國 立 交 通 大 學

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碩 士 論 文

KdV 方程式的 Darboux 變換

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摘　　要

這篇論文我們首先探討比較基本的 KdV 方程式的 Darboux 變換。接下來討論在 AKNS 系統上的 Darboux 變換，而這 AKNS 系統的好處在於降低計算的複雜度使我們計算 KdV 方程式較為便利。最後我們將給一個例子使用 AKNS 系統上的 Darboux 變換計算出一些 KdV 方程式的孤立子的解。我們也寫下當 Schördinger equation 的 potential energy 為孤立子解的 eigenvalue 和 eigenfunction 的表示式。

Darboux transformation of the KdV equation

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Abstract

In this thesis, we describe the classical form of Darboux transformation of KdV equation first. Next we discuss the Darboux transformation for AKNS system which is convenient for us when we calculate KdV equation. It can reduce the complexity successively. Finally, we give an example to figure out one and two soliton solutions of KdV equation by using Darboux transformation for AKNS system. We can also derive the eigenfunction and eigenvalue of Schrödinger equation.

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Contents

1 Introduction

1.1 soliton theory

It is customary to start something on soliton theory with John Scott Russell's description of "the great wave of translation". The story of the soliton thus starts in 1834 on the Canal near Edinburgh. Russell reported his discovery to the British Association in 1844.

In 1895 Korteweg and de Vrise derived their equation describing the propagation of waves variety of scale transformations which gave the Korteweg-de Vrise (KdV) equation can be written in simplified form:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

It is easy to find a traveling wave solution of the form u(x,t) with any soliton book:

$$u(x,t) = -2\kappa^2 sech^2[\kappa(x - x_0) - 4\kappa^3 t] \tag{2}$$

The obtained solution of the KdV equation describes the propagation of a solitary wave moving with constant velocity $4\kappa^2$, its amplitude and velocity are proportional to each other.

1.2 Darboux transformation and KdV

In 1882, G.Darboux have studied a second order linear differential equation (one dimension Schrödinger equation) of eigenvalue problem.

$$-\phi_{xx} + u(x)\phi = \kappa\phi \tag{3}$$

Where u(x) is the function given definitely and κ is a constant. Darboux have found the following facts : suppose u(x) and $\phi(x,\kappa)$ are satisfy (3), for any given constant λ_0, let $f(x) = \phi(x,\kappa_0)$, then f is a solution of (3) with $\kappa = \kappa_0$, and by

$$u^{(1)} = u - 2(lnf)_{xx} \tag{4}$$

$$\phi^{(1)}(x,\kappa) = \phi_x(x,\kappa) - \frac{f_x}{f}\phi(x,\kappa) \tag{5}$$

The function $u^{(1)}$ and $\phi^{(1)}$ must satisfying the same form of(3)

$$-\phi^{(1)}_{xx} + u^{(1)}(x)\phi^{(1)} = \kappa\phi^{(1)} \tag{6}$$

This means that the transformation (4) and (5) must be satisfy (3) by f(x) $= \phi(x,\kappa_0)$, where (u,ϕ) is transformed into ($u^{(1)}$,$\phi^{(1)}$). It is very important to notice that (u,ϕ) and ($u^{(1)}$,$\phi^{(1)}$) satisfy the same form of (3). This is the classical Darboux transformation.

$$(u,\phi) \rightarrow (u^{(1)},\phi^{(1)}) \tag{7}$$

where, $f \neq 0$ are invalid here.

The fundamental important of the KdV equation, in the contemporary view, is defined by the remarkable discovery of Gardner,Green,Kruskal and Miura [2]. In their work the ingenious idea to relate the solution u(x,t) of the KdV equation to the evolution the spectral data of the linear Schrödinger operator,

$$-\partial_x^2 + u(x,t)$$

was first introduced .

In 1968 Lax [1] explained in a very transparent way the greater part of the result of [2] by introducing the following operator :

$$L = -\partial_x^2 + u(x,t) \qquad A = -4\partial_x^3 + 6u\partial_x + 3u_x \tag{8}$$

He noted that the commutator of L and A gives exactly the RHS of (1), i.e.,

$$[A,L] = 6uu_x - u_{xxx} \tag{9}$$

where $[A,L] = AL - LA$ is a commutator, and the operators L and A are called a Lax pair (which proof in **Appendix A**). Hence, the KdV equation may be represented in the following form:

$$\partial_t L = [A,L] \tag{10}$$

usually called the Lax equation. Equation (10) is equivalent to the consistency condition for the following system of partial differential equations:

$$\begin{cases} \text{Ł}\phi = \lambda\phi \\ \phi_t = A\phi \end{cases} \longleftrightarrow \begin{cases} -\phi_{xx} + u\phi = \lambda\phi \\ \phi_t = -4\phi_{xxx} + 6u\phi_x + 3u_x\phi \end{cases} \tag{11}$$

The equivalence follows form the fact the system (11) means that

$$L_t\phi + L\phi_t = L_t\phi + LA\phi = \lambda_t\phi + \lambda\phi_t = \lambda A\phi = AL\phi \tag{12}$$

where λ is independent of t and which yield

$$L_t\phi = [A, L]\phi$$

Therefore $L_t = [A, L]$ if and only if $L_t = 6uu_x - u_{xxx}$, thus the KdV equation is of the form (1) and its scattering problem is of the form $L\phi = \lambda\phi$.

The first equation of (11) is covariant with respect to the action of Darboux transformation (4), (5) and it is easy to verify that the same is true for the second equation of (11). The covariance of the system (11) allows one to create new solutions of KdV equation, starting form some know solution u(x,t) for which we are able to solve (11) explicitly. When $\kappa = \kappa_0$, let f(x) $= \phi(x, \kappa_0)$ then $\phi^{(1)} = \phi_x(x, \kappa) - \frac{f_x}{f}\phi(x, \kappa)$, where $\phi(x, \kappa)$ and $\phi(x, \kappa_0)$ are some pair of solutions of (11), satisfies the system of differential equations of the same structure with u replaced by $u^{(1)}(x, \kappa)$, where $u^{(1)}(x, \kappa)$ is one soliton solution of KdV equation. The consistency of the system satisfied by $\phi^{(1)}$ shows that $u^{(1)}(x, \kappa)$ is a new solution of the KdV equation. Looking for the N-times repeated Darboux transformation we shall obtain an infinite family of the solutions of the KdV equation. For a given N these solutions are of the form first discovered by Wahlquist [3].

$$u^{(N)}(x, \kappa) = u - 2\partial_x^2 lnW(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \cdots \phi^{(N)}) \tag{13}$$

where $\phi^{(1)}, \cdots, \phi^{(N)}$, are fixed linearly independent solutions of (11) and the Wronskian determinant W of N functions $\phi^{(N)}$, $\phi^{(N)}$, $\cdots$, $\phi^{(N)}$ is defined by

$$W(\phi^{(1)}, \phi^{(2)}, \cdots, \phi^{(N)}) = det(A), \quad A_{ij} = \frac{d^{i-1}\phi_j}{dx^{i-1}} \quad i, j = 1, 2, \ldots, N$$

In particular, for example : taking u = 0 for the simplest starting solution of the KdV equation and choosing $\phi^{(1)}(x, \lambda)$ in the form

$$f(x) = \phi^{(1)}(x, \lambda_0) = \coth[\kappa(x - x_0) - 4\kappa^3]\,, \quad \lambda_0 = -\kappa^2 \tag{14}$$

we get the

$$u^{(1)} = -2\kappa^2 sech^2[\kappa(x - x_0) - 4\kappa^3 t] \tag{15}$$

Next, we want to introduce AKNS system and discuss Darboux transformation of AKNS system. Finally, we will give some argument whose result will tell us how to construct Darboux matrix.

2 AKNS and Darboux transformation

2.1 2×2 AKNS system

In order to popularize Lax pair of the MKdv equation to the general situation, V. E. Zakharov, A. B. Shabat [4] and M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. segur [5] study a kind of more general Lax pair, now we call AKNS system. In this paper, we only consider 2×2 AKNS system.

Consider Lax pair of AKNS form

$$\begin{aligned} \Phi_x &= U\Phi = \lambda J\Phi + P\Phi \\ \Phi_t &= V\Phi = \textstyle\sum_{j=0}^{m} V_j \lambda^{m-j}\Phi \end{aligned} \tag{16}$$

where

$$U = \begin{bmatrix} -i\lambda & q \\ r & i\lambda \end{bmatrix}, \quad V = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}, \quad J = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad P = \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix} \tag{17}$$

where V_j are 2×2 matrix, $q = q(x,t)$, $r = r(x,t)$ and A, B, C are function of x, t and λ with $\lambda_t = 0$. The integrability condition of (16) is $\Phi_{xt} = \Phi_{tx}$, which is

$$U_t - V_x + [U, V] = 0. \tag{18}$$

It is deduced from integrability condition that we have

$$\begin{aligned} A_x &= qC - rB \\ B_x &= q_t - 2i\lambda B - 2qA, \\ C_x &= r_t + 2i\lambda C + 2rA \end{aligned} \tag{19}$$

and

$$\begin{aligned} q_t &= B_x + 2i\lambda B + 2Aq \\ r_t &= C_x - 2i\lambda C - 2rA \end{aligned} \tag{20}$$

We can choose A, B, and C as a polynomial of λ

$$A = \sum_{j=0}^{n} a_j \lambda^{n-j}, \quad B = \sum_{j=0}^{n} b_j \lambda^{n-j}, \quad C = \sum_{j=0}^{n} c_j \lambda^{n-j} \tag{21}$$

Take (21) into (20) abd (19), then equating the coefficients of the powers of λ, we find

$$\begin{aligned}
&b_0 = c_0 = 0, \quad a_{0,x} = 0\\
&a_{j,x} = qc_j - rb_j\\
&b_{j,x} + 2ib_{j+1} + 2qa_j = 0\\
&c_{j,x} - 2ic_{j+1} - 2ra_j = 0\\
&q_t = b_{n,x} + 2qa_n\\
&r_t = c_{n,x} - 2ra_n
\end{aligned}$$

The following things will be explained in **Appendix B**. For $n = 3$, we have

$$\begin{aligned}
A &= \alpha_0\lambda^3 + \alpha_1\lambda^2 + (\frac{1}{2}\alpha_0 qr + \alpha_2)\lambda + \frac{1}{2}\alpha_1 qr - \frac{i}{4}\alpha_0(qr_x - q_x r) + \alpha_3\\
B &= i\alpha_0 q\lambda^2 + (i\alpha_1 q - \frac{1}{2}\alpha_0 q_x)\lambda + i\alpha_2 q + \frac{i}{2}\alpha_0 q^2 r - \frac{1}{2}\alpha_1 q_x - \frac{i}{4}\alpha_0 q_{xx}\\
C &= i\alpha_0 r\lambda^2 + (i\alpha_1 r + \frac{1}{2}\alpha_0 r_x)\lambda + i\alpha_2 r + \frac{i}{2}\alpha_0 r^2 q + \frac{1}{2}\alpha_1 r_x - \frac{i}{4}\alpha_0 r_{xx}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
q_t &= -\frac{i}{4}\alpha_0(q_{xxx} - 6rqq_x) - \frac{1}{2}\alpha_1(q_{xx} - 2q^2 r) + i\alpha_2 q_x + 2\alpha_3 q\\
r_t &= -\tfrac{i}{4}\alpha_0(r_{xxx} - 6qrr_x) + \frac{1}{2}\alpha_1(r_{xx} - 2r^2 q) + i\alpha_2 r_x - 2\alpha_3 r
\end{aligned} \tag{23}$$

Using (23) to obtained some special cases.

(1)KdV equation : Take $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_0 = -4i$, $q = -u$ and $r = -1$, we have

$$u_t - 6uu_x + u_{xxx} = 0$$

(2)MKdV equation : Take $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_0 = -4i$, $q = -u$ and $r = \pm u$, we have

$$q_t \pm 6q^2 q_x + q_{xxx} = 0$$

2.2 Darboux transformation of AKNS system

Now, we cconsider Lax pair of AKNS form of (16).

Definition 2.1 *Let D(x,t,λ) is 2×2 matrix, if P and Φ were given, where Φ is any solution of (16), that $\Phi^{(1)} = D\Phi$ satisfy the same linear equation form of (16).*

$$\begin{cases} \Phi_x^{(1)} = U^{(1)}\Phi^{(1)} = \lambda J\Phi^{(1)} + P^{(1)}\Phi^{(1)} \\ \\ \Phi_t^{(1)} = V^{(1)}\Phi^{(1)} = \sum_{j=0}^{m} V_j^{(1)}\lambda^{m-j}\Phi^{(1)} \end{cases} \tag{24}$$

where $P^{(1)}$ is 2×2 matrix with the diagonal element are zero, then we call the transformation (P, Φ) $\longrightarrow$ ($P^{(1)}$, $\Phi^{(1)}$) is Darboux transformation of AKNS system, and D(x, t, λ) is Darboux matrix

Note, the elements of $P^{(1)}$ should satisfy (20) by Definition 2.1. Next, we take $\Phi^{(1)} = D\Phi$ into the (24) then we can get the representation of $U^{(1)}$, $V^{(1)}$

$$\begin{aligned} U^{(1)} &= DUD^{-1} + D_x D^{-1} \\ V^{(1)} &= DVD^{-1} + D_t D^{-1} \end{aligned} \tag{25}$$

where $U^{(1)}$, $V^{(1)}$ must be satisfy the form

$$U_t^{(1)} - V_x^{(1)} + [U^{(1)}, V^{(1)}] = 0$$

this is integrability conditions of (24). By Definition 2.1, we can see the Darboux matrix that make the transformation $(U, V, \Phi) \Longrightarrow (U^{(1)}, V^{(1)}, \Phi^{(1)})$

Proposition 2.1 *If D is Darboux matrix of (16), $D^{(1)}$ is Darboux matrix of (24), then $D^{(1)}D$ is also Darboux matrix of (16).*

Now at first discuss about the Darboux matrix with λ is order one, there is no harm in supposing it have the $\lambda I - S$ form, so $D = \lambda I - S$, here S is a 2×2 matrix, and I is identity matrix.

Next, we want to construct the Darboux matrix, but it is important that we should know how to construct matrix S.

Lead out the differential equations satisfied of matrix S now, by the first equation of (24) and $\Phi^{(1)} = D\Phi$.

$$(\lambda J + P^{(1)})\Phi^{(1)} = (\lambda J + P^{(1)})(\lambda I - S)\Phi \tag{26}$$

$$RHS : (\lambda^2 IJ - \lambda JS + \lambda P^{(1)} I - P^{(1)} S)\Phi$$

$$LHS : \Phi_x^{(1)} = ((\lambda I - S)\Phi)_x = (\lambda I \Phi_x - S_x\Phi - S\Phi_x) = (\lambda I - S)(\lambda J + P)\Phi - S_x\Phi \tag{27}$$

(27) corresponding to the arbitrary solutions of (16) are established, so consider the coefficient of λ

$$P^{(1)} = P + [J, S] \tag{28}$$

this is the representation of P'.

Consider the constant term

$$S_x = P^{(1)}S - SP = PS - SP + JS^2 - SJS$$

then

$$S_x + [S, JS + P] = 0 \tag{29}$$

This is the first differential equation satisfied of S.

By the second equation of (24), we have

$$\begin{aligned}\Phi_t^{(1)} = \sum_{j=0}^{m} V_j^{(1)}\lambda^{m-j}\Phi^{(1)} &= \sum_{j=0}^{m} V_j^{(1)}\lambda^{m-j}(\lambda I - S)\Phi \\ &= ((\lambda I - S)\Phi)_t \\ &= \lambda I\Phi_t - S_t\Phi - S\Phi_t \\ &= (\lambda I - S)\sum_{j=0}^{m} V_j\lambda^{m-j}\Phi - S_t\Phi.\end{aligned}$$

consider the coefficients of λ^{m+1}, λ^m, ..., λ, then we can get the following:

$$\begin{aligned} V_0^{(1)} &= V_0 \\ V_{j+1}^{(1)} &= V_{j+1} + V_j^{(1)} S - S V_j \end{aligned} \tag{30}$$

by (30), we can get

$$V_j^{(1)} = V_j + \sum_{k=1}^{j} [V_{j-k}, S] S^{k-1}, \quad (1 \le j \le m) \tag{31}$$

At the same time, we can also get the second differential equation satisfied of S.

$$S_t = V_m^{(1)} - S V_m \tag{32}$$

Substitute (31) into (32),

$$\begin{aligned} S_t &= (V_m + \sum_{j=1}^{m} [V_{m-j}, S] S^{j-1}) S - S V_m \\ &= V_m S + \sum_{j=1}^{m} [V_{m-j}, S] S^j - S V_m \\ &= [V_m, S] + \sum_{j=1}^{m} [V_{m-j}, S] S^j \end{aligned}$$

then

$$S_t + [S, \sum_{j=0}^{m} V_j S^{m-j}] = 0 \tag{33}$$

Lemma 2.1 *λI-S is Darboux matrix of (16), when matrix S satisfy*

$$S_x + [S, JS + P] = 0 \tag{34}$$

$$S_t + [S, \sum_{j=0}^{m} V_j S^{m-j}] = 0 \tag{35}$$

and under the Darboux transformation constructed by $\lambda I - S$, $P^{(1)} = P + [J, S]$.

This shows that in order to construct Darboux matrix, we need to solve the S of the nonlinear partial differential equations (34), (35).

Next, following the lemma to give a method who to construct the once Darboux matrix. If the elements of P are satisfy (20), we take the different complex number λ_1, λ_2. Let $\Lambda = diag(\lambda_1, \lambda_2)$. Suppose h_i is the column vector of the solution of (16) with $\lambda = \lambda_i$. Define $H = (h_1, h_2)$. When $det(H) \neq 0$

$$S = H\Lambda H^{-1}, \quad \lambda I - S = \lambda I - H\Lambda H^{-1} \tag{36}$$

Lemma 2.2 *By (36), λI-S is Darboux matrix of (16)*

Next chapter will to work that how to construction Darboux matrix and KdV equation solution.

3 Darboux transformation of the KdV and AKNS hierarchy

We want to talk about and execute Darboux transformation of the KdV equation. Consider Lax pair of AKNS form of (16), where we take the (17) and choose $q = -u$ and $r = -1$.

$$U = \begin{bmatrix} -i\lambda & -u \\ -1 & i\lambda \end{bmatrix} \ , \ V = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \tag{37}$$

where A, B and C choose from (22) and take $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_0 = -4i$.

$$\begin{aligned} A &= -u_x + 2i\lambda u - 4i\lambda^3 \\ B &= -u_{xx} - 2u^2 + 2i\lambda u_x + 4\lambda^2 u \\ C &= 2u - 4\lambda^2 \end{aligned} \tag{38}$$

First, we construct Darboux matrix by (36). Let $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ be the solution of (16) with $\lambda = \lambda_0$. We can prove $\begin{pmatrix} \alpha - 2i\lambda_0\beta \\ \beta \end{pmatrix}$ is solution of (16) with $\lambda = -\lambda_0$. (prove is in **Appendix C**)

Let

$$\Lambda = \begin{bmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{bmatrix} \ , \ H = \begin{bmatrix} \alpha & \alpha - 2i\lambda_0\beta \\ \beta & \beta \end{bmatrix} \tag{39}$$

then

$$S = H\Lambda H^{-1} = \begin{bmatrix} -\lambda_0 - \frac{i}{\tau} & \frac{i}{\tau^2} + 2\lambda_0\frac{1}{\tau} \\ -i & \lambda_0 + \frac{i}{\tau} \end{bmatrix} \tag{40}$$

where $\tau = \dfrac{\beta(x,t,\lambda_0)}{\alpha(x,t,\lambda_0)}$. Then Darboux matrix is

$$D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} (\lambda I - S) = \begin{bmatrix} i\lambda + i\lambda_0 - \frac{1}{\tau} & \frac{1}{\tau^2} - 2\lambda_0 \frac{i}{\tau} \\ 1 & -i\lambda + i\lambda_0 - \frac{1}{\tau} \end{bmatrix} \tag{41}$$

First, it is easy to show

$$D^{-1} = \frac{1}{\lambda^2 - \lambda_0^2} \begin{bmatrix} -i\lambda + i\lambda_0 - \frac{1}{\tau} & -\frac{1}{\tau^2} + 2\lambda_0 \frac{i}{\tau} \\ \\ -1 & i\lambda + i\lambda_0 - \frac{1}{\tau} \end{bmatrix}, \text{ and}$$

$$D_x = \begin{bmatrix} u - \frac{1}{\tau^2} + 2\lambda_0 \frac{i}{\tau} & -2u\frac{1}{\tau} + 2\frac{1}{\tau^3} - 4\lambda_0 \frac{i}{\tau^2} + 2i\lambda_0 u - 2\lambda_0 \frac{i}{\tau^2} - 4\lambda_0^2 \frac{1}{\tau} \\ \\ 0 & u - \frac{1}{\tau^2} + 2\lambda_0 \frac{i}{\tau} \end{bmatrix}$$

so that

$$U^{(1)} = DUD^{-1} + D_x D^{-1} = \begin{bmatrix} -i\lambda & u - \frac{2}{\tau^2} + 4\lambda_0 \frac{i}{\tau} \\ -1 & i\lambda \end{bmatrix} = \begin{bmatrix} -i\lambda & -u^{(1)} \\ -1 & i\lambda \end{bmatrix} \tag{42}$$

$$V^{(1)} = DVD^{-1} + D_x D^{-1} = \begin{bmatrix} A[u^{(1)}] & B[u^{(1)}] \\ C[u^{(1)}] & -A[u^{(1)}] \end{bmatrix} \tag{43}$$

then we can get that

$$u^{(1)} = -u + 2\left(\frac{1}{\tau^2} - 2\lambda_0 \frac{i}{\tau}\right) \tag{44}$$

Where we can see property of D by (42) , D can make U and U' have the same lower left element -1 and diagonal elements except for upper right $-u$ change to $-u^{(1)}$. So D is the Darboux matrix.

$u^{(1)}$ can also be expressed as another form. We know $\alpha(x,t,\lambda_0) = -\beta_x(x,t,\lambda_0) + i\lambda_0\beta(x,t,\lambda_0)$.

Thus,

$$
\begin{aligned}
u^{(1)} &= -u + 2(\frac{1}{\tau^2} - 2\lambda_0\frac{i}{\tau}) \\
&= -u + 2((\frac{1}{\tau})_x + u) \\
&= u + 2(\frac{1}{\tau})_x \\
&= u + 2(\frac{\alpha(x,t,\lambda_0)}{\beta(x,t,\lambda_0)})_x \\
&= u + 2(\frac{i\lambda_0\beta(x,t,\lambda_0) - \beta_x(x,t,\lambda_0)}{\beta(x,t,\lambda_0)})_x \\
&= u - 2(\frac{\beta_x(x,t,\lambda_0)}{\beta(x,t,\lambda_0)})_x \\
&= u - 2(ln\beta(x,t,\lambda_0))_{xx}
\end{aligned}
$$

This is $u \to u^{(1)} = u - 2(\ln\beta(x,t,\lambda_0))_{xx}$. Similarly, we can use above method to do multisoliton solution.

We know $\begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix} = D(\lambda,\lambda_0)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ by way of the Definition 2.1.

We want to claim that the eigenfunction $\Phi^{(1)} = \begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix}$ which is also satisfy (16) with u replaced by $u^{(1)}$. So that

$$
\begin{aligned}
\begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix}_x &= \left(D(\lambda,\lambda_0) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right)_x \\
&= D(\lambda,\lambda_0)_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + D(\lambda,\lambda_0) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_x \\
&= D(\lambda,\lambda_0)_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + D(\lambda,\lambda_0) U \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \left[D(\lambda,\lambda_0)_x D^{-1}(\lambda,\lambda_0) + D(\lambda,\lambda_0) U D^{-1}(\lambda,\lambda_0) \right] D(\lambda,\lambda_0) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \left[D(\lambda,\lambda_0)_x D^{-1}(\lambda,\lambda_0) + D(\lambda,\lambda_0) U D^{-1}(\lambda,\lambda_0) \right] \begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix} \\
&= U^{(1)} \begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix}
\end{aligned}
$$

The thing deserve to be mentioned, the $\Phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $\beta(x,t,\lambda)$ satisfy Schrödinger equation which is

$$-\beta_{xx} + u\beta = \lambda^2 \beta \tag{45}$$

because

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_x = \begin{bmatrix} -i\lambda & -u \\ -1 & i\lambda \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{46}$$

$$\alpha_x = -i\lambda\alpha - u\beta \tag{47}$$

$$\beta_x = -\alpha + i\lambda\beta \tag{48}$$

Derivatives of (48) is $\beta_{xx} = -\alpha_x + i\lambda\beta_x$, then substitute (47) into $\beta_{xx} = -\alpha_x + i\lambda\beta_x$

. So, we can get (45).

Moreover, if we can choose some λ which make $\beta(x,t,\lambda_0) \to 0$ with $x \to \infty$. Then λ^2 is called eigenvalue of Schrödinger equation with corresponding eigenfunction $\beta(x,t,\lambda)$ and u is the potential energy.

Next, we will use the above argument to get example. In such examples, we will get two soliton solutions and find the eigenfunction and the eigenvalue of Schrödinger equation.

4 The solution of KdV equation and some result

Now, consider Lax pair of AKNS form

$$\Phi_x = \begin{bmatrix} -i\lambda & -u \\ -1 & i\lambda \end{bmatrix} \Phi$$
$$\Phi_t = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \Phi \tag{49}$$

where A, B, C come from (38). We take $u = 0$ because $u = 0$ is a trivial solution of KdV equation. First, we want to construct the Darboux matrix by (39), (40). To construct the Darboux matrix, we must solve $\Phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ which satisfies

$$\Phi_x = \begin{bmatrix} -i\lambda & 0 \\ -1 & i\lambda \end{bmatrix} \Phi$$
$$\Phi_t = \begin{bmatrix} -4i\lambda^3 & 0 \\ -4\lambda^2 & 4i\lambda^3 \end{bmatrix} \Phi \tag{50}$$

It is easy to get $\Phi_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} c_1\, e^{-i\lambda x - 4 i\lambda^3 t} \\ c_2 e^{i\lambda x + 4i\lambda^3 t} + \frac{c_1}{2i\lambda} e^{-i\lambda x - 4i\lambda^3 t} \end{pmatrix}$, where c_1, c_2 are constants.

First, we choose $\lambda = \lambda_0$. Define $\tau = \dfrac{\beta_1(x,t,\lambda_0)}{\alpha_1(x,t,\lambda_0)}$.

$$\tau = \frac{c_2 e^{i\lambda_0 x + 4i\lambda_0^3 t} + \dfrac{c_1}{2i\lambda_0} e^{-i\lambda_0 x - 4i\lambda_0^3 t}}{c_1\, e^{-i\lambda_0\, x - 4\, i{\lambda_0}^3 t}}$$

We get Darboux matrix by (41). It is easy to get $u^{(1)}$ by $u^{(1)} = 2(\frac{1}{\tau^2} - 2i\lambda_0 \frac{1}{\tau})$

$$u^{(1)} = \frac{-16\, {\kappa_0}^3 c_1\, c_2}{\left(2\, c_2\, \kappa_0 e^{\kappa_0\, \left(x - 4\, t {\kappa_0}^2\right)} + c_1\, e^{-\kappa_0\, \left(x - 4\, t {\kappa_0}^2\right)}\right)^2}. \tag{51}$$

Where $\kappa_0 = i\lambda_0$, and $\frac{c_1}{c_2} = 2\kappa_0 e^{2\kappa_0 x_0}$. We get

$$u^{(1)} = -2\kappa_0^2 sech^2[\kappa_0(x - x_0) - 4(\kappa_0)^3 t] \tag{52}$$

where x_0 is any real value. Next, we want to solve $\alpha^{(1)}$ and $\beta^{(1)}$ by $\Phi^{(1)} = D\Phi$.

$$\begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix} = \begin{bmatrix} i\lambda + i\lambda_0 - \frac{1}{\tau} & \frac{1}{\tau^2} - 2\lambda_0 \frac{i}{\tau} \\ 1 & -i\lambda + i\lambda_0 - \frac{1}{\tau} \end{bmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

where $\Phi_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ are different from $\Phi_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$.

More precisely, $\Phi_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} c_3\, e^{-i\lambda x - 4 i\lambda^3 t} \\ c_4 e^{i\lambda x + 4i\lambda^3 t} + \frac{c_3}{2i\lambda} e^{-i\lambda x - 4i\lambda^3 t} \end{pmatrix}$

where c_3, c_4 are constants. Note that $c_1 \neq c_3$ and $c_2 \neq c_4$. Thus,

$$\alpha^{(1)}(x, t, \lambda) = (i\lambda + i\lambda_0 - \frac{1}{\tau})\alpha_2(x, t, \lambda) + (\frac{1}{\tau^2} - 2\lambda_0 \frac{i}{\tau})\beta_2(x, t, \lambda) \tag{53}$$

$$\beta^{(1)}(x, t, \lambda) = \alpha_2(x, t, \lambda) + (-i\lambda + i\lambda_0 - \frac{1}{\tau})\beta_2(x, t, \lambda) \tag{54}$$

$$\beta^{(1)}(x, t, \lambda) = -\frac{2(\kappa_0 + \kappa)[\kappa c_1 c_4 e^{M_1} - \kappa_0 c_2 c_3 e^{-M_1}] + (\kappa_0 - \kappa)[c_1 c_3 e^{M_2} - 4c_2 c_3 \kappa\kappa_0 e^{-M_2}]}{2\kappa[2c_2\kappa_0 e^{\kappa_0(x-4t\kappa_0)} + c_1 e^{-\kappa_0(x-4t\kappa_0)}]} \tag{55}$$

where $M_1 = (\kappa_0 - \kappa)(4t\kappa_0^2 + 4t\kappa\kappa_0 - x + 4t\kappa^2)$, $M_2 = (\kappa_0 + \kappa)(4t\kappa_0^2 - 4t\kappa\kappa_0 - x + 4t\kappa^2)$, and $k = i\lambda$

$$\beta^{(1)}(x,t,\lambda_0) = -2\,\frac{\kappa_0\,(-c_3\,c_2 + c_1\,c_4)}{2\,c_2\,e^{-\kappa_0\,(4\,t\kappa_0{}^2 - x)}\kappa_0 + c_1\,e^{\kappa_0\,(4\,t\kappa_0{}^2 - x)}} \tag{56}$$

we can see that $\beta^{(1)}(x,\lambda_0) \to 0$ as $x \to \pm\infty$. We know that $\beta^{(1)}(x,t,\lambda_0)$ is the eigenfunction and λ_0^2 is the eigenvalue of this equation $-\beta^{(1)}_{xx}(x,t,\lambda_0)+u^{(1)}(x)\beta^{(1)}(x,t,\lambda_0) = \lambda_0^2\beta^{(1)}(x,t,\lambda_0)$ with $u^{(1)}$ is one soliton solution (52).

Next, we use the same method to do again. Now, we must to solve

$$\begin{aligned} \Phi^{(1)}_x &= U^{(1)}\Phi^{(1)} = \begin{bmatrix} -i\lambda & -u^{(1)} \\ -1 & i\lambda \end{bmatrix} \Phi^{(1)} \\ \Phi^{(1)}_t &= V^{(1)}\Phi^{(1)} \end{aligned}$$

By $\Phi^{(1)} = D\Phi$, we can easy to solve the Lax pair of AKNS form. Let $\lambda = \lambda_1$ (note that $\lambda_0 \neq \lambda_1$). We can also construct another $\tau^{(1)} = \dfrac{\alpha^{(1)}(x,t,\lambda_1)}{\beta^{(1)}(x,t,\lambda_1)}$ and Darboux matrix $D^{(1)}$.

$$D^{(1)} = \begin{bmatrix} i\lambda + i\lambda_1 - \frac{1}{\tau^{(1)}} & (\frac{1}{\tau^{(1)}})^2 - 2\lambda_1 \frac{i}{\tau^{(1)}} \\ 1 & -i\lambda + i\lambda_1 - \frac{1}{\tau^{(1)}} \end{bmatrix} \tag{57}$$

Therefore,

$$U^{(2)} = D^{(1)}U^{(1)}(D^{(1)})^{-1} + D^{(1)}_x(D^{(1)})^{-1} = \begin{bmatrix} -i\lambda & -u^{(2)} \\ -1 & i\lambda \end{bmatrix} \tag{58}$$

$$V^{(2)} = D^{(1)}V^{(1)}(D^{(1)})^{-1} + D^{(1)}_x(D^{(1)})^{-1} \tag{59}$$

Where

$$u^{(2)} = -u^{(1)} + 2((\frac{1}{\tau^{(1)}})^2 - 2\lambda_1 \frac{i}{\tau^{(1)}})$$

$$= \frac{16(\kappa_0+\kappa_1)[\frac{\kappa_0^3}{c_3c_4}(c_3^2e^{-U} + 4\kappa_1^2c_4^2e^U) + \frac{\kappa_1^3}{c_1c_2}(4\kappa_0^2c_2^2e^{-V} + c_1^2e^V) + 4\kappa_0\kappa_1(\kappa_0^2-\kappa_1^2)]}{\frac{2(\kappa_0+\kappa_1)}{\kappa_0-\kappa_1}[\frac{\kappa_0}{c_1c_4}e^{-(\kappa_0-\kappa_1)W_2} - \frac{\kappa_1}{c_2c_3}e^{(\kappa_0-\kappa_1)W_2}] + [\frac{4\kappa_0\kappa_1}{c_1c_3}e^{-(\kappa_0+\kappa_1)W_1} - \frac{1}{c_2c_4}e^{(\kappa_0+\kappa_1)W_1}]}$$

where we take this

$$\begin{aligned} \kappa_0 &= i\lambda_0 \\ \kappa_1 &= i\lambda_1 \\ U &= 2\kappa_1(x - 4t\kappa_1^2) \\ V &= -2\kappa_0(x - 4t\kappa_0^2) \\ W_1 &= 4\kappa_0^2t - 4t\kappa_0\kappa_1 + 4t\kappa_1^2 - x \\ W_2 &= 4\kappa_0^2t + 4t\kappa_0\kappa_1 + 4t\kappa_1^2 - x \end{aligned}$$

Next, we want to get $\alpha^{(2)}$, $\beta^{(2)}$ by $\Phi^{(2)} = D^{(1)}\Phi^{(1)} = D^{(1)}D\Phi$.

$$\Phi^{(2)} = \begin{bmatrix} i\lambda + i\lambda_1 - \frac{1}{\tau^{(1)}} & (\frac{1}{\tau^{(1)}})^2 - 2\lambda_1\frac{i}{\tau^{(1)}} \\ 1 & -i\lambda + i\lambda_1 - \frac{1}{\tau^{(1)}} \end{bmatrix} \begin{bmatrix} i\lambda + i\lambda_0 - \frac{1}{\tau} & \frac{1}{\tau^2} - 2\lambda_0\frac{i}{\tau} \\ 1 & -i\lambda + i\lambda_0 - \frac{1}{\tau} \end{bmatrix} \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix}$$

where $\Phi_3 = \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix}$ are different from $\Phi_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ and $\Phi_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$.

More precisely, $\Phi_3 = \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} c_5\, e^{-i\lambda x - 4i\lambda^3 t} \\ c_6 e^{i\lambda x + 4i\lambda^3 t} + \frac{c_5}{2i\lambda}e^{-i\lambda x - 4i\lambda^3 t} \end{pmatrix}$

where c_5, c_6 is constant. Note that $c_1 \neq c_3 \neq c_5$ and $c_2 \neq c_4 \neq c_6$

By the same way, we can evaluate and write $\beta^{(2)}(x,t,\lambda)$ in **Appendix D**.

In particular, we can write

$$\beta^{(2)}(x,t,\lambda_0) = \frac{2\kappa_0(c_2c_5 - c_1c_6)(\lambda_0^2 - \lambda_1^2)[2c_4\kappa_1 e^{\kappa_1(x-4t\kappa_1^2)} + c_3e^{-\kappa_1(x-4t\kappa_1^2)}]}{2(\kappa_1+\kappa_0)[\kappa_1c_1c_4e^{-S_1} - \kappa_0c_2c_3e^{S_1}] - (\kappa_1-\kappa_0)[4\kappa_0\kappa_1c_2c_4e^{S_2} - c_1c_3e^{-S_2}]} \tag{60}$$

and

$$\beta^{(2)}(x,t,\lambda_1) = \frac{-2\kappa_1(c_4c_5 - c_3c_6)(\kappa_1^2 - \kappa_0^2)[2c_2\kappa_0 e^{\kappa_0(x-4t\kappa_0^2)} + c_1e^{-\kappa_0(x-4t\kappa_0^2)}]}{2(\kappa_1+\kappa_0)[\kappa_1c_1c_4e^{-S_1} - \kappa_0c_2c_3e^{S_1}] - (\kappa_1-\kappa_0)[4\kappa_0\kappa_1c_2c_4e^{S_2} - c_1c_3e^{-S_2}]} \tag{61}$$

where

$$\begin{aligned} S_1 &= (\kappa_0 - \kappa_1)(4\lambda_0^2 t + 4t\lambda_0\lambda_1 + 4t\lambda_1^2 + x) \\ S_2 &= (\kappa_0 + \kappa_1)(4\lambda_0^2 t - 4t\lambda_0\lambda_1 + 4t\lambda_1^2 + x) \end{aligned}$$

we can see that $\beta^{(2)}(x,t,\lambda_0) \to 0$ and $\beta^{(2)}(x,t,\lambda_1) \to 0$ with $x \to \pm\infty$. We know that $\beta^{(2)}(x,t,\lambda_0)$, $\beta^{(2)}(x,t,\lambda_1)$ are the eigenfunctions and λ_0^2, λ_1^2 are the corresponding the eigenvalues. Moreover, they satisfy the corresponding equations

$$- \quad \beta_{xx}^{(2)}(x,t,\lambda_0) + u^{(2)}(x)\beta^{(2)}(x,t,\lambda_0) = \lambda_0^2\beta^{(2)}(x,t,\lambda_0) \tag{62}$$

$$- \quad \beta_{xx}^{(2)}(x,t,\lambda_1) + u^{(2)}(x)\beta^{(2)}(x,t,\lambda_1) = \lambda_1^2\beta^{(2)}(x,t,\lambda_1) \tag{63}$$

with $u^{(2)}$ is two soliton solution respectively. Finally, we can use the same method to get more solutions of KdV equation.

5 Conclusion

In this thesis, we start from trivial solution and apply the Darboux transformation for AKNS system. It can reduce the complexity, so it is convenient for us to figure out one and two soliton solutions of KdV equation. In addition, we write down the form of eigenfunction and eigenvalue of Schrödinger equation.

6 Appendix

Appendix A

Proof: where $D = \frac{\partial}{\partial x}$, $L = -D^2 + u(x,t)$ and $A = -4D^3 + 6uD + 3u_x$
Therefore, by elementary calculus,

$$L\phi = -D^2\phi + u\phi$$
$$AL\phi = (4D^5 - 4uD^3 - 12u_xD^2 - 12u_{xx}D - 4u_{xxx} - 6uD^3 + 6u^2D + 6uu_x - 3u_xD^2 + 3uu_x)\phi$$

and

$$LA\phi = (4D^5 - 6uD^3 - 12u_xD^2 - 6u_{xx}D - 3u_xD^2 - 6u_{xx}D - 3u_{xxx} - 4uD^3 + 6u^2D + 3uu_x)\phi$$

Therefore

$$(AL - LA)\phi = (-u_{xxx} + 6uu_x)\phi$$

for all sufficiently smooth ϕ. Therefore

$$(AL - LA) = -u_{xxx} + 6uu_x$$

and

$$L_t = u_t$$

Appendix B

we know $b_0 = c_0 = 0$, $a_{0,x} = 0$, so choose $a_0 = \alpha_0$

$$b_{j,x} + 2ib_{j+1} + 2qa_j = 0 \tag{64}$$

$$c_{j,x} - 2ic_{j+1} - 2ra_j = 0 \tag{65}$$

substitution $a_0 = \alpha_0$, $b_0 = c_0 = 0$ into (64) , (65), then we have $b_1 = iq\alpha_0$, $c_1 = ir\alpha_0$.

substitution $b_1 = iq\alpha_0$, $c_1 = ir\alpha_0$ into $a_{j,x} = qc_j - rb_j$, because $a_{1,x} = 0$ then $a_1 = \alpha_1$, by the same way

$$b_2 = i\alpha_1 q - \tfrac{1}{2}\alpha_0 q_x \quad \Longrightarrow \quad c_2 = i\alpha_1 r + \tfrac{1}{2}\alpha_0 r_x \quad \Longrightarrow \quad a_2 = \tfrac{1}{2}\alpha_0 qr + \alpha_2$$

then use again

$$b_3 = i\alpha_2 q + \frac{i}{2}\alpha_0 q^2 r - \frac{1}{2}\alpha_1 q_x - \frac{i}{4}\alpha_0 q_{xx}$$

$$c_3 = i\alpha_2 r + \frac{i}{2}\alpha_0 r^2 q + \frac{1}{2}\alpha_1 r_x - \frac{i}{4}\alpha_0 r_{xx}$$

$$a_3 = \frac{1}{2}\alpha_1 qr - \frac{i}{4}\alpha_0(qr_x - q_x r) + \alpha_3$$

Take a_i, b_i, c_i into (21), we can get (22)

Appendix C

Proof: since $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is solution of Lax pair (16) that when $\lambda = \lambda_0$

$$\alpha_x = -i\lambda_0\alpha - u\beta \tag{66}$$

$$\beta_x = -\alpha + i\lambda_0\beta \tag{67}$$

suppose $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ is solution of Lax pair (16) that when $\lambda = -\lambda_0$

$$\alpha_{1,x} = i\lambda_0\alpha_1 - u\beta_1 \tag{68}$$

$$\beta_{1,x} = -\alpha_1 - i\lambda_0\beta_1 \tag{69}$$

where choose $\beta = \beta_1$, we differentiate the (69) with respect to x then $\beta_{xx} = -\alpha_{1,x} - i\lambda_0\beta_x$, we differentiate the (67) with respect to x and substitution (66) into this equation, then $\beta_{xx} = -\lambda_0^2\beta + u\beta$

$$\begin{aligned}
\beta_{xx} &= -\alpha_{1,x} - i\lambda_0\beta_x \\
-\lambda_0^2\beta + u\beta &= -i\lambda_0\alpha_1 + u\beta - i\lambda_0\beta_x \\
-\lambda_0^2\beta + u\beta &= -i\lambda_0\alpha_1 + u\beta - i\lambda_0(-\alpha + i\lambda_0\beta) \\
\text{then} & \\
\alpha_1 &= \alpha - 2i\lambda_0\beta
\end{aligned}$$

so $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha - 2i\lambda_0\beta \\ \beta \end{pmatrix}$

Appendix D

We write $\beta^{(2)}(x,t,\lambda) = \dfrac{A}{2\kappa B}$, where

$$
\begin{aligned}
B = \; & - \; 2\,c_3\,e^{(-\kappa_1+\kappa_0)\left(-4t\kappa_0^2-4t\kappa_1\kappa_0-4t\kappa_1^2+x\right)}\kappa_1\,c_2\,\kappa_0 \\
& - \; e^{-(\kappa_0+\kappa_1)\left(-4t\kappa_0^2+4t\kappa_1\kappa_0-4t\kappa_1^2+x\right)}\kappa_1\,c_1 \\
& + \; 4\,\kappa_1^2\,c_2\,e^{(\kappa_0+\kappa_1)\left(-4t\kappa_0^2+4t\kappa_1\kappa_0-4t\kappa_1^2+x\right)}\kappa_0\,c_4 \\
& + \; 2\,\kappa_1^2\,c_1\,e^{-(-\kappa_1+\kappa_0)\left(-4t\kappa_0^2-4t\kappa_1\kappa_0-4t\kappa_1^2+x\right)}c_4 \\
& - \; 4\,c_2\,e^{(\kappa_0+\kappa_1)\left(-4t\kappa_0^2+4t\kappa_1\kappa_0-4t\kappa_1^2+x\right)}\kappa_0^2\,c_4\,\kappa_1 \\
& - \; 2\,c_2\,e^{(-\kappa_1+\kappa_0)\left(-4t\kappa_0^2-4t\kappa_1\kappa_0-4t\kappa_1^2+x\right)}\kappa_0^2\,c_3 \\
& + \; 2\,\kappa_0\,c_1\,e^{-(-\kappa_1+\kappa_0)\left(-4t\kappa_0^2-4t\kappa_1\kappa_0-4t\kappa_1^2+x\right)}c_4\,\kappa_1 \\
& + \; \kappa_0\,c_1\,e^{-(\kappa_0+\kappa_1)\left(-4t\kappa_0^2+4t\kappa_1\kappa_0-4t\kappa_1^2+x\right)}c_3.
\end{aligned}
$$

$$
\begin{aligned}
A = \; & \quad 2c_1c_3c_6(\kappa_0-\kappa_1)[\kappa^3+\kappa^2(\kappa_0+\kappa_1)+\kappa\kappa_0\kappa_1]e^{E_1} \\
& - 4c_2c_4c_5(\kappa_0-\kappa_1)[\kappa^2\kappa_0\kappa_1+\kappa\kappa_0\kappa_1(\kappa_0+\kappa_1)+\kappa_1^2\kappa_0^2]e^{-E_1} \\
& + c_1c_3c_5(\kappa_0-\kappa_1)[\kappa^2+\kappa(\kappa_0+\kappa_1)+\kappa_1\kappa_0]e^{E_2} \\
& - 8c_2c_4c_6(\kappa_0-\kappa_1)[\kappa^3\kappa_0\kappa_1+\kappa^2\kappa_0\kappa_1(\kappa_0+\kappa_1)+\kappa\kappa_0^2\kappa_1^2]e^{-E_2} \\
& - 4c_2c_3c_6(\kappa_0+\kappa_1)[\kappa^3\kappa_0-\kappa^2\kappa_0(\kappa_0-\kappa_1)+\kappa\kappa_0^2\kappa_1]e^{E_3} \\
& + 2c_1c_4c_5(\kappa_0+\kappa_1)[\kappa^2\kappa_1-\kappa\kappa_1(\kappa_0-\kappa_1)+\kappa_0\kappa_1^2]e^{-E_3} \\
& - 2c_2c_3c_5(\kappa_0+\kappa_1)[\kappa^2\kappa_0+\kappa\kappa_0(\kappa_0-\kappa_1)-\kappa_0^2\kappa_1]e^{E_4} \\
& + 4c_1c_4c_6(\kappa_0+\kappa_1)[\kappa^3\kappa_1+\kappa^2\kappa_1(\kappa_0-\kappa_1)-\kappa\kappa_0\kappa_1^2]e^{-E_4}
\end{aligned}
$$

where

$$
\begin{aligned}
E_1 &= x(\kappa-\kappa_0-\kappa_1)-4t(\kappa^3-\kappa_0^3-\kappa_1^3) \\
E_2 &= -x(\kappa+\kappa_0+\kappa_1)+4t(\kappa^3+\kappa_0^3+\kappa_1^3) \\
E_3 &= x(\kappa+\kappa_0-\kappa_1)-4t(\kappa^3+\kappa_0^3-\kappa_1^3) \\
E_4 &= -x(\kappa-\kappa_0+\kappa_1)+4t(\kappa^3-\kappa_0^3+\kappa_1^3)
\end{aligned}
$$

References

[1] P.D.Lax : Comm. Pure and Appl. Math 21, 467-490 (1968)

[2] C.S.Gardner, J.M. Green, M.D.Kruskal,R.M.Miura : Phys.Rev.Lett.19, 1095-1097 (1967)

[3] H. Wahlquist: in Bäcklund Transformations, ed. by R. miura, Lect. Notes Math. Vol. 515 pp.162-175

[4] V. E. Zakharov, A. B. Shabat : Ascheme for integrating the nonliner equations of mathematical physics by the method of the inverse scatting problem. Func. Anal. and its Appl., 1974, 8:226

[5] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. segur : Nonlinear evolution equations of physical significance. Rev. Lett., 1973, 31:125

[6] GU. Chaohao : Soliton theory and its applications, springer, 1991

[7] V. B Matveev, M. A. salle : Darboux Transformations and solitons, springer-Verlag, 1994

[8] GU. Chaohao, Hu. Hesheng, Zhou. Zixiang : Darboux transformation in soliton theory and its geometric applications, 1999

[9] GU. C H. on the Darboux form of Bäcklund Transformations. in integrable system, Nankai Lectures on Math. Phys. World Scientific, Singapore, 1989, 162

[10] Sattinger D H. Zurkowski V D. Gauge theory of Bäcklund Transformations II. Physica, 1987, 26D:225

[11] L. L. Chau, L. Ge, J. C. Shaw, and H. c. Yen : A unified derivation of Bäcklund Transformations for integrable Nonlinear equations, chinese journal of physis, Vol. 34, no. 5,1996

[12] M. J. Ablowitz and H. Seger : Soliton and inverse scattering trsnsform, SIAM, Philadelphia, 1981