國 立 交 通 大 學

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碩 士 論 文

二維網格模型中多符號的花樣生成問題

Two-Dimensional Patterns Generation Problem with Many Symbols

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摘　　　　要

此篇論文主要是研究在二維網格模型下三個符號的花樣生成問題。研究的主要目的是想找一些特別的置換矩陣 A_2 而能將熵明確地算出來。

Two-Dimensional Patterns Generation Problem with Many Symbols

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ABSTRACT

In this paper we discuss patterns generation problems with three symbols mainly. The main result is to find some special transition matrix $\mathbf{A}_2$ such that their spatial entropy can be solved explicitly.

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目　　　錄

1. Introduction

Lattices are important in scientifically modeling underlying spatial structures. Investigations in this field have covered phase transition [8], [11], [33], [34], [31], [32], [35], [42], [43], [44], [45], chemical reaction [6], [7], [23], biology [9], [10], [20], [21], [22], [28], [29], [30] and image processing and pattern recognition [15], [16], [17], [18], [19], [24]. In the field of lattice dynamical systems (LDS) and cellular neural networks (CNN), the complexity of the set of all global patterns recently attracted substantial interest. In particular, its spatial entropy has received considerable attention [1], [2], [5], [3], [4], [12], [14], [13], [25], [26], [27], [36], [37], [38], [39], [40], [41].

The one dimensional spatial entropy h can be found from an associated transition matrix $\mathbb{T}$. The spatial entropy h equals $\log\rho(\mathbb{T})$, where $\rho(\mathbb{T})$ is the maximum eigenvalue of $\mathbb{T}$.

In this paper, we study the two-dimensional patterns generation problems with many symbols. We first recall the results of two symbols in [3], [4].

1.1. Transition Matrices and Spatial Entropy

In two-dimensional situation, higher transition matrices have been discovered in [27] and developed systematically in [3] by studying the pattern generation problem. For simplicity, two symbols on 2×2 lattice $\mathbb{Z}_{2\times 2}$ are considered.

A transition matrix in the horizontal (or vertical) direction

$$\mathbf{A}_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \tag{1.1}$$

which is linked to a set of admissible local patterns on $\mathbb{Z}_{2\times 2}$ is considered, where $a_{ij} \in \{0, 1\}$ for $1 \le i, j \le 4$. The associated vertical (or horizontal) transition matrix $\mathbf{B}_2$ is given by

$$\mathbf{B}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}, \tag{1.2}$$

$\mathbf{A}_2$ and $\mathbf{B}_2$ are connected to each other as follows.

$$\mathbf{A}_2 = \left[\begin{array}{cc|cc} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ \hline b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{array}\right] = \begin{bmatrix} A_{2;1} & A_{2;2} \\ A_{2;3} & A_{2;4} \end{bmatrix}, \tag{1.3}$$

and

$$\mathbf{B}_2=\left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{21} & a_{22} \\ a_{13} & a_{14} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{41} & a_{42} \\ a_{33} & a_{34} & a_{43} & a_{44} \end{array}\right]=\begin{bmatrix} B_{2;1} & B_{2;2} \\ B_{2;3} & B_{2;4} \end{bmatrix}. \tag{1.4}$$

Notably if $\mathbf{A}_2$ represents the horizontal (or vertical) transition matrix then $\mathbf{B}_2$ represents the vertical (or horizontal) transition matrix. Results that hold for $\mathbf{A}_2$ are also valid for $\mathbf{B}_2$. Therefore, for simplicity, only $\mathbf{A}_2$ is presented herein.

The recursive formulae for n-th order transition matrices $\mathbf{A}_n$ defined on $\mathbb{Z}_{2\times n}$ are obtained in [4] as follows

$$\mathbf{A}_{n+1}=\left[\begin{array}{cc|cc} b_{11}A_{n;1} & b_{12}A_{n;2} & b_{21}A_{n;1} & b_{22}A_{n;2} \\ b_{13}A_{n;3} & b_{14}A_{n;4} & b_{23}A_{n;3} & b_{24}A_{n;4} \\ \hline b_{31}A_{n;1} & b_{32}A_{n;2} & b_{41}A_{n;1} & b_{42}A_{n;2} \\ b_{33}A_{n;3} & b_{34}A_{n;4} & b_{43}A_{n;3} & b_{44}A_{n;4} \end{array}\right], \tag{1.5}$$

whenever

$$\mathbf{A}_n=\begin{bmatrix} A_{n;1} & A_{n;2} \\ A_{n;3} & A_{n;4} \end{bmatrix}, \tag{1.6}$$

for $n \geq 2$, or equivalently,

$$\mathbf{A}_{n+1;\alpha}=\begin{bmatrix} b_{\alpha 1}A_{n;1} & b_{\alpha 2}A_{n;2} \\ b_{\alpha 3}A_{n;3} & b_{\alpha 4}A_{n;4} \end{bmatrix}, \tag{1.7}$$

for $\alpha \in \{1,2,3,4\}$.

The number of all admissible patterns defined on $\mathbb{Z}_{m\times n}$ which can be generated from $\mathbf{A}_2$ is now defined by

$$\begin{aligned} \Gamma_{m,n}(\mathbf{A}_2) &= |\mathbf{A}_n^{m-1}| \\ &= \text{ the sum of all entries in } 2^n \times 2^n \text{ matrix } \mathbf{A}_n^{m-1}. \end{aligned} \tag{1.8}$$

The spatial entropy $h(\mathbf{A}_2)$ is defined as

$$\begin{aligned} h(\mathbf{A}_2) &= \lim_{m,n\to\infty} \frac{1}{mn} \log\Gamma_{\mathrm{m,n}}(\mathbf{A}_2) \\ &= \lim_{m,n\to\infty} \frac{1}{mn} \log|\mathbf{A}_n^{m-1}|. \end{aligned} \tag{1.9}$$

The existence of the limit (1.9) has been shown in [3], [12], [13]], [27]. When $h(\mathbf{A}_2) > 0$, the number of admissible patterns grows exponentially with the lattice size $m \times n$. In this situation, spatial chaos arises. When $h(\mathbf{A}_2) = 0$, patterns formation occurs.

To compute the double limit in (1.9), $n \geq 2$ can be fixed initially and m is allowed to tend to infinity as in [3] and [27]; then Perron-Frobenius theorem is applied;

$$\lim_{m\to\infty} \frac{1}{m} \log |\mathbf{A}_n^{m-1}| = log\rho(\mathbf{A}_n), \tag{1.10}$$

which implies

$$h(\mathbf{A}_2) = \lim_{n\to\infty} \frac{1}{n} log\rho(\mathbf{A}_n), \tag{1.11}$$

where $\rho(\mathbf{A}_n)$ is the maximum eigenvalue of matrix $\mathbf{A}_n$.

1.2. Computation of Maximum Eigenvalue and Spatial Entropy

$\mathbf{A}_n$ is a $2^n \times 2^n$ matrix, so computing $\rho(\mathbf{A}_n)$ is usually quite difficult when n is large. However, for a class of $\mathbf{A}_2$, the recursive formulae for $\rho(\mathbf{A}_n)$ can be computed explicitly, along with a limiting equation to ρ^* =exp$(h(\mathbf{A}_2))$, as in [3]. This class of $\mathbf{A}_2$ has the form of $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$, i.e.,

$$\mathbf{A}_2 = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \tag{1.12}$$

where $A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix}$, $B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix}$ and $a, a_2, a_3, b, b_2, b_3 \in \{0, 1\}$. The results in [3] are recalled as follows.

Lemma 1.1. *Let A and B be non-negative and non-zero $m \times m$ matrices, respectively, and α and β are positive numbers. The maximum eigenvalue of $\begin{bmatrix} A & \alpha B \\ \beta B & A \end{bmatrix}$ is then the maximum eigenvalue of*

$$A + \sqrt{\alpha\beta}B. \tag{1.13}$$

Theorem 1.2. *Assume that $\mathbf{A}_2 = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ and $A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix}$ and $B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix}$ where $a, b, a_2, a_3, b_2, b_3 \in \{0, 1\}$. For $n \geq 2$, let λ_n be the largest eigenvalue of*

$$|\mathbf{A}_n - \lambda| = 0.$$

Then
$$\lambda_n = \alpha_{n-1} + \beta_{n-1},$$
where α_k *and* β_k *satisfy the following recursive relations:*
$$\begin{aligned}\alpha_{k+1} &= a\alpha_k + b\beta_k,\\ \beta_{k+1} &= \sqrt{(a_2\alpha_k + b_2\beta_k)(a_3\alpha_k + b_3\beta_k)},\end{aligned}$$
for $k \geq 0$, *and* $\alpha_0 = \beta_0 = 1$.
Furthermore, the spatial entropy $h(\mathbf{A}_2)$ *is equal to* $log\xi_*$, *where* ξ_* *is the maximum root of the following polynomials* $Q(\xi)$*:*
(I) if $a_2 = a_3 = 1$,
$$Q(\xi) \equiv 4\xi^2(\xi - a)^2 + (\gamma^2 - 4\delta)(\xi - a)^2 - \gamma^2\xi^2 - 2\gamma(2b - a\gamma)\xi - (2b - a\gamma)^2,$$
where
$$\gamma = b_2 + b_3 \text{ and } \delta = b_2 b_3.$$
(II) if $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 1$,
$$Q(\xi) \equiv \xi^3 - a\xi^2 - \delta\xi + a\delta - b.$$
Moreover, if $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 0$, then $h(\mathbf{A}_2) = 0$.

The proofs of above two theorems are shown in [3].

2. Three-Symbols Problems

In this section,we focus our study on three-symbols problems. We try to generalize the result of Theorem 1.2 to the three-symbols cases.

2.1. Transition Matrices and Spatial Entropy

By the same reason as two symbols on lattice $\mathbb{Z}_{2\times 2}$, we take a transition matrix $\mathbf{A}_2$ of three symbols on lattice $\mathbb{Z}_{2\times 2}$ as

$$
\begin{aligned}
\mathbf{A}_2 &= \left[\begin{array}{ccc|ccc|ccc}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ \hline
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\ \hline
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{array}\right] \\
&= \left[\begin{array}{ccc|ccc|ccc}
b_{11} & b_{12} & b_{13} & b_{21} & b_{22} & b_{23} & b_{31} & b_{32} & b_{33} \\
b_{14} & b_{15} & b_{16} & b_{24} & b_{25} & b_{26} & b_{34} & b_{35} & b_{36} \\
b_{17} & b_{18} & b_{19} & b_{27} & b_{28} & b_{29} & b_{37} & b_{38} & b_{39} \\ \hline
b_{41} & b_{42} & b_{43} & b_{51} & b_{52} & b_{53} & b_{61} & b_{62} & b_{63} \\
b_{44} & b_{45} & b_{46} & b_{54} & b_{55} & b_{56} & b_{64} & b_{65} & b_{66} \\
b_{47} & b_{48} & b_{49} & b_{57} & b_{58} & b_{59} & b_{67} & b_{68} & b_{69} \\ \hline
b_{71} & b_{72} & b_{73} & b_{81} & b_{82} & b_{83} & b_{91} & b_{92} & b_{93} \\
b_{74} & b_{75} & b_{76} & b_{84} & b_{85} & b_{86} & b_{94} & b_{95} & b_{96} \\
b_{77} & b_{78} & b_{79} & b_{87} & b_{88} & b_{89} & b_{97} & b_{98} & b_{99}
\end{array}\right] \\
&= \begin{bmatrix}
A_{2;1} & A_{2;2} & A_{2;3} \\
A_{2;4} & A_{2;5} & A_{2;6} \\
A_{2;7} & A_{2;8} & A_{2;9}
\end{bmatrix},
\end{aligned} \tag{2.1}
$$

which is linked to a set of admissible patterns on $\mathbb{Z}_{2\times 2}$, where $a_{i,j}, b_{i,j} \in \{0,1\}$.

The recursive formulae for $n+1$-th order transition matrices $\mathbf{A}_{n+1}$ defined on $\mathbb{Z}_{2\times(n+1)}$ are

$$
\mathbf{A}_{n+1} = \left[\begin{array}{ccc|ccc|ccc}
b_{11}A_{n;1} & b_{12}A_{n;2} & b_{13}A_{n;3} & b_{21}A_{n;1} & b_{22}A_{n;2} & b_{23}A_{n;3} & b_{31}A_{n;1} & b_{32}A_{n;2} & b_{33}A_{n;3} \\
b_{14}A_{n;4} & b_{15}A_{n;5} & b_{16}A_{n;6} & b_{24}A_{n;4} & b_{25}A_{n;5} & b_{26}A_{n;6} & b_{34}A_{n;4} & b_{35}A_{n;5} & b_{36}A_{n;6} \\
b_{17}A_{n;7} & b_{18}A_{n;8} & b_{19}A_{n;9} & b_{27}A_{n;7} & b_{28}A_{n;8} & b_{29}A_{n;9} & b_{37}A_{n;7} & b_{38}A_{n;8} & b_{39}A_{n;9} \\ \hline
b_{41}A_{n;1} & b_{42}A_{n;2} & b_{43}A_{n;3} & b_{51}A_{n;1} & b_{52}A_{n;2} & b_{53}A_{n;3} & b_{61}A_{n;1} & b_{62}A_{n;2} & b_{63}A_{n;3} \\
b_{44}A_{n;4} & b_{45}A_{n;5} & b_{46}A_{n;6} & b_{54}A_{n;4} & b_{55}A_{n;5} & b_{56}A_{n;6} & b_{64}A_{n;4} & b_{65}A_{n;5} & b_{66}A_{n;6} \\
b_{47}A_{n;7} & b_{48}A_{n;8} & b_{49}A_{n;9} & b_{57}A_{n;7} & b_{58}A_{n;8} & b_{59}A_{n;9} & b_{67}A_{n;7} & b_{68}A_{n;8} & b_{69}A_{n;9} \\ \hline
b_{71}A_{n;1} & b_{72}A_{n;2} & b_{73}A_{n;3} & b_{81}A_{n;1} & b_{82}A_{n;2} & b_{83}A_{n;3} & b_{91}A_{n;1} & b_{92}A_{n;2} & b_{93}A_{n;3} \\
b_{74}A_{n;4} & b_{75}A_{n;5} & b_{76}A_{n;6} & b_{84}A_{n;4} & b_{85}A_{n;5} & b_{86}A_{n;6} & b_{94}A_{n;4} & b_{95}A_{n;5} & b_{96}A_{n;6} \\
b_{77}A_{n;7} & b_{78}A_{n;8} & b_{79}A_{n;9} & b_{87}A_{n;7} & b_{88}A_{n;8} & b_{89}A_{n;9} & b_{97}A_{n;7} & b_{98}A_{n;8} & b_{99}A_{n;9}
\end{array}\right], \tag{2.2}
$$

where

$$
\mathbf{A}_n = \begin{bmatrix}
A_{n;1} & A_{n;2} & A_{n;3} \\
A_{n;4} & A_{n;5} & A_{n;6} \\
A_{n;7} & A_{n;8} & A_{n;9}
\end{bmatrix}, \tag{2.3}
$$

for any $n \geq 2$.

The definition of spatial entropy $h(\mathbf{A}_2)$ of three symbols on lattice $\mathbb{Z}_{2\times 2}$ is the same as two symbols which can also be proved as

$$h(\mathbf{A}_2) = \lim_{n\to\infty} \frac{1}{n} \log \rho(\mathbf{A}_n), \tag{2.4}$$

see [3].

2.2. Computation of Maximum Eigenvalues and Entropy

For three symbols, $\mathbf{A}_n$ is a $3^n \times 3^n$ matrix, so computing the maximum eigenvalue of $\mathbf{A}_n$ ($\rho(\mathbf{A}_n)$) is harder than it is for two symbols. We begin with the study of $\mathbf{A}_2$ of the form

$$\mathbf{A}_2 = \begin{bmatrix} A & B & C \\ B & C & A \\ C & A & B \end{bmatrix}, \tag{2.5}$$

where

$$A = \begin{bmatrix} a & a_{12} & a_{13} \\ a_{21} & a & a_{23} \\ a_{31} & a_{32} & a \end{bmatrix}, B = \begin{bmatrix} b & b_{12} & b_{13} \\ b_{21} & b & b_{23} \\ b_{31} & b_{32} & b \end{bmatrix}, C = \begin{bmatrix} c & c_{12} & c_{13} \\ c_{21} & c & c_{23} \\ c_{31} & c_{32} & c \end{bmatrix}, \tag{2.6}$$

and $a, b, c, a_{ij}, b_{ij}, c_{ij} \in \{0, 1\}, i, j \in \{1, 2, 3\}, i \neq j$. Consider

$$\mathbf{A}_n = \begin{bmatrix} A_n & B_n & C_n \\ B_n & C_n & A_n \\ C_n & A_n & B_n \end{bmatrix} \tag{2.7}$$

Let λ_n be the eigenvalue of $\mathbf{A}_n$, and $\mathbf{U}_n$ be the corresponding eigenvector of λ_n,
i.e.,

$$\mathbf{A}_n \mathbf{U}_n = \lambda_n \mathbf{U}_n, \tag{2.8}$$

where $\mathbf{U}_n = \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix}$. Therefore,

$$\begin{bmatrix} A_n & B_n & C_n \\ B_n & C_n & A_n \\ C_n & A_n & B_n \end{bmatrix} \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix} = \lambda_n \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix}. \tag{2.9}$$

Assume

$$u_n = v_n = w_n, \tag{2.10}$$

then (2.9) implies

$$(A_n + B_n + C_n)u_n = \lambda_n u_n. \tag{2.11}$$

Conversely, under the assumption (2.10), (2.11) implies (2.9). Therefore, (2.8) and (2.11) are equivalent.

We first prove the following lemma which is a generalization of Lemma 1.1.

Lemma 2.1. *Let* $\mathbf{A}_2$ *be given as in* (2.5) *and* (2.6)*. If*

$$\begin{aligned} a+b+c \quad &= a_{12}+b_{12}+c_{12} = a_{13}+b_{13}+c_{13} \\ &= a_{21}+b_{21}+c_{21} = a_{23}+b_{23}+c_{23} \\ &= a_{31}+b_{31}+c_{31} = a_{32}+b_{32}+c_{32} \end{aligned} \tag{2.12}$$

holds. Then

$$A_n + B_n + C_n = (a+b+c)\begin{bmatrix} A_{n-1} & B_{n-1} & C_{n-1} \\ B_{n-1} & C_{n-1} & A_{n-1} \\ C_{n-1} & A_{n-1} & B_{n-1} \end{bmatrix}. \tag{2.13}$$

Proof. Let $\mathbf{A}_2 = \begin{bmatrix} A_2 & B_2 & C_2 \\ B_2 & C_2 & A_2 \\ C_2 & A_2 & B_2 \end{bmatrix}$, where A_2, B_2, and C_2 are given in (2.6).

By (2.2),

$$\mathbf{A}_n = \begin{bmatrix} A_n & B_n & C_n \\ B_n & C_n & A_n \\ C_n & A_n & B_n \end{bmatrix}$$

$$= \left[\begin{array}{ccc|ccc|ccc} aA_{n-1} & a_{12}B_{n-1} & a_{13}C_{n-1} & bA_{n-1} & b_{12}B_{n-1} & b_{13}C_{n-1} & cA_{n-1} & c_{12}B_{n-1} & c_{13}C_{n-1} \\ a_{21}B_{n-1} & aC_{n-1} & a_{23}A_{n-1} & b_{21}B_{n-1} & bC_{n-1} & b_{23}A_{n-1} & c_{21}B_{n-1} & cC_{n-1} & c_{23}A_{n-1} \\ a_{31}C_{n-1} & a_{32}A_{n-1} & aB_{n-1} & b_{31}C_{n-1} & b_{32}A_{n-1} & bB_{n-1} & c_{31}C_{n-1} & c_{32}A_{n-1} & cB_{n-1} \\ \hline bA_{n-1} & b_{12}B_{n-1} & b_{13}C_{n-1} & cA_{n-1} & c_{12}B_{n-1} & c_{13}C_{n-1} & aA_{n-1} & a_{12}B_{n-1} & a_{13}C_{n-1} \\ b_{21}B_{n-1} & bC_{n-1} & b_{23}A_{n-1} & c_{21}B_{n-1} & cC_{n-1} & c_{23}A_{n-1} & a_{21}B_{n-1} & aC_{n-1} & a_{23}A_{n-1} \\ b_{31}C_{n-1} & b_{32}A_{n-1} & bB_{n-1} & c_{31}C_{n-1} & c_{32}A_{n-1} & cB_{n-1} & a_{31}C_{n-1} & a_{32}A_{n-1} & aB_{n-1} \\ \hline cA_{n-1} & c_{12}B_{n-1} & c_{13}C_{n-1} & aA_{n-1} & a_{12}B_{n-1} & a_{13}C_{n-1} & bA_{n-1} & b_{12}B_{n-1} & b_{13}C_{n-1} \\ c_{21}B_{n-1} & cC_{n-1} & c_{23}A_{n-1} & a_{21}B_{n-1} & aC_{n-1} & a_{23}A_{n-1} & b_{21}B_{n-1} & bC_{n-1} & b_{23}A_{n-1} \\ c_{31}C_{n-1} & c_{32}A_{n-1} & cB_{n-1} & a_{31}C_{n-1} & a_{32}A_{n-1} & aB_{n-1} & b_{31}C_{n-1} & b_{32}A_{n-1} & bB_{n-1} \end{array}\right].$$

Now

$$A_n+B_n+C_n = \begin{bmatrix} (a+b+c)A_{n-1} & (a_{12}+b_{12}+c_{12})B_{n-1} & (a_{13}+b_{13}+c_{13})C_{n-1} \\ (a_{21}+b_{21}+c_{21})B_{n-1} & (a+b+c)C_{n-1} & (a_{23}+b_{23}+c_{23})A_{n-1} \\ (a_{31}+b_{31}+c_{31})C_{n-1} & (a_{32}+b_{32}+c_{32})A_{n-1} & (a+b+c)B_{n-1} \end{bmatrix}.$$

By the assumption (2.12), (2.13) follows. The proof is complete.

Now, we can prove our first theorem.

Theorem 2.2. *Assume* (2.12) *holds, and* $a+b+c \geq 1$*, then*

$$h(\mathbf{A}_2) = \log(a+b+c). \tag{2.14}$$

Proof. Under the assumption (2.12) and by Lemma 2.1,

$$A_n + B_n + C_n = (a+b+c)\begin{bmatrix} A_{n-1} & B_{n-1} & C_{n-1} \\ B_{n-1} & C_{n-1} & A_{n-1} \\ C_{n-1} & A_{n-1} & B_{n-1} \end{bmatrix}$$

which implies

$$\lambda_n = (a+b+c)\lambda_{n-1}, \tag{2.15}$$

for any $n \geq 3$. Now

$$A_2 + B_2 + C_2 = (a+b+c)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

which implies

$$\lambda_2 = 3(a+b+c). \tag{2.16}$$

Combining (2.15) with (2.16),

$$\lambda_n = 3(a+b+c)^{n-1}. \tag{2.17}$$

Hence

$$\begin{aligned} h(\mathbf{A}_2) &= \lim_{n\to\infty} \frac{1}{n} \log \lambda_n \\ &= \lim_{n\to\infty} \frac{n-1}{n} \log(a+b+c) \\ &= \log(a+b+c), \end{aligned}$$

(2.14) follows. The proof is complete.

Remark 2.3. (i) It is of interest to study the case when $\mathbf{A}_2$ is of the form (2.5) but (2.12) fails. A lemma like Lemma 1.1 need to be established, some progress has been made.
(ii) Result of Theorem 2.2 also holds for any number of symbols provides that the assumptions like (2.12) hold.

Lemma 2.4. *Let A and B be non-negative and non-zero $n \times n$ matrices, respectively, and $a_1, a_2, a_3,$ and a_4 are positive numbers. The maximum eigenvalue of* $\begin{bmatrix} a_1A & a_2B & a_2B \\ a_3B & a_4B & a_1A \\ a_3B & a_1A & a_4B \end{bmatrix}$ *is then the maximum eigenvalue of*

$$a_1A + \frac{a_4 + \sqrt{a_4^2 + 8a_2a_3}}{2}B. \tag{2.18}$$

Proof. Consider

$$\begin{vmatrix} a_1A-\lambda & a_2B & a_2B \\ a_3B & a_4B-\lambda & a_1A \\ a_3B & a_1A & a_4B-\lambda \end{vmatrix} = 0.$$

There are two cases:
Case I. If $|a_1A-\lambda| = 0$, it is clear that $\lambda = a_1A$.
Case II. For $|a_1A-\lambda| \neq 0$, the last equation is equivalent to

$$\begin{vmatrix} a_1A-\lambda & a_2B & a_2B \\ 0 & (a_4B-\lambda)-a_2a_3B(a_1A-\lambda)^{-1}B & a_1A-a_2a_3B(a_1A-\lambda)^{-1}B \\ 0 & 0 & P \end{vmatrix} = 0,$$

where

$$\begin{aligned} P =& [(a_4B-\lambda)-a_2a_3B(a_1A-\lambda)^{-1}B]-[a_1A-a_2a_3B(a_1A-\lambda)^{-1}B] \\ & [(a_4B-\lambda)-a_2a_3B(a_1A-\lambda)^{-1}B]^{-1}[a_1A-a_2a_3B(a_1A-\lambda)^{-1}B], \end{aligned}$$

and we could simplify it to

$$|I-\{[a_1A-a_2a_3B(a_1A-\lambda)^{-1}B][(a_4B-\lambda)-a_2a_3B(a_1A-\lambda)^{-1}B]^{-1}\}^2| = 0.$$

Then, we have

$$\begin{aligned} & |I+\{[a_1A-a_2a_3B(a_1A-\lambda)^{-1}B][(a_4B-\lambda)-a_2a_3B(a_1A-\lambda)^{-1}B]^{-1}\}| = 0 \\ \text{or } & |I-\{[a_1A-a_2a_3B(a_1A-\lambda)^{-1}B][(a_4B-\lambda)-a_2a_3B(a_1A-\lambda)^{-1}B]^{-1}\}| = 0. \end{aligned}$$

Since A and B are non-negative and a_1, a_2, a_3, and a_4 are positive, verifying that the maximum eigenvalue λ of $\begin{bmatrix} a_1A & a_2B & a_2B \\ a_3B & a_4B & a_1A \\ a_3B & a_1A & a_4B \end{bmatrix}$ and $a_1A + \frac{a_4+\sqrt{a_4^2+8a_2a_3}}{2}B$ are equal is relatively easy. The proof is complete.

Theorem 2.5. *Assume that* $\boldsymbol{A}_2 = \begin{bmatrix} A & B & B \\ B & B & A \\ B & A & B \end{bmatrix}$ *and* $A = \begin{bmatrix} a_1 & a_2 & a_2 \\ a_3 & a_4 & a_1 \\ a_3 & a_1 & a_4 \end{bmatrix}$ *and* $B = \begin{bmatrix} b_1 & b_2 & b_2 \\ b_3 & b_4 & b_1 \\ b_3 & b_1 & b_4 \end{bmatrix}$ *where* $a_i, b_i \in \{0,1\}$, *and* $i \in \{1,2,3,4\}$. *For* $n \geq 2$, *let* λ_n *be the largest eigenvalue of*

$$|\boldsymbol{A}_n - \lambda| = 0.$$

Then
$$\lambda_n = \alpha_{n-1} + \beta_{n-1}, \tag{2.19}$$
where α_k *and* β_k *satisfy the following recursive relations:*
$$\alpha_k = a_1\alpha_{k-1} + b_1\beta_{k-1}, \tag{2.20}$$
$$\beta_k = \frac{1}{2}\{(a_4\alpha_{k-1} + b_4\beta_{k-1}) + [(a_4\alpha_{k-1} + b_4\beta_{k-1})^2 + 8(a_2\alpha_{k-1} + b_2\beta_{k-1})(a_3\alpha_{k-1} + b_3\beta_{k-1})]^{\frac{1}{2}}\}, \tag{2.21}$$
for $k = 1, 2, \ldots, n-1$, *and*
$$\alpha_0 = 1, \beta_0 = 2. \tag{2.22}$$
Furthermore, the spatial entropy $h(\boldsymbol{A}_2)$ *is equal to* $log\xi_*$, *where* ξ_* *is the maximum root of the following polynomials* $Q(\xi)$*:*
(I) if $a_1 = b_1 = 1$,
$$Q_1(\xi) \equiv \xi^4 - (2+b_4)\xi^3 + (1 - a_4 + 2b_4 - 2b_2b_3)\xi^2 + [(a_4 - b_4) - 2b_2(a_3 - b_3) - 2b_3(a_2 - b_2)]\xi - 2(a_2 - b_2)(a_3 - b_3). \tag{2.23}$$
(II) if $a_1 = 0, b_1 = 1$,
$$Q_2(\xi) \equiv \xi^4 - b_4\xi^3 - (a_4 + 2b_2b_3)\xi^2 - 2(a_2b_3 + a_3b_2)\xi - 2a_2a_3. \tag{2.24}$$
(III) if $a_1 = 1, b_1 = 0$,
$$Q_3(\xi) \equiv \xi^2 - b_4\xi - 2b_2b_3. \tag{2.25}$$
Proof. Since the structure of $\mathbf{A}_2$ is special, it is easy to show that for any $k \geq 2$, we get
$$\mathbf{H}_k = \begin{bmatrix} A_k & B_k & B_k \\ B_k & B_k & A_k \\ B_k & A_k & B_k \end{bmatrix},$$
and
$$\mathbf{H}_{k+1} = \begin{bmatrix} A_{k+1} & B_{k+1} & B_{k+1} \\ B_{k+1} & B_{k+1} & A_{k+1} \\ B_{k+1} & A_{k+1} & B_{k+1} \end{bmatrix},$$
here
$$A_{k+1} = \mathbf{H}_k \odot A = \begin{bmatrix} a_1A_k & a_2B_k & a_2B_k \\ a_3B_k & a_4B_k & a_1A_k \\ a_3B_k & a_1A_k & a_4B_k \end{bmatrix}, \tag{2.26}$$

and

$$B_{k+1} = \mathbf{H}_k \odot B = \begin{bmatrix} b_1 A_k & b_2 B_k & b_2 B_k \\ b_3 B_k & b_4 B_k & b_1 A_k \\ b_3 B_k & b_1 A_k & b_4 B_k \end{bmatrix}, \tag{2.27}$$

$A_2 = A$ and $B_2 = B$. We know that $|\mathbf{A}_{n+1} - \lambda_{n+1}| = 0$, so

$$|A_{n+1} + 2B_{n+1} - \lambda_{n+1}| = 0. \tag{2.28}$$

Let

$$\alpha_0 = 1 \text{ and } \beta_0 = 2. \tag{2.29}$$

By induction on k, $1 \leq k \leq n$, and using (2.26),(2.27),(2.28) and Lemma 2.4, it is straightforward to derive

$$|\alpha_k A_{n-k+1} + \beta_k B_{n-k+1} - \lambda_{n+1}| = 0, \tag{2.30}$$

with α_k and β_k satisfy (2.20) and (2.21). In particular,

$$\alpha_n = a_1\alpha_{n-1} + b_1\beta_{n-1}, \tag{2.31}$$

$$\begin{aligned} \beta_n =& \frac{1}{2}\{(a_4\alpha_{n-1} + b_4\beta_{n-1}) + [(a_4\alpha_{n-1} + b_4\beta_{n-1})^2 + \\ & 8(a_2\alpha_{n-1} + b_2\beta_{n-1})(a_3\alpha_{n-1} + b_3\beta_{n-1})]^{\frac{1}{2}}\}, \end{aligned} \tag{2.32}$$

and

$$\lambda_{n+1} = \alpha_n + \beta_n.$$

This proves the first part of the theorem.
The remainder of the proof, demonstrates that $h(\mathbf{A}_2) = log\lambda_*$ where λ_* is the maximum root of $Q(\lambda)$. There are three cases:
Case I. From (2.31), if $a_1 = b_1 = 1$, we have

$$\beta_{n-1} = \alpha_n - \alpha_{n-1}. \tag{2.33}$$

Substituting (2.33) into (2.21), yields

$$\begin{aligned} \alpha_{n+1} - \alpha_n =& \frac{1}{2}\{[(a_4 - b_4)\alpha_{n-1} + b_4\alpha_n] + [((a_4 - b_4)\alpha_{n-1} + b_4\alpha_n)^2 + \\ & 8((a_2 - b_2)\alpha_{n-1} + b_2\alpha_n)((a_3 - b_3)\alpha_{n-1} + b_3\alpha_n)]^{\frac{1}{2}}\}. \end{aligned} \tag{2.34}$$

Now, let

$$\xi_n = \frac{\alpha_n}{\alpha_{n-1}} \tag{2.35}$$

and after dividing (2.34) by α_n, we have

$$\xi_{n+1} - 1 = \frac{1}{2}\{[(a_4 - b_4)\frac{1}{\xi_n} + b_4] + [((a_4 - b_4)\frac{1}{\xi_n} + b_4)^2 + 8((a_2 - b_2)\frac{1}{\xi_n} + b_2)((a_3 - b_3)\frac{1}{\xi_n} + b_3)]^{\frac{1}{2}}\}. \tag{2.36}$$

(2.36) can be written as the following iteration map:

$$\xi_{n+1} = G_1(\xi_n), \tag{2.37}$$

where

$$G_1(\xi) = 1 + \frac{1}{2}\{[(a_4 - b_4)\frac{1}{\xi} + b_4] + [((a_4 - b_4)\frac{1}{\xi} + b_4)^2 + 8((a_2 - b_2)\frac{1}{\xi} + b_2)((a_3 - b_3)\frac{1}{\xi} + b_3)]^{\frac{1}{2}}\}. \tag{2.38}$$

We first observe the fixed point ξ_* of $G_1(\xi)$, i.e., $\xi_* = G_1(\xi_*)$, is a root of $Q(\xi)$.
Indeed, by letting $\xi_n = \xi_{n+1} = \xi_*$ in (2.36), we have

$$\xi_* - 1 = \frac{1}{2}\{[(a_4 - b_4)\frac{1}{\xi_*} + b_4] + [((a_4 - b_4)\frac{1}{\xi_*} + b_4)^2 + 8((a_2 - b_2)\frac{1}{\xi_*} + b_2)((a_3 - b_3)\frac{1}{\xi_*} + b_3)]^{\frac{1}{2}}\},$$

which gives us $Q(\xi_*) = 0$. It can be proven that the maximum fixed point of $G_1(\xi)$ or the maximum root ξ_* of $Q(\xi) = 0$ satisfies $1 \le \xi_* \le 2$ and

$$\xi_n \to \xi_* \text{ as } n \to \infty. \tag{2.39}$$

Details are omitted here for brevity. By (2.19),(2.33) and (2.35), we can also prove

$$\frac{\lambda_{n+1}}{\lambda_n} \to \xi_* \text{ as } n \to \infty. \tag{2.40}$$

Hence, $h(\mathbf{T}_2) = \log \xi_*$.
Table 3.1 is to evaluate the value of λ_*, where $4a_2+2a_3+a_4+1 = i$, $1 \le i \le 8$, and $4b_2 + 2b_3 + b_4 + 1 = j$, $1 \le j \le 8$.

$i \diagdown j$	1	2	3	4	5	6	7	8
8	2	2.2938	2.2056	2.5214	2.2056	2.5214	2.6180	3
7	1.7900	2	2	2.2599	2	2.2599	2.4142	2.7693
6	1.6180	2	2	2.3593	1.6180	2	2.3028	2.7321
5	1	1	1.6956	2	1	1	2	2.4142
4	1.6180	2	1.6180	2	2	2.3593	2.3028	2.7321
3	1	1	1	1	1.6956	2	2	2.4142
2	1.6180	2	1.6180	2	1.6180	2	2	1
1	1	1	1	1	1	1	1.4142	2

Table 3.1

Case II. If $a_1 = 0$ and $b_1 = 1$, then, from (2.31), we have

$$\alpha_n = \beta_{n-1}. \tag{2.41}$$

Again, substituting (2.41) into (2.21) and letting $\xi_n = \frac{\beta_n}{\beta_{n-1}}$ lead to

$$\begin{aligned}\xi_n =& \frac{1}{2}\{(a_4\frac{1}{\xi_{n-1}} + b_4 + [(a_4\frac{1}{\xi_{n-1}} + b_4)^2 + \\ & 8(a_2\frac{1}{\xi_{n-1}} + b_2)(a_3\frac{1}{\xi_{n-1}} + b_3)]^{\frac{1}{2}}\},\end{aligned} \tag{2.42}$$

i.e., $\xi_n = G_2(\xi_{n-1})$, where

$$\begin{aligned}G_2(\xi) =& \frac{1}{2}\{(a_4\frac{1}{\xi} + b_4 + [(a_4\frac{1}{\xi} + b_4)^2 + \\ & 8(a_2\frac{1}{\xi} + b_2)(a_3\frac{1}{\xi} + b_3)]^{\frac{1}{2}}\}.\end{aligned} \tag{2.43}$$

The maximum fixed point ξ_* of (2.43) is the maximum root of $Q(\xi) = 0$ in (2.24). It can be also be proven that (2.39) and (2.40) holds in this case. Table 3.2 is to show the value of λ_*, where $4a_2 + 2a_3 + a_4 + 1 = i$, $1 \le i \le 8$, and $4b_2 + 2b_3 + b_4 + 1 = j$, $1 \le j \le 8$.

$i \diagdown j$	1	2	3	4	5	6	7	8
8	1.4142	1.8536	1.6956	2.1234	1.6956	2.1234	2.2695	2.7321
7	1.1892	1.5437	1.4945	1.8737	1.4945	1.8737	2.0907	2.5346
6	1	1.6180	1.5214	2	1	1.6180	2	2.5115
5	0	1	1.2599	1.6956	0	1	1.7693	2.2695
4	1	1.6180	1	1.6180	1.5214	2	2	2.5115
3	0	1	0	1	1.2599	1.6956	1.7693	2.2695
2	1	1.6180	1	1.6180	1	1.6180	1.7321	2.3028
1	0	1	0	1	0	1	1.4142	2

Table 3.2

Case III. If $a_1 = 1$ and $b_1 = 0$, then, from (2.31), we have

$$\alpha_n = \alpha_{n-1}. \tag{2.44}$$

Repeating the above steps, hence we get

$$\xi_n = \frac{b_4 + \sqrt{b_4^2 + 8a_2a_3}}{2}. \tag{2.45}$$

The maximum fixed point ξ_* of (2.45) is the maximum root of $Q(\xi) = 0$ in (2.25). The proof is complete.
Table 3.3 is to show the value of λ_*.

b_2	b_3	b_4	λ_*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	$\sqrt{2}$
1	1	1	2

Table 3.3

3. Spatial Entropy of Cyclic Cases

In this section, we study the spatial entropy of $\mathbf{A}_2$ when $\mathbf{A}_2$ has certain cyclic structure. Consider

$$\mathbf{A}_2 = \begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix} \tag{3.1}$$

We first study A,B,and C with the following form, $A, B, C \in \{E, I, J, J'\}$, where

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, J' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.2}$$

Now, we have the following theorem for cyclic $\mathbf{A}_2$.

Theorem 3.1. *Assume* $\mathbf{A}_2$ *is of the form* (3.1). *Then*

$$\lambda_* \equiv \lim_{n\to\infty} \lambda_n^{\frac{1}{n}}, \tag{3.3}$$

satisfies the limiting equation $Q(\lambda)$ *as follows:*

		A	B	C	limiting equation Q(λ)	λ_*
	(i)	E	I	J	$Q_2(\lambda)$	λ_2^*
	(ii)	E	I	J'	$Q_2(\lambda)$	λ_2^*
(I)	(iii)	E	J	I	$Q_2(\lambda)$	λ_2^*
	(iv)	E	J	J'	$Q_1(\lambda)$	λ_1^*
	(v)	E	J'	I	$Q_3(\lambda)$	λ_3^*
	(vi)	E	J'	J	$Q_1(\lambda)$	λ_1^*

		A	B	C	limiting equation Q(λ)	λ_*
	(i)	I	E	J	$Q_2(\lambda)$	λ_2^*
(II)	(ii)	I	E	J'	$Q_1(\lambda)$	λ_1^*
	(iii)	I	J	E	$Q_1(\lambda)$	λ_1^*
	(iv)	I	J'	E	$Q_2(\lambda)$	λ_2^*

		A	B	C	limiting equation Q(λ)	λ_*
	(i)	J	E	I	$Q_3(\lambda)$	λ_3^*
(III)	(ii)	J	E	J'	$Q_2(\lambda)$	λ_2^*
	(iii)	J	I	E	$Q_1(\lambda)$	λ_1^*
	(iv)	J	J'	E	$Q_3(\lambda)$	λ_3^*

		A	B	C	limiting equation Q(λ)	λ_*
	(i)	J'	E	I	$Q_1(\lambda)$	λ_1^*
(IV)	(ii)	J'	E	J	$Q_3(\lambda)$	λ_3^*
	(iii)	J'	I	E	$Q_3(\lambda)$	λ_3^*
	(iv)	J'	J	E	$Q_2(\lambda)$	λ_2^*

where,

$$Q_1(\lambda) = \lambda - 1, \lambda_1^* = 1$$
$$Q_2(\lambda) = \lambda^2 - \lambda - 1, \lambda_2^* = \tfrac{1+\sqrt{5}}{2} \cong 1.618033988$$
$$Q_3(\lambda) = \lambda^3 - \lambda^2 - \lambda - 1, \lambda_3^* \cong 1.839286755.$$

Proof of Case(I)(i). Since the structure of $\mathbf{A}_2$ is similar to (2.5), it is easy to verify that (2.11) is also right to (3.1) and for any $k \geq 2$, we have

$$\mathbf{A}_k = \begin{bmatrix} E_k & I_k & J_k \\ J_k & E_k & I_k \\ I_k & J_k & E_k \end{bmatrix}.$$

By (2.11), $|\mathbf{A}_n - \lambda_n| = 0$, so

$$|E_n + I_n + J_n - \lambda_n| = 0. \tag{3.4}$$

Let

$$\alpha_0 = 1 \text{ and } \beta_0 = 1. \tag{3.5}$$

By induction on k, $1 \leq k \leq n$, and using (2.11), it is straightforward to derive

$$|\alpha_k E_{n-k} + \beta_k I_{n-k} + J_{n-k} - \lambda_n| = 0, \tag{3.6}$$

where α_k and β_k satisfy the following recursive relations:

$$\alpha_k = \alpha_{k-1} + \beta_{k-1}, \tag{3.7}$$

$$\beta_k = \alpha_{k-1} + 1, \tag{3.8}$$

and

$$\lambda_n = 3\alpha_{n-2} + \beta_{n-2} + 1. \tag{3.9}$$

By recursive formulae of (3.7) and (3.8), we get

$$\alpha_n = \frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^{n+1} + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^{n+1} + \frac{-5-3\sqrt{5}}{10}(1-\frac{1+\sqrt{5}}{2})^{n-1} + \frac{-5+3\sqrt{5}}{10}(1-\frac{1-\sqrt{5}}{2})^{n-1}, \tag{3.10}$$

$$\beta_n = \frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^{n+1} + \frac{5-\sqrt{5}}{10}(\frac{1-\sqrt{5}}{2})^{n+1} + \frac{-5-\sqrt{5}}{10}(1-\frac{1+\sqrt{5}}{2})^{n-1} + \frac{-5+\sqrt{5}}{10}(1-\frac{1-\sqrt{5}}{2})^{n-1} + 1. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we have

$$\lambda_n = \frac{11+5\sqrt{5}}{2}(\frac{1+\sqrt{5}}{2})^{n-3} + \frac{11-5\sqrt{5}}{2}(\frac{1-\sqrt{5}}{2})^{n-3} - 2. \tag{3.12}$$

By (1.11), it is obvious that $h(\mathbf{A}_2) = \lim_{n\to\infty} \frac{1}{n} \log \lambda_n = \log \frac{1+\sqrt{5}}{2}$. Thus it is clear that $Q(\lambda) = \lambda^2 - \lambda - 1$.
The proofs of the following cases is similar to **Case**(I)(i). The proof is complete.

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