國立交通大學

資訊科學與工程研究所

碩 士 論 文

條件式容錯超立方體下的邊泛迴圈之研究

Edge-bipancyclicity of conditional faulty hypercubes

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摘要

Xu et al.先前曾提過相關的論文研究，證明了當N大於四時，在n維度的超力方體下若壞邊的數量小餘n-1 時，通過任意指定的一個邊，仍可找到從 6 到 2^n這樣各種長度的迴圈，而限制條件在於並非所有壞邊皆集中在同一個點，意即每個點仍然存在有兩個好邊。

在這篇論文中，我們在相似的條件下，若壞邊不集中在同一個點，則壞邊個數能夠增加到 2n-5 個，且通過任意指定的一個邊，仍可找到長度從 6 到 $2^{n\text{的}}$各種的迴圈。此外我們仍證明了，當壞邊達到 2n-4 時，如此是無法被證明的，所以我們的結論是最佳的結果。

關鍵字：迴圈，邊泛迴圈，條件式容錯超力方體，容錯

Edge-bipancyclicity of conditional faulty hypercubes

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Abstract

Xu et al. showed that for any set of faulty edges F of an n-dimensional hypercube Q_n with $|F| \leqq n-1$, each edge of Q_n-F lies on a cycle of every even length from 6 to 2^n, $n \geqq 4$, provided not all edges in F are incident with the same vertex. In this paper, we find that under similar condition, the number of faulty edges can be much greater and the same result still holds. More precisely, we show that, for up to |F|=2n-5 faulty edges, each edge of the faulty hypercube Q_n-F lies on a cycle of every even length from 6 to 2^n with each vertex having at least two healthy edges adjacent to it, for $n \geqq 3$. Moreover, this result is optimal in the sense that the result can not be guaranteed, if there are 2n-4 faulty edges.

Keywords: cycles, Pancyclic, Conditional fault, Hypercube, Fault-tolerant

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Chapter 1

Introduction

The ring embedding problem, which deals with all the possible lengths of the cycles in a given graph, is investigated in a lot of interconnection networks [2, 3, 4].If a graph contains cycles of all lengths, it is called pancyclic [7]. Bipancyclicity is essentially a restriction of the concept of pancyclicity to cycle of even lengths. A bipartite graph is *vertex-bipancyclic* [6] if every vertex lies on a cycle of every even length from 4 to $|V(G)|$. Similarly, a bipartite graph is *edge-bipancyclic* if every edge lies on a cycle of every even length from 4 to $|V(G)|$. A bipartite graph is *k-edge-fault-tolerant edge-bipancyclic* if $G - F$ remains edge-bipancyclic for any set of faulty edges $F \subset E(G)$ with $|F| \leq k$. A path P is a sequence of adjacent vertices, written as $\langle v_0, v_1, ..., v_m \rangle$. The *length* of a path P, denoted by $l(P)$, is the number of edges in P. A hamiltonian cycle is a graph with a spanning cycle.In addition we call e a *healthy edge* when e is fault-free in a graph.

Chan and Lee [1] considered an injured n-dimensional hypercube where each vertex is incident with at least two healthy edges, and proved that it still contains a hamiltonian cycle even it has $(2n - 5)$ edge faults. Tsai [8] proved that such injured hypercube Q_n contains a cycle of every even length from 4 to 2^n, even if it has up to $(2n - 5)$ edge faults.

Recently, Xu et al. [9] showed that for any set of faulty edges F od Q^n with $|F| \leq n-1$, each edge of $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, $n \geq 4$, provided not all faulty edges are incident with the same vertex. We observe that not all faulty edges a re incident with the same vertex is equivalent to stating that each vertex has at least two healthy edges adjacent to it, if $|F| \leq n-1$. In this paper, we consider a set of faulty edges satisfies the condition that each vertex of $Q_n - F$ is incident with at least two healthy edges. Such a set of faulty edges F is called a set of conditional faulty edges and $Q_n - F$ is called a conditional faulty hypercube. We find that under this condition, the number of faulty edges can be much greater and the same result still holds. We show that, for up to $|F| = 2n-5$ conditional faulty edges, each edge of a faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to 2^n with each vertex having at least two healthy edges adjacent to it, for $n \geq 3$. We observe that, if $|F| < 2n-5$, we may arbitrarily delete some more edges to make a faulty edge set $F' \supseteq F$ and $|F'| = 2n-5$. If our result holds for F', it holds for F. From now on, we shall assume $|F| = 2n-5$.

The above result is optimal in the sense that result can not be guaranteed, if there are $2n-4$ conditional faulty edges. For example, take a cycle of length four in Q_n, let $\langle u_1, u_2, u_3, u_4 \rangle$ be the consecutive vertices on this cycle. Suppose that all the $(n-2)$ edges incident to vertex u_1 (respectively vertex u_3) are faulty except those two edges on the four cycle are healthy. There are $2(n-2)$ conditional faulty edges. (see Fig.1.1) Then there does not exist a hamiltonian cycle in this faulty Q_n, for $n \geq 3$.

We now give a formal definition of a hypercube. An n-dimensional hypercube is denoted by Q_n with the vertex set $V(Q_n)$ and the edge set $E(Q_n)$. Each vertex u of Q_n can be distinctly labeled by a n-bit binary strings, $u = u_{n-1}u_{n-2}...u_1u_0$. There is an edge

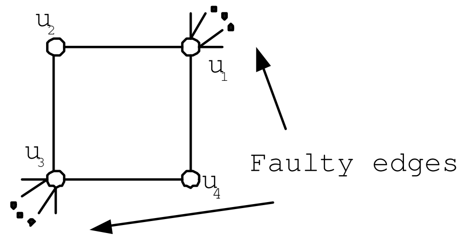

Figure 1.1: Illustration for the Q_n with (2n-4) edge fault.

between two vertices if and only if their binary labels differ in exactly one bit position. Let u and v be two adjacent vertices. If the binary labels of u and v differ in ith position, then the edge between them is said to be in dimension i and the edge (u, v) is called an ith dimension edge. Let i be a fixed position, we use Q^0_{n-1} to denote the subgraph of Q_n induced by $\{u \in V(Q_n)|u_i = 0\}$ and Q^1_{n-1} to denote the subgraph of Q_n induced by $\{u \in V(Q_n)|u_i = 1\}$. We say that Q_n is decomposed into Q^0_{n-1} and Q^1_{n-1} by dimension i, and Q^0_{n-1} and Q^1_{n-1} are $(n-1)$-dimensional subcube of Q_n induced by the vertices with the ith bit position being 0 and 1 respectively. Q^0_{n-1} and Q^1_{n-1} are all isomorphic to Q_{n-1}. For each vertex $u \in V(Q^0_{n-1})$, there is exactly one vertex in Q^1_{n-1}, denoted by $u^{(1)}$, such that $(u, u^{(1)}) \in E(Q_n)$. Conversely, for each $u \in V(Q^1_{n-1})$, there is one vertex in Q^0_{n-1}, denoted by $u^{(0)}$, such that $(u, u^{(0)}) \in E(Q_n)$. Let D_i be the set of all edges with one end in Q^0_{n-1} and the other in Q^1_{n-1}. These edges are called crossing edges in the ith dimension between between Q^0_{n-1} and Q^1_{n-1}. We also call D_i the set of all ith dimension edges. Consequently, $|D_i| = 2^{n-1}$ for all $0 \leq i \leq n-1$.

Chapter 2

Some Preliminaries

To prove our main theorem, we need some preliminary results.

Lemma 1 *[5] Q_n is edge-bipancyclic, and is (n-2)-edge-fault-tolerant edge-bipancyclic, for $n \geq 3$.*

Lemma 2 *[9] Each edge of $Q_4 - F$ lies on a cycle of every even length from 6 to $2^n = 16$ for any $F \subset E(Q_4)$ with $|F| = 3$, provided not all the faulty edges in F are incident with the same vertex.*

Lemma 3 *[9] Any two edges in Q_n are included in a hamiltonian cycle, for $n \geq 2$.*

Proof. We prove the lemma by induction on $n \geq 2$. Obviously, the lemma is true for $n = 2$. Assume that the lemma is true for every k with $2 \leq k < n$. Let e and e' be two edges in Q_n and express $Q_n = L_k \odot R_k$ such that none of e and e' is k-dimensional. Without loss of generality, we may assume $e \in L_k$. Furthermore, we can suppose that e' is in L_k, otherwise consider e'_L instead of e'. By the induction hypothesis, there exists a

Hamiltonian cycle C containing e and e' in L_k. Let u_Lv_L be an edge on C different from e and e'. The corresponding C' is a Hamiltonian cycle in R_k containing u_Rv_R, e_R and e'_R. Let $P = C - u_Lv_L$ and $P' = C' - u_Rv_R$. Then $P + u_Lv_L + P' + u_Rv_R$ is a Hamiltonian cycle in Q_n containing e and e'. □

The above lemma can be improved; In addition to the hamiltonian cycle, there are cycles of smaller lengthes passing through these two edges. We have the following lemma.

Lemma 4 *Any two edges in Q_n are included in a cycle of length 2^n, $2^n - 2$ and $2^n - 4$ respectively, for $n \geq 3$.*

Proof. Let e_1, e_2 be two arbitrary edges in Q_n. Since $n \geq 3$, we can decompose Q_n into two subcubes Q^0_{n-1} and Q^1_{n-1} by some dimension i, such that e_1 and e_2 are not crossing edges. We then consider the following two cases:

Case 1: *Both edges e_1 and e_2 are in the same subcube Q^i_{n-1}.* Without loss of generality, we assume that $\{e_1, e_2\} \subseteq E(Q^0_{n-1})$. By Lemma 3, there exists a hamiltonian cycle C_0 going through e_1 and e_2 in Q^0_{n-1}, and the length of C_0 is 2^{n-1}. Since $n \geq 3$, there is a third edge (u, v) other than e_1 and e_2 on cycle C_0. We write C_0 as $\langle u, P_0, v, u\rangle$. It follows from the definition of the hypercubes, $(u^{(1)}, v^{(1)})$ is an edge in Q^1_{n-1}. By Lemma 1, in Q^1_{n-1}, there exists a cycle C_1 of every even length $4 \leq l(C_1) \leq 2^{n-1}$ going through $(u^{(1)}, v^{(1)})$. We write C_1 as $\langle u^{(1)}, P_1, v^{(1)}, u^{(1)}\rangle$. Thus, $\langle u, P_0, v, v^{(1)}, P_1, u^{(1)}, u\rangle$ can form a cycle of length 2^n, $2^n - 2$ or $2^n - 4$ respectively passing through edges e_1 and e_2 in Q_n, if we adjust the length of P_1 properly.

Case 2: *e_1 and e_2 are in different subcubes.* Without loss of generality, we assume that $e_1 \in E(Q^0_{n-1})$, $e_2 \in E(Q^1_{n-1})$. By Lemma 1, there exists a cycle C_0 of every even length, $4 \leq l(C_0) \leq 2^{n-1}$ going through e_1 in Q^0_{n-1}. Since $n \geq 3$, we can choose an edge (u, v) on cycle C_0 and by definition, $(u^{(1)}, v^{(1)})$ is an edge in Q^1_{n-1} such that $(u, v) \neq e_1$, and $(u^{(1)}, v^{(1)}) \neq e_2$. We write C_0 as $\langle u, P_0, v, u\rangle$. By Lemma 3, there exists a cycle C_1 of length 2^{n-1} going through e_2 and $(u^{(1)}, v^{(1)})$. We write C_1 as $\langle u^{(1)}, P_1, v^{(1)}, u^{(1)}\rangle$. Thus, we can adjust the length of P_0 properly such that it produces a cycle $\langle u, P_0, v, v^{(1)}, P_1, u^{(1)}, u\rangle$ of length 2^n, $2^n - 2$ or $2^n - 4$ respectively going through edges e_1, e_2 in Q_n.

This proves the lemma. □

Lemma 5 *Let Q_n be an n-dimensional hypercube, $n \geq 2$, and let e_1 and e_2 be two edges in the same dimension i. Then there exists another dimension $j \neq i$ such that decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j, we have (1) neither e_1 nor e_2 is crossing edges, (2) not e_1 and e_2 are in the same subcube.*

Proof. Let $e_1 = (a, b)$ and $e_2 = (s, t)$ be two edges in the same dimension i. Let $a = a_n...a_i...a_1$ and $s = s_n...s_i...s_1$. Then $b = b_n...\overline{b_i}...b_1$ and $t = t_n...\overline{t_i}...t_1$. Since $e_1 \neq e_2$ and $n \geq 2$, there exists another dimension $j \neq i$, such that $a_j \neq s_j$. We decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j. Then, e_1 and e_2 are not crossing edges and in the different subcubes. □

In some special cases, Lemma 4 still holds even if there is one faulty edge. We need the following lemma later.

Lemma 6 *Let Q_n be an n-dimensional hypercube and e_0 be a faulty edge in the ith dimension. Any two healthy edges e_1 and e_2 in the ith dimension are included in a cycle of length 2^n, $2^n - 2$ and $2^n - 4$ respectively, for $n \geq 4$.*

Proof. Since e_0, e_1 and e_2 are all in the ith demension, and $n \geq 4$. By Lemma 5, we can choose a dimension j different from i such that e_1 and e_2 are in different subcubes. Without loss of generality, we may assume that e_1 is in Q^0_{n-1} and e_2 is in Q^1_{n-1} and e_0 is in Q^0_{n-1}. By Lemma 1, in $Q^0_{n-1} - \{e_0\}$, there exists a cycle C_0 of every length $4 \leq l(C_0) \leq 2^{n-1}$ going through e_1. Since $n \geq 4$, we can choose an edge (u, v) on cycle C_0 such that $(u, v) \neq e_1$, and $(u^{(1)}, v^{(1)}) \neq e_2$. We write C_0 as $\langle u, P_0, v, u\rangle$. By Lemma 3, in Q^1_{n-1}, there exists a hamiltonian cycle C_1 going through e_2 and $(u^{(1)}, v^{(1)})$. We write C_1 as $\langle u^{(1)}, P_1, v^{(1)}, u^{(1)}\rangle$. Thus, $\langle u, P_0, v, v^{(1)}, P_1, u^{(1)}, u\rangle$ is a cycle of length 2^n, $2^n - 2$ or $2^n - 4$ respectively, if we adjust the length of P_0 properly. □

Let F be a set of faulty edges of Q_n. Suppose that we decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j, and let $F_L = F \cap E(Q^0_{n-1})$, $F_R = F \cap E(Q^1_{n-1})$. Suppose that F is a set of conditional faulty edges of Q_n. If we arbitrarily decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by a dimension, F_L and F_R may not be conditional faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. However, we will show that it is always possible to find some suitable dimension such that decomposing by this dimension, both F_L and F_R are conditional faulty sets in Q^0_{n-1} and Q^1_{n-1} respectively.

Lemma 7 *Consider the n-dimensional hypercube Q_n, for $n \geq 4$. Let F be a set of conditional faulty edges with $|F| = 2n - 5$. There are at most two vertices in Q_n incident with (n-2) faulty edges.*

Proof. If there are three vertices in Q_n incident with (n-2) faulty edges, the number of faulty edge F is at least $3n - 8$. However, $(3n - 8) > (2n - 5)$ for all $n \geq 4$ which is a contradiction. □

Lemma 8 *Consider the n-dimensional hypercube Q_n, $n \geq 4$. Let F be a set of conditional faulty edges with $|F| = 2n - 5$. If there are two vertices x and y both incident with n-2 faulty edges, then x and y are adjacent in Q_n and the edge (x,y) is a faulty edge. Suppose that (x,y) is in dimension j. Then decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j, both F_L and F_R are sets of conditional faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. Moreover, $|F_L| \leq 2n - 6$ and $|F_R| \leq 2n - 6$.*

Proof. If there are two vertices x and y in Q_n incident with $(n-2)$ faulty edges, then these two vertices are connected by a faulty edge. Otherwise, $|F| = 2(n-2) = 2n-4 > 2n-5$ which is a contradiction. Suppose the edge (x, y) is in dimension j, we decompose Q_n into two subcubes. It is clearly that each vertex in Q^0_{n-1} and Q^1_{n-1} is still incident with at least two healthy edges, and F_L and F_R are both conditional faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. Then, $|F_L| = |F_R| = n - 3 \leq 2n - 6$, for $n \geq 4$. □

Lemma 9 *Consider an n-dimensional hypercube Q_n, for $n \geq 4$. Let F be a set of conditional faulty edges with $|F| = 2n - 5$. Suppose that there exists exactly one vertex x having (n-2) faulty edges indcident with it. Since $n - 2 \geq 2$, let e_1 and e_2 be two faulty edges incident with x, and let e_1 and e_2 be jth and kth dimension edges respectively. Then decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by either one of these two dimensions j and k, F_L and F_R are still sets of conditonal faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. Moreover, $|F_L| \leq 2n - 6$ and $|F_R| \leq 2n - 6$.*

Proof. If there exists only one vertex x having $(n-2)$ faulty edges incident with it, there are at least two faulty edges e_1 and e_2 incident with it, since $n \geq 4$. Obviously, these two faulty edges are in different dimensions. Without loss of generality, we may assume that e_1 is in dimension j and e_2 is in dimension k, for $j \neq k$. We can decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by either jth or kth dimension, and either e_1 or e_2 is a crossing edge. Therefore, each vertex in these two subcubes is incident with at least two healthy edges and $|F_L| \leq 2n-6$ and $|F_R| \leq 2n-6$. $\square$

Lemma 10 *Let Q_n be an n-dimensional hypercube, F be a set of faulty edges with $|F| \geq 2$, and e be a healthy edge, $n \geq 2$. Then there exists a dimension j, decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by this dimension, such that e is not a crossing edge and not all the faulty edges are in the same subcube.*

Proof. Suppose that $e = (u, v)$ is in dimension i. If there is a faulty edge f not in dimension i, say in dimension j. We decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j. Then f is a crossing edge but e is not, and all the faulty edges are not in the same subcube. Otherwise, all the faulty edges are in the same dimension i as e is in. We now choose any two faulty edges f_1 and f_2 in F. By Lemma 5, Q_n can be decomposed into Q^0_{n-1} and Q^1_{n-1} by some dimension $j \neq i$ such that edges f_1 and f_2 are not in the same subcube, and e_0 is not a crossing edge. $\square$

Chapter 3

Main theorem

We now prove our main result.

Theorem 1 *Let Q_n be an n-dimensional hypercube, and F be a set of conditional faulty edges with $|F| \leq 2n-5$. Then each edge of the conditional faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, for $n \geq 3$.*

Proof. We prove this lemma by induction on n. For $n = 3$, since $2n-5 = n-2$, by Lemma 1, the result is true. For $n = 4$, $2n-5 = n-1$, by Lemma 2, the result holds. Assume the lemma holds for $n-1$, for some $n \geq 5$, we shall show that it is true for n.

As we mentioned before, we may assume $|F| = 2n-5$. Let $e = (u, v)$ be an edge in $Q_n - F$. We shall find a cycle of every even length from 6 to 2^n passing through e in $Q_n - F$. Assume that e is an ith dimension edge, $e \in D_i$, for some $i \in \{1, 2, ..., n\}$. The proof is divided into three major cases:

Case 1: *There are two vertices x and y in Q_n incident with $(n-2)$ faulty edges.* By Lemma 8, (x, y) is an edge in Q_n and is a faulty edge. We denote this edge by e_f. Suppose

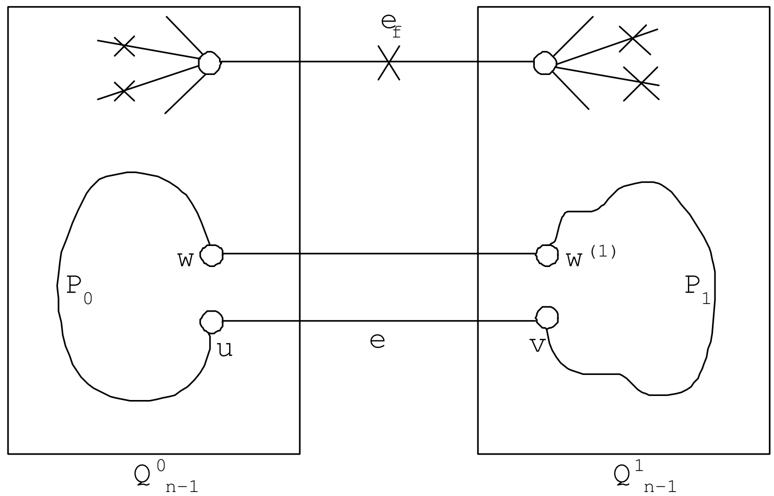

Figure 3.1: Illustration for of Theorem Case 1.1.

that e_f is a jth dimension edge. We decompose Q_n into Q_{n-1}^0 and Q_{n-1}^1 by dimension j. We then consider two further cases:

1.1 *$e_f = (x, y)$ and $e = (u, v)$ are in the same dimension.* Thus, $j = i$ and $e_f \in D_i$. (Fig. 3.1) In this case, e is an edge crossing Q_{n-1}^0 and Q_{n-1}^1. Without loss of generality, assume that $u \in V(Q_{n-1}^0)$ and $v \in V(Q_{n-1}^1)$. Since $n \geq 5$, u has a neighbor vertex $w \in Q_{n-1}^0$, by the definition of hypercube, $w^{(1)}$ is a neighbor of v such that the edge $(w, w^{(1)})$ is a healthy edge and $(w, w^{(1)})$ is a crossing edge between Q_{n-1}^0 and Q_{n-1}^1. By lemma 1, there exists a cycle C_0 in $Q_{n-1}^0 - F_L$ passing through (u, w) of every even length $4 \leq l(C_0) \leq 2^{n-1}$ and a cycle C_1 in $Q_{n-1}^1 - F_R$ going through $(v, w^{(1)})$ of every even length $4 \leq l(C_1) \leq 2^{n-1}$. We write C_0 as $\langle u, P_0, w, u \rangle$, and C_1 as $\langle v, P_1, w^{(1)}, v \rangle$. Thus, $\langle u, P_0, w, w^{(1)}, v, u \rangle$ is a cycle of length 6 with $l(P_0) = 3$. For every even l, $8 \leq l \leq 2^n$, we may choose C_0 and C_1 such that $l(C_0) = l(C_1) = \frac{l}{2}$. Thus, $\langle u, P_0, w, w^{(1)}, P_1, v, u \rangle$ can form a cycle of length l through e in $Q_n - F$, if we adjust the length of P_0 and P_1 properly.

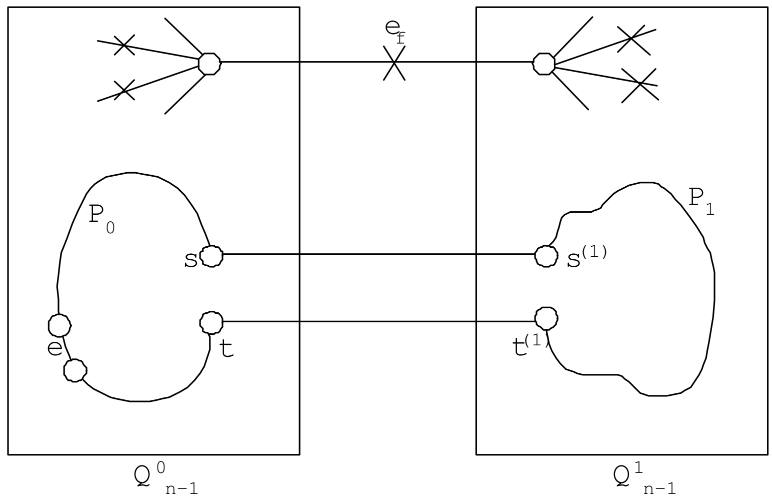

Figure 3.2: Illustration for of Theorem Case 1.2.

1.2 *e_f and e are in different dimension.* Thus, $j \neq i$ and $e_f \notin D_i$. (Fig 3.2) In this case, e is in Q^0_{n-1} or Q^1_{n-1}. Without loss of generality, we may assume that $e \in E(Q^0_{n-1})$. By Lemma 1, there exists a cycle C in $Q^0_{n-1} - F_L$ going through the edge e of every even length l, $6 \leq l \leq 2^{n-1}$. Let C_0 be a cycle of length $2^{n-1} - 2$ or 2^{n-1} passing through e in $Q^0_{n-1} - F_L$. Since $n \geq 5$, there exists an edge (s, t) on C_0 such that neither s nor t is adjacent to e_f and $(s, t) \neq e$. We write C_0 as $\langle s, P_0, t, s\rangle$. By definition, $(s^{(1)}, t^{(1)})$ is an edge in Q^1_{n-1}, and $(s, s^{(1)})$, $(t, t^{(1)})$ are healthy edges. By Lemma 1, there exists a cycle C_1 in $Q^1_{n-1} - F_R$ through $(s^{(1)}, t^{(1)})$ of every even length $4 \leq l(C_1) \leq 2^{n-1}$. We write C_1 as $\langle s^{(1)}, P_1, t^{(1)}, s^{(1)}\rangle$. For every even l, $2^{n-1} + 2 \leq l \leq 2^n$. $\langle s, P_0, t, t^{(1)}, P_1, s^{(1)}, s\rangle$ is a cycle of length l going through e in $Q_n - F$ if we adjust the length of P_0 and P_1 properly.

Case 2: *There is exactly one vertex in Q_n incident with $(n - 2)$ faulty edges.* Let x be the vertex having $(n - 2)$ faulty edges incident with it. Let f_1 and f_2 be two faulty edges incident with x, so f_1 and f_2 are in different dimensions j and k. By Lemma

9, decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by either jth or kth dimension, both $F_L = F \cap E(Q^0_{n-1})$ and $F_R = F \cap E(Q^1_{n-1})$ are sets of conditional faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. Between dimension j and k, we choose one to decompose Q_n into Q^0_{n-1} and Q^1_{n-1}, say dimension j, such that the required edge e is not a crossing edge. Therefore, there is a faulty edge crossing Q^0_{n-1} and Q^1_{n-1}, we denote this edge by e_f, and $e_f \in F \cap D_j$ is incident with x. Without loss of generality, we may assume that $x \in V(Q^0_{n-1})$.

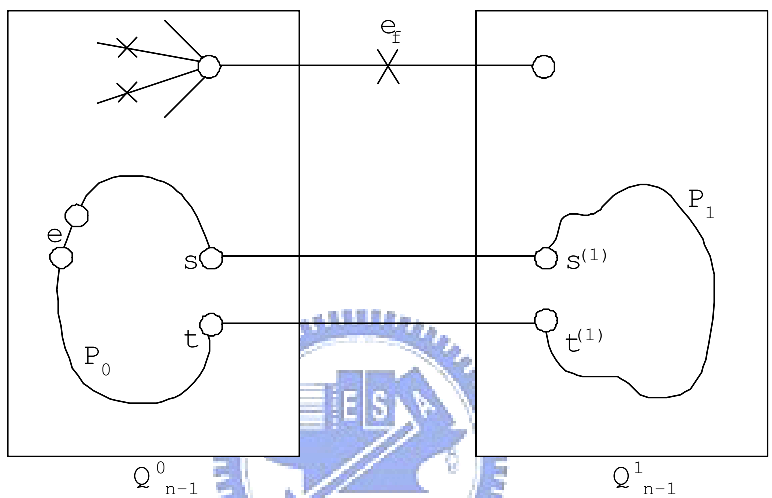

Figure 3.3: Illustration for of Theorem Case 2.1.

2.1: *Suppose* $|F_L| \leq 2n-7$ *and* $|F_R| \leq 2n-7$. (Fig. 3.3) Without loss of generality, we further assume that $e \in E(Q^0_{n-1})$. By induction hypothesis, there exists a cycle C in $Q^0_{n-1} - F_L$ of every even length $6 \leq l(C) \leq 2^{n-1}$ passing through e. Let C_0 be a cycle of length $2^{n-1} - 4 \leq l(C_0) \leq 2^{n-1}$ through e in $Q^0_{n-1} - F_L$. Since $|C_0 - e| \geq 2^{n-1} - 4 - 1 > 2(2n-5) = 2|F \cap D_j|$, for all $n \geq 5$. There exists an edge (s,t) on C_0 such that (s,t) is not e, and both $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. We write C_0 as $\langle s, P_0, t, s \rangle$. By induction hypothesis, there exists a cycle C_1 in $Q^1_{n-1} - F_R$ of every even length $6 \leq l(C_1) \leq 2^{n-1}$ passing through $(s^{(1)}, t^{(1)})$. We write C_1 as $\langle s^{(1)}, P_1, t^{(1)}, s^{(1)} \rangle$.

For every even l, $2^{n-1}+2 \leq l \leq 2^n$, $\langle s, P_0, t, t^{(1)}, P_1, s^{(1)}, s\rangle$ can form a cycle of length l going through e in $Q_n - F$, if we adjust the length of P_0 and P_1 properly.

2.2: *$|F_L| = 2n-6$ or $|F_R| = 2n-6$, say the former case.* In this case, $|F \cap D_j| = 1$ and $|F \cap E(Q^1_{n-1})| = |F_R| = 0$.

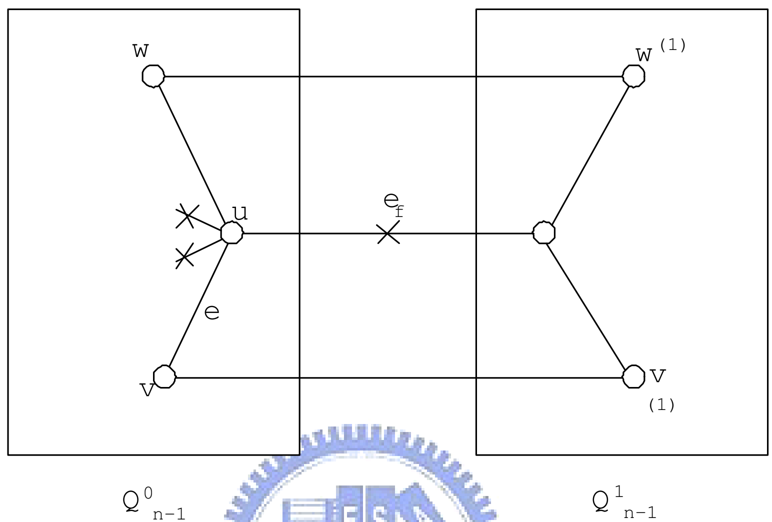

Figure 3.4: Illustration for of Theorem Case 2.2.1(a).

2.2.1: *e is in subcube Q^0_{n-1}.* To find a cycle of length 6 passing through $e = (u, v)$, we discuss the case that whether e is incident with x or not. If e is incident with x, without loss of generality, we assume that $u = x$. (Fig. 3.4) Thus, $(v, v^{(1)})$ is a healthy edge. Since F_L is a set of conditional faulty edges in Q^0_{n-1}, vertex $u = x$ has two healthy edges incident with it. Let w be a neighbor of u in Q^0_{n-1} such that (w, u) and $(w, w^{(1)})$ are healthy edges and $w \neq v$. Thus, $\langle u, v, v^{(1)}, u^{(1)}, w^{(1)}, w, u\rangle$ is a cycle of length 6 in $Q_n - F$. Otherwise, e is not incident with x, then $(u, u^{(1)})$ and $(v, v^{(1)})$ are healthy edges. (Fig. 3.5) By Lemma 1, there exists a cycle $C_1 = \langle u^{(1)}, P_1, v^{(1)}, u^{(1)}\rangle$ of length four in Q^1_{n-1} through the edge $(u^{(1)}, v^{(1)})$. Thus, $\langle u, u^{(1)}, P_1, v^{(1)}, v, u\rangle$ is a cycle of length 6 in $Q_n - F$, where $l(P_1) = 3$.

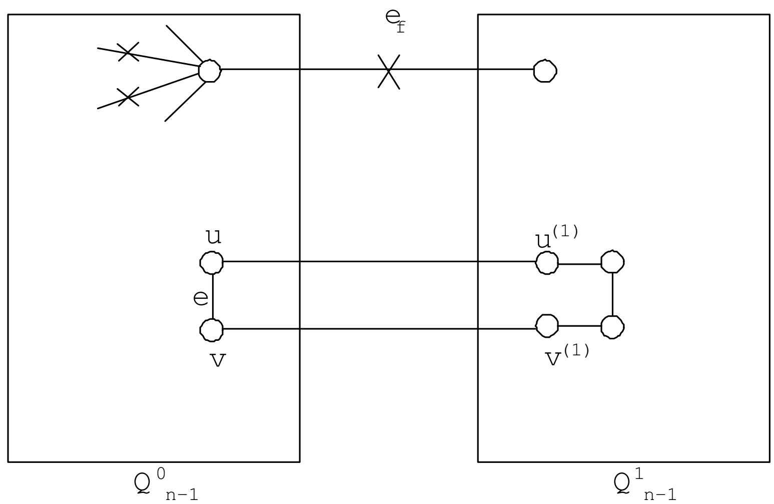

Figure 3.5: Illustration for of Theorem Case 2.2.1(b).

Let e_1 be a faulty edge in Q^0_{n-1} that is not adjacent to e_f. Though e_1 is a faulty edge, we treat it as a healthy edge temporarily, then the total number of faulty edge in Q^0_{n-1} is $2n-7$. By induction hypothesis, there exists a cycle C_0 of every length $6 \leq l(C_0) \leq 2^{n-1}$ going through e in $Q^0_{n-1} - \{F_L - \{e_1\}\}$. If C_0 passes e_1, we choose e_1 , or else, we choose any one edge on C_0 which is not adjacent to e_f. Let the chosen edge be denoted by (s,t). We write cycle C_0 as $\langle s, P_0, t, s\rangle$. Since $|F \cap D_j| = 1$ and $|F_R| = 0$, $(s, s^{(1)})$, $(t, t^{(1)})$ and $(s^{(1)}, t^{(1)})$ are all healthy edges. Thus, $\langle s, P_0, t, t^{(1)}, s^{(1)}, s\rangle$ is a cycle of length 8 in $Q_n - F$ if $l(P_0) = 6$. Suppose that $10 \leq l \leq 2^n$ and l is even. By Lemma 1, in Q^1_{n-1}, there exists a cycle C_3 of length $4 \leq l(C_3) \leq 2^{n-1}$ passing through $(s^{(1)}, t^{(1)})$. We write C_3 as $\langle s^{(1)}, P_3, t^{(1)}, s^{(1)}\rangle$. Thus, $\langle s, P_0, t, t^{(1)}, P_3, s^{(1)}, s, t\rangle$ is a cycle of length l through e in $Q_n - F$, if we adjust the length of P_0 and P_3 properly.

2.2.2: *e is in subcube Q^1_{n-1}*. (Fig. 3.6) By Lemma 1, there exists a cycle C of every even length $4 \leq l \leq 2^{n-1}$ passing through e in Q^1_{n-1}. Suppose that $2^{n-1}+2 \leq l \leq 2^n$ and

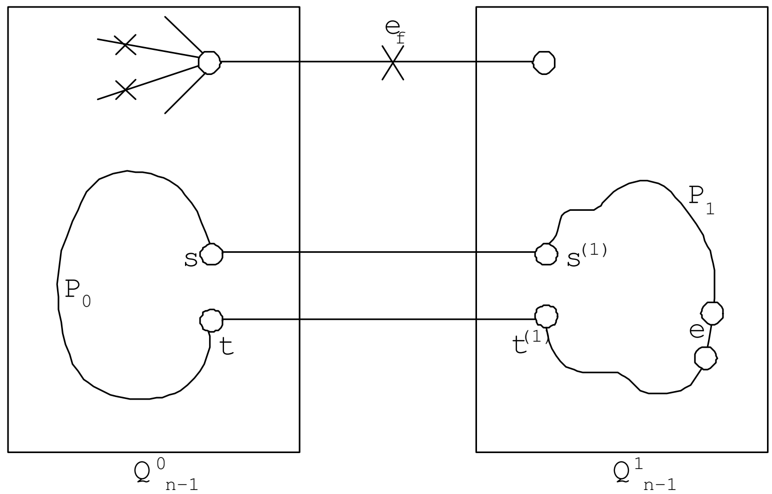

Figure 3.6: Illustration for of Theorem Case 2.2.2.

l is even. Since F_L is a set of conditional faulty edges, there are at most $(n-3)$ faulty edges adjacent to e_f in Q^0_{n-1}. For $n \geq 5$, $n-3 \geq 2$, we can choose a faulty edge $e_2 = (s,t)$ in Q^0_{n-1} such that e_2 is not adjacent to e_f and $(s^{(1)}, t^{(1)})$ is not e. Treating the edge e_2 as a healthy edge, by induction hypothesis, there exists a cycle C_0 of length $6 \leq l(C_0) \leq 2^{n-1}$ going through e_2 in $Q^0_{n-1} - F_L$. We write C_0 as $\langle s, P_0, t, s\rangle$, and observe that $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. By Lemma 4, there exists a cycle C_1 of every length $2^{n-1}-4$, $2^{n-1}-2$, or 2^{n-1} through $(s^{(1)}, t^{(1)})$ and e in Q^1_{n-1}. We write C_1 as $\langle s^{(1)}, P_1, t^{(1)}, s^{(1)}\rangle$. Thus, $\langle s, P_0, t, t^{(1)}, P_1, s^{(1)}, s\rangle$ is a cycle of even length l through e in $Q_n - F$, if we adjust the length of P_0 and P_1 properly.

Case 3: *Every vertex in Q_n is incident with at most $(n-3)$ faulty edges.* In this case, suppose that $e = (u, v)$ is in dimension i. By Lemma 10, Q_n can be decomposed into Q^0_{n-1} and Q^1_{n-1} by a dimension j different from i such that e is not a crossing edge and not all the faulty edges are in the same subcube. Then $|F_L| \leq 2n-6$ and $|F_R| \leq 2n-6$.

Next, we consider two further cases:

3.1: *At least one faulty edge is a jth dimension edge.* Thus, $|F \cap D_j| \neq 0$.

We then consider two cases: (a)$|F_L| \leq 2n-7$ and $|F_R| \leq 2n-7$, and (b)$|F_L| = 2n-6$ or $|F_R| = 2n-6$. The proof of this subcase is exactly the same as that of case 2.

3.2: *None of the faulty edges is a jth dimension edge.* Thus, $|F \cap D_j| = 0$.

3.2.1: $|F_L| \leq 2n-7$ and $|F_R| \leq 2n-7$. Without loss of generality, we may assume that $e \in E(Q^0_{n-1})$. By induction hypothesis, there exists a cycle C of even length $6 \leq l(C) \leq 2^{n-1}$ in $Q^0_{n-1} - F_L$ passing through e. Let C_0 be a cycle of even length $2^{n-1}-4 \leq l(C_0) \leq 2^{n-1}$ going through e in $Q^0_{n-1} - F_L$. There exists an edge (s,t) other than e in C_0. Since $|F \cap D_j| = 0$, $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. We write C_0 as $\langle s, P_0, t, s\rangle$. By induction hypothesis, there exists a cycle C_1 of every even length $6 \leq l(C_1) \leq 2^{n-1}$ in $Q^1_{n-1} - \{F_R - (s^{(1)}, t^{(1)})\}$ through $(s^{(1)}, t^{(1)})$. We write C_1 as $\langle s^{(1)}, P_1, t^{(1)}, s^{(1)}\rangle$. For even l, $2^{n-1}+2 \leq l \leq 2^n$, $\langle s, P_0, t, t^{(1)}, P_1, s^{(1)}, s\rangle$ can form a cycle of length l going through e in $Q_n - F$, if we adjust the length of P_0 and P_1 properly.

3.2.2: *Suppose $|F_L| = 2n-6$ or $|F_R| = 2n-6$, say the former case.* In this case, $|F_R| = 1$. We then consider two cases: (a) e is in subcube Q^0_{n-1}, and (b) e is in subcube Q^1_{n-1}.

(a) *$e = (u,v)$ is in subcube Q^0_{n-1}.* Since $|F \cap D_j| = 0$, both $(u, u^{(1)})$ and $(v, v^{(1)})$ are healthy edges. Let l be an even number with $6 \leq l \leq 2^{n-1}$. By Lemma 1, there exists a cycle C_1 of every even length from 4 to 2^{n-1} passing through $(u^{(1)}, v^{(1)})$ in $Q^1_{n-1} - \{F_R - (u^{(1)}, v^{(1)})\}$. We write C_1 as $\langle u^{(1)}, P_1, v^{(1)}, u^{(1)}\rangle$. No matter $(u^{(1)}, v^{(1)})$ is healthy or not, $\langle u, u^{(1)}, P_1, v^{(1)}, v, u\rangle$ forms a cycle of length l through e in $Q_n - F$. Suppose that

$2^{n-1}+2 \leq l \leq 2^n$. Let e_1 be a faulty edge in Q^0_{n-1}. We may treat e_1 as a healthy edges temporarily. By induction hypothesis, there exists a cycle C_0 of length $6 \leq l(C_0) \leq 2^{n-1}$ going through e in $Q^0_{n-1} - \{F_L - \{e_1\}\}$. If C_0 passes the edge e_1, we choose e_1 to be deleted. Otherwise, we choose another edge other than e on cycle C_0. Let the chosen edge be denoted by (s,t). We write the cycle C_0 as $\langle s, P_0, t, s\rangle$. Treating $(s^{(1)}, t^{(1)})$ as a healthy edge, by Lemma 1, there exists a cycle C_3 of every even length from 4 to 2^{n-1} passing through $(s^{(1)}, t^{(1)})$ in $Q^1_{n-1} - \{F_R - (s^{(1)}, t^{(1)})\}$. We write C_3 as $\langle s^{(1)}, P_3, t^{(1)}, s^{(1)}\rangle$, then $\langle s, P_0, t, t^{(1)}, P_3, s^{(1)}, s\rangle$ is the cycle of length l through e in $Q_n - F$.

(b): *e is in subcube Q^1_{n-1}*. Let e_1 be the only faulty edge in Q^1_{n-1}. By Lemma 1, there exists a cycle C of every even length from 6 to 2^{n-1} through e in $Q^1_{n-1} - \{e_1\}$. Suppose that $2^{n-1}+2 \leq l \leq 2^n$, and l is even. Let $e_0 = (s,t)$ be a faulty edge in Q^0_{n-1} such that $(s^{(1)}, t^{(1)}) \neq e$. By induction hypothesis, there exists a cycle C_0 of length $6 \leq l(C_0) \leq 2^{n-1}$ in $Q^0_{n-1} - \{F_L - \{e_0\}\}$ going through e. We write C_0 as $\langle s, P_0, t, s\rangle$. If $(s^{(1)}, t^{(1)}) = e_1$, treat e_1 as a healthy edge temporarily, by Lemma 4, there exists a cycle C_1 of length $2^{n-1}-4$, $2^{n-1}-2$, or 2^{n-1} respectively going through both $(s^{(1)}, t^{(1)})$ and e in Q^1_{n-1}. We write C_1 as $\langle s^{(1)}, P_1, t^{(1)}, s^{(1)}\rangle$. Thus, $\langle s, P_0, t, t^{(1)}, P_1, s^{(1)}, s\rangle$ can form a cycle of length l through e in $Q_n - F$, if we adjust the length of P_0 and P_1 properly. Otherwise, if $(s^{(1)}, t^{(1)}) \neq e_1$, by Lemma 6, there exists a cycle C_3 of length 2^{n-1}, $2^{n-1}-2$, or $2^{n-1}-4$, respectively, going through both e and $(s^{(1)}, t^{(1)})$ in $Q^1_{n-1} - \{e_1\}$. We write C_3 as $\langle s^{(1)}, P_3, t^{(1)}, s^{(1)}\rangle$. Thus, $\langle s, P_0, t, t^{(1)}, P_3, s^{(1)}, s\rangle$ can form a cycle of length l through e in $Q_n - F$, if we adjust the length of P_0 and P_3 properly.

This completes the proof. □

Chapter 4

Conclusions

Since every component in the network may have different reliability, it is important to consider properties of a network with some conditional faults. We consider the n-dimensional hypercube with some faulty edges such that each vertex is incident to at least two nonfaulty edges. We use induction to prove that the n-dimensional hypercube Q_n, $n \geq 3$, is $(2n-5)$-edge fault-tolerant conditional edge bipancyclic.

There exists an n-dimensional hypercube with (2n-4) edge faults, in which each vertex incident to at least two nonfaulty edges, such that for any pair of vertices does not exist path joining them. for example, let $u = 00\ldots0$ and $v = 100\ldots1$. One can consider the $(2n-4)$ faulty edge in $Q_n : e_i(u)$ and $e_i(v)$ for all $1 \leq i \leq n-2$ (see Fig. 1.1). obviously, vertices u and v each have exactly two nonfaulty edges incident to them. Hence the four edges $e_0(u), e_n - 1(u), e_0(v)$, and $e_n - 1(v)$ from a 4-cycle by themselves. Therefore, in this Q_n, $n \geq 3$, it is impossible to make a faulty a hamiltonian cycle joining any pair of vertices. On the other hand, it is also impossible to make a hamiltonian cycle for $n \geq 3$. There, our result are optimal.In our future work, there is a direction to be studied:

If hypercube Q_n is faulty free, there exist n cycles with the same length being L for each integer $4 \leq L \leq 2^n$. We will discuss that whether they are mutually independent or not.

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