

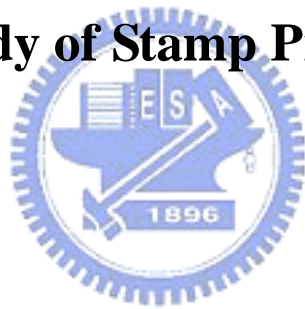
國立交通大學

應用數學系

碩士論文

郵票問題的研究

A Study of Stamp Problem



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指導老師：傅恆霖 教授

中華民國九十六年六月

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摘要

所謂的郵票問題是出版一套每一張面額皆不同的郵票，使得面額總和以內的每一個數都可以用彼此相連不分散的郵票湊出來。舉例來說，假如我們要發行一套最多三張郵票，面額分別是一元、二元和四元，則我們可以湊出一到七元的郵票，而且每一種組合的郵票彼此都是相連不分散的。由上述的概念，我們可以把郵票問題的想法延伸到 graph labeling 上，用以下的方式：找一個圖 G 的 labeling 函數 $\varphi: V(G) \rightarrow \mathbb{N}$ 使得對於 1 到值 φ 和內的每一個值 k ，必定存在一個值 φ 和為 k 的連通子圖 H 。這樣的 labeling 方式就稱為圖 G 的 IC-著色，圖 G 的所有 IC-著色中最大的值 φ 和就是圖 G 的 IC-指標。

在此篇論文當中，我們主要研究某些圖的 IC-著色和改善圖的指標的上界和下界。還有，我們也考慮當 $\varphi[V(G)] = [1, |V(G)|]$ 的情形，並研究是否圖 G 具有 IC-著色；如果有的話，圖 G 就稱為可和飽和的。

A Study of Stamp Problem

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Abstract

The so-called stamp problem is to produce a set of stamps with distinct values such that we can obtain consecutive values by using distinct combinations of stamps from the given set. For example, if we are allowed to use at most three stamps from a set of stamps with value set $\{1, 2, 4\}$, then we have consecutive values $1, 2, \dots, 7$.

By way of the notion mentioned above, we can extend the idea of stamp problem to graph labeling in the following way : find a labeling of G , $\varphi : V(G) \rightarrow N$, such that for each value $k \in [1, \sum_{v \in V(G)} \varphi(v)]$, there exists a connected subgraph H whose total value $\sum_{u \in H(G)} \varphi(u) = k$. The labeling obtained above is called an IC-coloring of G and the maximum value of an IC-coloring of G is the IC-index of G .

In this thesis, we mainly study the IC-coloring of certain graphs and improve their known upper bounds and/or lower bounds, respectively. Moreover, we also consider the case when $\varphi[V(G)] = [1, |V(G)|]$ and study whether the graph G has an IC-coloring, if so, G is said to be sum-saturable.

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1 Introduction

Consider the following simple mathematical puzzle from Bolt's book [2] : A country wishes to issue a block of four stamps, each with the shape of an equilateral triangle of unit side length. (illustrated in Figure 1) How should the values a, b, c and d be assigned so that one could get a connected group of stamps of total value k for each $k = 1, 2, \dots, 10$? This is the well-known Stamp Problem.

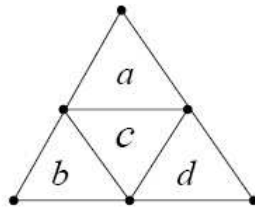


Figure 1: Stamp problem.

It is easily verified that the set $\{a, b, c, d\}$ must equal $\{1, 2, 3, 4\}$. And, it is equivalent to the problem of assigning positive integer labels to the vertices of $K(1, 3)$ in such a way that for each integer $k = 1, 2, \dots, 10$, there is a connected subgraph whose labels sum to k . Moreover, we can think about this question for various classes of graphs in such a way to assign integer labels to the vertices. For example, the complete graph K_n , the cycle C_n , the path P_n , \dots , and so on, and each leads to a graph labeling problem of the following type (Fink [3]).

Given a connected graph G having p vertices, is it possible to assign labels $1, 2, \dots, p$ to the vertices in such a way that for each $k = 1, 2, \dots, p(p+1)/2$, G contains a connected subgraph whose labels sum to k ?

We find out that not every graph could label vertices in such a way. A *labeling* of a graph G of order p is a bijection $\lambda : V(G) \rightarrow [1, p]$, the integer $\lambda(v)$ is called the λ -*label* of v , and for each subgraph H of G , the sum $\sum_{v \in V(H)} \lambda(v)$ will be denoted by $\lambda(H)$ and called the λ -*value* of H . We will use the notation Δ_p to represent $\sum_{j=1}^p j$. If λ is a labeling

of a connected graph G of order p , and if, for each $k \in [1, \Delta_p]$, G has a connected subgraph H of λ -value k , then we say λ is a *saturating labeling* of G , or G has a λ -*labeling*. Any graph for which a saturating labeling exists will be called a *sum-saturable* graph.

However, let's consider another coloring type to the vertices of a graph G . Given a function $f : V(G) \rightarrow N$ is called a *coloring* of G . Let H be a subgraph of G . We define $f_s(H) = \sum_{v \in V(H)} f(v)$. In particular, we denote $f_s(G)$ by $S(f)$ if $H = G$. A function $f : V(G) \rightarrow N$ is called an *IC-coloring* of G if for any integer $k \in [1, S(f)]$, there is a connected subgraph H of G such that $f_s(H) = k$. Note that any connected graph admits IC-coloring. For example, the function $f : V(G) \rightarrow N$ defined by $f(v) = 1$ for all $v \in V(G)$ is an IC-coloring with the sum $S(f) = n$. Here, for any integer $k \in [1, n]$, a connected subgraph H of G with exactly k vertices will satisfy the condition $f_s(H) = k$. The *IC-index* of a graph G , denoted by $M(G)$, is defined to be

$$M(G) = \max\{S(f) \mid f \text{ is an IC-coloring of } G\}.$$

Any IC-coloring $f : V(G) \rightarrow N$ for which $S(f) = M(G)$ will be called a *maximum IC-coloring* of G . A maximum IC-coloring of a graph is not necessarily unique, as illustrated in Figure 2.

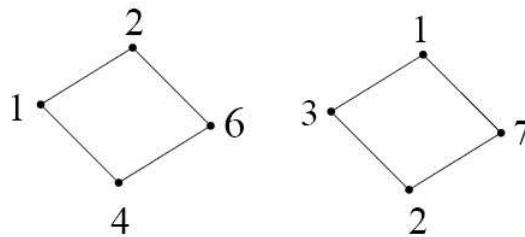


Figure 2: Two distinct maximum IC-colorings of C_4 .

However, in any IC-coloring of a graph G , one has to assign 1 to the vertices which is not cut-vertices; otherwise, the number $S(f) - 1$ can not be obtained from any connected subgraph of G . The problem of finding IC-coloring of finite graphs is related to the postage stamp problem in the number theory, which has been extensively studied in the literature [1, 4, 5, 6, 7]

1.1 Preliminaries

In this section, we first introduce the terminologies and definitions of graphs. For details, the readers may refer to the book “Introduction to Graph Theory” by D. B. West [10].

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its *endpoints*. A *loop* is an edge whose endpoints are equal. *Multiedges* are edges having the same pair of endpoints. A *simple graph* is a graph without loops or multiedges. In this thesis, all the graphs we consider are simple. The cardinality of the vertex set $V(G)$, $|V(G)|$, is called the *order* of G , and the cardinality of the edge set $E(G)$, $|E(G)|$, is called the *size* of G .

If $e = (u, v)$ (uv in short) is an edge of G , then e is said to be *incident* to u and v . We also say that u and v are *adjacent* to each other. For every $v \in V(G)$, $N(v)$ denotes the neighborhood of v , that is, all vertices of $N(v)$ are adjacent to v . The *degree* of v , $deg(v) = |N(v)|$, is the number of neighbors of v . A cut-vertex of a graph G is a vertex whose deletion will disconnect G .

An *induced subgraph* of a graph G is a subgraph obtained by deleting a set of vertices. A *spanning subgraph* of G is a subgraph H with $V(H) = V(G)$. A *matching* of size k in G is a subgraph of k pairwise vertex disjoint edges. If a matching covers all vertices of G , then it is a *perfect matching*.

A *factor* of a graph G is a spanning subgraph of G . A k -*factor* is a spanning subgraph with each degree equal to k . Then a 1-factor and a perfect matching are almost the same thing.

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. And a path of order n is denoted by P_n .

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle with n vertices is denoted by C_n . A *Hamiltonian graph* is a graph with a spanning cycle, also called a *Hamiltonian cycle*. A *wheel* with n spokes is obtained by the join operation $C_n + K_1$. It is denoted by W_n .

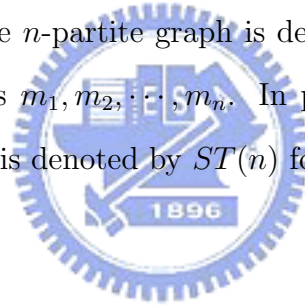
If G has a u, v -path, then the *distance* from u to v , written by $d(u, v)$, is the least

length of a u, v -path. The diameter of G is $\max_{u,v \in V(G)} d(u, v)$.

A graph with no cycle is *acyclic*. A *tree* is a connected acyclic graph. A *spanning tree* is a spanning subgraph that is a tree.

Trees of diameter three are called *double-star*. These graphs have two central vertices plus leaves. We will use $DS(m, n)$ to denote the double-star whose two central vertices have degrees m and n , respectively. (defined by [9])

A *complete graph* is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n . A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets called partite sets of G . A graph G is *m -partite* if $V(G)$ can be expressed as the union of m independent sets. A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have the sizes m and n , the complete bipartite graph is denoted by $K(m, n)$. Similarly, the complete n -partite graph is denoted by $K(m_1, m_2, \dots, m_n)$ when the n partite sets have the sizes m_1, m_2, \dots, m_n . In particular, the complete bipartite graph $K(1, n)$ is called *star* and is denoted by $ST(n)$ for any $n \geq 1$.



1.2 Known Results

1.2.1 Saturating labeling

Fink showed that for any $n \geq 4$, the path P_n is not sum-saturable[3]. However, all cycles, Hamiltonian graphs, and complete bipartite graphs are sum-saturable according to the following two results (Fink [3]).

Theorem 1.1. [3] *If G is a 2-connected graph having either a 1-factor or a near-one-factor, then G is sum-saturable.*

Theorem 1.2. [3] *A connected graph G of order p is sum-saturable if it has a vertex v for which*

(1) $\deg(v) \geq 1 + \lceil \log_2(p-1) \rceil$, and

(2) *there is a (proper) subset S of $N(v)$ such that $|S| = \lceil \log_2(p-1) \rceil$ and $G - S$ is connected.*

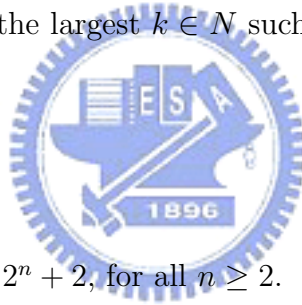
1.2.2 IC-coloring

In 1995, Penrice [8] introduced the concept of stamp covering of G as follows : for an integer $k > 0$, a labeling $f : V(G) \rightarrow N$ is called a k -labeling if for any integer j , $1 \leq j \leq k$, there exists a connected induced subgraph H of G with $f_s(H) = j$. Then, $M(G)$, the stamp covering number of G , is the largest $k \in N$ such that G has a k -labeling. He also showed that

(1) $M(K_n) = 2^n - 1$.

(2) $M(K(1, n)) = M(ST(n)) = 2^n + 2$, for all $n \geq 2$.

(3) For positive integer $n \geq 4$, $(n^2 + 6n - 4)/4 \leq M(P_n) \leq n(n + 1)/2 - 1$.



Around 10 years later, this notion was rediscovered by Salehi et al. [9] and the names IC-coloring and IC-index take place of stamp covering and stamp covering number, respectively. We list some of the results obtained so far.

Theorem 1.3. [9] *For any integer $n \geq 2$, the IC-index of the complete bipartite graph $K(2, n)$ is $3 \cdot 2^n + 1$.*

Theorem 1.4. [9] *For any integer $n \geq 3$, the IC-index of the wheel W_n satisfies the following inequalities : $2^n + 2 \leq M(W_n) \leq 2^n + n(n - 1) + 1$ for $n \geq 3$.*

Theorem 1.5. [3, 8, 9] *For any $n \geq 3$, $n(n + 1)/2 \leq M(C_n) \leq n(n - 1) + 1$. And $M(C_n) = n(n - 1) + 1$ when $n = 3, 4, 5, 6, 8, 9$.*

Observation 1.6. [9] If H is a subgraph of G , then $M(H) \leq M(G)$.

Observation 1.7. [9] If $v(G)$ is the number of connected induced subgraphs of G , then $M(G) \leq v(G)$.

Observation 1.8. [9] If $M(G) = \mu$ and $v \notin V(G)$, then $M(G + v) \geq 2\mu + 1$. The inequality is sharp when G is the complete graph. That is, $M(K_{n+1}) = 2M(K_n) + 1$. However, there are cases when equality does not hold. Consider the graph G in Figure 3. The graph has exactly 12 non-empty connected induced subgraphs with $M(G) = 12$. But, $M(G + v) \geq 27$.

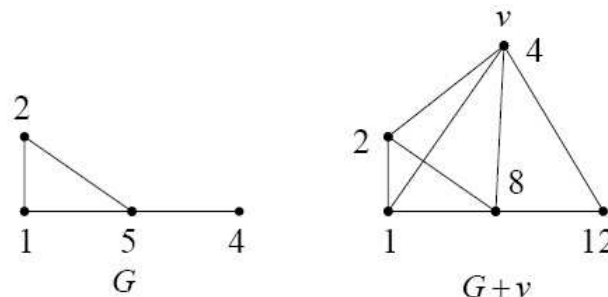


Figure 3: An example of G and $G + v$ with $M(G + v) \neq 2M(G) + 1$.

In this thesis, we shall first study the saturating labelings in Chapter 2. Mainly, constructive labelings are given for those graphs which have good structures, for example, complete multipartite graphs. Then, in Chapter 3, focus our effort in improving the upper and/or lower bounds of the IC-index of some graphs obtained in an earlier paper by Salehi et al.[9]

2 Saturating labelings of certain graphs

Though the complete graphs K_n , the complete bipartite graphs $K(m, n)$, the complete multipartite $K(m_1, m_2, \dots, m_n)$ and Petersen graph satisfy with the conditions mentioned in Theorem 1.1 and Theorem 1.2, we provide constructive proofs.

2.1 Constructive proofs of some graphs

Theorem 2.1. *For any complete graph K_n , K_n is sum-saturable.*

Proof. Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$, and define a labeling $\lambda : V(K_n) \rightarrow N$ by $\lambda(v_i) = i$, for each $i \in [1, n]$. Let $k \in [1, \Delta_n]$, where $\Delta_n = \sum_{i=1}^n i = \frac{n(n+1)}{2}$. It remains to show that λ is a saturating labeling.

- (1) When $1 \leq k \leq n$: The subgraph $\langle \lambda^{-1}(k) \rangle$ of K_n is a single vertex.
- (2) When $k = n + i$ for $1 \leq i \leq n - 1$: The subgraph of K_n is consisted of two vertices, which are $\lambda^{-1}(n)$, and $\lambda^{-1}(i)$ such that the λ -value of the subgraph is k .
- (3) When $k = n + (n - 1) + i$ for $1 \leq i \leq n - 2$: The subgraph of K_n is consisted of three vertices, which are $\lambda^{-1}(n)$, $\lambda^{-1}(n - 1)$, and $\lambda^{-1}(i)$ such that the λ -value of the subgraph is k .

We keep going on by this way:

- (n-1) When $k = n + (n - 1) + (n - 2) + \dots + 3 + i$ for $1 \leq i \leq 2$: The subgraph of K_n is consisted of $n - 1$ vertices, which are $\lambda^{-1}(n)$, $\lambda^{-1}(n - 1)$, $\lambda^{-1}(n - 2)$, \dots , $\lambda^{-1}(3)$, and $\lambda^{-1}(i)$ such that the λ -value of the subgraph is k .
- (n) When $k = n + (n - 1) + (n - 2) + \dots + 3 + 2 + 1$: The subgraph of K_n is K_n itself.

Since K_n is complete, any subgraph H induced by the vertices of K_n is also connected. Hence, K_n is sum-saturable. ■

Theorem 2.2. For all positive integers m and n , the complete bipartite graph $K(m, n)$ is sum-saturable.

Proof. Assume that $m \geq n$, and let $V_m = \{v_1, v_2, \dots, v_m\}$ and $V_n = \{v_{m+1}, v_{m+2}, \dots, v_{m+n}\}$ be the two partite sets of $K(m, n)$, with $|V_m| = m$ and $|V_n| = n$. Then we define a labeling $\lambda : V(K(m, n)) \rightarrow N$ by $\lambda(v_i) = i$, for each $i \in [1, m+n]$. Let $\Delta_{m+n} = \sum_{i=1}^{m+n} i = \frac{(m+n)(m+n+1)}{2}$.

We proceed by induction on n . If $|V_1| = 1$, $V_1 = \{v_{m+1}\}$ and $\lambda(v_{m+1}) = m+1$, and $\Delta_{m+1} = \sum_{i=1}^{m+1} i = \frac{(m+1)(m+2)}{2}$. We claim that $K(m, 1)$ is sum-saturable.

- (1) When $k \in [1, m+1]$: The subgraph $\langle \lambda^{-1}(k) \rangle$ of $K(m, 1)$ is a single vertex.
- (2) When $k = (m+1) + i$, for $1 \leq i \leq m$: The subgraph of $K(m, 1)$ is consisted of two vertices, which are $\lambda^{-1}(m+1)$, and $\lambda^{-1}(i)$ such that the λ -value of the subgraph is k .
- (3) When $k = (m+1) + m + i$, for $1 \leq i \leq m-1$: The subgraph of $K(m, 1)$ is consisted of three vertices, which are $\lambda^{-1}(m+1), \lambda^{-1}(m)$, and $\lambda^{-1}(i)$ such that the λ -value of the subgraph is k .

We keep going on by the same way:

- (m) When $k = (m+1) + m + (m-1) + \dots + 3 + i$ for $1 \leq i \leq 2$: The subgraph of $K(m, 1)$ is consisted of m vertices, which are $\lambda^{-1}(m+1), \lambda^{-1}(m), \lambda^{-1}(m-1), \dots, \lambda^{-1}(3)$, and $\lambda^{-1}(i)$ such that the λ -value of the subgraph is k .
- (m+1) When $k = (m+1) + m + \dots + 3 + 2 + 1 = \Delta_{m+1} = \sum_{i=1}^{m+1} i = \frac{(m+1)(m+2)}{2}$: The subgraph is $K(m, 1)$ itself.

And we can easily check that any subgraph listed above is connected since every subgraph listed above has two partite sets except subgraphs with only one vertex. Hence, $K(m, 1)$ is sum-saturable.

Suppose $K(m, k)$ is sum-saturable, then there exists a labeling λ of G such that $\lambda(v_i) = i$, for each $i \in [1, m+k]$. Suppose $|V_m| = m$ and $|V_n| = k$, and we use Δ_{m+k} to denote $\sum_{i=1}^{m+k} i = \frac{(m+k)(m+k+1)}{2}$.

Consider the case when $|V_n| = k + 1$. We assign a label $m + k + 1$ to the vertex v_{m+k+1} . Since $K(m, k)$ is sum-saturable, the subgraphs of $K(m, k)$ with labels 1 to Δ_{m+k} exist, and the subgraphs of $K(m, k)$ are connected. When $k = \Delta_{m+k} + i$, for $1 \leq i \leq m + k$, we can find the subgraph H_i of $K(m, n)$ is consisted of a set of $m + k$ vertices, which is $\{v_{m+k+1}, v_{m+k}, v_{m+k-1}, \dots, v_{m+1}, v_m, v_{m-1}, \dots, v_2, v_1\} \setminus \{v_{(m+k+1)-i}\}$ such that the λ -value of the subgraph is k . It is trivial that when k is equal to Δ_{m+k+1} , the subgraph is G itself. And we can easily check that the subgraphs mentioned above are connected since every subgraph H_i has also two partite sets. Thus, we know that the complete bipartite graph $K(m, n)$ is sum-saturable. \blacksquare

Corollary 2.3. *For any $n \geq 2$, the star $ST(n)$ is sum-saturable.*

Proof. $ST(n)$ is $K(1, n)$.

Corollary 2.4. *For any $n \geq 3$, the wheel W_n is sum-saturable.*

Proof. Consider $K(1, n)$ in Theorem 2.2.

Theorem 2.5. *For all positive integers m_1, m_2, \dots, m_n , the complete multipartite graph $K(m_1, m_2, \dots, m_n)$ is sum-saturable.*

Proof. Assume that $m_1 \geq m_2 \geq \dots \geq m_n$, and use $V_{m_1}, V_{m_2}, \dots, V_{m_n}$ to denote the partite sets of the complete multipartite graph $G = K(m_1, m_2, \dots, m_n)$ with $|V_{m_1}| = m_1$, $|V_{m_2}| = m_2$, \dots , $|V_{m_n}| = m_n$, and $m_1 + m_2 + \dots + m_n = p$. Let λ be the labeling that assigns the labels $1, 2, \dots, m_1$ to the vertices of V_{m_1} , and $(m_1 + 1), (m_1 + 2), \dots, (m_1 + m_2)$ to the vertices of V_{m_2} , and $[(m_1 + m_2) + 1], [(m_1 + m_2) + 2], \dots, [(m_1 + m_2) + m_3]$ to the vertices of V_{m_3} , \dots , and $[(m_1 + m_2 + \dots + m_{n-1}) + 1], [(m_1 + m_2 + \dots + m_{n-1}) + 2], \dots, [(m_1 + m_2 + \dots + m_{n-1}) + m_n]$ to the vertices of V_{m_n} .

First, we only focus on the two partite sets V_{m_1} and V_{m_2} . Aided by the preceding proof of Theorem 2.2, we can easily check that G has a connected subgraph with λ -value $k \in [1, \frac{(m_1+m_2)(m_1+m_2+1)}{2}]$. Then, we regard $V_{m_1} \cup V_{m_2}$ as a new partite set U_1 . Then, proceeding to do another two partite sets U_1 and V_{m_3} , we can also check that G has a connected subgraph with λ -value $k \in [1, \frac{(m_1+m_2+m_3)(m_1+m_2+m_3+1)}{2}]$.

Keeping going on by this technique mentioned above.

Finally, we view the union of $V_{m_1}, V_{m_2}, \dots, V_{m_{n-1}}$ as a new partite set $U_{m_{n-2}}$, and, consider the two partite sets $U_{m_{n-2}}$ and V_{m_n} , we can still easily check that G has a connected subgraph of λ -value $k \in [1, \Delta_p = \frac{p(p+1)}{2}]$.

Hence, we conclude that the complete multipartite graph $K(m_1, m_2, \dots, m_n)$ is sum-saturable. ■

Proposition 2.6. The Petersen graph is sum-saturable.

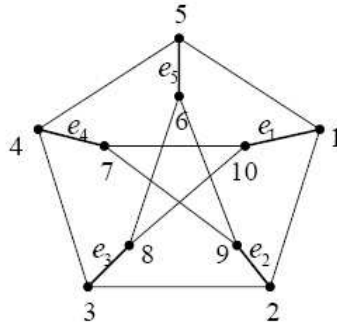


Figure 4: λ -label of vertices of Petersen graph.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_{10}\}$. and $F = \{e_1, e_2, e_3, e_4, e_5\}$. Let λ be a labeling of G satisfying the condition that for each $i \in [1, 5]$, one of the two vertices adjacent with the edge e_i has label i and the other has $11 - i$.

Let $k \in [1, \Delta_{10}]$, and let q and r be the nonnegative integers satisfying $k = 11q + r$, and $0 \leq r \leq 10$.

- (1) If $q = 0$, there exists a single vertex v_i such that $\lambda(v_i) = i$, for each $i \in [1, 10]$.
- (2) If $q \geq 1$ and $r \geq 1$, let $\lambda(v_j) = r$, we can find the subgraph H_1 of G containing q edges $\{e_{j+1}, e_{j+2}, \dots, e_{j+q}\}$ of F plus vertex v_j when $1 \leq j \leq 5$, where $j + q \pmod{5}$. Also, when $6 \leq j \leq 10$, we can find the subgraph H'_1 of G containing q edges $\{e_{11-j+2}, e_{11-j+4}, \dots, e_{11-j+2q}\}$ of F plus vertex v_j , where $11 - j + 2q \pmod{5}$ for $q \in [1, 4]$. One can check that H_1 and H'_1 are both connected subgraphs of G .

(3) If $q \geq 1$ and $r = 0$, the subgraph H_2 of G containing q edges $\{e_1, e_2, \dots, e_q\}$ of F such that $\lambda(H_2) = 11q$. And we can easily check that H_2 is a connected subgraph of G since G is 2-connected.

Thus, the Petersen graph is sum-saturable. ■

Fink [3] provided two sufficient conditions to check if a graph of order p is sum-saturable; such as the followings :

- (1) There exists v such that $\deg(v) \geq 1 + \lceil \log_2(p - 1) \rceil$, and
- (2) There is a (proper) subset S of $N(v)$ such that $|S| = \lceil \log_2(p - 1) \rceil$ and $G - S$ is connected.

In what follows, we shall check whether the double-star $DS(m, n)$ is sum-saturable using the two conditions.

Proposition 2.7. For $2 \leq m \leq n$, the double-star $DS(m, n)$ is sum-saturable except $DS(2, 2)$, which is a path P_4 .



Proof. Let u and v be the central vertices of $DS(m, n)$ with degree m and n , respectively. We first check the vertex v because it is the vertex with maximum degree in the double-star $DS(m, n)$. For $n \geq m$, we have $2n - 1 \geq m + n - 1$. One can check that $2^{n-1} \geq 2n - 1$ for $n \geq 4$. Hence, we get the inequality : $2^{n-1} \geq m + n - 1$. So, we have $n - 1 \geq \log_2(m + n - 1)$. It is equivalent that $n - 1 \geq \lceil \log_2(m + n - 1) \rceil$. Hence, the double-star $DS(m, n)$ fits the condition (1). That is,

$$n \geq 1 + \lceil \log_2(m + n - 1) \rceil, \text{ when } n \geq 4.$$

Now we check the condition (2). When we let $S \subseteq N(v) \setminus \{u\}$ with $|S| = \lceil \log_2(m + n - 1) \rceil$, $DS(m, n) - S$ leads $\deg(v)$ to be at least 1. Hence, $DS(m, n) - S$ is also connected. So, $DS(m, n)$ is sum-saturable when $n \geq 4$ and $n \geq m$.

Now, we check the double-star $DS(2, 3)$ (see the left graph in Figure 5). It also satisfies the sufficient condition (1) when we check vertex v . Now, let $S = \{y \mid \deg(y) = 1 \text{ for any}$

vertex $y \in N(v)$. Then, $G - S$ is connected satisfying condition (2) because $|S| = 2$ and $\deg(v) = 3$.

For $DS(3, 3)$, it follows by using the labelings in Figure 5.

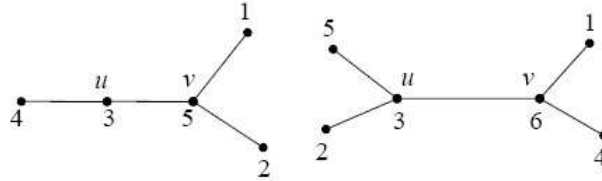


Figure 5: λ -labelings of $DS(2, 3)$, and $DS(3, 3)$.

■

2.2 Palmerworm $PW(n)$

It is worth of noting that we can not use Fink's conditions to show that $DS(3, 3)$ is sum-saturable. Hence, it seems that there exists some sum-saturable graphs in which $\deg(v) \leq \lceil \log_2(|V(G)| - 1) \rceil$ for any $v \in V(G)$. Fortunately, we find a graph called *palmerworm* is sum-saturable, which is illustrated in Figure 6. It is consisted with a path P_n and $n - 2$ vertices which are adjacent with the cut-vertices of P_n . We can check that $\deg(v) \leq \lceil \log_2(|V(G)| - 1) \rceil$ for any v . Here, we denote the palmerworm by $PW(n)$.

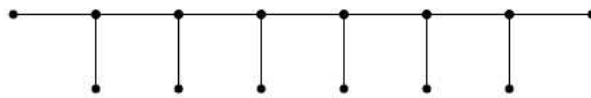


Figure 6: Palmerworm $PW(8)$ containing a path P_8 .

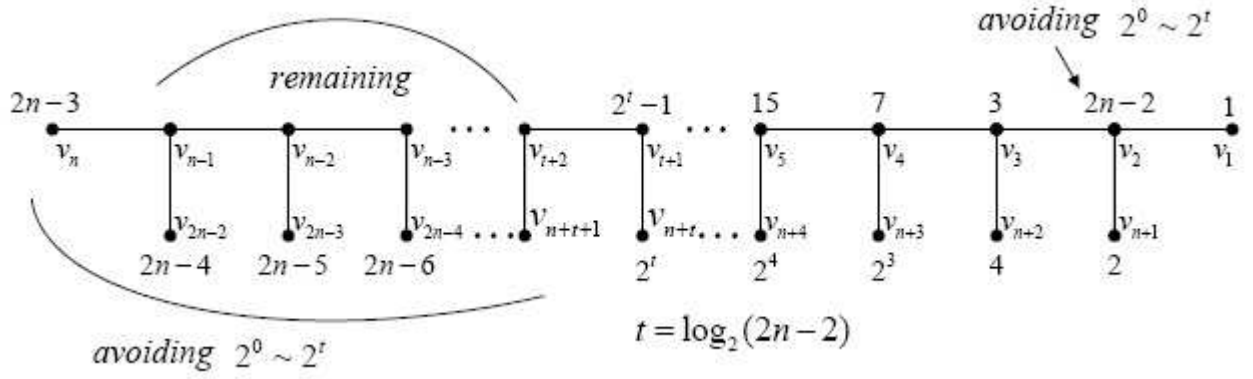


Figure 7: Labeling of Palmerworm $PW(n)$.

Theorem 2.8. *A graph $G = PW(n)$ is sum-saturable.*

Proof. Let $V(G) = \{v_1, v_2, v_3, \dots, v_{2n-2}\}$ be the vertices of $PW(n)$, and $t = \lfloor \log_2(2n - 2) \rfloor$. Then there exists a λ labeling of $PW(n)$ such as Figure 7. First, we label $1, 2, 4, \dots, 2^t$ to the vertices $v_1, v_{n+1}, v_{n+2}, \dots, v_{n+t}$, and $\lambda(v_3) = 1 + 2 = 3$, $\lambda(v_4) = 1 + 2 + 4 = 7$, \dots , $\lambda(v_{t+1}) = 1 + 2 + \dots + 2^{t-1} = 2^t - 1$, respectively. Secondly, the vertices $v_2, v_n, v_{2n-2}, v_{2n-3}, \dots, v_{n+t+1}$ are respectively labeled with $2n - 2, 2n - 3, 2n - 4, 2n - 5, \dots$, until vertex v_{n+t+1} has been labeled; but avoiding $2^0, 2^1, \dots, 2^t$ and the values given previously, i.e. if the same label of some vertex is w , then we give $w - 1$ to it and then going on. At last, the others are labeled with the remaining values randomly. Then we can easily check that $PW(n)$ is sum-saturable. \blacksquare

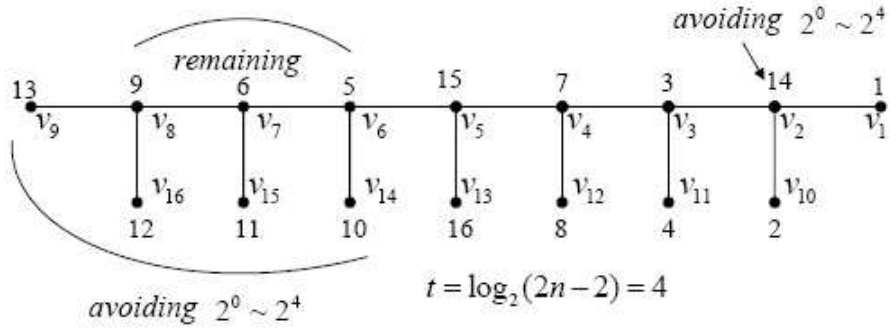


Figure 8: Labeling of Palmerworm $PW(9)$.

Example 2.9. A graph $G = PW(9)$ is sum-saturable, in which $\deg(v) \leq \lceil \log_2 16 \rceil = 4$ for any $v \in G$. Let $V(PW(9)) = \{v_1, v_2, \dots, v_{16}\}$. There exists a λ labeling as illustrated in Figure 8. It is necessary to check for any $k \in [1, \Delta_9 = 136]$, there exists a connected subgraph H such that $\lambda(H) = k$. When $k \in [1, 16]$, the subgraph is a single vertex. Next, when $k \in [14, 17], [17, 24], [24, 39], [39, 70], [44, 85], [50, 102]$ and $[59, 136]$, there still exists a connected subgraph H such that $\lambda(H) = k$. Hence, $PW(9)$ is sum-saturable.

Therefore, to find a necessary and sufficient condition for a graph to be sum-saturable is an interesting open problem.

3 Improved bounds of IC-index of some graphs

First, we consider the IC-index of the path P_n . Since $v(P_n) = n(n+1)/2$, the IC-index of P_n is less or equal to $n(n+1)/2$. And $M(P_n)$ has a lower bound $(2 + \lfloor n/2 \rfloor)(n - \lfloor n/2 \rfloor) + \lfloor n/2 \rfloor - 1$ [9]. The more n becomes, the more difficult it is to find out the bounds of IC-index of path P_n . Hence, we will search for the labels of vertices by using computer. Finally, we have a formulated type of the IC-coloring of path P_n .

3.1 Path P_n

Lemma 3.1. For any $n \in \mathbf{N}$, $M(P_n) \leq n(n+1)/2 - 1$.

Proof. Since P_n is a path of order n , let $V(P_n) = \{v_1, v_2, \dots, v_n\}$, where the vertices v_1, v_2, \dots, v_n are sorted in order. Observe that v_1 and v_n are the only two end-vertices of P_n , and every vertex other than v_1, v_n is a cut-vertex. Suppose f is a maximum IC-coloring of P_n such that $M(P_n) = \mu$. In order to produce $\mu - 1$, we have to let $f(v_1) = 1$ or $f(v_n) = 1$. Without loss of generality, we let $f(v_1) = 1$. And to produced $\mu - 2$, we have three cases : $f(v_2) = 1$, $f(v_n) = 1$ or $f(v_n) = 2$.

If $f(v_2) = 1$ or $f(v_n) = 1$, 1 is repeated, and hence $M(P_n) \leq n(n+1)/2 - 1$.

If $f(v_n) = 2$, we can also produce $\mu - 3$. To cover 3, 4, 5 and $\mu - 4$, $\mu - 5$, we consider the following cases :

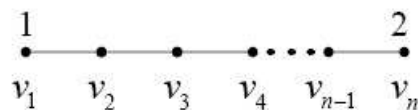


Figure 9: Label 1 and 2 assigned to the end-vertices of P_n

Case 1. $f(v_2) = 3$:

We can produce 3, 4, and $\mu - 4$. In order to produce $\mu - 5$, we have to let $f(v_{n-1}) = 3$ or $f(v_3) = 1$. Since either 3 or 1 is repeated, we have the upper bound $n(n+1)/2 - 1$.

Case 2. $f(v_{n-1}) = 3$:

We can produce 3, 5 and $\mu - 5$. To cover 4 and $\mu - 4$, we will have $f(v_2) = 3$, which is again repeated. Hence, $M(P_n) \leq n(n+1)/2 - 1$.

Case 3. $f(v_j) = 3$ for $j \neq 1$ and n :

We can only produce the value 3. To cover 4, $\mu - 4$ and $\mu - 5$, we will have three choices for the pair v_2 and v_{n-1} , which are (3, 3), (2, 2), (4, 2). Thus, number 1, 2, or 3 will be repeated in the three choices. Hence, we have improved the upper bound of $M(P_n)$. ■

Aided by using computer, we have the following results. Without loss of generality, let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and let the vertices are sorted from the left to the right. And the IC-indices of P_n are at least the sum of the vertices of P_n .

(1) $n = 3 : M(P_3) = 1 + 3 + 2 = 6$

(2) $n = 4 : M(P_4) \geq 9 = 1 + 1 + 4 + 3 = 1 + 3 + 3 + 2$

(3) $n = 5 : M(P_5) \geq 13 = 1 + 1 + 4 + 4 + 3 = 1 + 3 + 1 + 6 + 2 = 1 + 5 + 3 + 2 + 2$

(4) $n = 6 : M(P_6) \geq 17 = 1 + 1 + 1 + 5 + 5 + 4 = 1 + 1 + 4 + 4 + 4 + 3 = 1 + 1 + 6 + 4 + 3 + 2$

(5) $n = 7 : M(P_7) \geq 23 = 1 + 1 + 9 + 4 + 3 + 3 + 2 = 1 + 3 + 6 + 6 + 2 + 3 + 2$

(6) $n = 8 : M(P_8) \geq 29 = 1 + 4 + 4 + 7 + 7 + 3 + 2 + 1$

(7) $n = 9 : M(P_9) \geq 36 = 1 + 4 + 4 + 7 + 7 + 7 + 3 + 2 + 1$

(8) $n = 10 : M(P_{10}) \geq 43 = 1 + 4 + 4 + 7 + 7 + 7 + 7 + 3 + 2 + 1$

(9) $n = 11 : M(P_{11}) \geq 50 = 1 + 4 + 4 + 7 + 7 + 7 + 7 + 7 + 3 + 2 + 1$

(10) $n = 12 : M(P_{12}) \geq 56 = 1 + 1 + 5 + 5 + 9 + 9 + 9 + 9 + 4 + 2 + 1 + 1$

(11) $n = 13 : M(P_{13}) \geq 65 = 1 + 1 + 5 + 5 + 9 + 9 + 9 + 9 + 9 + 4 + 2 + 1 + 1$

(12) $n = 14 : M(P_{14}) \geq 74 = 1 + 1 + 5 + 5 + 9 + 9 + 9 + 9 + 9 + 9 + 4 + 2 + 1 + 1$

(13) $n = 15 : M(P_{15}) \geq 83 = 1 + 1 + 5 + 5 + 9 + 9 + 9 + 9 + 9 + 9 + 9 + 4 + 2 + 1 + 1$

(14) $n = 16 : M(P_{16}) \geq 91 = 1 + 1 + 1 + 6 + 6 + 11 + 11 + 11 + 11 + 11 + 11 + 5$
 $+ 2 + 1 + 1 + 1$

(15) $n = 17 : M(P_{17}) \geq 102 = 1 + 1 + 1 + 6 + 6 + 11 + 11 + 11 + 11 + 11 + 11 +$
 $11 + 5 + 2 + 1 + 1 + 1$

Hence, the lower bounds of the IC-index of a path P_n are $M(P_3) = 6, M(P_4) \geq 9, M(P_5) \geq 13, M(P_6) \geq 17, M(P_7) \geq 23$ for $4 \leq n \leq 7$. We see the lower bounds of $M(P_n)$ for $n \geq 7$ in this thesis is better than that of in paper [9]. From Lemma 3.1 and the results of IC-coloring of P_n presented above, we have the theorem below.

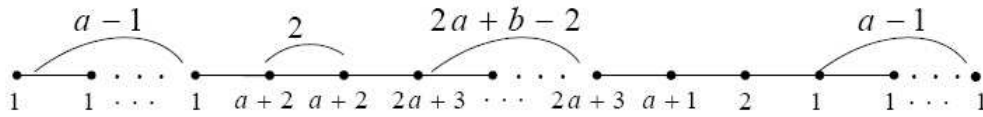


Figure 10: The IC-coloring of the path P_n

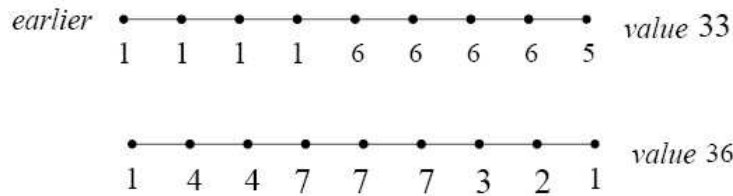


Figure 11: The improved lower bound of path P_9

Theorem 3.2. For any integer $n \geq 8$, if $n = 4a + b, 0 \leq b \leq 3$, where $a, b \in N$, then $4a^2 + 7a + 2ab + 3b - 1 \leq M(P_n) \leq n(n + 1)/2 - 1$.

Proof. The type of the IC-coloring of the path P_n is illustrated in Figure 10. Suppose f is a labeling function that from the left to the right side, the labels are $1, 1, \dots, 1, a + 2, a + 2, 2a + 3, 2a + 3, \dots, 2a + 3, a + 1, 2, 1, 1, \dots, 1$, respectively, where both sides of 1s occur $a - 1$ times, respectively, and $2a + 3$ repeats $2a + b - 2$ times. One can easily check that f is an IC-coloring of P_n . And this is a better and improved lower bound when $n \geq 8$

since $(4a^2 + 7a + 2ab + 3b - 1) - (n^2 + 6n - 4)/4 = a + b(6 - b)/4 \geq a \geq 2$. For example (see Figure 11), $M(P_9) \geq 33$ in [9]. But we have $M(P_9) \geq 36$ here.

3.2 Cycle C_n

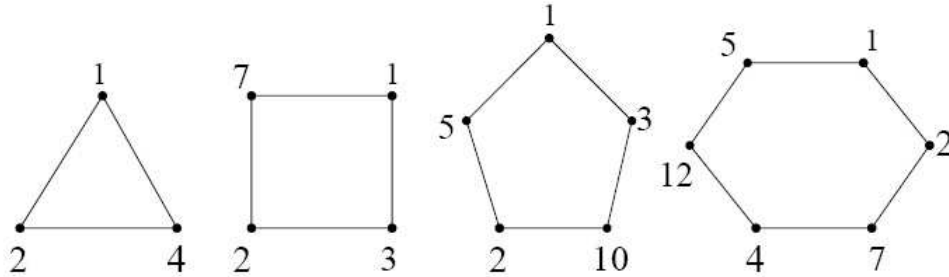


Figure 12: Maximal IC-colorings of C_3, C_4, C_5 and C_6 .

Let's refer to [8], we know that $M(C_n) = n(n - 1) + 1$ for $n = 3, 4, 5, 6, 8, 9$. And we also find $M(C_7) \geq 39$ by using computer. We also come up with a systematic way of finding an IC-coloring of C_n . Since the graph C_n has $n(n - 1) + 1$ connected subgraphs, $M(C_n) \leq n(n - 1) + 1$ by Observation 1.7. Also, Fink[3] has presented a saturating labeling of C_n with sum $n(n + 1)/2$ as the lower bound of $M(C_n)$. Again, with the help of computer, we have the following improved bounds about the cycles C_n for $n \geq 10$. Without loss of generality, let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and let the vertices are sorted clockwise. And suppose that there exists an IC-coloring f of C_n such that $S(f) = f(v_1) + f(v_2) + \dots + f(v_n)$.

(1) $n = 3 : M(C_3) = 7$.

(2) $n = 4 : M(C_4) = 13$.

(3) $n = 5 : M(C_5) = 21$.

(4) $n = 6 : M(C_6) = 31$.

(5) $n = 7 : 43 \geq M(C_7) \geq 39 = 1 + 1 + 7 + 11 + 6 + 10 + 3 = 1 + 3 + 2 + 7 + 8 + 8 + 10$
 $= 1 + 2 + 13 + 7 + 7 + 4 + 5 = 1 + 3 + 14 + 6 + 5 + 2 + 8 = 1 + 4 + 2 + 7 + 3 + 8 + 14 \geq 28$.

Note that 43 and 28 are the known upper and lower bounds of IC-index of $M(C_7)$.

(6) $n = 8 : M(C_8) = 57$

(7) $n = 9 : M(C_9) = 73$

(8) $n = 10 : 91 \geq M(C_{10}) \geq 62 = 1 + 3 + 2 + 10 + 13 + 13 + 13 + 1 + 3 + 3 \geq 10(10 + 1)/2$.

Note that 91 and 55 are the known upper and lower bounds of IC-index of $M(C_{10})$.

(9) $n = 11 : 111 \geq M(C_{11}) \geq 79 = 1 + 3 + 2 + 11 + 14 + 14 + 14 + 14 + 1 + 3 + 2 \geq 11(11 + 1)/2$.

Note that 111 and 66 are the known upper and lower bounds of IC-index of $M(C_{11})$.

(10) $n = 12 : 133 \geq M(C_{12}) \geq 87 = 1 + 3 + 2 + 12 + 15 + 15 + 15 + 15 + 1 + 3 + 3 + 2 \geq 12(12 + 1)/2$. Note that 133 and 78 are the known upper and lower bounds of IC-index of $M(C_{12})$.

From the results listed above, we improve the lower bounds of IC-indices of C_n . For example, the lower bound 87 of $M(C_{12})$ is better than 78 which is the trivial lower bound.

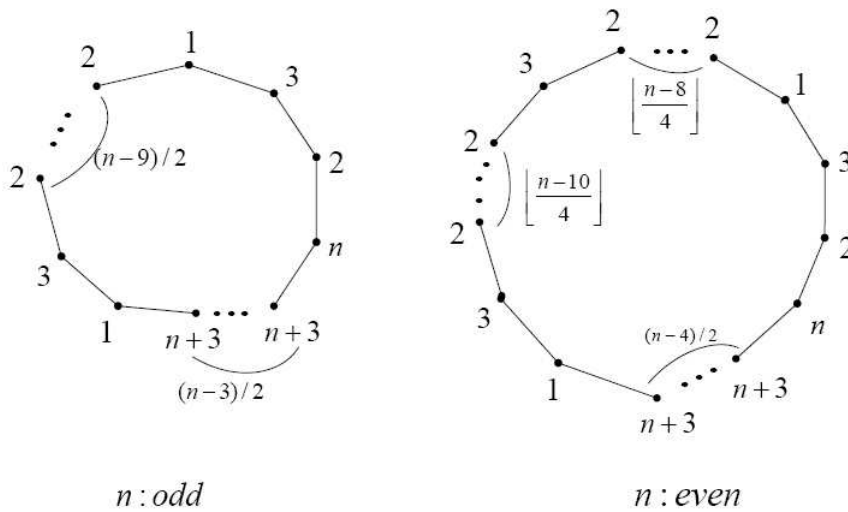


Figure 13: The IC-colorings of the cycle C_n

Theorem 3.3. For $n \geq 10$, the lower bound of $M(C_n)$ is :

(1) If n is odd, $M(C_n) \geq (n^2 + 4n - 7)/2$.

(2) If n is even, $M(C_n) \geq (n^2 + n + 14)/2 + 2\lfloor(n - 9)/2\rfloor$.

Proof.

Case 1. If n is odd, $M(C_n) \geq (n^2 + 4n - 7)/2$, which is obtained by labeling $V(C_n)$ with $1, 3, 2, n, (n + 3), (n + 3), \dots, (n + 3), 1, 3, 2, 2, \dots, 2$ in circular order, where $(n + 3)$ repeated $(n - 3)/2$ times and 2 repeated $(n - 9)/2$ times.

Case 2. If n is even, $M(C_n) \geq (n^2 + n + 14)/2 + 2\lfloor(n - 9)/2\rfloor$, which is obtained by labeling $V(C_n)$ with $1, 3, 2, n, (n + 3), (n + 3), \dots, (n + 3), 1, 3, 2, 2, \dots, 2, 3, 2, 2, \dots, 2$ in circular order, where $(n + 3)$ repeated $(n - 4)/2$ times and 2 repeated $\lfloor(n - 10)/4\rfloor$ and $\lfloor(n - 8)/4\rfloor$ times, respectively.

We can easily check that they are satisfied with the property of IC-coloring. Hence, we have an improved lower bound of IC-coloring of cycle C_n . ■

Though we could not decide the IC-index of C_n , we improve the lower bound of $M(C_n)$ which are better than the trivial lower bounds [9] : $M(C_n) \geq n(n + 1)/2$.

3.3 Double-stars $DS(m, n)$

Review that a double-star has two central vertices plus leaves. (Figure 14 is an example.) These graphs are trees of diameter three. For convenience, we use $DS(m, n)$ to denote a double-star whose two central vertices have degrees m and n , respectively. The following result deserved to be mentioned first.

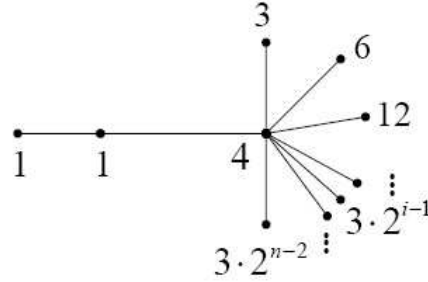


Figure 14: Double-star $DS(2, n)$

Lemma 3.4. [9] For $2 \leq m \leq n$, the IC-index of $DS(m, n)$ is at least $(2^{m-1} + 1)(2^{n-1} + 1)$.

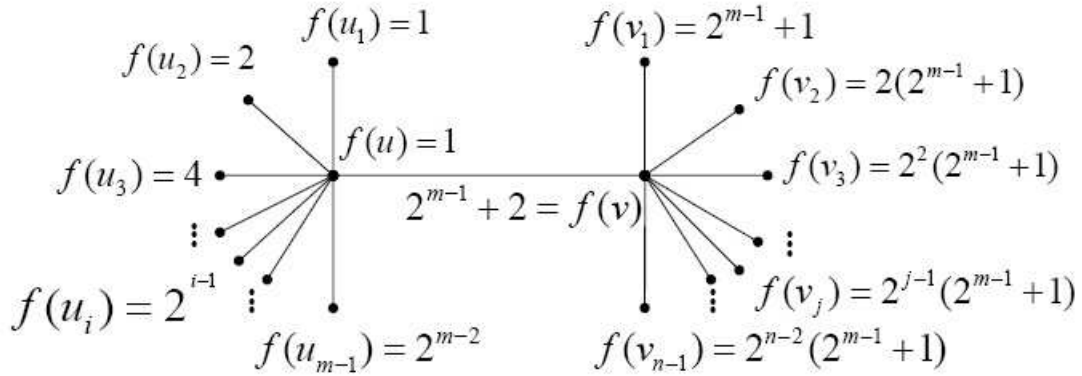


Figure 15: Lower bound of double-star $DS(m, n)$

Lemma 3.5. *If f is an IC-coloring of $DS(m, n)$, then the value of any central vertex must be the sum of some leaves.*

Proof. Let u and v be the only two central vertices of $DS(m, n)$, respectively; $f(u)$ and $f(v)$ be the labels of them. If $f(u) = a$ is not the sum of some leaves, then the subgraph H which satisfies $f_s(H) = S(f) - a$ must be a disconnected subgraph since u is a cut-vertex. Vertex v is the same case. Hence, we finish the lemma.

Proposition 3.6. For $2 \leq m \leq n$, $G = DS(m, n)$ has left and right central vertices, u and v , which are the cut-vertices, and the others are the leaves. Let $L = \{u_1, u_2, \dots, u_{m-1}\}$ and $R = \{v_1, v_2, \dots, v_{n-1}\}$ be the set of left and right leaves, respectively. Without loss of generality, we assume $f(u_1) \leq f(u_2) \leq \dots \leq f(u_{m-1})$ and $f(v_1) \leq f(v_2) \leq \dots \leq f(v_{n-1})$. If f is a maximum IC-coloring of $DS(m, n)$, then there does not exist

(1) $f(u_j) = \sum_{i \in \{1, 2, \dots, j-1\}} f(u_i)$ for $1 \leq i < j \leq m-1$, and

(2) $f(v_l) = \sum_{k \in \{1, 2, \dots, l-1\}} f(v_k)$ for $1 \leq l < k \leq n-1$.

Proof. (1) By Lemma 3.4, $M(DS(m, n)) \geq (2^{m-1} + 1)(2^{n-1} + 1)$. One can easily check that G has $(2^{m-1} + 1)(2^{n-1} + 1) + m + n - 3$ subgraphs.

First, we consider the left leaves. If $j = m-1$, and $i = 1, 2, \dots, m-2$. So, $f(u_{m-1}) = \sum_{i \in \{1, 2, \dots, m-2\}} f(u_i) = f(u_1) + f(u_2) + \dots + f(u_{m-2})$, then there are at least $2^{n-1} + 2$ subgraphs of the same sums. Observe that when we choose less u_i for $u_i \in \{u_1, u_2, \dots, u_{m-2}\}$, it produces more subgraphs of the same sums. For example, if $f(u_{m-1}) = \sum_{i \in \{1, 2, \dots, m-2\}} f(u_i) = f(u_{m-2})$, it produces $1 + 2^{m-3}2^n$ subgraphs of the same sums. Hence, it reduces the known lower bound $(2^{m-1} + 1)(2^{n-1} + 1)$ since $2 \leq m \leq n$. Thus, f is not a maximum IC-coloring of $DS(m, n)$.

The same interpretation for $j = 2, 3, \dots, m-2$

(2) The right leaves has the same result like the left ones .

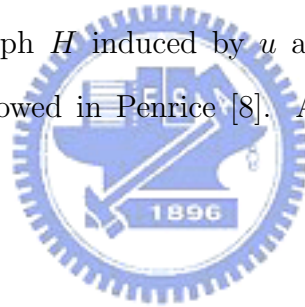
■

3.4 Corollaries of known results

We can decide the lower bounds of certain graphs by the results of the IC-index of star $ST(n)$ (proved by Penrice [8]) and the lower bound of the IC-index of double-star $DS(m, n)$ (proposed by [9]). The following two corollaries were proposed by Ebrahim Salehi, Sin-Min Lee and Mahdad Khatirinejad [9], but did not give proper proofs yet. Hence, we will give the proofs of them. Then, we will have a general lower bounds of IC-index of the certain graphs.

Corollary 3.7. [9] *If $\Delta = \Delta(G)$ is the maximum degree of a connected graph G , then $M(G) \geq 2^\Delta + 2$.*

Proof. Let u be the vertex with the maximum degree Δ of a connected graph G . Also, we use $N(u)$ to denote the set of the neighbors of vertex u . Hence, we have $|N(u)| = \Delta$. Observe that the subgraph H induced by u and $N(u)$ contains a star $ST(\Delta)$. So, we get $M(H) = 2^\Delta + 2$ showed in Penrice [8]. And by Observation 1.7, we have $M(G) \geq M(H) = 2^\Delta + 2$. ■



Corollary 3.8. *For any triangle-free graph G with more than two vertices, we have $M(G) \geq (2^{\Delta-1} + 1)(2^{\delta-1} + 1)$, where Δ and δ denote the maximum and minimum degrees of G , respectively.*

Proof. Let u and v be the vertices with the maximum degree Δ and the minimum degree δ , respectively. According to the adjacency of vertex u and v , we consider the following two cases :

Case 1. u and v are adjacent :

Observe the subgraph induced by the vertices set $N(u) \cup N(v)$ is the double-star $DS(\Delta, \delta)$. Hence, it satisfies the inequality : $M(G) \geq (2^{\Delta-1} + 1)(2^{\delta-1} + 1)$. The bound is very sharp. For example, a graph $ST(n)$ can be viewed as a double-star $DS(1, n)$. And they have the same IC-index $2^n + 2$.

Case 2. u and v are not adjacent :

Since the vertex v with the minimum degree δ is not adjacent with the vertex u , then vertex u must be adjacent with a vertex z such that $\deg(z) \geq \delta$. When we consider the double-star $DS(\Delta, \deg(z))$, it will fit the inequality : $M(G) \geq (2^{\Delta-1} + 1)(2^{\deg(z)-1} + 1)$. And since $\deg(z) \geq \delta$, we get the inequality : $M(G) \geq (2^{\Delta-1} + 1)(2^{\delta-1} + 1)$. ■

3.5 Improved lower bounds of some Graphs

The IC-index of complete bipartite graph $K(m, n)$ is not decided yet so far. But [9] has shown that for any integer $n \geq 2$, the IC-index of the complete bipartite graph $K(2, n)$ is $3 \cdot 2^n + 1$. It may give some information to prove $M(K(m, n))$. Here, we obtain a lower bound of $M(K(m, n))$. Note that when $m = 2$, it is equal to $M(K(2, n)) = 3 \cdot 2^n + 1$. When $m = n = 2$, $K(2, 2)$ is a C_4 , and we know $M(K(2, 2))$ is equal to $M(C_4) = 13$.

Lemma 3.9. *For any two integers $2 \leq m \leq n$, the IC-index of $K(m, n)$ is at least $1 + 2^{n-2}(m + 1)(m + 2)$.*

Proof. Let A and B be the two partite sets of $K(m, n)$, where $A = \{u_1, u_2, \dots, u_m\}$ and $B = \{v_1, v_2, \dots, v_n\}$. There exists an IC-coloring f such that $f(u_i) = 2i$ for $1 \leq i \leq m$, and $f(v_1) = 1, f(v_2) = 2m + 2, f(v_j) = 2^{j-3}\alpha$ for $3 \leq j \leq n$, where $\alpha = f(u_1) + f(u_2) + \dots + f(u_m) + f(v_2) = (m + 1)(m + 2)$. Hence, we can easily check that f is an IC-coloring. And the IC-index of $K(m, n)$ is at least $1 + 2^{n-2}(m + 1)(m + 2)$. ■

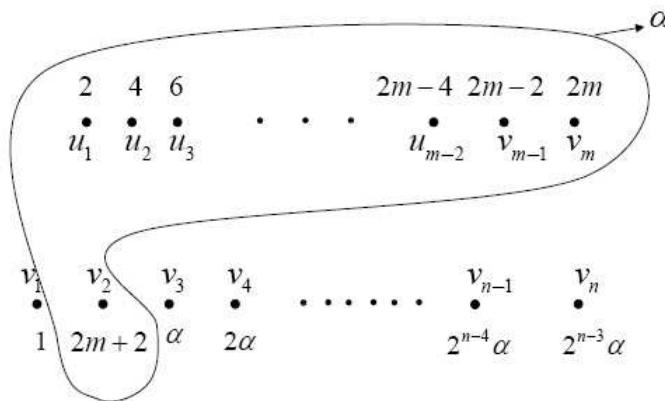


Figure 16: IC-coloring of $K(m, n)$

Lemma 3.10. For any integer $a \leq b \leq c$, the IC-index of the complete tripartite $K(a, b, c)$ is at least $2^c(1 + 2^{b-2}(a + 1)(a + 2))$.

Proof. Let A, B and C be the three partite sets of $K(a, b, c)$, where $A = \{u_1, u_2, \dots, u_a\}$ and $B = \{v_1, v_2, \dots, v_b\}$ and $C = \{w_1, w_2, \dots, w_c\}$. Let f be a labeling such that $f(u_i) = 2i$ for $1 \leq i \leq a$, and $f(v_1) = 1$, $f(v_2) = 2a + 2$, $f(v_j) = 2^{j-3}M$ for $3 \leq j \leq b$, and $f(w_k) = 2^{k-1}N$ for $1 \leq k \leq c$, where $M = f(u_1) + f(u_2) + \dots + f(u_m) + f(v_2) = (a+1)(a+2)$ and $N = f(u_1) + f(u_2) + \dots + f(u_a) + f(v_1) + f(v_2) + \dots + f(v_b) = 1 + 2^{b-2}(a + 1)(a + 2)$. Hence, we can easily check that f is an IC-coloring. And the IC-index of the complete tripartite $M(K(a, b, c))$ is at least $2^c(1 + 2^{b-2}(a + 1)(a + 2))$. ■

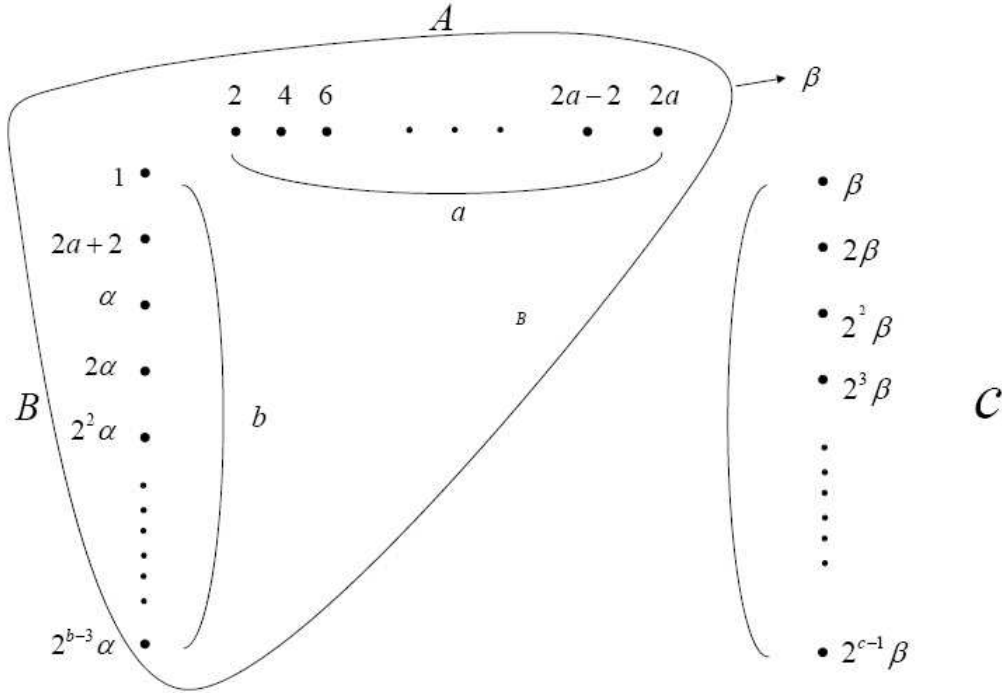


Figure 17: IC-coloring of the complete tripartite $K(a, b, c)$

Proposition 3.11. For any integer $m_1 \leq m_2 \leq \dots \leq m_n$, the IC-index of the complete multipartite $K(m_1, m_2, \dots, m_n)$ is at least $2^{m_n} 2^{m_{n-1}} \dots 2^{m_3} (1 + 2^{m_2-2}(m_1 + 1)(m_1 + 2))$.

Wheels with n spokes, denoted by W_n , are obtained by the join operator $C_n + K_1$. It is shown that the IC-index of W_n satisfies the following inequalities [9] : $2^n + 2 \leq M(W_n) \leq 2^n + n(n - 1) + 1$ for $n \geq 3$. Here, we have an improved lower bound of W_n .

Theorem 3.12. *For any integer $n \geq 3$, $M(W_n) \geq 2^n + 5$.*

Proof. Observe that if $n = 3$, W_3 is K_4 . Hence, $M(W_3)$ is equal to 15, which fits the upper bound of $M(W_n)$ tightly. Thus, we consider the case when $n \geq 4$. Let $V(W_n) = \{v_0, v_1, v_2, \dots, v_n\}$ for $n \geq 4$. Let v_0 be the central vertex, and vertices v_1, v_2, \dots, v_n are ordered clockwise. Then there exists an IC-coloring f such that $f(v_0) = 6$, $f(v_1) = 2$, $f(v_2) = 1$ and $f(v_i) = 2^{i-1}$ for $3 \leq i \leq n$. Thus, we can easily check that f is an IC-coloring, and the IC-index of W_n is at least $2^n + 5$. ■

Note that when $n = 3$, W_3 is K_4 . So $M(W_3) = 2^4 - 1 = 15$, which fits the upper bound $2^n + n(n - 1) + 1$. But we have $M(W_4) \geq 27$, $M(W_5) \geq 46$, and $M(W_6) \geq 76$, which are the better lower bounds than the results obtained in [9] and Theorem 3.12. Note that the values of central vertices of W_n for $n \leq 6$ are different from that of central vertices of W_n when $n \geq 7$.

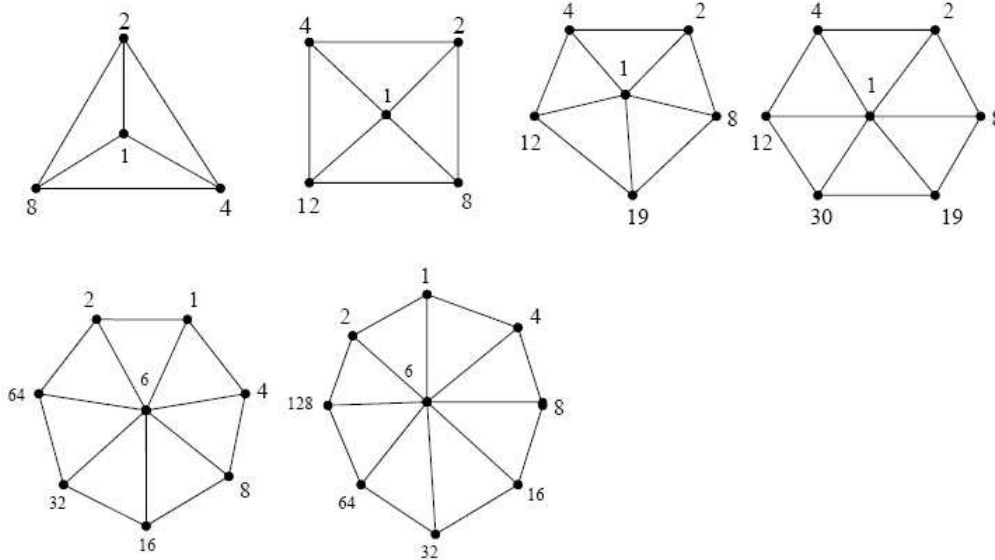


Figure 18: The central labels are 1 when $n \leq 6$ and 6 when $n \geq 7$.

4 Conclusion

From the study of the Stamp Problem, we have noticed that the most difficulty part of obtaining $M(G)$, the IC-index, is knowing the exact upper bound. Even we can figure out the number of distinct connected subgraphs of G , we have no idea whether some distinct connected subgraphs will come out with the same value from a labeling due to the structure of the graph G . Double-star is one of such graphs. So, to determine the IC-index of a general graph is not going to be an easy work. It's left a lot of works to do. But, certainly, we should be able to do a better job in the near future after knowing more inside of this topic.



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