國立交通大學

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碩 士 論 文

混色同構圖的平行族

Multicolored Parallelisms of Isomorphic Graphs

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摘要

在一個邊已著色的圖中，如果有一個子圖它每個邊的顏色皆不相同，則稱這種子圖為一個混色圖。在這篇論文中，首先我們先證明一個點數為 2m 的完全圖，其中 $m \neq 2$：在給一個 2m-1 個顏色的塗法後，可以將它分解成 m 個互相同構的混色懸掛樹。而對點數為 2m+1 的完全圖，我們也證明可以在著 2m+1 個顏色後將它分解成 m 個互相同構的混色哈米爾頓圈。第二部分，我們證明對於 2m 個點的完全圖，如果有一種 2m-1 個顏色的著色使得任兩種顏色均會形成一組 C_4 的分割，則這種著色的完全圖也可以分解成 m 個互相同構的混色懸掛樹。由這個結果，我們可以證明出在 K_{2m} 中，任意一種 2m-1 的邊著色，一定會存在兩個同構的混色懸掛樹。同樣地，對於點數為 2m+1 的完全圖，在任意的(2m+1)-邊著色下，也一定存在兩個同構的混色子圖，其中這個子圖是懸掛單圈圖。

Multicolored Parallelisms of Isomorphic Graphs

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Abstract

A subgraph in an *edge-colored* graph is *multicolored* if all its edges receive distinct colors. In this thesis, we first prove that a complete graph on $2m$ ($m \neq 2$) vertices K_{2m} can be properly edge-colored with $2m-1$ colors in such a way that the edges of K_{2m} can be partitioned into m multicolored isomorphic spanning trees. Then, for the complete graph on $2m+1$ vertices, we give a proper edge-coloring with $2m+1$ colors such that the edges of K_{2m+1} can be partitioned into m multicolored $Hamiltonian$ cycles. In the second part, we first prove that if K_{2m} admits a $(2m-1)$-edge-coloring such that any two colors induce a 2-factor with each component a 4-cycle, then K_{2m} can be decomposed into m isomorphic multicolored spanning trees. As a consequence, we prove the existence of two isomorphic multicolored spanning trees in K_{2m} for each $(2m-1)$-edge-coloring of K_{2m}. As to the complete graph of odd order, we find two multicolored *unicyclic* isomorphic subgraphs in K_{2m+1} for each $(2m+1)$-edge-coloring of K_{2m+1}.

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1 Introduction

Graph decomposition and graph coloring are two of the most important topics in the study of graph theory. Graph decomposition deals with the partition of the edge set of a graph G into subsets each induces a graph in the list of prescribed subgraphs of G and graph coloring studies the assignments of colors onto the vertex set of G or the edge set of G or both or some well-understood areas. Either one of them has made a strong impact to make graph theory more interesting and useful through the years.

The research on combining these two topics together starts at observing a subgraph in an edge-colored graph which has many colors. A subgraph whose edges are of distinct colors is known as a rainbow subgraph, see [9] for references. This research was developed from the edge-colorings of the complete graphs.

In 1991, Alon, Brualdi and Shader [2] first showed that in any edge-coloring of K_n such that each color class forms a complete bipartite graph, there is a spanning tree of K_n with distinct colors. Some years later, in 1996, Brualdi and Hollingsworth [3] proved the existence of two edge-disjoint *multicolored* spanning trees in any edge-coloring of K_{2n}. Then, they conjectured that a full partition into multicolored spanning trees is always possible. Not before long, in 2001, J. Krussel, S. Marshal and H. Verral [7] showed the existence of three multicolored spanning trees about above conjecture, and it stopped. No one could do a better job till now. How about adding a condition that these spanning trees are isomorphic mutually? In 2002, G. M. Constantine [5] proposed two conjectures. One of them is that any proper $(2n-1)$-edge-coloring of K_{2n} allows a partition of the edges into multicolored isomorphic spanning trees. The other one is a weaker version of above by giving an edge-coloring ourselves and partitioning it. Moreover, Constantine proved the latter conjecture on the order a power of two or five times a power of two.

It is not a coincidence that decomposing the complete graph with even order into spanning trees. Because it is easy to decompose K_{2n} into n Hamiltonian paths. But, how about the complete graph of odd order? Due to the chromatic index, it is natural to partition the graph into either *unicyclic* subgraphs or *Hamiltonian* cycles which is the best. In 2005, Constantine [6] partitioned K_{2n+1} into n multicolored Hamiltonian cycles by a given $(2n+1)$-edge-coloring if n is a prime. And he proposed a new conjecture that for any $(2n+1)$-edge-coloring of K_{2n+1}, the edges can be partition into multicolored unicyclic isomorphic subgraphs.

In this thesis, the main results are that for the complete graphs of even and odd order, we give each of them a proper edge-coloring and partitioned them into multicolored isomorphic spanning trees and multicolored Hamiltonian cycles, respectively. Furthermore, for an arbitrary edge-coloring of the complete graphs G using $\chi'(G)$ colors, we show that there exist two multicolored isomorphic spanning trees in G when $|V(G)| = 2n$ and there exist two multicolored unicyclic isomorphic subgraphs in G when $|V(G)| = 2n + 1$.

1.1 Preliminaries

In this section, we first introduce the terminologies and definitions of graphs. For details, the readers may refer to the book "Introduction to Graph Theory" by D. B. West.[8]

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its *endpoints*. A *loop* is an edge whose endpoints are equal. *Multiedges* are edges having the same pair of endpoints. A *simple graph* is a graph without loops or multiedges. In this thesis, all the graphs we consider are simple. The size of the vertex set $V(G)$, $|V(G)|$, is called the *order* of G, and the size of the edge set $E(G)$, $|E(G)|$, is called the *size* of G.

If $e = (u, v)$ (uv in short) is an edge of G, then e is said to be *incident* to u and v. We also say that u and v are *adjacent* to each other. For every $v \in V(G)$, $N(v)$ denotes the neighborhood of v, that is, all vertices of $N(v)$ are adjacent to v. The *degree* of v, $deg(v) = |N(v)|$, is the number of neighbors of v.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. A *spanning subgraph* of G is a subgraph H with $V(H) = V(G)$. A *matching* of size k in G is a subgraph of k pairwise disjoint edges. If a matching covers all vertices of G, then it is a *perfect matching.*

A *factor* of a graph G is a spanning subgraph of G. A *k-factor* is a spanning subgraph with each degree equal to k. Then a 1-factor and a perfect matching are almost the same thing.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle with n vertices is denoted by C_n. A *Hamiltonian graph* is a graph with a spanning cycle, also called a *Hamiltonian cycle*.

A graph with no cycle is *acyclic*. A *tree* is a connected acyclic graph. A *spanning tree* is a spanning subgraph that is a tree, and a graph with exactly one cycle is *unicyclic*.

A *complete graph* is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets called partite sets of G. A graph G is *m-partite* if $V(G)$ can be expressed as the union of m independent sets. A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have the sizes s and t, the complete bipartite graph is denoted by $K_{s,t}$. If the sets have the same size n, the complete bipartite graph is called *balanced*, which is denoted by $K_{n,n}$. Similarly, the complete m-partite graph is denoted by $K_{s_1,s_2,\ldots,s_m}$ and the balanced complete m-partite graph is denoted by $K_{m(n)}$ where each partite set has n vertices.

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is isomorphic to H", written $G \cong H$, if there is an isomorphism from G to H.

A *proper k-edge coloring* of a graph G is a mapping from $E(G)$ into a set of colors $\{1, 2, \ldots, k\}$ such that incident edges of G receive distinct colors. An *h-total-coloring* of a graph G is a mapping from $V(G) \cup E(G)$ into a set of colors $\{1, 2, \ldots, h\}$ such that (i) adjacent vertices in G receive distinct colors, (ii) incident edges in G receive distinct colors, and (iii) any vertex and its incident edges receive distinct colors. The *chromatic index* of a graph G, $\chi'(G)$, is the minimum number k for which G has a proper k-edge coloring.

A subgraph in an edge colored graph is said to be *multicolored* if no two edges have the same color.

Let S be an n-set. A *latin square* of order n based on S is an $n \times n$ array such that each element of S occurs in each row and each column exactly once. For example, $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}$ is a latin square of order 2 based on $\{0, 1\} = \mathbb{Z}_2$. Since this latin square corresponds to a group table of $\langle \mathbb{Z}_2, + \rangle$, the latin square is also known as a 2-group latin square.

For convenience, we denote a latin square of order n based on S by $L = [\ l_{i,j}\ ]$ where $l_{i,j} \in S$ and $i, j \in \mathbb{Z}_n$. Let $L = [\ l_{i,j}\ ]$ and $M = [\ m_{i,j}\ ]$ be two latin squares of order n. Then $L = [\ l_{i,j}\ ]$ and $M = [\ m_{i,j}\ ]$ are a pair of *orthogonal latin squares*, denoted by $L \perp M$, if and only if $\{(l_{i,j}, m_{i,j}) |\ 1 \le i, j \le n\} = S \times S$.

Let $L = [\ l_{i,j}\ ]$ and $M = [\ m_{i,j}\ ]$ be two latin squares of order l and m respectively. Then the direct product of L and M is a latin square of order $l \cdot m$: $L \times M = [\ h_{i,j}\ ]$ where $h_{x,y} = (\ l_{a,b}, m_{c,d}\ )$ provided that $x = ma + c$ and $y = mb + d$. For example, let L be the 2-group latin square, then $L \times L$ is a latin square of order 4 based on $\mathbb{Z}_2 \times \mathbb{Z}_2$ as in Figure 1.

	0	1	2	3
0	(0,0)	(0,1)	(1,0)	(1,1)
1	(0,1)	(0,0)	(1,1)	(1,0)
2	(1,0)	(1,1)	(0,0)	(0,1)
3	(1,1)	(1,0)	(0,1)	(0,0)

Figure 1: 2-group latin square of order 4

A *transversal* of a latin square of order n is a set of n entries from each column and each row such that these n entries are all distinct. For example, in $L \times L$, $\{h_{0,0}, h_{1,2}, h_{2,3}, h_{3,1}\}$ is a transversal. It is not difficult to see $L \times L$ does have 4 disjoint transversals. Clearly, if a latin square of order n has n disjoint transversals, then it has an orthogonal latin square.

A latin square $L = [l_{i,j}]$ is *commutative* if $l_{i,j} = l_{j,i}$ for each pair of distinct i and j and L is *idempotent* if $l_{i,i} = i$, $i = 1, 2, \cdots, n$. Furthermore, L is *circulant* if $l_{i,j} = l_{i-1,j+1}$ where the indeices i, j are taken modulo n. Now, let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and the edge v_iv_j is colored with $l_{i,j}$ where $L = [l_{i,j}]$ is an idempotent commutative latin square, then we obtain an n-edge-coloring of K_n. We note here that an idempotent commutative latin square of order n exists if and only if n is odd.

A similar idea shows that a latin square of order n corresponds to an n-edge-coloring of the complete bipartite graph $K_{n,n}$. Let $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ be the two partite sets of $K_{n,n}$ and the edge u_iv_j be colored with $l_{i,j}$ where $L = [l_{i,j}]$ is a latin square , we have a proper n-edge-coloring of $K_{n,n}$.

1.2 Known Results

First, we consider the total coloring and the edge coloring of the complete graph.

Theorem 1.1. [8] *$\forall n \in \mathbb{Z}$, $\chi'(K_{2n}) = 2n - 1$ and $\chi'(K_{2n+1}) = 2n + 1$.*

Theorem 1.2. [10] *If m is an odd positive integer, then K_m has an m-total coloring.*

Theorem 1.3. *If m is an positive integer, then $K_{m,m}$ has an m-edge coloring. In particular, if $V(K_{m,m}) = A \cup B$ where $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_m\}$, then by letting $\varphi(a_ib_j) = j - i \pmod m$ we have an m-edge coloring of $K_{m,m}$ using colors $0, 1, 2, \cdots, m - 1$.*

From theorem 1.1, it is natural to ask if there exists a partition of the edges of an edge-colored K_{2m} into multicolored subgraphs each has $2m - 1$ edges. Here are three conjectures related to this problem.

Conjecture 1.4. (Constantine, Weak version) [5] For any positive integer m, $m > 2$, there exists a proper (2m-1)-edge coloring of K_{2m} such that all edges can be partitioned into m isomorphic multicolored spanning trees.

Conjecture 1.5. (Brualdi-Hollingsworth) [3] If $m > 2$, then in any proper edge coloring of K_{2m} with $2m - 1$ colors, all edges can be partitioned into m multicolored spanning trees.

Conjecture 1.6. (Constantine, Strong version) [5] If $m > 2$, then in any proper edge coloring of K_{2m} with $2m - 1$ colors, all edges can be partitioned into m isomorphic multicolored spanning trees.

For the first conjecture, we give an example for $m = 3$ as follow:

Example 1.7.

	T_1	T_2	T_3
c_1:	35	46	12
c_2:	24	15	36
c_3:	25	34	16
c_4:	26	13	45
c_5:	14	23	56

Figure 2: 3 multicolored isomorphic spanning trees in K_6

By looking at the example on K_6 (see Figure 2), we can see the i-th row denotes the edges which are colored with c_i and the j-th column denotes the edges of a multicolored spanning tree for $1 \leq i \leq 5$ and $1 \leq j \leq 3$. Therefore, we have a parallelism as defined in Cameron [4], with an additional property due to color. Indeed, it is a double parallelism of K_n, one present in the rows of the array (perfect matchings) and the other in the columns that consist of edge disjoint isomorphic spanning trees. Due to this fact, we say that the complete graph K_{2m} admits a multicolored tree parallelism *(MTP)*, if there exists a proper $(2m-1)$-edge-coloring of K_{2m} for which all edges can be partitioned into m isomorphic multicolored spanning trees. The following known result provides an infinite number of complete graphs which admit MTP.

Theorem 1.8. [5, Constantine] *If $m \neq 1$ or 3 and K_{2m} admits an MTP, then for all $r \geq 1$, $K_{2^r m}$ admits an MTP.*

However, if the coloring is arbitrary, then the problem becomes very difficult. Only partial results have been obtained so far.

Theorem 1.9. [7, Krussel et al.] *If $m > 2$, then in any proper edge coloring of K_{2m} with $2m - 1$ colors, there exist three edge-disjoint multicolored spanning trees.*

Lemma 1.10. [3] *In any proper edge coloring of K_8 with 7 colors, all edges can be partitioned into 4 isomorphic multicolored spanning trees.*

Theorem 1.11. [3] *If $m > 2$, then in any proper edge coloring of K_{2m} with $2m-1$ colors, there exist two edge-disjoint multicolored spanning trees.*

On the other direction, we can also consider the complete graph of odd order. Since $\chi'(K_{2m+1}) = 2m + 1$, the maximal size of multicolored subgraph of $(2m+1)$-edge-colored K_{2m+1} is $2m+1$. So, it's natural to ask if there also exists a partition of the edges of K_{2m+1} into subgraphs of size $2m + 1$ which are colored with $2m + 1$ colors, and the following result is known.

Theorem 1.12. [6, Constantine] *If n is an odd prime, then K_n admits a multicolored Hamiltonian cycle parallelism (MHCP).*

In fact, Constantine proposed a stronger conjecture.

Conjecture 1.13. (Constantine) [6] Any proper coloring of the edges of a complete graph on an odd number of vertices allows a partition of the edges into multicolored isomorphic unicyclic subgraphs.

2 Multicolored Subgraph Parallelism

2.1 Existence of Multicolored Tree Parallelism

We prove Conjecture 1.4 in this section and the following lemma is essential.

Lemma 2.1. *The complete graph K_{12} admits an MTP.*

Proof. Let $V(K_{12}) = \{1, 2, \cdots, 12\}$ and the colors are $C_1, C_2, \cdots, C_{11}$. Let (i, j) be the edge with endpoints i and j. Figure 3 and Figure 4 show the construction of an MTP of K_{12}. ∎

	T_1	T_2	T_3	T_4	T_5	T_6
C_1 :	(2,11)	(1,12)	(6, 7)	(3, 8)	(4, 9)	(5,10)
C_2 :	(2, 9)	(5, 8)	(6,12)	(4,11)	(3,10)	(1, 7)
C_3 :	(4, 7)	(3, 9)	(6,10)	(1, 8)	(5,11)	(2,12)
C_4 :	(1,10)	(3,11)	(5, 9)	(6, 8)	(2, 7)	(4,12)
C_5 :	(2, 8)	(4,10)	(1,11)	(5, 7)	(6, 9)	(3,12)
C_6 :	(5,12)	(3, 7)	(4, 8)	(2,10)	(1, 9)	(6,11)
C_7 :	(3, 5)	(4, 6)	(1, 2)	(9,11)	(10,12)	(7, 8)
C_8 :	(2, 4)	(1, 5)	(3, 6)	(8,10)	(7,11)	(9,12)
C_9 :	(2, 5)	(3, 4)	(1, 6)	(8,11)	(9,10)	(7,12)
C_{10} :	(2, 6)	(1, 3)	(4, 5)	(8,12)	(7, 9)	(10,11)
C_{11} :	(1, 4)	(2, 3)	(5, 6)	(7,10)	(8, 9)	(11,12)

Figure 3: MTP of K_{12}

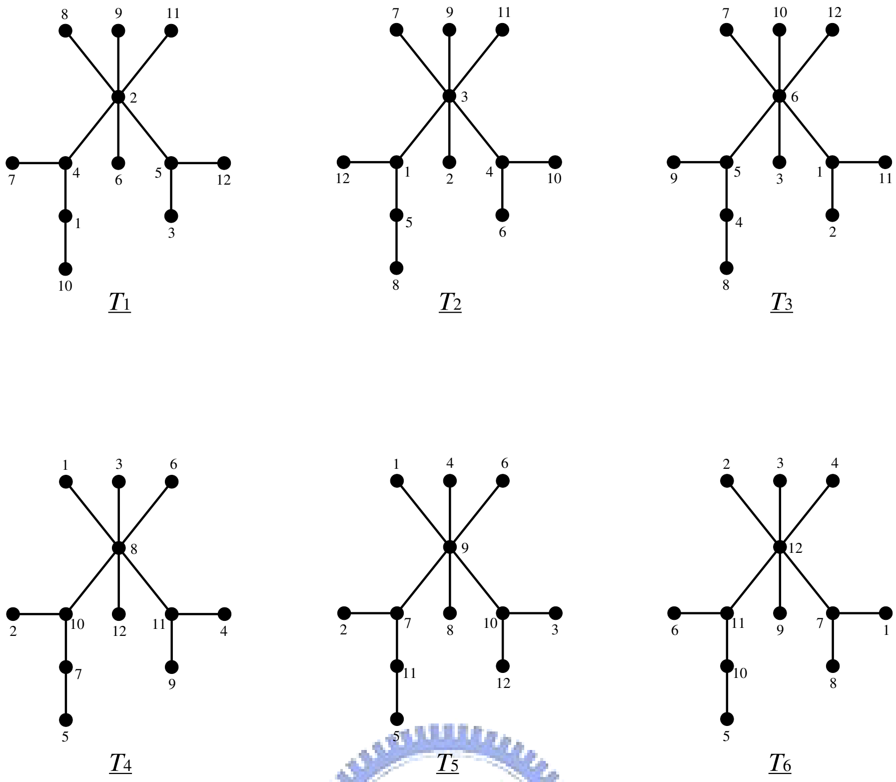

Figure 4: 6 multicolored isomorphic spanning trees in K_{12}

Now, we are ready for the main result.

Theorem 2.2. *For $m \neq 2$, K_{2m} admits an MTP.*

Proof. By Theorem 1.8, it suffices to prove that if m is an odd integer, then K_{2m} admits an MTP.

Let K_{2m} be defined on the set $A \cup B$ where $A = \{a_i \mid i \in \mathbb{Z}_m\}$ and $B = \{b_i \mid i \in \mathbb{Z}_m\}$. For convenience, let $G_1 =< A >$ and $G_2 =< B >$. Since m is odd, by Theorem 1.2, G_1 has a total coloring π which uses m colors, $1, 2, \cdots, m$. Now, define an edge-coloring φ of K_{2m} as follows:

(a) For each edge $a_j a_k \in E(G_1)$, let $\varphi(a_j a_k) = \pi(a_j a_k)$;

(b) For each edge $b_j b_k \in E(G_2)$, let $\varphi(b_j b_k) = \pi(a_j a_k)$;

(c) For each edge $a_i b_i, i \in \mathbb{Z}_m$, let $\varphi(a_i b_i) = \pi(a_i)$; and

(d) For each edge $a_j b_k, j \neq k$, let $\varphi(a_j b_k) = m + t$ where $t \equiv k - j \pmod{m}$ and $t \in \{1, 2, \cdots, m-1\}$.

Clearly, φ is a $(2m-1)$-edge-coloring of K_{2m}. It is left to decompose K_{2m} into m multicolored isomorphic spanning trees. First, for each $i \in \{1, 2, 3, \cdots, m\}$, let T_i be defined on the set $A \cup B$ and $E(T_i) = \{a_i a_{i+2t \ (mod \ m)}, b_i b_{i+2t-1 \ (mod \ m)}, b_i a_{i+2t-1 \ (mod \ m)}, a_{i+1} b_{i+2t \ (mod \ m)} \mid t = 1, 2, \cdots, \frac{m-1}{2}\} \cup \{a_i b_i\}$. Then, it is easy to check that T_i is a spanning tree of K_{2m} and also T_i is multicolored. Furthermore, T_i and T_j are isomorphic follows by the permutation of $A \cup B$ defined by mapping a_i into a_j and b_i into b_j respectively.

Now, if m is not an odd integer, then $2m = 2^t \cdot m'$ and $t \geq 2$. In case that $m' = 1$, t must be at least 3. Then it is direct consequence of Theorem 1.8. On the other hand, $m' \geq 3$. Thus $K_{2^t m'}$ admits an MTP by using doubling construction obtained in [5] except when $m' = 3$ and $t = 2$. Since this case can be handled by Lemma 2.1, we conclude the proof. ∎

We note here that the above theorem proves the weaker conjecture of Constantine and the result has been included in a paper written jointly with S. Akbari, A. Alipour and H. L. Fu [1] which is to appear in SIAM J. of Discrete Math.

2.2 Existence of Multicolored Hamiltonian Cycle Parallelism

To extend the study of parallelism to the other graph, K_{2m+1} deserves to be considered first. Since $\chi'(K_{2m+1}) = 2m + 1$, the multicolored subgraph we consider has $2m + 1$ edges. Thus, a multicolored Hamiltonian cycle in K_{2m+1} is the best candidate for the subgraphs. In this section, we shall prove that for each positive integer m, there exists a $(2m+1)$-edge-coloring of K_{2m+1} for which all edges can be partitioned into multicolored Hamiltonian cycles. Obviously, any two Hamiltonian cycles are isomorphic and therefore we have another parallelism if exists.

Definition 2.3. We call K_{2m+1} admits a multicolored Hamiltonian cycle parallelism ($MHCP$) if there exists a $(2m+1)$-edge-coloring of K_{2m+1} for which all edges can be partitioned into m multicolored Hamiltonian cycles.

For the convenience in the proof of our main result, we need a special circulant latin square M.

Definition 2.4. $M = [m_{i,j}]$ is a circulant latin square of order odd n with 1st row $(1, \frac{n+3}{2}, 2, \frac{n+5}{2}, 3, \cdots, \frac{n+n}{2}, \frac{n+1}{2})$.

Figure 5 shows M of order 7.

1	5	2	6	3	7	4
5	2	6	3	7	4	1
2	6	3	7	4	1	5
6	3	7	4	1	5	2
3	7	4	1	5	2	6
7	4	1	5	2	6	3
4	1	5	2	6	3	7

Figure 5: Circulant latin square of order 7

Using M, we have a proper n-edge-coloring of $K_{n,n}$ where $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ are the two partite sets of $K_{n,n}$. This coloring has an extra property that for $1 \leq j \leq n$, the edges in $\{u_1v_j, u_2v_{j+1}, u_3v_{j+2}, \cdots, u_nv_{j+n-1}\}$ form a perfect matching and they receive distinct colors. (Here, the indices i of v_i are taken modulo n and $i \in \{1, 2, \cdots, n\}$.)

We note here that if we permute the entries of M, we obtain another n-edge-coloring of $K_{n,n}$ which has the same property as above.

In order to prove the main theorem, we also need the following lemma.

Lemma 2.5. *Let v be a composite odd integer and n is the smallest prime which is a factor of v, say $v = mn$. Then $K_{m(n)}$ has an mn-edge-coloring such that the edge-colored $K_{m(n)}$ can be partitioned into $\frac{n(m-1)}{2}$ multicolored Hamiltonian cycles if K_m admits an MHCP.*

Proof. We prove the lemma by giving an mn-edge-coloring φ. Since K_m defined on $\{x_i \mid i \in \mathbb{Z}_m\}$ admits an $MHCP$, let μ be such an edge-coloring using the colors $0, 1, \cdots, m-1$. Let $V(K_{m(n)}) = \bigcup_{i=0}^{m-1} V_i$ where $V_i = \{x_{i,j} \mid j \in \mathbb{Z}_n\}$ and $L = [l_{h,k}]$ be a circulant latin square of order n as defined before Figure 5. Now, we have an mn-edge-coloring of $K_{m(n)}$ by letting $\varphi(x_{a,b}x_{c,d}) = l_{b,d} + \mu(x_a x_c) \cdot n$. Therefore, the edges in $K_{m(n)}$ joining a vertex of V_a to a vertex of V_c, denoted (V_a, V_c), are colored with the entries in $L + \mu(x_a x_c) \cdot n$. It is not difficult to see that φ is a proper coloring of $K_{m(n)}$. Now, it is left to show that the edges of $K_{m(n)}$ can be partitioned into multicolored Hamiltonian cycles.

Let $C = (c_0, c_1, c_2, \cdots, c_{m-1}) = (x_{\alpha(0)}, x_{\alpha(1)}, \cdots, x_{\alpha(m-1)})$ (α is a permutation of $\mathbb{Z}_m$) be a multicolored Hamiltonian cycle in K_m obtained from the $MHCP$ of K_m. Define $C_{m(n)}$ to be the subgraph induced by the set of edges in $\bigcup_{i=0}^{m-1} (V_{\alpha(i)}, V_{\alpha(i+1)})$. Now, if we let $S(r_0, r_1, \cdots, r_{m-1})$ be the set of perfect matchings in $(V_{\alpha(0)}, V_{\alpha(1)})$, $(V_{\alpha(1)}, V_{\alpha(2)})$,$\cdots$, $(V_{\alpha(m-2)}, V_{\alpha(m-1)})$ and $(V_{\alpha(m-1)}, V_{\alpha(0)})$ respectively where the perfect matching in $(V_{\alpha(i)}, V_{\alpha(i+1)})$, $i = 0, 1, 2, \cdots, m-1$, is the set of edges $x_{\alpha(i),a}x_{\alpha(i+1),b}$ with $b - a \equiv r_i \pmod{n}$, $r_i \in \mathbb{Z}_n$, then $S(r_0, r_1, \cdots, r_{m-1})$ is a 2-factor of $C_{m(n)}$. Moreover, by the edge-coloring we use for $K_{m(n)}$, $S(r_0, r_1, \cdots, r_{m-1})$ is indeed a multicolored 2-factor. Hence, we can partition the edges of $C_{m(n)}$ into n multicolored 2-factors due to the fact that $r_i \in \mathbb{Z}_n$. Note that $S(r_0, r_1, \cdots, r_{m-1})$ and $S(r'_0, r'_1, \cdots, r'_{m-1})$ are edge-disjoint 2-factors if and only if $r_i \neq r'_i$ for each $i \in \mathbb{Z}_m$.

So, the proof follows by selecting $(r_0, r_1, \cdots, r_{m-1}) \in \mathbb{Z}_n^m$ properly in order that each

2-factor $S(r_0, r_1, \cdots, r_{m-1})$ is a Hamiltonian cycle. Observe that if $\sum_{i=0}^{m-1} r_i$ is not a multiple of n, then $S(r_0, r_1, \cdots, r_{m-1})$ is a Hamiltonian cycle. (n is a prime.) Therefore, we let $(0, 0, \cdots, 0, 1), (1, 1, \cdots, 1, 2), \cdots$, and $(n-1, n-1, \cdots, n-1, 0)$ be the n m-tuples we need provided that n is not a factor of $m \cdot i + 1$ for $i = 0, 1, 2, \cdots, n-1$. On the other hand, assume that $n \mid m \cdot j + 1$ for some $j \in \mathbb{Z}_n$.(Here, note that such j occurs at most once.) If $j \in \{1, 2, \cdots, n-2\}$, then replace $(j, j, \cdots, j, j+1)$ and $(j+1, j+1, \cdots, j+1, j+2)$ with $(j, j, \cdots, j, j+1, j+1)$ and $(j+1, j+1, \cdots, j+1, j, j+2)$ respectively. Otherwise, if $j = n-1$, then replace $(n-2, n-2, \cdots, n-2, n-2, n-1)$ and $(n-1, n-1, \cdots, n-1, n-1, 0)$ with $(n-2, n-2, \cdots, n-1, n-1, n-1)$ and $(n-1, n-1, \cdots, n-2, n-2, 0)$ respectively. This implies that in either case, we have a partition of the edges of $C_{m(n)}$ into n edge-disjoint multicolored Hamiltonian cycles. Moreover, since $K_{m(n)}$ can be partitioned into $\frac{m-1}{2}$ $C_{m(n)}$'s, by a similar argument, we have a partition of the edges of $K_{m(n)}$ into $\frac{m-1}{2} \cdot n$ multicolored Hamiltonian cycles. ∎

As an example, if $m = n = 3$, then the three multicolored Hamiltonian cycles are $S(0, 0, 1) = (x_{0,0}, x_{1,0}, x_{2,0}, x_{0,1}, x_{1,1}, x_{2,1}, x_{0,2}, x_{1,2}, x_{2,2})$, $S(1, 1, 2) = (x_{0,0}, x_{1,1}, x_{2,2}, x_{0,1}, x_{1,2}, x_{2,0}, x_{0,2}, x_{1,0}, x_{2,1})$, $S(2, 2, 0) = (x_{0,0}, x_{1,2}, x_{2,1}, x_{0,2}, x_{1,1}, x_{2,0}, x_{0,1}, x_{1,0}, x_{2,2})$. In case that $m = 5$ and $n = 3$, then we have 6 multicolored Hamiltonian cycles. For each $C_{5(3)}$, we have three multicolored Hamiltonian cycles of type $S(0, 0, 0, 0, 1)$, $S(1, 1, 1, 2, 2)$, and $S(2, 2, 2, 1, 0)$.

Now, in order to partition the edges of an 9-edge-colored K_9 into 4 Hamiltonian cycles, we combine $S(0, 0, 1)$ with the three cliques (K_3) induced by the three partite sets V_1, V_2 and V_3, to obtain a 4-factor. Since these K_3's can be edge-colored with $\{4, 5, 6\}, \{7, 8, 9\}$ and $\{1, 2, 3\}$ respectively, we have an edge-colored 4-factor with each color occurs exactly twice. Thus, if this 4-factor can be partitioned into two multicolored Hamiltonian cycles, then we conclude that K_9 admits an $MHCP$. Figure 6 shows how this can be done.

Notice that in the induced subgraphs $< V_1 >, < V_2 >$ and $< V_3 >$ we have exactly one edge from each graph which is not included in the cycle with solid edges. Therefore, we may first color the edges in $< V_1 >, < V_2 >$ and $< V_3 >$ respectively and then adjust

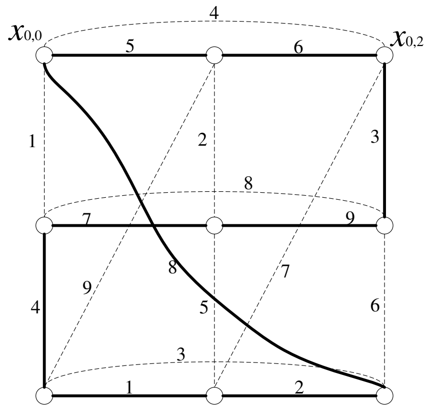

Figure 6: Two multicolored Hamiltonian cycles of K_9

the colors in (V_1, V_2), (V_2, V_3) and (V_3, V_1) respectively in order to obtain a multicolored Hamiltonian cycle. For example, if the color of $x_{0,0}x_{0,2}$ is 5 instead of 4, then we permute the entries in $\begin{array}{|c|c|c|}\hline 4 & 6 & 5 \\ \hline 6 & 5 & 4 \\ \hline 5 & 4 & 6 \\ \hline \end{array}$ by using (4,5), and thus the latin square used to color (V_2, V_3) becomes $\begin{array}{|c|c|c|}\hline 5 & 6 & 4 \\ \hline 6 & 4 & 5 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$. This is an essential trick we shall use when n is a larger prime.

Theorem 2.6. *For each odd integer $v \geq 3$, K_v admits an MHCP.*

Proof. The proof is by induction on v. By Theorem 1.12, the assertion is true for v is a prime. Therefore, we assume that v is a composite odd integer and the assertion is true for each odd order $u < v$. Let n be the smallest prime such that $v = n \cdot m$ and $V(K_v) = \bigcup_{i=1}^{m} V_i$ where $V_i = \{x_{i,j} \mid j \in \mathbb{Z}_n\}$, $i \in \mathbb{Z}_m$. By induction, K_m admits an $MHCP$ and hence $K_{m(n)}$ can be partitioned into $\frac{m-1}{2}$ $C_{m(n)}$'s each admits $MHCP$. Moreover, by Lemma 2.5, each $MHCP$ of $C_{m(n)}$ contains a multicolored Hamiltonian cycle $S(0, 0, \cdots, 0, 1)$. Here, the edge-coloring of $K_{m(n)}$ is induced by the coloring φ of K_m, i.e., if $v_i v_j$ is an edge of K_m with color $\varphi(v_i v_j) = t \in \mathbb{Z}_m$, then the colors of the edges in (V_i, V_j) are assigned by using $M + tn$ where M is a circulant latin square of order n as defined before Figure 5. We note here again that permuting the entries of a latin square $M + tn$ may give another coloring, but the coloring is still a proper coloring.

So, in order to obtain an $MHCP$ of K_v, we first give a v-edge-coloring of K_v and then adjust the coloring if it is necessary. Since $K_{m(n)}$ has an mn-edge-coloring φ, the edge-coloring μ of K_v can be defined as follows: (a) $\mu|_{K_{m(n)}} = \varphi$ and (b) $\mu|_{<V_i>} = \psi_i, i =$

$1, 2, \cdots, m$, where ψ_i is an n-edge-coloring of K_n such that K_n can be partitioned into $\frac{n-1}{2}$ multicolored Hamiltonian cycles. Moreover, the images of ψ_i are $1+tn, 2+tn, \cdots, n+tn$ where $t \in \mathbb{Z}_m$ and t is a color not occurs in the edges incident to $v_i \in V(K_m)$. (Here, the colors used to color the edges of K_m are $0, 1, 2, \cdots, m-1$.)

It is not difficult to check now μ is a v-edge-coloring of K_v. We shall revise μ by permuting the colors in (V_i, V_{i+1}) for some i and finally obtain the edge-coloring we need.

For convenience, let the edges of $K_{m(n)}$ be partitioned into $C^{(1)}_{m(n)}, C^{(2)}_{m(n)}, \cdots, C^{(\frac{m-1}{2})}_{m(n)}$ each contains a multicolored Hamiltonian cycle $E^{(1)}, E^{(2)}, \cdots, E^{(\frac{m-1}{2})}$ of type $S(0, 0, \cdots, 0, 1)$ and the edges of each K_n induced by $V_i, i = 1, 2, \cdots, m$, be partitioned into $\frac{n-1}{2}$ multicolored Hamiltonian cycles $D^{(1)}, D^{(2)}, \cdots, D^{(\frac{n-1}{2})}$. Since $m \geq n$, we consider the 4-factors $E^{(i)} \cup D^{(i)}$ where $i = 1, 2, \cdots, \frac{n-1}{2}$. Starting from $i = 1$, we shall partition the edges of $E^{(1)} \cup D^{(1)}$ into two Hamiltonian cycles such that both of them are multicolored. By the idea explained in Figure 6, we first obtain two Hamiltonian cycles from $E^{(1)} \cup D^{(1)}$ by a similar way, see Figure 7 for example. For the purpose of obtaining multicolored Hamiltonian cycles, we adjust the colors by permuting the colors in (V_i, V_{i+1}) to make sure the first cycle does contain each color exactly once. Then, the second one is clearly multicolored. Now, following the same process, we partition the edges of $E^{(2)} \cup D^{(2)}, \cdots,$ and $E^{(\frac{n-1}{2})} \cup D^{(\frac{n-1}{2})}$ into two multicolored Hamiltonian cycles respectively. We remark here that if permuting entries of a latin square is necessary, then we can keep doing the same trick since $C^{(1)}_{m(n)}, C^{(2)}_{m(n)}, \cdots, C^{(\frac{m-1}{2})}_{m(n)}$ are edge-disjoint subgraphs of $K_{m(n)}$. (The permutations are carried out independently.) This concludes that after all the permutations are done, we obtain a v-edge-coloring of K_v such that K_v can be partitioned into $\frac{v-1}{2}$ multicolored Hamiltonian cycles. ∎

As a conclusion, we use Figure 7 to explain how our idea works. The edge xy was colored with 26 originally by using the circulant latin square of order 5 mentioned before Figure 5. But, 26 occurs in the Hamiltonian cycle with solid edges already. Therefore, we use $(26, 30)$ to permute the square to obtain the edge-coloring we would like to have. After adjusting the colors of zw, $z'w'$ and ab respectively, we have two multicolored Hamiltonian cycles as desired.

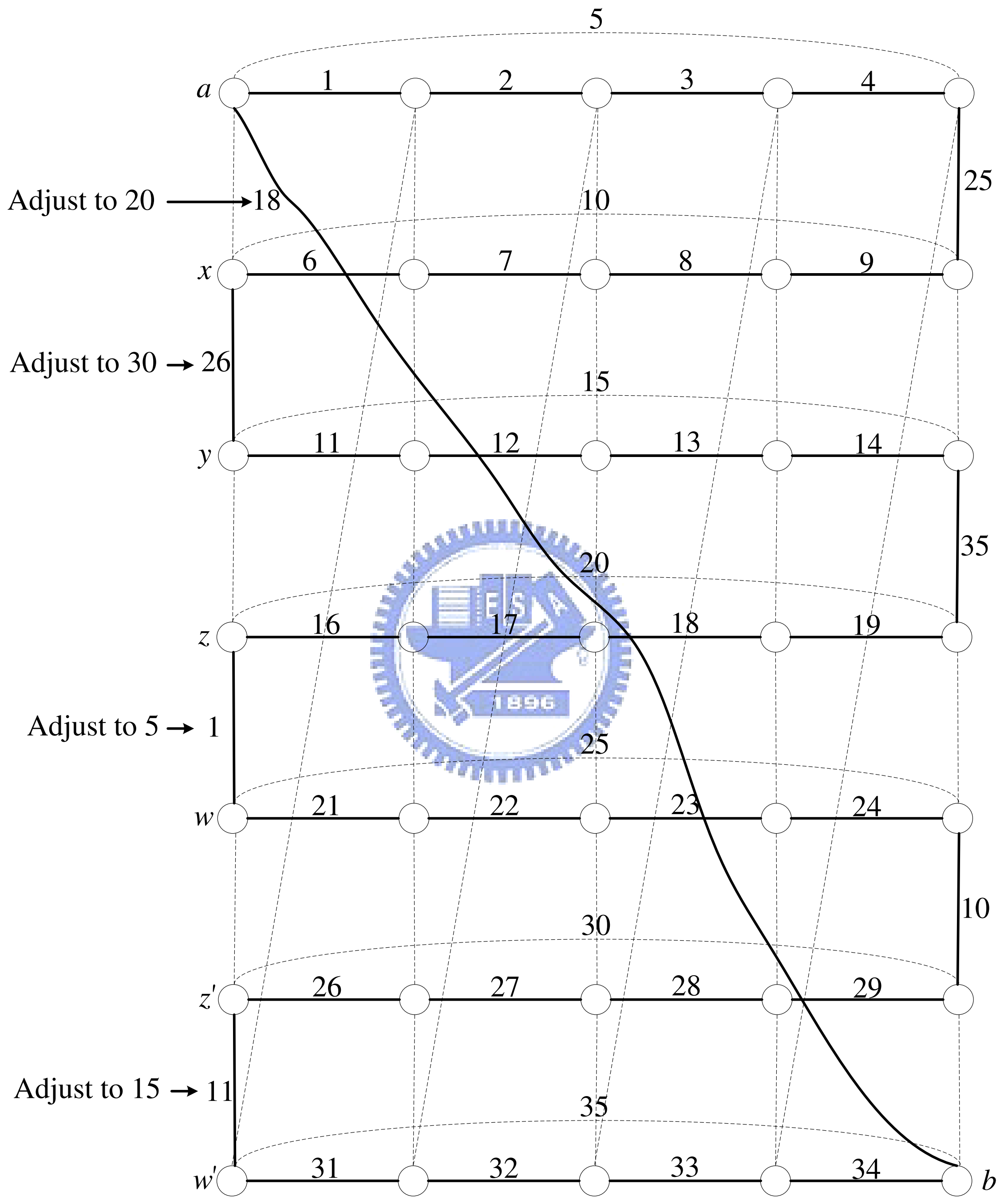

Figure 7: Two multicolored Hamiltonian cycles of K_{35}

3 The Existence of Multicolored (Rainbow) Subgraphs

3.1 Multicolored spanning trees in K_{2m}

Now, we consider a special edge-coloring of K_{2m} with $2m-1$ colors such that for any two colors form an C_4-factor. Let L be the 2-group latin square defined earlier in Section 1.2. In what follows, we show that $L^n = L \times L \times \cdots \times L$ based on $\mathbb{Z}_2{}^n$ has 2^n disjoint transversals for each $n \geq 2$.

Proposition 3.1. L^n has 2^n disjoint transversals for each $n \geq 2$.

Proof. The proof is by induction on n and by Figure 2, $n = 2$ is true.

	0	1	2	3
0	**(0,0)**	(0,1)	(1,0)	(1,1)
1	(0,1)	(0,0)	**(1,1)**	(1,0)
2	(1,0)	(1,1)	(0,0)	**(0,1)**
3	(1,1)	**(1,0)**	(0,1)	(0,0)

A_0

	0	1	2	3
0	(0,0)	**(0,1)**	(1,0)	(1,1)
1	(0,1)	(0,0)	(1,1)	**(1,0)**
2	(1,0)	(1,1)	**(0,0)**	(0,1)
3	**(1,1)**	(1,0)	(0,1)	(0,0)

A_1

	0	1	2	3
0	(0,0)	(0,1)	**(1,0)**	(1,1)
1	**(0,1)**	(0,0)	(1,1)	(1,0)
2	(1,0)	**(1,1)**	(0,0)	(0,1)
3	(1,1)	(1,0)	(0,1)	**(0,0)**

A_2

	0	1	2	3
0	(0,0)	(0,1)	(1,0)	**(1,1)**
1	(0,1)	**(0,0)**	(1,1)	(1,0)
2	**(1,0)**	(1,1)	(0,0)	(0,1)
3	(1,1)	(1,0)	**(0,1)**	(0,0)

A_3

Figure 8: 4 transversals in L^2

Assume that the assertion is true for each $k \geq 2$. Let $L^k = [l_{a,b}{}^{(k)}]$ and $L^{k+1} = \begin{array}{|c|c|}\hline L_0{}^k & L_1{}^k \\ \hline L_1{}^k & L_0{}^k \\ \hline\end{array}$. By definition of direct product, we have $L_0{}^k = [m_{a,b}]$ where $m_{a,b} = (0\ ,\ l_{a,b}{}^{(k)})$ (a $(k+1)$-dim. vector) and $L_1{}^k = [\overline{m}_{a,b}]$ where $\overline{m}_{a,b} = (1\ ,\ l_{a,b}{}^{(k)})$. We shall use the set of 2^k disjoint transversals in L^k to construct 2^{k+1} disjoint transversals in L^{k+1}.

Let $\{A_i \mid i = 0, 1, 2, \cdots, 2^k - 1\}$ be the set of disjoint transversals obtained in L^k by induction hypothesis. W.L.O.G. we may let A_i be the transversal which contains the entry $l_{0,i}{}^{(k)}$, $i = 0, 1, 2, \cdots, 2^k - 1$. Now, we shall use A_{2i} and A_{2i+1}, $i = 0, 1, 2, \cdots, 2^{k-1} - 1$, to construct four disjoint transversals in L^{k+1}. For convenience, we explain the construction by using A_0 and A_1.

Since A_0(respectively A_1) is a transversal in L^k, the corresponding entries in $L_0{}^k$ form a transversal, so are the corresponding entries in $L_1{}^k$. Let the corresponding transversals of A_0 in ${L_0}^k$ and ${L_1}^k$ be $\overline{A}_{0,0}$ and $\overline{A}_{1,0}$ respectively. Similarly, let the corresponding transversals of A_1 be $\overline{A}_{0,1}$ and $\overline{A}_{1,1}$ respectively. Note that for $0 \le r, s \le 1$, $\overline{A}_{r,s}$ has 2^k entries one from each row and from each column. Now, for $0 \le r, s \le 1$, we split $\overline{A}_{r,s}$ into two parts: ${\overline{A}_{r,s}}^{(u)}$ is the set of entries from the first to the 2^{k-1}-th row of $\overline{A}_{r,s}$, and ${\overline{A}_{r,s}}^{(l)}$ is the other half. By defining B_0, B_1, B_2 and B_3 as in Figure 9, we have four transversals in L^{k+1} as desired.

Since for $i = 1, 2, \cdots, 2^{k-1} - 1$, $\overline{A}_{2i}$ and $\overline{A}_{2i+1}$ can also be used to construct four transversals in L^{k+1}, we have a set of 2^{k+1} transversals in L^{k+1}. By the reason that $A_0, A_1, \cdots, A_{2^k-1}$ are disjoint transversals, we conclude the proof. ∎

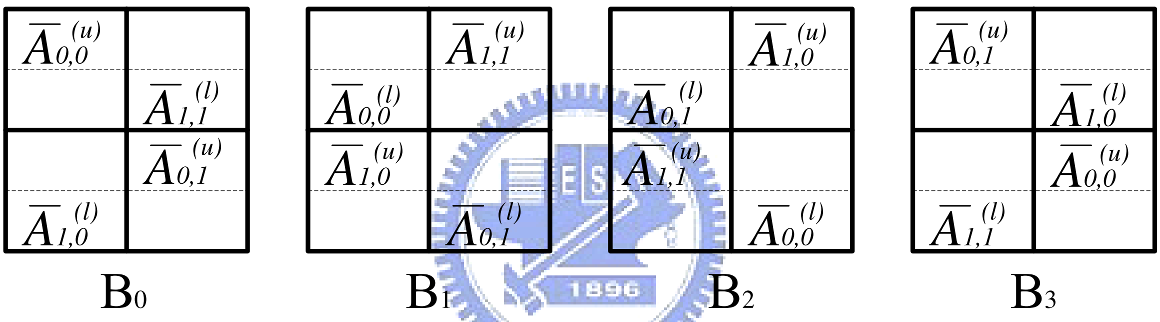

Figure 9: 4 transversals in L^{k+1} constructed from A_0 and A_1

Lemma 3.2. *Let μ be a $(2m-1)$-edge-coloring of K_{2m}, $m \ge 2$, such that for any two colors induce a 2-factor with each component a 4-cycle, then (a) $2m = 2^n$ for some $n \ge 2$ and (b) K_{2m} contains a clique K of order 2^k, $1 \le k \le n-1$ such that $\{\mu(e) \mid e \in E(K)\}$ is a $(2^k - 1)$-set, i.e., $\mu|_K$ is a $(2^k - 1)$-edge-coloring of K.*

Proof. First, we claim that (b) is true. The proof is by induction on n. Clearly, it is true when $n = 2$. By hypothesis, let H be a clique of order 2^h, $h < k$, and $\mu|_H$ is a $(2^h - 1)$-edge-coloring of H. W.L.O.G. let $V(H) = \{x_1, x_2, \cdots, x_{2^h}\}$ and the colors used in H be $\{c_1, c_2, \cdots, c_{2^h-1}\}$. Since μ is a $(2m-1)$-edge-coloring of K_{2m}, each color occurs around each vertex. Let c_{2^h} be a color not used in H. Then, we have a set H', $H' \cap H = \phi$, $H' = \{y_1, y_2, \cdots, y_{2^h}\}$ such that $\mu(x_i y_i) = c_{2^h}$ for $i = 1, 2, \cdots, 2^h$. Now, by the reason that any two colors induce a C_4-factor, we conclude that $\mu|_{H'}$ is also a $(2^h - 1)$-edge-coloring of H', moreover, $\mu(x_i x_j) = \mu(y_i y_j)$ for $1 \leq i \neq j \leq 2^h$. Therefore, the complete bipartite graph $K_{2^h,2^h} = (H, H')$ has a 2^h-edge-coloring following by the same reason. This implies that $\mu|_{H \cup H'}$ is a $(2^{h+1} - 1)$-edge-coloring of the clique induced by $H \cup H'$. So, we have the proof of (b).

Suppose $2m = 2^r \cdot p$ where p is an odd integer and $p \neq 1$. Using above argument, we can find the biggest clique G of order 2^s which uses $2^s - 1$ colors. Then we partition the vertices of K_{2m} into two sets X and Y where $X = V(G)$, and let $|Y| = q$. Here, we notice that $q < 2^s$. Consider these $2^s - 1$ colors used in coloring the edges of G, in total, there are $(2^s - 1)(2^{r-1} \cdot p)$ edges which use these colors. But, we have used these colors in G. Hence, there remains $2^{s-1}(2^s - 1)(\frac{q}{2} - 1)$ edges to be colored by using these colors. Since the edges between X and Y can't be colored with any of these colors, they have to be in Y. But, since $q < 2^s$, $2^{s-1}(2^s - 1)(\frac{q}{2} - 1) > \binom{q}{2}$, a contradiction. This implies that $p = 1$, and we have the proof of (a). ∎

Now, we are ready to prove the main result.

Theorem 3.3. *Let μ be a $(2m-1)$-edge-coloring of K_{2m}, $m > 2$, such that for any two colors induce a 2-factor with each component a 4-cycle. Then the edges of K_{2m} can be partitioned into m isomorphic multicolored spanning trees.*

Proof. By lemma 3.2, $2m = 2^n$ for some $n > 2$. We prove the theorem by induction on n. By Lemma 1.10, $n = 3$ is true.

Assume that the assertion is true for each $k \geq 3$ and consider $K_{2^{k+1}}$.

From the process of the proof of Lemma 3.2, it must exist two disjoint cliques of order 2^k with $2^k - 1$ colors in $K_{2^{k+1}}$. Let $V(K_{2^{k+1}}) = A \cup B$ where A, B are the vertex sets of the two cliques. Consider the colors of the edges between A and B. Let $A = \{a_0, a_1, \ldots, a_{2^k-1}\}$, $B = \{b_0, b_1, \ldots, b_{2^k-1}\}$, and define an array $M = [m_{i,j}]$ by $\mu(a_i b_j) = m_{i,j}$. It's clear that M is a latin square, furthermore, $M \cong L^k$. By Proposition 3.1, M has 2^k disjoint transversals. This implies that there are 2^k perfect matchings in the complete bipartite graph induced by $A \cup B$. Since the two cliques induced by A and B respectively have 2^{k-1} isomorphic spanning trees of order 2^k, respectively. Thus, by assigning a perfect matching to each spanning tree, we obtain 2^k spanning trees of order 2^{k+1}. Moreover, these spanning trees are isomorphic and multicolored. ∎

Now, we are ready to consider K_{2m} with an arbitrary $(2m-1)$-edge-coloring.

Theorem 3.4. *Let φ be an arbitrary (2m-1)-edge-coloring of K_{2m}. Then there exist two isomorphic multicolored spanning trees in K_{2m} for $m \geq 3$.*

Proof. Let $V(K_{2m}) = \{x_i | \ i \in \mathbb{Z}_{2m}\}$. We split the proof into two cases.

Case 1. There exists a 4-cycle (x_0, x_1, x_2, x_3) such that $\varphi(x_0x_1) = b$, $\varphi(x_2x_3) = c$, and $\varphi(x_0x_3) = \varphi(x_1x_2) = a$. Then the two isomorphic multicolored spanning trees can be obtained by the following figure.

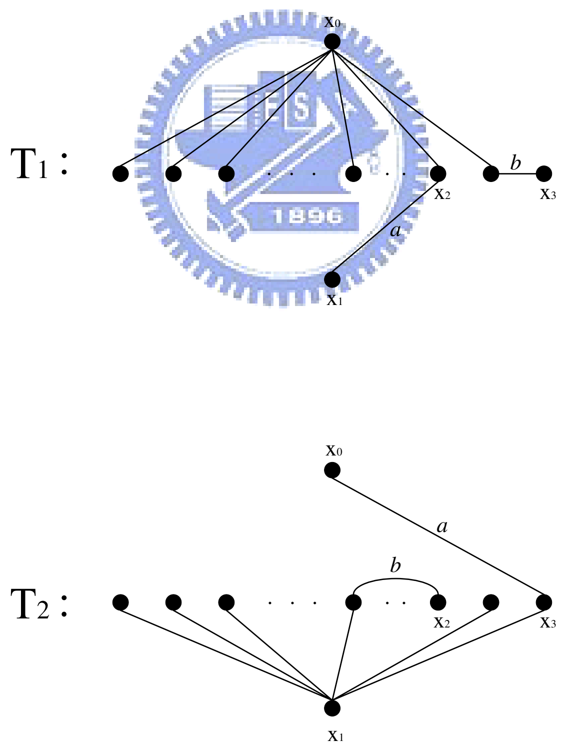

Figure 10: Two isomorphic spanning trees

Case 2. If any two colors of this edge-coloring induce a C_4-decomposition of K_{2m}, then we have the proof by Theorem 3.3. ∎

3.2 Multicolored unicyclic spanning subgraphs in K_{2m+1}

Since $\chi'(K_{2m+1}) = 2m + 1$, we consider K_{2m+1} with a proper $(2m + 1)$-edge-coloring.

Theorem 3.5. *For any positive integer m, given an arbitrary proper (2m+1)-edge-coloring of K_{2m+1}, there exists a pair of multicolored isomorphic unicyclic spanning subgraphs of K_{2m+1}.*

Proof. For each (2m+1)-edge-coloring K_{2m+1}, we observe that each vertex of K_{2m+1} is missing one color (exactly) of the color-set $\mathbb{Z}_{2m+1}$, and each color of the color-set occurs exactly m times. Therefore, if u and v are two distinct vertices, then their corresponding missing colors are distinct. So, without loss of generality, we may let $V(K_{2m+1}) = \mathbb{Z}_{2m+1}$, and at vertex $i \in \mathbb{Z}_{2m+1}$, the color missing is i.

Now, we can construct two multicolored subgraphs. In the first graph, we use the star with center 0 which has $2m$ edges. Then delete one edge $0x'$ which is colored "t" $\neq 0$. Let this star be H_1. Now, by adding an edge yy' (colored t) and an edge xx' (colored 0), we have the desired subgraph $G_1 = H_1 + yy' + xx'$. The second graph can be obtained by a similar way, which is from the star H_2 with center t by deleting one edge $0t$. Let 0t be of color a. Then, adding an edge (different from 0t) of color a and the edge 0x' we have the desired second subgraph $G_2 = H_2 + 0t + 0x'$.

Clearly, these two graphs are multicolored and the unique cycle in them are both a triangle. Since they are spanning subgraphs, we have a pair of multicolored isomorphic unicyclic spanning subgraphs of K_{2m+1}. ∎

Figure 11 depicts the construction of G_1 and G_2 in the proof.

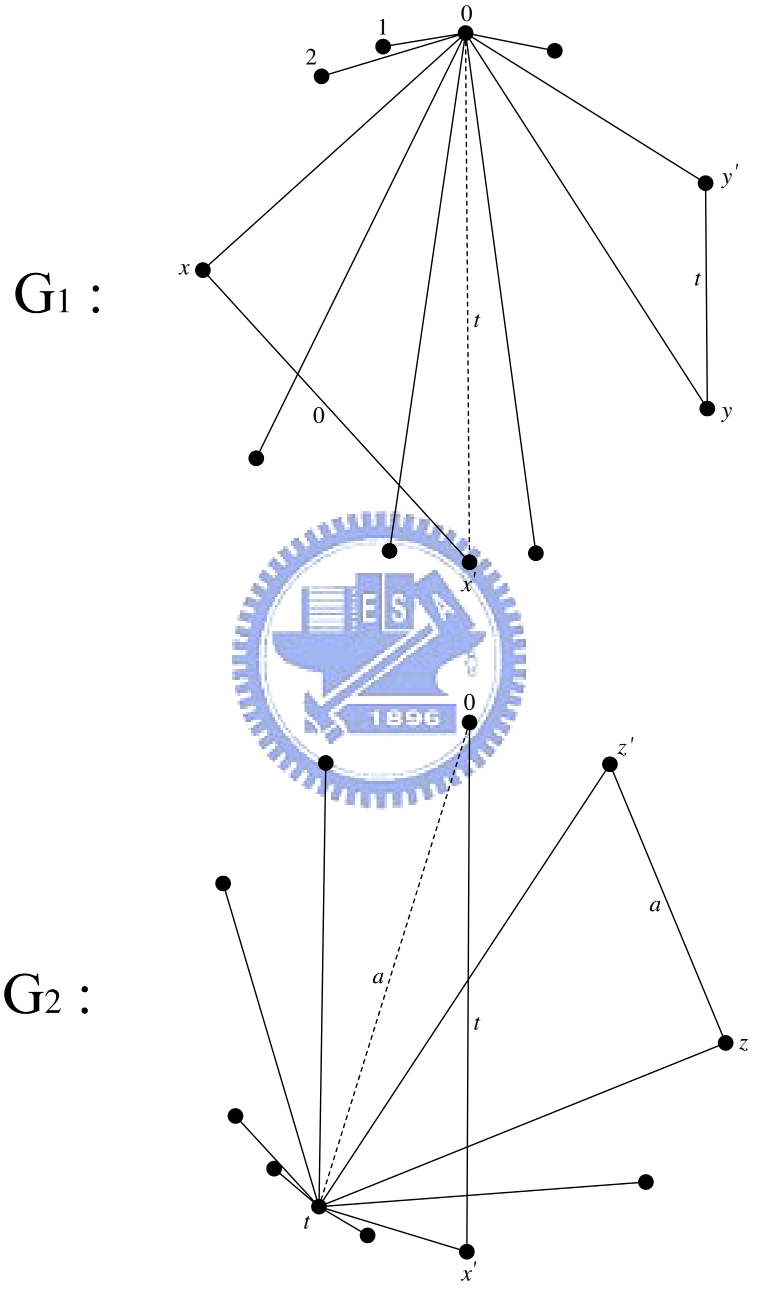

Figure 11: Two multicolored isomorphic unicyclic spanning subgraphs

4 Conclusion

In this thesis, we have obtained the following four main results:

1. A multicolored tree parallelism for K_{2m}, $m \geq 3$.

2. A multicolored Hamiltonian cycle parallelism for K_{2m+1}, $m \geq 2$.

3. The existence of two isomorphic multicolored spanning trees in an $(2m-1)$-edge-colored K_{2m}.

4. The existence of two isomorphic multicolored unicyclic spanning subgraphs in an $(2m+1)$-edge-colored K_{2m+1}.

From the results, we are able to prove the weaker conjecture (Conjecture 1.4) posed by Constantine. But, we are very far from verifying the other conjectures. Hopefully, this task can be done in the near future.

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