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傅利葉係數，黎阿普諾夫指數，不變測度及渾沌

Fourier Coefficients, Lyapunov Exponents, Invariant Measures and Chaos

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摘　　要

長久以來，在物理及工程上，常利用對一個複雜且不可預測的信號作光譜分析來判斷此信號是否渾沌。首先將此現象做數學分析的是陳鞏老師等人。他們是希望尋求一種關於渾沌動態系統以及傅利葉係數之間的關係。陳鞏老師等人找到了許多關於一個系統做 n 次疊代之後的傅利葉係數，可以使得這個系統的拓樸熵大於零的充分條件。在這篇論文當中，我們創新出一個針對定義在一個區間的函數，傅利葉係數，黎阿普諾夫指數和不變測度的關係。尤其我們是針對一個定義在馬可夫分割上的片段線性函數以及二次函數來討論這三種特徵量。

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ABSTRACT

A complex and unpredictable frequency spectrum of a signal has long been seen in physics and engineering as an indication of a chaotic signal. The first step to understand such phenomenon mathematically was taken up by Chen, Hsu, Huang and Roque-Sol. In particular, they look for possible connections between chaotic dynamical systems and the behavior of its Fourier coefficients. Among other things, they found variety of sufficient conditions on the Fourier coefficients of the n-th iterate f^n of an interval map f, for which the topological entropy of f is positive. In this thesis, we explore the relationship between the Fourier coefficients of an interval map and its Lyapunov exponent and invariant measure. Specifically, the relationships between those three quantities of two family of interval maps, piecewise linear maps admitting a Markov partition and quadratic family, are considered.

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首先感謝我的指導教授莊重老師，從一年級修老師的課開始，就給了我很多指導，讓第一次接觸到動態系統的我，對這門領域有了濃厚的興趣。後來進入老師門下，在做研究時碰到問題，老師總會給我幫助 ； 在每週定時的討論時，碰到看似不能繼續的困境時，老師也總可以為我想出柳暗花明的一條路，真的很謝謝老師這將近兩年來的照顧。

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目　錄

1. Introduction

Three common tasks are of great interest in a signal analysis. The first one is the elimination of a high frequency noise. It is usually done by first expressing f as a trigonometric series

$$f(t) = a_0 + \sum_k a_k \cos kt + b_k \sin kt$$

and then set the high frequency-coefficients (the a_k and b_k for large k) equal to zero. The second one is data compression, the idea here is to send a signal in such a way it requires minimal data transmission. This is done by expressing f as a trigonometric series, as above, and then send only those coefficients $a_k's$ and $b_k's$ that are greater (in absolute value) than a particular tolerance. The third is to decide if the signal is chaotic. This is usually done by "seeing" its frequency domain. If it is "complex" and "unpredictable", then it is an indication of a chaotic signal. Therefore, the knowledge of Fourier coefficients can give enough information to understand and control the main components of a given signal. Such concept was first established in the work of Chen, Hsu, Huang and Roque-Sol[4]. Among other things, they found variety of sufficient conditions on the Fourier coefficients of the n-th iterate f^n of an interval map f, for which the topological entropy of f is positive. For completeness, we record one of their main results in the following.

Theorem 1.1. (Main Theorem 1 of [4]) Let $f : [0,1] \to [0,1]$ be a C^0 function such that f has finitely many extremal points and there exists an integer-valued function $\phi : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ such that ϕ grows exponentially:

$$\lim_{n\to\infty} \frac{1}{n} \log \phi(n) \geq \alpha > 0, \text{ for some } \alpha > 0$$

and that

$$\lim_{n\to\infty} \frac{1}{n} \log |\phi(n) a^n_{\pm\phi(n)}| > 0, \text{ where } a^n_k \text{ is the } k^{th} \text{ Fourier coefficient of } f^n$$

then

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} \log V_I(f^n) \geq \alpha' > 0, \text{ for some } \alpha' > 0.$$

Consequently, f is chaotic in the sense of Li-Yorke.

However, in practical, other than a few selective cases, it is generally difficult to compute the Fourier coefficients of the n-th iterate of f^n. Moreover, the positivity of the topological entropy of a noninvertible map f, as often the cases in one-dimensional map, does not guarantee the chaotic behavior of f on an invariant set for which its measure is positive. Consider the quadratic map $f(x) = \mu x(1-x)$ on $[0, 1]$. Clearly, f has a large window begging near $\mu = 3.839$(see e.g., [11]). For those $\mu's$, the topological entropy of f is positive for having a periodic point of period s, where s is not of the form $s = 2^n$(see e.g.,[5]). While f has an attracting period three orbits for which its attracting set has a measure of 1. Thus, no computer can pick up the chaotic behavior of such f. Inspired by their work [4], the purpose of this work is to explore the relationship between the Fourier coefficients of an interval map and its Lyapunov exponent and invariant measure. The thread to connect three quantities is through Rokhlin formula(see Theorem 2.2.3). Moreover, due to the difficulty in getting the (absolutely continuous) invariant measures, we restrict to two family of interval maps : piecewise linear maps admitting a Markov partition and quadratic. In both cases, some sufficient conditions on their corresponding Fourier coefficients are obtained to ensure the positivity of their Lyapunov exponents. An example of the piecewise linear map admitting a Markov partition is given as an application to such results.

We conclude this introductory section by mentioning the organization of the thesis. Some definitions and basic results from Fourier theory and ergodic theory are recorded in Section 2. The main results are recorded in Section 3.

2. Preliminaries

2.1. **Fourier Series.**

We begin with giving some definitions and basic results, which can be found in [6, 7, 10].

A function is said to be T-periodic if it is defined for all real x and if there is some positive number T such that $f(x+T) = f(x)$ for all x.

Definition 2.1.1. Let f be an integrable function on $[a, b]$ with $f(a) = f(b)$. The k^{th} Fourier coefficient of f is defined by

$$a_k = \frac{1}{b-a}\int_a^b f(x)e^{\frac{-2\pi ikx}{b-a}}\,dx. \tag{2.1.1}$$

The Fourier series of f is given formally by

$$S(f)(x) = \sum_{k=-\infty}^{\infty} a_k e^{\frac{2\pi ikx}{b-a}}. \tag{2.1.2}$$

Theorem 2.1.1. (i) If f is continuous on $[a, b]$ and $a_k = 0$ for all $k \in Z$, then

$f = 0$. (ii) The Fourier series can be integrated term by term. That is, for all x we have $\int_0^x f(t)dt = \sum_{k=-\infty}^{\infty} a_k \int_0^x e^{2\pi ikt}dt$, the intergrated series being uniformly convergent on every interval, even if the Fourier series of f diverges.

Theorem 2.1.2. Suppose that f is a continuous function on $[a, b]$ with $f(a) = f(b)$ and that the Fourier series of f is absolutely convergent, $\sum_{k=-\infty}^{\infty} |a_k| < \infty$. Then the Fourier series converges uniformly to f, that is, $\lim_{K\to\infty} \sum_{k=-K}^{K} a_k e^{2\pi ikx} = f(x)$ uniformly in x.

Theorem 2.1.3. Let

$$B_1(x) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{k}. \tag{2.1.3}$$

(B_1 is called the Bernoulli function of order 1.)
Then $B_1(x) = x - [x] - \frac{1}{2}$ if x is not an integer.($[x]$ is the greatest integer $\leq x$.)

2.2. **Ergodic Theory.**

In this subsection, we will give definition of measure-preserving transformations and some of their basic properties(see e.g., [2, 3, 8, 13]).

Definition 2.2.1. Suppose (X_1, β_1, m_1), (X_2, β_2, m_2) are probability spaces. (a) A transformation $T : X_1 \to X_2$ is measurable if $T^{-1}(\beta_2) \subset \beta_1$(i.e., if $B_2 \in \beta_2$ then $T^{-1}(B_2) \in \beta_1$) (b) A transformation $T : X_1 \to X_2$ is measure-preserving if T is measurable and $m_1(T^{-1}(B_2)) = m_2(B_2)$ for any $B_2 \in \beta_2$.

Definition 2.2.2. Let (X, β, m) be a probability space. A measure-preserving transformation T of (X, β, m) is called ergodic if the only members B of β with $T^{-1}(B) = B$ satisfy $m(B) = 0$ or $m(B) = 1$.

The first major result in ergodic theory was proved in 1931 by G.D. Birkhoff.

Theorem 2.2.1. **(Birkhoff Ergodic Theorem)** Let (X, β, m) be a σ-finite space. Suppose $T : (X, \beta, m) \to (X, \beta, m)$ is measure-preserving and $f \in L^1(m)$. Then

(a) $\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x))$ converges a.e. to a function $f^* \in L^1(m)$.

(b) $f^* \circ T = f^*$ a.e.

(c) If $m(X) < \infty$, then $\int f^* dm = \int f dm$.
(d) If T is ergodic, then $f^* = \dfrac{1}{m(X)} \displaystyle\int f dm$ a.e.

In particular, if (X, β, m) is a probability space and T is ergodic we have

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} f(T^j(x))}{n} = \int f dm \text{ a.e., for all } f \in L^1(m).$$

Let $M(X)$ be the collection of all probability measures defined on the measurable space $(X, \beta(X))$, where $\beta(X)$ is the smallest σ-algebra containing all open subsets of X and the smallest σ-algebra containing all closed subsets of X. Let

$$M(X,T) = \{\mu \in M(X) | \mu(T^{-1}(B)) = \mu(B) \text{ for all } B \in \beta(X)\}. \qquad (2.2.1)$$

Theorem 2.2.2. If $T : X \to X$ is a continuous transformation of a compact space, then $M(X,T)$ is non-empty.

Definition 2.2.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function. For each point x_0 define the Lyapunov exponent of x_0, $\lambda(x_0)$, as follows:

$$\begin{aligned} \lambda(x_0) &= \overline{\lim_{n\to\infty}} \frac{1}{n} \log(|(f^n)'(x_0)|) \\ &= \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=0}^{n-1} \log|f'(x_j)|, \text{ where } x_j = f^j(x_0). \end{aligned} \qquad (2.2.2)$$

The following theorem is direct consequences of Theorem 2.2.1, Theorem 2.2.2, and (2.2.2).

Theorem 2.2.3. (Rokhlin Formula) Let $f : I = [0,1] \to I$ be continuous. Then for any $\mu \in M([0,1], f)$, we have that

$$\int_0^1 \lambda(x) d\mu(x) = \int_0^1 \log(|f'(x)|) d\mu(x). \qquad (2.2.3)$$

2.3. **Invariant Measures.**

In this subsection, we introduce the notions of the Frobenius-Perron Operator and (absolutely continuous) invariant measures (see [1, 2, 3], for more details).

Let $f : I \to I$, $A \subset I$, then we have that

$$\text{Prob}\{x \in A\} = \int_A \psi d\lambda,$$

where λ is the normalized Lebesgue measure on I, ψ is probability density function of x. If we want to know the probability of $f(x)$ lies in A, we must know the probability density function of $f(x)$. We denote the probability density function by $\phi(x)$. Thus, we have the following:

$$\begin{aligned}\text{Prob}\{f(x) \in A\} &= \text{Prob}\{x \in f^{-1}(A)\} \\ &= \int_{f^{-1}A} \psi d\lambda \\ &= \int_A \phi d\lambda.\end{aligned}$$

For f is nonsingular (i.e. $\lambda(A) = 0 \Rightarrow \lambda(f^{-1}(A)) = 0$), we construct an new measure μ defined by $\mu(A) = \int_{f^{-1}A} \psi d\lambda$. Hence, $\mu \ll \lambda$. Then, by the Radon-Nykodym Theorem, we can ensure the existence of ϕ.

Definition 2.3.1. **(Frobenius-Perron Operator)** Suppose $\psi(x) \in L^1$ and f defined on I is nonsingular. Then defined $P_f : L^1 \to L^1$ by

$$P_f\psi(x) = \frac{d}{dx}\int_{f^{-1}(A)} \psi d\lambda, \text{ for any measurable set} A \in I$$

.

By the following theorem, we can construct an invariant measure for f.

Theorem 2.3.1. $P_f\psi^* = \psi^*$ a.e., if and only if the measure $\mu(A) = \int \psi^* d\lambda$ is $f-$invariant, i.e., if and only if $\mu(f^{-1}(A)) = \mu(A)$ for all measurable sets A.

We next give a certain type of continuous piecewise linear maps that possesses an absolutely continuous invariant measure. Let $f : I = [0, 1] \to I$ is a continuous piecewise linear map such that

$$f'(x) = \beta_j,\ x \in I_j = [b_{j-1}, b_j],\ j = 1, ..., m.$$

$$\text{Here } 0 = b_0 < b_1 < b_2 < ... < b_{m-1} < b_m = 1. \tag{2.3.1}$$

For such f, Frobenius-Perron Operator , $P_f\psi(x)$, can be reduced to

$$P_f\psi = \sum_{j=1}^{m} \frac{\psi({f_j}^{-1}(x))}{|f'({f_j}^{-1}(x))|}\chi_{f(b_{j-1},b_j)}(x).$$

Let $B = \{b_0, b_1, ..., b_m\}$ be the set of endpoints of the interval I_j.

Definition 2.3.2. The intervals I_j, $j = 1, ..., m$, form a Markov partition if $f(B) \subset B$.

If $\{I_j\}$ forms a Markov partition. Then we can introduce a topological Markov chain corresponding to a graph G that has n vertices and edges joining vertices j and k if and only if $f(I_j) \supset int(I_k)$, interior of I_k. Denote by A the corresponding transition matrix. The following theorem gives an explicit construction of the density function of the absolutely continuous invariant measure admitted by a continuous piecewise linear map on a Markov partition.

Theorem 2.3.2. Suppose f is a continuous piecewise linear map satisfying (2.2.4) and admits a Markov partition. Let the matrix

$$B = diag(\frac{1}{|\beta_1|}, \frac{1}{|\beta_2|}, ..., \frac{1}{|\beta_m|})A. \tag{2.3.2}$$

Then the following hold true.
(i) $\lambda = 1$ is a left eigenvalue of B and its corresponding left eigenvector has nonnegative components.
(ii) Let $d = (d_1, d_2, ..., d_m)$ be the normalized left eigenvector, i.e., $\sum_{j=1}^{m} d_j \cdot length(I_j) = 1$, of left eigenvalue 1. Then $p(x) = d_j$, $x \in I_j$, is a density function of the absolutely continuous invariant measure admitted by f.

Theorem 2.3.3. (Tsujii, [11]) For any $\beta < 2$, there exists a subset E, containing 4, of parameter $[0, 4]$ with the properties:
$(i)Leb([4 - \epsilon, 4] - E) < \epsilon^{\beta}$ for sufficiently small $\epsilon > 0$,
(ii)The quadratic map $f_\mu(x) = \mu x(1-x)$ admits an absolutely continuous invariant measure m_μ, where $\mu \in E$, and
$(iii)m_\mu$ converges to the measure m_4 as a tends to 4 on the set E.

3. Main Results

We start with considering the following class of functions.

Definition 3.1. The notation $a_k = O(\frac{1}{|k^l|})$ as $|k| \to \infty$ means that there exist positive constants c_1 and c_2 such that

$$\frac{c_1}{|k^l|} \leq a_k \leq \frac{c_2}{|k^l|}, \text{ for all } k.$$

Definition 3.2. Let f be integrable on $[0, 1]$. Then f is said to be of the class $O(\frac{1}{k^2})$ if $a_k = O(\frac{1}{k^2})$ as $|k| \to \infty$. Here a_k is the k^{th} Fourier coefficient of f.

Proposition 3.1. Let $f \in C^2[0, 1]$, $f(0) = f(1)$, $f''(0) = f''(1)$ and $f'(0) \neq f'(1)$. Moreover, $f'(1) - f'(0) \neq \int_0^1 f''(x)e^{-2\pi ikx}dx$. Then f is of the class $O(\frac{1}{k^2})$.

Proof. The estimate on the Fourier coefficients is proved by integrating by parts twice for $k \neq 0$. Specifically, we have that

$$\begin{aligned} a_k &= \int_0^1 f(x)e^{-2\pi ikx}dx \\ &= \frac{f(x)e^{-2\pi ikx}}{-2\pi ik}\Big|_0^1 + \frac{1}{2\pi ik}\int_0^1 f'(x)e^{-2\pi ikx}dx \\ &= \frac{f'(x)e^{-2\pi ikx}}{4\pi^2k^2}\Big|_0^1 - \frac{1}{4\pi^2k^2}\int_0^1 f''(x)e^{-2\pi ikx}dx \\ &= \frac{f'(1) - f'(0)}{4\pi^2k^2} - \frac{1}{4\pi^2k^2}\int_0^1 f''(x)e^{-2\pi ikx}dx. \end{aligned}$$

Since

$$|\int_0^1 f''(x)e^{-2\pi ikx}dx| \leq \int_0^1 |f''(x)|dx \leq C,$$

we have

$$\frac{|f'(1) - f'(0) - \int_0^1 f''(x)e^{-2\pi ikx}dx|}{4\pi^2k^2} \leq |a_k| \leq \frac{|f'(1) - f'(0)| + C}{4\pi^2k^2}.$$

□

Remark 3.1. The quadratic map $f_\mu(x) = \mu x(1 - x)$ is of the class $O(\frac{1}{k^2})$.

Proposition 3.2. Let f be a continuous piecewise linear function such that (2.2.4) holds. Assume, further, that $f(0) = f(1)$. Then f is of the class $O(\frac{1}{k^2})$.

Proof. Assume that

$$f(x) = \beta_j(x) + c_j,\ x \in I_j,\ j = 1, 2,\ \ldots,\ m.$$

Then

$$\begin{aligned}
a_k &= \int_0^1 f(x)e^{-2\pi ikx}dx = \sum_{j=1}^{m}\int_{b_{j-1}}^{b_j}(\beta_j x + c_j)e^{-2\pi ikx}dx \\
&= -\frac{1}{2\pi ik}\sum_{j=1}^{m}((\beta_j x + c_j)e^{-2\pi ikx}|_{b_{j-1}}^{b_j}) + \frac{1}{4\pi^2k^2}\sum_{j=1}^{m}\beta_j(e^{-2\pi ikb_j} - e^{-2\pi ikb_{j-1}}) \\
&= \frac{1}{4\pi^2k^2}\sum_{j=1}^{m}\beta_j(e^{-2\pi ikb_j} - e^{-2\pi ikb_{j-1}}) \\
&= \frac{1}{4\pi^2k^2}\sum_{j=0}^{m}(\beta_j - \beta_{j+1})e^{-2\pi ikb_j},\ \text{where } \beta_0 = \beta_{m+1} = 0. \qquad (3.1)
\end{aligned}$$

The first term in the second equality above vanishes since f is continuous and $f(0) = f(1)$. □

Proposition 3.3. Let f be continuous on $[0, 1]$ with $f(0) = f(1)$ and be of the class $O(\frac{1}{k^2})$. Let $|k| \geq 1$. Suppose that the k^{th} Fourier coefficient a_k of f is equal to

$$\sum_{j=0}^{m}\frac{\widetilde{a_j}}{4\pi^2k^2}e^{-2\pi ikb_j}, \qquad (3.2)$$

where $\widetilde{a_j}, j = 0, 1, ..., m,$ are some constant numbers and $0 = b_0 < b_1 < b_2 < ... < b_{m-1} < b_m = 1$. Then the term by term differentiation of the Fourier series $\sum_{k=-\infty}^{\infty} a_k e^{2\pi ikx}$ of f converges and equals to $\sum_{j=0}^{m}\widetilde{a_j}(x - b_j - [x - b_j] - \frac{1}{2})$ except possibly at $x = b_j$, $j = 0, ..., m$.

Proof. Using Theorem 2.1.2, we see that $f(x) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikx}$. Now, consider that the series

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} 2\pi ika_k e^{2\pi ikx} &= \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{i}{2\pi k}(\sum_{j=0}^{m}\widetilde{a_j}e^{-2\pi ikb_j})e^{2\pi ikx} \\
&= \sum_{j=0}^{m}\frac{i\widetilde{a_j}}{2\pi}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{e^{2\pi ik(x-b_j)}}{k} \\
&= \sum_{j=0}^{m}\widetilde{a_j}(-\frac{1}{\pi})\sum_{k=1}^{\infty}\frac{\sin 2\pi k(x-b_j)}{k} \\
&= \sum_{j=0}^{m}\widetilde{a_j}(x - b_j - [x - b_j] - \frac{1}{2}). \qquad (3.3)
\end{aligned}$$

□

Remark 3.2. If f satisfies the assumption in (2.2.4). Then the Fourier coefficient a_k of f can be written as in (3.2) with $\widetilde{a_j} = (\beta_j - \beta_{j+1})$, $j = 0, 1, ..., m$. Here $\beta_0 = \beta_{m+1} = 0$. We shall call $\widetilde{a_j}$ the normalized j-mode of the the k^{th} Fourier coefficient a_k. Note that $\widetilde{a_j}$ is independent of k. Should no confusion arise, we shall just call $\widetilde{a_j}$ the normalized j-mode of the Fourier series of f.

Proposition 3.4. (i) If $f(x) = \mu x(1-x) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikx}$, then $f'(x) = \sum_{k=-\infty}^{\infty} 2\pi ika_k e^{2\pi ikx}$ except possibly at $x = 0$ or 1. (ii) Suppose f is a continuous piecewise linear function such that (2.2.4) holds and that $f(0) = f(1)$. Let $f(x) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikx}$. Then $f'(x) = \sum_{k=-\infty}^{\infty} 2\pi ika_k e^{2\pi ikx}$ except possibly at $\{b_i\}_{i=0}^{m}$.

Proof. Let $f(x) = \mu x(1-x)$. It follows from (3.2) and (3.3) that for ant $|k| \geq 1$

$$\widetilde{a_j} = -\mu, \; j = 0 \text{ or } 1. \tag{3.4}$$

Thus,

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} 2\pi ika_k e^{2\pi ikx} &= \sum_{j=0}^{1} \widetilde{a_j}(x - b_j - [x - b_j] - \frac{1}{2}) \\
&= -\mu(x - 0 - [x-0] - \frac{1}{2}) + (-\mu)(x - 1 - [x-1] - \frac{1}{2}) \\
&= -2\mu(x - \frac{1}{2}) \\
&= -2\mu x + \mu = f'(x).
\end{aligned}$$

The second equality holds except possibly at $x = 0$ or 1. If $f(x)$ is a continuous, piecewise linear function as assumed, then it follows from (3.1) and (3.2) that

$$\widetilde{a_k} = (\beta_k - \beta_{k+1}), k = 0, 1, ..., m. \tag{3.5}$$

Here $\beta_0 = \beta_{m+1} = 0$. Using (3.2) and (3.3), we get, except possibly at $\{b_j\}_{j=0}^m$, that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} 2\pi i k a_k e^{2\pi i k x} &= \sum_{j=0}^{m} (\beta_j - \beta_{j+1})(x - b_j - [x - b_j] - \frac{1}{2}) \\
&= \sum_{j=0}^{m} ((\beta_{j+1} - \beta_j) b_j + (\beta_{j+1} - \beta_j)[x - b_j]) \\
&= \sum_{j=1}^{m} (\beta_j (b_{j-1} - b_j) + \beta_j ([x - b_{j-1}] - [x - b_j])) \\
&= \sum_{j=1}^{m} ((f(b_{j-1}) - f(b_j)) + \beta_j \chi_{(b_{j-1}, b_j)}) \\
&= \sum_{j=1}^{m} \beta_j \chi_{(b_{j-1}, b_j)} \\
&= f'(x). \qquad (3.6)
\end{aligned}$$

Here $\chi_{(b_{j-1}, b_j)}$ is the characteristic function of (b_{j-1}, b_j). □

We are now ready to state our first main result.

Theorem 3.1. Let f be a continuous piecewise linear map defined on a Markov partition $I_i = [b_{i-1}, b_i]$, $i = 1, ..., m$, see (2.2.4). We assume further that $f(0) = f(1)$. Let $d = (d_1, d_2, ..., d_m)^T$ be the normalized left eigenvector of B, as given in (2.2.5), associated with the left eigenvalue 1. If the normalized s-mode $\widetilde{a_s}, s = 0, 1, ..., m$, of the Fourier series of f satisfy

$$\sum_{j=1}^{m} d_j \log(|\sum_{s=0}^{j-1} \widetilde{a_s}|) > 0, \qquad (3.7)$$

then there exists an invariant set $B \subset [0, 1]$ such that the measure $m(B) > 0$ and the Lyapunov exponent of $f_{|B}$ is positive. If, in addition, f is ergodic on $[0, 1]$, then the Lyapunov exponent of $f_{|I}$ is positive.

Proof. Using (3.5), we get that

$$\beta_j = -\sum_{s=0}^{j-1} \widetilde{a_s}. \qquad (3.8)$$

It then follows from (3.6), (3.7), (2.2.3) and Theorem 2.2.4 that $\int_0^1 \lambda(x) d\mu(x) = \sum_{j=1}^{m} d_j \log(|\sum_{s=0}^{j-1} \widetilde{a_s}|) > 0$. If f is ergodic, then $\lambda(x) \equiv \lambda$ a.e. and $\lambda > 0$. If f is not ergodic, then there exists a $B \subset I$ with $f^{-1}(B) = B$, $f^{-1}(I \setminus B) = I \setminus B$ and $1 > m(B) > 0$. Thus

$$\int_0^1 \lambda(x) d\mu(x) = \int_B \lambda(x) d\mu(x) + \int_{I \setminus B} \lambda(x) d\mu(x) > 0.$$

Without lost of generality, we may assume that $\int_B \lambda(x) d\mu(x) > 0$. Noting that $f_{|B}$ is ergodic on B, we conclude that $\lambda(x) \equiv \lambda$ a.e. on B and, hence, $\lambda > 0$. □

As an application to Theorem3.1, we consider an example of continuous piecewise linear map on a Markov partition. Specifically, let $\{b_i\}_{i=1}^4$ be such that $0 = b_0 < b_1 < b_2 < b_3 < b_4 = 1$ with

$$f(b_0) = b_2,\ f(b_1) = b_3,\ f(b_2) = b_4,\ f(b_3) = b_3 \text{ and } f(b_4) = 0. \tag{3.9}$$

Clearly, f admits a Markov partition on $I_i = [b_{i-1}, b_i]$, $i = 1, 2, 3, 4$.

Note that for such f, $f(0) \neq f(1)$. To apply our earlier results, we may extend f to an even periodic function $f_1(t)$ of period T=2. That is, $f_1(x) = f(-x)$, for $-1 \leq x \leq 0$ and $f_1(x) = f(x)$, for $0 \leq x \leq 1$. Then (3.1) becomes

$$\begin{aligned}
a_k &= \frac{1}{2}\int_{-1}^{1} f_1(x) e^{-\pi i k x} dx \\
&= \frac{1}{2}\left(\int_0^1 f(x) e^{\pi i k x} dx + \int_0^1 f(x) e^{-\pi i k x} dx\right) \\
&= \frac{1}{2\pi i k}\left(\sum_{j=1}^{m} (\beta_j x + c_j) e^{\pi i k x}\Big|_{b_{j-1}}^{b_j}\right) + \frac{1}{2\pi^2 k^2}\sum_{j=1}^{m} \beta_j (e^{\pi i k b_j} - e^{\pi i k b_{j-1}}) \\
&\quad - \frac{1}{2\pi i k}\left(\sum_{j=1}^{m} (\beta_j x + c_j) e^{-\pi i k x}\Big|_{b_{j-1}}^{b_j}\right) + \frac{1}{2\pi^2 k^2}\sum_{j=1}^{m} \beta_j (e^{-\pi i k b_j} - e^{-\pi i k b_{j-1}}) \\
&= \frac{1}{2\pi^2 k^2}\left(\sum_{j=1}^{m} \beta_j (e^{\pi i k b_j} - e^{\pi i k b_{j-1}}) + \beta_j (e^{-\pi i k b_j} - e^{-\pi i k b_{j-1}})\right) \\
&= \frac{1}{2\pi^2 k^2}\left(\sum_{j=0}^{m} (\beta_j - \beta_{j+1})(e^{\pi i k b_j} + e^{-\pi i k b_j})\right) \\
&=: \frac{1}{2\pi^2 k^2}\left(\sum_{j=0}^{m} \widetilde{a_j} (e^{\pi i k b_j} + e^{-\pi i k b_j})\right).
\end{aligned} \tag{3.10}$$

For such f, $\widetilde{a_j}$ is called the normalized $|j|$-mode of the k^{th} Fourier coefficient a_k, where $-m \leq j \leq m$. With such modification, we have the following corollary.

Corollary 3.1. Suppose the assumption that $f(0) = f(1)$ is dropped in Theorem 3.1. Then the assertion of Theorem 3.1 still holds true.

Proposition 3.5. Let f be a continuous piecewise linear function satisfying (3.9). Then the density function $p(x) = d_i$, $x \in I_i$, of the absolutely continuous invariant measure admitted by f is given by

$$d = (d_1, d_2, d_3, d_4) = K \cdot (\frac{1-b_3}{b_3}, \frac{1-b_3}{b_3}, \frac{(b_1-b_2+b_3)(1-b_3)}{b_3(b_3-b_2)}, 1),$$

where $K = \dfrac{b_3(b_3-b_2)}{-2{b_3}^2 + 2b_2b_3 - b_1b_3 + 3b_3 - 3b_2 + b_1}$ is a positive number.

Proof. Since f admits a Markov partition, we can apply Theorem 2.2.4. After some direct calculation, we can get that the slopes β_i, $i = 1,2,3,4$, of f on I_i are given by

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (\frac{b_3-b_2}{b_1}, \frac{1-b_3}{b_2-b_1}, \frac{b_3-1}{b_3-b_2}, \frac{-b_3}{1-b_3}).$$

For such f, we find that $f(I_1) \supset I_3$, $f(I_2) \supset I_4$, $f(I_3) \supset I_4$, and $f(I_4) \supset I_1 \cup I_2 \cup I_3$. Thus, the corresponding transition matrix A is given as follows.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let the matrix

$$B = diag(\frac{1}{|\beta_1|}, \frac{1}{|\beta_2|}, \frac{1}{|\beta_4|}, \frac{1}{|\beta_4|})A.$$

Then

$$d = (d_1, d_2, d_3, d_4) = K \cdot (\frac{1-b_3}{b_3}, \frac{1-b_3}{b_3}, \frac{(b_1-b_2+b_3)(1-b_3)}{b_3(b_3-b_2)}, 1)$$

is the normalized left eigenvector of the left eigenvalue 1. Using Theorem 2.2.4, we have that d is the density function of the absolutely continuous invariant measure admitted by f. □

Theorem 3.2. Let f be a continuous piecewise linear function satisfying (3.9). Assume that the normalized i-mode $\widetilde{a_i}$ of the Fourier coefficients satisfy one of the following:

$$|\sum_{j=0}^{2} \widetilde{a_j}| \geq 1 \tag{3.11a}$$

$$(2+|\sum_{j=0}^{2}\widetilde{a_j}|)<|\sum_{j=0}^{3}\widetilde{a_j}| \text{ and } |\sum_{j=0}^{2}\widetilde{a_j}|<1. \tag{3.11b}$$

Then the Lyapunov exponent of f is positive.

Proof. Let $f'(x)=\beta_j$, $x\in I_j$, and $p(x)=d_j$, $x\in I_j$ be the density function given in Theorem 2.2.4. Here $j=1,2,3,4$.

$$\begin{aligned}
\sum_{j=1}^{m} d_j \log|\sum_{s=0}^{j-1}\widetilde{a_s}| &= K\cdot\Big(\frac{1-b_3}{b_3}\log\frac{b_3-b_2}{b_1}+\frac{1-b_3}{b_3}\log\frac{1-b_3}{b_2-b_1}\\
&\quad+\frac{(b_1-b_2+b_3)(1-b_3)}{b_3(b_3-b_2)}\log\frac{1-b_3}{b_3-b_2}+\log\frac{b_3}{1-b_3}\Big)\\
&= K\cdot\log\Big(\frac{1-b_3}{b_3-b_2}\Big)^{\frac{b_1(1-b_3)}{b_3(b_3-b_2)}}\Big(\frac{(1-b_3)^2}{b_1(b_2-b_1)}\Big)^{\frac{1-b_3}{b_3}}\Big(\frac{b_3}{1-b_3}\Big)\\
&=: K\cdot\log A^aB^b(\frac{1}{b}),
\end{aligned}$$

where K is given as in Proposition3.5. To complete the proof of the theorem, we need to show that $A^aB^b\frac{1}{b}>1$ for those $\widetilde{a_j}$ satisfying (3.11). To this end, we break b_3 into three cases (i) $0<b_3\le\frac{1}{2}$ (ii) $\frac{1}{2}<b_3<\frac{1+b_2}{2}$ and (iii) $\frac{1+b_2}{2}\le b_3<1$. For case(i), it is clear that $A^a>1$. Now,

$$B^b\frac{1}{b}=B^b\cdot(\frac{1}{b})^b(\frac{1}{b})^{\frac{2b_3-1}{b_3}}=(\frac{b_3(1-b_3)}{b_1(b_2-b_1)})^b(\frac{1}{b})^{\frac{2b_3-1}{b_3}}.$$

Since the bases of the exponents above are greater than one and their corresponding powers are positive, we have $B^b(\frac{1}{b})>1$. Thus, $A^aB^b(\frac{1}{b})>1$.

For case(ii), we see that $A>1$ and $\frac{1}{b}>1$. Moreover

$$B^b\frac{1}{b}=B^b(\frac{1}{b})^b(\frac{1}{b})^{\frac{2b_3-1}{b_3}}=(\frac{b_3(1-b_3)}{b_1(b_2-b_1)})^b\frac{1}{b}^{\frac{2b_3-1}{b_3}}.$$

If $b_3\le\frac{2}{3}$, then $\frac{b_3(1-b_3)}{b_1(b_2-b_1)}\ge\frac{b_3}{3b_1(b_2-b_1)}\ge\frac{4b_3}{3{b_2}^2}>1$. We see that $B^b\frac{1}{b}>1$. If $b_3\ge\frac{2}{3}$, then $B^b\frac{1}{b}=B^b(\frac{1}{b})^{2b}(\frac{1}{b})^{\frac{3b_3-2}{b_3}}=(\frac{{b_3}^2}{b_1(b_2-b_1)})^b(\frac{1}{b})^{\frac{3b_3-2}{b_3}}>1$.

Obviously, $A^a > 1$. Thus, $A^a B^b(\frac{1}{b}) > 1$.

For case(iii), if $a + 2b < 1$, then $A^a B^b(\frac{1}{b}) = (\frac{A}{b})^a(\frac{B}{b^2})^b(\frac{1}{b})^{1-a-2b} > 1$. Note that $a = \frac{\beta_3}{\beta_4} = \frac{\sum_{j=0}^{2}\widetilde{a_j}}{\sum_{j=0}^{3}\widetilde{a_j}}$, and $b = -\frac{1}{\beta_4} = -\frac{1}{\sum_{j=0}^{3}\widetilde{a_j}}$. Thus $a + 2b < 1$ is equivalent to $\frac{1}{|\sum_{j=0}^{3}\widetilde{a_j}|}(2 + |\sum_{j=0}^{2}\widetilde{a_j}|) < 1$. Moreover, $b_3 \geq \frac{1+b_2}{2}$ is equivalent to $|\beta_3| \leq 1$ or $|\sum_{j=0}^{2}\widetilde{a_j}| \leq 1$. We just complete the proof of the theorem. □

Theorem 3.3. Let $\widetilde{a_1}(\mu)$ be the normalized 1-mode of the k^{th} Fourier coefficient of the quadratic map $f_\mu(x) = \mu x(1-x)$. There exist an $\epsilon > 0$ and a set E containing 4 such that

$$Leb([4-\epsilon, 4] - E) < \epsilon^\beta.$$

Here $\beta < 2$ and β can be small arbitrary close to 2. Moreover, if $|\widetilde{a_1}(\mu)| \in E$, then the Lyapunov exponents of f_μ is positive.

Proof. Applying Rokhlin formula and Proposition 3.4-(i) and recalling $\widetilde{a_1}(\mu) = -\mu$, we have that

$$\begin{aligned}
\text{Lyapunov exponents of } f_4 &= \int_0^1 \log|f_4'|d\mu(x) \\
&= \log 2|\widetilde{a_1}(4)| + \int_0^1 \log|x-\frac{1}{2}|d\mu(x) \\
&= \log 8 + \int_0^1 \frac{\log|x-\frac{1}{2}|}{\pi(x(1-x))^{\frac{1}{2}}}dx \\
&= \log 2 > 0.
\end{aligned}$$

The assertion of the theorem now follows from Theorem 2.2.5. □

Remark 3.3. A parameter μ is called a Misiurewicz point if the set $\{f_\mu^n(\frac{1}{2})\}_{n=1}^{\infty}$ is at a positive distance from the critical point $\frac{1}{2}$. In [9], Rychlik and Sorets proved that for μ being a Misiurewicz point and is near 4, the Lyapunov exponent of f_μ is positive.

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