國立交通大學

資訊科學與工程研究所

碩 士 論 文

三正則及連通圖中漢米爾頓性質之連線需求數目的研究

The Edge-Required-Hamiltonicity of the Cubic 3-Connected Hamiltonian Graphs

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摘要

給一個圖$G = (V, E)$以及邊集合$R \subseteq E$，其中R的邊為獨立路徑。如果一個圖G包含漢米爾頓迴路以及含有任何的需求邊R且$|R| \leq k$，則圖G稱為k−漢米爾頓需求邊。我們定義圖G的漢米爾頓需求邊且k為最大時，稱為$h_r(G)$。如果一個圖G-F 包含漢米爾頓但不包含壞邊F且$|F| \leq k$，則圖G稱為k−漢米爾頓容錯邊。我們定義圖G的漢米爾頓容錯邊且k為最大時，稱為$h_f(G)$。在這篇論文中，我們要證明如果圖G為三正則漢米爾頓圖，則$h_f(G) \leq 1$。如果圖G為三正則漢米爾頓圖且$h_f(G) = 1$，則 $1 \leq h_r(G) \leq 3$。我們將介紹一些$h_f(G) = 1$ 且$h_r(G) = i$其中$i = 1, 2, 3$ 的 3−連通漢米爾圖G，以及一些$h_f(G) = 0$ 且$h_r(G) = 1$ 的 3−連通漢米爾圖G。

關鍵字：漢米爾頓、漢米爾頓連結。

The Edge-Required-Hamiltonicity of the Cubic 3-Connected Hamiltonian Graphs

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Abstract

Given a graph $G = (V,E)$ and edge set $R \subseteq E$, where the edges of R form independent paths. A graph G is *k-edge-required-hamiltonian* if it contains a hamiltonian cycle including any R whenever $|R| \leq k$. We define *edge-required hamiltonicity* of G, denoted by $h_r(G)$, to be the maximum of such k. A graph G is *k-edge-fault-tolerant-hamiltonian* if $G - F$ is hamiltonian for any faulty edge set F with $|F| \leq k$. We define *edge-fault-tolerant hamiltonicity* of G, denoted by $h_f(G)$, to be the maximum of such k. In this thesis, we prove that $h_f(G) \leq 1$ if G is a cubic hamiltonian graph, $1 \leq h_r(G) \leq 3$ if G is a cubic hamiltonian graph with $h_f(G) = 1$. We present some cubic 3-connected hamiltonian graphs G with $h_f(G) = 1$ and $h_r(G) = i$ for $i = 1, 2, 3$, a cubic 3-connected hamiltonian graph G with $h_f(G) = 0$ and $h_r(G) = 0$, and a cubic 3-connected hamiltonian graph G with $h_f(G) = 0$ and $h_r(G) = 1$.

Keywords : hamiltonian, hamiltonian connected.

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Chapter 1

Introduction

For the graph definition and notation we follow [2]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$, vertex u(or v) is said to be incident with edge (u, v), u and v are called the ends of edge (u, v). Suppose that V' is a subset of V. The subgraph of G whose vertex set is $V - V'$ and whose edge set is the set of those edges of G that have both ends in $V - V'$ is called the subgraph of G induced by $V - V'$ and is denoted by $G - V'$. Suppose that E' is a subset of E. The subgraph of G whose vertex set is the set of ends of edges in $E - E'$ and whose edge set is $E - E'$ is called the subgraph of G induced by $E - E'$ and is denoted by $G - E'$. For any vertex $u \in V$, the *neighborhood* $N(u)$ of u is the set $\{v \mid (u, v) \in E\}$, and is called the *neighborhood* of u. For any vertex $x \in V$, $deg_G(x)$ denotes its degree in G. A graph G is cubic if $deg_G(x) = 3$ for any vertex x in G. A graph G is 3-connected if $G - V'$ is still connected for every vertex set $V' \subseteq V$ and $|V'| \leq 2$. A *path* P in G is represented by $\langle v_0, v_1, \cdots, v_k \rangle$, a sequence of distinct vertices of G, where every (v_i, v_{i+1}) belongs to E for $0 \leq i \leq k - 1$. We can write path $P = \langle v_0, v_1, \cdots, v_k \rangle$ as $\langle v_0, \cdots, v_i, P', v_j, \cdots, v_k \rangle$ or

$\langle v_0, \cdots, v_i \rangle \cup P' \cup \langle v_j, \cdots, v_k \rangle$, where $P' = \langle v_i, v_{i+1}, \cdots, v_j \rangle$ is a subpath of P. A *cycle* is nearly a path of length at least three with a difference that the first and the last vertices of this sequence are the same. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once and a graph is *hamiltonian* if it contains a hamiltonian cycle. A path of G is *hamiltonian path* if its vertices span $V(G)$, i.e., the path runs through all vertices once.

When searching a hamiltonian cycle (or path), we may ask the cycle to traverse several predetermined edges. These predetermined edges are called *required edges*. The idea of searching such kind of hamiltonian cycle is the motivation of this article. Proposed by William Hamilton, the original hamiltonian problem is a puzzle on the graph of the dodecahedron in which a path of length four is specified and the player is asked to extend the given path to a spanning cycle. This classical game can be treated as a special case of searching a hamiltonian cycle including required edges. Let us denote R the set of required edges and it must be *reasonable* to avoid creating any short cycle or *branch point* (a vertex of degree ≥ 3). In other words, a reasonable R is an edge set of independent paths.

A graph G is *k-edge-required-hamiltonian* if it contains a hamiltonian cycle including any reasonable R whenever $|R| \leq k$. We define *edge-required hamiltonicity* of G, denoted by $h_r(G)$, to be the maximum of such k. Those graphs G with $h_r(G) \geq 1$ is also known as edge-hamiltonian graphs [14]. Most of the previous studies of the edge-required hamiltonicity were concentrated on sufficient conditions [5, 7]. Recently, it is proved that $h_r(Q_n) = 2n - 3$ where Q_n is the n-dimensional hypercube with $n \geq 3$.

A dual concept to "required edges" is "faulty edges". Fault-tolerance is one of the most important properties for computer or network structures. A graph G is *k-edge-fault-tolerant-hamiltonian* if $G - F$ is hamiltonian for any *faulty edge set* F with $|F| \leq k$. Similarly, the *edge-fault-tolerant hamiltonicity* of G, denoted by $h_f(G)$, is defined to be the maximum of such k. There are some studies on edge-fault-tolerant hamiltonicity [15]. In particular, it is proved that $h_f(Q_n) = n - 2$ [3, 10].

We believe that the first step on studying edge-required-hamiltonicity is working on the family of cubic hamiltonian graphs. To exclude trivial cases, we further restricted our attention on cubic 3-connected hamiltonian graphs. In the following, we use Ω to denote the set of cubic 3-connected hamiltonian graphs.

In the following section, we will proved that $h_f(G) \leq 1$ and $h_r(G) \leq 3$ if G is in Ω. Moreover, $1 \leq h_r(G)$ if G is in Ω and $h_f(G) = 1$. Furthermore, $h_f(G) = 1$ if G is in Ω and $h_r(G) \geq 2$. Thus, we would like to know the existence of graph in Ω with $h_f(G) = 1$ and $h_r(G) = i$ for $i = 1, 2, 3$. For this reason, we give examples of graphs in Ω with $h_f(G) = 1$ and $h_r(G) = i$ for $i = 1, 2, 3$ in sections 3.1, 3.2, and 3.3. Again, we are interested in the existence of graphs in Ω with $h_f(G) = 0$ and $h_r(G) = 0$. An example is given in section 3.4. Finally, we are interested in the existence of graphs in Ω with $h_f(G) = 0$ and $h_r(G) = 1$. An example is given in section 3.5.

Chapter 2

Preliminaries

Lemma 1 *$h_f(G) \leq 1$ and $h_r(G) \leq 3$ if G is a graph in Ω.*

Proof. Suppose that $G = (V, E)$ is a graph in Ω. Let x be any vertex in G and $N_G(x) = \{u, v, w\}$. We set $F = \{(x, u), (x, v)\}$. Obviously, $deg_{G-F}(x) = 1$. Hence, there is no hamiltonian cycle in $G - F$. Therefore, $h_f(G) \leq 1$.

Let $N_G(x) = \{u, v, w\}$, $N_G(u) = \{x, u_1, u_2\}$ and $N_G(v) = \{x, v_1, v_2\}$. We set the required edge set $R = \{(u, u_1), (u, u_2), (v, v_1), (v, v_2)\}$. Thus, (x, u) and (x, v) is not on any hamiltonian cycle including the set R. Obviously, $deg_{G-\{(x,u),(x,v)\}}(x) = 1$. Hence, there is no hamiltonian cycle in $G - \{(x, u), (x, v)\}$. Therefore, $h_r(G) \leq 3$.

The lemma is proved. □

Lemma 2 *$h_r(G) \geq 1$ if G is a graph in Ω with $h_f(G) = 1$.*

Proof. Suppose that $G = (V, E)$ is a graph in Ω with $h_f(G) = 1$. Let (x, u) be any edge of G and $N_G(x) = \{u, v, w\}$. We set that $F = \{(x, v)\}$ be the faulty edge set. Since

$h_f(G) = 1$, there exists a hamiltonian cycle C in $G - F$. Obviously, $deg_{G-F}(x) = 2$. Thus, (x, u) is in C. Therefore, $h_r(G) \geq 1$.

The lemma is proved. $\Box$

Lemma 3 *$h_f(G) = 1$ if G is a graph in Ω with $h_r(G) \geq 2$.*

Proof. Suppose that $G = (V, E)$ is a graph in Ω and $h_r(G) \geq 2$. By Lemma 1, we know that $h_f(G) \leq 1$. Now, we want to show that $h_f(G) \neq 0$. Let (x, u) be any edge of G and $N_G(x) = \{u, v, w\}$. We set a required edge set $R = \{(x, v), (x, w)\}$. Since $h_r(G) \geq 2$, there exists a hamiltonian cycle C including the edge set R. Obviously, $(x, u) \notin C$. Hence, $h_f(G) = 1$.

The lemma is proved. $\Box$

Let G and K_4 be two graphs in Ω with $V(G) \cap V(K_4) = \emptyset$ where K_4 is a complete graph with four nodes. Note that K_4 is node symmetric. Let $x \in V(G)$ and $k \in V(K_4)$. Let $N(x) = \{x_1, x_2, x_3\}$ be an ordered set of the neighbors of x and $N(k) = \{k_1, k_2, k_3\}$ be the neighbors of k. *The 3-join* of G and K_4 at x and k, denoted by $J(G, x)$, is the graph with $V(J(G, x)) = (V(G) - \{x\}) \cup (V(K_4) - \{k\})$ and $E(J(G, x)) = (E(G) - \{(x, x_i) \mid 1 \leq i \leq 3\}) \cup (E(K_4) - \{(k, k_i) \mid 1 \leq i \leq 3\}) \cup \{(x_i, k_i) \mid 1 \leq i \leq 3\}$. A graph H is called *a 3-join of G and K_4* if $H = J(G, x)$ for some vertices $x \in V(G)$. It is easy to know that $J(G, x)$ is in Ω if G is in Ω. See Figure 2.1 for an illustration.

Lemma 4 *$h_f(J(G, x)) = h_f(G)$ if G is a graph in Ω.*

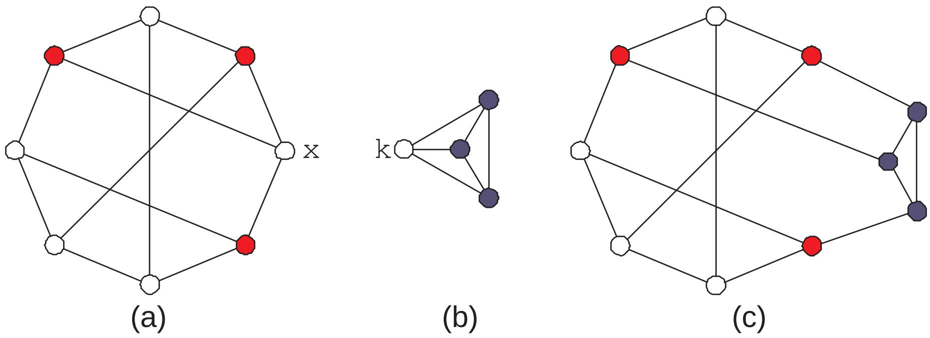

Figure 2.1: The graphs (a) G, (b) K_4, and (c) $J(G,x)$

Proof. Let G is a graph in Ω. Let $x \in V(G)$ and $k \in V(K_4)$. Assume that the neighbors of node x in G are $\{x_1, x_2, x_3\}$, the neighbors of node k in K_4 are $\{k_1, k_2, k_3\}$. By Lemma 1, we know that $h_f(G) \leq 1$ and $h_f(J(G,x)) \leq 1$.

Suppose that $h_f(G) = 1$. We can find a hamiltonian cycle in $G - F$ for any faulty edge set F with $|F| = 1$. Now, we want to show that for any faulty edge set F' with $|F'| = 1$, we can find a hamiltonian cycle C' in $J(G,x) - F'$.

Case 1. $F' = \{(x_1, k_1)\}$, $\{(x_2, k_2)\}$, or $\{(x_3, k_3)\}$. Without loss of generality, we assume that $F' = \{(x_3, k_3)\}$. Since $h_f(G) = 1$, we can find a hamiltonian cycle $\langle x_1, x, x_2, P, x_3, Q\rangle$ in $G - \{(x, x_3)\}$ where P and Q be two paths of G. Hence, we can find a hamiltonian cycle $\langle x_1, k_1, k_3, k_2, x_2, P, x_3, Q\rangle$ in $J(G,x) - \{(x_3, k_3)\}$.

Case 2. $F' = \{(k_1, k_2)\}$, $\{(k_1, k_3)\}$, or $\{(k_2, k_3)\}$. Without loss of generality, we assume that $F' = \{(k_1, k_2)\}$. Since $h_f(G) = 1$, we can find a hamiltonian cycle $\langle x_1, x, x_2, P, x_3, Q\rangle$ in $G - \{(x, x_3)\}$. Hence, we can find a hamiltonian cycle $\langle x_1, k_1, k_3, k_2, x_2, P, x_3, Q\rangle$ in $J(G,x) - \{(k_1, k_2)\}$.

Case 3. $F' = (u, v) \subseteq E(G) - \{(x_i, k_i) \mid 1 \leq i \leq 3\} - \{(k_1, k_2), (k_1, k_3), (k_2, k_3)\}$. Since $h_f(G) = 1$, we can find a hamiltonian cycle $\langle x_1, x, x_2, P, x_3, Q\rangle$ in $G - \{(u, v)\}$. Hence, we can find a hamiltonian cycle $\langle x_1, k_1, k_3, k_2, x_2, P, x_3, Q\rangle$ in $J(G, x) - \{(k_1, k_2)\}$.

Hence, we can find a hamiltonian cycle in $J(G, x) - F'$ with $|F'| = 1$. Therefore, $h_f(J(G, x)) = 1$ when $h_f(G) = 1$.

Suppose that $h_f(G) = 0$. Hence, there are not any hamiltonian cycle in $G - e$ for some edge e.

Case 1. $e = \{(x, x_1)\}$, $\{(x, x_2)\}$, or $\{(x, x_3)\}$. Without loss of generality, we assume that $e = \{(x, x_3)\}$. Assume that $h_f(J(G, x)) = 1$, then we can find a hamiltonian cycle $\langle x_1, k_1, k_3, k_2, x_2, P, x_3, Q\rangle$ in $J(G, x) - \{(k_1, k_2)\}$. Hence, we can find a hamiltonian cycle $\langle x_1, x, x_2, P, x_3, Q\rangle$ in $G - e$. We get a contradiction. Therefore, $h_f(J(G, x)) = 0$.

Case 2. $e \in E(G) - \{(x, x_1), (x, x_2), (x, x_3)\}$. Assume that $h_f(J(G, x)) = 1$, then we can find a hamiltonian cycle $\langle x_1, k_1, k_3, k_2, x_2, P, x_3, Q\rangle$ in $J(G, x) - e$. Hence, we can find a hamiltonian cycle $\langle x_1, x, x_2, P, x_3, Q\rangle$ in $G - e$. We get a contradiction. Therefore, $h_f(J(G, x)) = 0$.

The lemma is proved. □

Lemma 5 $h_r(J(G, x)) = \min\{2, h_r(G)\}$ *if G is a graph in Ω.*

Proof. By Lemma 1, we know that $h_r(G) \leq 3$. We have the following cases.

Case 1. $h_r(G) = 0$. Let $R \in E(G) - \{(x, x_1), (x, x_2), (x, x_3)\}$ be the required edge set of

$J(G,x)$ with $|R|=1$. Assume that $h_r(J(G,x))=1$, then we can find a hamiltonian cycle $\langle x_1,k_1,k_3,k_2,x_2,P,x_3,Q\rangle$ including R in $J(G,x)$. Hence, we can find a hamiltonian cycle $\langle x_1,x,x_2,P,x_3,Q\rangle$ including R in G. We get a contradiction. Therefore, $h_f(J(G,x))=0$.

Case 2. $h_r(G)=1$. Let $R\in E(G)-\{(x,x_1),(x,x_2),(x,x_3)\}$ be the required edge set of $J(G,x)$ with $|R|=2$. Assume that $h_r(J(G,x))=2$, then we can find a hamiltonian cycle $\langle x_1,k_1,k_3,k_2,x_2,P,x_3,Q\rangle$ including R in $J(G,x)$. Hence, we can find a hamiltonian cycle $\langle x_1,x,x_2,P,x_3,Q\rangle$ including R in G. We get a contradiction. Therefore, $h_f(J(G,x))=1$.

Case 3. $h_r(G)=2$. We have the following subcases:

Case 3.1. $R=\{(u_1,v_1),(u_2,v_2)\}\in E(G)-\{(x,x_1),(x,x_2),(x,x_3)\}$. We can find a hamiltonian cycle including the required edge set R in G. Without loss of generality, we assume that $\langle x_1,x,x_2,P,x_3,Q\rangle$ be the hamiltonian cycle in G. And we assume that $\langle k_1,k,k_2,k_3\rangle$ be the hamiltonian cycle in k_4. Obviously, $\langle x_1,k_1,k_3,k_2,x_2,P,x_3,Q\rangle$ is a hamiltonian cycle including the required edge set R in $J(G,x)$. Hence, $h_f(J(G,x))=2$.

Case 3.2. $R=\{(u_1,v_1),(u_2,v_2)\}\in\{(x_1,k_1),(x_2,k_2),(x_3,k_3),(k_1,k_2),(k_2,k_3),(k_1,k_4)\}$. Without loss of generality, we may assume that $R=\{(x_1,k_1),(x_2,k_2)\}$ or $\{(k_1,k_3),(k_2,k_3)\}$ or $\{(x_1,k_1),(x_1,k_3)\}$. Obviously, $\langle x_1,k_1,k_3,k_2,x_2,P,x_3,Q\rangle$ is a hamiltonian cycle including the required edge set R in $J(G,x)$. Hence, $h_f(J(G,x))=2$.

Case 3.3. $R=\{(u_1,v_1),(u_2,v_2)\}$. Let $(u_1,v_1)\in E(G)-\{(x,x_1),(x,x_2),(x,x_3)\}$ and $(u_2,v_2)\in\{(x_1,k_1),(x_2,k_2),(x_3,k_3),(k_1,k_2),(k_2,k_3),(k_1,k_4)\}$. Without loss of generality, we may assume that $R=\{(u_1,v_1),(x_1,k_1)\}$ or $\{(u_1,v_1),(k_1,k_3)\}$ and assume the path P

or the path Q including (u_1, v_1) . Obviously, $\langle x_1, k_1, k_3, k_2, x_2, P, x_3, Q\rangle$ is a hamiltonian cycle including the required edge set R in $J(G, x)$. Hence, $h_f(J(G, x)) = 2$.

Case 4. $h_r(G) = 3$. Let the node set $V_l = \{V(G) - V(x)\}$, $V_r = \{V(K_4) - V(x)\}$, edge cut set $S = \{(x_i, k_i)|1 \leq i \leq 3\}$. Assume that we can find a hamiltonian cycle C in $J(G, x)$. It is easy to know that $|C \cup S| = 2$. Hence, $h_f(J(G, x)) = 2$.

The lemma is proved. □

For integers n and k, $n \geq 3$ and $1 \leq k < n$. The *generalized Petersen graph* $P(n, k)$ is the graph with vertex set $\{i \mid 0 \leq i < n\} \cup \{i' \mid 0 \leq i < n\}$ and edge set $\{(i, i \oplus 1) \mid 0 \leq i < n\} \cup \{(i', (i \oplus k)') \mid 0 \leq i < n\} \cup \{(i, i') \mid 0 \leq i < n\}$ where $\oplus$ denotes addition in integer modulo n, Z_n. It is known that $P(n, k)$ is cubic, 3-connected, and hamiltonian. Hence, $P(n, k)$ is in Ω. The generalized Petersen graphs $P(7, 2)$ and $P(9, 3)$ are illustrated in Figure 2.2. In [1], the author had shown that $P(n, 2)$ is hamiltonian if and only if $n \neq 5$ (mod 6).

Lemma 6 $h_f(P(n, 1)) = 1$ *if* n *is a positive integer with* $n \geq 3$.

Proof. By Lemma 1, we know that $h_f(P(n, 1)) \leq 1$. Let F be any edge set of $P(n, 1)$ with $|F| = 1$. By the symmetric property of $P(n, 1)$, we may assume that $F = \{(0, 1)\}$, $\{(0', 1')\}$, or $\{(0, 0')\}$. Obviously, $\langle 1, 2, \ldots, n-1, 0, 0', (n-1)', \ldots, 1'\rangle$ is a hamiltonian cycle of $P(n, 1) - F$ if $F = \{(0, 1)\}$ or $\{(0', 1')\}$ and $\langle 2, 3, \ldots, 1, 1', 0', \ldots, 2'\rangle$ is a hamiltonian cycle of $P(n, 1) - F$ if $F = \{(0, 0')\}$. Hence, $h_f(P(n, 1)) \geq 1$.

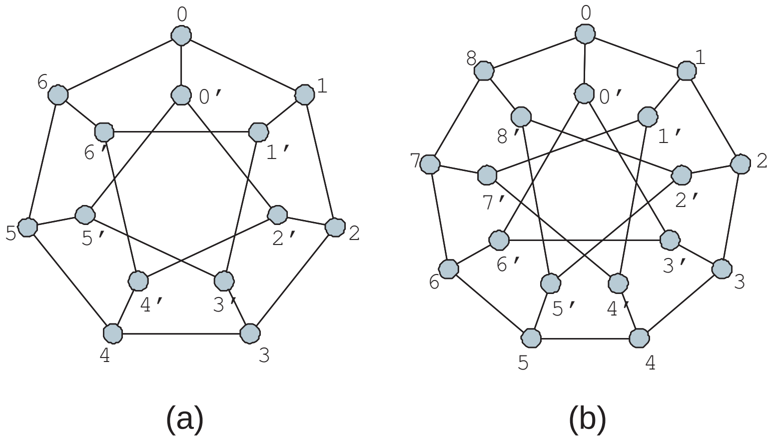

Figure 2.2: The graphs (a) P(7,2) and (b) P(9,3)

The lemma is proved. □

Lemma 7 $h_f(P(n,2)) = 1$ *if n is an even integer with $n \geq 6$.*

Proof. By Lemma 1, we know that $h_f(P(n,2)) \leq 1$. Let F be any edge set of $P(n,2)$ with $|F| = 1$. By the symmetric property of $P(n,2)$, we may assume that $F = \{(0,1)\}$, $\{((n-1)', 1')\}$, or $\{(2,2')\}$. Obviously, $\langle 0, 0', 2', \ldots, (n-2)', n-2, n-3, \ldots, 1, 1', 3', \ldots, (n-1)', n-1 \rangle$ is a hamiltonian cycle of $P(n,2) - F$. Hence, $h_f(P(n,2)) \geq 1$.

The lemma is proved. □

Lemma 8 $h_f(P(n,2)) = 1$ *if* $n = 1, 3 \pmod 6$ *with* $n > 6$.

Proof. By Lemma 1, we know that $h_f(P(n,2)) \leq 1$. Let F be any edge set of $P(n,2)$ with $|F| = 1$, $N_k = \langle k', (k+2)', k+2, k+3, k+4, (k+4)' \rangle$, and $M_k = \langle [k']_n, [(k+2)']_n, [k+2]_n, [k+3]_n, [k+4]_n, [(k+4)']_n \rangle$. We have the following cases.

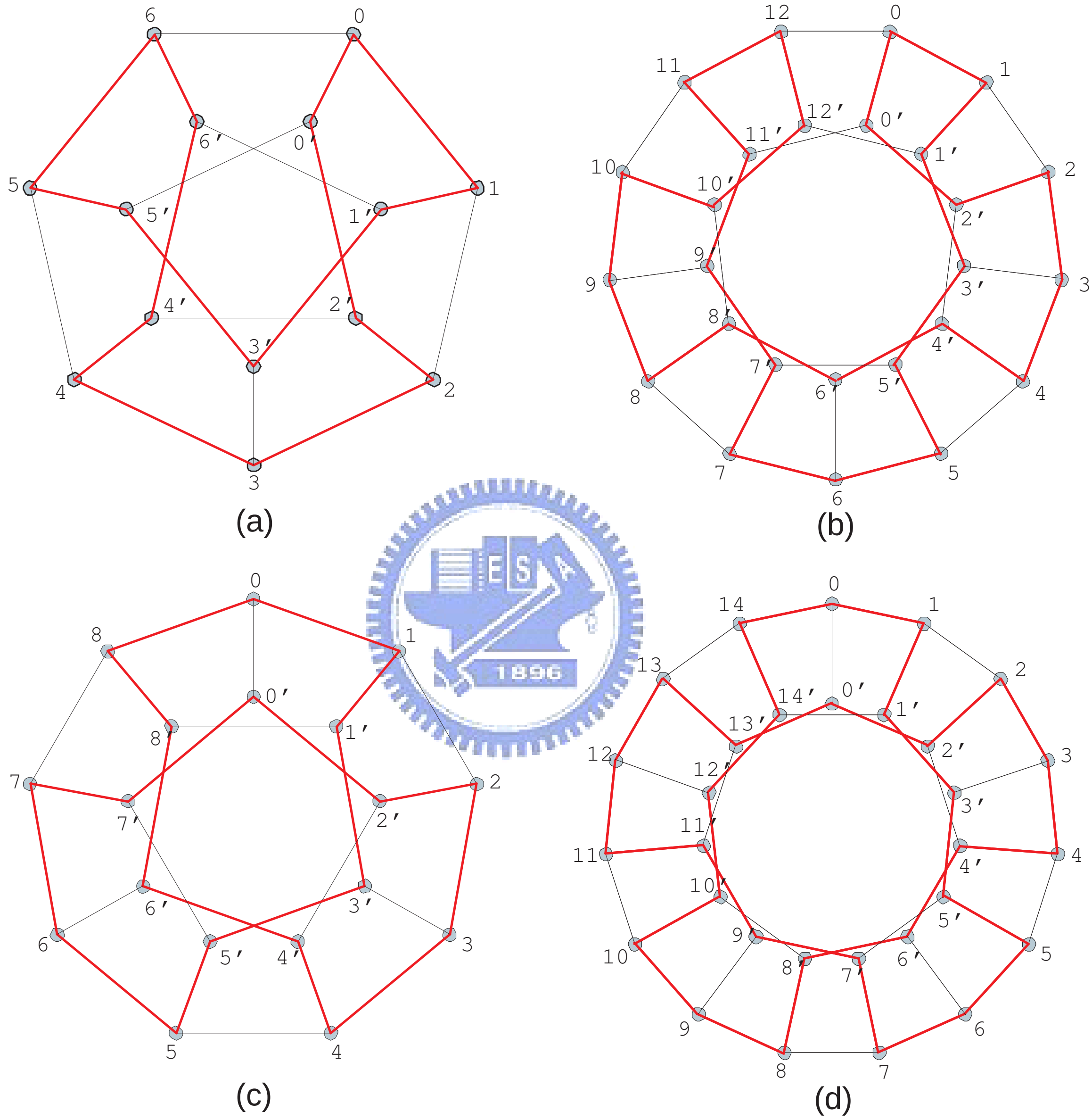

Figure 2.3: The graphs (a) P(7), (b) P(13), (c) P(9), and (d) P(15)

Case 1. n *is odd*, $n = 1 \pmod 6$ and $n \geq 7$. By the symmetric property of $P(n,2)$, we may assume that $F = \{(0, n-1)\}$, $\{(\frac{n-1}{2}, (\frac{n-1}{2})')\}$, or $\{(0', (n-2)')\}$. Obviously, $\langle 1', 1, 0, 0', 2', 2, 3, 4, 4', 6', 6, 5, 5', 3' \rangle$ is a hamiltonian cycle of $P(7,2) - F$ and $\langle 1', 1, 0, N_0, N_6, \ldots, N_{n-7}, (n-1)', n-1, n-2, (n-2)', N_{n-4}, N_{n-10}, \ldots, N_9, 3' \rangle$ is a hamiltonian cycle of $P(n,2) - F$ when $n > 7$. Hence, $h_f(P(n,2)) \geq 1$ when $n = 1 \pmod 6$ with $n \geq 7$. (See Figure 2.3(a) and 2.3(b) for an illustration of the case n *is odd*, $n = 1 \pmod 6$ and $n \geq 7$.)

Case 2. n *is odd*, $n = 3 \pmod 6$ and $n \geq 9$. By the symmetric property of $P(n,2)$, we may assume that $F = \{(0, 0')\}$, $\{(1,2)\}$, or $\{((n-1)', 1)\}$. Obviously, $\langle M_0, M_6, \ldots, M_{n-3}, M_3, M_9, \ldots, M_{n-6} \rangle$ is a hamiltonian cycle of $P(n,2) - F$ when $n \geq 9$. Hence, $h_f(P(n,2)) \geq 1$ when $n = 3 \pmod 6$ with $n \geq 9$. (See Figure 2.3(c) and 2.3(d) for an illustration of the case n *is odd*, $n = 3 \pmod 6$ and $n \geq 9$.)

The lemma is proved. □

For integer $n \geq 2$, the *project plane* $PJ(n)$ is the graph with vertex $\{i \mid 0 \leq i < 2n\}$ and edge set $\{(i, i \oplus 1) \mid 0 \leq i < 2n\} \cup \{(i, i+n) \mid 0 \leq i < n\}$ where $\oplus$ denotes addition in integer modulo $2n$, Z_{2n}. It is known that $PJ(n)$ is cubic, 3-connected, and hamiltonian. Hence, $PJ(n)$ is in Ω. The project plane graphs $PJ(8)$ and $PJ(10)$ are illustrated in Figure 2.4.

Lemma 9 $h_f(PJ(n)) = 1$.

Proof. By Lemma 1, we know that $h_f(PJ(n)) \leq 1$. Let F be any edge set of $PJ(n)$

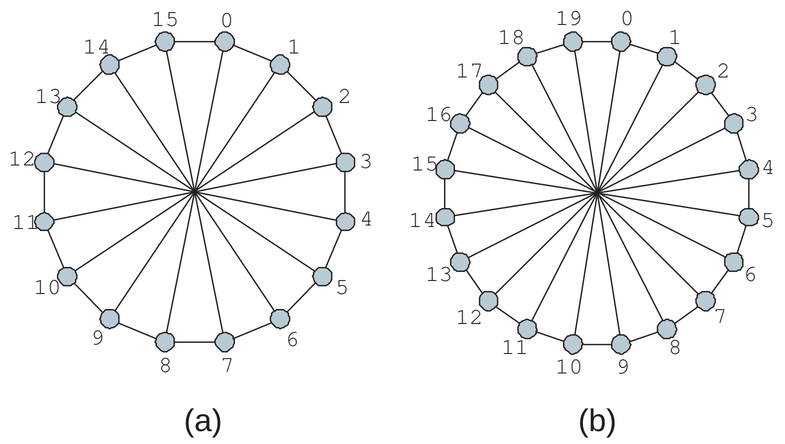

Figure 2.4: The graphs (a) PJ(8), (b) PJ(10)

with $|F| = 1$. By the symmetric property of $PJ(n)$, we may assume that $F = \{(0,1)\}$ or $\{(0,n)\}$. Obviously, $\langle 1, 2, \ldots, n, 0, n-1, \ldots, n+1\rangle$ is a hamiltonian cycle of $PJ(n) - F$ if $F = \{(0,1)\}$ and $\langle 2, 3, \ldots, n+1, 1, 0, n-1, \ldots, n+2\rangle$ is a hamiltonian cycle of $PJ(n) - F$ if $F = \{(0,n)\}$. Hence, $h_f(PJ(n)) \geq 1$.

The lemma is proved. □

For integer $n \geq 2$, the *ladder graph* $L(n)$ is the graph with vertex set $\{i \mid 0 \leq i \leq 2n-1\}$ and edge set $\{(i, 2n-i) \mid 1 \leq i < n\} \cup \{(i, i \oplus 1) \mid 0 \leq i \leq 2n-1\} \cup \{(0,n)\}$ where $\oplus$ denotes addition in integer modulo n, Z_n. It is known that $L(n)$ is cubic, 3-connected, and hamiltonian. Hence, $L(n)$ is in Ω. The ladder graphs $L(5)$ and $L(6)$ are illustrated in Figure 2.5.

Lemma 10 $h_f(L(n)) = 1$.

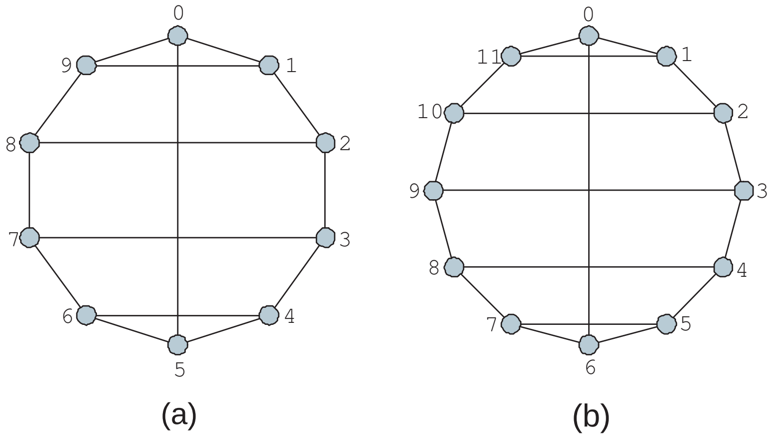

Figure 2.5: The graphs (a) L(5), (b) L(6)

Proof. By Lemma 1, we know that $h_f(L(n)) \leq 1$. Let F be any edge set of $L(n)$ with $|F| = 1$. By the symmetric property of $L(n)$, we may assume that $F = \{(0, n)\}$, $\{(1, 2n-1)\}$, $\{(2, 2n-2)\}$, ..., $\{(n-1, n+1)\}$, $\{(1, 2)\}$, $\{(3, 4)\}$, ..., or $\{(2n-1, 0)\}$. Obviously, $\langle 0, 1, 2, \ldots, 2n-1\rangle$ is a hamiltonian cycle of $L(n)-F$ if $F = \{(0, n)\}$, $\{(1, 2n-1)\}$, $\{(2, 2n-2)\}$, ..., or $\{(n-1, n+1)\}$, $\langle 0, 1, 2n-1, 2, \ldots, n-2, n+2, n+1, n\rangle$ is a hamiltonian cycle of $L(n) - F$ if n is odd and $F = \{(1, 2)\}$, $\{(3, 4)\}$, ..., $\{(2n-1, 0)\}$, and $\langle 0, 1, 2n-1, 2, \ldots, n+2, n-2, n-1, n\rangle$ is a hamiltonian cycle of $L(n) - F$ if n is even and $F = \{(1, 2)\}$, $\{(3, 4)\}$, ..., $\{(2n-1, 0)\}$. Therefore, $h_f(PJ(n)) \geq 1$.

The lemma is proved. □

Chapter 3

Examples

3.1 Examples of graph G in Ω with $h_f(G) = 1$ and $h_r(G) = 1$

Theorem 1 *$h_r(P(n,1)) = 1$ and $h_f(P(n,1)) = 1$ if n is odd and $n \geq 3$.*

Proof. By Lemma 6, we know that $h_f(P(n,1)) = 1$. Let R be any required edge set of $P(n,1)$ with $|R| = 1$. By the symmetric property of $P(n,1)$, we may assume that $R = \{(0,1)\}$, $\{(0',1')\}$, or $\{(0,0')\}$. Obviously, $\langle 0, 1, \ldots, n-1, (n-1)', (n-2)', \ldots, 0' \rangle$ is a hamiltonian cycle including the required edge set R. Hence, $h_r(P(n,1)) \geq 1$ if n is odd and $n \geq 3$. Now we prove that $h_r(P(n,1)) \leq 1$ for n is odd and $n \geq 3$. Let the required edge set $R = \{(1,1'), (n-1, (n-1)')\}$. We want to prove there is no hamiltonian cycle C of $P(n,1)$ including R. (See Figure 3.1(a) for an illustration of the case $n = 7$.)

We have the following two cases:

Case 1. $(0,0') \notin C$. Thus, the edge set $\{(0,1), (0,n-1), (0',1'), (0',(n-1)')\}$ are contained in C. We got a cycle $\langle 0, 1, 1', 0', (n-1)', n-1 \rangle$. Thus, there is no such

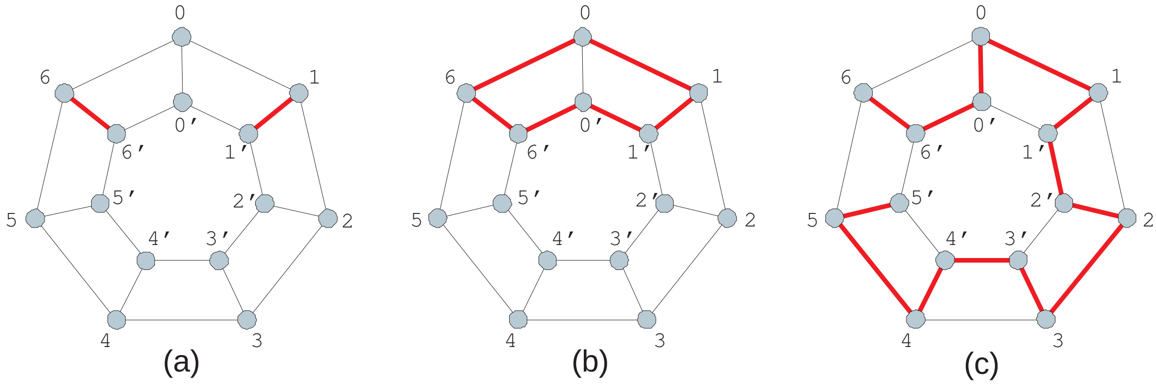

Figure 3.1: Illustrations for Theorem 1.

hamiltonian cycle. (See Figure 3.1(b) for an illustration of the case $n = 7$.)

Case 2. $(0, 0') \in C$. Obviously, either $(0, 1) \in C$ or $(0, n-1) \in C$. Without loss of generality, we assume $(0, 1) \in C$. Then C include the path $\langle n-1, (n-1)', 0', 0, 1, 1', 2', 2, 3, 3', \ldots, (n-3)', n-3, n-2, (n-2)' \rangle$. Note that $(n-1, (n-1)') \notin E(P(n, 1))$. Therefore, there is no such cycle. (See Figure 3.1(c) for an illustration of the case $n = 7$.)

Therefore, there is no hamiltonian cycle contains the required edge set R. Hence, $h_r(P(n, 1)) = 1$ when n is odd and $n \geq 3$.

The theorem is proved. □

Theorem 2 *$h_r(PJ(n)) = 1$ and $h_f(PJ(n)) = 1$ when n is even and $n \geq 2$.*

Proof. By Lemma 11, we know that $h_f(PJ(n)) = 1$. Let R be any required edge set of $PJ(n)$ with $|R| = 1$. By the symmetric property of $PJ(n)$, we may assume that $R = \{(0, 1)\}$, or $\{(0, n)\}$. Obviously, $\langle 0, 1, \ldots, n-1, 2n-1, 2n-2, \ldots, n \rangle$ is a hamiltonian

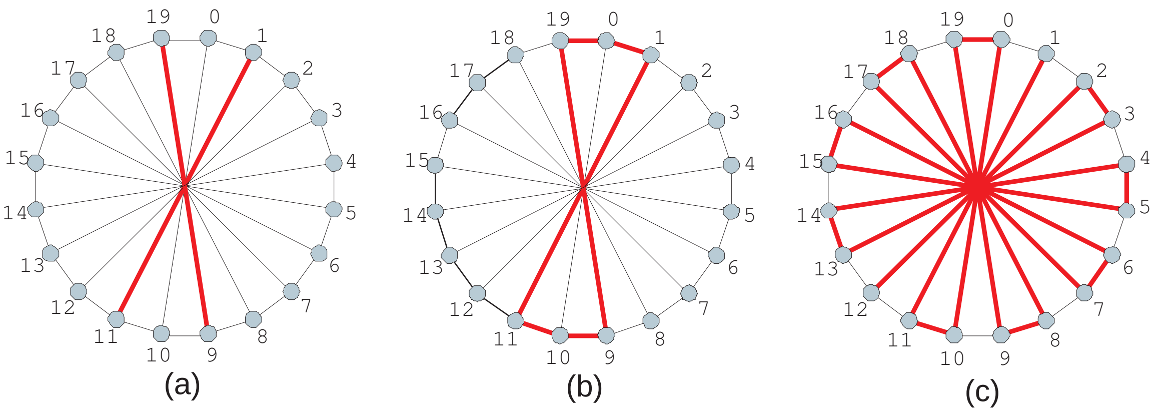

Figure 3.2: Illustrations for Theorem 2.

cycle including the required edge set R. Hence, $h_r(PJ(n)) \geq 1$ when n is even and $n \geq 2$. Now we prove that $h_r(PJ(n)) \leq 1$ when n is even and $n \geq 2$. Let the required edge set $R = \{(1, n+1), (n-1, 2n-1)\}$. We want to prove there is no hamiltonian cycle C of $PJ(n)$ including the edge set R. (See Figure 3.2(a) for an illustration of the case $n = 10$.)

We have the following two cases:

Case 1. $(0, n) \notin C$. The edge set $\{(0,1), (0, 2n-1), (n-1, n), (n, n+1)\}$ are contained in C. We got a cycle $\langle 0, 1, n+1, n, n-1, 2n-1 \rangle$. Thus, there is no such hamiltonian cycle. (See Figure 3.2(b) for an illustration of the case $n = 10$.)

Case 2. $(0, n) \in C$. Obviously, either $\{(0, 2n-1), (n, n+1)\} \in C$ or $\{(0, 1), (n, n-1)\} \in C$. Without loss of generality, we assume $\{(0, 2n-1), (n, n+1)\} \in C$. Then C include the path $\langle 1, n+1, n, 0, 2n-1, n-1, n-2, 2n-2, 2n-3, \ldots, 2, n+2 \rangle$. Therefore, there is no such hamiltonian cycle. (See Figure 3.2(c) for an illustration of the case $n = 10$.)

Therefore, there is no hamiltonian cycle contains R. Hence, $h_r(PJ(n)) = 1$ if n is even. The lemma is proved. □

3.2 Examples of graph G in Ω with $h_f(G) = 1$ and $h_r(G) = 2$

Lemma 11 *[16] A petersen graph $P(n, 2)$ is not hamiltonian if and only if $n = 5 \pmod 6$.*

Theorem 3 *$h_r(P(n, 2)) = 2$ and $h_f(P(n, 2)) = 1$ if $n = 1, 3 \pmod 6$.*

Proof. By Lemma 8, we know that $h_f(P(n, 2)) = 1$. Now we prove that $h_r(P(n, 2)) = 2$ for $n = 1, 3 \pmod 6$. Let R be any required edge set of $P(n, 2)$ with $|R| = 2$. We have the following cases:

Case 1. *n is odd*, $n = 1 \pmod 6$ and $n \geq 7$. Obviously, $\langle 1', 1, 0, 0', 2', 2, 3, 4, 4', 6', 6, 5, 5', 3'\rangle$ is a hamiltonian cycle of $P(7, 2)$ and $\langle 1', 1, 0, N_0, N_6, \ldots, N_{n-7}, (n-1)', n-1, n-2, (n-2)', N_{n-4}, N_{n-10}, \ldots, N_9, 3'\rangle$ is a hamiltonian cycle of $P(n, 2)$ when $n > 7$. It is easy to check that any two edge can be on the hamiltonian cycle. Hence, $h_r(P(n, 2)) = 2$ when $n = 1 \pmod 6$ with $n \geq 7$. (See Figure 2.3(a) and 2.3(b) for an illustration of the case *n is odd*, $n = 1 \pmod 6$ and $n \geq 7$.)

Case 2. *n is odd*, $n = 3 \pmod 6$ and $n \geq 9$. Obviously, $\langle M_0, M_6, \ldots, M_{n-3}, M_3, M_9, \ldots, M_{n-6}\rangle$ is a hamiltonian cycle of $P(n, 2)$ when $n \geq 9$. It is easy to check that any two edge can be on the hamiltonian cycle. Hence, $h_r(P(n, 2)) = 2$ when $n = 3 \pmod 6$

with $n \geq 9$. (See Figure 2.3(c) and 2.3(d) for an illustration of the case n *is odd*, $n = 3 \pmod 6$ and $n \geq 9$.)

Now we prove that $h_r(P(n,2)) = 3$ for $n = 1, 3 \pmod 6$. Let $M(v_0, v_1, v_2, v_3, v_4, v_5)$ is H that the path $P_0 = \langle v_0, x_0, x_1, \ldots, x_i, v_1\rangle$, $P_1 = \langle v_2, y_0, y_1, \ldots, y_j, v_3\rangle$, $P_2 = \langle v_4, z_0, z_1, \ldots, z_k, v_5\rangle$, and one method link the vertex $(v_0, v_1, v_2, v_3, v_4, v_5)$ can give a hamiltonian cycle in $P(n,2)$ if $n = 1, 3 \pmod 6$.

Suppose the $P(9,2)$ have a hamiltonian cycle. Let the required edge set $R = \{(1,2),(2,3),(3,4)\}$. Because the edges $(2,2'),(3,3')$ are not in C. Thus, the edges $(1',3'),(3',5'),(0',2'), and (2',4')$ are in C. And we can use $M(1,4,1',5',0',4')$ to give a hamiltonian cycle in $P(9,2)$. We can construct a hamiltonian cycle form $P(9,2)$ to $P(11,2)$, which insert two vertex x and y between 2 and 3 and insert two vertex x' and y' between $2'$ and $3'$. The $P(13,2)$ also have a hamiltonian cycle, but we know the $P(13,2)$ have not a hamiltonian cycle. This is contradiction. It is easy to check that $h_r(P(n,2)) \neq 3$ for $n = 1, 3 \pmod 6$. Therefore, $h_r(P(n,2)) \neq 3$ for $n = 1, 3 \pmod 6$. The theoerm is proved. □

Theorem 4 $h_r(L(n)) = 2$ *and* $h_f(L(n)) = 1$.

Proof. By Lemma 10, $h_f(L(n)) = 1$. Now, we prove $h_r(L(n)) = 2$.

Let us divide the edge set $E(L(n))$ into three sets A, B, and C where the edge sets $A = \{(i, i \oplus 1) \mid 0 \leq i \leq 2n-1\}$, $B = \{(i, 2n-i) \mid 1 \leq i < n\}$, and $C = \{(0,n)\}$. Obviously, $E(L(n)) = A \cup B \cup C$. Let the required edge set $R = \{p, q\}$. We have the following cases:

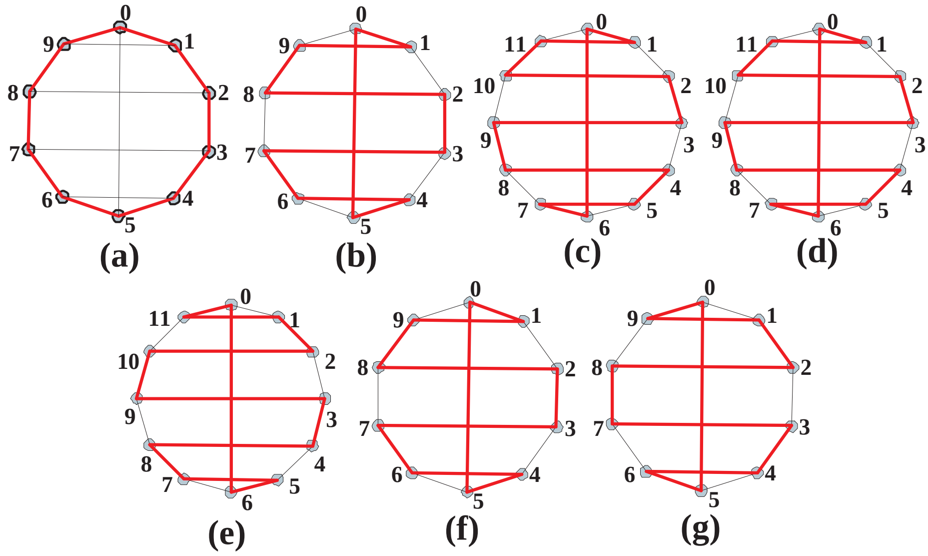

Figure 3.3: Illustrations for Theorem 3.3.

Case 1. $\{p, q\} \subseteq A$. Obviously, $\langle 0, 1, \ldots, 2n-2, 2n-1 \rangle$ forms a hamiltonian cycle including R. (See Figure 3.3(a) for an illustration of the case $n = 5$.)

Case 2. $\{p, q\} \subseteq B \cup C$. Suppose that n is odd. Obviously, cycle $C_1 = \langle 0, 1, 2n-1, 2n-2, 2, \ldots, n-2, n+2, n+1, n-1, n \rangle$ including R. Suppose that n is even. Cycle $C_2 = \langle 0, 1, 2n-1, 2n-2, 2, \ldots, n+2, n-2, n-1, n+1, n \rangle$ including R. (See Figures 3.3(b) and (c) for an illustration of the case $n = 5$ and 6.)

Case 3. $p \in A$ and $q \in B \cup C$. Without loss of generality, we assume that edge $q = (i, i+1)$ where $0 \leq i \leq n-1$. When n is even, cycles $C_1 = \langle 0, 1, 2n-1, 2n-2, 2, \ldots, n+2, n-2, n-1, n+1, n \rangle$ or $C_2 = \langle 0, 2n-1, 1, 2, 2n-2, \ldots, n-2, n+2, n+1, n-1, n \rangle$ including R. When n is odd, cycles $C_3 = \langle 0, 1, 2n-1, 2n-2, 2, \ldots, n-2, n+2, n+1, n-1, n \rangle$ or $C_4 = \langle 0, 2n-1, 1, 2, 2n-2, \ldots, n+2, n-2, n-1, n+1, n \rangle$ including R. (See Figure

3.3(d),(e),(f), and (g) for an illustration of the case $n = 5$ and 6.)

Hence, $h_r(L(n)) \geq 2$.

Assume that there exists a hamiltonian cycle C including the required edge set $R' = \{(0,1), (0, 2n-1), (2, 2n-2)\}$. Thus, the edge set $\{(1,2), (2n-2, 2n-1)\}$ are contained in C. We got a cycle $\langle 0, 1, 2, 2n-2, 2n-1 \rangle$. Thus, there is no such hamiltonian cycle. Hence, $h_r(L(n)) \leq 2$.

Therefore, $h_r(L(n)) = 2$. The theorem is proved. □

3.3 Examples of graph G in Ω with $h_f(G) = 1$ and $h_r(G) = 3$

Theorem 5 *$h_r(P(n,1)) = 3$ and $h_f(P(n,1)) = 1$ when n is even.*

Proof. By Lemma 6, we know that $h_f(P(n,1)) = 1$. Now, we want to show that $h_r(P(n,1)) = 3$. Let us divide the edge set $E(P(n,1))$ into two sets A and B where the edge sets $A = \{(i, i \oplus 1) \mid 0 \leq i \leq n-1\} \cup \{(i', (i \oplus 1)') \mid 1 \leq i < n-1\}$ and $B = \{(i, i') \mid 1 \leq i < n\}$. Obviously, $E(P(n,1)) = A \cup B$. Let the required edge set $R = \{p, q, r\}$. We have the following cases:

Case 1. $\{p, q, r\} \subseteq A$. Without loss of generality, we assume that $\{p, q, r\} \cap \{(0, n-1), (0', (n-1)')\} = \emptyset$. The hamiltonian cycle $\langle 0, 1, 2, \ldots, n-1, (n-1)', (n-2)', \ldots, 1', 0' \rangle$ including the required edge set R.

Case 2. $\{p, q\} \subseteq A$ and $\{r\} \subseteq B$. Without loss of generality, we assume that the edge $r = (0, 0')$. We have the following subcases:

Case 2.1. $\{p, q\} \cap \{(0, 1), (0', 1')\} = \emptyset$. The hamiltonian cycle $\langle 1, 2, \ldots, n-1, 0, 0', (n-1)', (n-2)', \ldots, 1' \rangle$ including the required edge set R.

Case 2.2. $\{p, q\} = \{(0, 1), (0', 1')\}$. The hamiltonian cycle $\langle 0, 1, \ldots, n-1, (n-1)', (n-2)', \ldots, 1', 0' \rangle$ including the required edge set R.

Case 2.3. $\{p, q\} \cap \{(0, 1), (0', 1')\} = \{(0, 1)\}$ or $\{(0', 1')\}$. Without loss of generality, we set $p = (0, 1)$. The hamiltonian cycle $\langle 0, 1, 2, \ldots, n-1, (n-1)', (n-2)', \ldots, 1', 0' \rangle$ including the required edge set R when $q \neq (0', (n-1)')$. And the hamiltonian cycle $\langle 0, 1, 1', 2', 2, \ldots, n-2, n-1, (n-1)', 0' \rangle$ including the required edge set R when $q = (0', (n-1)')$.

Case 3. $\{p\} \subseteq A$ and $\{q, r\} \subseteq B$. Without loss of generality, we assume that the edge $p = (0, n-1)$. The hamiltonian cycle $\langle 0, 0', 1', 1, 2, 2', \ldots, n-2, (n-2)', (n-1)', n-1 \rangle$ including the required edge set R.

Case 4. $\{p, q, r\} \subseteq B$. The hamiltonian cycle $\langle 0, 0', 1', 1, 2, 2', \ldots, n-2, (n-2)', (n-1)', n-1 \rangle$ including the required edge set R.

Hence, $h_r(P(n, 1)) = 3$ when n is even. The theorem is proved. □

Theorem 6 *$h_r(PJ(n)) = 3$ and $h_f(PJ(n)) = 1$ when n is odd.*

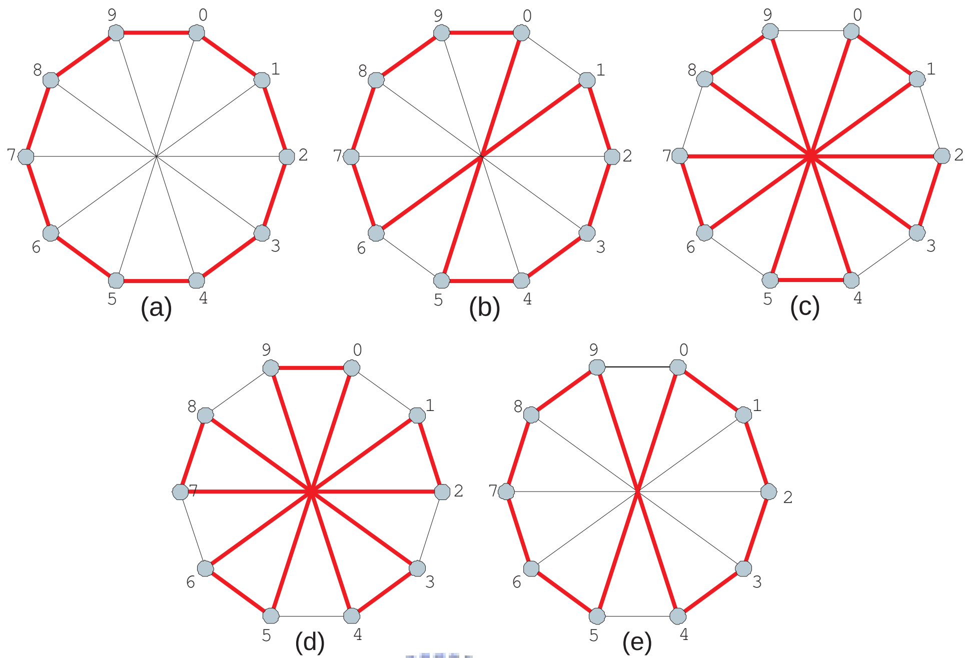

Figure 3.4: Illustrations for Theorem 3.4.

Proof. By Lemma 11, we know that $h_f(PJ(n)) = 1$. By Lemma 1, we know $h_r(PJ(n)) \leq 3$. Now, we want to prove that for any required edge set $R = \{p, q, r\}$, we can find a hamiltonian cycle including the edges R.

Let the edge sets $A = \{(i, i \oplus 1) \mid 0 \leq i \leq 2n - 1\}$ and $B = \{(i, i + n) \mid 1 \leq i < n\}$. Obviously, $E(PJ(n)) = A \cup B$. We have the following cases:

Case 1. $\{p, q, r\} \subseteq A$. There is a hamiltonian cycle $\langle 0, 1, \ldots, 2n - 2, 2n - 1 \rangle$ including R. (See Figure 3.4(a) for an illustration of the case $n = 5$.)

Case 2. $\{p, q\} \subseteq A$ and $r \in B$. Without loss of generality, we set edge $r = (0, n)$.

Case 2.1. $\{p,q\}\cap\{(0,1),(n,n+1)\}=\emptyset$. There is a hamiltonian cycle $\langle 1,2,\ldots,n,0,2n-1,\ldots,n+1\rangle$ including R. (See Figure 3.4(b) for an illustration of the case $n=5$.)

Case 2.2. $\{p,q\}\cap\{(0,1),(n,n+1)\}=\{(0,1)\}$. Without loss of generality, we set $p=(0,1)$. There is a hamiltonian cycle $\langle 0,1,\ldots,n-1,2n-1,2n-2,\ldots,n\rangle$ including R when $q\neq(n-1,n)$. And there is a hamiltonian $\langle 0,1,n+1,n+2,2,3,n+3,n+4,\ldots,2n-2,2n-1,n-1,n\rangle$ including R when $q=(n-1,n)$. (See Figure 3.4(c) and Figure 3.4(e) for an illustration of the case $n=5$)

Case 2.3. $\{p,q\}\cap\{(0,1),(n,n+1)\}=\{(n,n+1)\}$. Without loss of generality, we set $p=(n,n+1)$. There is a hamiltonian cycle $\langle 0,1,\ldots,n-1,2n-1,2n-2,\ldots,n\rangle$ including R when $q\neq(0,2n-1)$. And there is a hamiltonian $\langle 0,n,n+1,1,n+2,n+3,3,4,\ldots,n-2,n-1,2n-1\rangle$ including R when $q=(0,2n-1)$. (See Figure 3.4(d) and Figure 3.4(e) for an illustration of the case $n=5$)

Case 2.3. $\{p,q\}=\{(0,1),(n,n+1)\}$. There is a hamiltonian cycle $\langle 0,1,\ldots,n-1,2n-1,2n-2,\ldots,n\rangle$ including R. (See Figure 3.4(e) for an illustration of the case $n=5$.)

Case 3. $p\in A$ and $\{q,r\}\subseteq B$. Without loss of generality, we set $p=(0,1)$. There is a hamiltonian cycle $\langle 0,1,n+1,n+2,2,3,n+3,n+4,\ldots,2n-2,2n-1,n-1,n\rangle$ including R. (See Figure 3.4(c) for an illustration of the case $n=5$.)

Case 4. $\{p,q,r\}\subseteq B$. There is a hamiltonian cycle $\langle 0,1,n+1,n+2,2,3,n+3,n+4,\ldots,2n-2,2n-1,n-1,n\rangle$ including R. (See Figure 3.4(c) for an illustration of the

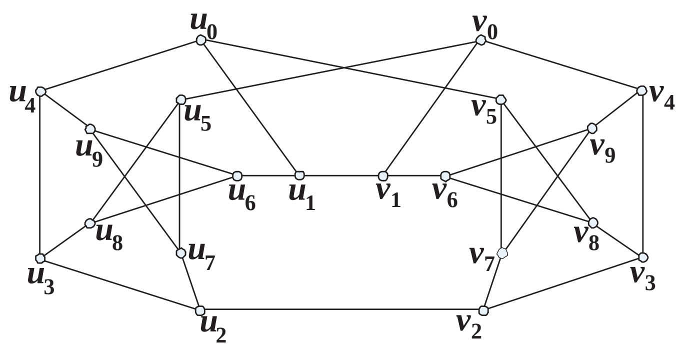

Figure 3.5: The graph M.

case $n = 5$.)

Hence, $h_r(PJ(n)) = 3$ when n is odd. The theorem is proved. □

3.4 Examples of graph G in Ω with $h_f(G) = 0$ and $h_r(G) = 0$

In this section, we will prove the the graph M in Figure 3.5 is in Ω with $h_f(M) = 0$ and $h_r(M) = 0$.

Theorem 7 *Graph M is in Ω. $h_f(M) = 0$ and $h_r(M) = 0$.*

Proof. It is easy to check that $\kappa(M) = 3$. In Figure 3.6, we give a hamiltonian cycle indicated by redden edges. Therefore, M is in Ω.

By Lemma 1, we know that $h_f(M) \leq 1$. Let the fault edge set $F = \{(u_2, v_2)\}$. We want to show that there is no any hamiltonian cycle in $M - F$. Let the node set $V_l = \{u_0, u_1, \ldots, u_9\}$, $V_r = \{v_0, v_1, \ldots, v_9\}$, edge cut set $S = \{(u_0, v_5), (v_0, u_5), (u_1, v_1)\}$.

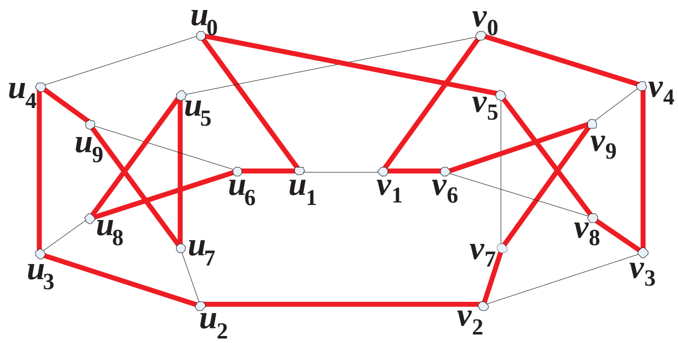

Figure 3.6: A hamiltonian cycle in M.

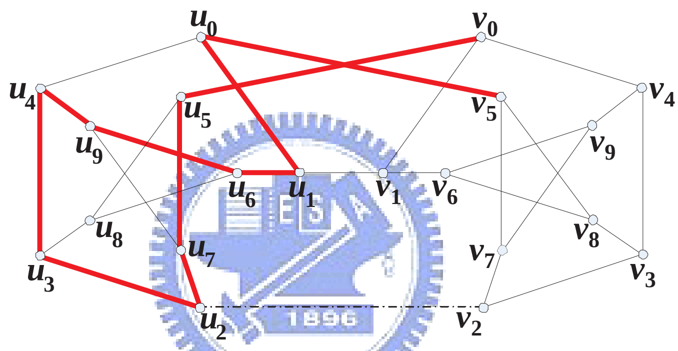

Figure 3.7: Illustration for Theorem 7, Case A1

Assume that we can find a hamiltonian cycle C in $M - F$. It is easy to know that $|C \cap S| = 2$. Now, we consider the edges (u_0, v_5), (u_5, v_0), and (u_1, v_1) in C or not in the following cases.

Case A1. $(u_0, v_5), (u_5, v_0) \in C$. Because the edge (u_1, v_1) is not in C, we implies that the edges (u_2, u_3), (u_2, u_7), (u_1, u_0), and (u_1, u_6) are in C. And then (u_3, u_4) and (u_4, u_9) are in C. Therefore, (u_5, u_7) and (u_6, u_9) are in C. We got a path joining nodes u_0 and u_5 in M_{V_l} but we lost node u_8. Hence, we can not find any hamiltonian cycle C in $M - F$

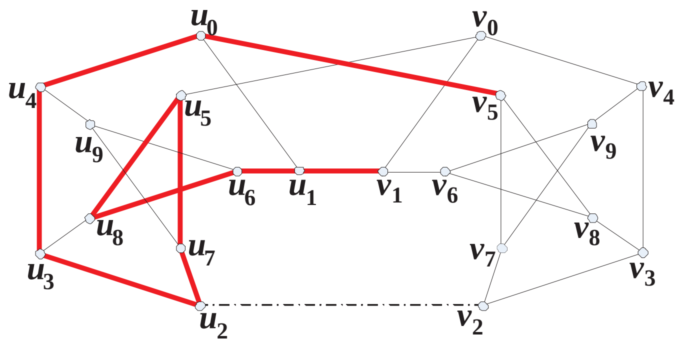

Figure 3.8: Illustration for Theorem 7, Case A2

with edges (u_0, v_5) and (u_5, v_0) are in C. (See Figure 3.7 for an illustration.)

Case A2. (u_0, v_5), $(u_1, v_1) \in C$ or (u_5, v_0), $(u_1, v_1) \in C$. Without loss of generality, we consider (u_0, v_5),$(u_1, v_1) \in C$. Because the edge (u_5, v_0) is not in C, we implies that the edges (u_2, u_3), (u_2, u_7), (u_5, u_7), (u_5, u_8), (u_3, u_4) are in C. And then (u_0, u_4) and (u_1, u_6) are in C. Thus, (u_6, u_8) is in C. We got a path joining nodes u_0 and u_1 in M_{V_l} but we lost node u_9. Hence, we can not find any hamiltonian cycle C in $M - F$ with edges (u_0, v_5) and (u_1, v_1) are in C but (u_5, v_0) is not in C. (See Figure 3.8 for an illustration.)

Hence, we can not find a hamiltonian cycle in $M - F$. Therefore, $h_f(M) = 0$.

By Lemma 3, we know that $h_r(M) \leq 1$. Let $R = (u_1, v_1)$ be the required edge set of M with $|R| = 1$. We want to show that we can not find any hamiltonian cycle in M including the required edge set R. Assume that C be the hamiltonian cycle in M including the required edge set R. Let the cut edge set $S = \{(u_0, v_5), (u_5, v_0), (u_2, v_2)\}$. It is easy to know that $|C \cap S| = 1$ or 3, because edge (u_1, v_1) is in C. We consider the

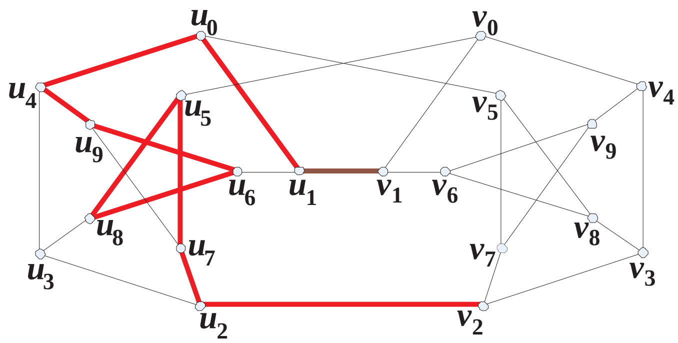

Figure 3.9: Illustration for Theorem 7, Case B1

edges (u_0, v_5), (u_5, v_0), and (u_2, v_2) in C or not in the following cases.

Case B1. $(u_2, v_2) \in C$. Because the edges (u_5, v_0), (u_0, v_5) are not in C. Thus, the edges (u_0, u_1), (u_0, u_4), (u_5, u_7), (u_5, u_8) are in C. And then (u_6, u_8), (u_6, u_9) are in C. Hence, (u_2, u_7), (u_4, u_9) are in C. Now, we got a path joining nodes u_1 and u_2 but we lost node u_3. Thus, there is no such hamiltonian cycle C with edges (u_1, v_1) and (u_2, v_2) are in C. (See Figure 3.9 for an illustration.)

Case B2. $(u_5, v_0) \in C$. Because the edges (u_0, v_5), (u_2, v_2) are not in C. Thus, the edges (v_2, v_7), (v_2, v_3), (v_5, v_7), (v_5, v_8) are in C. And then the edges (v_1, v_6), (v_0, v_4), (v_4, v_3) are in C. Thus, (v_6, v_8) is in C. Now, we got a path joining nodes v_0 and v_1 but we lost node v_9. Thus, there is no such hamiltonian cycle C with edges (u_1, v_1) and (u_5, v_0) are in C. (See Figure 3.10 for an illustration.)

Case B3. $\{(u_5, v_0), (u_0, v_5), (u_2, v_2)\} \in C$. We have the following subcases:

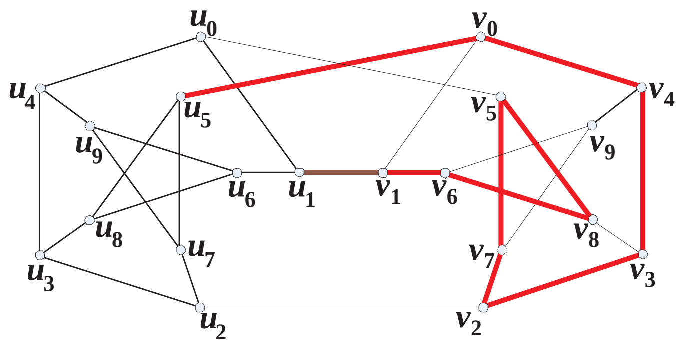

Figure 3.10: Illustration for Theorem 7, Case B2

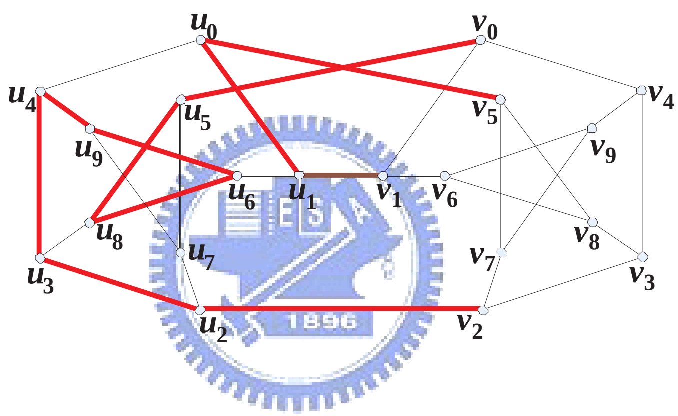

Figure 3.11: Illustration for Theorem 7, Case B3.1

Case B3.1. $\{(u_0, u_1)\} \in C$ or $\{(v_0, v_1)\} \in C$. Without loss of generality, we assume that $\{(u_0, u_1)\} \in C$. Because the edges (u_1, u_6) and (u_0, u_4) are not in C, the edges (u_6, u_9), (u_6, u_8), (u_4, u_9), (u_4, u_3) are in C. Thus, the edges (u_5, u_8) and (u_3, u_2) are in C. We got a path joining node u_5 and u_2 but we lost node u_7. Thus, there is no such hamiltonian cycle C. (See Figure 3.11 for an illustration.)

Case B3.2. $\{(u_0, u_1), (u_0, u_1)\} \notin C$. Hence, the edges (u_1, u_6) and (v_1, v_6) are in C. We have the following subcases:

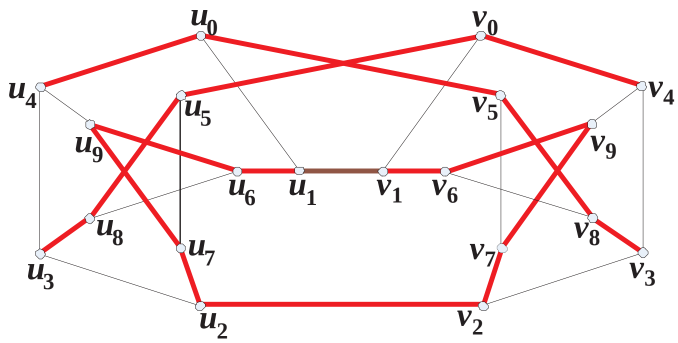

Figure 3.12: Illustration for Theorem 7, Case B3.2.1

Case B3.2.1. $\{(u_6, u_9), (v_6, v_9)\} \in C$. Because the edges (u_0, u_1), (v_0, v_1) are not in C. Thus, the edges (u_0, u_4), (u_1, u_6), (v_0, v_4), (v_1, v_6) are in C. And then the edges $(u_8, u_5), (u_8, u_3), (v_8, v_5), (v_8, v_3)$ are in C. Thus, $(u_7, u_2), (u_7, u_9), (v_7, v_2), (v_7, v_9)$ are in C. Now, we got a cycle $\langle u_1, u_6, u_9, u_7, u_2, v_2, v_7, v_9, v_6, v_1\rangle$ in M. Thus, there is no such hamiltonian cycle. (See Figure 3.12 for an illustration.)

Case B3.2.2. $\{(u_6, u_9), (v_6, v_8)\}$ or $\{(u_6, u_8), (v_6, v_9)\} \in C$. Without loss of generality, we consider $\{(u_6, u_9), (v_6, v_8)\} \in C$. Because the edges (u_0, u_1), (v_0, v_1) are not in C. Thus, the edges (u_0, u_4), (u_1, u_6), (v_0, v_4), (v_1, v_6) are in C. And then the edges $(u_8, u_5), (u_8, u_3), (v_9, v_4), (v_9, v_7)$ are in C. Thus, $(u_7, u_2), (u_7, u_9), (v_3, v_2), (v_3, v_8)$ are in C. Now, we got a cycle $\langle u_1, u_6, u_9, u_7, u_2, v_2, v_3, v_8, v_6, v_1\rangle$ in M. Thus, there is no such hamiltonian cycle. (See Figure 3.13 for an illustration.)

Case B3.2.3. $\{(u_6, u_8), (v_6, v_8)\} \in C$. Because the edges (u_0, u_1), (v_0, v_1) are not in C. Thus, the edges (u_0, u_4), (u_1, u_6), (v_0, v_4), (v_1, v_6) are in C. And then the edges $(u_9, u_4), (u_9, u_7), (v_9, v_4), (v_9, v_7)$ are in C. Thus, $(u_3, u_2), (u_3, u_8), (v_3, v_2), (v_3, v_8)$ are in

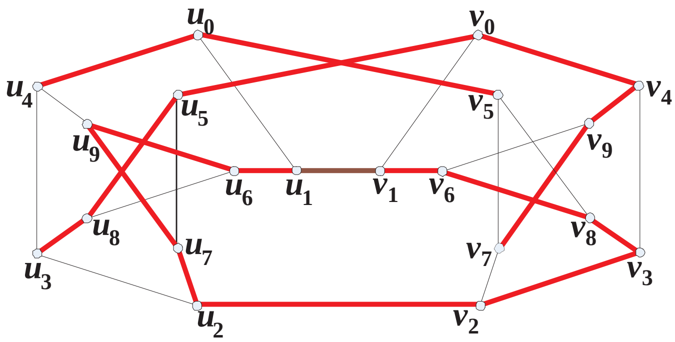

Figure 3.13: Illustration for Theorem 7, Case B3.2.2

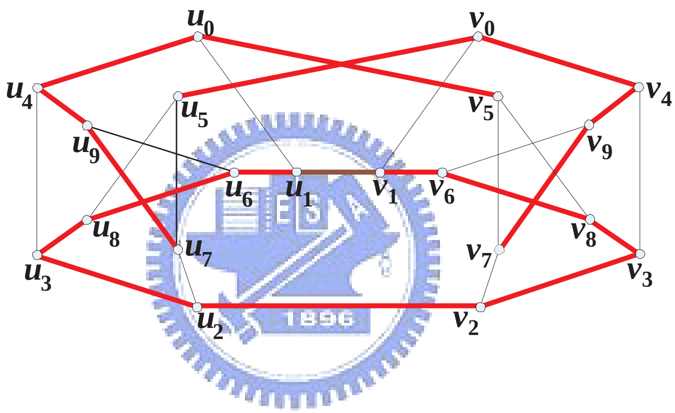

Figure 3.14: Illustration for Theorem 7, Case B3.2.3

C. Now, we got a cycle $\langle u_1, u_6, u_8, u_3, u_2, v_2, v_3,\ v_8, v_6, v_1\rangle$ in M. Thus, there is no such hamiltonian cycle. (See Figure 3.14 for an illustration.)

Hence, we can not find any hamiltonian cycle in M including the required edge set $R = \{(1, 1')\}$. Therefore, $h_r(M) = 0$. $\square$

Theorem 8 *Graph $J(M, x)$ is in Ω. $h_f(J(M, x)) = 0$ and $h_r(J(M, x)) = 0$.*

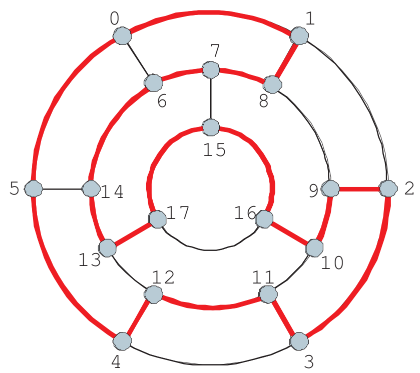

Figure 3.15: The graph N.

Proof. By Lemma 4 and Lemma 5, we know that $h_f(J(G,x)) = 0$ and $h_r(J(G,x)) = 0$. Hence, $h_f(J(M,x)) = 0$ and $h_r(J(M,x)) = 0$. □

3.5 Examples of graph G in Ω with $h_f(G) = 0$ and $h_r(G) = 1$

In this section, we will prove the the graph N in Figure 3.15 is in Ω with $h_f(N) = 0$ and $h_r(N) = 1$.

Theorem 9 *Graph N is in Ω such that $h_f(N) = 0$ and $h_r(N) = 1$.*

Proof. It is proved in [11] that graph $N - \{(0,1)\}$ is not hamiltonian. Hence, $h_f(N) = 0$.

By Lemma 3, we know that $h_r(N) \leq 1$. Let C be the hamiltonian cycle indicated by darken edges in N as shown in Figure 3.15. It is easy to check that any edge can be on the hamiltonian cycle. Hence, $h_r(N) = 1$. □

Theorem 10 *Graph $J(N, x)$ is in Ω. $h_f(J(N, x)) = 0$ and $h_r(J(N, x)) = 1$.*

Proof. By Lemma 4 and Lemma 5, we know that $h_f(J(G, x)) = 0$ and $h_r(J(G, x)) = 1$. Hence, $h_f(J(N, x)) = 0$ and $h_r(J(N, x)) = 1$. □

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