$.

We define $[x]_\beta \in F^n$ by

$$[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

(b) Let $b_{ij}(i, j = 1, \ldots, n)$ be the scalars such that

$$v_j = \Sigma_{i=1}^n b_{ij} u_i \text{ for } 1 \leq j \leq n.$$

We define the $n \times n$ matrix $[1]_\beta^\gamma$ by $[1]_\beta^\gamma = (b_{ij})$.

Now, we are going to observe the root system of each type of complex semisimple Lie algebras. In what follows, we still denote elements in Euclidean space by column vectors written in the forms $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ or $(x_1 \cdots x_n)^t$.

Type $A_n(n \geq 1)$

Consider the hyperplane

$$V = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}^{\perp} = \{ \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} | x_1 + \cdots + x_{n+1} = 0 \} \subseteq \mathbb{R}^{n+1},$$

as well as

$$\Phi = \{ v \in V \cap \mathbb{Z}^n | \|v\|^2 = 2 \} = \{ e_i - e_j | 1 \leq i \neq j \leq n+1 \}.$$

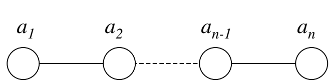
Then Φ is a root system of type A_n, and

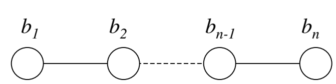
$$\Delta = \Delta(A_n) = \{ \alpha_1 = e_1 - e_2, \ldots, \alpha_n = e_n - e_{n+1} \}$$

is the simple system of Φ.

In V, since σ_{α_i} permutes e_i, e_{i+1} and leaves all other $e_j{}'s$ fixed, σ_{α_i} corresponds to the transposition $(i\ i{+}1)$ in the symmetric group S_{n+1}. These transpositions generate S_{n+1}, so we obtain $W \cong S_{n+1}$.

The behavior of W on the standard basis is so simple and direct that we can easily judge if two vectors in V are equivalent with respect to W. Given two vectors $a, b \in V$, we have $[a]_{std}$ and $[b]_{std}$ in $\mathbb{R}^{n+1}$ by choosing standard basis to represent them. To ask if there exists an element of Weyl group translating a to b is equivalent to ask if there exists a permutation of coordinates translating $[a]_{std}$ to $[b]_{std}$. Therefore, we have the following result.

Lemma 4.2 

$\sim$ 

if and only if there exists $\sigma \in S_{n+1}$ such that $\sigma . M(a_1 \cdots a_n)^t = M(b_1 \cdots b_n)^t$, where

$$M = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{pmatrix}_{(n+1) \times n}.$$

Type $B_n (n \geq 2)$

We just follow the idea that we did in type A_n. Let $V = \mathbb{R}^n$, as well as

$$\Phi = \{v \in V \cap \mathbb{Z}^n | \|v\|^2 = 1 \text{ } or \text{ } \|v\|^2 = 2\} = \{\pm e_i | 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j | 1 \leq i \leq j \leq n\}.$$

Then Φ is a root system of B_n and

$$\Delta = \Delta(B_n) = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}$$

is the simple system of Φ. In V, σ_{α_i} permutes e_i, e_{i+1} for $i = 1, \ldots, n-1$, and σ_{α_i} changes the sign of e_n. These generate all permutations and sign changes of standard coordinates, and can be described by $\mathbb{Z}_2^n \rtimes S_n$. Hence we have the following result.

Lemma 4.3 

if and only if there exists $\sigma \in \mathbb{Z}_2^n \rtimes S_n$ *such that* $\sigma.[1]^{std}_{\Delta(B_n)}(a_1 \cdots a_n)^t = [1]^{std}_{\Delta(B_n)}(b_1 \cdots b_n)^t$, *where*

$$[1]^{std}_{\Delta(B_n)} = \begin{pmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & 1 & \\ & & -1 & 1 \end{pmatrix}_{n \times n}.$$

(i.e. they are different from a permutation and some sign changes in standard coordinates representation.)

Type $C_n (n \geq 3)$

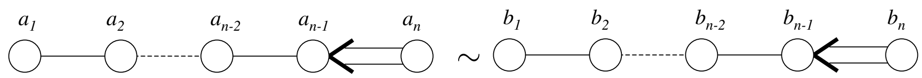
The case of C_n is almost the same as B_n. Consider $V = \mathbb{R}^n$, then

$$\Phi = \{\pm 2e_i\} \cup \{\pm e_i \pm e_j | 1 \leq i < j \leq n\}$$

is a root system of type C_n and

$$\Delta = \Delta(C_n) = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n\}$$

is the simple system of Φ. In V, σ_{α_i} permutes e_i, e_{i+1} for $i = 1, \ldots, n-1$, and σ_{α_i} changes the sign of e_n. These generate the same group action on standard coordinates as type C_n. Therefore, the Weyl group action in type C_n is the same as that in type B_n. Hence we can make use of the same method to judge whether two vectors are equivalent.

Lemma 4.4 

if and only if there exists $\sigma \in \mathbb{Z}_2^n \rtimes S_n$ *s.t.* $\sigma.[1]^{std}_{\Delta(C_n)}(a_1 \cdots a_n)^t = [1]^{std}_{\Delta(C_n)}(b_1 \cdots b_n)^t$, *where*

$$[1]^{std}_{\Delta(C_n)} = \begin{pmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & 1 & \\ & & -1 & 2 \end{pmatrix}_{n \times n}.$$

Type $D_n (n \geq 4)$

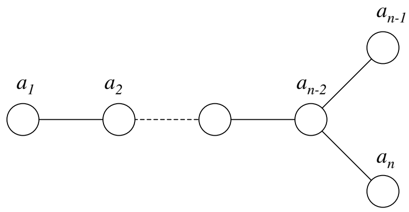
Consider $V = \mathbb{R}^n$, as well as a root system

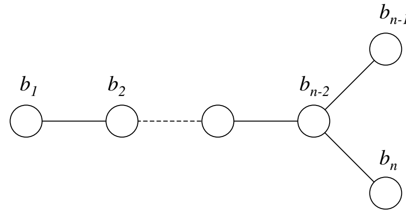
$$\Phi = \{\pm e_i \pm e_j | 1 \leq i < j \leq n\}$$

corresponding to the simple system

$$\Delta = \Delta(D_n) = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}.$$

In V, σ_{α_i} permutes e_i, e_{i+1} , for $i = 1, \ldots, n-1$; σ_{α_n} permutes e_{n-1}, e_n and changes their sign simultaneously. These generate all permutations and all sign changes of even number. Such kind of sign changes can be described by $\mathbb{Z}_2^{n-1}$, since the n-th component in coordinate is determined by the other $n-1$ components. Therefore, $W \cong \mathbb{Z}_2^{n-1} \rtimes S_n$ and we have the following result.

Lemma 4.5 

$\sim$ 

if and only if there exists $\sigma \in \mathbb{Z}_2^{n-1} \rtimes S_n$ *such that* $\sigma.[1]^{std}_{\Delta(D_n)}(a_1 \cdots a_n)^t = [1]^{std}_{\Delta(D_n)}(b_1 \cdots b_n)^t$, *where*

$$[1]^{std}_{\Delta(D_n)} = \begin{pmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & 1 & 1 \\ & & -1 & 1 \end{pmatrix}_{n\times n}.$$

5 Equivalence Relation:Exceptional cases

In this section, we keep the same work as Section 4 for exceptional cases of complex semisimple Lie algebras. For type E, we have not find a method good enough to study the orbit of Weyl group yet. So here we only discuss the type F_4 and G_2 .

Type F_4

Let $V = \mathbb{R}^4$, and the root system

$$\Phi = \{\pm e_i \pm e_j | 1 \leq i < j \leq 4\} \cup \{\pm e_i | i = 1,2,3,4\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\},$$

as well as the simple system

$$\Delta = \{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}.$$

When a vector are represented by standard basis, it become more complex after σ_{α_4} moving it. Hence it is not a good idea to follow the same method to observe the orbits of $W(F_4)$. Instead, we make use of the relation between $W(F_4)$ and $W(D_4)$. The relation of the Weyl group of F_4 and D_4 is associated by the automorphism of their root system which we are going to discuss.

Definition 5.1 Let Ψ be a root system. Define

$$Aut(\Psi) = \{\phi : \Psi \to \Psi | \phi \text{ is linear and } < \alpha, \beta > = < \phi(\alpha), \phi(\beta) > \text{ for all } \alpha, \beta \in \Psi\}.$$

Consider Φ', the root system of D_4, observe that the 24 long roots in Φ form a root system Φ' of type D_4. In what follows, we are going to show that $W(\Phi) = Aut(\Phi')$. Consider $W(\Phi') \subset W(\Phi) = Aut(\Phi')$, other automorphisms of Φ' arise naturally from $Aut(Dyn(D_4))$. Finally, $W(\Phi) = Aut(Dyn(\Phi')) \ltimes W(\Phi') = S_3 \ltimes W(D_4)$. Next, we explain that more precisely.

Definition 5.2 A *lattice* is a discrete subgroup of Euclidean space and contains the origin. Define the lattices L_1, L_2 in $\mathbb{R}^n$:

(a) $L_1 = \{\Sigma_{i=1}^n a_i e_i \in \mathbb{Z}^n | \Sigma_{i=1}^n a_i \text{ } is \text{ } even\}$ is a subgroup of $\mathbb{Z}^n$.

(b) $L_2 = \mathbb{Z}^n + \mathbb{Z}\frac{1}{2}(\Sigma_{i=1}^n e_i) = \{v + \frac{k}{2}\Sigma_{i=1}^n e_i | v \in \mathbb{Z}^n, k \in \mathbb{Z}\}$.

Lemma 5.3 *Let Φ' be the root system of D_4 that we have defined previously. Then*

(a)$Aut(\Phi')$ preserves $(\ ,\ )$ in Φ'

(b)$Aut(\Phi')$ preserves $<\ ,\ >$ in L_1

(c)$Aut(\Phi')$ preserves $<\ ,\ >$ in L_2

Proof. (a) Let $\phi \in Aut(\Phi')$, $\alpha, \beta \in \Phi'$. Then

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} =< \alpha, \beta >=< \phi(\alpha), \phi(\beta) >= \frac{2(\phi(\alpha),\phi(\beta))}{(\phi(\beta),\phi(\beta))}.$$

Since $(\beta, \beta) = (\phi(\beta), \phi(\beta))$, $(\alpha, \beta) = (\phi(\alpha), \phi(\beta))$.

(b) Let $\phi \in Aut(\Phi')$, $\alpha, \beta_1, \beta_2 \in \Phi, c \in \mathbb{R}$. Then

$(\phi(\alpha), \phi(c\beta_1 + \beta_2))$

$= (\phi(\alpha), c\phi(\beta_1) + \phi(\beta_2)) = c(\phi(\alpha), \phi(\beta_1)) + (\phi(\alpha), \phi(\beta_2)) = c(\alpha, \beta_1) + (\alpha, \beta_2)$

$= (\alpha, c\beta_1 + \beta_2)$.

Since all elements in L_1 are linear combination of Φ', $Aut(\Phi')$ preserves the inner product $(\cdot, \cdot)$ of L_1. Therefore, it also preserves $<\ ,\ >$ in L_1.

(c) Let $\phi \in \Phi'$, $\lambda \in L_2$. It is obvious that $2\lambda \in L_1$. By (b), we have

$< \phi(\lambda), \phi(\lambda) >=< 2\phi(\lambda), 2\phi(\lambda) >=< \phi(2\lambda), \phi(2\lambda) >=< 2\lambda, 2\lambda >=< \lambda, \lambda >$. □

Proposition 5.4 *Let Φ and Φ' be the root systems of F_4 and D_4 defined as previous respectively, then $Aut(\Phi) = Aut(\Phi')$.*

Proof. Recall that Φ have disjoint three parts:

$$\Phi = \{\pm e_i \pm e_j | 1 \le i < j \le 4\} \cup \{\pm e_i | i = 1, 2, 3, 4\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

Observe that for all $\tau \in Aut(\Phi')$, τ is stable on these three parts respectively. Hence τ is stable on Φ. In addition, based on the last lemma and the fact that $\Phi \subseteq L_2$, τ preserves $< \ , \ >$ in Φ. It follows that $Aut(\Phi') \subseteq Aut(\Phi)$. Conversely, $Aut(\Phi) \subseteq Aut(\Phi')$ because the elements in Φ' are exactly the long roots of Φ. □

The next Corollary is followed by Proposition 5.4 and the fact that

$$Aut(\Phi) = Aut(Dyn(\Phi)) \ltimes W(\Phi).$$

Corollary 5.5 $W(F_4) = S_3 \ltimes W(D_4)$

Proof. $Aut(Dyn(F_4)) = 1$ implies that

$$Aut(F_4) = Aut(Dyn(F_4)) \ltimes W(F_4) = W(F_4).$$

On the other hand,

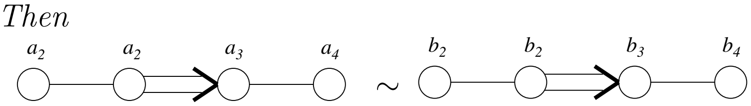
$$Aut(D_4) = Aut(Dyn(D_4)) \ltimes W(D_4) = S_3 \ltimes W(D_4).$$

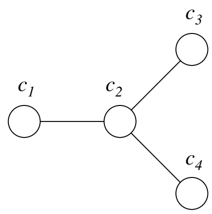
Apply Proposition 5.4 , we have $W(F_4) = S_3 \ltimes W(D_4)$. □

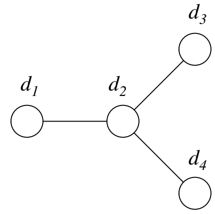
Recall that $\Delta(D_4) = \{\beta_1 = e_1 - e_2, \beta_2 = e_2 - e_3, \beta_3 = e_3 - e_4, \beta_4 = e_3 + e_4\}$ and $Aut(Dyn(D_4))$ is the set of all bijections of $\{\beta_1, \beta_3, \beta_4\}$. Hence $Aut(Dyn(D_4))$ can naturally be described by S_3 consisting of all permutations of $\{\beta_1, \beta_3, \beta_4\}$. The group action $Aut(Dyn(D_4))$ on $W(D_4)$ is defined by $\tau.\sigma_{\beta_i} = \sigma_{\beta_{\tau(i)}}$, for all $\beta_i \in \Delta(D_4)$, for all $\tau \in S_3$. When $[a]_{\Delta(F_4)} \sim_{F_4} [b]_{\Delta(F_4)}$, it means $[a]_{\Delta(D_4)} \sim_{D_4} \tau.[b]_{\Delta(D_4)}$ for some $\tau \in S_3$. Now, apply Lemma 4.5, we have the following property.

Lemma 5.6 *Let* $(c_1\ c_2\ c_3\ c_4)^t = [1]^{\Delta(D_4)}_{\Delta(F4)}(a_1\ a_2\ a_3\ a_4)^t$ *and* $(d_1\ d_2\ d_3\ d_4)^t = [1]^{\Delta(D_4)}_{\Delta(F4)}(b_1\ b_2\ b_3\ b_4)^t$

Then



if and only if $\tau.$ 

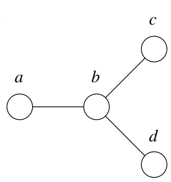
$\sim$ 

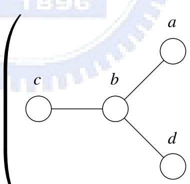
, for some $\tau \in S_3$ *denoting the set of all permutations of* $\{\beta_1, \beta_3, \beta_4\} \subseteq \Delta(D_4)$

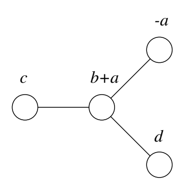
if and only if $\sigma.([1]^{std}_{\Delta(D_4)}(\tau.(c_1\ c_2\ c_3\ c_4)^t)) = [1]^{std}_{\Delta(D_4)}(d_1\ d_2\ d_3\ d_4)^t$*, for some* $\sigma \in S_4 \ltimes \mathbb{Z}_2^3$*, some* $\tau \in S_3$*, where*

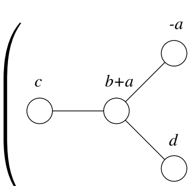
$$[1]^{\Delta(D_4)}_{\Delta(F_4)} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

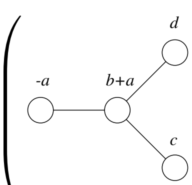
Example 5.7 Let $(a\ b\ c\ d)^t = [1]^{\Delta(D_4)}_{\Delta(F4)}(x\ y\ z\ t)^t$, $(a\ b\ d\ c)^t = [1]^{\Delta(D_4)}_{\Delta(F4)}(x'\ y'\ z'\ t')^t$ and $\Delta(D_4) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$. Then

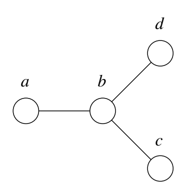
$((1\ 3), \sigma_{\beta_3}).$ (

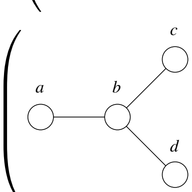
) $= \sigma_{\beta_3}$ (

) $=$ 

$((1\ 4\ 3), \sigma_{\beta_1}).$ (

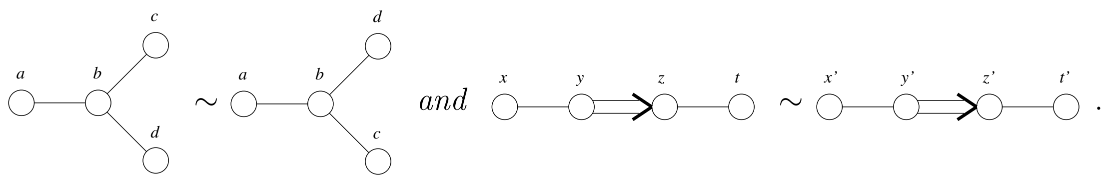
) $= \sigma_{\beta_1}$ (

) $=$ 

$= ((3\ 4), 1).$ (

)

This corresponds to the operation in $S_3 \ltimes W(D_4)$, i.e. $((1\ 4\ 3), \sigma_{\beta_1})((1\ 3), \sigma_{\beta_3}) = ((1\ 4\ 3)(1\ 3), \sigma_{\beta_1} \circ ((1\ 4\ 3).\sigma_{\beta_3})) = ((3\ 4), \sigma_{\beta_1} \circ \sigma_{\beta_1}) = ((3\ 4), 1)$.Therefore,



Type G_2

Consider

$$V = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^{\perp} = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 + x_2 + x_3 = 0\} \subseteq \mathbb{R}^3.$$

Define

$$\begin{aligned} \Phi &= \{v \in V \cap \mathbb{Z}^n | \|v\|^2 = 2 \ or \ \|v\|^2 = 6\} \\ &= \{\pm(e_i - e_j) | 1 \le i < j \le 3\} \cup \{\pm(2e_i - e_j - e_k) | \{i, j, k\} = \{1, 2, 3\}\} \\ &= \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3), \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\} \end{aligned}$$

Then Φ is a root system of type G_2, and

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3\}$$

is the simple system of Φ.

In V, σ_{α_1} permutes e_1, e_2; σ_{α_2} permutes e_2, e_3 and changes sign of any vector in V. Recall that the definition of group action of $\mathbb{Z}_2^3 \rtimes S_3$ on $\mathbb{R}^3$, we have $\sigma_{\alpha_1} = ((0\ 0\ 0)^t, (1\ 2))$ and $\sigma_{\alpha_2} = ((1\ 1\ 1)^t, (2\ 3))$ in V. Since W is generated by σ_{α_1} and σ_{α_2}, W has an embedding in $\mathbb{Z}_2^3 \rtimes S_3$. That is,

$$W = < \sigma_{\alpha_1}, \sigma_{\alpha_2} > \cong < \left((0\ 0\ 0)^t, (1\ 2)\right), \left((1\ 1\ 1)^t, (2\ 3)\right) >$$

(In this thesis, $< \cdot >$ in a Weyl group always means the group generation).

This subgroup can actually be described by $< (0\,0\,0)^t, (1\,1\,1)^t > \times < (1\,2), (2\,3) >$ since $(0\ 0\ 0)^t$ and $(1\ 1\ 1)^t$ are both fixed by any element in S_3. Besides, note that the additive group $<(0\ 0\ 0)^t, (1\ 1\ 1)^t> \cong \mathbb{Z}_2$ which acts on V and denotes sign change(simultaneously sign change on all standard coordinates of V). Therefore,

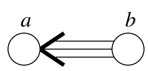
$$W \cong <(0,(1\ 2)),(1,(2\ 3))> = \mathbb{Z}_2 \times S_3.$$

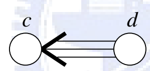
In fact,

$$W \cong D_6 = \{\sigma^0, \dots, \sigma^5, \tau, \sigma\tau, \dots, \sigma^5\tau\}$$

with $|\sigma| = 6$, $\tau^2 = 1$, and $\sigma\tau\sigma = \tau$, where $\sigma = (1,(1\,2\,3)), \tau = (0,(1\,2))$. We make use of the former structure ($\mathbb{Z}_2 \times S_3$) in that it is more helpful to check the equivalence relation for vectors in V.

Considering all the factors above, for two vectors a, b in V, $[a]_{\Delta(G_2)} \sim [b]_{\Delta(G_2)}$ means that the difference between $[a]_{std}$ and $[b]_{std}$ in $\mathbb{R}^3$ are their arrange or sign. Hence we have the following result.

Lemma 5.8 

$\sim$ 

if and only if $\exists \sigma \in \mathbb{Z}_2 \times S_3$ *such that* $\sigma.M\binom{a}{b} = M\binom{c}{d}$, *where* $M = \begin{pmatrix} 1 & -2 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$.

Corollary 5.9 *Let* $x \in V$*, suppose that* $x = a\alpha_1 + b\alpha_2 = x_1e_1 + x_2e_2 + x_3e_3$*, for some scalars* a, b, x_1, x_2, x_3 *in* $\mathbb{R}$*. Then the equivalence class of* x *is*

$$[x] = \{\pm(x_1\ x_2\ x_3)^t, \pm(x_1\ x_3\ x_2)^t, \pm(x_2\ x_1\ x_3)^t, \pm(x_2\ x_3\ x_1)^t, \pm(x_3\ x_1\ x_2)^t, \pm(x_3\ x_2\ x_1)^t\} \subseteq \mathbb{R}^3.$$

參 考 文 獻

[1] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York 1972.

[2] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.

[3] N. Bourbaki, *Groupes et algèbres de Lie*, Masson, Paris, 1981.

[4] Anthony W. Knapp, *Lie Groups Beyond an Itroduction*, 2nd edn., Boston :Birkhauser, 2002.