國 立 交 通 大 學

應用數學系

碩 士 論 文

圖的臨界圓石分配數

Critical Pebbling Numbers of Graphs

研 究 生：蔡宗翰

指導老師：傅恆霖　教授

中 華 民 國 九 十 五 年 六 月

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研 究 生：蔡宗翰　　Student：Tzung-Han Tsai

指導教授：傅恆霖　教授　　Advisor：Dr. Hung-Lin Fu

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A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of Master

in

Applied Mathematics
June 2006

Hsinchu, Taiwan, Republic of China

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摘　要

給定一個圖 G，我們將圓石放在圖 G 上的點，一步圓石移動定義為從 G 上的一點 v 拿兩顆圓石然後放一個到與 v 相鄰的點。如果 G 上的一個圓石分配方式 D，我們可以經由一序列的圓石移動後，對於任一 G 中被選定的點會存在至少有一顆圓石，則我們稱這個分配方式 D 為可以解的。一連通圖 G 上的圓石分配數 $f(G)$ 是最少的圓石數，使得對於在 G 上含有 $f(G)$ 個圓石數的每種分配方式都是可以解的。如果 G 上的一個圓石分配方式 D，我們可以經由一序列的圓石移動後，對於 G 中被選定的點 r 會存在至少有一顆圓石，則我們稱這個分配方式 D 為 r-可以解的。一連通圖 G 上的臨界圓石分配數 $c_r(G)$ 是對於 G 上任何有選定分配方式中最小 r-可以分解的最大圓石數。

在此篇論文當中，我們首先從圓石分配數中的研究當中，整理出一些已知的性質，然後我們也得到一些關於臨界圓石數的新結果，其中包括圈，樹圖，彼得森圖及一些乘積圖等等的臨界圓石數。

Critical Pebbling Numbers of Graphs

Student: Tzung-Han Tsai
Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan 30050

Advisor: Hung-Lin Fu
Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan 30050

Abstract

Given a distribution of pebbles on the vertices of a graph G, a *pebbling step* (or *pebbling move*) takes two pebbles from one vertex and place one pebble on an adjacent vertex. A distribution D of pebbles on the graph G is called *solvable* if, starting from D, it is possible to place a pebble on any given vertex using a sequence of pebbling steps. The *pebbling number f(G)* of a connected graph is the smallest number of pebbles such that every distribution with *f(G)* pebbles on G is solvable. A distribution D is *r-solvable* if there exists a sequence of pebbling steps which begin from D and end in at least one pebble on the vertex r. The *r-critical pebbling number* $c_r(G)$ is the largest size of a minimally r-solvable rooted distribution on G for any r. In this thesis, we first derive several known results from the study of pebbling numbers, and then we also obtain several new results on critical pebbling numbers. The results are (1) $c_r(C_m) = f(C_m) - 1$ if $m \equiv 3 \pmod 4$, and $f(C_m)$ otherwise; (2) $c_r(T) = 2^d$, where T is a tree and d is the diameter of T; (3) $c_r(P) = 6$, where P is the Petersen graph; (4) $c_r(Q_n) = 2^n$, where Q_n is an n-cube; and (5) $c_r(C_5 \Box C_5) = 25$.

Acknowledgement

首先感謝指導教授傅恆霖老師，這兩年來細心的指導以及諄諄教誨。老師很有耐心的與我討論問題，協助我解決問題，讓我在學習研究的過程中受益良多。

其次，感謝黃大原老師、陳秋媛老師、翁志文老師等，在課業上的熱心幫助和生活上的關心，也要感謝黃明輝學長、嚴志弘學長、張嘉芬學姊、郭志銘學長、陳宏賓學長、詹棨丰學長的許多幫助，使得我在研究的過程中與資料的取得順利不少，以及元勳、俊全、怡菁、采瑩、秋美、書于、景堯、冠成這些同學，和他們一起遊玩、一起討論課業，讓我在交大的這兩年留下美好的回憶。

最後要感謝我的家人，總是掛念我的生活、學業、健康，感謝他們背後的支持，陪伴我一起走過這段充實的研究生涯。

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1 Introduction

The concept of pebbling graphs was first suggested by Lagarias and Saks [17, 22] to solve a number theoretic conjecture proposed by Erdös and Lemke using the different method which can be found in [18]. In 1987, Lagarias and Saks proposed the following question. Suppose 2^n pebbles are distributed onto vertices of an n-cube, is it always possible to move one pebble to a specified vertex by a sequence of pebbling steps from any distribution of 2^n pebbles? Note here that a pebbling step (or pebbling move) takes two pebbles from one vertex and place one pebble on an adjacent vertex. Later, in [4], Chung was able to show that the above game can be done and successfully used this tool to prove the conjecture. This sets up further study of pebbling numbers, and has been developed by many others including Hurlbert who published a survey of pebbling results in [15].

Moreover, several related topics of pebbling numbers, such as optimal pebbling number, cover pebbling number, and critical pebbling number are introduced and studied. Since then, the study of pebbling numbers is getting familiar to the researchers especially graph theorists.

In this thesis, we first derive several known results on the study of pebbling numbers, and then we obtain several new results on critical pebbling numbers. The results are (1) $c_r(C_m) = f(C_m) - 1$ if $m \equiv 3 \pmod 4$, and $f(C_m)$ otherwise; (2) $c_r(T) = 2^d$, where T is a tree and d is the diameter of T; (3) $c_r(P) = 6$, where P is the Petersen graph; (4) $c_r(Q_n) = 2^n$, where Q_n is an n-cube; and (5) $c_r(C_5 \Box C_5) = 25$.

2 Preliminaries

First, we introduce that the terminologies and definitions of graphs. For details, the readers may refer to the book "Introduction to Graph Theory" by D. B. West [28].

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its *endpoints*. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges. When u and v are the endpoints of an edge, they are *adjacent* and are *neighbors*.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. A *spanning subgraph* of G is a subgraph H with $V(H) = V(G)$. A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A graph G is *connected* if each pair of vertices in G belongs to a path; otherwise, G is *disconnected*. In this thesis, we consider all of the graphs which are simple and connected.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A graph with no cycle is *acyclic*. A *tree* is a connected acyclic graph.

A *complete graph* is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets called partite sets of G. A graph G is *r-partite* if $V(G)$ can be expressed as the union of r independent sets. A *complete multipartite graph* is a simple graph of G whose vertices can be partitioned into sets so that u is adjacent to v if and only if u and v belong to different sets of the partition. We use $K_{n_1,n_2,\ldots,n_r}$ to denote the *complete r-partite graph* with partite sets of sizes $n_1, n_2, ..., n_r$. Note that if $k = 2$, it is called *complete bipartite graph*.

A *k-dimensional cube (hypercube or k-cube)* Q_k is the simple graph whose vertices are the k-tuples with entries in {0,1} and whose edges are the pairs of k-tuples that differ in exactly one position.

The *wheel graph* W_n is composed of a cycle consisting of n vertices, $v_1, v_2, \ldots, v_n$, which are all connected to hub vertex v_0, for a total of $v = n + 1$ vertices.

The *Petersen graph* is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.

For any two graphs G and H, we define the *Cartesian product* of G and H, denoted $G\Box H$, is the graph with vertex set

$$V(G\Box H) = \{(u,v)\colon\ u \in V(G),\ v \in V(H)\},$$

and edge set

$$E(G\Box H) = \{((u,v),(u',v'))\colon\ u = u' \text{ and } (v,v') \in E(H) \text{ or } v = v' \text{ and } (u,u') \in E(G)\}.$$

Clearly, 1-cube is K_2, 2-cube is $K_2\Box K_2$, and n-cube Q_n is $Q_{n-1}\Box K_2$. The m-by-n *grid* is the Cartesian product $P_m\Box P_n$. Figure 1, shows $P_3\Box P_4$.

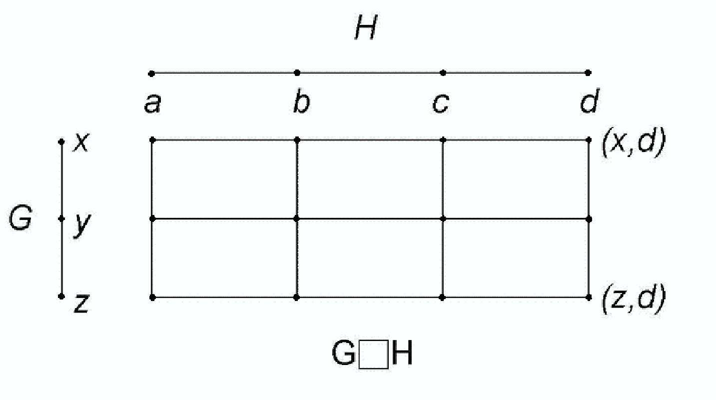

Figure 1: $P_3\Box P_4$.

A *complete m-ary tree* with height h, denoted by T_h^m, is an m-ary tree satisfying that v has m children for each vertex v not in the h-th level.

A tree T is called a *caterpillar* if the deletion of all pendent vertices of the tree results in a path P'. For convenience, we shall call a path P with maximum length which contains P' a *body* of the caterpillar, and all the edges which are incident to pendent vertices are the *legs* of the caterpillar T. Furthermore, the vertex $v \in V(P)$ is a *joint* of T provided that $deg_T(v) \geq 3$ or v is adjacent to the end vertices, Figure 2 is an example.

Suppose G is a connected graph. A *pebbling distribution* (simply called *distribution*) D on G is a function $D : V(G) \to N \cup \{0\}$ which assigns to every vertex of G a natural number of pebbles or zero pebble. If D is a distribution on G and a is a vertex of G, then $D(a)$ be the number of pebbles on a in the distribution D. The *size* of the distribution

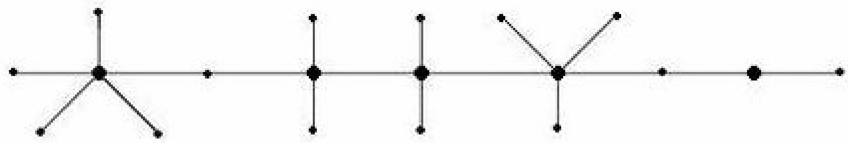

Figure 2: A caterpillar with 5 joints.

D is the number of pebbles in D, $|D| = \sum_{a\in V(G)} D(a)$. If there are some pebbles on the vertices of a graph G, a *pebbling step* is that one can remove two pebbles from one vertex and place one pebble on an adjacent vertex. We say that a pebble can be moved to a vertex r, the rooted vertex, if we can apply pebbling steps repeatedly (if necessary) so that in the resulting distribution the vertex r has one pebble.

A distribution D is *r-solvable* if there exists a sequence of pebbling steps which begin from D and end in at least one pebble on the vertex r. If D is r-solvable for any r, D is called *solvable*. A distribution D is *unsolvable* if there is some vertex r for which D is not r-solvable.

A *rooted distribution* is a distribution which identifies a vertex r of G as the rooted vertex of G. We say that the rooted distribution D is *solved* if it has at least one pebble on r, and *solvable* if there exists a sequence of pebbling steps staring with D and ending with a solved distribution. We say that an un-rooted distribution is a *global distribution*.

A rooted distribution D is *minimally r-solvable* if D is r-solvable but the removal of any one pebble makes D not r-solvable. A global distribution D is *minimally solvable* if D is solvable but the removal of any one pebble makes D unsolvable. A rooted distribution D is *maximally r-unsolvable* if D is not r-solvable but the addition of any one pebble makes D r-solvable. A global distribution D is *maximally unsolvable* if D is unsolvable but the addition of any one pebble makes D solvable.

The *pebbling number f(G)* of a graph G is one greater than the largest size of a maximally unsolvable global distribution on G. Equivalently, *f(G)* is one greater than the largest size of a maximally r-unsolvable rooted distribution on G for any r. Or equivalently, is the smallest integer m, such that for any distribution of m pebbles to the vertices of G, one pebble can be moved to a specified vertex, i.e. every distribution of m pebbles on G is solvable.

A distribution is a *k-pebbling* if it is possible to move k pebbles to any given vertex r after a sequence of pebbling steps. Let $f_k(T)$ denote the smallest integer n such that every distribution which at least n pebbles is k-pebbling.

We say that G satisfies the *2-pebbling property* if two pebbles can be moved to a given vertex when the total starting number of pebbles are $2f(G)-q+1$, where q is the number of vertices with at least one pebble.

The *optimal pebbling number* $f'(G)$ of a graph G is the smallest size of a minimally solvable global distribution on G. Equivalently, $f'(G)$ is the least number k such that there exists a solvable distribution of k pebbles on G.

We say that a distribution D on G is *cover solvable* if there exists a sequence of pebbling steps starting with D and ending with a pebble on every vertex of G simultaneously.

The *cover pebbling number* $\gamma(G)$ of G is the smallest integer n, such that for any distribution of n pebbles to the vertices of G, one pebble can be moved to every vertex simultaneously, i.e. every distribution of n pebbles on G is cover solvable.

Let a weight function ω be given that assigns an integer $\omega(v)$ to every vertex v of G. We say that ω is *positive* if $\omega(v) > 0$ for all v. The *weighted cover pebbling numbers* $\gamma_\omega(G)$ is the smallest integer k, such that for any distribution of k pebbles to the vertices of G, $\omega(v)$ pebbles can be moved to every vertex v simultaneously.

So, we know that the cover pebbling number $\gamma(G)$ is the weighted cover pebbling numbers $\gamma_\omega(G)$ with $\omega(v) = 1$ for all v.

The *r-critical pebbling number* $c_r(G)$ is the largest size of a minimally r-solvable rooted distribution on G for any r. If a minimally r-solvable rooted distribution on G has $c_r(G)$ pebbles, then we call it an *r-ceiling* distribution.

The distance between the vertices a and b denoted by $d(a,b)$. The *pebbling step* $[a,b]$ is an operation which remove two pebbles from the vertex a and place one pebble on the adjacent vertex b. The pebbling step $[a,b]$ is *greedy* if $d(a,r) > d(b,r)$, and the rooted distribution D is *greedy* if there is a solution of D which uses only greedy pebbling steps.

The *weight* of the rooted distribution D is defined as $\omega(D) = \sum_{v\in V(G)} \frac{D(v)}{2^{d(v,r)}}$. Therefore, the *weight* $\omega(G)$ of G is the largest weight of any r-ceiling distribution on G.

In this thesis, we first derive several known results from the study of pebbling numbers, and then we also obtain several new results on critical pebbling numbers. The results are

(1) $c_r(C_m) = f(C_m) - 1$ if $m \equiv 3 \pmod 4$, and $f(C_m)$ otherwise; (2) $c_r(T) = 2^d$, where T is a tree and d is the diameter of T; (3) $c_r(P) = 6$, where P is the Petersen graph; (4) $c_r(Q_n) = 2^n$, where Q_n is an n-cube; and (5) $c_r(C_5 \square C_5) = 25$.

3 The Pebbling Numbers

We review that the *pebbling number f(G)* of a graph G is one greater than the largest size of a maximally unsolvable global distribution on G. Equivalently, *f(G)* is one greater than the largest size of a maximally r-unsolvable rooted distribution on G for any r. Or equivalently, is the smallest integer m, such that for any distribution of m pebbles to the vertices of G, one pebble can be moved to a specified vertex, i.e. every distribution of m pebbles on G is solvable. We start with some properties of the pebbling number of a graph G, $f(G)$.

3.1 Some Preliminary Results

First, we include some facts about *f(G)*.

(F_1) **Breadth Lower Bound :** $f(G) \geq n(G)$, where $n(G)$ denotes the number of vertices of G.

It follows by the fact that we place one pebble on every vertex of G except one.

Note that graphs whose pebbling number equals $n(G)$ are said to be of Class 0, and equals $n(G) + 1$ are said to be of Class 1.

(F_2) **Depth Lower Bound :** $f(G) \geq 2^{diam(G)}$, where $diam(G)$ denotes the diameter of G.

It follows by the fact that we place all the pebbles on a vertex which is at distance $diam(G)$ from another vertex.

(F_3) **Pigeonhole Upper Bound :** $f(G) \leq (n(G) - 1)(2^{diam(G)} - 1) + 1$.

By pigeonhole principle, at least one vertex has $2^{diam(G)}$ pebbles and thus the distribution is solvable.

Combining (F_1), (F_2) and (F_3), we conclude that if $d = diam(G)$ and $n = n(G)$, then

$$max\{n, 2^d\} \leq f(G) \leq (2^d - 1)(n - 1) + 1.$$

(F_4) **Cut Lower Bound :** $f(G) > n(G)$, when G contains a cut vertex a.

Suppose not. Let A and B be two components of $G - a$, with v in A and r in B. Then by placing 3 pebbles on v and 1 pebble on every other vertex except a and r, we have a distribution which is not solvable.

Note that Class 0 graphs are 2-connected.

(F_5) **Spanning Subgraph :** $f(H) \geq f(G)$, where H is spanning subgraph of G.

From the above properties (F_1), (F_2), $\cdots$, (F_5), we are able to obtain several nice results :

- $f(K_n) = n$.
- $f(P_n) = 2^{n-1}$.
- $f(C_5) = 5$ and $f(C_6) = 8$.
- $f(P) = 10$ where P is the Petersen graph [5].
- If L_n is the line graph of the complete graph K_n, then $f(L_n) = \binom{n}{2}$ [26].

Theorem 3.1.1. [21] *For* $k \geq 1$, $f(C_{2k}) = 2^k$ *and* $f(C_{2k+1}) = 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$.

Indeed, finding $f(T)$ needs some extra effort. We define a maximum path partition P of a tree T. Suppose T is a tree with a given vertex v. T can be viewed as a directed tree denoted by T_v^* with edges directed toward the given vertex v, v is also called the rooted vertex. Consider a partition $P = (P_1, P_2, \ldots, P_t)$ of the edges of T into P_1, P_2, $\ldots$, P_m such that $p_i \geq p_{i+1}$ where $p_i = |P_i|$, which is a set of nonoverlapping directed paths and the union of which is T. A partition is said to *majorize* another if the nonincreasing sequence of the path size majorize that of other; that is, $(p_1, p_2, \ldots, p_t) \geq (p'_1, p'_2, \ldots, p'_n)$ if and only if $p_i \geq p'_i$, where $i = min\{j : a_j \neq b_j\}$. A path partition is *maximum* if no other path partition majorizes it. Now, we have the following beautiful result by D. Moews.

Theorem 3.1.2. [20] *Let T be a tree and r be the rooted vertex of T. If $(p_1, p_2, \ldots, p_t)$ is the nonincreasing sequence of a maximum path partition $P = (P_1, P_2, \ldots, P_t)$, then* $f(T) = (\sum_{i=1}^{t} 2^{p_i}) - t + 1$ *and* $f_k(T) = k2^{p_1} + (\sum_{i=2}^{t} 2^{p_i}) - t + 1$.

3.2 2-Pebbling and Products

Let Q_n be the n-dimensional cube (simply called n-cube). Chung [4] gives the 2-Pebbling property in order to prove the following theorems.

Review that G satisfies the *2-pebbling property* if two pebbles can be moved to a given vertex when the total starting number of pebbles are $2f(G) - q + 1$, where q is the number of vertices with at least one pebble. Note that both n-cube and paths satisfy the 2-pebbling property.

Theorem 3.2.1. [4] *Suppose G satisfies the 2-pebbling property. Then the following holds:*
(1) $f(G \square K_t) \leq tf(G)$.
(2) If $f(G \square K_t) = tf(G)$, then $G \square K_t$ satisfies the 2-pebbling property.

The conjecture about the pebbling number of the cartesian product of graph was posed by Graham.

Conjecture 3.2.2. (Graham) $f(G \square H) \leq f(G)f(H)$.

For the n-cube Q_n, Chung [4] proved $f(Q_n) = 2^n$. The above conjecture has been verified for several classes of graphs. First, by Theorem 3.2.1, we have $f(K_{n_1} \square K_{n_2} \square \cdots \square K_{n_t}) = n_1 n_2 \ldots n_t$.

Theorem 3.2.3. [4] $f(P_{n_1+1} \square P_{n_2+1} \square \cdots \square P_{n_t+1}) = 2^{n_1+n_2+\ldots+n_t}$.

Moreover, the conjecture also holds for, a cycle by a cycle [13, 14, 21], complete bipartite graphs [8], fans and wheels [9], and trees [20].

3.3 Diameter, Connectivity and Class 0 Graphs

Theorem 3.3.1. [21] *Let G be a graph with diameter 2, then G has the 2-pebbling property.*

Theorem 3.3.2. [21] *If $diam(G) = 2$, then $f(G) = n(G)$ or $n(G) + 1$.*

Bukh recently obtained an upper bound of diameter 3 graphs as following.

Theorem 3.3.3. [2] *If diam(G) = 3, then $f(G) \leq 3n/2$, which is best possible.*

Hurlbert et. al. can characterize Class 0 graphs of diameter two which are decided by connectivity.

Theorem 3.3.4. [6] *If diam(G) = 2 and $\kappa(G) \geq 3$, then G is of Class 0.*

Theorem 3.3.5. [6] *If G is of Class 0, then $\kappa(G) \geq 2$. In particular, if $diam(G) = 2$ and $\kappa(G) = 1$, then G is of Class 1.*

4 The Optimal Pebbling Numbers

We review that the *optimal pebbling number* $f'(G)$ of a graph G is the smallest size of a minimally solvable global distribution on G. Equivalently, $f'(G)$ is the least number k such that there exists a solvable distribution of k pebbles on G.

4.1 The Known Results

For optimal pebbling numbers, it seems that to find its upper bounds is easier than to find lower bounds. Clearly, in order to obtain an upper bound, it suffices to give a distribution and show that it is solvable. But, finding a tight lower bound, we have to show that every distribution with less pebbles is not solvable.

It is easy to see that $f'(G) \leq 2^d$, where d is a diameter of G. The following general upper bound for graphs on n vertices was proved in [3].

Corollary 4.1.1. *If G is a connected graph of n vertices, then $f'(G) \leq \lceil 2n/3 \rceil$.*

It was shown that the tight upper bound for paths $f'(P_{3t+r}) = 2t + r$, where $0 \leq r \leq 2$ [21], and cycles $f'(C_{3t+r}) = 2t + r$, where $0 \leq r \leq 2$ [3].

The Graham's conjecture on pebbling number is still an open problem, but the analogous statement for optimal pebbling number is easier.

Theorem 4.1.2. [23] *For any two graphs G and H, $f'(G \square H) \leq f'(G)f'(H)$.*

Even $f(Q_n)$ is known, $f'(Q_n)$ is far from being determined.

Theorem 4.1.3. [19] $f'(Q_n) = (\frac{4}{3})^{n+O(logn)}$.

More known results about optimal pebbling are as following.

Theorem 4.1.4. [11] *If T_h^m is an m-ary tree satisfying that v has m children for each vertex v not in the h-th level, then $f'(T_h^m) = 2^h$ for each $m \geq 3$.*

We review that a tree T is called a *caterpillar* if the deletion of all pendent vertices of the tree results in a path P'. For convenience, we shall call a path P with maximum length which contains P' a *body* of the caterpillar, and all the edges which are incident to pendent vertices are the *legs* of the caterpillar T. Furthermore, the vertex $v \in V(P)$ is a *joint* of T provided that $deg_T(v) \geq 3$ or v is adjacent to the end vertices.

Theorem 4.1.5. [10] *Let T be a caterpillar with P a body of T and $|V(P)| = n$. Let $\alpha(v) = 2$ if v is a joint of T and $\alpha(v) = 1$ otherwise. Let $P'_1, P'_2, \cdots, P'_m$ be 2-maximal subpaths of P with respect to α and P_i be a subpath between P'_i and P'_{i+1} for $i = 1, 2, \ldots, m-1$. Then $f'(T) = n - m - \sum_{i=1}^{m-1} \lfloor |v(P_i)|/3 \rfloor$.*

4.2 Bounds with Minimum Degree

By Corollary 4.1.1, we have proved that $f'(G) \leq \lceil 2n/3 \rceil$ for any connected graph G with n vertices, with equality holds for paths and cycles. We will the find $f'(G)$ by adding the consideration of minimum degree k.

Proposition 4.2.1. [3] Among n vertices graphs with minimum degree k, $f'(G)$ can be as large as $2n/(k+1)$.

Conjecture 4.2.2. Given $k \in N$, there exists n_0 such that for all $n \geq n_0$, every connected n vertices graph G with minimum degree k satisfies $f'(G) \leq \lceil 2n/(k+1) \rceil$.

If the case $k = 2$, the above Corollary 4.1.1 satisfies the above conjecture. It would be interesting to deal with the case $k = 3$. We find the probabilistic argument in Alon [1] as follows.

Theorem 4.2.3. *If G is a connected graph with n vertices and minimum degree k, then $f'(G) \leq \frac{2n}{k+1}(1 + ln\frac{k+1}{2})$.*

5 The Cover Pebbling Numbers

We review that the *cover pebbling number* $\gamma(G)$ of G is the smallest integer n such that for any distribution of n pebbles to the vertices of G, one pebble can be moved to every vertex simultaneously, i.e. every distribution of n pebbles on G is cover solvable.

Let a weight function ω be given that assigns an integer $\omega(v)$ to every vertex v of G. We say that ω is *positive* if $\omega(v) > 0$ for all v. The *weighted cover pebbling number* $\gamma_\omega(G)$ is the smallest integer k, such that for any distribution of k pebbles to the vertices of G, $\omega(v)$ pebbles can be moved to every vertex v simultaneously.

5.1 The Known Results

First, by observation, we have $\gamma(G) \geq 2^d(n-d+1)-1$, for G be a graph with n vertices and diameter d. Therefore, we have $\gamma(K_n) = 2n-1$, and $\gamma(P_n) = 2^n-1$. If we denote that the total weight by $|\omega| = \sum_v \omega(v)$ and $min\omega = min_v \omega(v)$, then $\gamma_\omega(K_n) = 2|\omega| - min\omega$, (see [7]), for every positive weight function ω.

The *fuse* $F_l(n)$ ($l \geq 2$ and $n \geq 3$) is the graph with n vertices $v_1, v_2, \ldots, v_n$, such that the first l vertices form a path (or *wick*) from v_1 to v_l, and the remaining n - l vertices are independent vertices (or *sparks*) and adjacent only to v_l.

Theorem 5.1.1. [7] $\gamma(F_l(n)) = (n-l+1)2^l - 1$.

Example 5.1.2. $F_2(n)$ is a star with n vertices and $\gamma(F_2(n)) = 4n-5$.

Let T be an arbitrary tree and let *V(T)* be the vertex set of T. For $v \in V(T)$, we define

$$s(v) = \sum_{u \in V(T)} 2^{d(u,v)} \text{ and } s(T) = \max_{v \in V(G)} s(v),$$

where $d(u,v)$ denotes the distance between u and v.

Suppose a positive weight function ω of T is given, we define

$$s_\omega(v) = \sum_{u \in V(T)} \omega(u) 2^{d(u,v)} \text{ and } s_\omega(T) = \max_{v \in V(T)} s_\omega(v).$$

Clearly, $\gamma_\omega(T) \geq s_\omega(T)$, for any T and any positive weight function ω. In fact, this lower bound is tight.

Theorem 5.1.3. [7] *For any tree T and any positive weight function ω, we have*

$$\gamma_\omega(T) = s_\omega(T).$$

Theorem 5.1.4. *(Stacking Theorem)* [25] *For any graph G and any positive weight function ω, we have*

$$\gamma_\omega(G) = s_\omega(G).$$

In the case of n-cube, we have the following result.

Theorem 5.1.5. [16] $\gamma(Q_n) = 3^n$.

The complete multipartite graphs and wheel graphs were considered in [27].

Theorem 5.1.6. $\gamma(K_{n_1,n_2,\ldots,n_r}) = 4n_1 + 2n_2 + \cdots + 2n_r - 3$, *where* $n_1 \geq n_2 \geq \cdots \geq n_r$.

We review that the wheel graph W_n is composed of a cycle consisting of n vertices, $v_1, v_2, \ldots, v_n$, which are all connected to hub vertex v_0, for a total of $v = n+1$ vertices.

Theorem 5.1.7. *For $n \geq 3$, $\gamma(W_n) = 4n - 5 = 4v - 9$.*

The following cases, we discuss the cycles and certain graph products which considered in [24].

Theorem 5.1.8. $\gamma(C_n) = 2^r + 2^{n-r+1} - 3$, *where $r = n/2$ if n is even and $r = (n+1)/2$ if n is odd.*

5.2 Graph Products

There are some known covering pebbling numbers of the graph products.

(1) For any graph G, $\gamma(P_n \Box G) \leq (2n-1)\gamma(G)$.

(2) $\gamma(C_n \Box G) \leq \begin{cases} (2^{(n/2)+1} + 2^{(n/2)} - 3)\gamma(G) & \text{if } n \text{ is even.} \\ (2^{(n+1)/2} + 2^{(n+1)/2} - 3)\gamma(G) & \text{if } n \text{ is odd.} \end{cases}$

In particular, taking G to be a single vertex, and we have the above Theorem 5.1.8.

(3) We define that G is *good* if $\gamma(G) = \sum_{u \in V(G)} 2^{d(u,v)}$ for some vertex $v \in V(G)$.

(i) Suppose G and H are *good*. $\gamma(G \Box H) = \gamma(G)\gamma(H) \Longleftrightarrow G \Box H$ is *good*.

(ii) $H = (\Box_i P_{n_i})\Box(\Box_j C_{m_j})$ is *good*.

- $\gamma(H) = \prod_i \gamma(P_{n_i}) \prod_j \gamma(C_{m_j})$.

  In particular,

  $\gamma(\Box_i P_{n_i}) = \prod_i (2^{n_i} - 1)$.

  $\gamma(\Box_j C_{m_j}) = \prod_j (2^{r_{m_j}} + 2^{m_j - r_{m_j} + 1} - 3)$, where $r_{m_j} = \lceil m_j/2 \rceil$.

- $\gamma(H\Box K_n) = \gamma(H)\gamma(K_n)$.
- $\gamma(H\Box T) = \gamma(H)\gamma(T)$ for any tree T.

Example 5.2.1. K_n, P_n, C_n, and T are good.

Since Q_n is isomorphic to $\Box_n P_2$, thus $\gamma(Q_n) = \gamma(\Box_n P_2) = \prod_n (2^2 - 1) = 3^n$. Then we can also use this result to prove Theorem 5.1.5.

6 The Critical Pebbling Numbers

We review that the *r-critical pebbling number* $c_r(G)$ is the largest size of a minimally r-solvable rooted distribution on G for any r. If a minimally r-solvable rooted distribution on G has $c_r(G)$ pebbles, then we call it an *r-ceiling* distribution.

6.1 Some Preliminary Results

Lemma 6.1.1. *Suppose that G is a graph with diameter d, then $c_r(G) \geq 2^d$.*

Proof.

Let a and b be two vertices with distance d. Let D be the rooted distribution where $a = r$ and $D(b) = 2^d$. Clearly, this distribution is minimally r-solvable. So, we have $c_r(G) \geq 2^d$. ∎

Lemma 6.1.2. [12] *If G is a graph, then $f(G) \geq c_r(G)$.*

It is easy to show that $c_r(K_n) = 2$, $c_r(P_n) = 2^{n-1}$ and $c_r(K_{n_1,n_2,\ldots,n_r}) = 4$.

Table 1 shows that the numbers of the four pebbling parameters for four simple graphs.

	K_5	P_4	C_7	$K_{3,2}$
$f(G)$	5	8	11	5
$f'(G)$	2	3	5	3
$\gamma(G)$	9	15	29	13
$c_r(G)$	2	8	10	4

Table 1: Some pebbling parameters of graphs.

A solution of the rooted distribution D is called *r-critical* if it leaves one pebble on r and no pebbles on any other vertex. A distribution D is *r-excessive* if D is r-solvable and not r-critical, moreover, D is r-insufficient if it is not r-solvable. Note that for an r-excessive distribution E, an r-critical distribution C and an r-insufficient distribution I, we have that $|I| \geq |C| \geq |E|$. Figure 3 is an example of this three distributions.

Theorem 6.1.3. [20, 12] $f(K_{1,n}) = n + 2$ *and* $c_r(K_{1,n}) = 4$ *for* $n \geq 4$.

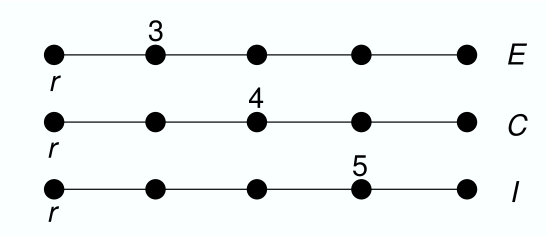

Figure 3: Three rooted distributions on the graph P_5.

By Lemma 6.1.2, $f(G) \geq c_r(G)$ for any graph G. As a matter of fact Theorem 6.1.3 gives an example of a family of graphs G such that $|f(G) - c_r(G)|$ can be arbitrarily large.

Now, we consider the *fan* F_k which is obtained by joining a vertex to path P_k.

Theorem 6.1.4. [12] *$f(F_k) = k + 1$ and $c_r(F_k) = k$ for $k \geq 4$.*

The example of an r-ceiling and an r-insufficient rooted distribution on the fan F_5 are shown in Figure 4.

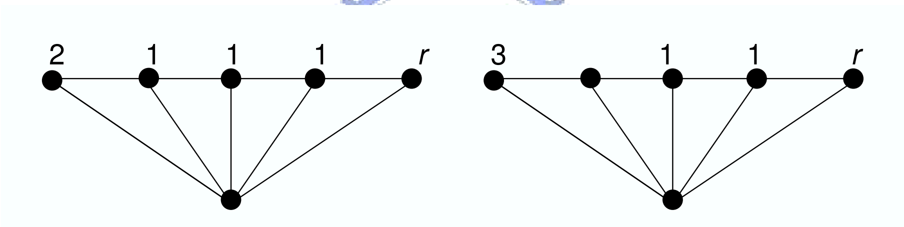

Figure 4: An r-ceiling and an r-insufficient rooted distribution on the fan F_5.

Corollary 6.1.5. [12] *If k is a positive integer, there exist graphs with r-critical pebbling number k if and only if $k \neq 3$.*

6.2 Weight and Greedy Pebbling Step

We review that the *weight* of the rooted distribution D is defined as $\omega(D) = \sum_{v \in V(G)} \frac{D(v)}{2^{d(v,r)}}$. Therefore, the *weight* $\omega(G)$ of G is the largest weight of any r-ceiling distribution on G.

Theorem 6.2.1. [12] *The fan F_k has weight $\frac{k+1}{4}$.*

Corollary 6.2.2. [12] *There exist graphs with diameter 2 that have arbitrarily large weight.*

Lemma 6.2.3. [12] *If the rooted distribution E is obtained from the rooted distribution D, then $\omega(E) \le \omega(D)$ with equality holds only when a greedy pebbling step is used.*

Proof.

Suppose E is obtained from D by the pebbling step $[a, b]$. If $[a, b]$ is greedy and $d(a, r) = d$, then $d(b, r) = d - 1$. E has two fewer pebbles on a and one additional pebble on b. So,

$$\omega(E) = \omega(D) - \tfrac{2}{2^d} + \tfrac{1}{2^{d-1}} = \omega(D).$$

If $[a, b]$ is not greedy and $d(a, r) = d$, then $d(b, r) = e \ge d$. So,

$$\omega(E) = \omega(D) - \tfrac{2}{2^d} + \tfrac{1}{2^e} < \omega(D).$$

∎

Lemma 6.2.4. *If D is r-critical and greedy, then $\omega(D) = 1$.*

Corollary 6.2.5. *If $\omega(D) < 1$, then D is r-insufficient.*

Corollary 6.2.6. *For any graph G, $\omega(G) \ge 1$.*

Lemma 6.2.7. [12] *If D is a rooted distribution on G, P is a path in G with end vertex r, E is the rooted distribution on P induced from D, and $\omega(E) > 1$, then D is r-excessive.*

Proof.

Suppose that there are no pebbles on r and at most one pebble on every other vertex on P. So, we have $\omega(E) < 1$. Assume that E is not solved, then there exists a vertex a of P such that $\omega(a) > 1$. Therefore, we may put pebble from a towards r. Since this pebbling step is greedy, by Lemma 6.2.3, the new rooted distribution E' obtained from this pebbling step still satisfies $\omega(E') > 1$. Then we continue in this way until we reach a solved rooted distribution F. Since $\omega(F) > 1$, it yields F is r-excessive, so E is r-excessive. Now, we use the same pebbling steps on D and conclude that D is also r-excessive. ∎

6.3 The Main Results

Theorem 6.3.1. $c_r(C_m) = \begin{cases} f(C_m) - 1 & \textit{if } m \equiv 3 \pmod 4, \textit{ and} \\ f(C_m) & \textit{otherwise.} \end{cases}$

Proof.

Let $C_m = (r, a_1, a_2, \cdots, a_{m-1})$, see Figure 5. Consider the two paths $r, a_1, \cdots, a_{m-1}$ and $r, a_{m-1}, \cdots, a_1$.

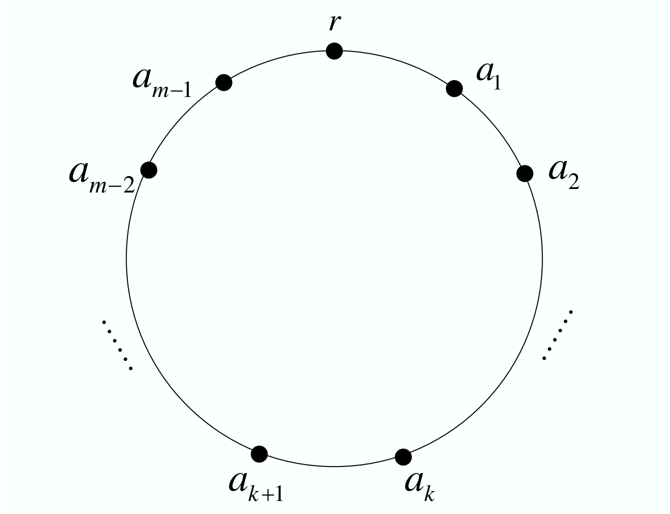

Figure 5: The graph C_m.

Suppose that D is an r-critical rooted distribution on C_m. Then, by Lemma 6.2.7, we have the following inequalies:

$$\begin{cases} \displaystyle\sum_{i=1}^{m-1} \frac{D(a_i)}{2^i} \leq 1 \\ \displaystyle\sum_{i=1}^{m-1} \frac{D(a_i)}{2^{m-i}} \leq 1. \end{cases}$$

$$\Longrightarrow \begin{cases} \displaystyle\sum_{i=1}^{m-1} 2^{m-1-i} D(a_i) \leq 2^{m-1} \ldots (1) \\ \displaystyle\sum_{i=1}^{m-1} 2^{i-1} D(a_i) \leq 2^{m-1} \ldots (2) \end{cases}$$

Summing up (1) and (2), we have

$$\sum_{i=1}^{m-1} (2^{m-1-i} + 2^{i-1}) D(a_i) \leq 2^m.$$

We assume that $m = 2k+1$. If D is an r-critical rooted distribution on C_{2k+1} with the number of pebbles at least $f(C_{2k+1})$, then

$$(2^k + 2^{k+1}) \sum_{i=1}^{2k} D(a_i) + \sum_{i=1, i \neq \{k,k+1\}}^{2k} (2^{2k-i} + 2^{i-1}) D(a_i) \leq 2^{2k+1}.$$

Since $\sum_{i=1}^{2k} D(a_i) \geq 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1 = f(C_{2k+1})$,

$$(3 \cdot 2^{k-1})(2\lfloor \tfrac{2^{k+1}}{3} \rfloor + 1) + \sum_{i=1, i \neq \{k,k+1\}}^{2k} (2^{2k-i} + 2^{i-1}) D(a_i) \leq 2^{2k+1}.$$

If k is odd, we have $\lfloor \frac{2^{k+1}}{3} \rfloor = \frac{2^{k+1}-1}{3}$. This implies that

$$(3 \cdot 2^{k-1})(2 \cdot \tfrac{2^{k+1}-1}{3} + 1) + \sum_{i=1, i \neq \{k,k+1\}}^{2k} (2^{2k-i} + 2^{i-1}) D(a_i) \leq 2^{2k+1}.$$

But, $(3 \cdot 2^{k-1})(2 \cdot \frac{2^{k+1}-1}{3} + 1) = 2^k(2^{k+1} - 1) + 3 \cdot 2^{k-1} = 2^{2k+1} - 2^k + 3 \cdot 2^{k-1} > 2^{2k+1}$.

This is a contradiction. Thus, there is no r-critical distribution D on odd cycle with more than $f(C_{2k+1})$ pebbles, and we conclude that $c_r(C_{2k+1}) \leq f(C_{2k+1}) - 1$.

Moreover, by placing $\frac{2^{k+1}-1}{3} - 1$ and $\frac{2^{k+1}-1}{3} + 1$ pebbles on vertices a_k and a_{k+1}, respectively, we have an r-critical rooted distribution. Thus $c_r(C_{2k+1}) = f(C_{2k+1}) - 1$, if k is odd.

Now we consider the case when k is even.

Since $c_r(C_{2k+1}) \leq f(C_{2k+1}) = 2\lfloor \frac{2^{k+1}}{3} \rfloor + 1$, and the distribution of $\lfloor \frac{2^{k+1}}{3} \rfloor$ and $\lfloor \frac{2^{k+1}}{3} \rfloor + 1$ pebbles on vertices a_k and a_{k+1}, respectively, we have an r-critical rooted distribution. Thus $c_r(C_{2k+1}) = f(C_{2k+1})$, if k is even.

For the case $m = 2k$, we also let a_i, $i = 1, 2, \ldots, 2k - 1$ be non-rooted vertices which starting from a vertex adjacent to rooted vertex r and continuing around the cycle. By a similar argument as above, we have $c_r(C_{2k}) \leq f(C_{2k}) = 2^k$ and an r-critical rooted distribution which use 2^k pebbles on vertex a_k. Thus, $c_r(C_{2k}) = f(C_{2k})$. ∎

Theorem 6.3.2. *Let T be a tree with diameter d. Then $c_r(T) = 2^d$.*

Proof.

Let T be a tree with diameter d. Therefore, there exists two vertices $r, r' \in V(T)$ such that $d(r, r') = d$. Let r be the rooted vertex of T and the other of vertices are labeled by $a_{j,i}$ where $d(a_{j,i}, r) = j$, $i \in \{1, 2, \ldots\}$ and $j \in \{1, 2, \ldots, d\}$.

Since D is an r-critical rooted distribution on T, $\sum_{m=1}^{d} \frac{D(A_m)}{2^m} \leq 1$ where $D(A_m) = \sum_{k=1} D(a_{m,k})$. For otherwise, if $\sum_{m=1}^{d} \frac{D(A_m)}{2^m} > 1$, then there will be at least one pebble which can not be used. This implies that D is not r-critical. So, $\sum_{m=1}^{d} \frac{D(A_m)}{2^m} \leq 1$ and thus $\sum_{m=1}^{d} D(A_m) \leq \sum_{m=1}^{d} 2^{d-m} D(A_m) \leq 2^d$. Hence, $c_r(T) \leq 2^d$.

By Lemma 6.1.1, we have $c_r(T) \geq 2^d$. This concludes that $c_r(T) = 2^d$. ∎

Theorem 6.3.3. *Let P be the Petersen graph. Then $c_r(P) = 6$.*

Proof.

Let P be the Petersen graph with rooted vertex r and the other 9 vertices are labeled by a_i where $d(a_i, r) = 1$, $\forall i = \{1, 2, 3\}$, and b_j where $d(b_j, r) = 2$, $\forall j = \{1, 2, 3, 4, 5, 6\}$, see Figure 6.

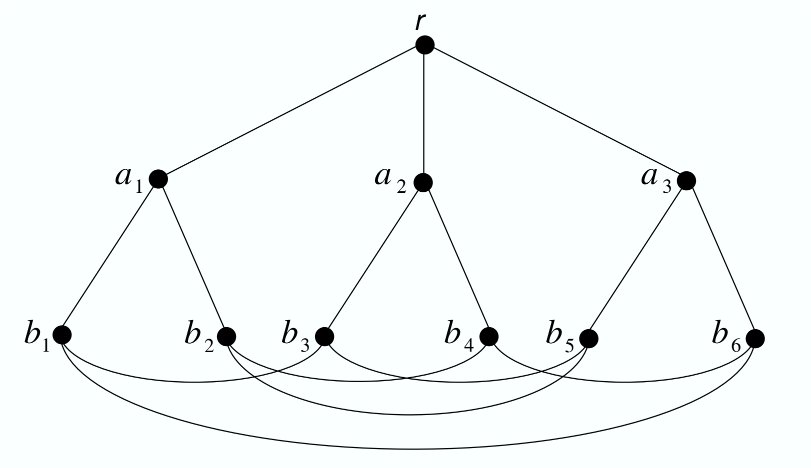

Figure 6: Petersen graph.

If D is an r-critical rooted distribution, then the following properties hold:

1. $D(a_i) \leq 1$ and $D(b_j) \leq 3$. For otherwise, $D(a_i) \geq 2$ or $D(b_j) \geq 4$, then it can not satisfy the definition of r-critical pebbling number.

2. If $D(a_i) = 1$, then $D(a_k) = 0$ for $i \neq k$.

3. If $D(b_j) \geq 2$, then $|b_j| \leq 3$. For otherwise, $|b_j| \geq 4$ and $D(b_j) \geq 2$, then we will find that there are two vertices b_j such that $D(a_i) \geq 2$ by a pebbling step, this contradicts property 1.

4. If $D(b_j) \geq 2$, then $D(a_i) = 0$ where a_i is adjacent to b_j. For otherwise, $D(a_i) = 1$ and $D(b_j) \geq 2$, then we have $D(a_i) = 2$ by a pebbling step, this contradicts property 1.

5. There exists at least one vertex $v \in V(D)$ such that $D(v)$ is even .

6. If $D(v) = 3$, then $|v| \leq 1$, and $D(v) \geq 2$ where $|v| \leq 3$.

Now, we put the pebbles on the vertices. $(b_1, b_3, b_5) = (2, 1, 3)$ or $(2, 2, 2)$. Then will be a pebble on the rooted vertex r by a sequence of pebbling steps. So $c_r(P) \geq 6$.

Therefore, we claim that $c_r(P) \leq 6$. Thus, it suffices to show that 7 pebbles on P can not satisfy the above properties.

Moreover, we partition 7 into positive integers not greater than 3 (as the cases).

Case 1 : (3,3,1)

It contradicts property 5.

Case 2 : (3,2,2)

First, without loss of generality, we put 2 pebbles on vertex b_1 and put 3 pebbles on one of the vertices in $\{b_2, b_3, \cdots, b_6\}$. Clearly, 3 pebbles are not on b_2, b_3 or b_6, since it can not satisfy the definition of r-critical pebbling number. So, we suppose 3 pebbles on b_4 and b_5 is treated by the same way. Then the last 2 pebbles will be put on b_2, b_3, b_5 or b_6. However, if we put 2 pebbles on either one of them, it will not satisfy the definition of r-critical pebbling number.

Case 3 : (3,2,1,1)

First, without loss of generality, we put 2 pebbles are placed on b_1 and 3 pebbles on b_4. Consider 1 pebble on a_3, b_2, b_3, b_5 or b_6. However, if we put 1 pebble on a_3, b_2 or b_3, it will not satisfy the definition of r-critical pebbling number. So, we can only put 1 pebble on b_5 and b_6, respectively. But, by a sequence of pebbling steps, 1 pebble on b_5 will not be used. It will not satisfy the definition of r-critical pebbling number.

Case 4 : (3,1,1,1,1)

It contradicts property 5.

Case 5 : (2,2,2,1)

First, without loss of generality, we put 2 pebbles on vertex b_1. Since $b_1, b_2, \ldots, b_6$ form a cycle. So, there are three cases such that three vertices have 2 pebbles. The three subcases are (b_1, b_3, b_5), (b_1, b_2, b_3), and (b_1, b_4, b_5). But, these subcases which have 6 pebbles will put a pebble on r by a sequence of pebbling steps. Then the 7th pebble is unnecessary. This means that the rooted distribution D is excessive. Thus, it will not satisfy the definition of r-critical pebbling number.

Case 6 : (2,2,1,1,1)

First, without loss of generality, we put 2 pebbles on vertex b_1. We know that b_2, b_3 and b_6 can not be put 2 pebbles on either one of them, since it can not satisfy the definition of r-critical pebbling number. So, we suppose 2 pebbles on b_4 (respectively on b_5). Then, the other three pebbles will be put on a_3, b_2, b_3, b_5 or b_6 one for each of three vertices. Again, this can not satisfy the definition of r-critical pebbling number.

Case 7 : (2,1,1,1,1,1)

First, without loss of generality, we put 2 pebbles on vertex b_1. Then, a_1 must has no pebble on it, otherwise it can not satisfy the definition of r-critical pebbling number. Consider b_3 and b_6. One of them has 1 pebble and the other has no pebble. We suppose that b_3 has 1 pebble, then a_2 must have no pebble on it. So, we just remain only 4 vertices a_3, b_2, b_4, and b_5 to put 1 pebble on one of them. But, it will not satisfy the definition of r-critical pebbling number either.

Case 8 : (1,1,1,1,1,1,1)

It contradicts property 5.

This concluses the proof. ∎

Theorem 6.3.4. $c_r(Q_n) = 2^n$.

Proof.

By Lemma 6.1.2, $f(Q_n) \geq c_r(Q_n)$. Since $f(Q_n) = 2^n$ (see [4]), $c_r(Q_n) \leq 2^n$. Now, by Lemma 6.1.1, since Q_n is a graph with diameter n, so we have $c_r(G) \geq 2^n$. Therefore, $c_r(Q_n) = 2^n$. ∎

Theorem 6.3.5. $c_r(C_5 \square C_5) = 25$.

Proof.

By Lemma 6.1.2, $f(C_5 \square C_5) \geq c_r(C_5 \square C_5)$. Since $f(C_5 \square C_5) = 25$ (see [14]), $c_r(C_5 \square C_5) \leq 25$. Since we have an r-critical rooted distribution on $C_5 \square C_5$, see Figure 7. Therefore, $c_r(C_5 \square C_5) = 25$. ∎

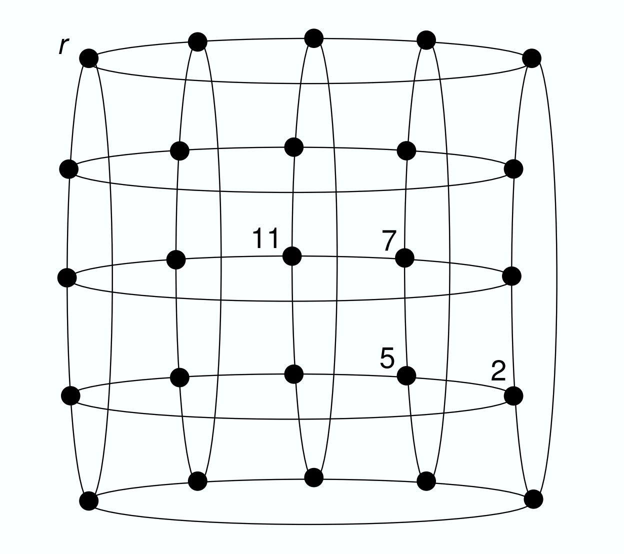

Figure 7: An r-critical rooted distribution on $C_5 \square C_5$.

Finally, we consider that the analogous statement about Graham's Conjecture for r-critical pebbling number, $c_r(G \square H) \leq c_r(G)c_r(H)$. But $c_r(C_3) = 2$, and $c_r(C_3 \square C_3) \geq 5$. So the inequality can not be satisfied. Therefore, the analog of Graham's Conjecture on pebbling number does not hold.

7 Conclusion

In this thesis, we mainly study the critical pebbling number of several classes of graphs and we are able to obtain several new results. The results are (1) $c_r(C_m) = f(C_m) - 1$ if $m \equiv 3 \pmod 4$, and $f(C_m)$ otherwise; (2) $c_r(T) = 2^d$, where T is a tree and d is the diameter of T; (3) $c_r(P) = 6$, where P is the Petersen graph; (4) $c_r(Q_n) = 2^n$, where Q_n is an n-cube; and (5) $c_r(C_5 \square C_5) = 25$. It takes no time to realize that finding the critical pebbling number of a graph is not easy at all. More properties have to be discovered. We also wish that we can do a better job in the near future.

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