國立交通大學應用數學系

博 士 論 文

一維波方程系統中的等向與非等向混沌震動

Isotropic and Nonisotropic Chaotic Vibrations of the 1D Wave System

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摘要

波方程在一維閉區間 I 的右端有梵德波形式的非線性邊界條件。而在閉區間 I 的左端，當參數 $\eta > 0$ 時其邊界條件為能量的輸入；當 $\eta = 0$ 時其邊界條件為齊次諾曼條件。這個波方程系統的解與一個區間函數的疊代有關，所以我們定義當這個區間函數有李－約克的渾沌現象時稱這個波方程系統有混沌震動。因為在我們討論的波方程系統中兩特徵線的運動速度為任意兩正數 c_1 與 c_2，所以系統中有著等向性與非等向性的混沌震動現象產生。在這篇論文中，我們將討論波方程系統中混沌震動現象的產生分別與參數 η 、 c_1 、 c_2 變化之間的關聯。

Isotropic and Nonisotropic Chaotic Vibrations of the 1D Wave System

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Abstract

A wave equation on a one-dimensional interval I has a van der Pol type nonlinear boundary condition at the right end. At the left end, the boundary condition is energy-injecting if the parameter $\eta > 0$ and is the homogeneous Neumann condition if $\eta = 0$. The solution of the wave system is corresponding to the iteration of one interval map, so we say the wave system is chaotic if the interval map is chaotic in the sense of Li-Yorke. The system which we consider contains both isotropic and nonisotropic chaotic vibrations, since the two associated families of characteristics travel with two speeds c_1, c_2 for any given positive c_1, c_2. In this paper, we discuss that the chaotic vibrations of the 1D wave system occur when the paramerters η, c_1, c_2 vary separately.

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僅以此論文，獻給最疼愛我的爺爺—胡大洲先生

從中研院數學所研究助理、清大數學碩士、交大應用數學博士這七年的歲月，我前後跟著台灣數學界最具引響力的三位數學家學習到一個研究者應有的態度—充分利用時間並且努力做到最好。在中研院兩年的時間，李國偉教授教我許多做人處事的道理，我在李老師的身上看到高尚的人格，李老師眼睛看到的都是別人的優點，他熱心幫忙協助他身邊所有需要幫助的的人，所以李老師的身邊總是有許多受到他鼓舞的年輕人，而我就是其中的一個。很難相信一位像李老師這樣有名的數學家，會願意花時間在一個大學剛畢業，甚麼都不懂的我身上，耐心的指導我數學觀念，也是從那時候開始，我才漸漸知道數學是甚麼，也對數學產生興趣。

清華大學碩士班指導教授王懷權老師在我心目中就像是我的親人一樣，從準備博士班考試一直到博士班畢業，他不斷的在我最需要的時候給予我最大的幫助，在碩士班受他指導時，最常聽王老師說到他讀書時一大早起床為了比其他同學多讀一些書的故事，也是這樣努力才讓王老師有今日如此不凡的成就。

交通大學博士班指導教授林文偉老師就像是一個充滿

俠義心腸的劍客一樣，在我人生最絕望的時候，願意伸出他的手幫助我，這一年來在星期五論文研討時，聽到老師對我們的叮嚀—要每秒鐘都要想數學才有機會做到別人所不能的問題，他也有著極大的耐心來指導每一位學生。在這三位老師身上，我看到了身為數學家的風範，也看到我將來人生最佳的模範。

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感謝你對我禱告的垂聽，希望你能照顧我在天國的長

輩，爺爺與雯玲的外婆，感謝主。

胡忠澤 98.6.6

Contents

Chapter 1

Introduction

The imbalance of the boundary energy flow due to energy injection at one end and a nonlinear van der Pol boundary condition at the other end of the spatial one-dimensional interval can cause isotropic chaotic vibration of the linear wave equation. Such chaotic vibration is isotropic with respect to space and time because the two associated families of characteristics both propagate with the same speed (see Chen et al. [2]). In [2], the 1D wave system is considered:

$$\omega_{tt} - \omega_{xx} = 0,\ 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$\begin{cases} \omega_t(0,t) = -\eta\omega_x(0,t), & \eta > 0,\ \eta \neq 1,\ t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in (0,1],\ \beta > 0,\ t > 0, \end{cases}$$

and the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]),\ \omega_t(x,0) = \psi(x) \in C^0([0,1]).$$

In the 1D wave equation $\omega_{tt} - \omega_{xx} = 0$, two families of characteristics travel with the same speed $c_1 = c_2 = 1$. The boundary condition at the left endpoint $x = 0$ is energy-injecting and the boundary condition at the right endpoint $x = 1$ is a van der Pol condition. In [2], Chen et al. proved the

1D wave system is chaotic when the parameter η enters the region

$$\left[\frac{3\sqrt{3}-1-\alpha}{3\sqrt{3}+1+\alpha}, 1\right) \cup \left(1, \frac{3\sqrt{3}+1+\alpha}{3\sqrt{3}-1-\alpha}\right], \text{ for any given } \alpha \in (0,1],\ \beta > 0.$$

In [8, 12, 13], the 1D wave system is considered:

$$\omega_{tt} - \omega_{xx} = 0,\ 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$\begin{cases} \omega_x(0,t) = -\eta\omega_t(0,t), & \eta > 0,\ \eta \neq 1,\ t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in (0,1],\ \beta > 0,\ t > 0, \end{cases}$$

and the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]),\ \omega_t(x,0) = \psi(x) \in C^0([0,1]).$$

Huang et al. characterized the dynamical behavior in terms of the growth of the total variation of the interval map and proved that for any given $\alpha \in (0,1]$, there exist four constants $\underline{\eta_0}$, $\underline{\eta_H}$, $\overline{\eta_H}$ and $\overline{\eta_0}$ with

$$0 < \underline{\eta_0} < \underline{\eta_H} < 1 < \overline{\eta_H} < \overline{\eta_0} < \infty$$

such that the total variation of the interval map remains bounded, is unbounded, is unbounded exponentially when the parameter η belongs to $(0,\underline{\eta_0})\cup(\overline{\eta_0},\infty)$, $(\underline{\eta_0},\underline{\eta_H})\cup(\overline{\eta_H},\overline{\eta_0})$, and $(\underline{\eta_H},1)\cup(1,\overline{\eta_H})$, respectively. In particular, the last case corresponds to chaos in the 1D wave system. Notice that the boundary condition at the left endpoint in this system is $\omega_x(0,t) = -\eta\omega_t(0,t)$ which is different from $\omega_t(0,t) = -\eta\omega_x(0,t)$ in [2].

By including a mixed partial derivative linear transport term in the wave equation, nonlinearity in the van der Pol boundary condition can also cause nonisotropic chaotic vibration (without energy injection from the other end). Such chaotic vibration is nonisotropic with respect to space and time because the two associated families of characteristics travel with different speeds c_1, c_2

which satisfy $c_1c_2 = 1$ (see Chen et al. [5]). In [5], the 1D wave system is considered:

$$\omega_{tt} + v\omega_{tx} - \omega_{xx} = 0,\, v > 0,\, 0 < x < 1,\, t > 0,$$

with the boundary conditions

$$\omega_x(0,t) = 0,\, t > 0,$$

and

$$\omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t),\, \alpha \in \left(0, \frac{v+\sqrt{v^2+4}}{2}\right],\, \beta > 0,\, t > 0,$$

and the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]),\, \omega_t(x,0) = \psi(x) \in C^0([0,1]).$$

In the 1D wave equation $\omega_{tt}+v\omega_{tx}-\omega_{xx} = 0$, two families of characteristics travel with different speeds $c_1 = \frac{-v+\sqrt{v^2+4}}{2}$ and $c_2 = \frac{v+\sqrt{v^2+4}}{2}$ which satisfy $c_1c_2 = 1$ and $c_2 - c_1 = v$. The boundary condition at the left endpoint $x = 0$ is the homogeneous Neumann condition and the boundary condition at the right endpoint $x = 1$ is a van der Pol condition. In [5], Chen et al. proved the 1D wave system is chaotic when the parameters (v, α) enter a certain subregion of

$$S = \left\{(v,\alpha) \in \mathbb{R}^2 \mid 0 < v < \infty, 0 < \alpha \leq \frac{v+\sqrt{v^2+4}}{2}\right\}.$$

In [9], Huang proved that there exist three subregions S_1^0, S_1^1 and S_2 of S such that the growth of the total variation of the interval map remains bounded, is unbounded, is unbounded exponentially when the parameters (v, α) belong to S_1^0, S_1^1, and S_2, respectively. In particular, the last case corresponds to chaos in the 1D wave system.

In chapter 3 of this paper, the 1D wave system is considered:

$$\omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0,\, d \in \mathbb{R},\, c > 0,\, 0 < x < 1,\, t > 0, \tag{1.1}$$

with the boundary conditions

$$\omega_x(0,t) = -\eta\omega_t(0,t),\, \eta \geq 0,\, \eta \neq \frac{1}{c_2},\, t > 0, \tag{1.2}$$

and

$$\omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t),\ \alpha \in \left(0, \frac{1}{c_1}\right],\ \beta > 0,\ t > 0. \tag{1.3}$$

And with the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]),\ \omega_t(x,0) = \psi(x) \in C^0([0,1]). \tag{1.4}$$

Remark 1.0.1 *We denote the parameters*

$$c_1 = \frac{d + \sqrt{d^2 + 4c^2}}{2} \text{ and } c_2 = \frac{-d + \sqrt{d^2 + 4c^2}}{2}$$

in (1.2) and (1.3).

In the 1D wave equation $\omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0$, two families of characteristics travel with speeds c_1 and c_2. If $d = 0$, the speeds $c_1 = c_2 = c$; if $d \neq 0$, the speeds $c_1 \neq c_2$. The boundary condition at the left endpoint $x = 0$ is energy-injecting when $\eta > 0$ and is the homogeneous Neumann condition when $\eta = 0$. The boundary condition at the right endpoint $x = 1$ is a van der Pol condition which is a well-known self-regulating mechanism in automatic control.
If $d = 0$, $c^2 = 1$ in (1.1) and $\eta > 0$ in (1.2), we have the 1D wave equation

$$\omega_{tt} - \omega_{xx} = 0,\ 0 < x < 1,\ t > 0, \tag{1.5}$$

with the boundary conditions

$$\begin{cases} \omega_x(0,t) = -\eta\omega_t(0,t), & \eta > 0,\ \eta \neq 1,\ t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in (0,1],\ \beta > 0,\ t > 0, \end{cases} \tag{1.6}$$

and the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]),\ \omega_t(x,0) = \psi(x) \in C^0([0,1]). \tag{1.7}$$

If $d = -v$, $c^2 = 1$ in (1.1) and $\eta = 0$ in (1.2), we have the 1D wave equation

$$\omega_{tt} + v\omega_{tx} - \omega_{xx} = 0,\ v > 0,\ 0 < x < 1,\ t > 0, \tag{1.8}$$

with the boundary conditions

$$\omega_x(0,t) = 0,\ t > 0, \tag{1.9}$$

and

$$\omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t),\ \alpha \in \left(0, \frac{v+\sqrt{v^2+4}}{2}\right],\ \beta > 0,\ t > 0, \tag{1.10}$$

and the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]),\ \omega_t(x,0) = \psi(x) \in C^0([0,1]). \tag{1.11}$$

Thus, the 1D wave system (1.1)-(1.4) contains the 1D wave systems in [8, 12, 13] and [5, 9]. Furthermore, the system (1.1)-(1.4) contains both isotropic and nonisotropic chaotic vibrations since the two associated families of characteristics travel with two speeds c_1, c_2 for any given positive c_1, c_2. In section 3, we show the chaotic region of the 1D wave system. And based on this region, we show the chaotic region of the paramerter η when the other parameters are fixed. Furthermore, we show the system is chaotic if $c_1 \to \infty$, or $c_1 \to 0^+$, or $c_2 \to \infty$.

In chapter 4, the 1D wave system is considered:

$$\begin{cases} \omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0, & d \in \mathbb{R},\ c > 0,\ 0 < x < 1,\ t > 0, \\ \omega_t(0,t) + \eta\omega_x(0,t) = 0, & \eta > 0,\ \eta \neq c_2,\ t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^{2m+1}(1,t), & \alpha \in \left(0, \frac{1}{c_1}\right],\ \beta, t > 0,\ m \in \mathbb{N}, \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases} \tag{1.12}$$

In this system, the boundary condition at the left endpoint $x = 0$ is energy-injecting and the boundary condition at the right endpoint $x = 1$ has odd-degree nonlinearity. In section 4, we show the 1D wave system (1.12) is chaotic when the parameter η satisfies either

$$c_2 < \eta \leq \frac{c_2\left[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)\right]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)}$$

or

$$\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \leq \eta < c_2$$

for any given parameters c, d, α, β, m satisfy inequality

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) > 0,$$

and when the parameter η satisfies either

$$\eta > c_2 \text{ or } \frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_1(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2)+\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \leq \eta < c_2$$

for any given parameters c, d, α, β, m satisfy the inequality

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_2(1+\alpha c_2) \leq 0.$$

And we show the 1D wave system (1.12) is chaotic for any given c_1 if the parameters η, c_2, α, β, m satisfy

$$\eta > c_2 \text{ and } 2mc_2(1+\alpha c_2)-\sqrt[2m]{2m+1}(2m+1)(\eta-c_2) \geq 0.$$

And the 1D wave system (1.12) is chaotic for sufficiently small c_1 if the parameters η, c_2, α, β, m satisfy some conditions or c_1 is sufficiently large if the parameters η, c_2, α, β, m satisfy some other conditions.

It is easy to see that the 1D wave system (1.12) contains the 1D wave system in [2]. Thus, we consider the 1D wave systems in [2, 5, 8, 9, 12, 13] as three examples of the systems (1.1)-(1.4) and (1.12) in chapter 5. And in chapter 6, we use two methods to detect the chaos in 1D wave systems (1.1)-(1.4) and (1.12) (see Li et al. [14, 15]).

Chapter 2

Preliminary

We list some definitions and background facts that a reader should know in this chapter.

Definition 2.0.2 (Topologically Transitive) *A map $f : X \to X$ is (topologically) transitive on an invariant set Y provided the forward orbit of some point p is dense in Y. The Birkhoff Transitivity Theorem proves that a map f is transitive on Y if and only if, given any pair of open sets $U, V \subset Y$ there exists $k > 0$ such that $f^k(U) \cap V \neq \varnothing$.*

Intuitively, a topologically transitive map has points which eventually move under iteration from one arbitrarily small neighborhood to any other. Consequently, the dynamical system cannot be decomposed into two disjoint open sets which are invariant under the map.

Definition 2.0.3 (Sensitive Dependence on Initial Conditions) *A map $f : X \to X$ has sensitive dependence on initial conditions if there exists $\delta > 0$ (independent of the point) such that, for each point $x \in X$ and any neighborhood N of x, there exists $y \in N$ such that $d(f^n(x), f^n(y)) \geq \delta$ for some $n \geq 0$.*

Intuitively, a map possesses sensitive dependence on initial conditions if there exist points arbitrarily close to x which eventually separate from x by at least δ under iteration of f. We emphasize that not all points near x need eventually separate from x under iteration, but there must be at least one such point in every neighborhood of x.

Definition 2.0.4 (Expansive) *A map f on a metric space X is said to be expansive provided there is an $r > 0$ (independent of the point) such that, for each pair of points $x, y \in X$ there is a $k \geq 0$ such that $d\left(f^k(x), f^k(y)\right) \geq r$.*

If f is expansive and X is a perfect metric space, then it has sensitive dependence on initial conditions.

Definition 2.0.5 (Chaotic in the Sense of Devaney) *Let V be a set. $f : V \to V$ is said to be chaotic on V if*

1. f has sensitive dependence on initial conditions.

2. f is topologically transitive.

3. periodic points are dense in V.

Definition 2.0.6 (Chaotic in the Sense of Robinson) *A map f on a metric space X is said to be chaotic on an invariant set Y or exhibits chaos provided (i) f is transitive on Y and (ii) f has sensitive dependence on initial conditions on Y.*

The paper of Banks, Brooks, Cairns, Davis, and Stacey (1992) proves that any map which (i) is transitive on Y and (ii) has dense periodic points also must have sensitive dependence on initial conditions.

Definition 2.0.7 (Chaotic in the Sense of Li-Yorke) *A continuous map f on the compact metric space (X, d) is said to be chaotic on a nonempty and invariant set X_0 in the sense of Li-Yorke if there is an uncountable set $S \subset X_0$ such that*

(i) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$, for all x, $y \in S$ and $x \neq y$.

(ii) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$, for all x, $y \in S$.

Theorem 2.0.8 *[17, Li and Yorke, 1975]Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous, and there is a point a such that either (i) $f^3(a) \leq a < f(a) < f^2(a)$ or (ii) $f^3(a) \geq a > f(a) > f^2(a)$. Then, f has points of all periods.*

Definition 2.0.9 *In order to state the result of Sharkovskii, we need to introduce a new ordering on the positive integers using the symbol $\rhd$, called the Sharkovskii ordering. First, the odd integers greater than one are put in the backward order:*

$$3 \rhd 5 \rhd 7 \rhd 9 \rhd 11 \cdots .$$

Next, all the integers which are two times an odd integer are added to the ordering, and then the *odd integers times increasing powers of two:*

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \cdots$$
$$\rhd 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd 2^n\cdot 7 \rhd \cdots \rhd 2^{n+1}\cdot 3 \rhd 2^{n+1}\cdot 5 \rhd 2^{n+1}\cdot 7 \rhd \cdots.$$

Finally, all the powers of two are added to the ordering in decreasing powers:

$$3 \rhd 5 \rhd \cdots \rhd 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd \cdots \rhd \cdots \rhd 2^{n+1} \rhd 2^n \rhd \cdots \rhd 2^2 \rhd 2 \rhd 1.$$

We have now given an ordering between all positive integers. This ordering seems strange but it turns out to the be ordering which expresses which periods imply which other periods as given in the theorem of Sharkovskii (Sharkovskii, 1964).

Theorem 2.0.10 (Sharkovskii) *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function from an interval I into the real line. Assume f has a point of period n and $n \rhd k$. Then, f has a point of period k. (By period, we mean least period.)*

Theorem 2.0.11 (Period Doubling Bifurcation) *Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^r function jointly in both variables with $r \geq 3$, and that f satisfies the following conditions.*
(1) The point x_0 is a fixed point for $\mu = \mu_0$: $f(x_0, \mu_0) = x_0$.
(2) The derivative of f_{μ_0} at x_0 is -1: $f'_{\mu_0}(x_0) = -1$. Since this derivative is not equal to 1, there is a curve of fixed points $x(\mu)$ for μ near μ_0.
(3) The derivative of $f'_\mu(x(\mu))$ with respect to μ is nonzero (the derivative is varying along the family of fixed points):

$$\alpha = \left[\frac{\partial^2 f}{\partial\mu\partial x} + \left(\frac{1}{2}\right)\left(\frac{\partial f}{\partial\mu}\right)\left(\frac{\partial^2 f}{\partial x^2}\right)\right]\bigg|_{(x_0,\mu_0)} \neq 0.$$

(4) The graph of $f^2_{\mu_0}$ has nonzero cubic term in its tangency with the diagonal (the quadratic term is zero):

$$\beta = \left(\frac{1}{3!}\frac{\partial^3 f}{\partial x^3}(x_0,\mu_0)\right) + \left(\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x_0,\mu_0)\right)^2 \neq 0.$$

Then, there is a period doubling bifurcation at (x_0, μ_0). More specifically, there is a differentiable curve of fixed points, $x(\mu)$, passing through x_0 at μ_0, and the stability of the fixed point changes at μ_0. (Which side of μ_0 is attracting depends on the sign of α.) There is also a differentiable curve γ passing through (x_0, μ_0) so that $\gamma \setminus \{(x_0, \mu_0)\}$ is the union of hyperbolic period 2 orbits. The curve

γ is tangent to the line $\mathbb{R} \times \{\mu_0\}$ at (x_0, μ_0), so γ is the graph of a function of x, $\mu = m(x)$ with $m'(x_0) = 0$ and $m''(x_0) = -2\beta/\alpha \neq 0$. The stability type of the period 2 orbit depends on the sign of β: if $\beta > 0$, then the period 2 orbit is attracting; and if $\beta < 0$, then the period 2 orbit is repelling.

Definition 2.0.12 (Homoclinic Point) *Let a map $f \in C(I, I)$. A point $x \in I$ is called homoclinic point of f if there exists a periodic point p of period n with $x \neq p$, $x \in W^u(p, f^n)$ and $f^{nm}(x) = p$ for some positive integer m. We call such point p a periodic point associated with a homoclinic point x and denote by $P_h(f)$ the set of all such periodic points.*

In [10, Corollary 9.1], Chen et al. proved the results as below.

Lemma 2.0.13 *[10, Corollary 9.1]Let $f \in C(I, I)$. Suppose that f is piecewise monotone with finitely many extremal points on I. Then the following conditions are equivalent.*
(1) f has a periodic point whose period is not a power of 2.
(2) f has a homoclinic point. That is, $P_h(f) \neq \emptyset$.
(3) f has positive topological entropy.
(4) The total variation $V_I(f^n)$ of f on I grows exponentially as $n \to \infty$.
Furthermore, each of the above conditions implies that f is chaotic in the sense of Li-Yorke.

Remark 2.0.14 *In [10, Corollary 9.1], Chen et al. have the conclutions as below.*
If f is piecewise monotone with finitely many extremal points on I, then
Chaos in the sense of Devaney$\Rightarrow$sensitive dependence on initial conditions$\Rightarrow$exponential grows of the total variation $V_I(f^n)$ with respect to n as $n \to \infty$ $\Rightarrow$positive topological entropy$\Leftrightarrow$existence of a periodic point of a period being not a power of 2 $\Leftrightarrow$existence of a homoclinic point$\Leftrightarrow$Chaos in the sense of Li-Yorke.

Chapter 3

The 1D wave system (1.1)-(1.4)

In this chapter, the 1D wave system (1.1)-(1.4) is considered:

$$\omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0,\ d \in \mathbb{R},\ c > 0,\ 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$\omega_x(0,t) = -\eta\omega_t(0,t),\ \eta \geq 0,\ \eta \neq \frac{1}{c_2},\ t > 0,$$

and

$$\omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t),\ \alpha \in \left(0, \frac{1}{c_1}\right],\ \beta > 0,\ t > 0.$$

And with the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]),\ \omega_t(x,0) = \psi(x) \in C^0([0,1]).$$

3.1 Chaotic vibrations of the system (1.1)-(1.4)

The general solution of (1.1) is

$$\omega(x,t) = u(c_1 t + x) + v(c_2 t - x), \tag{3.1}$$

where u, v are arbitrary C^2-function. Substituting (3.1) in (1.2) and (1.3), we have

$$u'(c_1 t) = \frac{1-\eta c_2}{1+\eta c_1} v'(c_2 t),\ t > 0, \tag{3.2}$$

and

$$\begin{aligned} &\beta \left(c_1 u'(c_1 t + 1) + c_2 v'(c_2 t - 1)\right)^3 + \left(\tfrac{1}{c_1} - \alpha\right)\left(c_1 u'(c_1 t + 1) + c_2 v'(c_2 t - 1)\right) \\ &- \left(1 + \tfrac{c_2}{c_1}\right) v'(c_2 t - 1) = 0,\ t > 0. \end{aligned} \tag{3.3}$$

When $\eta = \frac{1}{c_2}$ in (3.2), we have

$$u'(c_1 t) = 0 \text{ for } t > 0 \Rightarrow u(c_1 t + x) = C \text{ for } t > 0.$$

Thus, we consider the case $\eta \neq \frac{1}{c_2}$. And depends on (3.2), we can use v' to replace u' in (3.3) to derive one difference equation as follows.

By using the substitution

$$z(c_1 t) = \begin{cases} v'\left(c_2 t - 1\right), & 0 \le t \le \frac{1}{c_2}, \\ \frac{1+\eta c_1}{1-\eta c_2} u'\left(c_1 t - \frac{c_1}{c_2}\right), & t > \frac{1}{c_2}, \end{cases}$$

we have the difference equation

$$\begin{aligned} &\beta \left(c_1 \tfrac{1-\eta c_2}{1+\eta c_1} z(\tau + \Delta) + c_2 z(\tau)\right)^3 + \left(\tfrac{1}{c_1} - \alpha\right)\Big(c_1 \tfrac{1-\eta c_2}{1+\eta c_1} z(\tau + \Delta) + \\ &c_2 z(\tau)\Big) - \left(1 + \tfrac{c_2}{c_1}\right) z(\tau) = 0, \end{aligned} \tag{3.4}$$

where $\tau = c_1 t$, $\Delta = 1 + \frac{c_1}{c_2}$.

And the initial condition of (3.4) is

$$z(c_1 t) = \begin{cases} \frac{\psi(1-c_2 t) - c_1 \varphi'(1-c_2 t)}{c_1 + c_2}, & 0 \le t \le \frac{1}{c_2}. \\ \frac{1+\eta c_1}{1-\eta c_2} \frac{\psi\left(c_1 t - \frac{c_1}{c_2}\right) + c_2 \varphi'\left(c_1 t - \frac{c_1}{c_2}\right)}{c_1 + c_2}, & \frac{1}{c_2} < t \le \frac{1}{c_1} + \frac{1}{c_2}. \end{cases}$$

Remark 3.1.1 *In this paper, we assume that the initial value $\varphi(x)$ and $\psi(x)$ are chosen such that*

$z(\tau)$ is continuous on $\left[0, 1+\frac{c_1}{c_2}\right]$ and satisfies the compatibility condition

$$\beta\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}z(\Delta)+c_2z(0)\right)^3+\left(\tfrac{1}{c_1}-\alpha\right)\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}z(\Delta)+c_2z(0)\right)$$

$$-\left(1+\tfrac{c_2}{c_1}\right)z(0)=0.$$

Definition 3.1.2 *We denote the range of $z(\tau)$ on $[0,\Delta]$ to be the compact interval Λ; i.e., $\Lambda = z([0,\Delta])$.*

We show the dependence of $z(\tau+\Delta)$ on $z(\tau)$ is given implicitly by one C^1-function f_λ as follows.

Lemma 3.1.3 (Existence and Uniqueness of the Solution) *Let the parameters c_1, c_2, η, α, β be fixed in (3.4) with $c_1>0$, $c_2>0$, $\eta\geq 0$, $\eta\neq\frac{1}{c_2}$, $\alpha\in\left(0,\frac{1}{c_1}\right]$ and $\beta>0$. Then there exists one C^1-function f_λ such that*

$$f_\lambda(z(t)) = z(t+\Delta) \text{ for all } t>0,$$

where $\lambda=(c_1,c_2,\eta,\alpha,\beta)$.

Proof. Let

$$H_\lambda(u,v)=\beta\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}u+c_2v\right)^3+\left(\tfrac{1}{c_1}-\alpha\right)\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}u+c_2v\right)$$

$$-\left(1+\tfrac{c_2}{c_1}\right)v=0,$$

where $u=z(\tau+\Delta)$, $v=z(\tau)$, $\lambda=(c_1,c_2,\eta,\alpha,\beta)$.
(i) If $\alpha=\frac{1}{c_1}$, then $H_\lambda(u,v)=0$ implies

$$c_1\frac{1-\eta c_2}{1+\eta c_1}u=\sqrt{\left(1+\frac{c_2}{c_1}\right)\frac{v}{\beta}}-c_2v.$$

Hence, the C^1-function f_λ exists.
(ii) If $\alpha\in\left(0,\frac{1}{c_1}\right)$, then

$$\frac{\partial}{\partial u}H_\lambda(u,v)=3\beta c_1\frac{1-\eta c_2}{1+\eta c_1}\left(c_1\frac{1-\eta c_2}{1+\eta c_1}u+c_2v\right)^2+(1-\alpha c_1)\frac{1-\eta c_2}{1+\eta c_1}\neq 0.$$

By the implicit function theorem, the C^1-function f_λ exists. ■

Definition 3.1.4 *We denote $f_\lambda(z(\tau)) = z(\tau+\Delta)$ to be the function, which satisfies (3.4) for all $\tau \geq 0$, where $\lambda = (\eta, c_1, c_2, \alpha, \beta)$.*

Since

$$f_\lambda(z(\tau)) = z(\tau+\Delta) \text{ for all } \tau > 0,$$

we can use the map f_λ and the interval Λ to generate $z(\tau)$ for all $\tau > 0$. And the corresponding solution of the 1D wave system (1.1)-(1.4) is calculated via the formulae

$$\omega(x,t) = \int_{\frac{1}{c_2}}^{t+\frac{x}{c_1}+\frac{1}{c_2}} c_1 \frac{1-\eta c_2}{1+\eta c_1} z(c_1\tau)d\tau + \int_0^{t-\frac{x}{c_2}+\frac{1}{c_2}} c_2 z(c_1\tau)d\tau.$$

Definition 3.1.5 (Chaotic Vibration) *The solution of the 1D wave system is said to be chaotic if the map $f_\lambda : \Lambda \to \mathbb{R}$ is chaotic in the sense of Li-Yorke; i.e., there exists one nonempty invariant subset $\Lambda_0 \subseteq \Lambda$ such that f_λ is chaotic in the sense of Li-Yorke on Λ_0 (see Definition 2.0.7).*

Remark 3.1.6 *In this paper, we say the 1D wave system is chaotic if its solution is chaotic.*

3.2 The chaotic region of the system (1.1)-(1.4)

In this section, we want to show the chaotic region of the solution of the 1D wave system (1.1)-(1.4). First, we consider (3.4) as below:

$$H_\lambda(x,y) = \beta\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}y + c_2 x\right)^3 + \left(\tfrac{1}{c_1}-\alpha\right)\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}y + c_2 x\right) - \left(1+\tfrac{c_2}{c_1}\right)x$$

$$= 0, \text{ where } c_1,\ c_2 > 0,\ \eta \geq 0,\ \eta \neq \tfrac{1}{c_2},\ \alpha \in \left(0, \tfrac{1}{c_1}\right] \text{ and } \beta > 0.$$

Definition 3.2.1 *We denote*

$$v_c = \frac{c_1 - 2\alpha c_1 c_2 + 3c_2}{3c_2\,(c_1+c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}} \text{ and } M = \frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}$$

in the following lemmas.

We show the local maximum, minimum and piecewise monotonicity of the function f_λ which satisfies (3.4) as below.

Lemma 3.2.2 (Local Maximum, Minimum and Piecewise Monotonicity) *The function f_λ is odd and f_λ has local extrema at (v_c, M) and $(-v_c, -M)$. Furthermore, f_λ is strictly monotonic on $(-\infty, -v_c)$, $(-v_c, v_c)$ and (v_c, ∞).*

Proof. Since $H(-x, f_\lambda(-x)) = H(-x, -f_\lambda(x)) = 0$, we have $f_\lambda(-x) = -f_\lambda(x)$. Thus f_λ is odd. Then use

$$\tfrac{d}{dx}H(x,y) = 3\beta\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}y + c_2 x\right)^2\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}y' + c_2\right)$$

$$+\left(\tfrac{1}{c_1}-\alpha\right)\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}y' + c_2\right) - \left(1+\tfrac{c_2}{c_1}\right) = 0,$$

and carry out the computations, we have the results. ■

We show the x-axis Intercepts, fixed points and intersections with the line $y = -x$ of the function f_λ as below.

Lemma 3.2.3 (x-axis Intercepts) *The function f_λ intersects the x-axis at the points*

$$\left(-\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, 0\right),\ (0,0),\ \left(\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, 0\right).$$

Proof. The results can be directly comfirmed by computing $H(x, 0) = 0$. ■

Lemma 3.2.4 (Intersections with the Line $y = x$) *The function f_λ intersects the line $y = x$ at the points*

$$\left(-\frac{1+\eta c_1}{c_1+c_2}\sqrt{\frac{\eta+\alpha}{\beta}}, -\frac{1+\eta c_1}{c_1+c_2}\sqrt{\frac{\eta+\alpha}{\beta}}\right),\ (0,0)\ \textit{and}\ \left(\frac{1+\eta c_1}{c_1+c_2}\sqrt{\frac{\eta+\alpha}{\beta}}, \frac{1+\eta c_1}{c_1+c_2}\sqrt{\frac{\eta+\alpha}{\beta}}\right).$$

Proof. The results can be directly comfirmed by computing $H(x, x) = 0$. ■

Definition 3.2.5 *We denote the point*

$$B = \frac{1+\eta c_1}{|2\eta c_1 c_2 + c_2 - c_1|}\sqrt{\frac{2+2\alpha\eta c_1 c_2 + \eta(c_1 - c_2) + \alpha(c_2 - c_1)}{\beta(2\eta c_1 c_2 + c_2 - c_1)}}$$

in the following lemmas.

Lemma 3.2.6 (Intersections with the Line $y = -x$) *Let the parameters η, c_1, c_2, α, β be fixed in (3.4) with $\frac{2+2\alpha\eta c_1 c_2+\eta(c_1-c_2)+\alpha(c_2-c_1)}{\beta(2\eta c_1 c_2+c_2-c_1)} > 0$, then the function f_λ intersects the line $y = -x$ at the points $(-B, B)$, $(0,0)$, $(B, -B)$. Otherwise, the function f_λ intersects the line $y = -x$ only at the point $(0,0)$.*

Proof. The results can be directly comfirmed by computing $H(x,-x)=0$. ■

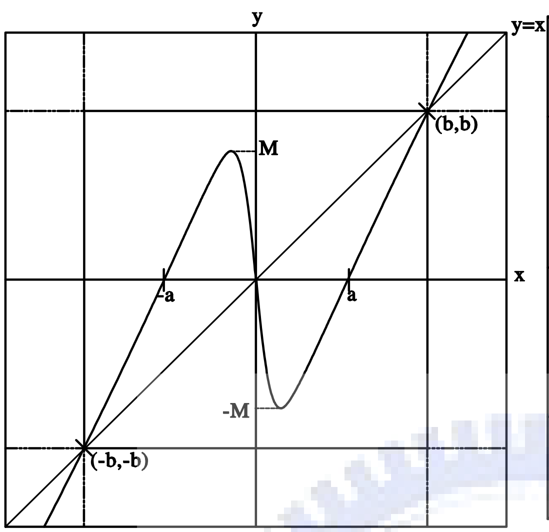

Figure 1: The map f_λ with $\eta > \frac{1}{c_2}$ and $|M| \le \frac{1+\eta c_1}{c_1+c_2}\sqrt{\frac{\eta+\alpha}{\beta}}$.

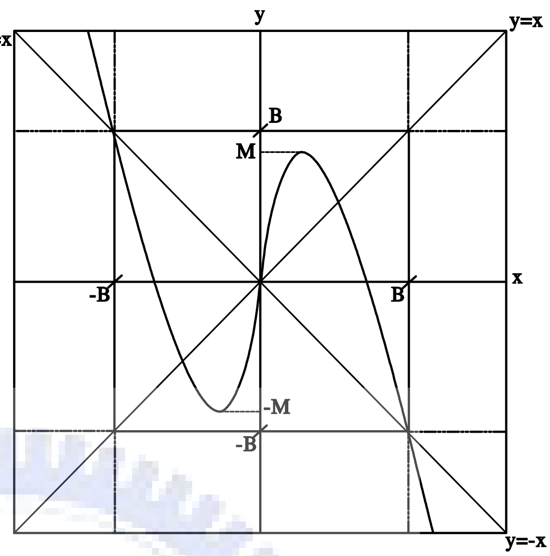

Figure 2: The map f_λ with $0 < \eta < \frac{1}{c_2}$ and $|M| \le B$.

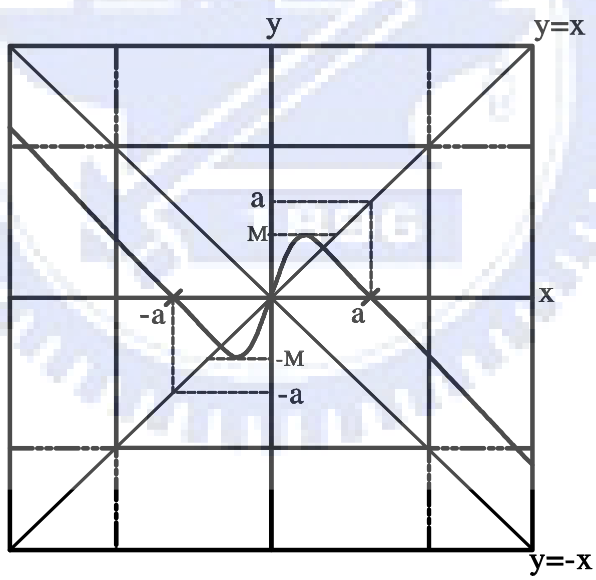

Figure 3: The map f_λ with $0 < \eta < \frac{1}{c_2}$.

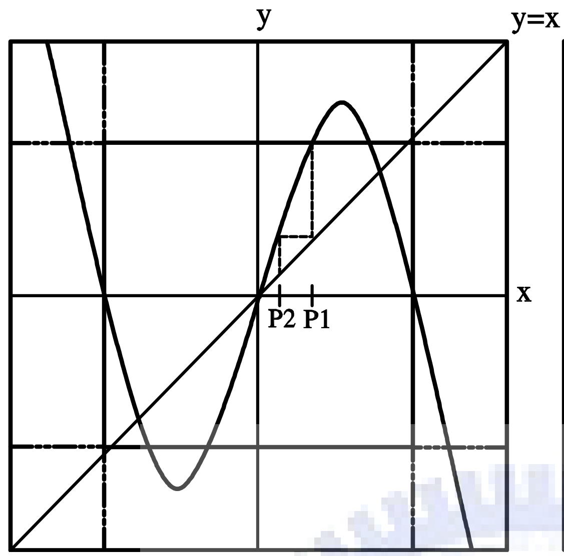

Figure 4: $f_\lambda^3(p_2) < p_2 < f_\lambda(p_2) < f_\lambda^2(p_2)$.

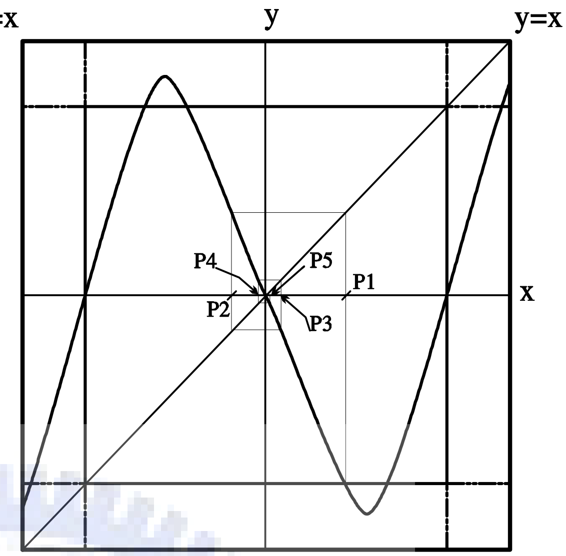

Figure 5: $f_\lambda^6(p_5) < p_5 < f_\lambda^2(p_5) < f_\lambda^4(p_5)$.

We show the function f_λ has bounded invariant interval or bounded invariant cantor-like subset in following lemmas.

Lemma 3.2.7 (Bounded Invariant Interval) *Let the parameters η, c_1, c_2, α, β be fixed in (3.4).*
(i) If $\eta > \frac{1}{c_2}$ and $|M| = \left|\frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right| \leq \frac{1+\eta c_1}{c_1+c_2}\sqrt{\frac{\eta+\alpha}{\beta}}$, then the iterates of every point in the set

$$U \equiv \left(-\infty, -\frac{1+\eta c_1}{c_1+c_2}\sqrt{\frac{\eta+\alpha}{\beta}}\right) \cup \left(\frac{1+\eta c_1}{c_1+c_2}\sqrt{\frac{\eta+\alpha}{\beta}}, \infty\right)$$

escape to $\pm\infty$, while those of any point in $\mathbb{R}\backslash\overline{U}$ are attracted to the bounded invariant interval

$$\left[-\left|\frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right|, \left|\frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right|\right]$$

of f_λ, i.e., $[-|M|, |M|]$ of f_λ.
(ii) If $0 < \eta < \frac{1}{c_2}$ and f_λ intersects the line $y = -x$ at three points and $|M| \leq B$, then the iterates of every point in the set $U \equiv (-\infty, -B) \cup (B, \infty)$ escape to $\pm\infty$, while those of any point in $\mathbb{R}\backslash\overline{U}$ are attracted to the bounded invariant interval $[-|M|, |M|]$ of f_λ.
(iii) If $0 < \eta < \frac{1}{c_2}$ and f_λ intersects the line $y = -x$ at $(0,0)$, then the iterates of every point in $\mathbb{R}$ are attracted to the bounded invariant interval $[-|M|, |M|]$ of f_λ.

Proof. The results of (i) and (ii) follow easily from the above lemmas and other piecewise monotonic properties of f_λ, as can be directly comfirmed by graphical analysis (see Figure 1 and Figure 2).

We omit the details.

(iii) If $0 < \eta < \frac{1}{c_2}$ and f_λ intersects the line $y = -x$ only at $(0,0)$, then $|f_\lambda(x)| < |x|$ for all

$$x \in \left(-\infty, -\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}\right) \cup \left(\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, \infty\right).$$

Thus, $|f_\lambda^n(x)|$ is strctly decreasing for $n \leq n_0$, where

$$f_\lambda^{n_0}(x) \in \left(-\infty, -\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}\right) \cup \left(\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, \infty\right)$$

and

$$f_\lambda^{n_0+1}(x) \notin \left(-\infty, -\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}\right) \cup \left(\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, \infty\right).$$

Hence, the iterates of every point in $\mathbb{R}$ are attracted to the bounded invariant interval $[-|M|, |M|]$ of f_λ (see Figure 3). ■

Lemma 3.2.8 (Bounded Cantor-like Invariant Subset) *The bounded invariant interval*

$$\left[-\left|\frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right|, \left|\frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right|\right]$$

no longer exists in the case (i) and (ii) of the Lemma 3.2.7 if the condition

$$|M| \leq \frac{1+\eta c_1}{c_1+c_2}\sqrt{\frac{\eta+\alpha}{\beta}} \text{ or } |M| \leq B$$

is violated. Instead, we have a bounded Cantor-like invariant set.

Proof. The method of proof is now standard, see [18, Sec. 1.7], for example. ■

We have the chaotic region of the function f_λ as below.

Lemma 3.2.9 *Let the parameters η, c_1, c_2, α, β be fixed in (3.4) and satisfy the inequality*

$$\left|\frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\right|\sqrt{\frac{1+\alpha c_2}{3\beta c_2}} \geq \frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, \tag{3.5}$$

then the interval map f_λ is chaotic in the sense of Li-Yorke if the domain of f_λ contains the interval

$$\left[-\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, \frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}\right].$$

Proof. (i) If $0 < \eta < \frac{1}{c_2}$, then

$$f_\lambda(v_c) = \frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}} \text{ is the local maximum.}$$

Since f_λ is strictly increasing on $[0, v_c]$ and $f_\lambda(v_c) \geq \frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}$, there exists one unique point $p_1 \in (0, v_c]$ such that $f_\lambda(p_1) = \frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}$. Similarly, there exists one unique point $p_2 \in (0, p_1)$ such that $f_\lambda(p_2) = p_1$. Hence, we have

$$0 = f_\lambda^3(p_2) < p_2 < f_\lambda(p_2) < f_\lambda^2(p_2) \text{ (see Figure 4).}$$

Thus, f_λ has points of all periods which implies chaos [by Li and Yorke, 1975].
(ii) If $\eta > \frac{1}{c_2}$, then

$$f_\lambda(v_c) = \frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}} \text{ is the local minimum and}$$

$$f_\lambda(-v_c) = -\frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}} \text{ is the local maximum.}$$

Since f_λ is strictly decreasing on $[-v_c, v_c]$ and $f_\lambda(v_c) \leq -\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}$, there exist one unique point $p_1 \in (0, v_c]$ such that $f_\lambda(p_1) = -\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}$. And since f_λ is odd, there exists one unique point $p_2 \in (-p_1, 0)$ such that $f_\lambda(p_2) = p_1$. Similarly, there exists one unique point $p_3 \in (0, -p_2)$ such that $f_\lambda(p_3) = p_2$ and then there exists one unique point $p_4 \in (-p_3, 0)$ such that $f_\lambda(p_4) = p_3$. Then there exists one unique point $p_5 \in (0, -p_4)$ such that $f_\lambda(p_5) = p_4$. Hence, we have

$$0 = f_\lambda^6(p_5) < p_5 < f_\lambda^2(p_5) < f_\lambda^4(p_5) \text{ (see Figure 5).}$$

Thus, f_λ^2 has points of all periods which implies chaos [by Li and Yorke, 1975]. Hence f_λ is chaotic in the sense of Li and Yorke. ■

Thus, we have the main theorem as below.

Theorem 3.2.10 (Chaotic Region of the 1D Wave System) *Let the parameters η, c_1, c_2, α, β be fixed in the 1D wave system (1.1)-(1.4) and satisfy the inequality*

$$\left|\frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)}\right| \geq \frac{\sqrt{3}}{c_2}, \tag{3.6}$$

and if the 1D wave system has initial conditions of type I, then the 1D wave system is chaotic.

Now we want to show the chaotic region of η when c_1, c_2, α, β are fixed. There are two different cases as follows.

Proposition 3.2.11 *Let the parameters c_1, c_2, α, β be fixed and satisfy the inequality*

$$3\sqrt{3}c_1 + (3\sqrt{3} - 2)c_2 - 2\alpha c_2^2 > 0.$$

Then the inequality (3.6) holds if and only if η satisfies either

$$\frac{3\sqrt{3}c_1 + (3\sqrt{3} - 2)c_2 - 2\alpha c_2^2}{(3\sqrt{3} + 2)c_1 c_2 + 2\alpha c_1 c_2^2 + 3\sqrt{3}c_2^2} \leq \eta < \frac{1}{c_2}$$

or

$$\frac{1}{c_2} < \eta \leq \frac{2c_2 + 2\alpha c_2^2 + 3\sqrt{3}(c_1 + c_2)}{(3\sqrt{3} - 2)c_1 c_2 + 3\sqrt{3}c_2^2 - 2\alpha c_1 c_2^2}.$$

Proof. (i) If $\eta < \frac{1}{c_2}$, then the inequality (3.6) is equivalent to

$$\eta c_2 \left[2c_1(1 + \alpha c_2) + 3\sqrt{3}(c_1 + c_2)\right] \geq 3\sqrt{3}(c_1 + c_2) - 2c_2(1 + \alpha c_2).$$

And since

$$3\sqrt{3}(c_1 + c_2) - 2c_2(1 + \alpha c_2) > 0,$$

the inequality (3.6) is equivalent to

$$\frac{1}{c_2} > \eta \geq \frac{3\sqrt{3}(c_1 + c_2) - 2c_2(1 + \alpha c_2)}{c_2\left[2c_1(1 + \alpha c_2) + 3\sqrt{3}(c_1 + c_2)\right]}.$$

(ii) If $\eta > \frac{1}{c_2}$, then the inequality (3.6) is equivalent to

$$2c_2(1 + \alpha c_2) + 3\sqrt{3}(c_1 + c_2) \geq \eta c_2 \left[3\sqrt{3}(2m + 1)(c_1 + c_2) - 2c_1(1 + \alpha c_2)\right].$$

Furthermore, the inequality (3.6) is equivalent to

$$\frac{2c_2(1 + \alpha c_2) + 3\sqrt{3}(c_1 + c_2)}{c_2\left[3\sqrt{3}(c_1 + c_2) - 2c_1(1 + \alpha c_2)\right]} \geq \eta > \frac{1}{c_2}.$$

By (i) and (ii), the inequality (3.6) holds if and only if η satisfies either

$$\frac{3\sqrt{3}c_1 + (3\sqrt{3}-2)c_2 - 2\alpha c_2^2}{(3\sqrt{3}+2)c_1c_2 + 2\alpha c_1 c_2^2 + 3\sqrt{3}c_2^2} \le \eta < \frac{1}{c_2}$$

or

$$\frac{1}{c_2} < \eta \le \frac{2c_2 + 2\alpha c_2^2 + 3\sqrt{3}(c_1+c_2)}{(3\sqrt{3}-2)c_1c_2 + 3\sqrt{3}c_2^2 - 2\alpha c_1 c_2^2}.$$

■

Proposition 3.2.12 *Let the parameters c_1, c_2, α, β be fixed and satisfy the inequality*

$$3\sqrt{3}c_1 + (3\sqrt{3}-2)c_2 - 2\alpha c_2^2 \le 0.$$

Then the inequality (3.6) holds if and only if η satisfies either

$$0 \le \eta < \frac{1}{c_2} \ or \ \frac{1}{c_2} < \eta \le \frac{2c_2 + 2\alpha c_2^2 + 3\sqrt{3}(c_1+c_2)}{(3\sqrt{3}-2)c_1c_2 + 3\sqrt{3}c_2^2 - 2\alpha c_1 c_2^2}.$$

Proof. If $\eta < \frac{1}{c_2}$, then the inequality (3.6) is equivalent to

$$\eta c_2 \left[2c_1(1+\alpha c_2) + 3\sqrt{3}(c_1+c_2)\right] \ge 3\sqrt{3}(c_1+c_2) - 2c_2(1+\alpha c_2).$$

Since

$$3\sqrt{3}(c_1+c_2) - 2c_2(1+\alpha c_2) \le 0,$$

we can conclude that the inequality (3.6) always holds. Thus the inequality (3.6) holds if and only if η satisfies either

$$\eta < \frac{1}{c_2} \text{ or } \frac{2c_2 + 2\alpha c_2^2 + 3\sqrt{3}(c_1+c_2)}{(3\sqrt{3}-2)c_1c_2 + 3\sqrt{3}c_2^2 - 2\alpha c_1 c_2^2} \ge \eta > \frac{1}{c_2}.$$

■

We show the system is chaotic if $c_1 \to \infty$, or $c_1 \to 0^+$, or $c_2 \to \infty$ as follows.

Proposition 3.2.13 *Let the parameters η, c_2, β be fixed and satisfy either*

$$\frac{3\sqrt{3}}{2+3\sqrt{3}} \le \eta c_2 < 1 \ or \ \frac{3\sqrt{3}}{3\sqrt{3}-2} \ge \eta c_2 > 1,$$

then the inequality (3.6) holds if c_1 is sufficiently large (while α is sufficiently small).

Proof. Since $\alpha \in \left(0, \frac{1}{c_1}\right]$, we can see the parameter $\alpha \to 0^+$ if the parameter $c_1 \to \infty$.

$$\lim_{c_1 \to \infty} \left| \frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)} \right| \geq \frac{\sqrt{3}}{c_2} \text{ implies } \left| \frac{2\eta}{3(1-\eta c_2)} \right| \geq \frac{\sqrt{3}}{c_2}.$$

Then we have the results by considering two different cases which one is $\eta > \frac{1}{c_2}$ and the other is $\eta < \frac{1}{c_2}$. ■

Proposition 3.2.14 *Let the parameters η, c_2, α, β be fixed and satisfy the inequality*

$$2(1+\alpha c_2) \left| \frac{1}{1-\eta c_2} \right| \geq 3\sqrt{3},$$

then there exists one positive $\varepsilon << 1$ such that the inequality (3.6) holds for all $c_1 < \varepsilon$.

Proof.

$$\lim_{c_1 \to 0^+} \left| \frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)} \right| \geq \frac{\sqrt{3}}{c_2} \text{ implies } \left| \frac{2(1+\alpha c_2)}{3c_2(1-\eta c_2)} \right| \geq \frac{\sqrt{3}}{c_2}.$$

■

Proposition 3.2.15 *Let the parameters η, c_1, α, β be fixed and satisfy the inequality*

$$2\alpha + 2\alpha\eta c_1 - 3\sqrt{3}\eta \geq 0,$$

then there exists one positive real number M such that the inequality (3.6) holds for all $c_2 > M$.

Proof.

$$\lim_{c_2 \to \infty} \left| \frac{2c_2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)} \right| \geq \sqrt{3} \text{ implies } 2\alpha(1+\eta c_1) \geq 3\sqrt{3}\eta.$$

■

Since $\lim_{c_2 \to 0^+} \left| \frac{2(1+\alpha c_2)(1+\eta c_1)}{3(c_1+c_2)(1-\eta c_2)} \right| = \frac{2(1+\eta c_1)}{3c_1}$ and $\lim_{c_2 \to 0^+} \frac{\sqrt{3}}{c_2} = \infty$, we can see the graph of the map f_λ is very flat. Thus, there exists no chaos if c_2 is sufficiently small.

3.3 Main results of the system (1.1)-(1.4)

Definition 3.3.1 (Initial Conditions of Type I) *We say the 1D wave system (1.1)-(1.4) has initial conditions of type I if the initial conditions satisfy the compatibility condition and the union* *of the ranges of*

$$F_0(x) \equiv \tfrac{\psi(x)-c_1\varphi'(x)}{c_1+c_2} \;\; on \;\; [0,1] \;\; and$$

$$F_1(x) \equiv \tfrac{1+\eta c_1}{1-\eta c_2}\tfrac{\psi(x)+c_2\varphi'(x)}{c_1+c_2} \;\; on \; [0,1]$$

contains the interval

$$I \equiv \left[-\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, \frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}\right];$$

i.e., $I \subseteq \Lambda$ *(see Remark 3.1.1 and Definition 3.1.2).*

Remark 3.3.2 *In the following theorems, we can compute*

$$c_1 = \frac{d+\sqrt{d^2+4c^2}}{2} \;\; and \; c_2 = \frac{-d+\sqrt{d^2+4c^2}}{2}$$

for any given c *and* d*. Conversely, we can compute* $d = c_1 - c_2$ *and* $c = \sqrt{c_1c_2}$ *for any given* c_1 *and* c_2*.*

Theorem 3.3.3 *Suppose that the parameters* c, d, α, β *are to be fixed in the 1D wave system (1.1)-(1.4) and satisfy the inequality*

$$3\sqrt{3}c_1 + (3\sqrt{3}-2)c_2 - 2\alpha c_2^2 > 0.$$

If the 1D wave system has initial conditions of type I and if η *satisfies either*

$$\frac{3\sqrt{3}c_1 + (3\sqrt{3}-2)c_2 - 2\alpha c_2^2}{(3\sqrt{3}+2)c_1c_2 + 2\alpha c_1c_2^2 + 3\sqrt{3}c_2^2} \leq \eta < \frac{1}{c_2}$$

or

$$\frac{1}{c_2} < \eta \leq \frac{2c_2 + 2\alpha c_2^2 + 3\sqrt{3}(c_1+c_2)}{(3\sqrt{3}-2)c_1c_2 + 3\sqrt{3}c_2^2 - 2\alpha c_1c_2^2},$$

then the 1D wave system is chaotic.

Proof. The results follow easily from Theorem 3.2.10 and Proposition 3.2.11. ■

Example 3.3.4 *Consider the one-dimensional wave system as below:*

$$\begin{cases} \omega_{tt} - \omega_{xx} = 0, & 0 < x < 1,\, t > 0. \\ \omega_x(0,t) + \eta\omega_t(0,t) = 0, & \eta > 0,\, \eta \neq 1,\, t > 0. \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in (0,1],\, \beta > 0,\, t > 0. \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

Suppose that the parameters α, β are to be fixed and the 1D wave system has initial conditions of type $I = \left[-\sqrt{\frac{1+\alpha}{\beta}}, \sqrt{\frac{1+\alpha}{\beta}}\right]$. If η satisfies either

$$1 < \eta \leq \frac{3\sqrt{3}+1+\alpha}{3\sqrt{3}-1-\alpha} \text{ or } \frac{3\sqrt{3}-1-\alpha}{3\sqrt{3}+1+\alpha} \leq \eta < 1,$$

then the wave system is chaotic (isotropic chaotic vibration of the linear wave system). In [8, 12, 13], Huang et al. showed the same results as above.

Theorem 3.3.5 *Suppose that the parameters c, d, α, β are to be fixed in the 1D wave system (1.1)-(1.4) and satisfy the inequality*

$$3\sqrt{3}c_1 + (3\sqrt{3}-2)c_2 - 2\alpha c_2^2 \leq 0.$$

If the 1D wave system has initial conditions of type I and if η satisfies either

$$0 \leq \eta < \frac{1}{c_2} \text{ or } \frac{1}{c_2} < \eta \leq \frac{2c_2 + 2\alpha c_2^2 + 3\sqrt{3}(c_1+c_2)}{(3\sqrt{3}-2)c_1c_2 + 3\sqrt{3}c_2^2 - 2\alpha c_1 c_2^2}.$$

then the 1D wave system is chaotic.

Proof. The results follow easily from Theorem 3.2.10 and Proposition 3.2.12. ■

Example 3.3.6 *Consider the one-dimensional wave system as below:*

$$\begin{cases} \omega_{tt} + 2\omega_{tx} - 3\omega_{xx} = 0, & 0 < x < 1,\, t > 0. \\ \omega_x(0,t) + \eta\omega_t(0,t) = 0, & \eta \geq 0,\, \eta \neq 1/3,\, t > 0. \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in \left[\frac{2\sqrt{3}-1}{3}, 1\right],\, \beta > 0,\, t > 0. \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

Suppose that the parameters α, β are to be fixed and the 1D wave system has initial conditions of *type I* $= \left[-\frac{1}{3}\sqrt{\frac{1+3\alpha}{3\beta}}, \frac{1}{3}\sqrt{\frac{1+3\alpha}{3\beta}}\right]$. *If* η *satisfies either*

$$0 \leq \eta < \frac{1}{3} \text{ or } \frac{1}{3} < \eta \leq \frac{2\sqrt{3}+1+3\alpha}{6\sqrt{3}-1-3\alpha},$$

then the wave system is chaotic (nonisotropic chaotic vibration of the linear wave system).

Theorem 3.3.7 *Suppose that the parameters* η, c_2, α, β *are to be fixed in the 1D wave system (1.1)-(1.4) and satisfy either*

$$\frac{3\sqrt{3}}{2+3\sqrt{3}} \leq \eta c_2 < 1 \text{ or } \frac{3\sqrt{3}}{3\sqrt{3}-2} \geq \eta c_2 > 1.$$

If the 1D wave system has initial conditions of type I, then the 1D wave system is chaotic for c_1 *is sufficiently large (while* α *is sufficiently small).*

Proof. The results follow easily from Theorem 3.2.10 and Proposition 3.2.13. ■

Theorem 3.3.8 *Suppose that the parameters* η, c_2, α, β *are to be fixed in the 1D wave system (1.1)-(1.4) and satisfy the inequality*

$$2(1+\alpha c_2)\left|\frac{1}{1-\eta c_2}\right| \geq 3\sqrt{3}.$$

If the 1D wave system has initial conditions of type I, then the 1D wave system is chaotic for c_1 *is sufficiently small.*

Proof. The results follow easily from Theorem 3.2.10 and Proposition 3.2.14. ■

Theorem 3.3.9 *Suppose that the parameters* η, c_2, α, β *are to be fixed in the 1D wave system (1.1)-(1.4) and satisfy the inequality*

$$2\alpha + 2\alpha\eta c_1 - 3\sqrt{3}\eta \geq 0.$$

If the 1D wave system has initial conditions of type I, then the 1D wave system (1.12) is chaotic for c_2 *is sufficiently large.*

Proof. The results follow easily from Theorem 3.2.10 and Proposition 3.2.15. ■

Example 3.3.10 *Consider the one-dimensional wave system (1.8)-(1.11) as below:*

$$\begin{cases} \omega_{tt} + v\omega_{tx} - \omega_{xx} = 0, & v > 0,\ 0 < x < 1,\ t > 0. \\ \omega_x(0,t) = 0, & t > 0. \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in \left(0, \frac{v+\sqrt{v^2+4}}{2}\right],\ \beta > 0,\ t > 0. \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

Suppose that the parameter β is to be fixed and the parameters v, α satisfy the inequality

$$\left|\frac{2(1+\alpha c_2)}{3(c_1+c_2)}\right| \geq \frac{\sqrt{3}}{c_2},$$

where $c_1 = \frac{-v+\sqrt{v^2+4}}{2}$ and $c_2 = \frac{v+\sqrt{v^2+4}}{2}$. If the 1D wave system has initial conditions of type I where $I = \left[-\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, \frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}\right]$, then the wave system is chaotic (nonisotropic chaotic vibration of the linear wave system). In [5], Chen et al. showed the same results as above in the proof of theorem 3.2.

Chapter 4

The 1D wave system (1.12)

In this chapter, the 1D wave system (1.12) is considered:

$$\begin{cases} \omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0, & d \in \mathbb{R},\, c > 0,\, 0 < x < 1,\, t > 0, \\ \omega_t(0,t) + \eta\omega_x(0,t) = 0, & \eta > 0,\, \eta \neq c_2,\, t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^{2m+1}(1,t), & \alpha \in \left(0, \frac{1}{c_1}\right],\, \beta, t > 0,\, m \in \mathbb{N}, \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

4.1 Chaotic vibrations of the system (1.12)

The general solution of $(1.12)_1$ is

$$\omega(x,t) = u(c_1 t + x) + v(c_2 t - x), \tag{4.1}$$

where u, v are arbitrary C^2-function. Substituting (4.1) in $(1.12)_2$ and $(1.12)_3$ we have

$$u'(c_1 t) = -\frac{c_1 - \eta}{c_2 + \eta} v'(c_2 t),\, t > 0, \tag{4.2}$$

and

$$\begin{aligned} &\beta(c_1 u'(c_1 t + 1) + c_2 v'(c_2 t - 1))^{2m+1} + \left(\tfrac{1}{c_1} - \alpha\right)(c_1 u'(c_1 t + 1) \\ &+ c_2 v'(c_2 t - 1)) - \left(1 + \tfrac{c_2}{c_1}\right) v'(c_2 t - 1) = 0,\, t > 0. \end{aligned} \tag{4.3}$$

When $\eta = c_2$ in (4.2), we have

$$u'(c_1 t) = 0 \text{ for } t > 0 \Rightarrow u(c_1 t + x) = C \text{ for } t > 0.$$

Thus, we consider the case $\eta \neq c_2$. And depends on (4.2), we can use v' to replace u' in (4.3) to derive one difference equation as follows.

By using the substitution

$$z(c_1 t) = \begin{cases} v'\left(\frac{c_2}{c_1}\left(c_1 t - \frac{c_1}{c_2}\right)\right), \ 0 \le t \le \frac{1}{c_2}, \\ \\ \frac{\eta + c_1}{\eta - c_2} u'\left(c_1 t - \frac{c_1}{c_2}\right), \ t > \frac{1}{c_2}, \end{cases}$$

we have the difference equation

$$\begin{aligned} \beta\left(c_1 \tfrac{\eta - c_2}{\eta + c_1} z(\tau + \Delta) + c_2 z(\tau)\right)^{2m+1} + \left(\tfrac{1}{c_1} - \alpha\right)\Big(c_1 \tfrac{\eta - c_2}{\eta + c_1} z(\tau + \Delta) \\ + c_2 z(\tau)\Big) - \left(1 + \tfrac{c_2}{c_1}\right) z(\tau) = 0, \end{aligned} \tag{4.4}$$

where $\tau = c_1 t$, $\Delta = 1 + \frac{c_1}{c_2}$.

And the initial condition of (4.4) is

$$z(c_1 t) = \begin{cases} \frac{\psi(1 - c_2 t) - c_1 \varphi'(1 - c_2 t)}{c_1 + c_2}, \ 0 \le t \le \frac{1}{c_2}. \\ \\ \frac{\eta + c_1}{\eta - c_2} \frac{\psi\left(c_1 t - \frac{c_1}{c_2}\right) + c_2 \varphi'\left(c_1 t - \frac{c_1}{c_2}\right)}{c_1 + c_2}, \ \frac{1}{c_2} < t \le \frac{1}{c_1} + \frac{1}{c_2}. \end{cases}$$

Remark 4.1.1 *In this paper, we assume that the initial value $\varphi(x)$ and $\psi(x)$ are chosen such that $z(\tau)$ is continuous on $[0, 1 + \frac{c_1}{c_2}]$ and satisfy the compatibility condition*

$$\beta(c_1 \tfrac{\eta - c_2}{\eta + c_1} z(\Delta) + c_2 z(0))^{2m+1} + (\tfrac{1}{c_1} - \alpha)(c_1 \tfrac{\eta - c_2}{\eta + c_1} z(\Delta)$$

$$+ c_2 z(0)) - (1 + \tfrac{c_2}{c_1}) z(0) = 0.$$

Definition 4.1.2 *In the following theorems of this paper, we denote the range of $z(\tau)$ on $[0, \Delta]$ to be the compact interval Λ, i.e. $\Lambda = z([0, \Delta])$.*

We show the dependence of $z(\tau+\Delta)$ on $z(\tau)$ is given implicitly by one C^1-function f_λ as follows.

Lemma 4.1.3 (Existence and Uniqueness of the Solution) *Let the parameters c_1, c_2, η, α, β be fixed in (4.4) with $c_1 > 0$, $c_2 > 0$, $\eta \geq 0$, $\eta \neq c_2$, $\alpha \in \left(0, \frac{1}{c_1}\right]$ and $\beta > 0$. Then there exists one C^1-function f_λ such that*

$$f_\lambda(z(t)) = z(t+\Delta) \text{ for all } t > 0,$$

where $\lambda = (c_1, c_2, \eta, \alpha, \beta)$.

Proof. Let

$$H_\lambda(u,v) = \beta\left(c_1\tfrac{\eta-c_2}{\eta+c_1}u + c_2 v\right)^{2m+1} + \left(\tfrac{1}{c_1} - \alpha\right)\left(c_1\tfrac{\eta-c_2}{\eta+c_1}u + c_2 v\right)$$
$$-\left(1+\tfrac{c_2}{c_1}\right)v = 0,$$

where $u = z(\tau+\Delta)$, $v = z(\tau)$, $\lambda = (c_1, c_2, \eta, \alpha, \beta)$.

(i) If $\alpha = \frac{1}{c_1}$, then $H_\lambda(u,v) = 0$ implies

$$c_1\frac{\eta-c_2}{\eta+c_1}u = \sqrt[2m+1]{\left(1+\frac{c_2}{c_1}\right)\frac{v}{\beta}} - c_2 v.$$

Hence, the C^1-function f_λ exists.

(ii) If $\alpha \in \left(0, \frac{1}{c_1}\right)$, then

$$\frac{\partial}{\partial u}H_\lambda(u,v) = 3\beta c_1\frac{\eta-c_2}{\eta+c_1}\left(c_1\frac{\eta-c_2}{\eta+c_1}u + c_2 v\right)^{2m} + (1-\alpha c_1)\frac{\eta-c_2}{\eta+c_1} \neq 0.$$

By the implicit function theorem, the C^1-function f_λ exists. ∎

Definition 4.1.4 *We denote $f_\lambda(z(\tau)) = z(\tau+\Delta)$ to be the function satisfies (4.4) for all $\tau \geq 0$, where $\lambda = (\eta, c_1, c_2, \alpha, \beta, m)$.*

Since

$$f_\lambda(z(\tau)) = z(\tau+\Delta) \text{ for all } \tau > 0,$$

we can use the map f_λ and the interval Λ to generate $z(\tau)$ for all $\tau > 0$. And the corresponding solution of the 1D wave system (1.12) is calculated via the formulae

$$\omega(x,t) = \int_{\frac{1}{c_2}}^{t+\frac{x}{c_1}+\frac{1}{c_2}} c_1 \frac{\eta - c_2}{\eta + c_1} z(c_1\tau)d\tau + \int_0^{t-\frac{x}{c_2}+\frac{1}{c_2}} c_2 z(c_1\tau)d\tau.$$

Definition 4.1.5 (Chaotic Vibration) *The solution of the 1D wave system is said to be chaotic if the map $f_\lambda : \Lambda \to \mathbb{R}$ is chaotic in the sense of Li-Yorke; i.e., there exists one nonempty invariant subset $\Lambda_0 \subseteq \Lambda$ such that f_λ is chaotic in the sense of Li-Yorke on Λ_0 (see Definition 2.0.7).*

Remark 4.1.6 *In this paper, we say the 1D wave system is chaotic if its solution is chaotic.*

4.2 The chaotic region of the system (1.12)

In this section, we consider (4.4) as below:

$$H(x,y) = \beta\left(c_1\tfrac{\eta-c_2}{\eta+c_1}y + c_2 x\right)^{2m+1} + \left(\tfrac{1}{c_1} - \alpha\right)\left(c_1\tfrac{\eta-c_2}{\eta+c_1}y + c_2 x\right)$$

$$-\left(1 + \tfrac{c_2}{c_1}\right)x = 0,$$

where η, c_1, c_2, β are positve ($\eta \neq c_2$), $0 < \alpha \leq \frac{1}{c_1}$ and $m \in \mathbb{N}$. And we have the results as follows.

Definition 4.2.1 *We denote*

$$v_c = \frac{c_1}{c_1+c_2}\left[\frac{1+\alpha c_2}{c_2(2m+1)} + \frac{1}{c_1} - \alpha\right]\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}$$

and

$$M = \frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}$$

in the following lemmas.

We show the local maximum, minimum and piecewise monotonicity of the function h which satisfies (3.4) as below.

Lemma 4.2.2 (Local Maximum, Minimum and Piecewise Monotonicity) *Let $y = h(x)$ be the unique function which satisfies (4.4). Then the fuction h is odd and h has local extrema at*

(v_c, M) *and* $(-v_c, -M)$. *Furthermore, the function* h *is strictly monotonic on* $(-\infty, -v_c)$, $(-v_c, v_c)$ *and* (v_c, ∞).

Proof. Since $H(-x, h(-x)) = H(-x, -h(x)) = 0$, the function h is odd. Then use

$$\tfrac{d}{dx}H(x,y) = (2m+1)\beta\left(c_1\tfrac{\eta-c_2}{\eta+c_1}y + c_2x\right)^{2m}\left(c_1\tfrac{\eta-c_2}{\eta+c_1}y' + c_2\right)$$

$$+\left(\tfrac{1}{c_1}-\alpha\right)\left(c_1\tfrac{\eta-c_2}{\eta+c_1}y' + c_2\right) - \left(1+\tfrac{c_2}{c_1}\right) = 0,$$

and carry out the computations, we have the results. ■

We show the x-axis Intercepts, fixed points and intersections with the line $y = -x$ of the function h as below.

Lemma 4.2.3 **(x-axis Intercepts)** *The function* h *intersects the* x*-axis at the points*

$$\left(-\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, 0\right),\ (0,0),\ \left(\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, 0\right).$$

Proof. Straightforward verification by computing

$$H(x,0) = \beta(0 + c_2x)^{2m+1} + \left(\frac{1}{c_1}-\alpha\right)(0+c_2x) - \left(1+\frac{c_2}{c_1}\right)x = 0,$$

we have $x\left(\beta c_2^{2m+1}x^{2m} - \alpha c_2 - 1\right) = 0$ which implies $x = 0, \pm\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}$. ■

Lemma 4.2.4 **(Intersections with the Line $y = x$)** *The function* h *intersects the line* $y = x$ *at the points*

$$\left(-\frac{\eta+c_1}{\eta(c_1+c_2)}\sqrt[2m]{\frac{1+\alpha\eta}{\beta\eta}}, -\frac{\eta+c_1}{\eta(c_1+c_2)}\sqrt[2m]{\frac{1+\alpha\eta}{\beta\eta}}\right),\ (0,0),$$

and

$$\left(\frac{\eta+c_1}{\eta(c_1+c_2)}\sqrt[2m]{\frac{1+\alpha\eta}{\beta\eta}}, \frac{\eta+c_1}{\eta(c_1+c_2)}\sqrt[2m]{\frac{1+\alpha\eta}{\beta\eta}}\right).$$

Proof. Straightforward verification by computing $H(x, x) = 0$. ■

Definition 4.2.5 *We denote the point*

$$B = \frac{\eta+c_1}{|2c_1c_2 + (c_2-c_1)\eta|}\sqrt[2m]{\frac{2\eta + 2\alpha c_1c_2 + (c_1-c_2) + \alpha\eta(c_2-c_1)}{\beta[2c_1c_2 + (c_2-c_1)\eta]}}$$

in the following lemmas.

Lemma 4.2.6 (Intersections with the Line $y = -x$) *Let the parameters η, c_1, c_2, α, β, m be fixed in (4.4) with*

$$\eta > c_2 \text{ and } 2c_1c_2 + (c_2 - c_1)\eta \neq 0.$$

Then the function h intersects the line $y = -x$ at the points

$$(-B, B),\ (0, 0),\ (B, -B),$$

if (i) $c_1 \leq c_2$ or if (ii) $c_1 > c_2$ and $2c_1c_2+(c_2-c_1)\eta > 0$ or if (iii) $c_1 > c_2$ and $2c_1c_2+(c_2-c_1)\eta < 0$ and $\frac{2\eta+c_1-c_2}{2c_1c_2+(c_2-c_1)\eta} > -\alpha$.
Furthermore, if the parameters are not in these three cases then the function h intersects the line $y = -x$ only at the point $(0, 0)$.

Proof. Straightforward verification by computing $H(x, -x) = 0$ and the three cases provide that

$$\frac{2\eta + 2\alpha c_1 c_2 + (c_1 - c_2) + \alpha\eta\,(c_2 - c_1)}{\beta[2c_1c_2 + (c_2 - c_1)\eta]} \text{ is positve.}$$

Otherwise, $\frac{2\eta+2\alpha c_1c_2+(c_1-c_2)+\alpha\eta(c_2-c_1)}{\beta[2c_1c_2+(c_2-c_1)\eta]}$ is zero or negative. ■

We show the function h has bounded invariant interval or bounded invariant cantor-like subset in following lemmas.

Lemma 4.2.7 (Bounded Invariant Interval) *Let the parameters η, c_1, c_2, α, β, m be fixed in (4.4).*
(i) If $0 < \eta < c_2$ and $|M| = \left|\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}\right| \leq \frac{\eta+c_1}{\eta(c_1+c_2)}\sqrt[2m]{\frac{1+\alpha\eta}{\beta\eta}}$, then the iterates of every point in the set

$$U \equiv \left(-\infty, -\frac{\eta + c_1}{\eta\,(c_1 + c_2)}\sqrt{\frac{1 + \alpha\eta}{\beta\eta}}\right) \cup \left(\frac{\eta + c_1}{\eta\,(c_1 + c_2)}\sqrt{\frac{1 + \alpha\eta}{\beta\eta}}, \infty\right)$$

escape to $\pm\infty$, while those of any point in $\mathbb{R}\backslash\overline{U}$ are attracted to the bounded invariant interval

$$\left[-\left|\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}\right|, \left|\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}\right|\right]$$

of h, i.e., $[-|M|, |M|]$ of h.
(ii) If $\eta > c_2$ and h intersects the line $y = -x$ at three points and $|M| \leq B$, then the iterates of

every point in the set $U \equiv (-\infty, -B) \cup (B, \infty)$ *escape to* $\pm\infty$*, while those of any point in* $\mathbb{R}\backslash\overline{U}$ *are attracted to the bounded invariant interval* $[-|M|, |M|]$ *of* h*.*

(iii) If $\eta > c_2$ *and* h *intersects the line* $y = -x$ *at* $(0,0)$*, then the iterates of every point in* $\mathbb{R}$ *are attracted to the bounded invariant interval* $[-|M|, |M|]$ *of* h*.*

Proof. The results of (i) and (ii) follow easily from the above lemmas and other piecewise monotonic properties of h, as can be directly comfirmed by graphical analysis (see Figure 6 and Figure 7). We omit the details.

(iii) If $\eta > c_2$ and h intersects the line $y = -x$ only at $(0,0)$, then $|h(x)| < |x|$ for all

$$x \in \left(-\infty, -\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}\right) \cup \left(\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, \infty\right).$$

Thus, $|h^n(x)|$ is strctly decreasing for $n \leq n_0$, where

$$h^{n_0}(x) \in \left(-\infty, -\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}\right) \cup \left(\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, \infty\right)$$

and

$$h^{n_0+1}(x) \notin \left(-\infty, -\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}\right) \cup \left(\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, \infty\right).$$

Hence, the iterates of every point in $\mathbb{R}$ are attracted to the bounded invariant interval $[-|M|, |M|]$ of h (see Figure 8). ■

Lemma 4.2.8 (Bounded Cantor-like Invariant Subset) *The bounded invariant interval*

$$\left[-\left|\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}\right|, \left|\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}\right|\right]$$

no longer exists in the case (i) and (ii) of the Lemma 4.2.7 if the condition

$$|M| \leq \frac{\eta+c_1}{\eta(c_1+c_2)}\sqrt[2m]{\frac{1+\alpha\eta}{\beta\eta}} \text{ or } |M| \leq B$$

is violated. Instead, we have a bounded Cantor-like invariant set.

Proof. The method of proof is now standard, see [18, Sec. 1.7], for example. ■

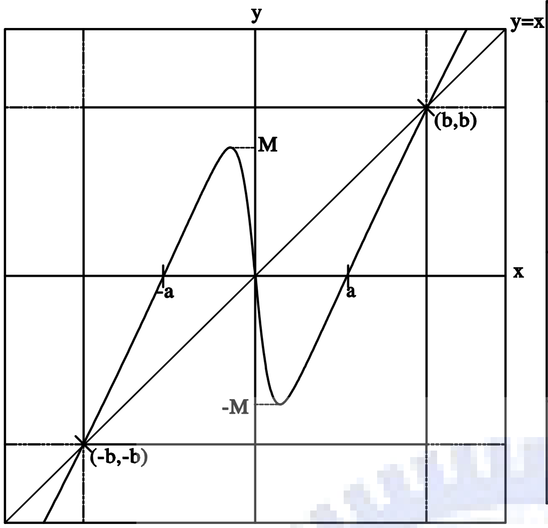

Figure 6: The map h with $\eta < c_2$ and $|M| \leq b$.

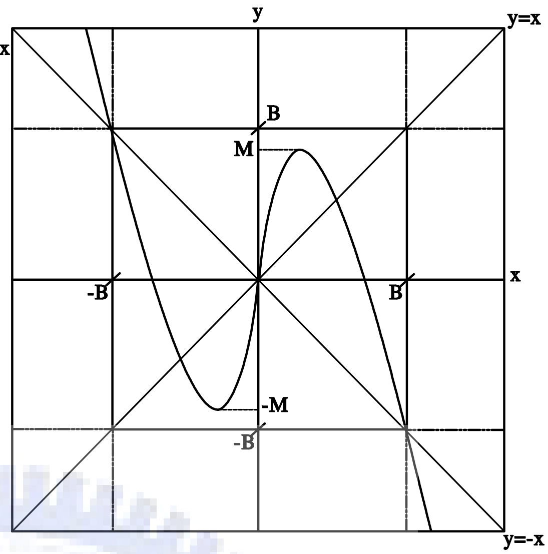

Figure 7: The map h with $\eta > c_2$ and $|M| \leq B$.

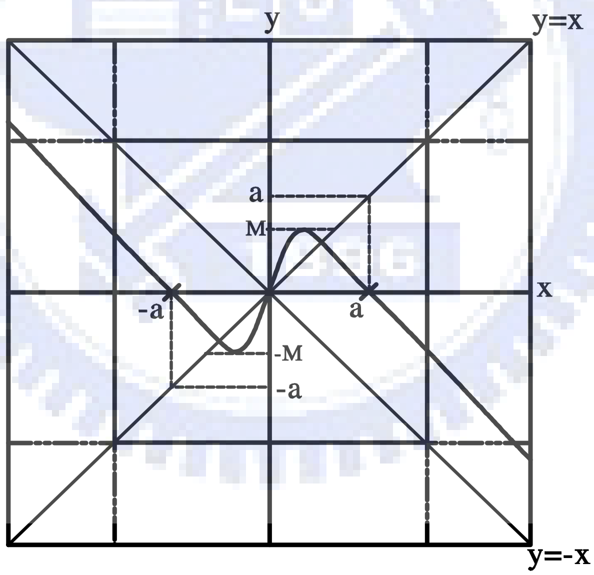

Figure 8: The map h with $\eta > c_2$.

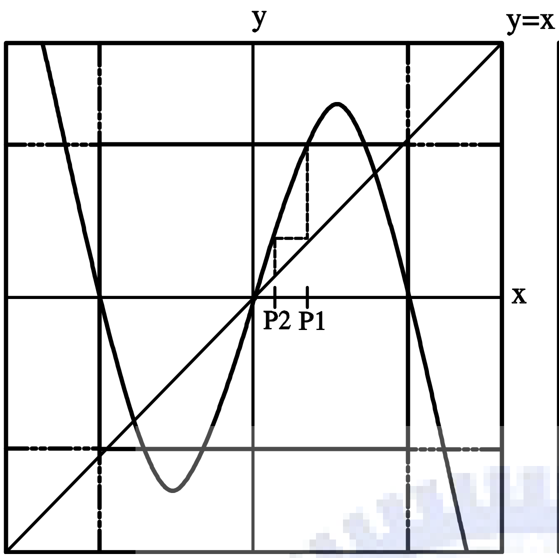

Figure 9: $h^3(p_2) < p_2 < h(p_2) < h^2(p_2)$.

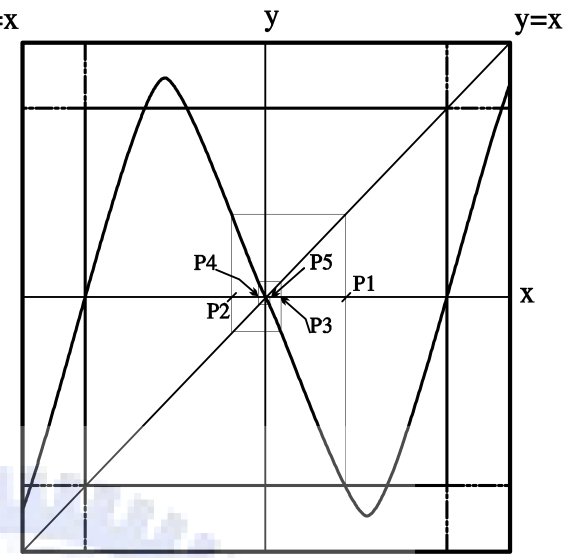

Figure 10: $h^6(p_5) < p_5 < h^2(p_5) < h^4(p_5)$.

We have the chaotic region of the function h as below.

Lemma 4.2.9 (Chaotic Region of the 1D Wave System) *Let the parameters η, c_1, c_2, α, β, m be fixed in (4.4) and satisfy the inequality*

$$\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\left|\frac{\eta+c_1}{\eta-c_2}\right|\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}} \geq \frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, \tag{4.5}$$

then the interval map h is chaotic in the sense of Li-Yorke if the domain of h contains the interval

$$\left[-\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, \frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}\right].$$

Proof. (i) If $\eta > c_2$, then

$$h(v_c) = \frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}} \text{ is the local maximum.}$$

Since h is strictly increasing on $[0, v_c]$ and $h(v_c) \geq \frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}$, there exists one unique point $p_1 \in (0, v_c]$ such that $h(p_1) = \frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}$. Similarly, there exists one unique point $p_2 \in (0, p_1)$ such that $h(p_2) = p_1$. Hence we have

$$0 = h^3(p_2) < p_2 < h(p_2) < h^2(p_2) \text{ (see Figure 9).}$$

Thus h has points of all periods which implies chaos [by Li and Yorke, 1975].

(ii) If $0 < \eta < c_2$, then

$$h(v_c) = \frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}} \text{ is the local minimum and}$$

$$h(-v_c) = -\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}} \text{ is the local maximum.}$$

Since h is strictly decreasing on $[-v_c, v_c]$ and $h(v_c) \leq -\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}$, there exist one unique point $p_1 \in (0, v_c]$ such that $h(p_1) = -\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}$. And since h is odd, there exists one unique point $p_2 \in (-p_1, 0)$ such that $h(p_2) = p_1$. Similarly, there exists one unique point $p_3 \in (0, -p_2)$ such that $h(p_3)$
$= p_2$ and then there exists one unique point $p_4 \in (-p_3, 0)$ such that $h(p_4) = p_3$. Then there exists one unique point $p_5 \in (0, -p_4)$ such that $h(p_5) = p_4$. Hence we have

$$0 = h^6(p_5) < p_5 < h^2(p_5) < h^4(p_5) \text{ (see Figure 10).}$$

Thus $g = h^2$ has points of all periods which implies chaos [by Li and Yorke, 1975]. Hence h is chaotic in the sense of Li and Yorke. ■

Thus, we have the main theorem as below.

Theorem 4.2.10 (Chaotic Region of the 1D Wave System) *Let the parameters η, c_1, c_2, α, β, m be fixed in the 1D wave system (1.12) and satisfy the inequality*

$$\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\left|\frac{\eta+c_1}{\eta-c_2}\right| \geq \frac{\sqrt[2m]{2m+1}}{c_2}, \tag{4.6}$$

and if the 1D wave system has initial conditions of type I, then the 1D wave system is chaotic.

Now we want to show the chaotic region of η when c_1, c_2, α, β, m are fixed. There are two different cases as follows.

Proposition 4.2.11 *Let the parameters c_1, c_2, α, β, m be fixed and satisfy the inequality*

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) > 0.$$

Then the inequality (4.6) holds if and only if η satisfies either

$$c_2 < \eta \leq \frac{c_2\left[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)\right]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)}$$

or

$$0 < \frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \leq \eta < c_2.$$

Proof. (i) If $\eta > c_2$, then the inequality (4.6) is equivalent to

$$c_2\left[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)\right] \geq$$

$$\eta\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)\right].$$

And since

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) > 0,$$

the inequality (4.6) is equivalent to

$$c_2 < \eta \leq \frac{c_2\left[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)\right]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)}.$$

(ii) If $\eta < c_2$, then the inequality (4.6) is equivalent to

$$\eta[2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)] \geq$$

$$c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)\right].$$

Furthermore, the inequality (4.6) is equivalent to

$$\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \leq \eta < c_2.$$

And since

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2) \geq$$

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\frac{c_2}{c_1}) > 0,$$

we have

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2)+\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)}>0.$$

By (i) and (ii), the inequality (4.6) holds if and only if η satisfies either

$$c_2<\eta\le\frac{c_2\left[2mc_1(1+\alpha c_2)+\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)\right]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_2(1+\alpha c_2)}$$

or

$$0<\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_1(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2)+\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)}\le\eta<c_2.$$

■

Proposition 4.2.12 *Let the parameters c_1, c_2, α, β, m be fixed and satisfy the inequality*

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_2(1+\alpha c_2)\le 0.$$

Then the inequality (4.6) holds if and only if η satisfies either

$$\eta>c_2 \text{ or } 0<\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_1(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2)+\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)}\le\eta<c_2.$$

Proof. If $\eta>c_2$, then the inequality (4.6) is equivalent to

$$c_2\left[2mc_1(1+\alpha c_2)+\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)\right]\ge$$

$$\eta\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_2(1+\alpha c_2)\right].$$

Since

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_2(1+\alpha c_2)\le 0,$$

we can conclude that the inequality (4.6) always holds. Thus the inequality (4.6) holds if and only if η satisfies either

$$\eta>c_2 \text{ or } 0<\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)-2mc_1(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2)+\sqrt[2m]{2m+1}(2m+1)(c_1+c_2)}\le\eta<c_2.$$

■

Now we want to show the chaotic region of c_1 when η, c_2, α, β, m are fixed. There are three different cases as follows.

Proposition 4.2.13 *Let the parameters η, c_2, α, β, m be fixed and satisfy the inequality*

$$\eta > c_2 \text{ and } 2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) \geq 0,$$

then the inequality (4.6) holds for any c_1.

Proof. If $\eta > c_2$, then the inequality (4.6) is equivalent to

$$c_1 \left[2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)\right] \geq$$

$$c_2 \left[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)\right].$$

Since

$$2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) \geq 0,$$

we have

$$\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2) < 0.$$

Thus the inequality (4.6) holds for any c_1. ∎

Proposition 4.2.14 *Let the parameters η, c_2, α, β, m be fixed and satisfy the inequality*

$$\eta > c_2 \text{ and } 2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) < 0.$$

If

$$\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)} > 0$$

and if c_1 satisfies

$$0 < c_1 \leq \min\left\{\frac{1}{\alpha}, \frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)}\right\},$$

then the inequality (4.6) holds.

Proof. If $\eta > c_2$, then the inequality (4.6) is equivalent to

$$c_1\left[2mc_2(1+\alpha c_2)-\sqrt[2m]{2m+1}(2m+1)\left(\eta-c_2\right)\right]\geq$$

$$c_2\left[\sqrt[2m]{2m+1}(2m+1)\left(\eta-c_2\right)-2m\eta(1+\alpha c_2)\right].$$

Since

$$2mc_2(1+\alpha c_2)-\sqrt[2m]{2m+1}(2m+1)(\eta-c_2)<0,$$

then the inequality (4.6) is equivalent to

$$c_1\leq\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)\left(\eta-c_2\right)-2m\eta(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2)-\sqrt[2m]{2m+1}(2m+1)\left(\eta-c_2\right)}.$$

And since

$$\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)\left(\eta-c_2\right)-2m\eta(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2)-\sqrt[2m]{2m+1}(2m+1)\left(\eta-c_2\right)}>0,$$

the inequality (4.6) holds if

$$0<c_1\leq\min\{\frac{1}{\alpha},\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)\left(\eta-c_2\right)-2m\eta(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2)-\sqrt[2m]{2m+1}(2m+1)\left(\eta-c_2\right)}\}.$$

■

Proposition 4.2.15 *Let the parameters η, c_2, α, β, m be fixed and satisfy the inequality*

$$\eta<c_2 \text{ and } 2mc_2(1+\alpha c_2)-\sqrt[2m]{2m+1}(2m+1)(c_2-\eta)>0.$$

Then the inequality (4.6) holds if and only if c_1 satisfies

$$c_1\geq\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_2-\eta)-2m\eta(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2)-\sqrt[2m]{2m+1}(2m+1)(c_2-\eta)}.$$

Proof. If $\eta < c_2$, then the inequality (4.6) is equivalent to

$$c_1\left[2mc_2(1+\alpha c_2)-\sqrt[2m]{2m+1}(2m+1)\left(c_2-\eta\right)\right]\geq$$

$$c_2\left[\sqrt[2m]{2m+1}(2m+1)\left(c_2-\eta\right)-2m\eta(1+\alpha c_2)\right].$$

Since

$$2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) > 0,$$

then the inequality (4.6) is equivalent to

$$c_1 \geq \frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_2-\eta) - 2m\eta(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2-\eta)}.$$

■

4.3 Main results of the system (1.12)

Definition 4.3.1 (Initial Conditions of Type I) *We say the 1D wave system (1.12) has initial conditions of type I if the initial conditions satisfy the compatibility condition and the union of the ranges of*

$$F_0(x) \equiv \tfrac{\psi(x) - c_1\varphi'(x)}{c_1+c_2} \text{ on } [0,1] \text{ and}$$

$$F_1(x) \equiv \tfrac{\eta+c_1}{\eta-c_2}\tfrac{\psi(x)+c_2\varphi'(x)}{c_1+c_2} \text{ on } [0,1]$$

contains the interval

$$I \equiv \left[-\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, \frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}\right];$$

i.e., $I \subseteq \Lambda$ (see Remark 4.1.1 and Definition 4.1.2).

Remark 4.3.2 *In the following theorems, we can compute*

$$c_1 = \left(d + \sqrt{d^2+4c^2}\right)/2 \text{ and } c_2 = \left(-d + \sqrt{d^2+4c^2}\right)/2$$

for any given c and d. Conversely, we can compute $d = c_1 - c_2$ and $c = \sqrt{c_1c_2}$ for any given c_1 and c_2.

Theorem 4.3.3 *Suppose that the parameters c, d, α, β, m are to be fixed in the 1D wave system (1.12) and satisfy the inequality*

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) > 0.$$

If the 1D wave system has initial conditions of type I and if η satisfies either

$$c_2 < \eta \leq \frac{c_2\left[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)\right]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)}$$

or

$$\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \leq \eta < c_2,$$

then the 1D wave system (1.12) is chaotic.

Proof. The results follow easily from Theorem 4.2.10 and Proposition 4.2.11. ∎

Example 4.3.4 *Consider the one-dimensional wave system (1.5)-(1.7) as below:*

$$\begin{cases} \omega_{tt} - \omega_{xx} = 0, & 0 < x < 1,\ t > 0. \\ \omega_x(0,t) + \eta\omega_t(0,t) = 0, & \eta > 0,\ \eta \neq 1,\ t > 0. \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in (0,1],\ \beta > 0,\ t > 0. \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

Suppose the parameters α, β are to be fixed and the 1D wave system has initial conditions of type I, where $I = \left[-\sqrt{\frac{1+\alpha}{\beta}}, \sqrt{\frac{1+\alpha}{\beta}}\right]$. If η satisfies either

$$1 < \eta \leq \frac{3\sqrt{3}+1+\alpha}{3\sqrt{3}-1-\alpha} \ \ or\ \ \frac{3\sqrt{3}-1-\alpha}{3\sqrt{3}+1+\alpha} \leq \eta < 1,$$

then the wave system is chaotic. In [2], Chen et al. showed the same result as above.

Theorem 4.3.5 *Suppose that the parameters c, d, α, β, m are to be fixed in the 1D wave system (1.12) and satisfy the inequality*

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) \leq 0.$$

If the 1D wave system has initial conditions of type I and if η satisfies either

$$\eta > c_2 \ \ or\ \ \frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)\right]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \leq \eta < c_2,$$

then the 1D wave system (1.12) is chaotic.

Proof. The results follow easily from Theorem 4.2.10 and Proposition 4.2.12. ∎

Example 4.3.6 *Consider the 1D wave system as below:*

$$\begin{cases} \omega_{tt} + 2\omega_{tx} - 3\omega_{xx} = 0, & 0 < x < 1,\ t > 0, \\ \omega_t(0,t) + \eta\omega_x(0,t) = 0, & \eta > 0,\ \eta \neq 3,\ t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in \left[\frac{2\sqrt{3}-1}{3}, 1\right],\ \beta > 0,\ t > 0, \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

Suppose that the parameters α, β are to be fixed and the 1D wave system has initial conditions of type I, where $I = \left[-\frac{1}{3}\sqrt{\frac{1+3\alpha}{3\beta}}, \frac{1}{3}\sqrt{\frac{1+3\alpha}{3\beta}}\right]$. If η satisfies either

$$\eta > 3 \text{ or } \frac{6\sqrt{3} - 1 - 3\alpha}{2\sqrt{3} + 1 + 3\alpha} \leq \eta < 3,$$

then the 1D wave system (1.12) is chaotic.

Theorem 4.3.7 *Suppose that the parameters η, c_2, α, β, m are to be fixed in the 1D wave system (1.12) and satisfy the inequality*

$$\eta > c_2 \text{ and } 2mc_2(1 + \alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) \geq 0.$$

If the 1D wave system has initial conditions of type I, then the 1D wave system (1.12) is chaotic for any c_1.

Proof. The results follow easily from Theorem 4.2.10 and Proposition 4.2.13. ∎

Theorem 4.3.8 *Suppose that the parameters η, c_2, α, β, m are to be fixed in the 1D wave system (1.12) and satisfy the inequality*

$$\eta > c_2 \text{ and } 2mc_2(1 + \alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) < 0.$$

If the 1D wave system has initial conditions of type I and if

$$\frac{c_2\left[\sqrt[2m]{2m+1}(2m+1)\,(\eta - c_2) - 2m\eta(1 + \alpha c_2)\right]}{2mc_2(1 + \alpha c_2) - \sqrt[2m]{2m+1}(2m+1)\,(\eta - c_2)} > 0,$$

then for any c_1 satisfies

$$0 < c_1 \leq \min \left\{ \frac{1}{\alpha}, \frac{c_2 \left[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2) \right]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)} \right\}$$

the 1D wave system (1.12) is chaotic.

Proof. The results follow easily from Theorem 4.2.10 and Proposition 4.2.14. ∎

Theorem 4.3.9 *Suppose that the parameters η, c_2, α, β, m are to be fixed in the 1D wave system (1.12) and satisfy the inequality*

$$\eta < c_2 \text{ and } 2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2 - \eta) > 0.$$

If the 1D wave system has initial conditions of type I and if for any c_1 satisfies

$$c_1 \geq \frac{c_2 \left[\sqrt[2m]{2m+1}(2m+1)(c_2 - \eta) - 2m\eta(1+\alpha c_2) \right]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2 - \eta)},$$

then the 1D wave system (1.12) is chaotic.

Proof. The results follow easily from Theorem 4.2.10 and Proposition 4.2.15. ∎

Chapter 5

Three examples

In this chapter, we consider the 1D wave systems in [2, 5, 8, 12, 13].

5.1 One special case of the system (1.12)

In [2], Chen et al. consider the 1D wave system as below:

$$\omega_{tt} - \omega_{xx} = 0,\, 0 < x < 1,\, t > 0, \tag{5.1}$$

with the boundary conditions

$$\begin{cases} \omega_t(0,t) = -\eta\omega_x(0,t), & \eta > 0,\, \eta \neq 1,\, t > 0, \\ \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in (0,1],\, \beta > 0,\, t > 0, \end{cases} \tag{5.2}$$

and the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]),\, \omega_t(x,0) = \psi(x) \in C^0([0,1]). \tag{5.3}$$

Actually, this is the case of $d = 0$, $c^2 = 1$ in $(1.12)_1$ and $m = 1$ in $(1.12)_3$ in section 4. Thus, the function f_λ is the unique real solution of the cubic equation

$$\beta\left(\frac{\eta-1}{\eta+1}y + x\right)^3 + (1-\alpha)\left(\frac{\eta-1}{\eta+1}y + x\right) - 2x = 0.$$

And we have the following results by letting $c_1 = c_2 = m = 1$ in section 4.2. We show the local maximum, minimum and piecewise monotonicity of the function f_λ as below.

Lemma 5.1.1 (Local Maximum, Minimum and Piecewise Monotonicity) *The fuction f_λ is odd and f_λ has local extrema at (v_c, M) and $(-v_c, -M)$. Furthermore, the function f_λ is strictly monotonic on $(-\infty, -v_c)$, $(-v_c, v_c)$ and (v_c, ∞), where*

$$v_c = \frac{2-\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}} \text{ and } M = \frac{1+\alpha}{3}\frac{\eta+1}{\eta-1}\sqrt{\frac{1+\alpha}{3\beta}}.$$

We show the x-axis Intercepts, fixed points and intersections with the line $y = -x$ of the function f_λ as below.

Lemma 5.1.2 (x-axis Intercepts) *The function f_λ intersects the x-axis at the points*

$$\left(-\sqrt{\frac{1+\alpha}{\beta}}, 0\right),\ (0,0),\ \left(\sqrt{\frac{1+\alpha}{\beta}}, 0\right).$$

Lemma 5.1.3 (Intersections with the Line $y = x$) *The function f_λ intersects the line $y = x$ at the points*

$$\left(-\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, -\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}\right),\ (0,0),\ \left(\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, \frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}\right).$$

Lemma 5.1.4 (Intersections with the Line $y = -x$) *The function f_λ intersects the line $y = -x$ at the points*

$$\left(-\frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}, \frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}\right),\ (0,0),\ \left(\frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}, -\frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}\right).$$

We show the function f_λ has bounded invariant interval or bounded invariant cantor-like subset in following lemmas.

Lemma 5.1.5 (Bounded Invariant Interval) *Let the parameters $0 < \alpha \leq 1$, $\beta > 0$, and $\eta > 0$, $\eta \neq 1$.*
(i) If $0 < \eta < 1$ and $|M| = \left|\frac{1+\alpha}{3}\frac{\eta+1}{\eta-1}\sqrt{\frac{1+\alpha}{3\beta}}\right| \leq \frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}$, then the iterates of every point in the set

$$U \equiv \left(-\infty, -\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}\right) \cup \left(\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, \infty\right)$$

escape to $\pm\infty$, *while those of any point in* $\mathbb{R}\backslash\overline{U}$ *are attracted to the bounded invariant interval*

$$\left[-\left|\frac{1+\alpha}{3}\frac{\eta+1}{\eta-1}\sqrt{\frac{1+\alpha}{3\beta}}\right|,\left|\frac{1+\alpha}{3}\frac{\eta+1}{\eta-1}\sqrt{\frac{1+\alpha}{3\beta}}\right|\right]$$

of f_λ, *i.e.,* $[-|M|,|M|]$ *of* f_λ.

(ii) If $\eta > 1$ *and* $|M| \le \frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}$, *then the iterates of every point in the set*

$$U \equiv \left(-\infty, -\frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}\right) \cup \left(\frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}, \infty\right)$$

escape to $\pm\infty$, *while those of any point in* $\mathbb{R}\backslash\overline{U}$ *are attracted to the bounded invariant interval*

$$\left[-\left|\frac{1+\alpha}{3}\frac{\eta+1}{\eta-1}\sqrt{\frac{1+\alpha}{3\beta}}\right|,\left|\frac{1+\alpha}{3}\frac{\eta+1}{\eta-1}\sqrt{\frac{1+\alpha}{3\beta}}\right|\right]$$

of f_λ, *i.e.,* $[-|M|,|M|]$ *of* f_λ.

Lemma 5.1.6 (Bounded Cantor-like Invariant Subset) *The bounded invariant interval*

$$\left[-\left|\frac{1+\alpha}{3}\frac{\eta+1}{\eta-1}\sqrt{\frac{1+\alpha}{3\beta}}\right|,\left|\frac{1+\alpha}{3}\frac{\eta+1}{\eta-1}\sqrt{\frac{1+\alpha}{3\beta}}\right|\right]$$

no longer exists in the case (i) and (ii) of the above lemma if the condition

$$|M| \le \frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}} \text{ or } |M| \le \frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}$$

is violated. Instead, we have a bounded Cantor-like invariant set.

Thus, we have the main theorem as below.

Theorem 5.1.7 (Chaotic Region of the 1D Wave System) *Let parameter* η *enters the region*

$$\left[\frac{3\sqrt{3}-1-\alpha}{3\sqrt{3}+1+\alpha}, 1\right) \cup \left(1, \frac{3\sqrt{3}+1+\alpha}{3\sqrt{3}-1-\alpha}\right], \text{ for any given } \alpha \in (0,1],\ \beta > 0.$$

Then the interval map f_λ *is chaotic in the sense of Li-Yorke if the domain of* f_λ *contains the interval* $\left[-\sqrt{\frac{1+\alpha}{\beta}}, \sqrt{\frac{1+\alpha}{\beta}}\right]$.

Definition 5.1.8 *We denote*

$$\underline{\eta_H} = \frac{3\sqrt{3} - 1 - \alpha}{3\sqrt{3} + 1 + \alpha} \text{ and } \overline{\eta_H} = \frac{3\sqrt{3} + 1 + \alpha}{3\sqrt{3} - 1 - \alpha}.$$

We list the Period-Doubling Birfulcation Theorem in [2] as follows.

Lemma 5.1.9 (Correspondence of Period-2^n Orbits to a Unimodal Map) *Let $0 < \alpha \leq 1$, $\beta > 0$ and $0 < \eta < 1$. Assume that α, β and η satisfy*

$$|M| = \frac{1+\alpha}{3}\frac{1+\eta}{1-\eta}\sqrt{\frac{1+\alpha}{3\beta}} \leq \sqrt{\frac{1+\alpha}{\beta}}.$$

Assume that $x_0 \in [-|M|, |M|]$ is a periodic point of prime period-2^n, for some $n \in \{2, 3, 4, ...\}$. Then $|x_0|$ is also a periodic point of $-f_\lambda$ of prime period-2^n such that all the points on the orbit $\left\{-f_\lambda^j(|x_0|) \mid j = 0, 1, 2, ..., 2^n - 1\right\}$ are positive.
Conversely, let $x_0 > 0$ be a periodic point of prime period-2^n of $-f_\lambda$ for some $n \in \{2, 3, 4, ...\}$. Then $\left\{-f_\lambda^j(|x_0|) \mid j = 0, 1, 2, ..., 2^n - 1\right\}$ is the full orbit of x_0 of the map f_λ of prime period-2^n. The period-2^n orbit, $n \geq 2$, of f_λ is atracting (resp., repelling) if and only if the corresponding period-2^n orbit of $-f_\lambda$ is atracting (resp., repelling).

Theorem 5.1.10 (Period-Doubling Bifurcaion Theorem for f_λ, $0 < \eta < 1$) *Let $0 < \alpha \leq 1$, $\beta > 0$ be fixed, and let $\eta : 0 < \eta < \underline{\eta_H}$ be a varying parameter. Let $h(x, \lambda) = -f_\lambda(x)$. Then*
(i) $x_0(\eta) = \frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}$ is a curve of fixed points of $h : h(x_0(\eta), \eta) = x_0(\eta)$.
(ii) The algebraic equation

$$\frac{1}{2}\left(\frac{1+\alpha\eta}{3\beta\eta}\right)^{1/2}\left[\frac{1+(3-2\alpha)\eta}{3\eta}\right] = \frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}$$

has a unique solution η_0, for any given α and β. (Actually, η_0 is independent of β.) We have

$$\frac{\partial}{\partial x}h(x, \eta)\Big|_{(x_0(\eta_0), \eta_0)} = -1.$$

(iii) For $\eta = \eta_0$, we have

$$A \equiv \left[\frac{\partial^2 h}{\partial\eta\partial x} + \frac{1}{2}\left(\frac{\partial h}{\partial \eta}\right)\frac{\partial^2 h}{\partial x^2}\right]\Big|_{(x_0(\eta_0), \eta_0)} \neq 0.$$

(iv) For $\eta = \eta_0$, we have

$$B \equiv \left[\frac{1}{6}\frac{\partial^3 h}{\partial x^3} + \frac{1}{4}\left(\frac{\partial^2 h}{\partial x^2}\right)^2 \right]\Bigg|_{(x_0(\eta_0),\eta_0)} > 0.$$

Consequently, there is period-doubling bifurcation at $(x_0(\eta_0), \eta_0)$. The stability type of the bifurcated period-2 orbit is attracting.

Theorem 5.1.11 (Period-Doubling Bifurcaion Theorem for f_λ, $\eta > 1$) *Let $0 < \alpha \leq 1$, $\beta > 0$ be fixed, and let $\eta : \overline{\eta}_H < \eta < \infty$ be a varying parameter. Let $h(x, \lambda) = f_\lambda(x)$. Then*
(i) $x_0(\eta) = \frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}$ is a curve of fixed points of $h : h(x_0(\eta), \eta) = x_0(\eta)$.
(ii) The algebraic equation

$$\frac{1}{6}\left(\frac{\alpha+\eta}{3\beta}\right)^{1/2}[3+\eta-2\alpha] = \frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}$$

has a unique solution η_0, for any given α and β. (Actually, η_0 is independent of β.) We have

$$\frac{\partial}{\partial x}h(x,\eta)\Bigg|_{(x_0(\eta_0),\eta_0)} = -1.$$

(iii) For $\eta = \eta_0$, we have

$$A \equiv \left[\frac{\partial^2 h}{\partial\eta\partial x} + \frac{1}{2}\left(\frac{\partial h}{\partial \eta}\right)\frac{\partial^2 h}{\partial x^2} \right]\Bigg|_{(x_0(\eta_0),\eta_0)} \neq 0.$$

(iv) For $\eta = \eta_0$, we have

$$B \equiv \left[\frac{1}{6}\frac{\partial^3 h}{\partial x^3} + \frac{1}{4}\left(\frac{\partial^2 h}{\partial x^2}\right)^2 \right]\Bigg|_{(x_0(\eta_0),\eta_0)} > 0.$$

Consequently, there is period-doubling bifurcation at $(x_0(\eta_0), \eta_0)$. The stability type of the bifurcated period-2 orbit is attracting.

Definition 5.1.12 *For any given $\alpha \in (0, 1]$, denoete by $\underline{\eta_0}$ the unique real solution of the algebraic equation*

$$\frac{1}{2}\left(\frac{1+\alpha\eta}{3\beta\eta}\right)^{1/2}\left[\frac{1+(3-2\alpha)\eta}{3\eta}\right] = \frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}},$$

and denoete by $\overline{\eta_0}$ *the unique real solution of the algebraic equation*

$$\frac{1}{6}\left(\frac{\alpha+\eta}{3\beta}\right)^{1/2}[3+\eta-2\alpha]=\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}.$$

Definition 5.1.13 *For any given* $\alpha \in (0,1]$, *denoete by* $\underline{\eta_B}$ *the unique real solution of the algebraic equation*

$$\frac{2\eta}{1-\eta}\left(\frac{\eta}{1+\alpha\eta}\right)^{1/2}=\frac{3\sqrt{3}}{(1+\alpha)^{3/2}},$$

and $\overline{\eta_B}=\frac{1}{\underline{\eta_B}}$.

We have

$$0<\underline{\eta_0}<\underline{\eta_H}<\underline{\eta_B}<1<\overline{\eta_B}<\overline{\eta_H}<\overline{\eta_0}<\infty.$$

Lemma 5.1.14 *For any given* $\alpha \in (0,1]$, *assume that either* $0<\eta\le\underline{\eta_B}$ *or* $\overline{\eta_B}\le\eta<\infty$. *Then* f_λ *has invariant intervals* $\left[-\sqrt{\frac{1+\alpha}{\beta}},\sqrt{\frac{1+\alpha}{\beta}}\right]$ *and* $[-|M|,|M|]$. *Furthermore,*

(i) if $\eta\in(0,\underline{\eta_0})\cup(\overline{\eta_0},\infty)$, *then* f_λ *has no periodic point of period larger than or equal to* 2;

(ii) if $\eta\in(\underline{\eta_0},\underline{\eta_H})\cup(\overline{\eta_H},\overline{\eta_0})$, *then* f_λ *has at least a periodic point of period* 2 *and two fixed points;*

(iii) f_λ *has period-doubling cascades as* η *is increasing in* $(\overline{\eta_H},\overline{\eta_0})$ *or is decreasing in* $(\underline{\eta_0},\underline{\eta_H})$ *and there exists two critical parameters* $\overline{\eta_\infty}$ *and* $\underline{\eta_\infty}$ *with* $\overline{\eta_0}>\overline{\eta_\infty}$ *and* $\underline{\eta_\infty}=\frac{1}{\overline{\eta_\infty}}$ *such that* f_λ *has a homoclinic point when* $\eta\in(\underline{\eta_\infty},\underline{\eta_H}]\cup[\overline{\eta_H},\overline{\eta_\infty})$.

For example, if $\alpha=0.5$ and $\beta=1$, simulation results shows that (see [12])

$$\underline{\eta_0}\approx 0.433,\ \overline{\eta_0}=\frac{1}{\underline{\eta_0}}\approx 2.312,\ \underline{\eta_H}\approx 0.552,\ \overline{\eta_H}=\frac{1}{\underline{\eta_H}}\approx 1.812,$$
$$\underline{\eta_B}=\frac{2}{3},\ \overline{\eta_B}=\frac{1}{\underline{\eta_B}}=1.5,\ \underline{\eta_\infty}\approx 0.5249,\ \overline{\eta_\infty}=\frac{1}{\underline{\eta_\infty}}\approx 1.905.$$

Theorem 5.1.15 *Consider the 1D wave system (5.1)-(5.3).*

(i) if $\eta\in(0,\underline{\eta_0})\cup(\overline{\eta_0},\infty)$ *and initial conditions* $\varphi(x)$, $\psi(x)$ *are piecewise monotone with finitely many extremal points on* $[0,1]$ *such that the ranges of* φ *and* ψ *are contained in* $\left[-\sqrt{\frac{1+\alpha}{\beta}},\sqrt{\frac{1+\alpha}{\beta}}\right]$ *and* $[-|M|,|M|]$, *respectively. Then the total variation of* f_λ *on* $[-|M|,|M|]$ *remains bounded.*

(ii) If $\eta\in(\underline{\eta_0},\underline{\eta_\infty}]\cup[\overline{\eta_\infty},\overline{\eta_0})$ *and the ranges of* φ *and* ψ *contain* $\left[-\sqrt{\frac{1+\alpha}{\beta}},\sqrt{\frac{1+\alpha}{\beta}}\right]$ *and* $[-|M|,|M|]$, *respectively. Then the total variation of* f_λ *on* $\left[-\sqrt{\frac{1+\alpha}{\beta}},\sqrt{\frac{1+\alpha}{\beta}}\right]$ *and* $[-|M|,|M|]$ *is unbounded.*

(iii) If $\eta\in(\underline{\eta_\infty},\underline{\eta_H}]\cup[\overline{\eta_H},\overline{\eta_\infty})$ *and the ranges of* φ *and* ψ *contain* $\left[-\sqrt{\frac{1+\alpha}{\beta}},\sqrt{\frac{1+\alpha}{\beta}}\right]$ *and* $[-|M|,|M|]$, *respectively. Then the total variation of* f_λ *on* $\left[-\sqrt{\frac{1+\alpha}{\beta}},\sqrt{\frac{1+\alpha}{\beta}}\right]$ *and* $[-|M|,|M|]$ *is unbounded exponentially.*

5.2 Main results of the system (1.5)-(1.7)

In [8, 12, 13], Huang et al. consider the 1D wave system (1.5)-(1.7) as below:

$$\omega_{tt} - \omega_{xx} = 0,\, 0 < x < 1,\, t > 0,$$

with the boundary conditions

$$\begin{cases} \omega_x(0,t) = -\eta\omega_t(0,t), & \eta > 0,\, \eta \neq 1,\, t > 0, \\ \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in (0,1],\, \beta > 0,\, t > 0, \end{cases}$$

and the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]),\, \omega_t(x,0) = \psi(x) \in C^0([0,1]).$$

Actually, this is the case of in (1.1) and $\eta > 0$ in (1.2) in section 3. Thus, the function f_λ is the unique real solution of the cubic equation

$$\beta\left(\frac{1-\eta}{1+\eta}y + x\right)^3 + (1-\alpha)\left(\frac{1-\eta}{1+\eta}y + x\right) - 2x = 0.$$

And we have the following results by letting $c_1 = c_2 = 1$ in section 3.2. We show the local maximum, minimum and piecewise monotonicity of the function f_λ as below.

Lemma 5.2.1 (Local Maximum, Minimum and Piecewise Monotonicity) *The fuction f_λ is odd and f_λ has local extrema at (v_c, M) and $(-v_c, -M)$. Furthermore, the function f_λ is strictly monotonic on $(-\infty, -v_c)$, $(-v_c, v_c)$ and (v_c, ∞), where*

$$v_c = \frac{2-\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}} \text{ and } M = \frac{(1+\alpha)(1+\eta)}{3(1-\eta)}\sqrt{\frac{1+\alpha}{3\beta}}.$$

We show the x-axis Intercepts, fixed points and intersections with the line $y = -x$ of the function f_λ as below.

Lemma 5.2.2 (x-axis Intercepts) *The function f_λ intersects the x-axis at the points*

$$\left(-\sqrt{\frac{1+\alpha}{\beta}}, 0\right),\, (0,0),\, \left(\sqrt{\frac{1+\alpha}{\beta}}, 0\right).$$

Lemma 5.2.3 (Intersections with the Line $y = x$) *The function f_λ intersects the line $y = x$ at the points*

$$\left(-\frac{1+\eta}{2}\sqrt{\frac{\eta+\alpha}{\beta}}, -\frac{1+\eta}{2}\sqrt{\frac{\eta+\alpha}{\beta}}\right), (0,0), \left(\frac{1+\eta}{2}\sqrt{\frac{\eta+\alpha}{\beta}}, \frac{1+\eta}{2}\sqrt{\frac{\eta+\alpha}{\beta}}\right).$$

Lemma 5.2.4 (Intersections with the Line $y = -x$) *The function f_λ intersects the line $y = -x$ at the points*

$$\left(-\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, \frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}\right), (0,0), \left(\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, -\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}\right).$$

We show the function f_λ has bounded invariant interval or bounded invariant cantor-like subset in following lemmas.

Lemma 5.2.5 (Bounded Invariant Interval) *Let the parameters $0 < \alpha \leq 1$, $\beta > 0$, and $\eta > 0$, $\eta \neq 1$.*
(i) If $\eta > 1$ and $|M| = \left|\frac{(1+\alpha)(1+\eta)}{3(1-\eta)}\sqrt{\frac{1+\alpha}{3\beta}}\right| \leq \frac{1+\eta}{2}\sqrt{\frac{\eta+\alpha}{\beta}}$, then the iterates of every point in the set

$$U \equiv \left(-\infty, -\frac{1+\eta}{2}\sqrt{\frac{\eta+\alpha}{\beta}}\right) \cup \left(\frac{1+\eta}{2}\sqrt{\frac{\eta+\alpha}{\beta}}, \infty\right)$$

escape to $\pm\infty$, while those of any point in $\mathbb{R}\backslash\overline{U}$ are attracted to the bounded invariant interval

$$\left[-\left|\frac{(1+\alpha)(1+\eta)}{3(1-\eta)}\sqrt{\frac{1+\alpha}{3\beta}}\right|, \left|\frac{(1+\alpha)(1+\eta)}{3(1-\eta)}\sqrt{\frac{1+\alpha}{3\beta}}\right|\right]$$

of f_λ, i.e., $[-|M|, |M|]$ of f_λ.
(ii) If $0 < \eta < 1$ and $|M| \leq \frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}$, then the iterates of every point in the set

$$U \equiv \left(-\infty, -\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}\right) \cup \left(\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, \infty\right)$$

escape to $\pm\infty$, while those of any point in $\mathbb{R}\backslash\overline{U}$ are attracted to the bounded invariant interval

$$\left[-\left|\frac{(1+\alpha)(1+\eta)}{3(1-\eta)}\sqrt{\frac{1+\alpha}{3\beta}}\right|, \left|\frac{(1+\alpha)(1+\eta)}{3(1-\eta)}\sqrt{\frac{1+\alpha}{3\beta}}\right|\right]$$

of f_λ, i.e., $[-|M|, |M|]$ of f_λ.

Lemma 5.2.6 (Bounded Cantor-like Invariant Subset) *The bounded invariant interval*

$$\left[-\left|\frac{(1+\alpha)(1+\eta)}{3(1-\eta)}\sqrt{\frac{1+\alpha}{3\beta}}\right|,\left|\frac{(1+\alpha)(1+\eta)}{3(1-\eta)}\sqrt{\frac{1+\alpha}{3\beta}}\right|\right]$$

no longer exists in the case (i) and (ii) of the above lemma if the condition

$$|M|\leq\frac{1+\eta}{2}\sqrt{\frac{\eta+\alpha}{\beta}}\ \text{or}\ |M|\leq\frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}$$

is violated. Instead, we have a bounded Cantor-like invariant set.

Thus, we have the main theorem as below.

Theorem 5.2.7 (Chaotic Region of the 1D Wave System) *Let parameter η enters the region*

$$\left[\frac{3\sqrt{3}-1-\alpha}{3\sqrt{3}+1+\alpha},1\right)\cup\left(1,\frac{3\sqrt{3}+1+\alpha}{3\sqrt{3}-1-\alpha}\right],\ \text{for any given}\ \alpha\in(0,1],\ \beta>0.$$

Then the interval map f_λ is chaotic in the sense of Li-Yorke if the domain of f_λ contains the interval $\left[-\sqrt{\frac{1+\alpha}{\beta}},\sqrt{\frac{1+\alpha}{\beta}}\right]$.

Definition 5.2.8 *We denote*

$$\underline{\eta_H}=\frac{3\sqrt{3}-1-\alpha}{3\sqrt{3}+1+\alpha}\ \text{and}\ \overline{\eta_H}=\frac{3\sqrt{3}+1+\alpha}{3\sqrt{3}-1-\alpha}.$$

The Period-Doubling Birfulcation Theorem is similar to section 1 of this chapter and we have the results as follows.

Lemma 5.2.9 (Correspondence of Period-2^n Orbits to a Unimodal Map) *Let $0<\alpha\leq 1$, $\beta>0$ and $\eta>1$. Assume that α, β and η satisfy*

$$|M|=\frac{(1+\alpha)(\eta+1)}{3(\eta-1)}\sqrt{\frac{1+\alpha}{3\beta}}\leq\sqrt{\frac{1+\alpha}{\beta}}.$$

Assume that $x_0\in[-|M|,|M|]$ is a periodic point of prime period-2^n, for some $n\in\{2,3,4,...\}$. Then $|x_0|$ is also a periodic point of $-f_\lambda$ of prime period-2^n such that all the points on the orbit $\left\{-f_\lambda^j(|x_0|)\mid j=0,1,2,...,2^n-1\right\}$ are positive.
Conversely, let $x_0>0$ be a periodic point of prime period-2^n of $-f_\lambda$ for some $n\in\{2,3,4,...\}$.

Then $\left\{-f_\lambda^j(|x_0|) \mid j = 0,1,2,...,2^n-1\right\}$ *is the full orbit of* x_0 *of the map* f_λ *of prime period-*2^n*. The period-*2^n *orbit,* $n \geq 2$*, of* f_λ *is atracting (resp., repelling) if and only if the corresponding period-*2^n *orbit of* $-f_\lambda$ *is atracting (resp., repelling).*

Theorem 5.2.10 (Period-Doubling Bifurcaion Theorem for f_λ**,** $\eta > 1$**)** *Let* $0 < \alpha \leq 1$*,* $\beta > 0$ *be fixed, and let* $\eta : \overline{\eta}_H < \eta < \infty$ *be a varying parameter. Let* $h(x,\lambda) = -f_\lambda(x)$*. Then*
(i) $x_0(\eta) = \frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}$ *is a curve of fixed points of* $h : h(x_0(\eta),\eta) = x_0(\eta)$*.*
(ii) The algebraic equation

$$\frac{1}{6}\left(\frac{\alpha+\eta}{3\beta}\right)^{1/2}[3+\eta-2\alpha] = \frac{1+\eta}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}$$

has a unique solution η_0*, for any given* α *and* β*. (Actually,* η_0 *is independent of* β*.) We have*

$$\left.\frac{\partial}{\partial x}h(x,\eta)\right|_{(x_0(\eta_0),\eta_0)} = -1.$$

(iii) For $\eta = \eta_0$*, we have*

$$A \equiv \left.\left[\frac{\partial^2 h}{\partial\eta\partial x} + \frac{1}{2}\left(\frac{\partial h}{\partial \eta}\right)\frac{\partial^2 h}{\partial x^2}\right]\right|_{(x_0(\eta_0),\eta_0)} \neq 0.$$

(iv) For $\eta = \eta_0$*, we have*

$$B \equiv \left.\left[\frac{1}{6}\frac{\partial^3 h}{\partial x^3} + \frac{1}{4}\left(\frac{\partial^2 h}{\partial x^2}\right)^2\right]\right|_{(x_0(\eta_0),\eta_0)} > 0.$$

Consequently, there is period-doubling bifurcation at $(x_0(\eta_0),\eta_0)$*. The stability type of the bifurcated period-2 orbit is attracting.*

Theorem 5.2.11 (Period-Doubling Bifurcaion Theorem for f_λ**,** $0 < \eta < 1$**)** *Let* $0 < \alpha \leq 1$*,* $\beta > 0$ *be fixed, and let* $\eta : 0 < \eta < \underline{\eta}_H$ *be a varying parameter. Let* $h(x,\lambda) = f_\lambda(x)$*. Then*
(i) $x_0(\eta) = \frac{1+\eta}{2}\sqrt{\frac{\eta+\alpha}{\beta}}$ *is a curve of fixed points of* $h : h(x_0(\eta),\eta) = x_0(\eta)$*.*
(ii) The algebraic equation

$$\frac{1}{2}\left(\frac{1+\alpha\eta}{3\beta\eta}\right)^{1/2}\left[\frac{1+(3-2\alpha)\eta}{3\eta}\right] = \frac{1+\eta}{2}\sqrt{\frac{\alpha+\eta}{\beta}}$$

has a unique solution η_0, for any given α and β. (Actually, η_0 is independent of β.) We have

$$\frac{\partial}{\partial x}h(x,\eta)\Big|_{(x_0(\eta_0),\eta_0)} = -1.$$

(iii) For $\eta = \eta_0$, we have

$$A \equiv \left[\frac{\partial^2 h}{\partial\eta\partial x} + \frac{1}{2}\left(\frac{\partial h}{\partial\eta}\right)\frac{\partial^2 h}{\partial x^2}\right]\Big|_{(x_0(\eta_0),\eta_0)} \neq 0.$$

(iv) For $\eta = \eta_0$, we have

$$B \equiv \left[\frac{1}{6}\frac{\partial^3 h}{\partial x^3} + \frac{1}{4}\left(\frac{\partial^2 h}{\partial x^2}\right)^2\right]\Big|_{(x_0(\eta_0),\eta_0)} > 0.$$

Consequently, there is period-doubling bifurcation at $(x_0(\eta_0),\eta_0)$. The stability type of the bifurcated period-2 orbit is attracting.

Remark 5.2.12 *The 1D wave system (5.1)-(5.3) and the 1D wave system (1.5)-(1.7) are very similar. The only difference is the boundary condition at the left endpoint $x = 0$ in system (1.5)-(1.7) is $\omega_x = -\eta\omega_t$ but the boundary condition at the left endpoint in system (5.1)-(5.3) is $\omega_t = -\eta\omega_x$. In fact, Huang et al. consider the 1D wave system (5.1)-(5.3) in [8, 12, 13]. However, the results are the same if we consider the 1D wave system (1.5)-(1.7).*

5.3 Main results of the system (1.8)-(1.11)

In [5], Chen et al. consider the 1D wave system (1.8)-(1.11) as below:

$$\omega_{tt} + v\omega_{tx} - \omega_{xx} = 0,\ v > 0,\ 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$\omega_x(0,t) = 0,\ t > 0,$$

and

$$\omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t),\ \alpha \in \left(0, (v+\sqrt{v^2+4})/2\right],\ \beta > 0,\ t > 0,$$

and the initial conditions

$$\omega(x,0)=\varphi(x)\in C^1([0,1]),\, \omega_t(x,0)=\psi(x)\in C^0([0,1]).$$

Actually, this is the case of $d=-v$, $c^2=1$ in (1.1) and $\eta=0$ in (1.2) in section 3. Thus, the function f_λ is the unique real solution of the cubic equation

$$\beta\left(c_1y+c_2x\right)^3+\left(c_2-\alpha\right)\left(c_1y+c_2x\right)-\left(1+c_2/c_1\right)x=0,$$

where $c_1=\left(-v+\sqrt{v^2+4}\right)/2$ and $c_2=\left(v+\sqrt{v^2+4}\right)/2$. And we have the following results by letting $c_1=\left(-v+\sqrt{v^2+4}\right)/2$, $c_2=\left(v+\sqrt{v^2+4}\right)/2$ and $\eta=0$ in section 3.2. We show the local maximum, minimum and piecewise monotonicity of the function f_λ as below.

Lemma 5.3.1 (Local Maximum, Minimum and Piecewise Monotonicity) *The fuction f_λ is*

odd and f_λ has local extrema at (v_c, M) and $(-v_c,-M)$. Furthermore, the function f_λ is strictly decreasing on $(-\infty,-v_c)$ and (v_c,∞), but strictly increasing on $(-v_c,v_c)$, where

$$v_c=\frac{1-2\alpha c_2+3c_2^2}{3c_2\left(1+c_2^2\right)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\ \text{and}\ M=\frac{2(1+\alpha c_2)}{3(c_1+c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}.$$

We show the x-axis Intercepts, fixed points and intersections with the line $y=-x$ of the function f_λ as below.

Lemma 5.3.2 (x-axis Intercepts) *The function f_λ intersects the x-axis at the points*

$$\left(-\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}},0\right),\ (0,0),\ \left(\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}},0\right).$$

Lemma 5.3.3 (Intersections with the Line $y=x$) *The function f_λ intersects the line $y=x$ at the points*

$$\left(-\frac{1}{c_1+c_2}\sqrt{\frac{\alpha}{\beta}},\frac{1}{c_1+c_2}\sqrt{\frac{\alpha}{\beta}}\right),\ (0,0),\ \left(\frac{1}{c_1+c_2}\sqrt{\frac{\alpha}{\beta}},-\frac{1}{c_1+c_2}\sqrt{\frac{\alpha}{\beta}}\right).$$

Lemma 5.3.4 (Intersections with the Line $y=-x$) *The function f_λ intersects the line $y=$*

$-x$ *at the points*

$$\left(-\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}},\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}}\right),\ (0,0)$$

and

$$\left(\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}},-\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}}\right).$$

Remark 5.3.5 *By the way we point out that the result in [5] should be the function f_λ intersects the line $y=-x$ at the points*

$$\left(-\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}},\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}}\right),\ (0,0)$$

and

$$\left(\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}},-\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}}\right).$$

We show the function f_λ has bounded invariant interval or bounded invariant cantor-like subset in following lemmas.

Lemma 5.3.6 (Bounded Invariant Interval) *Let the parameters $0<\alpha\le c_2$, $\beta>0$. If*

$$|M|=\left|\frac{2(1+\alpha c_2)}{3(c_1+c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right|\le\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}},$$

then the iterates of every point in the set

$$U\equiv\left(-\infty,-\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}}\right)\cup\left(\frac{1}{c_2-c_1}\sqrt{\frac{2+\alpha\,(c_2-c_1)}{\beta(c_2-c_1)}},\infty\right)$$

escape to $\pm\infty$, while those of any point in $\mathbb{R}\backslash\overline{U}$ are attracted to the bounded invariant interval

$$\left[-\left|\frac{2(1+\alpha c_2)}{3(c_1+c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right|,\left|\frac{2(1+\alpha c_2)}{3(c_1+c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right|\right]$$

of f_λ, i.e., $[-|M|,|M|]$ of f_λ.

Lemma 5.3.7 (Bounded Cantor-like Invariant Subset) *The bounded invariant interval*

$$\left[-\left|\frac{2(1+\alpha c_2)}{3(c_1+c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right|,\left|\frac{2(1+\alpha c_2)}{3(c_1+c_2)}\sqrt{\frac{1+\alpha c_2}{3\beta c_2}}\right|\right]$$

no longer exists in the above lemma if the condition

$$|M| \leq \frac{1}{c_2 - c_1}\sqrt{\frac{2 + \alpha\left(c_2 - c_1\right)}{\beta(c_2 - c_1)}}$$

is violated. Instead, we have a bounded Cantor-like invariant set.

Thus, we have the main theorem as below.

Theorem 5.3.8 (Chaotic Region of the 1D Wave System) *Let parameters c_1, c_2 satisfy the inequality*

$$\frac{2(1+\alpha c_2)}{3(c_1+c_2)} \geq \sqrt{3}/c_2, \text{ for any given } \alpha \in (0, c_2],\ \beta > 0.$$

Then the interval map f_λ is chaotic in the sense of Li-Yorke if the domain of f_λ contains the interval $\left[-\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, \frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}\right]$.

We list the Period-Doubling Birfulcation Theorem in [5] as follows.

Theorem 5.3.9 (Period-Doubling Bifurcaion Theorem for f_λ) *Let $0 < \alpha \leq \sqrt{2}$, $\beta > 0$ be fixed, and define $\overline{v}_{1,\alpha}$ by*

$$\overline{v}_{1,\alpha} = \varepsilon, \text{ where } \varepsilon \text{ is any small positive number,}$$

Let $v \in [\overline{v}_{1,\alpha}, \infty)$ be a varying parameter. Let $h(x,v) = f_v(x)$. Then

(i) α satisfies $0 < \alpha \leq c_2(v)$ for all $v \in [\overline{v}_{1,\alpha}, \infty)$.

(ii) $x_0(v) = [1/\left(c_1(v) + c_2(v)\right)]\sqrt{\alpha/\beta}$ is a curve of fixed points of $h : h(x_0(v), v) = x_0(v)$.

(iii) For $v_0 = 1/\alpha$, we have $v_0 > \overline{v}_{1,\alpha}$, $c_2(v_0) \geq \alpha$, and

$$\left.\frac{\partial}{\partial x}h(x,\eta)\right|_{(x_0(v_0),v_0)} = -1.$$

(iv) For $v_0 = 1/\alpha$, we have

$$A \equiv \left.\left[\frac{\partial^2 h}{\partial v \partial x} + \frac{1}{2}\left(\frac{\partial h}{\partial v}\right)\frac{\partial^2 h}{\partial x^2}\right]\right|_{(x_0(v_0),v_0)} \neq 0.$$

(v) For $v_0 = 1/\alpha$*, we have*

$$B \equiv \left[\frac{1}{6}\frac{\partial^3 h}{\partial x^3} + \frac{1}{4}\left(\frac{\partial^2 h}{\partial x^2}\right)^2\right]\Bigg|_{(x_0(v_0),v_0)} > 0.$$

Consequently, there is period-doubling bifurcation at $(x_0(v_0), v_0)$*. The stability type of the bifurcated period-*2 *orbit is attracting.*

In [9], Huang proved there exist three subregions S_1^0, S_1^1 and S_2 of S such that the growth of the total variation of the interval map remains bounded, is unbounded, is unbounded exponentially when the parameters (v, α) belong to S_1^0, S_1^1, and S_2, respectively. And since Huang proved the results by using the result in [5] which contains an error, so we do not list the results in [9] here.

Chapter 6

Two methods to detect chaos

In this chapter, we show that when the chaotic vibrations corresponding to the 1D wave systems (1.1)-(1.4) and (1.12) occur by using the main theorems in [14] and [15]. The 1D wave system (1.1)-(1.4) is considered as below:

$$\begin{cases} \omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0, & d \in \mathbb{R},\, c > 0,\, 0 < x < 1,\, t > 0, \\ \omega_x(0,t) = -\eta\omega_t(0,t), & \eta \geq 0,\, \eta \neq \frac{1}{c_2},\, t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in \left(0, \frac{1}{c_1}\right],\, \beta > 0,\, t > 0, \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]), \end{cases}$$

where

$$c_1 = \left(d + \sqrt{d^2 + 4c^2}\right)/2 \text{ and } c_2 = \left(-d + \sqrt{d^2 + 4c^2}\right)/2.$$

And the 1D wave system (1.12) is considered as below:

$$\begin{cases} \omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0, & d \in \mathbb{R},\, c > 0,\, 0 < x < 1,\, t > 0, \\ \omega_t(0,t) + \eta\omega_x(0,t) = 0, & \eta > 0,\, \eta \neq c_2,\, t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^{2m+1}(1,t), & \alpha \in \left(0, \frac{1}{c_1}\right],\, \beta, t > 0,\, m \in \mathbb{N}, \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

First, we discretize (3.4) as follows. Given one $\tau \in (0, \Delta]$, we denote $z(\tau + n\Delta) = z_n$ in (3.4).

Then we have

$$\beta\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}z_{n+1}+c_2z_n\right)^3+\left(\tfrac{1}{c_1}-\alpha\right)\left(c_1\tfrac{1-\eta c_2}{1+\eta c_1}z_{n+1}+c_2z_n\right)$$
$$-\left(1+\tfrac{c_2}{c_1}\right)z_n=0,\text{ where } n\in\mathbb{N}\cup\{0\},\ \Delta=1+\tfrac{c_1}{c_2}. \tag{6.1}$$

And we discretize (4.4) as follows. Given one $\tau\in(0,\Delta]$, we denote $z(\tau+n\Delta)=z_n$ in (4.4). Then we have

$$\beta\left(c_1\tfrac{\eta-c_2}{\eta+c_1}z_{n+1}+c_2z_n\right)^{2m+1}+\left(\tfrac{1}{c_1}-\alpha\right)\left(c_1\tfrac{\eta-c_2}{\eta+c_1}z_{n+1}+c_2z_n\right)$$
$$-\left(1+\tfrac{c_2}{c_1}\right)z_n=0,\text{ where } n\in\mathbb{N}\cup\{0\},\ \Delta=1+\tfrac{c_1}{c_2}. \tag{6.2}$$

6.1 Chaos in the 1D wave systems (1.1)-(1.4) and (1.12)

Definition 6.1.1 *Let ℓ_∞ be the space of bounded real sequences endowed with the norm*

$$\|\underline{y}\|=\sup\{|y_n|:n\in\mathbb{Z}\}\ \text{for } \underline{y}=(y_n),\ y_n\in\mathbb{R},$$

i.e., with the topology of uniform convergence and let $\sigma:\ell_\infty\to\ell_\infty$ be the shift map, i.e., $\sigma(\underline{y})=\underline{y}'$ with $y'_n=y_{n+1}$, $n\in\mathbb{Z}$.

Definition 6.1.2 *In the following theorems, we will consider mainly subsets of ℓ_∞ endowed with the product (or Tichonov) topology on $\mathbb{R}^{\mathbb{Z}}$, i.e., with the topology of pointwise convergence. In such a case we will supply the notation of the appropriate sets with subscript prod, for example: $\ell_{\infty,prod}$, B_{prod}, etc.*

Let us consider a difference equation of the form

$$\Phi_\lambda(y_n,y_{n+1},\cdots,y_{n+m})=0 \tag{6.3}$$

where λ is a parameter from a metric space E and the function Φ_λ is defined on a closed subset Q^{m+1}
$\subset\mathbb{R}^{m+1}$, where $Q=[s_1,s_2]\setminus V$ for some real numbers $s_1<s_2$ and some open (possibly empty) set $V\subset[s_1,s_2]$. We assume that for each $\lambda\in E$ the function $\Phi_\lambda:Q^{m+1}\to\mathbb{R}$ is C^1 and is continuous in λ on E and also that the partial derivatives $\partial_i\Phi_\lambda(x_1,\cdots,x_{m+1})$, $i=1,...,m+1$,

$(x_1, \cdots, x_{m+1}) \in Q^{m+1}$are continuous in λ on E, where $\partial_i \Phi_\lambda$ is the partial derivative of Φ_λ with respect to the ith variable.

Given a $\lambda \in E$, let Y_λ be the set of solutions of the difference equation (6.3), i.e., the set of sequences $\underline{y} = (y_n) = (\cdots, y_{-1}, y_0, y_1, \cdots)$ such that for any $n \in \mathbb{Z}$

1. $y_n \in Q$; and

2. $(m+1)$ consecutive components $y_n, y_{n+1}, \cdots, y_{n+m}$ of $\underline{y}$ satisfy (6.3)

It is easy to see that Y_λ is a closed subset of the space ℓ_∞. Note that $Y_{\lambda,prod}$ is also a closed subset of $[s_1, s_2]^{\mathbb{Z}}_{prod}$, and since the latter space is compact (by the Tichonov theorem), $Y_{\lambda,prod}$ is compact. Note that the shift map σ, being considered as a map from ℓ_∞ to ℓ_∞, is an isometric linear operator, while $\sigma : \ell_{\infty,prod} \to \ell_{\infty,prod}$ is a homeomorphism. It is evident that for any $\lambda \in E$, the set $Y_{\lambda,prod}$ is σ-invariant and the restriction $\sigma|Y_{\lambda,prod}$ is a homeomorphism on a compact space. Thus we can define the *topological entropy* for solutions of the difference equation (6.3) as $h_{top}(\sigma|Y_{\lambda,prod})$.

Lemma 6.1.3 *[14, Main theorem] Let*

$$\Phi_\lambda(y_n, y_{n+1}, \cdots, y_{n+m}) = 0 \tag{6.4}$$

be a difference equation with parameter $\lambda \in [\lambda_0, \lambda_1]$ and let the function $\Phi_\lambda : Q^{m+1} \to \mathbb{R}$, where $Q = [s_1, s_2] \backslash V$ for some numbers $s_1 < s_2$ and some open set $V \subset [s_1, s_2]$, be such that it is C^1 for each λ and is continuous in λ and so are the partial derivatives $\partial_i \Phi_\lambda$, $i = 1, \cdots, m+1$. Suppose that for $\lambda = \lambda_0$, the function Φ_{λ_0} depends on only one variable: $\Phi_{\lambda_0}(x_1, x_2, \cdots, x_{m+1}) = \varphi(x_N)$, where N is an integer with $1 \le N \le m+1$ and $\varphi : Q \to \mathbb{R}$ is a C^1-function with k simple zeros in the interior of Q.
Then there exists one $\overline{\delta} > 0$ such that for any $\lambda \in [\lambda_0, \lambda_0 + \overline{\delta})$ there is a closed σ-invariant subset Γ_λ of Y_λ, the set of solutions for (6.4) in the product topology, such that $\sigma|\Gamma_\lambda$ is topologically conjugate to $\sigma|\Sigma_k$, the full shift on k symbols; in particular, $h_{top}(\sigma|Y_\lambda) \ge \log k$.

First we consider (6.1) as below:

$$\Psi_\lambda(z_n, z_{n+1}) = \beta \left(c_1 \tfrac{1-\eta c_2}{1+\eta c_1} z_{n+1} + c_2 z_n \right)^3 + \left(\tfrac{1}{c_1} - \alpha \right) \left(c_1 \tfrac{1-\eta c_2}{1+\eta c_1} z_{n+1} \right.$$

$$\left. + c_2 z_n \right) - \left(1 + \tfrac{c_2}{c_1} \right) z_n = 0, \text{ where } n \in \mathbb{N} \cup \{0\}.$$

Let the parameters c_1, c_2, α, β be fixed while $\lambda = |1 - \eta c_2| \to 0$, then we have the function

$$\varphi(z_n) = z_n[\beta c_2^3 z_n^2 - (\alpha c_2 + 1)] = 0.$$

Since φ is a C^1 function with three simple zeros in the interior of $[s_1, s_2]$ ($s_1 < -\frac{1}{c_2}\sqrt{\frac{\alpha c_2+1}{\beta c_2}}$ and $s_2 > \frac{1}{c_2}\sqrt{\frac{\alpha c_2+1}{\beta c_2}}$), we have the results as follows.

Proposition 6.1.4 *Let the parameters c_1, c_2, α, β, m be fixed in the difference equation (6.1) and consider the function $\Phi_\lambda : [s_1, s_2] \times [s_1, s_2] \to \mathbb{R}$ where $s_1 < -\frac{1}{c_2}\sqrt{\frac{\alpha c_2+1}{\beta c_2}}$ and $s_2 > \frac{1}{c_2}\sqrt{\frac{\alpha c_2+1}{\beta c_2}}$. Then there exists one $\overline{\delta} > 0$ such that for any $\eta \in \left(\frac{1}{c_2} - \overline{\delta}, \frac{1}{c_2}\right) \cup \left(\frac{1}{c_2}, \frac{1}{c_2} + \overline{\delta}\right)$, there is a closed σ-invariant subset Γ_λ of Y_λ, the set of solutions for (6.1) in the product topology, such that $\sigma|\Gamma_\lambda$ is topologically conjugate to $\sigma|\Sigma_3$, the full shift on 3 symbols; in particular, $h_{top}(\sigma|Y_\lambda) \geq \log 3$.*

And then we consider (6.2) as below:

$$\Phi_\lambda(z_n, z_{n+1}) = \beta\left(c_1\frac{\eta-c_2}{\eta+c_1}z_{n+1} + c_2 z_n\right)^{2m+1} + \left(\frac{1}{c_1} - \alpha\right)\left(c_1\frac{\eta-c_2}{\eta+c_1}z_{n+1} + c_2 z_n\right) - \left(1 + \frac{c_2}{c_1}\right)z_n = 0, \text{ where } n \in \mathbb{N} \cup \{0\}.$$

Let the parameters c_1, c_2, α, β, m be fixed while $\lambda = |\eta - c_2| \to 0$, then we have the function

$$\varphi(z_n) = z_n[\beta c_2^{2m+1} z_n^{2m} - (\alpha c_2 + 1)] = 0.$$

Since φ is a C^1 function with three simple zeros in the interior of $[s_1, s_2]$ ($s_1 < -\frac{1}{c_2}\sqrt[2m]{\frac{\alpha c_2+1}{\beta c_2}}$ and $s_2 > \frac{1}{c_2}\sqrt[2m]{\frac{\alpha c_2+1}{\beta c_2}}$), we have the results as follows.

Proposition 6.1.5 *Let the parameters c_1, c_2, α, β, m be fixed in the difference equation (6.2) and consider the function $\Phi_\lambda : [s_1, s_2] \times [s_1, s_2] \to \mathbb{R}$ where $s_1 < -\frac{1}{c_2}\sqrt[2m]{\frac{\alpha c_2+1}{\beta c_2}}$ and $s_2 > \frac{1}{c_2}\sqrt[2m]{\frac{\alpha c_2+1}{\beta c_2}}$. Then there exists one $\overline{\delta} > 0$ such that for any $\eta \in (c_2 - \overline{\delta}, c_2) \cup (c_2, c_2 + \overline{\delta})$, there is a closed σ-invariant subset Γ_λ of Y_λ, the set of solutions for (6.2) in the product topology, such that $\sigma|\Gamma_\lambda$ is topologically conjugate to $\sigma|\Sigma_3$, the full shift on 3 symbols; in particular, $h_{top}(\sigma|Y_\lambda) \geq \log 3$.*

Lemma 6.1.6 *[15, Main theorem] Let*

$$\Phi_\lambda(y_n, y_{n+1}, \cdots, y_{n+m}) = 0 \tag{6.5}$$

be a difference equation with parameter $\lambda \in [\lambda_0, \lambda_1]$ *and let the function* $\Phi_\lambda : Q^{m+1} \to \mathbb{R}$*, where* $Q = [s_1, s_2] \setminus V$ *for some numbers* $s_1 < s_2$ *and* V *is a finite union of open intervals in* $[s_1, s_2]$ *be such that it is* C^1 *for each* λ *and continuous in* λ *and so are the partial derivatives* $\partial_i \Phi_\lambda$*,* $i = 1, ..., m+1$*. Assume that* Φ_{λ_0} *is a function in two variables:*

$$\Phi_{\lambda_0}(x_1, x_2, \cdots, x_{m+1}) = \chi(x_{M+N}, x_N)$$

where M*,* N *are integers with* $1 \leq N \leq m$*,* $1 \leq M+N \leq m$*. Assume, in addition, that for the equation* $\chi(y,x) = 0$ *there is a branch* $y = \varphi(x)$*, where* $\varphi : Q \to \mathbb{R}$ *is a* C^2 *function with positive topological entropy.*
Then for any $\varepsilon > 0$ *there exists one* $\overline{\delta} > 0$ *such that for any* $\lambda \in [\lambda_0, \lambda_0 + \overline{\delta})$ *there is a closed* σ*-invariant subset* Γ_λ *of* Y_λ *(*Y_λ *is the set of solutions of the difference equation (6.5)), the set of solutions for (6.5) in the product topology, such that* $h_{top}(\sigma|\Gamma_\lambda) > \frac{1}{M}(h_{top}(\varphi) - \varepsilon)$*.*

Let the parameters η, c_2, α, β be fixed while $c_1 \to 0^+$ in the difference equation (6.1) as below.

$$\beta\left(c_1\frac{1-\eta c_2}{1+\eta c_1}z_{n+1} + c_2 z_n\right)^3 + \frac{1-\eta c_2}{1+\eta c_1}z_{n+1} - \alpha c_1\frac{1-\eta c_2}{1+\eta c_1}z_{n+1}$$

$$-\alpha c_2 z_n - z_n = 0, \text{ where } n \in \mathbb{N}\cup\{0\},\ \Delta = 1 + \tfrac{c_1}{c_2}.$$

We have the equation

$$z_{n+1} = \varphi(z_n) = \frac{1+\alpha c_2}{1-\eta c_2} z_n \left(1 - \frac{\beta c_2^2}{\alpha c_2 + 1} z_n^2\right). \tag{6.6}$$

Substituting $x_n = \left(\frac{\beta c_2^2}{\alpha c_2+1}\right)^{\frac{1}{2}} z_n$ in (6.6), we have

$$x_{n+1} = \varphi(x_n) = \frac{1+\alpha c_2}{1-\eta c_2} x_n (1 - x_n^2).$$

Hence we consider the function

$$h(x) = \mu x(1-x^2) \text{ on } [0,1].$$

Since $h'\left(\sqrt{\frac{1}{3}}\right) = 0$ and $h''\left(\sqrt{\frac{1}{3}}\right) < 0$, we can see that the point $\sqrt{\frac{1}{3}}$ is nondegenerate and $h\left(\sqrt{\frac{1}{3}}\right)$ is the local maximum. Moreover, $h\left(\sqrt{\frac{1}{3}}\right) \geq 1$ if $\mu \geq \frac{3\sqrt{3}}{2}$. Thus, we have the results as follows.

Lemma 6.1.7 *Consider the C^2-function $h(x) = \mu x(1-x)$. If $\mu \geq \frac{3\sqrt{3}}{2}$, then $h_{top}(h)$ is positive.*

Proof. The function h is strictly increasing on $\left[0, \sqrt{\frac{1}{3}}\right]$ and is continuous on $[0,1]$ with $h(0) = h(1) = 0$ and $h\left(\sqrt{\frac{1}{3}}\right) \geq 1$. By IVT, there exists one unique point $x_1 \in \left(0, \sqrt{\frac{1}{3}}\right]$ such that $h(x_1) = 1$. Since $h : [0, x_1] \to [0,1]$ is one to one, onto and strictly increasing, there exists one point $x_2 \in (0, x_1)$ such that $h(x_2) = x_1$. Denote $p = x_2$, we have

$$0 = h^3(p) < p < h(p) < h^2(p).$$

Thus, h has points of all periods which implies chaos. Hence, $h_{top}(h)$ is positive. ∎

Therefore, we can conclude that if the parameters η, α, c_2 satisfy the inequality

$$\frac{1+\alpha c_2}{1-\eta c_2} \geq \frac{3\sqrt{3}}{2},$$

then φ has positive topological entropy ($s_1 < 0$ and $s_2 > \frac{1}{c_2}\sqrt{\frac{\alpha c_2+1}{\beta c_2}}$). Thus, we have the results as follows.

Proposition 6.1.8 *Suppose the parameters c_2, η, α, β are to be fixed and satisfy the inequality*

$$\frac{1+\alpha c_2}{1-\eta c_2} \geq \frac{3\sqrt{3}}{2}$$

in the difference equation (6.1) and consider the function $\Phi_\lambda : [s_1, s_2] \times [s_1, s_2] \to \mathbb{R}$, where $s_1 < 0$ and $s_2 > \frac{1}{c_2}\sqrt{\frac{\alpha c_2+1}{\beta c_2}}$. If c_1 is sufficiently small, then there is a closed σ-invariant subset Γ_λ of Y_λ (Y_λ is the set of solutions of the difference equation (6.1)), the set of solutions for (6.1) in the product topology, such that $h_{top}(\sigma|\Gamma_\lambda)$ is positive.

Let the parameters η, c_2, α, β, m be fixed while $c_1 \to 0^+$ in the difference equation (6.2) as below.

$$\beta\left(c_1\tfrac{\eta-c_2}{\eta+c_1}z_{n+1} + c_2 z_n\right)^{2m+1} + \tfrac{\eta-c_2}{\eta+c_1}z_{n+1} - \alpha c_1\tfrac{\eta-c_2}{\eta+c_1}z_{n+1}$$

$$-\alpha c_2 z_n - z_n = 0, \text{ where } n \in \mathbb{N}\cup\{0\},\ \Delta = 1 + \tfrac{c_1}{c_2}.$$

Let the parameters η, c_2, α, β, m be fixed while $c_1 \to 0^+$ in the difference equation (6.2). We have the equation

$$z_{n+1} = \varphi(z_n) = \frac{\eta(\alpha c_2+1)}{\eta - c_2} z_n \left(1 - \frac{\beta c_2^{2m+1}}{\alpha c_2 + 1} z_n^{2m}\right). \tag{6.7}$$

Substituting $x_n = \left(\frac{\beta c_2^{2m+1}}{\alpha c_2+1}\right)^{\frac{1}{2m}} z_n$ in (6.7), we have

$$x_{n+1} = \varphi(x_n) = \frac{\eta(\alpha c_2 + 1)}{\eta - c_2} x_n(1 - x_n^{2m}).$$

Hence we consider the function

$$h(x) = \mu x(1 - x^r) \text{ on } [0, 1], \, r \geq 2.$$

Since $h'\left(\sqrt[r]{\frac{1}{r+1}}\right) = 0$ and $h''\left(\sqrt[r]{\frac{1}{r+1}}\right) < 0$, we can see that the point $\sqrt[r]{\frac{1}{r+1}}$ is nondegenerate and $h\left(\sqrt[r]{\frac{1}{r+1}}\right)$ is the local maximum. Moreover, $h\left(\sqrt[r]{\frac{1}{r+1}}\right) \geq 1$ if $\mu \geq \frac{(r+1)\sqrt[r]{r+1}}{r}$. Thus, we have the results as follows.

Lemma 6.1.9 *Consider the C^2-function $h(x) = \mu x(1-x^r)$, where $r \geq 2$. If $\mu \geq \frac{(r+1)\sqrt[r]{r+1}}{r}$, then $h_{top}(h)$ is positive.*

Proof. The function h is strictly increasing on $\left[0, \sqrt[r]{\frac{1}{r+1}}\right]$ and is continuous on $[0, 1]$ with $h(0) = h(1) = 0$ and $h\left(\sqrt[r]{\frac{1}{r+1}}\right) \geq 1$. By IVT, there exists one unique point $x_1 \in \left(0, \sqrt[r]{\frac{1}{r+1}}\right]$ such that $h(x_1) = 1$. Since $h : [0, x_1] \to [0, 1]$ is one to one, onto and strictly increasing, there exists one point $x_2 \in (0, x_1)$ such that $h(x_2) = x_1$. Denote $p = x_2$, we have

$$0 = h^3(p) < p < h(p) < h^2(p).$$

Thus, h has points of all periods which implies chaos. Hence, $h_{top}(h)$ is positive. ■

Therefore, we can conclude that for any given $m \in \mathbb{N}$ if the parameters η, α, c_2 satisfy the inequality

$$\frac{\eta(\alpha c_2 + 1)}{\eta - c_2} > \frac{(2m+1)\sqrt[2m]{2m+1}}{2m},$$

then φ has positive topological entropy ($s_1 < 0$ and $s_2 > \frac{1}{c_2}\sqrt[2m]{\frac{\alpha c_2+1}{\beta c_2}}$). Thus, we have the results as follows.

Proposition 6.1.10 *Suppose the parameters c_2, η, α, β, m are to be fixed and satisfy the inequality*

$$\frac{\eta(\alpha c_2 + 1)}{\eta - c_2} > \frac{(2m+1)\sqrt[2m]{2m+1}}{2m}$$

in the difference equation (6.2) and consider the function $\Phi_\lambda : [s_1, s_2] \times [s_1, s_2] \to \mathbb{R}$*, where* $s_1 < 0$ *and* $s_2 > \frac{1}{c_2} \sqrt[2m]{\frac{\alpha c_2+1}{\beta c_2}}$*. If* c_1 *is sufficiently small, then there is a closed* σ*-invariant subset* Γ_λ *of* Y_λ *(*Y_λ *is the set of solutions of the difference equation (6.2)), the set of solutions for (6.2) in the product topology, such that* $h_{top}(\sigma|\Gamma_\lambda)$ *is positive.*

It is easy to see that $\sigma|\Gamma_\lambda$ is topologically conjugate to $f_\lambda|\Lambda_\lambda$ as

$$\begin{array}{ccc} \Gamma_\lambda & \overset{\pi_0}{\to} & \Lambda_\lambda \\ \sigma \downarrow & & \downarrow f_\lambda \\ \Gamma_\lambda & \underset{\pi_0}{\to} & \Lambda_\lambda \end{array} ,$$

where π_0 is a projection and $\pi_0(\Gamma_\lambda) = \Lambda_\lambda$. Hence, $h_{top}(f_\lambda|\Lambda_\lambda)$ is positive if $h_{top}(\sigma|\Gamma_\lambda)$ is positive. Thus, we have the results as follows.

Theorem 6.1.11 *Let the parameters* c_1*,* c_2*,* α*,* β *be fixed in the 1D wave system (1.1)-(1.4) and the initial condition* I *contains the interval*

$$\left[-\frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}, \frac{1}{c_2}\sqrt{\frac{1+\alpha c_2}{\beta c_2}}\right] \text{ (see Definition 3.1.2).}$$

Then there exists one $\overline{\delta} > 0$ *such that if* $\eta \in \left(\frac{1}{c_2} - \overline{\delta}, \frac{1}{c_2}\right) \cup \left(\frac{1}{c_2}, \frac{1}{c_2} + \overline{\delta}\right)$ *, the 1D wave system is chaotic.*

Proof. The results follow easily from Lemma 6.1.3 and Proposition 6.1.4. ■

Example 6.1.12 *Consider the 1D wave system (1.5)-(1.7) as below:*

$$\begin{cases} \omega_{tt} - \omega_{xx} = 0, & 0 < x < 1,\, t > 0, \\ \omega_x(0,t) + \eta\omega_t(0,t) = 0, & \eta > 0,\, \eta \neq 1,\, t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), & \alpha \in (0,1],\, \beta > 0,\, t > 0, \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

Let the parameters α*,* β *be fixed and suppose that the initial condition* I *contains the interval*

$$\left[-\sqrt{\frac{1+\alpha}{\beta}}, \sqrt{\frac{1+\alpha}{\beta}}\right].$$

Then there exists one $\overline{\delta} > 0$ *such that the 1D wave system (1.5)-(1.7) is chaotic if*

$$\eta \in (1-\overline{\delta}, 1) \cup (1, 1+\overline{\delta}).$$

Theorem 6.1.13 *Let the parameters* c_1, c_2, α, β, m *be fixed in the 1D wave system (1.12) and the initial condition* I *contains the interval*

$$\left[-\frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, \frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}\right] \text{ (see Definition 4.1.2).}$$

Then there exists one $\overline{\delta} > 0$ *such that if* $\eta \in (c_2 - \overline{\delta}, c_2) \cup (c_2, c_2 + \overline{\delta})$ *, the 1D wave system is chaotic.*

Proof. The results follow easily from Lemma 6.1.3 and Proposition 6.1.5. ■

Theorem 6.1.14 *Let the parameters* c_2, η, α, β *be fixed and satisfy the inequality*

$$\frac{1+\alpha c_2}{1-\eta c_2} \geq 3\sqrt{3}/2$$

in the 1D wave system (1.1)-(1.4) and the initial condition I *contains the interval* $\left[0, \frac{1}{c_2}\sqrt{\frac{\alpha c_2+1}{\beta c_2}}\right]$. *Then the 1D wave system is chaotic if* c_1 *is sufficiently small.*

Proof. The results follow easily from Lemma 6.1.6, Lemma6.1.7 and Proposition 6.1.8. ■

Theorem 6.1.15 *Let the parameters* c_2, η, α, β, m *be fixed and satisfy the inequality*

$$\frac{\eta(\alpha c_2 + 1)}{\eta - c_2} > \frac{(2m+1)\sqrt[2m]{2m+1}}{2m}$$

in the 1D wave system (1.12) and the initial condition I *contains the interval* $\left[0, \frac{1}{c_2}\sqrt[2m]{\frac{\alpha c_2+1}{\beta c_2}}\right]$. *Then the 1D wave system is chaotic if* c_1 *is sufficiently small.*

Proof. The results follow easily from Lemma 6.1.6, Lemma6.1.9 and Proposition 6.1.10. ■

6.2 Further discussions

Furthermore, we consider the 1D wave system as below:

$$\begin{cases} \omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0, & d \in \mathbb{R},\ c > 0,\ 0 < x < 1,\ t > 0, \\ \omega_t(0,t) + \eta\omega_x(0,t) = 0, & \eta > 0,\ \eta \neq c_2,\ t > 0, \\ \omega_x(1,t) = h_\mu(\omega_t(1,t)), & \mu = (a_1, \cdots, a_k),\ t > 0, \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), & \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases} \tag{6.8}$$

Now we want to find the condition to ensure the existence and uniqueness of the solution of the 1D wave system (6.8).

Lemma 6.2.1 *Consider the 1D wave system (6.8). Let the function $h_\mu(x)$ be a real-valued function defined on $\mathbb{R}$ where $\mu = (a_1, \cdots, a_k)$. Suppose that there exists one set $A \subseteq \mathbb{R}^k$ such that for each $\mu \in A$ the function $h_\mu(x)$ is C^1, onto and the derivative of h_μ satisfies that $h'_\mu(x) \neq \frac{1}{c_1}$ for all $x \in \mathbb{R}$, then there exists one C^1-function $\hat{h}_\lambda$ such that $\hat{h}_\lambda(z(t)) = z(t + \Delta)$ for all $t > 0$, where*

$$\lambda = (\eta, c_1, c_2, a_1, \cdots, a_k) \in G \times (0,\infty) \times (0,\infty) \times A (G = (0,\infty)\backslash\{c_2\}),\ \Delta = 1 + c_1/c_2.$$

Proof. Let

$$H_\lambda(u,v) = h_\mu\left(c_1\frac{\eta - c_2}{\eta + c_1}u + c_2 v\right) - \frac{\eta - c_2}{\eta + c_1}u + c_2 v = 0,$$

where

$$\lambda = (\eta, c_1, c_2, a_1, \cdots, a_k) \in G \times (0,\infty) \times (0,\infty) \times A\ (G = (0,\infty)\backslash\{c_2\}).$$

Since $h'_\mu(x) \neq \frac{1}{c_1}$ and h_μ is onto for each μ, the function $f_\mu(x) \equiv h_\mu(x) - \frac{x}{c_1}$ is one-to-one and onto. Hence, for each $v_0 \in \mathbb{R}$ there exists one unique x_0 such that $f_\mu(x_0) = \frac{c_1 + c_2}{c_1}v_0$. And for this unique x_0 there exists one unique u_0 such that $c_1\frac{\eta - c_2}{\eta + c_1}u_0 + c_2 v_0 = x_0$. Therefore, we denote

$$S = \{(v,u) : H_\lambda(u,v) = 0\}.$$

Since

$$\frac{\partial}{\partial u}H_\lambda(u,v) = c_1\frac{\eta - c_2}{\eta + c_1}h'_\mu\left(c_1\frac{\eta - c_2}{\eta + c_1}u + c_2 v\right) - \frac{\eta - c_2}{\eta + c_1} \neq 0,$$

there exists one C^1-function $\hat{h}_\lambda$ and the graph of $\hat{h}_\lambda$ is S by the implicit function theorem. ■

First we discretelize the difference equation

$$\frac{\eta - c_2}{\eta + c_1} z(\tau + \Delta) - z(\tau) = h_\mu \left(c_1 \frac{\eta - c_2}{\eta + c_1} z(\tau + \Delta) + c_2 z(\tau) \right) \tag{6.9}$$

as follows.

Given one $\tau \in (0, \Delta]$, we denote $z(\tau + n\Delta) = z_n$ $(n = 0, 1, \cdots)$ in (6.9). Then we have

$$h_\mu \left(c_1 \frac{\eta - c_2}{\eta + c_1} z_{n+1} + c_2 z_n \right) - \frac{\eta - c_2}{\eta + c_1} z_{n+1} + c_2 z_n = 0, \text{ where } n \in \mathbb{N} \cup \{0\}.$$

Definition 6.2.2 *We denote the difference equation*

$$\Psi_\lambda(z_n, z_{n+1}) = h_\mu \left(c_1 \frac{\eta - c_2}{\eta + c_1} z_{n+1} + c_2 z_n \right) - \frac{\eta - c_2}{\eta + c_1} z_{n+1} + c_2 z_n = 0, \tag{6.10}$$

where $\lambda = (\eta, c_1, c_2, a_1, \cdots, a_k)$.

Theorem 6.2.3 *Consider the 1D wave system (6.8). Let the function* $h_\mu : \mathbb{R} \to \mathbb{R}$ *be* C^1*, onto for each* μ *and be continuous in* μ *and so is the derivative* $h_\mu^{'}(x)$*, where* $\mu = (a_1, \cdots, a_k)$*. Suppose that for each* μ *the derivative* $h_\mu^{'}(x) \neq \frac{1}{c_1}$ *for all* $x \in \mathbb{R}$ *and there exist two distinct values* $a_\mu, b_\mu \in \mathbb{R}$ *such that* $h_\mu(a_\mu) + a_\mu = h_\mu(b_\mu) + b_\mu = 0$*, then for each* $(c_1, c_2, a_1, \cdots, a_k)$ *there exists one corresponding* $\overline{\delta} > 0$ *such that for any* $\eta \in (c_2 - \overline{\delta}, c_2) \cup (c_2, c_2 + \overline{\delta})$ *there is a compact* $\hat{h}_\lambda$*-invariant subset* Λ_λ *such that* $h_{top}(\hat{h}_\lambda | \Lambda_\lambda)$ *is positive. Furthermore, the 1D wave system (6.8) is chaotic if* $\Lambda_\lambda \subseteq I$*.*

Proof. By Lemma 6.2.1, there exists a C^1-function $\hat{h}_\lambda$ such that $\hat{h}_\lambda(z(t)) = z(t + \Delta)$ for all $t > 0$ where $\lambda = (\eta, c_1, c_2, a_1, \cdots, a_k)$.
Let

$$\Omega = \{(\eta, c_1, c_2, a_1, \cdots, a_k) : \eta = c_2\}.$$

For each $\lambda_0 \in \Omega$, we have

$$\Psi_{\lambda_0}(z_n, z_{n+1}) = \varphi(z_n) = h_\mu(c_2 z_n) + c_2 z_n.$$

The function φ is C^1 and has at least two simple zeros in the interior of $[a_\mu - \varepsilon, b_\mu + \varepsilon]$ for every $\varepsilon > 0$, since there exist two distinct values a_μ, $b_\mu \in \mathbb{R}$ such that

$$h_\mu(a_\mu) + a_\mu = h_\mu(b_\mu) + b_\mu = 0.$$

By Lemma 6.1.3, there exists one $\overline{\delta}_{\lambda_0} > 0$ such that for any $\eta \in (c_2 - \overline{\delta}_{\lambda_0}, c_2) \cup (c_2, c_2 + \overline{\delta}_{\lambda_0})$ there is a closed σ-invariant subset Γ_λ of Y_λ, the set of solutions for (6.10) in the product topology, such that $\sigma|\Gamma_\lambda$ is topologically conjugate to $\sigma|\Sigma_k$, the full shift on k symbols ($k \geq 2$); in particular, $h_{top}(\sigma|Y_\lambda)$
$\geq \log k$.
We can see $\sigma|\Gamma_\lambda$ is topologically conjugate to $\hat{h}_\lambda|\Lambda_\lambda$ as

$$\begin{array}{ccc} \Gamma_\lambda & \overset{\pi_0}{\rightarrow} & \Lambda_\lambda \\ \sigma\downarrow & & \downarrow \hat{h}_\lambda \\ \Gamma_\lambda & \underset{\pi_0}{\rightarrow} & \Lambda_\lambda \end{array},$$

where π_0 is a projection and $\pi_0(\Gamma_\lambda) = \Lambda_\lambda$. Hence $h_{top}(\hat{h}_\lambda|\Lambda_\lambda)$ is positive, since $h_{top}(\sigma|\Gamma_\lambda)$ is positive. ■

Example 6.2.4 *Consider the 1D wave system (6.8) which the function*

$$h_\mu(x) = a_1 x - \sum_{i=2}^{m} a_i x^{2i-1},\ a_1 \in (0, \frac{1}{c_1}),\ a_i > 0 \text{ for } i \geq 2,\ m \in \mathbb{N}.$$

The function $h_\mu : \mathbb{R} \to \mathbb{R}$ is C^1, onto for each μ and is continuous in μ and so is the derivative $h_\mu^{'}(x)$, where $\mu = (a_1, \cdots, a_m, m)$. And for each μ

$$h_\mu^{'}(x) = a_1 - 3a_2x^2 - \cdots - (2m-1)a_m x^{2m-2} \neq \frac{1}{c_1} \text{ for all } x \in \mathbb{R}.$$

Since

$$h_\mu(x) + x \geq x[(1 + a_1) - (a_2 + \cdots + a_m)x] \text{ for } x \in (0, 1],$$

we have

$$h_\mu(p_\mu) + p_\mu \geq 0 \text{ where } p_\mu = \min\{1, \frac{1 + a_1}{a_2 + \cdots + a_m}\}.$$

It is easy to see there exists one point q_μ *which is large enough such that* $h_\mu(q_\mu) + q_\mu < 0$*. Therefore, there exist two points* 0 *and* $x_\mu \in [p_\mu, q_\mu)$ *such that*

$$h_\mu(0) + 0 = h_\mu(x_\mu) + x_\mu = 0.$$

By Theorem 6.2.3, for each $(c_1, c_2, a_1, \cdots, a_m, m)$ *there exists one corresponding* $\overline{\delta} > 0$ *such that for any* $\eta \in (c_2 - \overline{\delta}, c_2) \cup (c_2, c_2 + \overline{\delta})$ *there is a compact* $\hat{h}_\lambda$*-invariant subset* Λ_λ *such that* $h_{top}(\hat{h}_\lambda | \Lambda_\lambda)$ *is positive. Furthermore, the 1D wave system is chaotic if* $\Lambda_\lambda \subseteq I$*.*

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