國立交通大學

資訊工程學系

博 士 論 文

連結網路上之漢彌爾頓性質

Some Hamiltonian Properties on Interconnection Networks

研 究 生：滕元翔

指導教授：譚建民　博士

徐力行　博士

中華民國 九十七 年 一 月

連結網路上之漢彌爾頓性質
Some Hamiltonian Properties on Interconnection Networks

研 究 生：滕元翔　　Student：Yuan-Hsiang Teng

指導教授：譚建民博士　　Advisor：Dr. Jimmy J.M. Tan

徐力行博士　　Dr. Lih-Hsing Hsu

國 立 交 通 大 學 資 訊 學 院

資 訊 工 程 系

博 士 論 文

A Thesis

Submitted to Department of Computer Science

College of Computer Science

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Computer Science

January 2008

Hsinchu, Taiwan, Republic of China

中華民國九十七年元月

誌謝

首先非常感謝我的指導教授譚建民教授與徐力行教授這幾年來的照顧與指導，除了對於這篇論文的指導以及日常生活的關心，平日在研討時也常常提供我寶貴的意見與經驗，使我學會許多做研究的方法與態度，這些都是我將來做學問時豐富的資產。

在這裡，我也要感謝我的論文計畫指導委員陳榮傑老師與梁婷老師；校內口試委員蔡錫鈞老師；校外口試委員李家同老師、陳健輝老師、許健平老師、王有禮老師。非常感謝各位老師在口試時對於論文及研究的指導。

感謝實驗室的所有伙伴。晃哥、寬哥、峰哥、嚕嚕米、阿鋼、韓老師、學弟小碩二們，因為你們，使我在實驗室的生活裡，能找到許多笑點、驚喜與樂趣。也謝謝系上耀萱助理，她泡的茶應該很棒。

在交大待了十年，要感謝許多曾經幫助過我的師長。另外，在此要特別感謝國中導師廖蘭英老師與大學微積分老師陳秋媛老師。兩位老師除了對我在數學科目的啟發，更是常常關心我的生活，使我在成長與學習的過程中，更添順利。這篇論文也獻給您們，感謝老師。

感謝小愛貝比帶給我的酸甜苦辣，我相信，甜的成份比較多。最後要謝謝我的母親，含辛茹苦地養育我、妹妹和多多，照顧我日常起居。由於您們大力的支持，讓我可以在求學的過程中無後顧之憂。在此謝謝您們無私的付出。

連結網路上之漢彌爾頓性質

研 究 生：滕元翔　　　　　指導教授：譚建民　博士

徐力行　博士

國立交通大學資訊工程系

摘要

在這篇論文當中，我們研究了一些漢彌爾頓問題，像是相互獨立漢彌爾頓以及泛可放置漢彌爾頓。在圖G中，我們用n來標記頂點的數目並用e來標記邊的數目。我們用$\bar{e}$來標記G的補圖中邊的數目。假設G爲一圖並$\bar{e}$≤n-4且n≥4。我們證明除了n=5且$\bar{e}$=1的情形之外，在G中任一對不同的頂點之間有至少n-2-$\bar{e}$條相互獨立漢彌爾頓路徑。假設G任兩不相鄰點的分支度和至少n+2。令u和v爲G的任兩相異點。我們證明若(u,v)∈E(G)，則u和v之間有至少$deg_G(u)$ + $deg_G(v)$ - n條相互獨立漢彌爾頓路徑且其他情形下，u和v之間有至少$deg_G(u)$ + $deg_G(v)$ – n+2條相互獨立漢彌爾頓路徑。

排列圖$A_{n,k}$爲星狀圖的一般化。它比星狀圖在大小上更爲彈性。已有些研究著重在排列圖的漢彌爾頓性與泛圈性。我們提出新的概念稱爲泛可放置漢彌爾頓。一漢彌爾頓圖G爲泛可放置若對G中任兩相異點x和y，以及對任意整數l滿足d(x,y)≦l≦|V(G)|-d(x,y)，G存在一漢彌爾頓圈C使得x和y在C上的距離爲l。一圖G爲泛連通圖若存在一條長爲l之路徑連接兩相異點x和y且d(x,y)≦l≦|V(G)|-1。我們證明$A_{n,k}$ 爲泛可放置漢彌爾頓且泛連通若k≧1且n-k≧2。

假設 m 和 n 爲正偶數且 n≥4。已知每個蜂巢矩形圓環面 HReT(m,n)爲三正則二分圖。我們證明在任何 HReT(m,n)中，存在三條內部不相交衍生路徑連接 x 和 y，當 x 和 y 分屬不同的分割集合。對任意一對 x 和 y 屬於同一分割集合，存在一頂點 z 在沒有 x 和 y 的分割集合中，

使得 $G-\{z\}$中存在有三條內部不相交衍生路徑連接 x 和 y。對任三點 x，y 和 z 屬於同一分割集合，$G-\{z\}$中存在有三條內部不相交衍生路徑連接 x 和 y，若且唯若 $n \geq 6$ 或 $m=2$。

關鍵字: 漢彌爾頓、漢彌爾頓連結、漢彌爾頓路徑、泛可放置漢彌爾頓、泛連通性、連通性、排列圖、蜂巢圓環面

Some Hamiltonian Properties on Interconnection Networks

Student: Yuan-Hsiang Teng Advisor: Dr. Jimmy J. M. Tan
Dr. Lih-Hsing Hsu

Department of Computer Science
College of Computer Science
National Chiao Tung University

Abstract

In this thesis, we study some variant of hamiltonian problems, such as mutually independent hamiltonicity and panpositionable hamiltonicity. We use n to denote the number of vertices and use e to denote the number of edges in graph G. We use $\bar{e}$ to denote the number of edges in the complement of G. Suppose that G is a graph with $\bar{e} \leq n-4$ and $n \geq 4$. We prove that there are at least $n-2-\bar{e}$ mutually independent hamiltonian paths between any pair of distinct vertices of G except $n=5$ and $\bar{e}=1$. Assume that G is a graph with the degree sum of any two non-adjacent vertices being at least n+2. Let u and v be any two distinct vertices of G. We prove that there are $\deg_G(u) + \deg_G(v) - n$ mutually independent hamiltonian paths between u and v if $(u,v) \in E(G)$ and there are $\deg_G(u) + \deg_G(v) - n + 2$ mutually independent hamiltonian paths between u and v if otherwise.

The arrangement graph $A_{n,k}$ is a generalization of the star graph. It is more flexible in its size than the star graph. There are some results concerning hamiltonicity and pancyclicity of the arrangement graphs. We propose a new concept called panpositionable hamiltonicity. A hamiltonian graph G is panpositionable if for any two different vertices x and y of G and for any integer l satisfying $d(x,y) \leqq l \leqq |V(G)|-d(x,y)$, there exists a hamiltonian cycle C of G such that the relative distance between x and y on C is l. A graph G is panconnected if there exists a path of length l joining any two different vertices x and y with $d(x,y) \leqq l \leqq |V(G)|-1$. We show that $A_{n,k}$ is panpositionable hamiltonian and panconnected if $k \geqq 1$ and $n-k \geqq 2$.

Assume that *m* and *n* are positive even integers with n≥4. It is known that every honeycomb rectangular torus HReT(m,n) is a 3-regular bipartite graph. We prove that in any HReT(m,n), there exist three internally-disjoint spanning paths joining *x* and *y* whenever *x* and *y* belong to different partite sets. For any pair of vertices *x* and *y* in the same partite set, there exists a vertex z in the partite set not containing *x* and *y*, such that there exist three internally-disjoint spanning paths of G-{z} joining *x* and *y*. For any three vertices *x*, *y*, and *z* of the same partite set there exist three internally-disjoint spanning paths of G-{z} joining *x* and y if and only if n≥6 or m=2.

Keywords: hamiltonian, hamiltonian connected, hamiltonian path, panpositionable hamiltonian, panconnectivity, connectivity, arrangement graph, honeycomb torus.

Contents

List of Figures

Chapter 1

Introduction

The research about interconnection networks is important for parallel and distributed computer systems. The layouts of processors and links in distributed computer systems are usually represented by a network. The network topology is a crucial factor for an interconnection network since it determines the performance of the network and the distributed system. Many interconnection network topologies have been proposed in literature for the purpose of connecting a large number of processing elements and the designing of a parallel computing systems [1, 11, 13, 16, 19, 25, 30, 39, 40, 42].

There are several requirements in designing a good topology for an interconnection network, such as connectivity and hamiltonicity. The hamiltonian property is one of the major requirements in designing an interconnection network. The hamiltonian property is fundamental to the deadlock-free routing algorithms of distributed systems [33, 46]. A high-reliability network design can be based on constructing a hamiltonian cycle in an interconnection network. Many related works can be referred in recent research [14, 20, 24, 28, 42, 50].

In practice, the processors or links in a network may be failure. Thus the fault tolerant hamiltonian property and the fault tolerant hamiltonian connected property become an important issue on network topologies. Many results about the fault tolerant hamiltonicity have been proposed in literature [6, 20, 23, 24, 27, 29, 34, 38, 42, 45, 47]. For example, Hsieh et al. [20] and Hsu et al. [24] studied the fault tolerant hamiltonian property of the arrangement graph to enhance the reliability of the specific interconnection network.

Further attempts at hamiltonian problems led researches into the study of super-hamiltonian graphs, such as pancyclic graphs and panconnected graphs. The concept of pancyclic graphs is proposed by Bondy [5], and the concept of panconnected graphs is proposed by Alavi and Williamson [3]. There are some studies concerning panconnectivity and pancyclicity of some interconnection networks [7, 21, 49, 48]. For example, Yang et al. study the pancyclic problem on faulty Möbius cubes in [48].

In this thesis, we study some hamiltonian problems, such as mutually independent hamiltonicity, panpositionable hamiltonicity, and globally 3^*-connectivity. We say a set of hamiltonian paths are *mutually independent* if any two distinct paths in the set are independent. Similarly, a set of hamiltonian cycles are mutually independent if any two hamiltonian cycles in the set are independent. Some related studies can be referred in the literature [32, 41, 43]. We also propose a new concept called *panpositionable hamiltonicity*. A hamiltonian graph G is *panpositionable* if for any two different vertices x and y of G and for any integer l satisfying $d(x, y) \leq l \leq |V(G)| - d(x, y)$, there exists a hamiltonian cycle C of G such that the relative distance between x and y on C is l. One example, the alternating group graph is proved to be panpositionable hamiltonian [44]. If there exist three internally-disjoint paths joining x and y such that the three paths span all the vertices in G, we say that G is *globally* 3^**-connected*. In [4], Albert et al. first studied some cubic 3-connected graphs with this property. Such graphs are called *globally* 3^**-connected graphs*. In the following section, we give some definitions of basic terms used in our thesis.

1.1 Basic Terms

Computer network topologies are usually represented by graphs where vertices represent processors and edges represent links between processors. In this thesis, a network is represented as an undirected graph. For the graph definitions and notation, we follow [18].

Let $G = (V, E)$ be a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* is a sequence of vertices such that two consecutive vertices are adjacent. A path is represented by $\langle v_0, v_1, v_2, ..., v_n \rangle$. The *length* of a path P is the number of edges in P, denoted by $L(P)$. We sometimes write the path $\langle v_0, v_1, v_2, ..., v_k \rangle$ as $\langle v_0, P_1, v_i, v_{i+1}, ..., v_j, P_2, v_t, ..., v_k \rangle$, where P_1 is the path $\langle v_0, v_1, ..., v_i \rangle$ and P_2 is the path $\langle v_j, v_{j+1}, ..., v_t \rangle$. It is possible to write a path $\langle v_0, v_1, P, v_1, v_2, ..., v_k \rangle$ if $L(P) = 0$. We use $d_G(u, v)$, or simply $d(u, v)$ if there is no ambiguity, to denote the distance between u and v in a graph G, i.e., the length of shortest path joining u and v in G. We use $d_C(u, v)$ and $D_C(u, v)$ to denote the shorter and the longer distance between u and v on a cycle C of G respectively. It is possible that $D_C(u, v) = d_C(u, v)$ if the lengths of the two disjoint paths joining u and v in C are equal. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last one.

A *hamiltonian path* is a path such that its vertices are distinct and span V. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. A *hamiltonian cycle* is a cycle such that its vertices are distinct except for the first vertex and the last vertex and span V. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph $G = (V, E)$ is *1-edge hamiltonian* if $G - e$ is hamiltonian for any $e \in E$, and a graph $G = (V, E)$ is *1-node hamiltonian* if $G - v$ is hamiltonian for any $v \in V$. Obviously, any 1-edge hamiltonian graph is hamiltonian. A graph $G = (V, E)$ is *1-hamiltonian* if $G - f$ is hamiltonian for any $f \in E \cup V$.

1.2 Organization of the Thesis

In the follows, we describe the organization of this thesis. In Chapter 2, we discuss about the mutually independent hamiltonian paths on simple graphs under some conditions. We show that if $\bar{e} \leq n - 4$ and $n \geq 4$, there are at least $n - 2 - \bar{e}$ mutually independent hamiltonian paths between any pair of distinct vertices of G except $n = 5$ and $\bar{e} = 1$; here n is the number of vertices, e is the number of edges in a graph G, and $\bar{e}$ is the number of edges in the complement of G.

In Chapter 3, we study the panpositionable hamiltonicity of the arrangement graph $A_{n,k}$. We show that the arrangement graph is panpositionable hamiltonian for all $k \geq 1$ and $n - k \geq 2$, and we find that it is closely related to its panconnected and pancyclic properties. By applying our result, we can show that the arrangement graph is panconnected and pancyclic. We also derive some relationship between the panpositionable hamiltonicity and the other useful properties for a interconnection network.

In Chapter 4, we focus on the connectivity problem. Assume that m and n are positive even integers with $n \geq 4$. It is known that every honeycomb rectangular torus HReT(m, n) is a 3-regular bipartite graph. We prove that in any HReT(m, n), there exist three internally-disjoint spanning paths joining x and y whenever x and y belong to different partite sets. Moreover, for any pair of vertices x and y in the same partite set, there exists a vertex z in the partite set not containing x and y, such that there exist three internally-disjoint spanning paths of $G - \{z\}$ joining x and y. Furthermore, for any three vertices x, y and z of the same partite set there exist three internally-disjoint spanning paths of $G - \{z\}$ joining x and y if and only if $n \geq 6$ or $m = 2$. We present our conclusion in chapter 5.

Chapter 2

Mutually Independent Hamiltonian Property

As we discussed in the previous chapter, there are many studies on hamiltonian connected graphs. In this chapter, we are interested in another aspect of hamiltonian connected graphs. Let $P_1 = \langle v_1, v_2, v_3, \ldots, v_n \rangle$ and $P_2 = \langle u_1, u_2, u_3, \ldots, u_n \rangle$ be any two hamiltonian paths of G. We say that P_1 and P_2 are *independent* if $u_1 = v_1, u_n = v_n$, and $u_i \neq v_i$ for $1 < i < n$. We say a set of hamiltonian paths $P_1, P_2, \ldots, P_s$ of G are *mutually independent* if any two distinct paths in the set are independent. In [32], it is proved that there exist $(k-2)$ mutually independent hamiltonian paths between any two vertices from different bipartite sets of the star graph S_k if $k \geq 4$. The concept of mutually independent hamiltonian arises from the following application. If there are k pieces of data needed to be sent from u to v, and the data needed to be processed at every node (and the process takes times), then we want mutually independent hamiltonian paths so that there will be no waiting time at a processor. The existence of mutually independent hamiltonian paths is useful for communication algorithms. Motivated by this result, we begin the study on graphs with mutually independent hamiltonian paths between every pair of distinct vertices.

In this chapter, we use n to denote the number of vertices and use e to denote the number of edges in graph G. We use $\bar{e}$ to denote the number of edges in the complement of G. Suppose that G is a graph with $\bar{e} \leq n-4$ and $n \geq 4$. We will prove that there are at least $n-2-\bar{e}$ mutually independent hamiltonian paths between any pair of distinct vertices of G except $n=5$ and $\bar{e}=1$.

Moreover, assume that G is a graph with the degree sum of any two non-adjacent vertices being at least $n+2$. Let u and v be any two distinct vertices of G. We will prove that there are $\deg_G(u) + \deg_G(v) - n$ mutually independent hamiltonian paths between

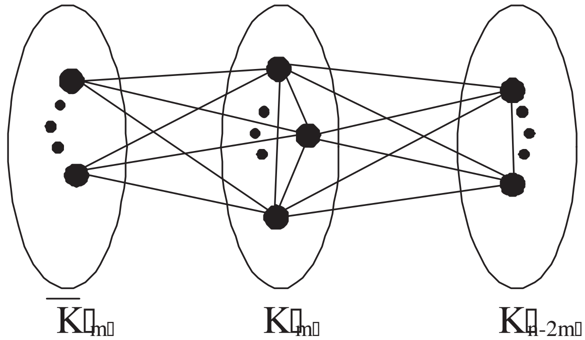

Figure 2.1: The graph $C_{m,n}$.

u and v if $(u,v) \in E(G)$ and there are $\deg_G(u) + \deg_G(v) - n + 2$ mutually independent hamiltonian paths between u and v if otherwise.

2.1 Preliminaries for Mutually Independent Hamiltonian Property

Throughout this chapter, we use $[i]$ to denote i **mod** $(n-2)$. Let G and H be two graphs. We use $G+H$ to denote the disjoint union of G and H. We use $G \vee H$ to denote the graph obtained from $G+H$ by joining each vertex of G to each vertex of H. For $1 \leq m < n/2$, let $C_{m,n}$ denote the graph $(\bar{K}_m + K_{n-2m}) \vee K_m$, see Figure 2.1. The following theorem is proved by Chvátal [10].

Theorem 1. *[10] Assume that G is a graph with $n \geq 3$ and $\bar{e} \leq n-3$. Then G is hamiltonian. Moreover, the only non-hamiltonian graphs with $\bar{e} \leq n-2$ are $C_{1,n}$ and $C_{2,5}$.*

The following lemma is obvious.

Lemma 1. *Let u and v be two distinct vertices of G. Then there are at most $\min\{\deg_G(u), \deg_G(v)\}$ mutually independent hamiltonian paths between u and v if $(u,v) \notin E(G)$, and there are at most $\min\{\deg_G(u), \deg_G(v)\} - 1$ mutually independent hamiltonian paths between u and v if $(u,v) \in E(G)$.*

Theorem 2. *Let n be a positive integer with $n \geq 3$. There are $n-2$ mutually independent hamiltonian paths between every two distinct vertices of K_n.*

Proof. Let s and t be two distinct vertices of K_n. We relabel the remaining $(n-2)$ vertices of K_n as $0, 1, 2, \ldots, n-3$. For $0 \leq i \leq n-3$, we set P_i as $\langle s, [i], [i+1], [i+$

$2], \ldots, [i+(n-3)], t\rangle$. It is easy to see that $P_0, P_1, \ldots, P_{n-3}$ form $(n-2)$ mutually independent hamiltonian paths joining s and t. □

Here are some theorems about the hamiltonian property.

Theorem 3. *[37] Assume that G is a graph with $\bar{e} \leq n-4$ and $n \geq 4$. Then G is hamiltonian connected.*

Theorem 4. *[37] Assume that G is a graph with the sum of any two distinct non-adjacent vertices being at least n with $n \geq 3$. Then G is hamiltonian.*

Theorem 5. *[17] Assume that G is a graph with the sum of any two distinct non-adjacent vertices being at least $n+1$ with $n \geq 3$. Then G is hamiltonian connected.*

2.2 Mutually Independent Hamiltonian Paths

In this section, we will prove that there are $\deg_G(u) + \deg_G(v) - n$ mutually independent hamiltonian paths between u and v if $(u, v) \in E(G)$ and there are $\deg_G(u) + \deg_G(v) - n + 2$ mutually independent hamiltonian paths between u and v if otherwise. The following result strengthens that of Theorem 3.

Lemma 2. *Assume that G is a graph with $n \geq 4$ and $\bar{e} = n-4$. Then there are two independent hamiltonian paths between any two distinct vertices of G except $n = 5$.*

Proof. For $n = 4$, G is isomorphic to K_4. By Theorem 2, there are two independent hamiltonian paths between any two distinct vertices of G. Assume that $n = 5$. Then G is isomorphic to $K_5 - \{f\}$ for some edge f. Without loss of generality, we assume that $V(G) = \{1, 2, 3, 4, 5\}$ and $f = (1, 2)$. It is easy to check that $P_1 = \langle 3, 2, 5, 1, 4\rangle$ and $P_2 = \langle 3, 1, 5, 2, 4\rangle$ are the only two hamiltonian paths between 3 and 4, but P_1 and P_2 are not independent.

Now, we assume that $n \geq 6$. Let s and t be any two distinct vertices of G. Let H be the subgraph of G induced by the remaining $(n-2)$ vertices of G. We have the following two cases:

Case 1: Suppose that H is hamiltonian. We can relabel the vertices of H with $\{0, 1, 2, \ldots, n-3\}$ so that $\langle 0, 1, 2, \ldots, n-3, 0\rangle$ forms a hamiltonian cycle of H. Let Q denote the set $\{i \mid (s, [i+1]) \in E(G) \text{ and } (i, t) \in E(G)\}$. Since $\bar{e} = n-4$, $|Q| \geq n-2-(n-4) = 2$. There are at least two elements in Q. Let q_1 and q_2 be the two elements in Q. For $j = 1, 2$, we set P_j as $\langle s, [q_j+1], [q_j+2], \ldots, [q_j], t\rangle$. Then P_1 and P_2 are two independent hamiltonian paths between s and t.

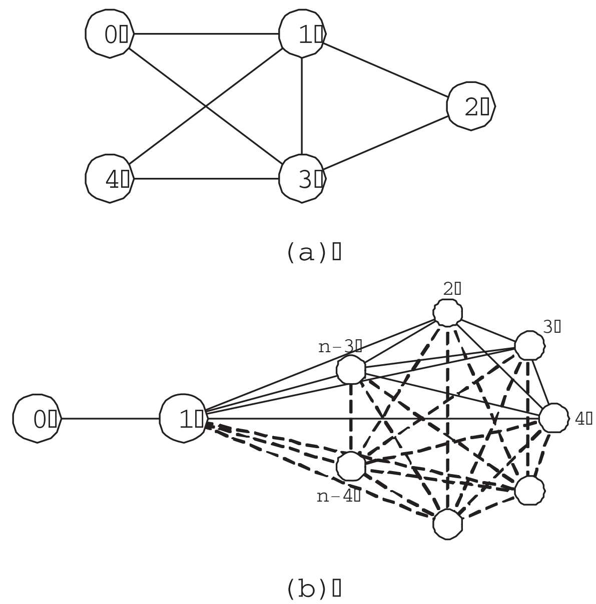

Figure 2.2: (a) The graph $C_{2,5}$; (b) The graph $C_{1,n-2}$.

Case 2: Suppose that H is non-hamiltonian. There are exactly $(n-2)$ vertices in H. By Theorem 1, there are exactly $(n-4)$ edges in the complement of H, and H is isomorphic to $C_{1,n-2}$ or $C_{2,5}$. Since $\bar{e} = n-4$, we know that $(s, v) \in E(G)$ and $(t, v) \in E(G)$ for every vertex v in H. We can construct two independent hamiltonian paths between s and t as following subcases:

Subcase 2.1: Suppose that H is isomorphic to $C_{2,5}$. We label the vertices of $C_{2,5}$ with $\{0, 1, 2, 3, 4\}$ as shown in Figure 2.2(a). Let $P_1 = \langle s, 0, 1, 2, 3, 4, t \rangle$ and $P_2 = \langle s, 2, 3, 4, 1, 0, t \rangle$. Then P_1 and P_2 form the required independent paths.

Subcase 2.2: Suppose that H is isomorphic to $C_{1,n-2}$. We label the vertices of $C_{1,n-2}$ with $\{0, 1, \ldots, n-3\}$ as shown in Figure 2.2(b). Let $P_1 = \langle s, 0, 1, 2, \ldots, n-3, t \rangle$ and $P_2 = \langle s, 2, 3, \ldots, n-3, 1, 0, t \rangle$. Then P_1 and P_2 form the required independent paths. □

We can further strengthen Theorem 3.

Theorem 6. *Assume that G is a graph with $n \geq 4$ and $\bar{e} \leq n-4$. Then there are $n-2-\bar{e}$ mutually independent hamiltonian paths between every two distinct vertices of G except $n = 5$ and $\bar{e} = 1$.*

Proof. With Lemma 2, the theorem holds for $\bar{e} = n-4$. Now, we need to prove that the theorem holds for $\bar{e} = n-4-r$ with $1 \leq r \leq n-4$. Let s and t be two distinct vertices of G. Let H be the subgraph of G induced by the remaining $(n-2)$ vertices of G. Then there are exactly $(n-2)$ vertices in H, and there are at most $n-4-r$ edges in the complement of H with $1 \le r \le n-4$. By Theorem 1, H is hamiltonian. We can label the vertices of H with $\{0,1,2,\ldots,n-3\}$ so that $\langle 0,1,2,\ldots,n-3,0\rangle$ forms a hamiltonian cycle of H. Let Q denote the set $\{i \mid (s,[i+1]) \in E(G) \text{ and } (t,i) \in E(G)\}$. Since $\bar{e} = n-4-r$ with $1 \le r \le n-4$, we know that $|Q| \ge n-2-(n-4-r) = n-2-\bar{e}$ for $1 \le r \le n-4$. Hence, there are at least $n-2-\bar{e}$ elements in Q. Let $q_1, q_2, \ldots, q_{n-2-\bar{e}}$ be the elements in Q. For $j = 1,2,\ldots,n-2-\bar{e}$, we set $P_j = \langle s,[q_j+1],[q_j+2],\ldots,[q_j],t\rangle$. It is not difficult to see that $P_1, P_2, \ldots, P_{n-2-\bar{e}}$ are mutually independent paths between s and t. □

The following result, in a sense, generalizes that of Theorem 5.

Theorem 7. *Assume that G is a graph such that $\deg_G(x) + \deg_G(y) \ge n+2$ for any two vertices x and y with $(x,y) \notin E(G)$. Let u and v be two distinct vertices of G. Then there are $\deg_G(u) + \deg_G(v) - n$ mutually independent hamiltonian paths between u and v if $(u,v) \in E(G)$, and there are $\deg_G(u) + \deg_G(v) - n + 2$ mutually independent hamiltonian paths between u and v if $(u,v) \notin E(G)$.*

Proof. Let s and t be two distinct vertices of G, and H be the subgraph of G induced by the remaining $(n-2)$ vertices of G. Let u' and v' be any two distinct vertices in H. We have $\deg_H(u') + \deg_H(v') \ge n+2-4 = n-2 = |V(H)|$. By Theorem 4, H is hamiltonian. We can label the vertices of H with $\{0,1,\ldots,n-3\}$, so that $\langle 0,1,2,\ldots,n-3,0\rangle$ forms a hamiltonian cycle of H. Let S denote the set $\{i \mid (s,[i+1]) \in E(G)\}$ and T denote the set $\{i \mid (i,t) \in E(G)\}$. Clearly, $|S \cup T| \le n-2$. We have the following two cases:

Case 1: $(s,t) \in E(G)$. Suppose that $|S \cap T| \le \deg_G(s) + \deg_G(t) - n - 1$. We have $\deg_G(s) + \deg_G(t) - 2 = |S| + |T| = |S \cup T| + |S \cap T| \le \deg_G(s) + \deg_G(t) - n - 1 + n - 2$. This is a contradiction. Thus, there are at least $w = \deg_G(s) + \deg_G(t) - n$ elements in $S \cap T$. Let $q_1, q_2, \ldots, q_w$ be the elements in $S \cap T$. For $j = 1,2,\ldots,w$, we set $P_j = \langle s,[q_j+1],[q_j+2],\ldots,[q_j],t\rangle$. So $P_1, P_2, \ldots, P_w$ are mutually independent paths between s and t.

Case 2: $(s,t) \notin E(G)$. Assume that $|S \cap T| \le \deg_G(s) + \deg_G(t) - n + 2 - 1$. We obtain $\deg_G(s) + \deg_G(t) = |S| + |T| = |S \cup T| + |S \cap T| \le \deg_G(s) + \deg_G(t) - n + 2 - 1 + n - 2$. This is a contradiction. Thus, there are at least $w = \deg_G(s) + \deg_G(t) - n + 2$ elements in $S \cap T$. Let $q_1, q_2, \ldots, q_w$ be the elements in $S \cap T$. For $j = 1,2,\ldots,w$, we set $P_j = \langle s,[q_j+1],[q_j+2],\ldots,[q_j],t\rangle$, and $P_1, P_2, \ldots, P_w$ are mutually independent paths between s and t. □

Example. Let G be the graph $(K_1 \cup K_{n-d-1}) \vee K_d$ where d is an integer with $4 \le$

$d < n-1$. So $\bar{e} = n-1-d \le n-4$. Let x be the vertex corresponding to K_1, y be an arbitrary vertex in K_d, and z be a vertex in K_{n-d-1}. Then $\deg_G(x) = d$, $\deg_G(y) = n-1$, $\deg_G(z) = n-2$, $(x, y) \in E(G)$, $(y, z) \in E(G)$, and $(x, z) \notin E(G)$. By Theorem 6, there are $n-2-\bar{e} = n-2-(n-1-d) = d-1$ mutually independent hamiltonian paths between any two distinct vertices of G. By Lemma 1, there are at most $d-1$ mutually independent hamiltonian paths between x and y. Hence, the result in Theorem 6 is optimal.

Consider the same example as above, it is easy to check that any two vertices u and v in G, $\deg_G(u) + \deg_G(v) \ge n+2$. Let x and y be the same vertices as described above, by Theorem 7, there are $\deg_G(x) + \deg_G(y) - n = d + (n-1) - n = d-1$ mutually independent hamiltonian paths between x and y. By Lemma 1, there are at most $d-1$ mutually independent hamiltonian paths between x and y. Hence, the result in Theorem 7 is also optimal.

Combining Theorems 5 and 7, we have the following Corollary.

Corollary 1. *Let r be a positive integer. Assume that G is a graph such that $\deg_G(x) + \deg_G(y) \ge n + r$ for any two distinct vertices x and y. Then there are at least r mutually independent hamiltonian paths between any two distinct vertices of G.*

Chapter 3

Panpositionable Hamiltonian Property

In this chapter, we will introduce the new concept called panpositionable hamiltonicity by using the arrangement graph as an example. We will show that the arrangement graph is panpositionable hamiltonian and panconnected. Moreover, we will compare the difference between the three concepts, panpositionable hamiltonicity, panconnectivity and pancyclicity.

3.1 Panpositionable Hamiltonicity, Panconnectivity and Pancyclicity

Further attempts at hamiltonian problems led researches into the study of super-hamiltonian graphs, such as panconnected graphs and pancyclic graphs. The definition of panconnectivity and pancyclicity is described as follows. A graph G is *pancyclic* if it contains a cycle of length l for each l satisfying $3 \leq l \leq |V(G)|$. The concept of pancyclic graphs is proposed by Bondy [5]. A graph G is *panconnected* if there exists a path of length l joining any two different vertices x and y with $d(x, y) \leq l \leq |V(G)| - 1$. The concept of panconnected graphs is proposed by Alavi and Williamson [3]. There are some studies concerning panconnectivity and pancyclicity of some interconnection network [7, 21, 49].

We propose a new concept called *panpositionable hamiltonicity*. A hamiltonian graph G is *panpositionable* if for any two different vertices x and y of G and for any integer l satisfying $d(x, y) \leq l \leq |V(G)| - d(x, y)$, there exists a hamiltonian cycle C of G such that the relative distance between x and y on C is l; more precisely, $d_C(x, y) = l$ if $l \leq \lfloor \frac{|V(G)|}{2} \rfloor$ or $D_C(x, y) = l$ if $l > \frac{|V(G)|}{2}$. Given a hamiltonian cycle C, if $d_C(x, y) = l$, we have $D_C(x,y) = |V(G)| - d_C(x,y)$. Therefore, a graph is panpositionable hamiltonian if for any integer l with $d(x,y) \leq l \leq \frac{|V(G)|}{2}$, there exists a hamiltonian cycle C of G with $d_C(x,y) = l$.

Similar to the importance of hamiltonicity for the communication between processors in an interconnection network, panpositionable hamiltonicity allows more flexible communication in a hamiltonian network. The panpositionable hamiltonian property inherits the hamiltonian property and advances it further. We first give an example to show that a panconnected graph G is not necessarily panpositionable hamiltonian.

Let $n, s_1, s_2, \ldots, s_r$ be integers with $1 \leq s_1 < s_2 < \cdots < s_r$. The circulant graph $C(n; s_1, s_2, \ldots, s_r)$ is a graph with vertex set $\{0, 1, \ldots, n-1\}$. Two vertices i and j are adjacent if and only if $i - j = \pm s_k \pmod{n}$ for some k where $1 \leq k \leq r$. We can check that $C(n; 1, 2)$ is panconnected by brute force for $n \in \{5, 6, 7, 8, 9, 10\}$. Now we will prove that $C(10; 1, 2)$ is not panpositionable hamiltonian.

Theorem 8. *The circulant graph $C(n; 1, 2)$ is not panpositionable hamiltonian for $n = 10$.*

Proof. Figure 3.1 shows the structure of $C(10; 1, 2)$. Consider vertex 0 and vertex 2, with $d(0,2) = 1$. We prove by contradiction that $C(10; 1, 2)$ does not contain a hamiltonian cycle HC with $d_{HC}(0,2) = 5$. Suppose to the contrary that HC is a hamiltonian cycle of $C(10; 1, 2)$ with $d_{HC}(0,2) = 5$. There are three possible paths, $P_1 = \langle 0, 8, 9, 1, 3, 2 \rangle$, $P_2 = \langle 0, 9, 1, 3, 4, 2 \rangle$ and $P_3 = \langle 0, 1, 3, 5, 4, 2 \rangle$, of length 5 joining vertex 0 and vertex 2. If HC contains P_1, then the edges $(0,1), (0,2), (0,9)$ can not belong to HC. If HC contains P_2 or P_3, then the edges $(2,0), (2,1), (2,3)$ can not belong to HC. Hence for $n = 10$, there does not exist any hamiltonian cycle in $C(10; 1, 2)$ such that the distance on the cycle between vertex 0 and vertex 2 is 5. So $C(10; 1, 2)$ is not panpositionable hamiltonian.

□

In fact, the circulant graph $C(n; 1, 2)$ is panconnected for every $n \geq 5$, but it is not panpositionable hamiltonian for some values of n. Therefore, the panpositionable hamiltonian property is a stronger property for an interconnection network. In the following sections, we will try to find the panpositionable hamiltonicity of the arrangement graphs.

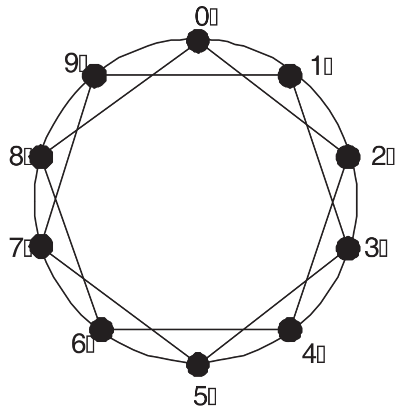

Figure 3.1: The circulant graph $C(10; 1, 2)$.

3.2 The Arrangement Graphs

3.2.1 The Basic Properties of the Arrangement Graphs

The arrangement graph [13] was proposed by Day and Tripathi as a generalization of the star graph. It is more flexible in its size than the star graph. Let n and k be two positive integers with $n > k$. And, let $\langle n \rangle$ and $\langle k \rangle$ denote the sets $\{1, 2, ..., n\}$ and $\{1, 2, ..., k\}$, respectively. Then, the vertex set of the arrangement graph $A_{n,k}$, $V(A_{n,k}) = \{p \mid p = p_1p_2...p_k$ with $p_i \in \langle n \rangle$ for $1 \leq i \leq k$ and $p_i \neq p_j$ if $i \neq j\}$ and the edge set of $A_{n,k}$, $E(A_{n,k}) = \{(p, q) \mid p, q \in V(A_{n,k}), p$ and q differ in exactly one position $\}$. Figure 3.2 illustrates $A_{4,2}$. By the definition of the arrangement graph, $A_{n,k}$ is a regular graph of degree $k(n-k)$ with $\frac{n!}{(n-k)!}$ vertices. The diameter of $A_{n,k}$ is $\lfloor \frac{3k}{2} \rfloor$. The arrangement graph $A_{n,1}$ is isomorphic to the complete graph K_n, and $A_{n,n-1}$ is isomorphic to the n-dimensional star graph. Moreover, $A_{n,k}$ is vertex symmetric and edge symmetric [13].

Let i and j be two positive integers with $1 \leq i, j \leq n$. And, let $V(A_{n,k}^{(j:i)}) = \{p \mid p = p_1p_2...p_k$ and $p_j = i\}$. It is the set of all vertices with the j-th position being i. For a fixed position j, $\{V(A_{n,k}^{(j:i)}) \mid 1 \leq i \leq n\}$ forms a partition of $V(A_{n,k})$. Let $A_{n,k}^{(j:i)}$ denote the subgraph of $A_{n,k}$ induced by $V(A_{n,k}^{(j:i)})$. It is easy to see that each $A_{n,k}^{(j:i)}$ is isomorphic to $A_{n-1,k-1}$. Thus, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$. Each $A_{n,k}^{(j:i)}$ represents a *subcomponent* of $A_{n,k}$, and we say that $A_{n,k}$ is decomposed into subcomponents according to the j-th position. Let I be a subset of $\{1, 2, ..., n\}$. We use $A_{n,k}^{(j:I)}$ to denote the subgraph of $A_{n,k}$ induced by $\bigcup_{i \in I} V(A_{n,k}^{(j:i)})$. $A_{n,k}^{(j:I)}$ is called an *incomplete* arrangement graph if $|I| < n$. We observe that each $A_{n,k}^{(j:i)}$ can be recursively decomposed

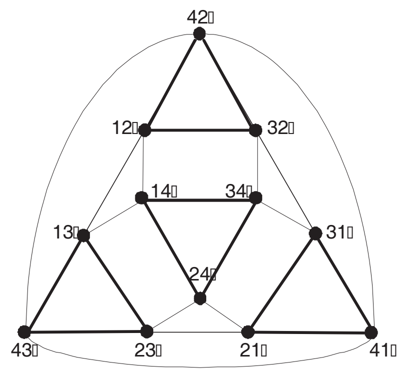

Figure 3.2: The arrangement graph $A_{4,2}$.

into its smaller subcomponents. For simplicity, if there is no ambiguity, we shall concentrate on the last position, and we use $A^i_{n,k}$ and $A^I_{n,k}$ to denote $A^{(k:i)}_{n,k}$ and $A^{(k:I)}_{n,k}$ respectively, where k is the last position, and $E^{i,j}$ to denote the set of edges between $A^i_{n,k}$ and $A^j_{n,k}$. Let F be a faulty set which may include faulty edges, faulty vertices, or both. The *good edge set* $GE^{i,j}(F)$ is the set of edges $(u, v) \in E^{i,j}$ such that $\{u, v, (u, v)\} \cap F = \emptyset$. We need some basic properties of the arrangement graph. The following proposition follows directly from the definition of the arrangement graphs.

Proposition 1. *Let n, k be two positive integers with $n, k \geq 2$, and let i and j be two distinct elements of $\langle n \rangle$. Suppose that H is one subcomponent of $A^j_{n,k}$ with the $(k-1)$-th position being h and the k-th position being j for some $h \in \langle n \rangle - \{j\}$. Then $|E^{i,j}| = \frac{(n-2)!}{(n-k-1)!}$, and the number of edges between $A^i_{n,k}$ and H is $\frac{(n-3)!}{(n-k-1)!}$. Moreover, if (u, v) and (u', v') are distinct edges in $E^{i,j}$, then $\{u, v\} \bigcap \{u', v'\} = \emptyset$, and $(u, u') \in E(A^i_{n,k})$ if and only if $(v, v') \in E(A^j_{n,k})$.*

Let $u \in V(A^i_{n,k})$ for some $i \in \langle n \rangle$. We say that v is a *neighbor* of u if v is adjacent to u. Let I be a subset of $\{1, 2, ..., n\}$, and we use $N^I(u)$ to denote the set of all neighbors of u which are in $A^I_{n,k}$. Particularly, we use $N^*(u)$ and $N^i(u)$ as an abbreviation of $N^{\langle n \rangle - \{i\}}(u)$ and $N^{\{i\}}(u)$ respectively. We call vertices in $N^*(u)$ the *outer neighbors* of u. It follows from the definitions, $|N^i(u)| = (k-1)(n-k)$ and $|N^*(u)| = (n-k)$. We say that vertex u is adjacent to subcomponent $A^j_{n,k}$ if u has an outer neighbor in $A^j_{n,k}$. Then, we define the *adjacent subcomponent* $AS(u)$ of u as $\{j \mid u$ is adjacent to $A^j_{n,k}\}$. We have the following proposition:

Proposition 2. *Suppose that $k \geq 2$, $n - k \geq 2$, and $i \in \langle n \rangle$. Let u and v be two distinct vertices in $A^i_{n,k}$.*

(a) If $d(u, v) = 1$, then $|AS(u) \cap AS(v)| = n - k - 1$.

(b) If $d(u,v) \leq 2$, then $AS(u) \neq AS(v)$.

Proof. Let $u = u_1u_2...u_k$, $v = v_1v_2...v_k$, and $u_k = v_k = i$. If $d(u,v) = 1$, we have $u_s \neq v_s$ for some $s \in \langle k-1 \rangle$, and $u_t = v_t$ for all $t \neq s$. Then, $AS(u) = \langle n \rangle - \{u_1, u_2, ..., u_s, ..., u_k\}$ and $AS(v) = \langle n \rangle - \{v_1, v_2, ..., v_s, ..., v_k\}$. Thus $AS(u) \cap AS(v) = \langle n \rangle - \{u_1, u_2, ..., u_s, ..., u_k, v_s\}$ and $|AS(u) \cap AS(v)| = n-(k+1) = n-k-1$. Since $u_s \neq v_s$, $v_s \in AS(u)$ but $v_s \notin AS(v)$.

If $d(u,v) = 2$, there exists a vertex $w \in V(A^i_{n,k})$ such that $d(u,w) = d(w,v) = 1$. Let $w = w_1w_2...w_k$. And, let s' and t' be two indices such that $w_{s'} \neq u_{s'}$ and $v_{t'} \neq w_{t'}$. Clearly, $s' \neq t'$ or $d(u,v) = 1$. Hence $w_{s'}$ is not in $\{u_1, u_2, ..., u_k\}$ but in $\{v_1, v_2, ..., v_k\}$. Thus $w_{s'} \in AS(u)$ but $w_{s'} \notin AS(v)$. Hence, the statement follows. □

Day and Tripathi [13] presented a shortest path routing algorithm for the arrangement graph, and gave some characterizations of the minimum length path between two arbitrary vertices in $A_{n,k}$. We can derive the following lemma directly from their routing algorithm.

Lemma 3. *Let $u = u_1u_2...u_k$ and $v = v_1v_2...v_k$ be two vertices in $A_{n,k}$. There exists a way of decomposing $A_{n,k}$ into subcomponents such that one of the following three cases holds.*

(a) If $u_x = v_x = i$ for some position $x \in \langle k \rangle$ and $i \in \langle n \rangle$, we decompose $A_{n,k}$ into subcomponents according to the x-th position. Then u and v belong to the same subcomponent and $u, v \in V(A^{(x:i)}_{n,k})$. Moreover, a shortest path from u to v in $A_{n,k}$ is completely contained in $A^{(x:i)}_{n,k}$

(b) If $u_x \neq v_x$ for every $x \in \langle k \rangle$ and $\{u_1, u_2, ..., u_k\} \neq \{v_1, v_2, ..., v_k\}$, there exists a position $u_y \notin \{v_1, v_2, ..., v_k\}$ for some $y \in \langle k \rangle$, say the y-th position. We decompose $A_{n,k}$ into subcomponents according to the y-th position, then u and v belong to different subcomponents, say $u \in V(A^{(y:i)}_{n,k})$ and $v \in V(A^{(y:j)}_{n,k})$ for some $i \neq j \in \langle n \rangle$. Moreover, a minimum length path connecting u and v has the form $\langle u, P, u', v \rangle$, in which $u' \in V(A^{(y:i)}_{n,k})$, and P is a path completely contained in $A^{(y:i)}_{n,k}$.

(c) If $u_x \neq v_x$ for every $x \in \langle k \rangle$ and $\{u_1, u_2, ..., u_k\} = \{v_1, v_2, ..., v_k\}$, decomposing $A_{n,k}$ into subcomponents according to any position, say y-th position, $y \in \langle k \rangle$, then u and v belong to different subcomponents, say $u \in V(A^{(y:i)}_{n,k})$ and $v \in V(A^{(y:j)}_{n,k})$ for some $i \neq j \in \langle n \rangle$. Moreover, a minimum length path connecting u and v has the form $\langle u, P, u', v', v \rangle$, in which $u' \in V(A^{(y:i)}_{n,k})$, $v' \in V(A^{(y:j)}_{n,k})$, and P is a path completely contained in $A^{(y:i)}_{n,k}$.

Example. Suppose that u and v are two vertices in $A_{7,5}$. If $u = 12345$ and $v = 13452$, then $u, v \in V(A^{(1:1)}_{7,5})$. A minimum length path connecting u and v is $\langle 12345, 12\underline{6}45, 13\underline{6}45, 1364\underline{2}, 136\underline{5}2, 13\underline{4}52 \rangle$ which is completely contained in $A^{(1:1)}_{7,5}$, and case (a) holds. If $u =$

12345 and $v = 26453$, then $u \in V(A_{7,5}^{(1:1)})$ and $v \in V(A_{7,5}^{(1:2)})$. A minimum length path connecting u and v is $\langle 12345, 1234\underline{6}, 123\underline{5}6, 12\underline{4}56, 1245\underline{3}, 1\underline{6}453, \underline{2}6453 \rangle$, and case (b) holds. If $u = 12345$ and $v = 23451$, then $u \in V(A_{7,5}^{(1:1)})$ and $v \in V(A_{7,5}^{(1:2)})$. A minimum length path connecting u and v is $\langle 12345, 1234\underline{6}, 123\underline{5}6, 12\underline{4}56, 1\underline{3}456, \underline{2}3456, 2345\underline{1} \rangle$, and case (c) holds.

3.2.2 The Hamiltonicity of the Arrangement Graphs

Hsu et al. studied the fault hamiltonicity and fault hamiltonian connectivity of the arrangement graphs in [24]. Some results are listed as follows.

Theorem 9. *[24] Let n and k be two positive integers with $n - k \geq 2$. Then $A_{n,k}$ is $k(n-k)-2$ fault tolerant hamiltonian and $k(n-k)-3$ fault tolerant hamiltonian connected.*

The above theorem states that with up to $k(n-k)-2$ faulty edges and faulty vertices $A_{n,k}$ still has a hamiltonian cycle, and with up to $k(n-k)-3$ faulty edges and faulty vertices $A_{n,k}$ is still hamiltonian connected.

Lemma 4. *[24] Suppose that*

1. *$k \geq 3$ and $n - k \geq 2$,*
2. *t is a fixed position with $1 \leq t \leq k$,*
3. *$I \subseteq \langle n \rangle$ with $|I| \geq 2$,*
4. *$F \subseteq V(A_{n,k}) \cup E(A_{n,k})$, and*
5. *$A_{n,k}^{(t:l)} - F$ is hamiltonian connected for each $l \in I$ and $|F(A_{n,k}^{(t:I)})| \leq k(n-k)-3$.*

Then, for any $x \in V(A_{n,k}^{(t:i)})$ and $y \in V(A_{n,k}^{(t:j)})$ with $i \neq j \in I$, there is a hamiltonian path of $A_{n,k}^{(t:I)} - F$ joining x and y.

The following lemma considers the hamiltonian connectivity of the incomplete arrangement graphs $A_{n,2}$. The lemma states that for any two vertices x and y in different subcomponents of the incomplete arrangement graphs $A_{n,2}$, there exists a hamiltonian path joining them if $n \geq 5$. The result holds even when there is one faulty vertex or one faulty edge if $n \geq 6$.

Lemma 5. *Suppose that $n \geq 5$, t is a fixed position with $1 \leq t \leq 2$, $F \subseteq V(A_{n,2})$, and $I \subseteq \langle n \rangle$ with $|I| \geq 2$.*

(a) If $n \geq 5$, then for any $x \in V(A_{n,2}^{(t:i)})$ and $y \in V(A_{n,2}^{(t:j)})$ with $i \neq j \in I$, there is a hamiltonian path of $A_{n,2}^{(t:I)}$ joining x and y.

(b) If $n \geq 6$ and $|F| \leq 1$, then for any $x \in V(A_{n,2}^{(t:i)})$ and $y \in V(A_{n,2}^{(t:j)})$ with $i \neq j \in I$, there is a hamiltonian path of $A_{n,2}^{(t:I)} - F$ joining x and y.

Proof. Because of the symmetric property of $A_{n,2}$, without loss of generality, we may assume that $t = 2$. By Proposition 1, $|E^{i,j}| = \frac{(n-2)!}{(n-2-1)!} = n-2 \geq 3$ if $n \geq 5$, and $n-2 \geq 4$ if $n \geq 6$ for every $i, j \in I$, and $\{u, v\} \bigcap \{u', v'\} = \emptyset$ if (u, v) and (u', v') are distinct edges in $E^{i,j}$. Hence the number of good edge $|GE^{i,j}| \geq 3$ if $n \geq 5$, or $n \geq 6$ with $|F| \leq 1$. We then prove this lemma by induction on $|I|$. Suppose that $|I| = 2$, and $I = \{i, j\}$ for some i, j. Since $|GE^{i,j}| \geq 3$, there exists an edge $(u, v) \in GE^{i,j}$ such that $u \neq x \in V(A_{n,2}^{i})$ and $v \neq y \in V(A_{n,2}^{j})$. By Theorem 9, for each $l \in I$, $A_{n,2}^{l} - F$ is hamiltonian connected if $|F| \leq 1$. There is a hamiltonian path P_1 of $A_{n,2}^{i} - F$ from x to u and a hamiltonian path P_2 of $A_{n,2}^{j} - F$ from v to y. Thus $\langle x, P_1, u, v, P_2, y \rangle$ forms a hamiltonian path of $A_{n,2}^{I} - F$ from x to y.

Assume that the statement is true for all I' with $2 \leq |I'| < |I|$. There exists an $i' \in I$ with $i' \neq i, j$. Since $|GE^{i',j}| \geq 3$, we can find an edge $(u, v) \in GE^{i',j}$ with $u \in V(A_{n,2}^{i'})$ and $v \neq y \in V(A_{n,2}^{j})$. Then there is a hamiltonian path P_1 of $A_{n,2}^{I-\{j\}} - F$ from x to u and a hamiltonian path P_2 of $A_{n,2}^{j} - F$ from v to y. Thus $\langle x, P_1, u, v, P_2, y \rangle$ forms a hamiltonian path of $A_{n,2}^{I} - F$ from x to y. Hence the lemma follows. □

3.2.3 The Disjoint Paths in an Arrangement Graphs

In this subsection, we will show that there exist two vertex disjoint paths spanning all the vertices in an incomplete arrangement graph with one vertex fault tolerant.

Lemma 6. *Suppose that*

1. $k \geq 3$, $n - k \geq 2$,
2. $I \subseteq \langle n \rangle$ *with* $|I| \geq 2$,
3. $F \subseteq V(A_{n,k}^{I})$ *with* $|F| \leq 1$, *and*
4. $x_1 \in V(A_{n,k}^{i_1}) - F$ *and* $x_2 \in V(A_{n,k}^{i_2}) - F$ *with* $i_1 \neq i_2 \in I$.

Then, for any pair of distinct vertices $\{y_1, y_2\}$ in $V(A_{n,k}^{I}) - F$, there exist two disjoint paths, one joining x_1 and y_i for some $i \in \{1, 2\}$, and the other joining x_2 and y_j with $i \neq j$, such that these two paths span all the vertices in $A_{n,k}^{I} - F$.

Proof. Let $i_1, i_2, ..., i_{|I|}$ be $|I|$ distinct indices of $\langle n \rangle$. We prove this lemma by finding two disjoint paths P_1 and P_2 in $A^I_{n,k} - F$ such that P_1 joins x_1 and y_i, and P_2 joins x_2 and y_j with $i \neq j$. Moreover, P_1 and P_2 span all the vertices in $A^I_{n,k} - F$. According to the location of y_1 and y_2, we have the following cases:

Case 1: Suppose that y_1 and y_2 are located in different subcomponents.

Subcase 1.1: Suppose that x_1, x_2, y_i and y_j are located in four different subcomponents. $y_i \in V(A^{i_3}_{n,k})$ and $y_j \in V(A^{i_4}_{n,k})$ with $|I| \geq 4$. See Figure 3.3(a) for an illustration. By Lemma 4, we can find a hamiltonian path P_1 from x_1 to y_i in $A^{\{i_1,i_3\}}_{n,k} - F$. Similarly, we can find a hamiltonian path P_2 from x_2 to y_j in $A^{I-\{i_1,i_3\}}_{n,k} - F$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A^I_{n,k} - F$.

Subcase 1.2: Suppose that one of y_1, y_2 and one of x_1, x_2 are located in the same subcomponent. Without loss of generality, we may assume that x_1 and y_i are located in the same subcomponent, and x_2 and y_j are located in different subcomponents. $y_i \in V(A^{i_1}_{n,k})$ and $y_j \in V(A^{i_3}_{n,k})$ with $|I| \geq 3$. See Figure 3.3(b) for an illustration. By Theorem 9, since $A^{i_1}_{n,k} - F$ is hamiltonian connected, we can find a hamiltonian path P_1 from x_1 to y_i in $A^{i_1}_{n,k} - F$. By Lemma 4, we can find a hamiltonian path P_2 from x_2 to y_j in $A^{I-\{i_1\}}_{n,k} - F$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A^I_{n,k} - F$.

Subcase 1.3: Suppose that x_1 and y_i are located in the same subcomponent for some $i \in \{1, 2\}$, and x_2 and y_j are located in the same subcomponent with $i \neq j$. $y_i \in V(A^{i_1}_{n,k})$ and $y_j \in V(A^{i_2}_{n,k})$ with $|I| \geq 2$. See Figure 3.3(c) for an illustration. Without loss of generality, we may assume that $i = 1$ and $j = 2$. By Theorem 9, since $A^{i_1}_{n,k} - F$ is hamiltonian connected, we can find a hamiltonian path P_1 from y_1 to x_1 in $A^{i_1}_{n,k} - F$. If $|I| \geq 3$, since $|N^*(y_2)| > 2$, we can find an edge $(y_2, y_2') \in E^{i_2,j}$ such that $y_2' \in V(A^j_{n,k})$ for some $j \in I - \{i_1, i_2\}$. By Lemma 4, we can find a hamiltonian path P_2' from y_2' to x_2 in $A^{I-\{i_1\}}_{n,k} - \{y_2\} \cup F$. Let $P_2 = \langle y_2, y_2', P_2', x_2 \rangle$. If $|I| = 2$, by Theorem 9, there is a hamiltonian path P_2' from y_2 to b_2 in $A^{i_2}_{n,k} - F$. Let $P_2 = \langle y_2, P_2', x_2 \rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A^I_{n,k} - F$.

Case 2: Suppose that y_i and y_j are located in the same subcomponent.

Subcase 2.1: Suppose that $y_1, y_2 \in V(A^{i_1}_{n,k})$ or $y_1, y_2 \in V(A^{i_2}_{n,k})$ with $|I| \geq 2$. See Figure 3.3(d) for an illustration. Without loss of generality, we consider the former case and assume that $i = 1$ and $j = 2$. By Theorem 9, $A^{i_1}_{n,k} - (\{y_2\} \cup F)$ is hamiltonian connected, hence we can find a hamiltonian path P_1 from y_1 to x_1 in $A^{i_1}_{n,k} - \{y_2\} \cup F$. If $|I| \geq 3$, since $|N^*(y_2)| > 2$, we can find an edge $(y_2, y_2') \in E^{i_1,j}$ such that $y_2' \in V(A^j_{n,k})$ for some $j \in$

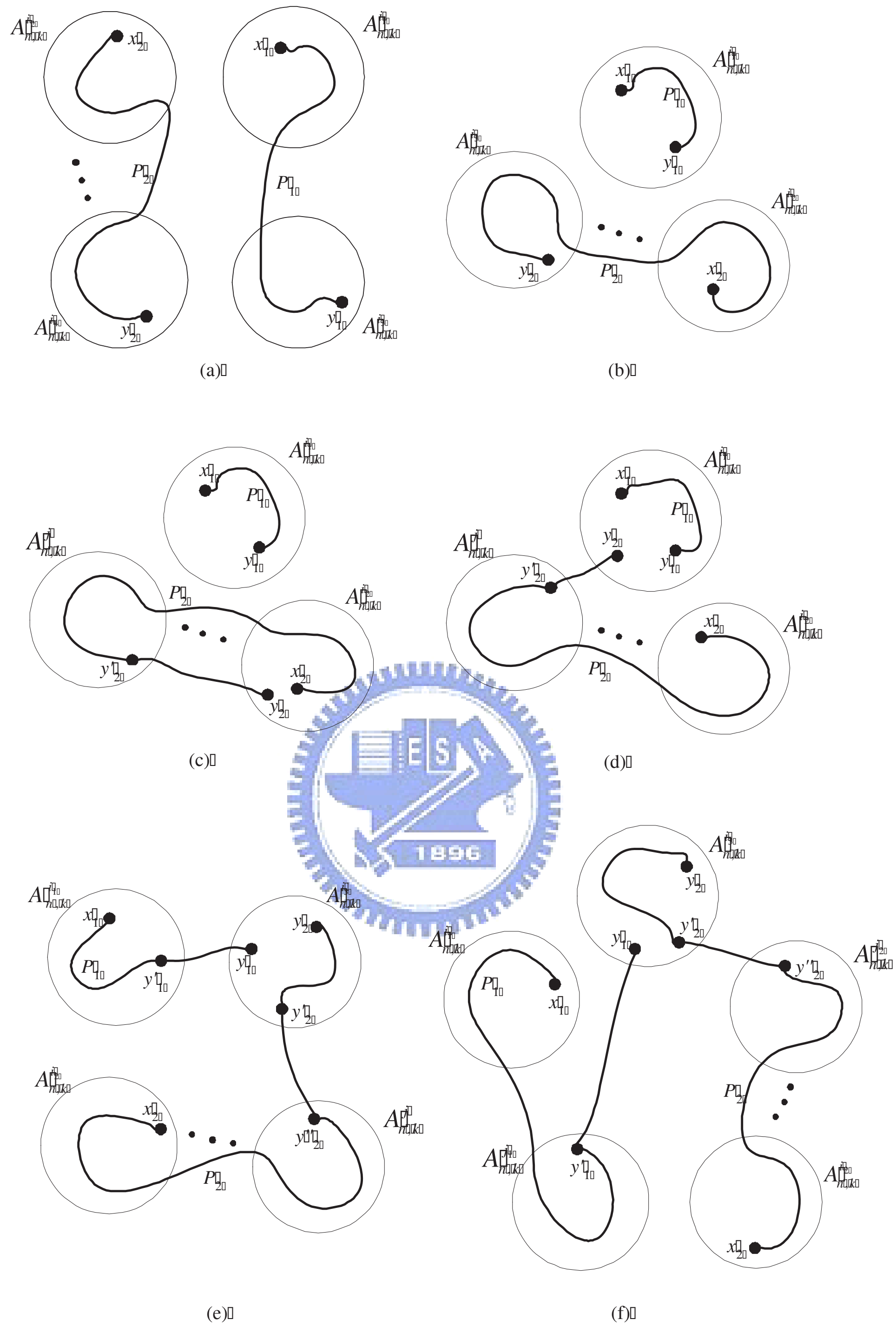

Figure 3.3: Illustrations for Lemma 6. Notice that $|F| \leq 1$ in each $A^I_{n,k}$.

$I - \{i_1, i_2\}$. By Lemma 4, we can find a hamiltonian path $P_2^{'}$ from $y_2^{'}$ to x_2 in $A_{n,k}^{I-\{i_1\}} - F$. If $|I| = 2$, there exists an edge $(y_2, y_2^{'}) \in E^{i_1,i_2}$ such that $y_2^{'} \in V(A_{n,k}^{i_2})$. By Theorem 9, there is a hamiltonian path $P_2^{'}$ from $y_2^{'}$ to x_2 in $A_{n,k}^{i_2} - F$. Let $P_2 = \langle y_2, y_2^{'}, P_2^{'}, x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A_{n,k}^{I} - F$.

Subcase 2.2: Suppose that $y_1, y_2 \in V(A_{n,k}^{i_3})$. Without loss of generality, we consider two subcases:

Subcase 2.2.1: Suppose that there exists some $i_1 \in AS(y_1)$ for $i \in \{1, 2\}$ with $|I| \geq 3$. Without loss of generality, we may assume that $i = 1$. See Figure 3.3(e) for an illustration. Since $x_1 \in AS(y_1)$, we can find an edge $(y_1, y_1^{'}) \in E^{i_1,i_3}$ such that $y_1^{'} \in V(A_{n,k}^{i_1})$ and $x_1 \neq y_1^{'}$. By Theorem 9, we can find a hamiltonian path $P_1^{'}$ from $y_1^{'}$ to x_1 in $A_{n,k}^{i_1} - F$. Let $P_1 = \langle y_1, y_1^{'}, P_1^{'}, x_1\rangle$. Let $y_2^{'} \neq y_1 \in V(A_{n,k}^{i_3})$. By Theorem 9, since $A_{n,k}^{i_3} - \{y_1\} \cup F$ is hamiltonian connected, we can find a hamiltonian path $P_2^{''}$ from y_2 to $y_2^{'}$ in $A_{n,k}^{i_3} - \{y_1\} \cup F$. If $|I| \geq 4$, since $|N^*(y_2^{'})| > 2$, we can find an edge $(y_2^{'}, y_2^{''}) \in E^{i_3,j}$ such that $y_2^{''} \in V(A_{n,k}^{j})$ for some $j \in I - \{i_1, i_2, i_3\}$. By Lemma 4, we can find a hamiltonian path $P_2^{'}$ from $y_2^{''}$ to x_2 in $A_{n,k}^{I-\{i_1,i_3\}} - F$. If $|I| = 3$, there exists an edge $(y_2^{'}, y_2^{''}) \in E^{i_3,i_2}$ such that $y_2^{''} \in V(A_{n,k}^{i_2})$. By Theorem 9, there is a hamiltonian path $P_2^{'}$ from $y_2^{''}$ to x_2 in $A_{n,k}^{i_2} - F$. Let $P_2 = \langle y_2, P_2^{''}, y_2^{'}, y_2^{''}, P_2^{'}, x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A_{n,k}^{I} - F$.

Subcase 2.2.2: Suppose that $\{i_1, i_2\} \bigcap \{AS(y_1) \cup AS(y_2)\} = \emptyset$ with $|I| \geq 4$. See Figure 3.3(f) for an illustration. Since $|N^*(y_1)| > 2$, we can find an edge $(y_1, y_1^{'}) \in E^{i_1,j_1}$ such that $y_1^{'} \in V(A_{n,k}^{j_1})$ for some $j_1 \in I - \{i_1, i_2, i_3\}$. By Lemma 4, we can find a hamiltonian path $P_1^{'}$ from $y_1^{'}$ to x_1 in $A_{n,k}^{\{i_1,j_1\}} - F$. Let $P_1 = \langle y_1, y_1^{'}, P_1^{'}, x_1\rangle$. Let $y_2^{'} \in V(A_{n,k}^{i_3})$ and $y_2^{'} \in N^{i_3}(y_1)$. By Proposition 2, we have $AS(y_1) \neq AS(y_2^{'})$. By Theorem 9, since $A_{n,k}^{i_3} - \{y_1\} \cup F$ is hamiltonian connected, we can find a hamiltonian path $P_2^{''}$ from y_2 to $y_2^{'}$ in $A_{n,k}^{i_3} - \{y_1\} \cup F$. If $|I| \geq 5$, since $|N^*(y_2^{'})| > 2$, we can find an edge $(y_2^{'}, y_2^{''}) \in E^{i_3,j_2}$ such that $y_2^{''} \in V(A_{n,k}^{j_2})$ for some $j_2 \in I - \{i_1, i_2, i_3, j_1\}$. By Lemma 4, we can find a hamiltonian path $P_2^{'}$ from $y_2^{''}$ to x_2 in $A_{n,k}^{I-\{i_1,i_3,j_1\}} - F$. If $|I| = 4$, since $|N^*(y_2^{'})| > 2$, we can find an edge $(y_2^{'}, y_2^{''}) \in E^{i_3,i_2}$ such that $y_2^{''} \in V(A_{n,k}^{i_2})$. Since $A_{n,k}^{i_2} - F$ is hamiltonian connected, there is a hamiltonian path $P_2^{'}$ from $y_2^{''}$ to x_2 in $A_{n,k}^{i_2} - F$. Let $P_2 = \langle y_2, P_2^{''}, y_2^{'}, y_2^{''}, P_2^{'}, x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A_{n,k}^{I} - F$.

Thus the lemma follows. □

3.3 Panpositionable Hamiltonicity of the Arrangement Graphs $A_{n,2}$

In this section, we will prove that the arrangement graph $A_{n,2}$ is panpositionable hamiltonian for all $n-k \geq 2$. The basic idea is to study $A_{n,1}$ and $A_{4,2}$ first, and then to prove the general case by induction.

Lemma 7. *The arrangement graph $A_{n,1}$ is panconnected and panpositionable hamiltonian for all $n \geq 3$.*

Proof. Since $A_{n,1}$ is isomorphic to the complete graph K_n, the lemma follows trivially. □

Lemma 8. *The arrangement graph $A_{4,2}$ is panpositionable hamiltonian.*

Proof. Let s and t be any two vertices of $A_{4,2}$ in Figure 3.2. The arrangement graph is vertex symmetric and edge symmetric, and the diameter of $A_{n,k}$ is $\lfloor \frac{3k}{2} \rfloor$ by Day and Tripathi [13]. Hence the diameter of $A_{4,2}$ is 3. We prove this lemma by considering the distance between s and t. Without loss of generality, we may assume that $s = 42$ and $t = 32$ if $d(s,t) = 1$. Assume that $s = 42$ and $t = 31$ if $d(s,t) = 2$. And, assume that $s = 42$ and $t = 24$ if $d(s,t) = 3$. Obviously, if $d_{HC}(s,t) = x$, we also have $D_{HC}(s,t) = |V(HC)| - x$. Hence, we only need to prove that for each $l \in \{d(s,t), d(s,t)+1, ..., \frac{|A_{4,2}|}{2}\}$, we can construct a hamiltonian cycle of $A_{4,2}$ such that the distance between s and t on the cycle is l. The corresponding hamiltonian cycle HC in $A_{4,2}$ are listed below.

$d(s,t)$	$d_{HC}(s,t)$	The cycle HC
1	1	$\langle 42, 32, 31, 41, 21, 24, 34, 14, 12, 13, 23, 43, 42 \rangle$
1	2	$\langle 42, 12, 32, 31, 34, 14, 13, 43, 23, 24, 21, 41, 42 \rangle$
1	3	$\langle 42, 41, 31, 32, 34, 14, 24, 21, 23, 43, 13, 12, 42 \rangle$
1	4	$\langle 42, 41, 31, 34, 32, 12, 13, 14, 24, 21, 23, 43, 42 \rangle$
1	5	$\langle 42, 12, 14, 24, 34, 32, 31, 41, 21, 23, 13, 43, 42 \rangle$
1	6	$\langle 42, 41, 43, 23, 21, 31, 32, 34, 24, 14, 13, 12, 42 \rangle$
2	2	$\langle 42, 32, 31, 21, 41, 43, 13, 23, 24, 34, 14, 12, 42 \rangle$
2	3	$\langle 42, 32, 34, 31, 41, 21, 24, 23, 43, 13, 14, 12, 42 \rangle$
2	4	$\langle 42, 12, 32, 34, 31, 41, 21, 23, 24, 14, 13, 43, 42 \rangle$
2	5	$\langle 42, 32, 12, 14, 34, 31, 41, 21, 24, 23, 13, 43, 42 \rangle$
2	6	$\langle 42, 12, 13, 14, 34, 32, 31, 41, 21, 24, 23, 43, 42 \rangle$
3	3	$\langle 42, 32, 34, 24, 21, 31, 41, 43, 23, 13, 14, 12, 42 \rangle$
3	4	$\langle 42, 32, 31, 21, 24, 34, 14, 12, 13, 23, 43, 41, 42 \rangle$
3	5	$\langle 42, 32, 31, 41, 21, 24, 34, 14, 12, 13, 23, 43, 42 \rangle$
3	6	$\langle 42, 41, 21, 31, 32, 34, 24, 23, 43, 13, 14, 12, 42 \rangle$

Thus the lemma holds. □

Lemma 9. *The arrangement graph $A_{n,2}$ is panpositionable hamiltonian for all $n \geq 4$.*

Proof. By Lemma 8, the result holds for $n = 4$. Suppose that $n \geq 5$, and s and t are two distinct vertices of $A_{n,2}$. Then for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{|V(A_{n,2})|}{2}\}$, we shall find a hamiltonian cycle of $A_{n,2}$ such that the distance between s and t on the cycle is l.

We would like to make a remark here. Throughout this chapter, the proof idea of the panpositionable hamiltonian property of the arrangement graph is essentially similar to Case 1 described below except for some minor adjustments.

Case 1: s and t belong to the same subcomponent $A^i_{n,2}$. See Figure 3.4. Suppose that $s, t \in V(A^i_{n,2})$ for some $i \in \langle n \rangle$. Since $A^i_{n,2}$ is isomorphic to the complete graph K_{n-1}, we have $d(s,t) = 1$. For each $l_0 \in \{1, 2, 3, ..., n-2\}$, we can construct a hamiltonian cycle HC_i of $A^i_{n,2}$ such that the distance between s and t on the cycle is l_0. Node t has two distinct neighbors on cycle HC_i. Let u and v be two neighbors of t on HC_i. Let $HC_i = \langle s, LP, u, t, v, RP, s\rangle$ and $P_0 = \langle s, LP, u, t\rangle$. Without loss of generality, let $L(P_0) = l_0$. Since $|N^*(t)| = n-2 \geq 3$ for $n \geq 5$, we can find a subcomponent $A^{h_t}_{n,2}$ different from $A^i_{n,2}$, and a vertex $t' \in V(A^{h_t}_{n,2})$ such that $(t, t') \in E^{i,h_t}$ for some $h_t \in \langle n \rangle - \{i\}$. By Proposition 2, $d(t,u) = 1$, hence we have $|AS(t) \cap AS(u)| = n-3 \geq 2$ for $n \geq 5$. It means that we can find a subcomponent $A^{j_1}_{n,2}$ which $j_1 \in \langle n \rangle - \{i, h_t\}$, such that there exist two disjoint edges (u, p_1) and (t, q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A^{j_1}_{n,2})$. Since $|N^*(v)| = n-2 \geq 3$ for $n \geq 5$, we can find a subcomponent $A^{h_v}_{n,2}$, and a vertex $v' \in V(A^{h_v}_{n,2})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n \rangle - \{i, h_t, j_1\}$. By Lemma 5(a), there exists a hamiltonian path HP of $A^{\langle n \rangle - \{i\}}_{n,2}$ joining t' and v'. Thus $\langle s, P_0, t, t', HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{1, 2, 3, ..., n-2\}$, the distance between s and t on the cycle is l_0.

Now we present an algorithm to expand the path $P_0 = \langle s, LP, u, t\rangle$ between s and t to various lengths. The idea is to expand the path by inserting the vertices of $A^{j_1}_{n,2}$ into P_0. We now describe the details.

If we want to insert p_1 and q_1 to P_0, let $P_1 = \langle s, LP, u, p_1, q_1, t\rangle$. See Figure 3.5(a) for an illustration. Thus we have $L(P_1) = l_0 + 2$. We can expand the path P_1 to a longer path as follows. By Theorem 9, there is a hamiltonian path HP_1 from p_1 to q_1 in $A^{j_1}_{n,2}$. So we can join all the vertices of $A^{j_1}_{n,k}$ to P_1, let $P^*_1 = \langle s, LP, u, p_1, HP_1, q_1, t\rangle$. Hence $L(P^*_1) = l_0 + n - 1$. Since $1 \leq l_0 \leq n-2$, we have $3 \leq L(P_1) \leq n$ and $n \leq L(P^*_1) \leq 2n-3$. Therefore, for each $l_1 \in \{1, 2, 3, ..., 2n-3\}$, we can construct a path $PP_1 \in \{P_0, P_1, P^*_1\}$

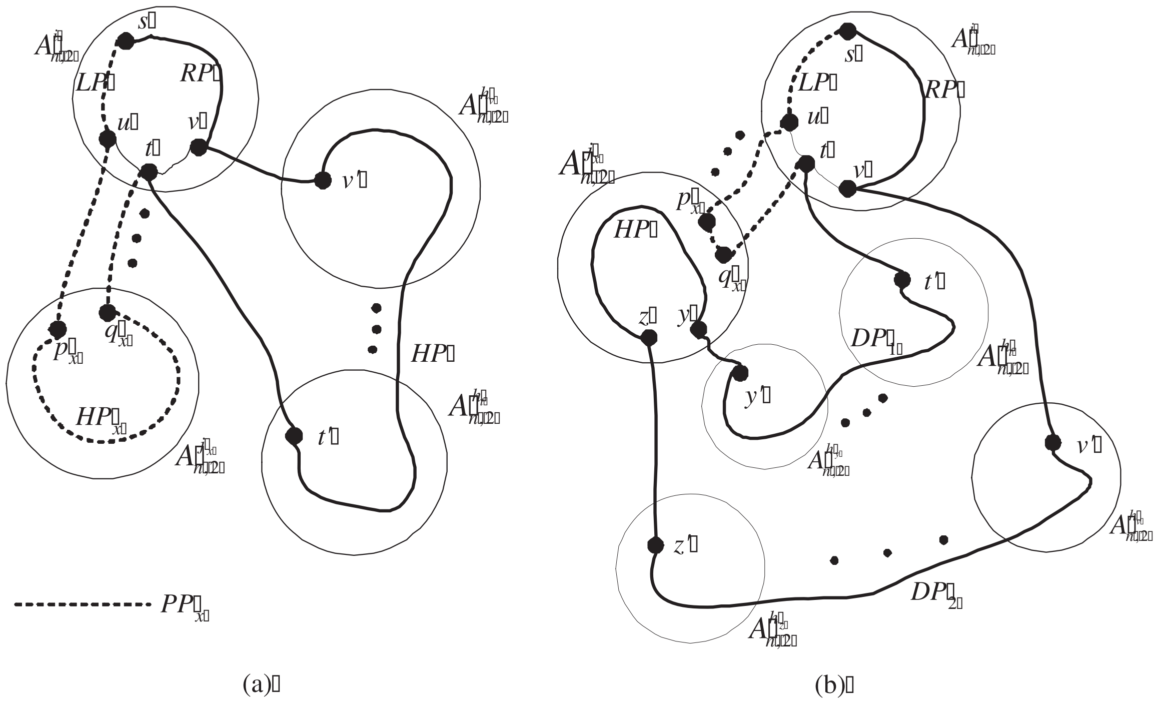

Figure 3.4: Lemma 9, Case 1.

from s to t such that the distance between s and t on the path is l_1.

Using the same idea, we can expand the path HP_1. Let u_1 and t_1 be two adjacent vertices on HP_1. That is, $HP_1 = \langle p_1, LP_1, u_1, t_1, RP_1, q_1 \rangle$. By Proposition 1 and 2, there exist two distinct edges (u_1, p_2) and (t_1, q_2) in E^{j_1,j_2} for some $j_2 \in \langle n \rangle - \{i, h_t, h_v, j_1\}$ such that $(p_2, q_2) \in E(A^{j_2}_{n,2})$. See Figure 3.5(b) for an illustration. Let $P_2 = \langle s, LP, u, p_1, LP_1, u_1, p_2, q_2, t_1, RP_1, q_1, t \rangle$. Thus we have $L(P_2) = l_0+n+1$. By Theorem 9, there is a hamiltonian path HP_2 from p_2 to q_2 in $A^{j_2}_{n,2}$. Let $P^*_2 = \langle s, LP, u, p_1, LP_1, u_1, p_2, HP_2, q_2, t_1, RP_1, q_1, t \rangle$. Hence we have $L(P^*_2) = l_0 + 2n - 2$. Since $1 \leq l_0 \leq n - 2$, we have $n + 2 \leq L(P_2) \leq 2n - 1$ and $2n - 1 \leq L(P^*_2) \leq 3n - 4$. Therefore, for each $l_2 \in \{1, 2, 3, ..., 3n - 4\}$, we can construct a path $PP_2 \in \{P_0, P_1, P^*_1, P_2, P^*_2\}$ from s to t such that the distance between s and t on the path is l_2 if $n \geq 5$. The maximal value of l_2 is $3n - 4$. If $n = 5$, then we have $3n - 4 \geq \frac{|V(A_{n,2})|}{2} = \frac{n(n-1)}{2}$.

We can use the algorithm repeatly for $n \geq 6$. For each $3 \leq x \leq \lfloor \frac{n}{2} \rfloor$, let u_{x-1} and t_{x-1} be the two adjacent vertices on HP_{x-1}. That is, $HP_{x-1} = \langle p_{x-1}, LP_{x-1}, u_{x-1}, t_{x-1}, RP_{x-1}, q_{x-1} \rangle$. By Proposition 1 and Proposition 2, there exist two distinct edges (u_{x-1}, p_x) and (t_{x-1}, q_x) in E^{j_{x-1},j_x} for some $j_x \in \langle n \rangle - \{i, h_t, h_v, j_1, ..., j_{x-1}\}$ such that $(p_x, q_x) \in E(A^{j_x}_{n,2})$. Let $P_x = \langle s, LP, u, p_1, LP_1,\ u_1, ..., u_{x-1}, p_x, q_x, t_{x-1}, ..., t_1, RP_1, q_1, t \rangle$. Thus we

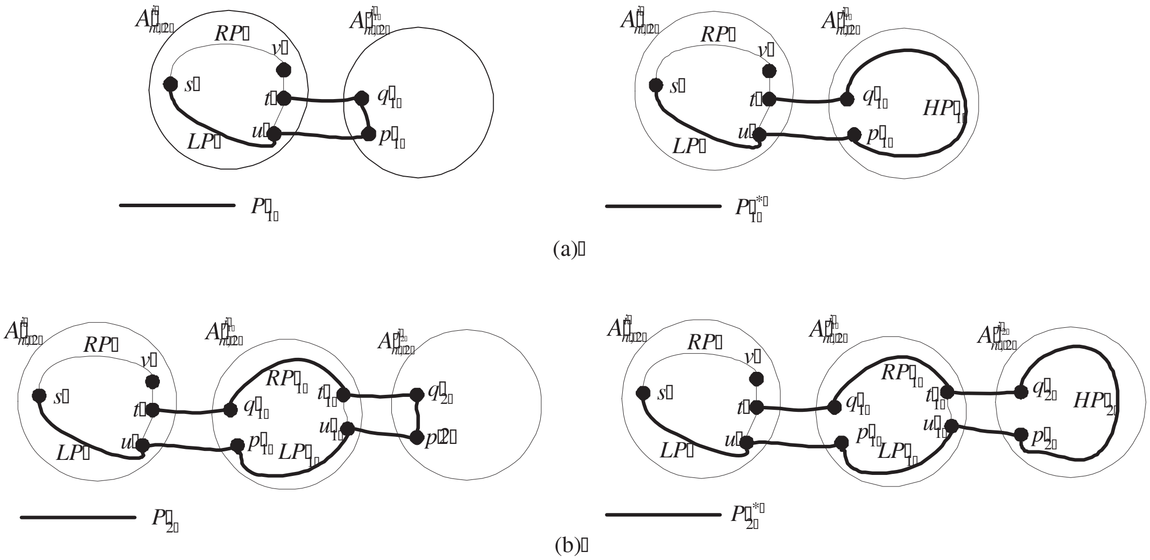

Figure 3.5: The paths P_1, P_1^*, P_2, and P_2^*.

have $L(P_x) = l_0+(x-1)(n-1)+2$. By Theorem 9, there is a hamiltonian path HP_x from p_x to q_x in $A_{n,2}^{j_x}$. Let $P_x^* = \langle s, LP, u, p_1, LP_1, u_1, ..., u_{x-1}, p_x, HP_x, q_x, t_{x-1}, ..., t_1, RP_1, q_1, t\rangle$. Hence we have $L(P_x^*) = l_0 + (x-1)(n-1) + n - 1$. Since $1 \le l_0 \le n-2$, we have $(x-1)(n-1)+3 \le L(P_x) \le (x-1)(n-1)+n$ and $(x-1)(n-1)+n \le L(P_x^*) \le (x-1)(n-1)+2n-3$. Therefore, for each $l_x \in \{1, 2, 3, ..., (x-1)(n-1)+2n-3\}$, we can construct a path $PP_x \in \{P_0, P_1, P_1^*, ..., P_x, P_x^*\}$ from s to t such that the distance of s and t on the path is l_x if $n \ge 6$. The maximal value of l_x is $(\lfloor \frac{n}{2} \rfloor - 1)(n-1)+2n-3$, and $(\lfloor \frac{n}{2} \rfloor - 1)(n-1)+2n-3 \ge \frac{|V(A_{n,2})|}{2} = \frac{n(n-1)}{2}$. To construct a hamiltonian cycle, we consider the two subcases:

Subcase 1.1: Suppose that $PP_x \in \{P_0, P_1^*, ..., P_x^*\}$ for each $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3.4(a) for an illustration. By Lemma 5(a), there exists a hamiltonian path HP of $A_{n,2}^{\langle n\rangle - \{i, j_1, ..., j_x\}}$ joining t' and v' which $t' \in V(A_{n,2}^{h_t})$ and $v' \in V(A_{n,2}^{h_v})$. Thus $\langle s, PP_x, t, t', HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{1, 2, 3, ..., \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 1.2: Suppose that $PP_x \in \{P_1, ..., P_x\}$ for each $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3.4(b) for an illustration. Assume that $H_1, H_2 \in \langle n\rangle - \{i, j_1, ..., j_x\}$ and $H_1 \cap H_2 = \emptyset$. Let $h_t, h_y \in H_1$ and $h_v, h_z \in H_2$. Let $F \subseteq V(A_{n,2}^{j_x})$ and $F = \{p_x, q_x\}$. Let y, z be two distinct vertices in $A_{n,2}^{j_x} - F$. Since $|N^*(y)| = |N^*(z)| = n-2 \ge \lceil \frac{n}{2} \rceil$ for $n \ge 5$, there exist two distinct edges $(y, y') \in E^{j_x, h_y}$ and $(z, z') \in E^{j_x, h_z}$ such that $y' \ne t' \in V(A_{n,2}^{h_y})$ and $z' \ne v' \in V(A_{n,2}^{h_z})$, respectively. $A_{n,2}^{j_x} - F$ is isomorphic to K_{n-3}, hence there is a hamiltonian path HP from y to z in $A_{n,2}^{j_x} - F$. By Theorem 9 and Lemma 5(a), there exist a hamiltonian path DP_1 from t' to y' in $A_{n,2}^{H_1}$ and a hamiltonian path DP_2 from v' to z' in $A_{n,2}^{H_2}$. Thus $\langle s, PP_x, t, t', DP_1, y', y, HP, z, z', DP_2, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{1, 2, 3, ..., \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Case 2: s and t belong to different subcomponents of $A_{n,2}$. Suppose that $s \in V(A_{n,2}^i)$ and $t \in V(A_{n,2}^{h_t})$ for $i \neq h_t \in \langle n\rangle$. Each subcomponent of $A_{n,2}$ is isomorphic to the complete graph K_{n-1}, and $|E^{i,h_t}| > 0$, we have $d(s,t) = 1$, $d(s,t) = 2$ or $d(s,t) = 3$. In the case of $d(s,t) = 1$, suppose that $s = s_1s_2...s_{k-1}i$ and $t = t_1t_2...t_{k-1}h_t$ are adjacent, and $s_x = t_x$ for each $1 \leq x \leq k-1$. We may decompose $A_{n,2}$ into subcomponents according to the first position such that s and t belong to the same subcomponent. Hence the case for $d(s,t) = 1$ is the same as Case 1. In the following, we discuss the other two cases.

Subcase 2.1: Suppose that $d(s,t) = 2$. See Figure 3.6 for an illustration. Without loss of generality, let (t', t) be an edge in E^{i,h_t} such that $t' \in V(A_{n,2}^i)$ and $t' \in N^*(t)$. Since $A_{n,2}^i$ is isomorphic to complete graph K_{n-1}, we have $d(s, t') = 1$. For each $l_0 \in \{1, 2, 3, ..., n-2\}$, we can construct a hamiltonian cycle HC_i of $A_{n,2}^i$ such that the distance between s and t' on the cycle is l_0. Let u and v be two neighbors of t' on HC_i, and $HC_i = \langle s, LP, u, t', v, RP, s\rangle$. Let $P_0 = \langle s, LP, u, t', t\rangle$. Without loss of generality, we may assume that $L(P_0) = l_0 + 1$.

By Proposition 2, $d(t', u) = 1$, hence we have $|AS(t') \cap AS(u)| = n - 3 \geq 2$ if $n \geq 5$. It means that we can find an index $j_1 \in \langle n\rangle - \{i, h_t\}$, such that there exist two disjoint edges (u, p_1) and (t', q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A_{n,2}^{j_1})$. Since $|N^*(v)| = n-2 \geq 3$ if $n \geq 5$, we can find a vertex $v' \in V(A_{n,2}^{h_v})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, h_t, j_1\}$. If we want to join p_1 and q_1 to P_0, let $P_1 = \langle s, LP, u, p_1, q_1, t', t\rangle$. Then we have $L(P_1) = l_0 + 3$. By Theorem 9, there is a hamiltonian path HP_1 from p_1 to q_1 in $A_{n,2}^{j_1}$. Let $P_1^* = \langle s, LP, u, p_1, HP_1, q_1, t', t\rangle$. Hence we have $L(P_1^*) = l_0 + n$. Since $1 \leq l_0 \leq n-2$, we have $4 \leq L(P_1) \leq n+1$ and $n+1 \leq L(P_1^*) \leq 2n-2$. Therefore, for each $l_1 \in \{2, 3, 4, ..., 2n-2\}$, we can construct a path $PP_1 \in \{P_0, P_1, P_1^*\}$ from s to t such that the distance between s and t on the path is l_1.

Recursively, for each $2 \leq x \leq \lfloor \frac{n}{2} \rfloor$, let u_{x-1} and t_{x-1} be two adjacent vertices on HP_{x-1}. That is, $HP_{x-1} = \langle p_{x-1}, LP_{x-1}, u_{x-1}, t_{x-1}, RP_{x-1}, q_{x-1}\rangle$. By Proposition 1 and Proposition 2, there exist two distinct edges (u_{x-1}, p_x) and (t_{x-1}, q_x) in E^{j_{x-1},j_x} for some $j_x \in \langle n\rangle - \{i, h_t, h_v, j_1, ..., j_{x-1}\}$. And, $(p_x, q_x) \in E(A_{n,2}^{j_x})$. Let $P_x = \langle s, LP, u, p_1, LP_1, u_1, ..., u_{x-1}, p_x, q_x, t_{x-1}, ..., t_1, RP_1, q_1, t', t\rangle$. Thus we have $L(P_x) = l_0 + (x-1)(n-1) + 3$. By Theorem 9, there is a hamiltonian path HP_x from p_x to q_x in $A_{n,2}^{j_x}$. Let $P_x^* = \langle s, LP, u, p_1, LP_1, u_1, ..., u_{x-1}, p_x, HP_x, q_x, t_{x-1}, ..., t_1, RP_1, q_1, t', t\rangle$. Hence we have $L(P_x^*) = l_0 + (x-1)(n-1) + n$. Since $1 \leq l_0 \leq n-2$, we have $(x-1)(n-1) + 4 \leq L(P_x) \leq$

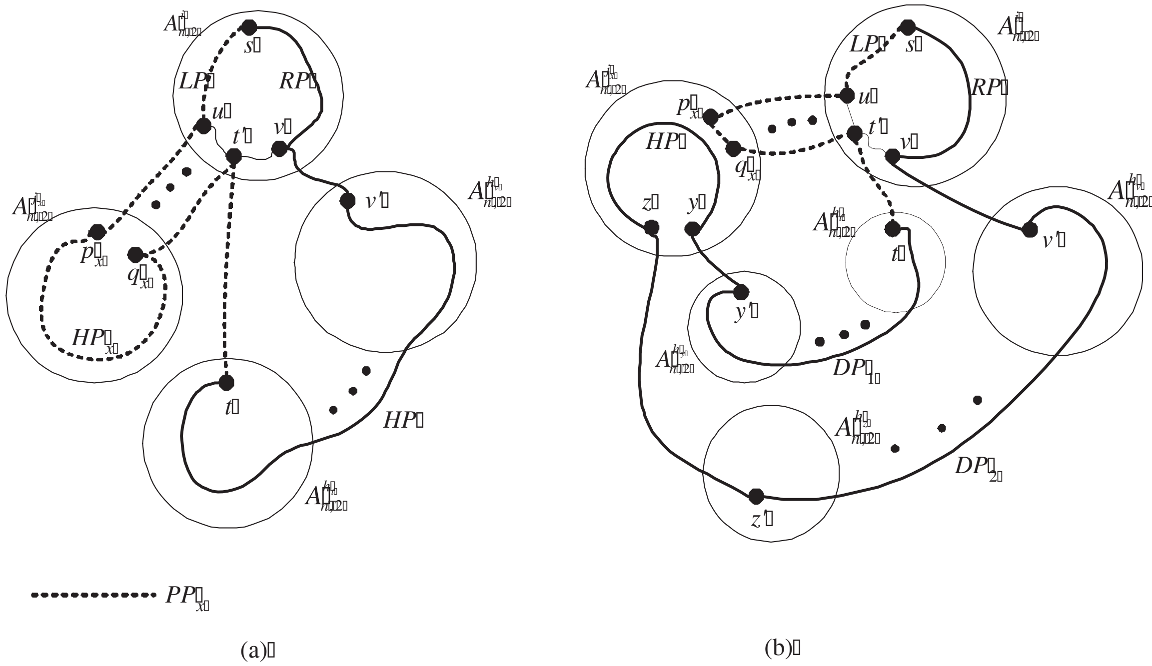

Figure 3.6: Lemma 9, Case 2.1.

$(x-1)(n-1)+n+1$ and $(x-1)(n-1)+n+1 \le L(P_x^*) \le (x-1)(n-1)+2n-2$. Therefore, for each $l_x \in \{2,3,4,...,(x-1)(n-1)+2n-2\}$, we can construct a path $PP_x \in \{P_0, P_1, P_1^*, ..., P_x, P_x^*\}$ from s to t such that the distance between s and t on the path is l_x if $n \ge 5$. The maximal value of l_x is $(\lfloor \frac{n}{2} \rfloor - 1)(n-1)+2n-2$, and $(\lfloor \frac{n}{2} \rfloor - 1)(n-1)+2n-2 \ge \frac{|V(A_{n,2})|}{2} = \frac{n(n-1)}{2}$. To construct a hamiltonian cycle, we consider the two subcases:

Subcase 2.1.1: Suppose that $PP_x \in \{P_0, P_1^*, ..., P_x^*\}$ for each $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3.6(a) for an illustration. By Lemma 5(a), there exists a hamiltonian path HP of $A_{n,2}^{\langle n \rangle - \{i,j_1,...,j_x\}}$ joining t and v'. Thus $\langle s, PP_x, t', t, HP, v', v,\ RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{2,3,4,..., \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.1.2: Suppose that $PP_x \in \{P_1, ..., P_x\}$ for each $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3.6(b) for an illustration. Assume that $H_1, H_2 \subseteq \langle n \rangle - \{i, j_1, ..., j_x\}$ and $H_1 \cap H_2 = \emptyset$. Let $h_t, h_y \in H_1$ and $h_v, h_z \in H_2$. Let $F \subseteq V(A_{n,2}^{j_x})$ and $F = \{p_x, q_x\}$. Let y and z be two distinct vertices in $A_{n,2}^{j_x} - F$. Since $|N^*(y)| = |N^*(z)| = n-2 \ge \lceil \frac{n}{2} \rceil$ for $n \ge 5$, there exist two distinct edges $(y,y') \in E^{j_x,h_y}$ and $(z,z') \in E^{j_x,h_z}$ such that $y' \ne t \in V(A_{n,2}^{h_y})$ and $z' \ne v' \in V(A_{n,2}^{h_z})$, respectively. $A_{n,2}^{j_x} - F$ is isomorphic to K_{n-3}, hence there is a hamiltonian path HP from y to z in $A_{n,2}^{j_x} - F$. By Theorem 9 and Lemma 5(a), there

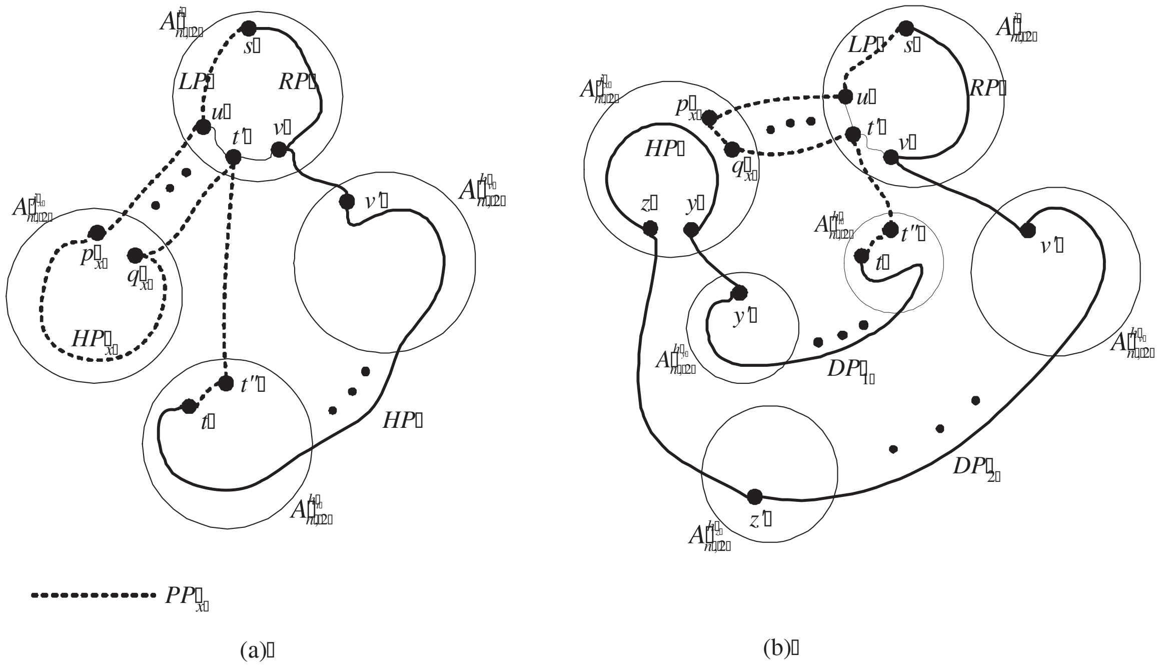

Figure 3.7: Lemma 9, Case 2.2.

exist a hamiltonian path DP_1 from t to y' in $A^{H_1}_{n,2}$ and a hamiltonian path DP_2 from v' to z' in $A^{H_2}_{n,2}$. Thus $\langle s, PP_x, t', t, DP_1, y', y, HP, z, z', DP_2, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{2, 3, 4, ..., \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.2: Suppose that $d(s,t) = 3$ and $n \geq 6$. See Figure 3.7 for an illustration. We shall discuss the subcase $d(s,t) = 3$ and $n = 5$ later in Subcase 2.3. Let (t', t'') be an edge in E^{i,h_t} such that $t' \in V(A^i_{n,2})$, $t'' \in V(A^{h_t}_{n,2})$, $t'' \in N(t)$, and $t'' \in N^*(t')$. Since $A^i_{n,2}$ is isomorphic to complete graph K_{n-1}, we have $d(s,t') = 1$. For each $l_0 \in \{1, 2, 3, ..., n-2\}$, we can construct a hamiltonian cycle HC_i of $A^i_{n,2}$ such that the distance of s and t' on the cycle is l_0. Suppose that u and v are two distinct vertices in $V(A^i_{n,2})$, and u and v are two neighbors of t' on HC_i. Let $HC_i = \langle s, LP, u, t', v, RP, s\rangle$. Let $P_0 = \langle s, LP, u, t', t'', t\rangle$. Hence, without loss of generality, we have $L(P_0) = l_0 + 2$.

By Proposition 2, $d(t', u) = 1$, we have $|AS(t') \cap AS(u)| = n - 3 \geq 2$ if $n \geq 6$. It means that we can find an index $j_1 \in \langle n\rangle - \{i, h_t\}$, such that there exist two disjoint edges (u, p_1) and (t', q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A^{j_1}_{n,2})$. Since $|N^*(v)| = n - 2 \geq 3$ if $n \geq 5$, we can find a vertex $v' \in V(A^{h_v}_{n,2})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, h_t, j_1\}$. If we want to join p_1 and q_1 to P_0, let $P_1 = \langle s, LP, u, p_1, q_1, t', t'', t\rangle$. Thus we have $L(P_1) = l_0 + 4$. By Theorem 9, there is a hamiltonian path HP_1 from p_1 to q_1 in $A^{j_1}_{n,2}$. Let $P^*_1 = \langle s, LP, u, p_1, HP_1, q_1, t', t'', t\rangle$. Hence we have $L(P^*_1) = l_0 + n + 1$.

Since $1 \le l_0 \le n-2$, we have $5 \le L(P_1) \le n+2$ and $n+2 \le L(P_1^*) \le 2n-1$. Therefore, for each $l_1 \in \{3,4,5,...,2n-1\}$, we can construct a path $PP_1 \in \{P_0, P_1, P_1^*\}$ from s to t such that the distance between s and t on the path is l_1.

Similarly, for each $2 \le x \le \lfloor \frac{n}{2} \rfloor$, let u_{x-1} and t_{x-1} be the two adjacent vertices on HP_{x-1}. That is, $HP_{x-1} = \langle p_{x-1}, LP_{x-1}, u_{x-1}, t_{x-1}, RP_{x-1}, q_{x-1} \rangle$. By Proposition 1 and Proposition 2, there exist two distinct edges (u_{x-1}, p_x) and (t_{x-1}, q_x) in E^{j_{x-1}, j_x} for some $j_x \in \langle n \rangle - \{i, h_t, h_v, j_1, ..., j_{x-1}\}$. And, $(p_x, q_x) \in E(A_{n,2}^{j_x})$. Let $P_x = \langle s, LP, u, p_1, LP_1, u_1, ..., u_{x-1}, p_x, q_x, t_{x-1}, ..., t_1, RP_1, q_1, t', t'', t \rangle$. Thus we have $L(P_x) = l_0 + (x-1)(n-1) + 4$. By Lemma 9, there is a hamiltonian path HP_x from p_x to q_x in $A_{n,2}^{j_x}$. Let $P_x^* = \langle s, LP, u, p_1, LP_1, u_1, ..., u_{x-1}, p_x, HP_x, q_x, t_{x-1}, ..., t_1, RP_1, q_1, t', t'', t \rangle$. Hence we have $L(P_x^*) = l_0 + (x-1)(n-1) + n + 1$. Since $1 \le l_0 \le n-2$, we have $(x-1)(n-1)+5 \le L(P_x) \le (x-1)(n-1)+n+2$ and $(x-1)(n-1)+n+2 \le L(P_x^*) \le (x-1)(n-1)+2n-1$. Therefore, for each $l_x \in \{3,4,5,...,(x-1)(n-1)+2n-1\}$, we can construct a path $PP_x \in \{P_0, P_1, P_1^*, ..., P_x, P_x^*\}$ from s to t such that the distance between s and t on the path is l_x if $n \ge 5$. The maximal value of l_x is $(\lfloor \frac{n}{2} \rfloor - 1)(n-1) + 2n - 1$, and $(\lfloor \frac{n}{2} \rfloor - 1)(n-1) + 2n - 1 \ge \frac{|V(A_{n,2})|}{2} = \frac{n(n-1)}{2}$. To construct a hamiltonian cycle, we consider the two subcases:

Subcase 2.2.1: Suppose that $PP_x \in \{P_0, P_1^*, ..., P_x^*\}$ for each $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3.7(a) for an illustration. Let $F_t \subseteq V(A_{n,2}^{h_t})$ and $F_t = \{t''\}$. By Lemma 5(b), there exists a hamiltonian path HP of $A_{n,2}^{\langle n \rangle - \{i, j_1, ..., j_x\}} - F_t$ joining t and v'. Thus $\langle s, PP_x, t', t'', t, HP, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{3,4,5,..., \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.2.2: Suppose that $PP_x \in \{P_1, ..., P_x\}$ for each $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3.7(b) for an illustration. Assume that $H_1, H_2 \in \langle n \rangle - \{i, j_1, ..., j_x\}$ and $H_1 \cap H_2 = \emptyset$. Let $h_t, h_y \in H_1$ and $h_v, h_z \in H_2$. Let $F_j \subseteq V(A_{n,2}^{j_x})$ and $F_j = \{p_x, q_x\}$. Let y and z be two distinct vertices in $A_{n,2}^{j_x} - F_j$. Since $|N^*(y)| = |N^*(z)| = n-2 \ge \lceil \frac{n}{2} \rceil$ for $n \ge 5$, there exist two distinct edges $(y, y') \in E^{j_x, h_y}$ and $(z, z') \in E^{j_x, h_z}$ such that $y' \ne t, t'' \in V(A_{n,2}^{h_y})$ and $z' \ne v' \in V(A_{n,2}^{h_z})$, respectively. $A_{n,2}^{j_x} - F_j$ is isomorphic to K_{n-3}, hence there is a hamiltonian path HP from y to z in $A_{n,2}^{j_x} - F_j$. By Theorem 9 and Lemma 5(b), there exist a hamiltonian path DP_1 from t to y' in $A_{n,2}^{H_1} - F_t$ and a hamiltonian path DP_2 from v' to z' in $A_{n,2}^{H_2}$. Thus $\langle s, PP_x, t', t'', t, DP_1, y', y, HP, z, z', DP_2, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{3,4,5,..., \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.3: Suppose that $d(s,t) = 3$ and $n = 5$. Let s and t be two distinct vertices of $A_{5,2}$ in Figure 3.8. By the vertex and edge symmetric properties, we may assume that

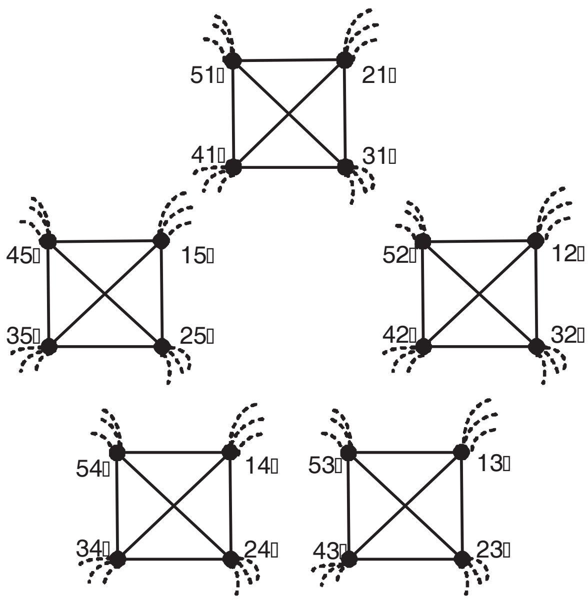

Figure 3.8: The arrangement graph $A_{5,2}$.

$s = 12$ and $t = 21$ for $d(s,t) = 3$. The corresponding hamiltonian cycle HC in $A_{5,2}$ are listed below.

$d_{HC}(s,t)$	The cycle HC
3	$\langle 21, 23, 13, 12, 15, 25, 35, 45, 43, 53, 54, 14, 24, 34, 32, 42, 52, 51, 41, 31, 21 \rangle$
4	$\langle 21, 31, 32, 42, 12, 52, 53, 13, 23, 43, 41, 51, 54, 14, 24, 34, 35, 45, 15, 25, 21 \rangle$
5	$\langle 21, 31, 32, 42, 52, 12, 13, 53, 23, 43, 41, 51, 54, 14, 24, 34, 35, 45, 15, 25, 21 \rangle$
6	$\langle 21, 31, 41, 42, 32, 52, 12, 13, 23, 43, 53, 51, 54, 14, 24, 34, 35, 45, 15, 25, 21 \rangle$
7	$\langle 21, 31, 41, 51, 52, 42, 32, 12, 13, 23, 43, 53, 54, 14, 24, 34, 35, 45, 15, 25, 21 \rangle$
8	$\langle 21, 31, 41, 51, 53, 52, 42, 32, 12, 13, 43, 23, 24, 14, 54, 34, 35, 45, 15, 25, 21 \rangle$
9	$\langle 21, 31, 41, 51, 53, 43, 42, 32, 52, 12, 13, 23, 24, 14, 54, 34, 35, 45, 15, 25, 21 \rangle$
10	$\langle 21, 31, 41, 51, 53, 13, 43, 42, 32, 52, 12, 15, 45, 35, 34, 54, 14, 24, 23, 25, 21 \rangle$

Hence the lemma follows. □

3.4 Panpositionable Hamiltonicity and Panconnectivity of the Arrangement Graphs $A_{n,k}$

3.4.1 Panpositionable Hamiltonicity of the Arrangement Graphs $A_{n,k}$

In this section, we show that the arrangement graph $A_{n,k}$ is panpositionable hamiltonian for $k \geq 1$ and $n-k \geq 2$.

Theorem 10. *The arrangement graph $A_{n,k}$ is panpositionable hamiltonian for all $k \geq 1$ and $n-k \geq 2$.*

Proof. We prove this theorem by induction on k. By Lemma 7, $A_{n,1}$ is panpositionable hamiltonian for all $n \geq 3$. By Lemma 9, $A_{n,2}$ is panpositionable hamiltonian for all $n \geq 4$. Suppose that the result holds for $A_{n,k-1}$ for some $k \geq 3$ and for all $n-(k-1) \geq 2$. We observe that $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$, and each $A_{n-1,k-1}$ is panpositionable hamiltonian by inductive hypothesis, for all $n-k \geq 2$. Let s and t be two distinct vertices of $A_{n,k}$. Then for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{|V(A_{n,k})|}{2}\}$, we shall find a hamiltonian cycle of $A_{n,k}$ such that the distance between s and t on the cycle is l. The basic idea of our construction is similar to that presented in Lemma 9.

Case 1: s and t belong to the same subcomponent $A^i_{n,k}$. See Figure 3.9 for an illustration. Suppose that $s,t \in V(A^i_{n,k})$ for some $i \in \langle n \rangle$. Since $A^i_{n,k}$ is isomorphic to $A_{n-1,k-1}$, by inductive hypothesis, for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., |V(A^i_{n,k})| - d(s,t)\}$, we can construct a hamiltonian cycle HC_i of $A^i_{n,k}$ such that the distance between s and t on the cycle is l_0. Let u and v be the two neighbors of t on HC_i. Let $HC_i = \langle s, LP, u, t, v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t\rangle$. Without loss of generality, let $L(P_0) = l_0$. By Proposition 2, $d(t,u) = 1$, we have $|AS(t) \cap AS(u)| = n-k-1 \geq 1$ if $n-k \geq 2$. It means that we can find a subcomponent $A^{j_1}_{n,k}$ which $j_1 \in \langle n \rangle - \{i\}$, such that there exist two disjoint edges (u,p_1) and (t,q_1) in E^{i,j_1}. By Proposition 1, $(p_1,q_1) \in E(A^{j_1}_{n,k})$. Since $|N^*(t)| = n-k \geq 2$, we can find a subcomponent $A^{h_t}_{n,k}$ different from $A^i_{n,k}$ and $A^{j_1}_{n,k}$, and a vertex $t' \in V(A^{h_t}_{n,k})$ such that $(t,t') \in E^{i,h_t}$ for some $h_t \in \langle n \rangle - \{i, j_1\}$. By Proposition 2, $d(t,v) \leq 2$ hence $AS(t) \supseteq \{j_1, h_t\}$ and $AS(t) \neq AS(v)$, and $|N^*(v)| = n-k \geq 2$, we can find another subcomponent $A^{h_v}_{n,k}$, and a vertex $v' \in V(A^{h_v}_{n,k})$ such that $(v,v') \in E^{i,h_v}$ for some $h_v \in \langle n \rangle - \{i, j_1, h_t\}$. By Lemma 4, there exists a hamiltonian path HP of $A^{\langle n \rangle - \{i\}}_{n,k}$ joining t' and v'. Thus $\langle s, P_0, t, t', HP, v', v,\ RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., |V(A^i_{n,k})| - d(s,t)\}$, the distance between s and t on the cycle is l_0.

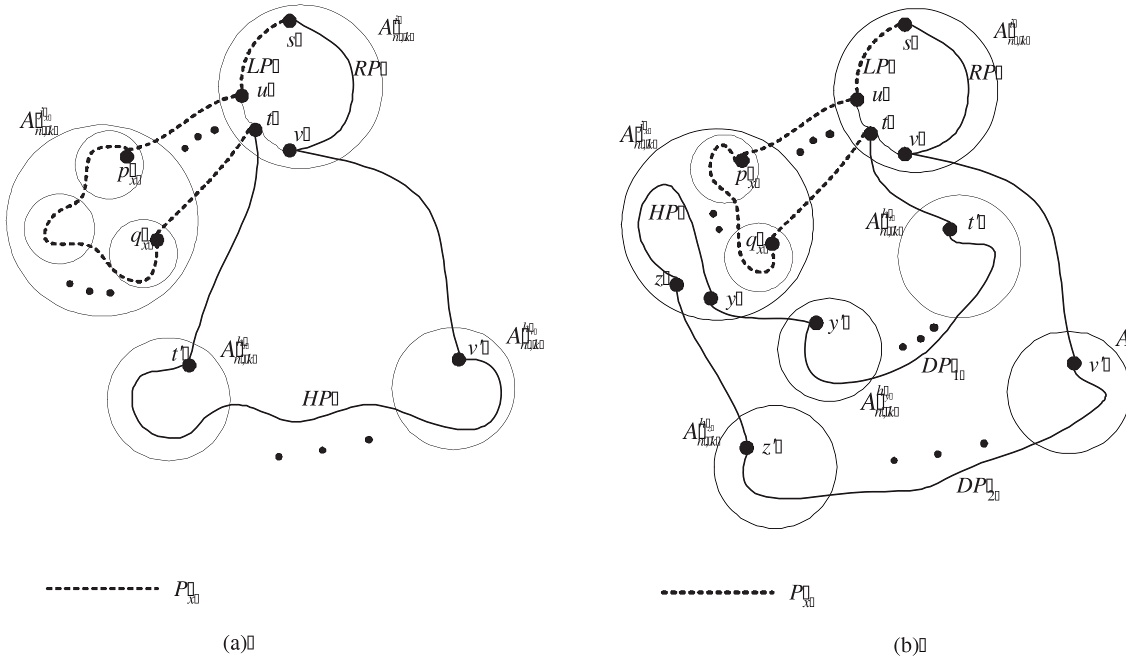

Figure 3.9: Theorem 10, Case 1.

Now we present an algorithm called *st-expansion* to expand the path P_0 between s and t to various lengths. We describe the detail as follows.

We can insert one subcomponent of $A_{n,k}^{j_1}$ to P_0 as follows. See Figure 3.10(a) for an illustration. Because p_1 and q_1 are adjacent, we may regard them as in the same subcomponent of $A_{n,k}^{j_1}$, say C. C is isomorphic to $A_{n-2,k-2}$. By Theorem 9, there is a hamiltonian path HP_1 of C joining p_1 and q_1 with $L(HP_1) = |V(A_{n-2,k-2})| - 1$. We can insert more than one subcomponent of $A_{n,k}^{j_1}$ to P_0 as following. See Figure 3.10(b) for an illustration. We regard p_1 and q_1 as in different subcomponents of $A_{n,k}^{j_1}$. By Lemma 4, there is a hamiltonian path HP_1 joining p_1 and q_1 with $L(HP_1) = m|V(A_{n-2,k-2})| - 1$, where m is the number of the subcomponents of $A_{n,k}^{j_1}$ we wanted to insert. Thus we can construct a path HP_1 between p_1 and q_1 such that $L(HP_1) = I_1|V(A_{n-2,k-2})| - 1$ for each integer I_1 with $1 \leq I_1 \leq n-1$. Let $P_1 = \langle s, LP, u, p_1, HP_1, q_1, t\rangle$. Thus we have $L(P_1) = l_0 + I_1|V(A_{n-2,k-2})| = l_0 + \frac{I_1(n-2)!}{(n-k)!}$. Since $d(s,t) \leq l_0 \leq |V(A_{n,k}^i)| - d(s,t)$, we have $\frac{I_1(n-2)!}{(n-k)!} + d(s,t) \leq L(P_1) \leq \frac{I_1(n-2)!}{(n-k)!} + \frac{(n-1)!}{(n-k)!} - d(s,t)$. For each $1 \leq I_1 \leq n-1$, $\frac{(I_1-1)(n-2)!}{(n-k)!} + \frac{(n-1)!}{(n-k)!} - d(s,t) \geq \frac{I_1(n-2)!}{(n-k)!} + d(s,t)$ if $n \geq 5$. Therefore, for each $l_1 \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{2(n-1)!}{(n-k)!} - d(s,t)\}$, we can construct a path P_1 from s to t such that the distance between s and t on the path is l_1.

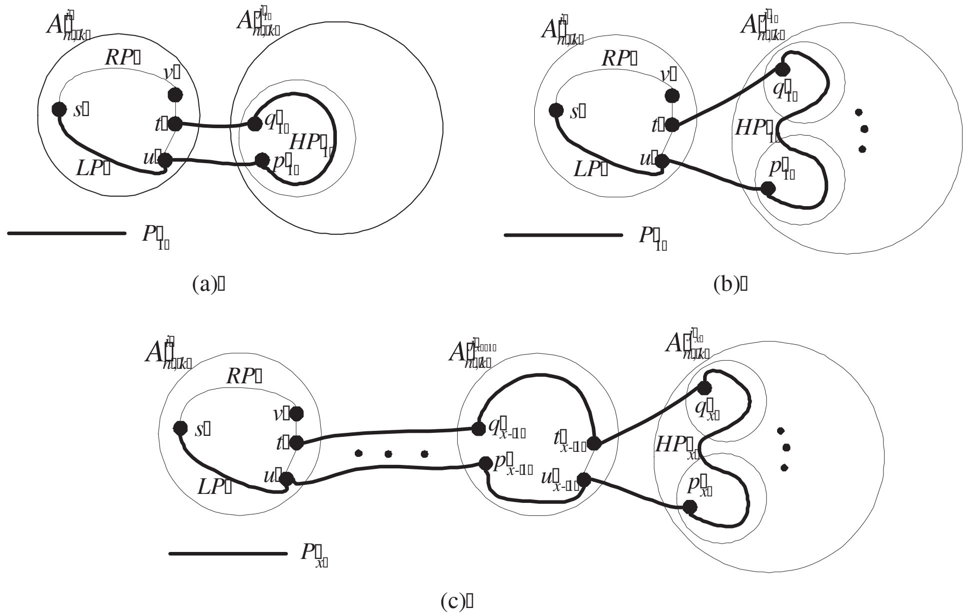

Figure 3.10: st-expansion.

Similar as above, we can expand the path between s and t more. For each $2 \le x \le \lfloor \frac{n}{2} \rfloor$, let u_{x-1} and t_{x-1} be two adjacent vertices on HP_{x-1}, where HP_{x-1} is a hamiltonian path of $A_{n,k}^{j_{x-1}}$ joining p_{x-1} and q_{x-1}. By Proposition 1 and Proposition 2, there exist two distinct edges (u_{x-1}, p_x) and (t_{x-1}, q_x) in E^{j_{x-1}, j_x} for some $j_x \in \langle n \rangle - \{i, h_t, h_v, j_1, ..., j_{x-1}\}$ such that $(p_x, q_x) \in E(A_{n,k}^{j_x})$. See Figure 3.10(c) for an illustration. We can insert one subcomponent of $A_{n,k}^{j_x}$ to P_0 as follows. Because p_x and q_x are adjacent, we may regard them as in the same subcomponent of $A_{n,k}^{j_x}$, say C. C is isomorphic to $A_{n-2,k-2}$. By Theorem 9, there is a hamiltonian path HP_x of C joining p_x and q_x with $L(HP_x) = |V(A_{n-2,k-2})| - 1$. We can insert more than one subcomponent of $A_{n,k}^{j_x}$ to P_0 as follows. We regard p_x and q_x as in different subcomponents of $A_{n,k}^{j_x}$. By Lemma 4, there is a hamiltonian path HP_x joining p_x and q_x with $L(HP_x) = m|V(A_{n-2,k-2})| - 1$, where m is the number of the subcomponents of $A_{n,k}^{j_x}$ we wanted to insert. Thus we can construct a path HP_x between p_x and q_x such that $L(HP_x) = I_x|V(A_{n-2,k-2})| - 1$ for each integer I_x with $1 \le I_x \le n-1$. Let $P_x = \langle s, LP, u, p_1, ..., p_x, HP_x, q_x, ..., q_1, t \rangle$. Thus we have $L(P_x) = l_0 + (x-1)|V(A_{n-1,k-1})| + I_x|V(A_{n-2,k-2})| = l_0 + \frac{(x-1)(n-1)!}{(n-k)!} + \frac{I_x(n-2)!}{(n-k)!}$. Since $d(s,t) \le l_0 \le |V(A_{n,k}^i)| - d(s,t)$, we have $\frac{(x-1)(n-1)!}{(n-k)!} + \frac{I_x(n-2)!}{(n-k)!} + d(s,t) \le L(P_x) \le \frac{I_x(n-2)!}{(n-k)!} + \frac{x(n-1)!}{(n-k)!} - d(s,t)$. For each $1 \le I_x \le n-1$, $\frac{(I_x-1)(n-2)!}{(n-k)!} + \frac{x(n-1)!}{(n-k)!} - d(s,t) \ge \frac{I_x(n-2)!}{(n-k)!} + \frac{(x-1)(n-1)!}{(n-k)!} + d(s,t)$ if $n \ge 5$. Therefore, for each $l_x \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{(x+1)(n-1)!}{(n-k)!} - d(s,t)\}$, we can construct a path P_x from s to t such that the distance between s and t on the path is l_x by using st-expansion. Notice that the maximal value of l_x is $\frac{(\lfloor \frac{n}{2} \rfloor + 1)(n-1)!}{(n-k)!} - d(s,t)$, which is greater than $\frac{n!}{2(n-k)!}$, and $\frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$. Hence for any integer l with $d(s,t) \leq l \leq \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t with the length of the path being l. We will use st-expansion for the remaining cases of the proof.

To construct a hamiltonian cycle, we consider the following two subcases:

Subcase 1.1: All the vertices of $A_{n,k}^{\{j_1,\ldots,j_x\}}$ are on the path P_x for some $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$. See Figure 3.9(a) for an illustration. By Lemma 4, there is a hamiltonian path HP of $A_{n,k}^{\langle n \rangle - \{i,j_1,\ldots,j_x\}}$ joining t' and v' which $t' \in V(A_{n,k}^{h_t})$ and $v' \in V(A_{n,k}^{h_v})$. Thus $\langle s, P_x, t, t', HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 1.2: Not all the vertices of $A_{n,k}^{\{j_1,\ldots,j_x\}}$ are on the path P_x for some $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$. See Figure 3.9(b) for an illustration. Then we can find two adjacent vertices y and z in $A_{n,k}^{j_x}$ which are not on the path P_x. Let $F \subseteq V(P_x)$. By Proposition 1 and Proposition 2, there exist two distinct edges $(y, y') \in E^{j_x,h_y}$ and $(z, z') \in E^{j_x,h_z}$ such that $y' \neq t' \in V(A_{n,k}^{h_y})$ and $z' \neq v' \in V(A_{n,k}^{h_z})$, respectively. If $A_{n,k}^{j_x} - F$ is isomorphic to $A_{n-2,k-2}$, by Theorem 9, there is a hamiltonian path HP from y to z in $A_{n,k}^{j_x} - F$. If $A_{n,k}^{j_x} - F$ contains more than one subcomponents of $A_{n,k}^{j_x}$, by Lemma 4 if $k-1 > 2$, and by Lemma 5(a) if $k-1 = 2$, there is a hamiltonian path HP from y to z in $A_{n,k}^{j_x} - F$. By Lemma 6, there exist two disjoint paths DP_1 and DP_2, such that DP_1 joins t' and y', and DP_2 joins v' and z'. Moreover, the two paths span all of the vertices in $A_{n,k}^{\langle n \rangle - \{i,j_1,\ldots,j_x\}}$. Thus $\langle s, P_x, t, t', DP_1, y', y, HP, z, z', DP_2, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Case 2: s and t belong to different subcomponents of $A_{n,k}$. Suppose that $s \in V(A_{n,k}^i)$ and $t \in V(A_{n,k}^j)$ for any $i \neq j \in \langle n \rangle$. By Lemma 3, there exists a minimum length path connecting s and t with the form $\langle s, MP, t'', t\rangle$ or $\langle s, MP, t'', t', t\rangle$, where MP is a path in $A_{n,k}^i$, $t'' \in V(A_{n,k}^i)$, and $t' \in V(A_{n,k}^j)$. Moreover, by considering the subcases of $n-k > 2$ and $n-k = 2$, we have the following four subcases:

Subcase 2.1: Suppose that $n-k > 2$, and the minimum length path connecting s and t has the form $\langle s, MP, t'', t\rangle$. Then $d(s,t) = d(s,t'') + 1$. See Figure 3.11(a) for an illustration. Since $A_{n,k}^i$ is isomorphic to $A_{n-1,k-1}$, by inductive hypothesis, for each $l_0 \in \{d(s,t''), d(s,t'')+1, d(s,t'')+2, \ldots, |V(A_{n,k}^i)| - d(s,t'')\}$, we can construct a hamiltonian cycle HC_i of $A_{n,k}^i$ such that the distance between s and t'' on the cycle is l_0. Let u and v be the two neighbors of t'' on HC_i. Let $HC_i = \langle s, LP, u, t'', v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t'', t\rangle$. Without loss of generality, let $L(P_0) = l_0 + 1$. By Proposition 2,

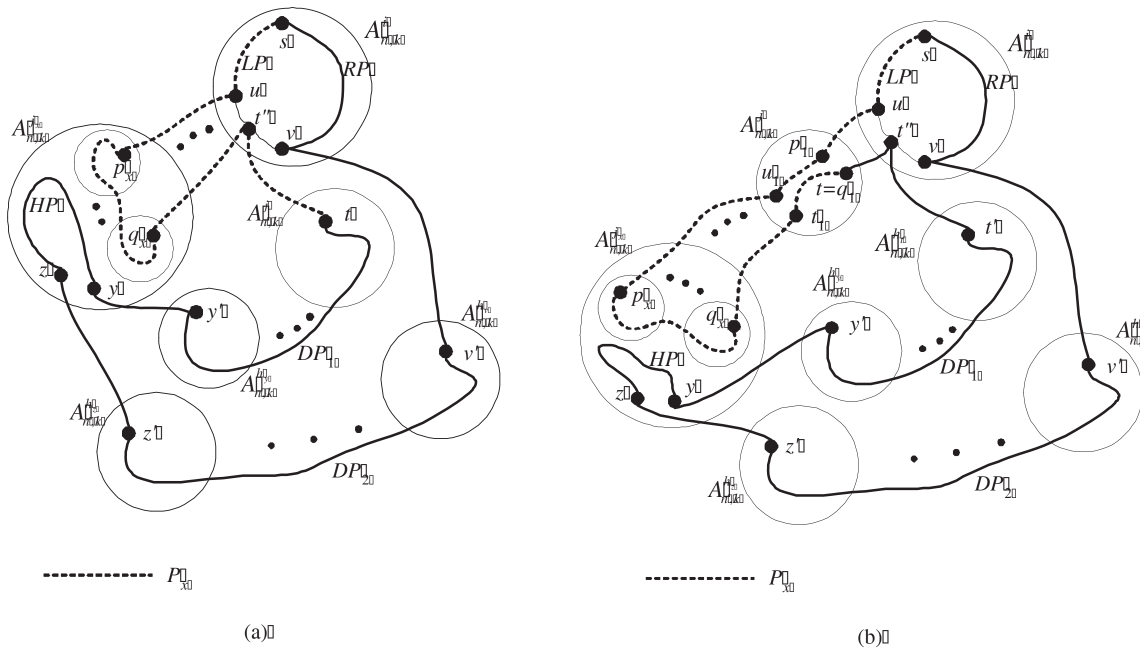

Figure 3.11: Theorem 10, Subcase 2.1 and Subcase 2.2.

$d(t'', u) = 1$, we have $|AS(t'') \cap AS(u)| = n - k - 1 > 1$ if $n - k > 2$. It means that we can find a subcomponent $A^{j_1}_{n,k}$ which $j_1 \in \langle n \rangle - \{i, j\}$, such that there exist two disjoint edges (u, p_1) and (t'', q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A^{j_1}_{n,k})$. By Proposition 2, $d(t'', v) \leq 2$ hence $AS(t'') \supseteq \{j, j_1\}$, and $AS(t'') \neq AS(v)$, and $|N^*(v)| = n - k > 2$, we can find a subcomponent $A^{h_v}_{n,k}$, and a vertex $v' \in V(A^{h_v}_{n,k})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n \rangle - \{i, j, j_1\}$. By Lemma 4, there exists a hamiltonian path HP of $A^{\langle n \rangle - \{i\}}_{n,k}$ joining t and v'. Thus $\langle s, P_0, t, HP, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., |V(A^i_{n,k})| - d(s,t) + 1\}$, the distance between s and t on the cycle is l_0.

Similar to Case 1, by using st''-expansion, for any integer l'' with $d(s, t'') \leq l'' \leq \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t'' with the length of the path being l''. Since $d(s, t'') = d(s, t) - 1$, for any integer l with $d(s, t) \leq l \leq \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t with the length of the path being l.

To construct a hamiltonian cycle, we consider the following two subcases:

Subcase 2.1.1: All the vertices of $A^{\{j_1,...,j_x\}}_{n,k}$ are on the path P_x for some $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$. By Lemma 4, there is a hamiltonian path HP of $A^{\langle n \rangle - \{i,j_1,...,j_x\}}_{n,k}$ joining t and v' which $t \in V(A^j_{n,k})$ and $v' \in V(A^{h_v}_{n,k})$. Thus $\langle s, P_x, t'', t, HP, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.1.2: Not all the vertices of $A_{n,k}^{\{j_1,...,j_x\}}$ are on the path P_x for some $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3.11(a) for an illustration. Then we can find two adjacent vertices y and z in $A_{n,k}^{j_x}$ which are not on the path P_x. Let $F \subseteq V(P_x)$. By Proposition 1 and Proposition 2, there exist two distinct edges $(y, y') \in E^{j_x,h_y}$ and $(z, z') \in E^{j_x,h_z}$ such that $y' \neq t' \in V(A_{n,k}^{h_y})$ and $z' \neq v' \in V(A_{n,k}^{h_z})$, respectively. If $A_{n,k}^{j_x} - F$ is isomorphic to $A_{n-2,k-2}$, by Theorem 9, there is a hamiltonian path HP from y to z in $A_{n,k}^{j_x} - F$. If $A_{n,k}^{j_x} - F$ contains more than one subcomponents of $A_{n,k}^{j_x}$, by Lemma 4, there is a hamiltonian path HP from y to z in $A_{n,k}^{j_x} - F$. By Lemma 6, there exist two disjoint paths DP_1 and DP_2, such that DP_1 joins t and y', and DP_2 joins v' and z'. Moreover, the two paths span all the vertices in $A_{n,k}^{\langle n \rangle - \{i,j_1,...,j_x\}}$. Thus $\langle s, P_x, t'', t, DP_1, y', y, HP, z, z', DP_2, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.2: Suppose that $n - k = 2$, and the minimum length path connecting s and t has the form $\langle s, MP, t'', t\rangle$. Then $d(s,t) = d(s,t'') + 1$. See Figure 3.11(b) for an illustration. Since $A_{n,k}^i$ is isomorphic to $A_{n-1,k-1}$, by inductive hypothesis, for each $l_0 \in \{d(s,t''), d(s,t'')+1, d(s,t'')+2, ..., |V(A_{n,k}^i)| - d(s,t'')\}$, we can construct a hamiltonian cycle HC_i of $A_{n,k}^i$ such that the distance between s and t'' on the cycle is l_0. Let u and v be the two neighbors of t'' on HC_i. Let $HC_i = \langle s, LP, u, t'', v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t'', t\rangle$. Without loss of generality, let $L(P_0) = l_0 + 1$. By Proposition 2, $d(t'', u) = 1$, we have $|AS(t'') \cap AS(u)| = n - k - 1 = 1$ if $n - k = 2$. It means that we can find a subcomponent $A_{n,k}^{j_1}$ which $j_1 \in \langle n \rangle - \{i\}$. If $t \notin V(A_{n,k}^{j_1})$, the proof is exactly the same as Case 2.1. So we consider the case that $t \in V(A_{n,k}^{j_1})$, that is, $j_1 = j$. Let $q_1 = t$. There exist two disjoint edges (u, p_1) and (t'', q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A_{n,k}^{j_1})$. By Proposition 2, $d(t'', v) \le 2$ hence $AS(t'') = \{j_1\}$, and $AS(t'') \neq AS(v)$. Since $|N^*(t'')| = n - k = 2$, we can find a subcomponent $A_{n,k}^{h_t}$, and a vertex $t' \in V(A_{n,k}^{h_t})$ such that $(t'', t') \in E^{i,h_t}$ for some $h_t \in \langle n \rangle - \{i, j_1\}$. Since $|N^*(v)| = n - k = 2$ and $AS(t'') \neq AS(v)$, we can find a subcomponent $A_{n,k}^{h_v}$, and a vertex $v' \in V(A_{n,k}^{h_v})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n \rangle - \{i, j_1, h_t\}$. By Lemma 4, there exists a hamiltonian path HP of $A_{n,k}^{\langle n \rangle - \{i\}}$ joining t and v'. Thus $\langle s, P_0, t, HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., |V(A_{n,k}^i)| - d(s,t)+1\}$, the distance between s and t on the cycle is l_0.

By using st''-expansion, for any integer l'' with $d(s,t'') \le l'' \le \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t'' with the length of the path being l''. Therefore, for any integer l with $d(s,t) \le l \le \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t with the length of the path being l.

To construct a hamiltonian cycle, we consider the following two subcases:

Subcase 2.2.1: All the vertices of $A_{n,k}^{\{j_1,...,j_x\}}$ are on the path P_x for some $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$. By Lemma 4, there is a hamiltonian path HP of $A_{n,k}^{\langle n \rangle - \{i,j_1,...,j_x\}}$ joining t' and v' where $t' \in V(A_{n,k}^{h_t})$ and $v' \in V(A_{n,k}^{h_v})$. Thus $\langle s, P_x, t, t'', t', HP, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.2.2: Not all the vertices of $A_{n,k}^{\{j_1,...,j_x\}}$ are on the path P_x for some $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$. See Figure 3.11(b) for an illustration. Then we can find two adjacent vertices y and z in $A_{n,k}^{j_x}$ which are not on the path P_x. Let $F \subseteq V(P_x)$. By Proposition 1 and Proposition 2, there exist two distinct edges $(y, y') \in E^{j_x,h_y}$ and $(z, z') \in E^{j_x,h_z}$ such that $y' \neq t' \in V(A_{n,k}^{h_y})$ and $z' \neq v' \in V(A_{n,k}^{h_z})$, respectively. If $A_{n,k}^{j_x} - F$ is isomorphic to $A_{n-2,k-2}$, by Theorem 9, there is a hamiltonian path HP from y to z in $A_{n,k}^{j_x} - F$. If $A_{n,k}^{j_x} - F$ contains more than one subcomponents of $A_{n,k}^{j_x}$, by Lemma 4, there is a hamiltonian path HP from y to z in $A_{n,k}^{j_x} - F$. By Lemma 6, there exist two disjoint paths DP_1 and DP_2, such that DP_1 joins t' and y', and DP_2 joins v' and z'. Moreover, the two paths span all of the vertices in $A_{n,k}^{\langle n \rangle - \{i,j_1,...,j_x\}}$. Thus $\langle s, P_x, t, t'', t', DP_1, y', y, HP, z, z', DP_2, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.3: Suppose that $n - k > 2$, and the minimum length path connecting s and t has the form $\langle s, MP, t'', t', t \rangle$. Then $d(s,t) = d(s,t'') + 2$. See Figure 3.12(a) for an illustration. Since $A_{n,k}^i$ is isomorphic to $A_{n-1,k-1}$, by inductive hypothesis, for each $l_0 \in \{d(s,t''), d(s,t'')+1, d(s,t'')+2, ..., |V(A_{n,k}^i)| - d(s,t'')\}$, we can construct a hamiltonian cycle HC_i of $A_{n,k}^i$ such that the distance between s and t'' on the cycle is l_0. Let u and v be the two neighbors of t'' on HC_i. Let $HC_i = \langle s, LP, u, t'', v, RP, s \rangle$, and let $P_0 = \langle s, LP, u, t'', t', t \rangle$. Without loss of generality, let $L(P_0) = l_0 + 2$. By Proposition 2, $d(t'', u) = 1$, we have $|AS(t'') \cap AS(u)| = n - k - 1 > 1$ if $n - k > 2$. It means that we can find a subcomponent $A_{n,k}^{j_1}$ which $j_1 \in \langle n \rangle - \{i, j\}$, such that there exist two disjoint edges (u, p_1) and (t'', q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A_{n,k}^{j_1})$. By Proposition 2, $d(t'', v) \leq 2$ hence $AS(t'') \supseteq \{j, j_1\}$, and $AS(t'') \neq AS(v)$, and $|N^*(v)| = n - k > 2$, we can find a subcomponent $A_{n,k}^{h_v}$, and a vertex $v' \in V(A_{n,k}^{h_v})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n \rangle - \{i, j, j_1\}$. Let $F \subseteq V(A_{n,k})$ and $F' = \{t'\}$. By Lemma 4, there exists a hamiltonian path HP of $A_{n,k}^{\langle n \rangle - \{i\}} - F'$ joining t and v'. Thus $\langle s, P_0, t, HP, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., |V(A_{n,k}^i)| - d(s,t)+2\}$, the distance between s and t on the cycle is l_0.

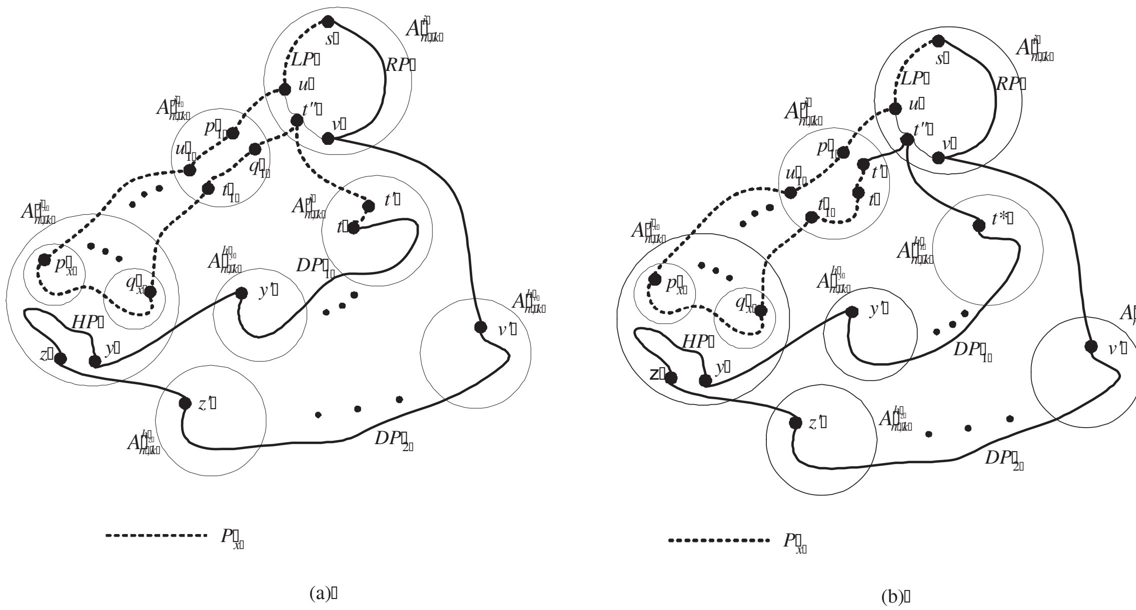

Figure 3.12: Theorem 10, Subcase 2.3 and Subcase 2.4.

By using st''-expansion, for any integer l'' with $d(s,t'') \le l'' \le \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t'' with the length of the path being l''. Since $d(s,t'') = d(s,t) - 2$, for any integer l with $d(s,t) \le l \le \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t with the length of the path being l.

To construct a hamiltonian cycle, we consider two subcases:

Subcase 2.3.1: All the vertices of $A_{n,k}^{\{j_1,\ldots,j_x\}}$ are on the path P_x for some $1 \le x \le \lfloor \frac{n}{2} \rfloor$. By Lemma 4, there is a hamiltonian path HP of $A_{n,k}^{\langle n\rangle - \{i,j_1,\ldots,j_x\}} - F'$ joining t and v' which $F' = \{t'\}$, $t \in V(A_{n,k}^{j})$ and $v' \in V(A_{n,k}^{h_v})$. Thus $\langle s, P_x, t'', t', t, HP, v', v,\ RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.3.2: Not all the vertices of $A_{n,k}^{\{j_1,\ldots,j_x\}}$ are on the path P_x for some $1 \le x \le \lfloor \frac{n}{2} \rfloor$. See Figure 3.12(a) for an illustration. Then we can find two adjacent vertices y and z in $A_{n,k}^{j_x}$ which are not on the path P_x. Let $F \subseteq V(P_x)$. By Proposition 1 and Proposition 2, there exist two distinct edges $(y,y') \in E^{j_x,h_y}$ and $(z,z') \in E^{j_x,h_z}$ such that $y' \ne t' \in V(A_{n,k}^{h_y})$ and $z' \ne v' \in V(A_{n,k}^{h_z})$, respectively. If $A_{n,k}^{j_x} - F$ is isomorphic to $A_{n-2,k-2}$, by Theorem 9, there is a hamiltonian path HP from y to z in $A_{n,k}^{j_x} - F$. If $A_{n,k}^{j_x} - F$ contains more than one subcomponents of $A_{n,k}^{j_x}$, by Lemma 4, there is a hamiltonian path HP from y to z in $A_{n,k}^{j_x} - F$. By Lemma 6, there exist two disjoint paths DP_1 and DP_2, such that DP_1 joins t and y', and DP_2 joins v' and z'. Moreover, the two paths span all the vertices in $A_{n,k}^{\langle n\rangle-\{i,j_1,...,j_x\}} - F'$ which $F' = \{t'\}$. Thus $\langle s, P_x, t'', t', t, DP_1, y', y, HP, z, z', DP_2, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.4: Suppose that $n-k=2$, and the minimum length path connecting s and t has the form $\langle s, MP, t'', t', t\rangle$. Then $d(s,t) = d(s,t'')+2$. See Figure 3.12(b) for an illustration. Since $A_{n,k}^i$ is isomorphic to $A_{n-1,k-1}$, by inductive hypothesis, for each $l_0 \in \{d(s,t''), d(s,t'')+1, d(s,t'')+2, ..., |V(A_{n,k}^i)| - d(s,t'')\}$, we can construct a hamiltonian cycle HC_i of $A_{n,k}^i$ such that the distance between s and t'' on the cycle is l_0. Let u and v be the two neighbors of t'' on HC_i. Let $HC_i = \langle s, LP, u, t'', v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t'', t', t\rangle$. Without loss of generality, let $L(P_0) = l_0+2$. By Proposition 2, $d(t'', u) = 1$, we have $|AS(t'') \cap AS(u)| = n-k-1 = 1$ if $n-k=2$. It means that we can find a subcomponent $A_{n,k}^{j_1}$ which $j_1 \in \langle n\rangle - \{i\}$. If $t, t' \notin V(A_{n,k}^{j_1})$, the proof is exactly the same as Subcase 2.3. So we consider the case that $t, t' \in V(A_{n,k}^{j_1})$, that is, $j_1 = j$. There exist two disjoint edges (u, p_1) and (t'', t') in E^{i,j_1}. By Proposition 1, $(p_1, t') \in E(A_{n,k}^{j_1})$. By Proposition 2, $d(t'', v) \leq 2$ hence $AS(t'') = \{j_1\}$, and $AS(t'') \neq AS(v)$. Since $|N^*(t'')| = n-k=2$, we can find a subcomponent $A_{n,k}^{h_t}$, and a vertex $t^* \in V(A_{n,k}^{h_t})$ such that $(t'', t^*) \in E^{i,h_t}$ for some $h_t \in \langle n\rangle - \{i, j_1\}$. Since $|N^*(v)| = n-k = 2$ and $AS(t'') \neq AS(v)$, we can find a subcomponent $A_{n,k}^{h_v}$, and a vertex $v' \in V(A_{n,k}^{h_v})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, j_1, h_t\}$. Let $F \subseteq V(A_{n,k})$ and $F' = \{t^*\}$. By Lemma 4, there exists a hamiltonian path HP of $A_{n,k}^{\langle n\rangle-\{i\}}$ joining t and v'. Thus $\langle s, P_0, t, HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., |V(A_{n,k}^i)| - d(s,t)+1\}$, the distance between s and t on the cycle is l_0.

Now we modify st-expansion slightly to expand the path P_0 between s and t to various lengths. We describe the detail as follows.

For $n=5$, that is, $A_{5,3}$, we have $d(s,t)=4$ in this subcase. As we describe above, $\langle s, LP, u, t'', t, HP, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{4, 5, 6, ..., 12\}$, the distance between s and t on the cycle is l_0. Let $F_j \subseteq V(A_{n,k}^{j_1})$ and $F_j = \{t'\}$. By Theorem 9, we can find a hamiltonian path HP_1 of $A_{n,k}^{j_1} - F_j$ joining p_1 and t. Let $P_1 = \langle s, LP, u, p_1, HP_1, t\rangle$. We have $11 \leq L(P_1) \leq 19$. Therefore, for each $l_1 \in \{4, 5, 6, ..., 19\}$, we can construct a path P_1 from s to t such that the distance between s and t on the path is l_1 in $A_{5,3}$. Suppose that $n \geq 6$. We can insert one subcomponent of $A_{n,k}^{j_1}$, which is isomorphic to $A_{n-2,k-2}$, to P_0 as follows. Because $d(p_1, t) = 2$ which is less than the diameter of $A_{n-2,k-2}$, and by the symmetric property of the arrangement graph, we may regard p_1 and t as in the same subcomponent of $A_{n,k}^{j_1}$, say C. By Lemma 4, there is a hamiltonian path HP_1 of $C - F_j$ joining p_1 and t with $L(HP_1) = |V(A_{n-2,k-2})| - 2$. Let

C^* be the m subcomponents of $A^{j_1}_{n,k}$ we wanted to insert to P_0, where m is the number of the subcomponents of $A^{j_1}_{n,k}$. We regard p_1 and t as in different subcomponents of $A^{j_1}_{n,k}$. By Lemma 4, there is a hamiltonian path HP_1 of $C^* - F_j$ joining p_1 and t with $L(HP_1) = m|V(A_{n-2,k-2})| - 2$. Thus we can construct a path HP_1 between p_1 and t such that $L(HP_1) = I_1|V(A_{n-2,k-2})| - 2$ for each integer I_1 with $1 \le I_1 \le n-1$. Let $P_1 = \langle s, LP, u, p_1, HP_1, t\rangle$. Thus we have $L(P_1) = l_0 + I_1|V(A_{n-2,k-2})| - 2 = l_0 + \frac{I_1(n-2)!}{(n-k)!} - 2$. Since $d(s,t) - 2 \le l_0 \le |V(A^i_{n,k})| - d(s,t) + 2$, we have $\frac{I_1(n-2)!}{(n-k)!} + d(s,t) - 4 \le L(P_1) \le \frac{I_1(n-2)!}{(n-k)!} + \frac{(n-1)!}{(n-k)!} - d(s,t)$. For each $1 \le I_1 \le n-1$, $\frac{(I_1-1)(n-2)!}{(n-k)!} + \frac{(n-1)!}{(n-k)!} - d(s,t) \ge \frac{I_1(n-2)!}{(n-k)!} + d(s,t) - 4$ if $n \ge 6$. Therefore, for each $l_1 \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{2(n-1)!}{(n-k)!} - d(s,t)\}$, we can construct a path P_1 from s to t such that the distance between s and t on the path is l_1. Then, similar to st-expansion we described in Case 1, we can expand the path between s and t such that for each $l_x \in \{d(s,t), d(s,t)+1, d(s,t)+2, ..., \frac{(x+1)(n-1)!}{(n-k)!} - d(s,t)\}$, we can construct a path P_x from s to t such that the distance between s and t on the path is l_x. Hence for any integer l with $d(s,t) \le l \le \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t with the length of the path being l.

To construct a hamiltonian cycle, the proof is the same as that given in Subcase 2.2.1 and Subcase 2.2.2 by replacing vertex t' in Subcase 2.2 with vertex t^* in this subcase.

Hence the theorem is proved. □

3.4.2 Panconnectivity of the Arrangement Graphs $A_{n,k}$

In this subsection, we will prove that the arrangement graph $A_{n,k}$ is panconnected for all $n \ge 3$ and $n - k \ge 2$ by applying the above theorem.

Theorem 11. *The arrangement graph $A_{n,k}$ is panconnected for all $n \ge 3$ and $n - k \ge 2$.*

Proof. For $k = 1$, by Lemma 7, $A_{n,1}$ is panconnected for all $n \ge 3$. Chiang and Chen [8] showed that the $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n, and Chang et al. [7] proved that AG_n is panconnected for all $n \ge 4$. Hence the result holds for $n \ge 4$ and $k = n-2$. Now we prove that $A_{n,k}$ is panconnected for all $n \ge 5$ and $n - k > 2$. Suppose that u and v are any two distinct vertices in $A_{n,k}$. By Theorem 10, $A_{n,k}$ is panpositionable hamiltonian. That is, for each integer l with $d(u,v) \le l \le |V(A_{n,k})| - d(u,v)$, we can construct a path P of length l joining u and v.

For each integer l with $|V(A_{n,k})| - d(u,v) + 1 \le l \le |V(A_{n,k})| - 1$, we can construct a path P of length l joining u and v as following. The diameter of $A_{n,k}$ is $\lfloor \frac{3k}{2} \rfloor$, and we have

$d(u,v) \leq \lfloor \frac{3k}{2} \rfloor$. By Theorem 9, $A_{n,k}$ is $k(n-k)-3$ fault tolerant hamiltonian connected. For $n \geq 5$ and $n-k>2$, we have $k(n-k)-3 \geq \lfloor \frac{3k}{2} \rfloor - 1$. That means that for each integer l with $|V(A_{n,k})| - d(u,v) + 1 \leq l \leq |V(A_{n,k})| - 1$, we can construct a path P of length l joining u and v by regarding the vertices not in P as faulty vertices. Therefore, for each integer l with $d(u,v) \leq l \leq |V(A_{n,k})| - 1$, there is a path of length l joining u and v in $A_{n,k}$. The theorem is proved. □

Example. There are 60 vertices in $A_{5,3}$, and the diameter of $A_{5,3}$ is 4. Let u and v be two vertices in $A_{5,3}$ with $d(u,v) = 4$. By the panpositionable hamiltonian property, we can find a path joining u and v with length $l \in \{4, 5, 6, ..., 56\}$. Let $F \subseteq V(A_{5,3}) - \{u, v\}$. We can find three paths of length 57, 58, and 59 joining u and v with $|F| = 2$, $|F| = 1$, and $|F| = 0$ respectively.

By choosing two adjacent vertices u and v and applying the above theorem, we can obtain the following corollary immediately.

Corollary 2. *The arrangement graph $A_{n,k}$ is pancyclic for all $n \geq 3$ and $n-k \geq 2$.*

3.5 The Spanning Diameter of the Arrangement Graphs

Another important issue in the design of an interconnection network is connectivity. The *connectivity* of G, $\kappa(G)$ is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. Let $G = (V, E)$ be a graph with connectivity $\kappa(G) = \kappa$. It follows from Menger's Theorem [36] that there are l *internally node-disjoint* (abbreviated as disjoint) paths joining any two vertices u and v when $l \leq \kappa(G)$. A *container* $C(u,v)$ between two distinct vertices u and v in G is a set of disjoint paths between u and v. The *width* of a $C(u,v)$, written as $w(C(u,v))$, is its cardinality. A w-*container* is a container of width w. The *length* of a $C(u,v)$, written as $l(C(u,v))$, is the length of the longest path in $C(u,v)$. The w-*wide distance* between u and v, $\delta_w(u,v)$, is $\min\{l(C(u,v)) \mid C(u,v)$ is w-container$\}$.

In this section, we are interesting in a particular type of containers. A w-container $C(u,v)$ is a w^*-*container* if every vertex of G is incident with a path in $C(u,v)$. A graph G is w^*-*connected* if there exists a w^*-container between any two distinct vertices u and v. Obviously, a graph G is 1^*-connected if and only if it is hamiltonian connected. Moreover, a graph G is 2^*-connected if it is hamiltonian. The study of w^*-connected graph is motivated by the globally 3^*-connected graphs proposed by Albert, Aldred, and Holton [4]. A globally 3^*-connected graph is a 3-regular 3^*-connected graph. We also define w^*-*distance* between any two vertices u and v, $d_w^{s_L}(u,v)$, to be $\min\{l(C(u,v)) \mid C(u,v)$ is w^*-container$\}$. The w^{*_L}-*spanning diameter* of G, denoted by $D_w^{s_L}(G)$, as the maximum number of $d_w^{s_L}(u,v)$. Lin et al. studied the spanning diameter of the star graphs in [31]. It is proved that $D_{\kappa(S_n)}^{s_L}(S_n) = \frac{n!}{n-2} + 1$ and $D_2^{s_L}(S_n) = \frac{n!}{2} + 1$.

In this section, we will discuss about the spanning diameter of the arrangement graphs $A_{n,k}$. We will prove that $D_2^{s_L}(A_{n,k}) = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$ if $k \geq 2$ and $n-k \geq 2$ by applying the panpositionable hamiltonian property of the arrangement graphs. Assume that x and y are any two distinct vertices in the arrangement graph $A_{n,k}$ with $k \geq 2$ and $n-k \geq 2$. Now we prove that there exist two internally-disjoint paths P_1 and P_2 joining x and y such that $P_1 \cup P_2$ spans $A_{n,k}$ and $L(P_i) = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$ for $i = 1, 2$.

Theorem 12. *Suppose that $k \geq 2$ and $n-k \geq 2$. Then $d_2^{s_L}(x,y) = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$ for any two vertices x and y in the arrangement graph $A_{n,k}$. That is, the 2^{*_L}-diameter $D_2^{s_L}(A_{n,k}) = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$.*

Proof. By Theorem 10, for any two different vertices x and y in the arrangement graph $A_{n,k}$ and for any integer l satisfying $d(x,y) \leq l \leq |V(A_{n,k})| - d(x,y)$, there exists a hamiltonian cycle of $A_{n,k}$ such that the relative distance of x and y on the cycle is l. Since the diameter of $A_{n,k}$ is $\lfloor \frac{3k}{2} \rfloor$, $d(x,y) \leq \lfloor \frac{3k}{2} \rfloor$. Then $\lfloor \frac{3k}{2} \rfloor \leq \frac{|V(A_{n,k})|}{2} \leq |V(A_{n,k})| - \lfloor \frac{3k}{2} \rfloor$. Let $l = \frac{|V(A_{n,k})|}{2}$, we can find a hamiltonian cycle $C = \langle x, P_1, y, P_2, x \rangle$ of $A_{n,k}$ such that the distance between x and y on C is $\frac{|V(A_{n,k})|}{2}$. Obviously, P_1 and P_2 forms a 2^*-container. Moreover, $L(P_1) = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$, and $P_2 = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$. Hence the statement follows. □

For a graph G with even vertices, $D_2^{s_L}(G) \geq \frac{|V(G)|}{2}$. The arrangement graph $A_{n,k}$ with $k \geq 2$ has even vertices, thus our result about the 2^{*_L}-diameter of $A_{n,k}$ is optimal.

Chapter 4

The Globally Bi-3^*-Connected Property of the Honeycomb Rectangular Torus

We discuss another property about the connectivity of an interconnection network called globally 3^*-connected property. Suppose that x and y are two vertices in a graph G. If there exist three internally-disjoint paths joining x and y such that these three paths span all the vertices in G, we say that G is globally 3^*-connected. In this chapter, we will show that in any honeycomb rectangular torus HReT(m, n), there exist three internally-disjoint spanning paths joining x and y whenever x and y belong to different partite sets. Moreover, for any pair of vertices x and y in the same partite set, there exists a vertex z in the partite set not containing x and y, such that there exist three internally-disjoint spanning paths of $G - \{z\}$ joining x and y. Furthermore, for any three vertices x, y and z of the same partite set there exist three internally-disjoint spanning paths of $G - \{z\}$ joining x and y if and only if $n \geq 6$ or $m = 2$.

4.1 Honeycomb Rectangular Torus

We give a review of the idea of w^*-*container*, and introduce the concept of globally bi-3^*-connected graphs in the following subsection. Then we give the definition of the honeycomb rectangular torus in subsection 4.1.2.

4.1.1 Globally Bi-3^*-Connected Graphs

As we introduced in section 3.5, a k-container $C_k(x,y)$ in a graph G is a set of k internally vertex-disjoint paths between x and y. A k^*-container $C_{k^*}(x,y)$ in a graph G is a k-container such that every vertex of G is on some path in $C_k(x,y)$. Let G be a k-connected graph, it follows from Menger's Theorem [36] that there exists a k-container between any two different vertices of G. A graph G is k^*-connected if there exists a k^*-container between any two distinct vertices in G. Obviously, a graph G is 1^*-connected if and only if it is hamiltonian connected. Moreover, a graph G is 2^*-connected if it is hamiltonian. The study of k^*-connected graph is motivated by the 3^*-connected graphs proposed by Albert et al. [4]. In [4], Albert et al. first studied those cubic 3-connected graphs such that there exists a 3^*-container between any pair of vertices. Such graphs are called *globally* 3^**-connected graphs*.

Since every globally 3^*-connected graph is cubic, it contains an even number of vertices. Assume that $G = (V_1 \cup V_2, E)$ is a cubic 3-connected bipartite graphs with bipartition V_1 and V_2 such that $|V_1| \geq |V_2| \geq 2$. Let x and y be two distinct vertices in V_2. Assume that there exists a 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in G. Suppose that there are a_i vertices of V_1 in P_i for $i = 1,2,3$. Obviously, there are $a_i + 1$ vertices of V_2 in P_i for $i = 1,2,3$. Hence, there are $a_1 + a_2 + a_3$ vertices of V_1 incidence with $P_1 \cup P_2 \cup P_3$ and there are $(a_1+1)+(a_2+1)+(a_3+1)-4 = a_1+a_2+a_3-1$ vertices of V_2 incidence with $P_1 \cup P_2 \cup P_3$. Therefore, any cubic 3-connected bipartite graph is not globally 3^*-connected.

For this reason, we say that a cubic bipartite graph $G = (V_1 \cup V_2, E)$ is *globally bi-*3^**-connected* if there exists a 3^*-container between any pair of vertices of the different partite sets. Obviously, $|V_1| = |V_2|$ in any globally bi-3^*-connected with bipartition V_1 and V_2. Furthermore, a globally bi-3^*-connected graph is *hyper* if there exists a $C_{3^*}(x,y)$ in $G-\{z\}$ for any three vertices x,y, and z of the same partite set of G. A globally bi-3^*-connected graph is *strong* if for any x and y in the same partite set of G, there exists a vertex z of the same partite set as the one that contains x and y such that $G - \{z\}$ has a $C_{3^*}(x,y)$. Obviously, any globally bi-3^*-connected is strong if it is hyper. The concept of globally bi-3^*-connected, hyper globally bi-3^*-connected, and strong globally bi-3^*-connected was proposed by Kao et al. [26]. It is proved that $G - \{e\}$ is hamiltonian for any $e \in E(G)$ if G is globally bi-3^*-connected. Moreover, $G - \{x,y\}$ is hamiltonian for any $x \in V_1$ and $y \in V_2$ if G is hyper globally bi-3^*-connected.

4.1.2 Honeycomb Rectangular Torus HReT(m,n)

Assume that m and n are positive even integers with $n \geq 4$. The honeycomb rectangular torus HReT(m,n), introduced by Stojmenovic [40], is an alternative to existing networks

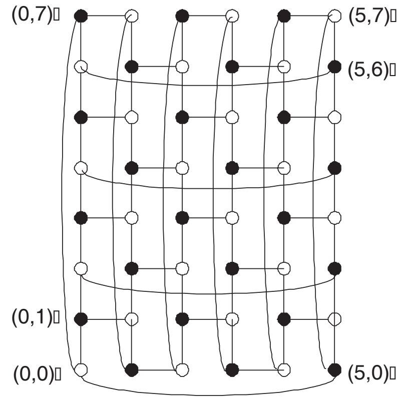

Figure 4.1: The honeycomb rectangular torus HReT(6,8).

such as mesh-connected networks in parallel and distributed computing. There are many studies on the properties of HReT(m, n) [9, 35, 40]. Stojmenovic [40] showed that the network cost of the honeycomb rectangular torus, which is defined as the product of degree and the diameter, is better than the other families based on mesh-connected computers and tori. Megson et al. [35] established the hamiltonian property of honeycomb torus. In particular, Cho and Hsu [9] proved that HReT$(m, n) - e$ is hamiltonian for any edge $e \in E(\text{HReT}(m, n))$. Furthermore, HReT$(m, n) - \{x, y\}$ is hamiltonian for any $x \in V_0$ and $y \in V_1$ if $n \geq 6$.

For any two positive integers r and s, we use $[r]_s$ to denote r (**mod** s). We use the brick drawing, proposed in [40], to define the honeycomb rectangular torus. The honeycomb rectangular torus HReT(m, n) is the graph with the vertex set $\{(i, j) \mid 0 \leq i < m, 0 \leq j < n\}$ such that (i, j) and (k, l) are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = [l \pm 1]_n$;
2. $j = l$ and $k = [i + 1]_m$ if $i + j$ is odd; and
3. $j = l$ and $k = [i - 1]_m$ if $i + j$ is even.

For example, the graph HReT$(6, 8)$ is shown in Figure 4.1. It is easy to see that HReT(m, n) is a bipartite graph with bipartition V_0 and V_1 where $V_0 = \{(i, j) \mid i + j$ is even$\}$ and $V_1 = \{(i, j) \mid i + j$ is odd$\}$. Moreover, $|V_0| = |V_1|$.

Based on Menger's Theorem [36], the 3-connected property of the honeycomb rectangular torus HReT(m, n) can be derived. In this chapter, we study the globally bi-3*-connected property of the honeycomb rectangular torus HReT(m, n). We prove that any honeycomb rectangular torus HReT(m,n) is strongly globally bi-3*-connected. Moreover, HReT(m,n) is hyper globally bi-3*-connected if and only if $n \geq 6$ or $m = 2$.

4.2 A Basic Algorithm

In this section, we present an algorithm. The purpose of this algorithm is to extend a 3*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ of HReT(m,n) to a 3*-container of HReT$(m+2,n)$.

Algorithm 1. For $0 \leq i \leq m-1$, let $f_i : V(\text{HReT}(m,n)) \to V(\text{HReT}(m+2,n))$ be a function so assigned

$$f_i(k,l) = \begin{cases} (k,l) & \text{if } i \geq k \geq 0 \\ (k+2,l) & \text{otherwise.} \end{cases}$$

For $0 \leq i \leq m-1$ and $0 \leq j,k \leq n-1$, let $Q_i(j,[j+k]_n)$ denote the path $\langle (i,[j]_n),(i,[j+1]_n),(i,[j+2]_n),...,(i,[j+k]_n)\rangle$ in HReT(m,n). Suppose that $C_3(x,y)$ is a 3-container of HReT(m,n) containing at least one edge joining vertices of column i to vertices of column $[i+1]_m$; i.e., $((i,j),([i+1]_m,j))$ in $E(C_3(x,y))$ for some $0 \leq j \leq n-1$. Let $0 \leq k_0 < k_1 < ... < k_t \leq n-1$ be the indices such that $((i,k_j),(i+1,k_j)) \in E(C_3(x,y))$. We construct $C'_{3,i}(x,y)$ as follows:

Let $\overline{C_{3,i}(x,y)}$ be the image of $C_3(x,y) - \{((i,k_j),(i+1,k_j)) \mid 0 \leq k_j \leq n-1\}$ under f_i. We set $j' = [j]_{(t+1)}$ and define A_j as

$$\langle (i,[k_j]_n),([i+1]_{m+2},[k_j]_n),Q_{[i+1]_{m+2}}([k_j]_n,[k_{j'}-1]_n),([i+1]_{m+2},[k_{j'}-1]_n),$$

$$([i+2]_{m+2},[k_{j'}-1]_n),Q^{-1}_{[i+2]_{m+2}}([k_j]_n,[k_{j'}-1]_n),([i+2]_{m+2},[k_j]_n),([i+3]_{m+2},[k_j]_n)\rangle.$$

Obviously, A_j is a path joining $(i,[k_j]_n)$ and $(i+3,[k_j]_n)$ for $0 \leq j < t$. It is easy to see that edges of $\overline{C_{3,i}(x,y)}$ together with edges of A_j, with $0 \leq j \leq t$ form a 3-container $C'_{3,i}(x,y)$ of HReT$(m+2,n)$. For example, a 3*-container $C_{3^*}((0,0),(2,2))$ of HReT$(4,12) - \{(1,7)\}$ is shown in Figure 4.2(a). The corresponding $C'_{3,1}((0,0),(2,2))$ is shown in Figure 4.2(b). We have the following lemma.

Lemma 10. *Suppose that $C_3(x,y)$ is a 3-container of HReT(m,n) containing at least one edge joining vertices of column i to vertices of column $[i+1]_m$. Then $C'_{3,i}(x,y)$ forms a 3-container of HReT$(m+2,n)$ containing at least one edge joining the vertices of column l*

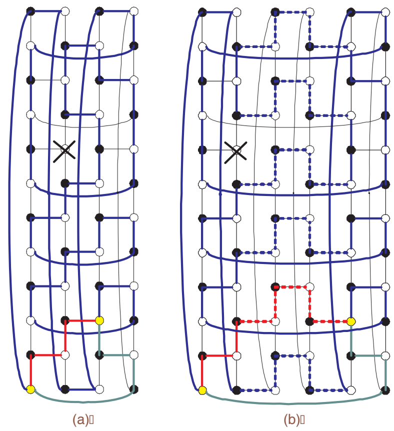

Figure 4.2: Illustrations for Algorithm 1.

to the vertices of column $[l+1]_m$ *for any* $l \in \{i, [i+1]_m, [i+2]_m\}$*. Moreover,* $C'_{3^*,i}(x, y)$ *is a* 3^**-container of* $HReT(m+2, n)$ *if* $C_{3^*}(x, y)$ *is a* 3^**-container of* $HReT(m, n)$*. Furthermore,* $C'_{3^*,i}(x, y)$ *is a* 3^**-container of* $HReT(m + 2, n) - \{f_i(z)\}$ *if* $C_{3^*}(x, y)$ *is a* 3^**-container of* $HReT(m, n) - \{z\}$*.*

Lemma 11. *Suppose that* $C_3(x, y)$ *is a 3-container of* $HReT(2, n)$ *containing at least one edge in* $\{((0, j), (1, j)) \mid j \text{ is odd}\}$ *and at least one edge in* $\{((0, j), (1, j)) \mid j \text{ is even}\}$*. Then* $C'_{3,i}(x, y)$ *with* $i \in \{0, 1\}$ *forms a 3-container of* $HReT(4, n)$ *containing at least one edge joining the vertices of column* l *to the vertices of column* $l+1$ *for any* $l \in \{0, 1, 2, 3\}$*. Moreover,* $C'_{3^*,i}(x, y)$ *is a* 3^**-container of* $HReT(m+2, n)$ *if* $C_{3^*}(x, y)$ *is a* 3^**-container of* $HReT(m, n)$*. Furthermore,* $C'_{3^*,i}(x, y)$ *is a* 3^**-container of* $HReT(m + 2, n) - \{f_i(z)\}$ *if* $C_{3^*}(x, y)$ *is a* 3^**-container of* $HReT(m, n) - \{z\}$*.*

With Lemma 10 and Lemma 11, we say a 3-container $C_3(x, y)$ of HReT$(2, n)$ is *regular* if $C_3(x, y)$ contains at least one edge in $\{((0, j), (1, j)) \mid j \text{ is odd}\}$ and at least one edge in $\{((0, j), (1, j)) \mid j \text{ is even}\}$. Assume that $m \geq 4$. We say a 3-container $C_3(x, y)$ of HReT(m, n) is *regular* if $C_3(x, y)$ contains at least one edge joining vertices in column i to vertices in column $[i + 1]_m$ for $0 \leq i \leq m - 1$. We have the following lemma.

Lemma 12. *Suppose that* $C_{3^*}(x, y)$ *is a regular* 3^**-container for* $HReT(m, n)$*. Then* $C'_{3^*,i}(x, y)$ *is a regular* 3^**-container for* $HReT(m + 2, n)$ *for every* $0 \leq i < m$*. Moreover, suppose that* $C_{3^*}(x, y)$ *is a regular* 3^**-container for* $HReT(m, n) - \{z\}$*. Then* $C'_{3^*,i}(x, y)$ *is a regular* 3^**-container for* $HReT(m + 2, n) - \{f_i(z)\}$ *for every* $0 \leq i < m$*.*

4.3 The Globally Bi-3^*-Connected Property of Honeycomb Rectangular Torus HReT(2,n)

We first discuss the globally bi-3^*-connected property of the honeycomb rectangular torus HReT(m,n) for $m = 2$. Then we show the globally bi-3^*-connected properties of HReT(m,n) for $m = 2$ and general m in sections 4.4 and 4.5, respectively.

For $h = \{0,1\}$ and $0 \le j,k \le n-1$, let $R_h(j,[j+k]_n)$ denote the path $\langle (h,[j]_n),(h,[j+1]_n),([h+1]_m,[j+1]_n),([h+1]_m,[j+2]_n),(h,[j+2]_n),...,([h+1]_m,[j+k-1]_n),(h,[j+k-1]_n),(h,[j+k]_n)\rangle$ in HReT$(2,n)$.

Lemma 13. *Let x and y be any two vertices of HReT$(2,n) = (V_0 \cup V_1, E)$ with $x \in V_0$ and $y \in V_1$. Then there exists a regular 3^*-container $C_{3^*}(x,y)$ of HReT$(2,n)$. Hence HReT$(2,n)$ is globally bi-3^*-connected.*

Proof. Without loss of generality, we may assume that $x = (0,0)$ and $y = (i,j)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1,P_2,P_3\}$ in HReT$(2,n)$. We have the following cases:

Case 1: $i = 0$ and j is odd. The corresponding paths are:

$$\begin{aligned} P_1 &= \langle (0,0), Q_0(0,j), (0,j)\rangle; \\ P_2 &= \langle (0,j), R_0(j,0), (0,0)\rangle; \\ P_3 &= \langle (0,0), (1,0), Q_1(0,j), (1,j), (0,j)\rangle. \end{aligned}$$

Case 2: $i = 1$ and j is even.

Case 2.1: $j = 0$. The corresponding paths are:

$$\begin{aligned} P_1 &= \langle (0,0), Q_0(0,n-2), (0,n-2), (1,n-2), Q_1^{-1}(0,n-2), (1,0)\rangle; \\ P_2 &= \langle (0,0), (1,0)\rangle; \\ P_3 &= \langle (0,0), (0,n-1), (1,n-1), (1,0)\rangle. \end{aligned}$$

Case 2.2: $j > 0$. The corresponding paths are:

$$\begin{aligned} P_1 &= \langle (0,0), Q_0(0,j), (0,j), (1,j)\rangle; \\ P_2 &= \langle (1,j), (1,j+1), (0,j+1), R_0(j+1,0), (0,0)\rangle; \\ P_3 &= \langle (0,0), (1,0), Q_1(0,j), (1,j)\rangle. \end{aligned}$$

Hence HReT$(2,n)$ is globally bi-3^*-connected. See Figure 4.3 for illustrations. □

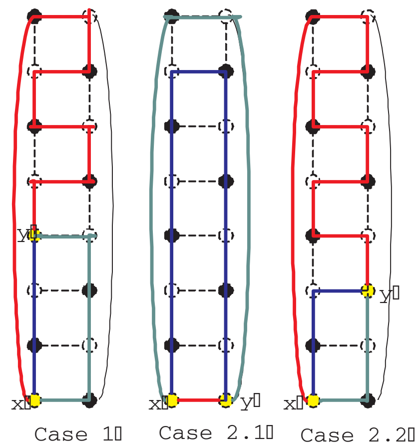

Figure 4.3: Illustrations for Lemma 13.

Lemma 14. *Let x, y, and z be any three different vertices of $HReT(2,n) = (V_0 \cup V_1, E)$ in V_0. Then there exists a regular 3^*-container $C_{3^*}(x,y)$ of $HReT(2,n) - \{z\}$. Hence $HReT(2,n)$ is hyper globally bi-3^*-connected.*

Proof. Without loss of generality, we may assume that $x = (0,0)$, $y = (i,j)$, and $z = (k,l)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in HReT$(2,n) - \{z\}$. We have the following cases:

Case 1: $i = 0$. Then j is even.

Case 1.1: $k = 0$. Then l is even. By the symmetric property of HReT$(2,n)$, we may assume that $l < j$. The corresponding paths are:

$$\begin{aligned} P_1 &= \langle (0,j), Q_0(j,0), (0,0) \rangle; \\ P_2 &= \langle (0,0), R_0(0,l-1), (0,l-1), (1,l-1), (1,l), (1,l+1), (0,l+1), \\ &\quad R_0(l+1,j), (0,j) \rangle; \\ P_3 &= \langle (0,j), (1,j), Q_1(j,0), (1,0), (0,0) \rangle. \end{aligned}$$

Case 1.2: $k = 1$. Then l is odd. By the symmetric property of HReT$(2,n)$, we may assume that $l < j$. The corresponding paths are:

$$\begin{aligned} P_1 &= \langle (0,j), Q_0(j,0), (0,0) \rangle; \\ P_2 &= \langle (0,0), R_0(0,l), (0,l), R_0(l,j), (0,j) \rangle; \\ P_3 &= \langle (0,j), (1,j), Q_1(j,0), (1,0), (0,0) \rangle. \end{aligned}$$

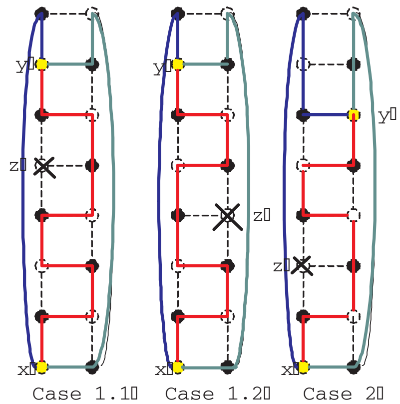

Figure 4.4: Illustrations for Lemma 14.

Case 2: $i = 1$. Then j is odd. $k = 0$. Then l is even. By the symmetric property of HReT$(2, n)$, we may assume that $l < j$. The corresponding paths are:

$$\begin{aligned} P_1 &= \langle (1,j), (0,j), Q_0(j,0), (0,0)\rangle; \\ P_2 &= \langle (0,0), R_0(0,l-1), (0,l-1), (1,l-1), (1,l), (1,l+1), (0,l+1), \\ &\quad R_0(l+1,j-1), (0,j-1), (1,j-1), (1,j)\rangle; \\ P_3 &= \langle (1,j), Q_1(j,0), (1,0), (0,0)\rangle. \end{aligned}$$

Hence HReT$(2, n)$ is hyper globally bi-3^*-connected. See Figure 4.4 for illustrations. □

4.4 The Globally Bi-3^*-Connected Property of Honeycomb Rectangular Torus HReT(4,n)

In this section, we need the following path patterns. For $0 \le i \le m-1$ and $0 \le j, k \le n-1$, we set

$$\begin{aligned} S_i^L(j) &= \langle ([i]_m, [j]_n), ([i-1]_m, [j]_n), ([i-1]_m, [j+1]_n), ([i-2]_m, [j+1]_n), \\ &\quad ([i-2]_m, [j+2]_n), ([i-3]_m, [j+2]_n), ([i-3]_m, [j+3]_n), \\ &\quad ([i-4]_m, [j+3]_n), ([i-4]_m, [j+2]_n)\rangle; \\ S_i^R(j) &= \langle ([i]_m, [j]_n), ([i+1]_m, [j]_n), ([i+1]_m, [j+1]_n), ([i+2]_m, [j+1]_n), \end{aligned}$$

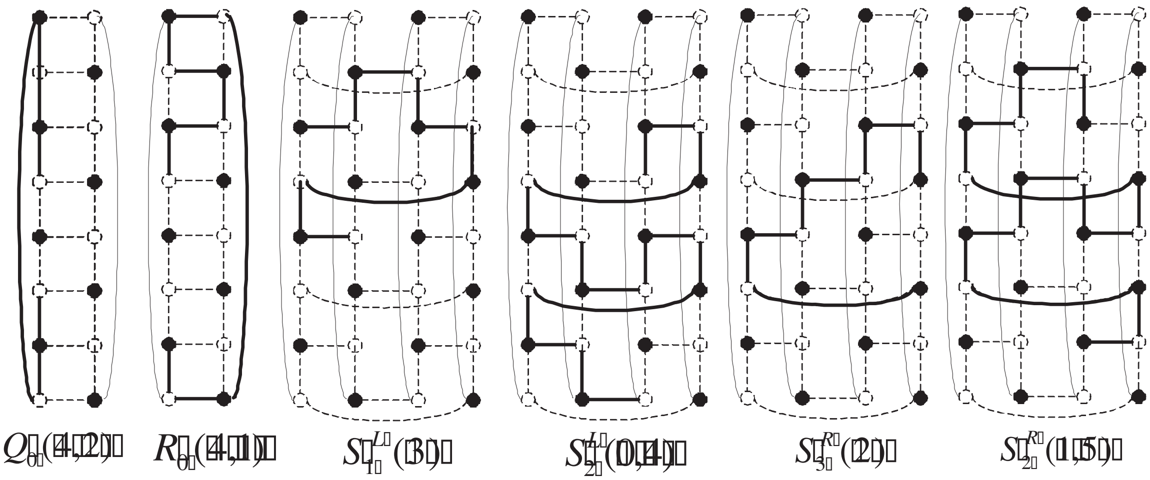

Figure 4.5: The path patterns $Q_0(4,2)$, $R_0(4,1)$, $S^L_1(3)$, $S^L_2(0,4)$, $S^R_3(2)$, and $S^R_2(1,5)$.

$$
\begin{aligned}
& ([i+2]_m,[j+2]_n),([i+3]_m,[j+2]_n),([i+3]_m,[j+3]_n),\\
& ([i+4]_m,[j+3]_n),([i+4]_m,[j+2]_n)\rangle;\\
S^L_i(j,k) \;=\; & \langle([i]_m,[j]_n),S^L_{[i]_m}(j),([i-4]_m,[j+2]_n),S^L_{[i-4]_m}([j+2]_n),\\
& ([i-8]_m,[j+4]_n),\ldots,([i-2(k-j-2)]_m,[k-2]_n),\\
& S^L_{[i-2(k-j-2)]_m}([k-2]_n),([i-2(k-j)]_m,[k]_n)\rangle;\text{ and}\\
S^R_i(j,k) \;=\; & \langle([i]_m,[j]_n),S^R_{[i]_m}(j),([i+4]_m,[j+2]_n),S^R_{[i+4]_m}([j+2]_n),\\
& ([i+8]_m,[j+4]_n),\ldots,([i+2(k-j-2)]_m,[k-2]_n),\\
& S^R_{[i+2(k-j-2)]_m}([k-2]_n),([i+2(k-j)]_m,[k]_n)\rangle.
\end{aligned}
$$

See Figure 4.5 for illustrations.

Lemma 15. *Let x and y be any two vertices of $HReT(4,n)=(V_0\cup V_1,E)$ with $x\in V_0$ and $y\in V_1$. Then there exists a regular 3^*-container $C_{3^*}(x,y)$ of $HReT(4,n)$. Hence $HReT(4,n)$ is globally bi-3^*-connected.*

Proof. Without loss of generality, we may assume that $x=(0,0)$ and $y=(i,j)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y)=\{P_1,P_2,P_3\}$ in HReT$(4,n)$. By the symmetric property of HReT$(4,n)$, we may assume that $i\in\{0,1,2\}$. Hence we have the following cases:

Case 1: Suppose that $i\in\{0,1\}$. By Lemma 13, there exists a regular 3^*-container $C_{3^*}((0,0),(i,j))$ of HReT$(2,n)$. By Lemma 12, $C'_{3^*,1}((0,0),(i,j))$ forms a 3^*-container of

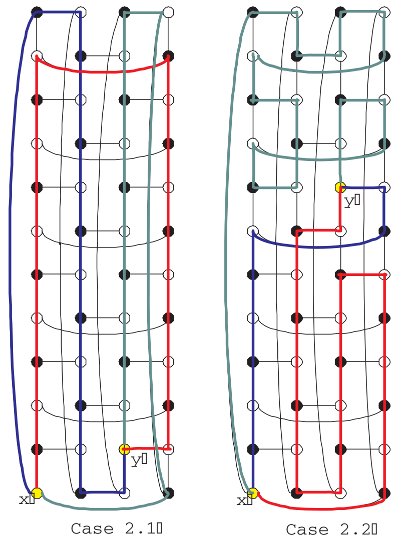

Figure 4.6: Illustrations for Lemma 15.

HReT(4, n).

Case 2: $i = 2$. Then j is odd.

Case 2.1: Suppose that $j = 1$. The corresponding paths are:

$$\begin{aligned} P_1 &= \langle (0,0), (0, n-1), (1, n-1), Q_1^{-1}(0, n-1), (1,0), (2,0), (2,1) \rangle; \\ P_2 &= \langle (0,0), Q_0(0, n-2), (0, n-2), (3, n-2), Q_3^{-1}(1, n-2), (3,1), (2,1) \rangle; \\ P_3 &= \langle (0,0), (3,0), (3, n-1), (2, n-1), Q_2^{-1}(1, n-1), (2,1) \rangle. \end{aligned}$$

Case 2.2: Suppose that $j \neq 1$. The corresponding paths are:

$$\begin{aligned} P_1 &= \langle (0,0), Q_0(0, j-1), (0, j-1), (3, j-1), (3, j), (2, j) \rangle; \\ P_2 &= \langle (0,0), (3,0), Q_3(0, j-2), (3, j-2), (2, j-2), Q_2^{-1}(0, j-2), (2,0), (1,0), \\ &\quad Q_1(0, j-1), (1, j-1), (2, j-1), (2, j) \rangle; \\ P_3 &= \langle (0,0), (0, n-1), S_L^{-1}(j+3, n-1), (0, j+3), (0, j+2), (1, j+2), (1, j+1), \\ &\quad (1, j), (0, j), (0, j+1), (3, j+1), (3, j+2), (2, j+2), (2, j+1), (2, j) \rangle. \end{aligned}$$

Hence HReT$(4, n)$ is globally bi-3*-connected. See Figure 4.6 for illustrations. □

Lemma 16. *Let x, y, and z be any three different vertices of $HReT(4,6) = (V_0 \cup V_1, E)$ in V_0. Then there exists a regular 3^*-container $C_{3^*}(x, y)$ of $HReT(4,6) - \{z\}$. Hence $HReT(4,6)$ is hyper globally bi-3^*-connected.*

Proof. Without loss of generality, we may assume that $x = (0,0)$, $y = (i,j)$, and $z = (k,l)$. The corresponding regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in $\text{HReT}(4,6) - \{z\}$ are listed below.

y	z	$C_{3*}(x,y)$
$(0,2)$	$(2,2)$	$\langle(0,0),(0,1),(0,2)\rangle$ $\langle(0,0),(0,5),(1,5),(1,0),Q_1(0,4),(1,4),(2,4),(2,3),(3,3),(3,2),(0,2)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,0),(2,5),(3,5),(3,4),(0,4),(0,3),(0,2)\rangle$
$(0,2)$	$(2,4)$	$\langle(0,0),(0,1),(0,2)\rangle$ $\langle(0,2),(0,3),(0,4),(3,4),(3,5),(2,5),(2,0),(1,0),Q_1(0,5),(1,5),(0,5),(0,0)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,2),(2,3),(3,3),(3,2),(0,2)\rangle$
$(0,4)$	$(0,2)$	$\langle(0,0),(0,5),(0,4)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,4),(2,5),(3,5),(3,4),(0,4)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,1),(2,0),(1,0),(1,5),(1,4),(1,3),(0,3),(0,4)\rangle$
$(0,4)$	$(1,1)$	$\langle(0,0),(0,5),(0,4)\rangle$ $\langle(0,0),(0,1),(0,2),(3,2),(3,3),(2,3),(2,2),(1,2),(1,3),(0,3),(0,4)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,0),(1,0),(1,5),(1,4),(2,4),(2,5),(3,5),(3,4),(0,4)\rangle$
$(1,3)$	$(0,2)$	$\langle(0,0),(0,5),(0,4),(0,3),(1,3)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(1,3)\rangle$ $\langle(0,0),(3,0),Q_3(0,5),(3,5),(2,5),Q_2^{-1}(0,5),(2,0),(1,0),(1,5),(1,4),(1,3)\rangle$
$(1,5)$	$(0,2)$	$\langle(0,0),(0,5),(1,5)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(1,3),(0,3),(0,4),(3,4),(3,5),(2,5),(2,4),(1,4),(1,5)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),Q_2^{-1}(0,3),(2,0),(1,0),(1,5)\rangle$
$(1,1)$	$(2,0)$	$\langle(0,0),(0,1),(1,1)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,2),(1,2),(1,1)\rangle$ $\langle(0,0),(0,5),(0,4),(3,4),(3,5),(2,5),(2,4),(2,3),(3,3),(3,2),(0,2),(0,3),(1,3),(1,4),(1,5),(1,0),(1,1)\rangle$
$(1,1)$	$(2,2)$	$\langle(1,1),Q_1(1,4),(1,4),(2,4),(2,3),(3,3),(3,2),(0,2),(0,3),(0,4),(3,4),(3,5),(2,5),(2,0),(2,1),(3,1),(3,0),(0,0)\rangle$ $\langle(0,0),(0,1),(1,1)\rangle$ $\langle(0,0),(0,5),(1,5),(1,0),(1,1)\rangle$
$(1,1)$	$(2,4)$	$\langle(1,1),(1,0),(2,0),(2,5),(3,5),(3,4),(0,4),(0,3),(0,2),(3,2),(3,3),(2,3),(2,2),(2,1),(3,1),(3,0),(0,0)\rangle$ $\langle(0,0),(0,1),(1,1)\rangle$ $\langle(0,0),(0,5),(1,5),Q_1^{-1}(1,5),(1,1)\rangle$
$(1,3)$	$(2,0)$	$\langle(0,0),(0,1),(1,1),(1,0),(1,5),(1,4),(1,3)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,2),(1,2),(1,3)\rangle$ $\langle(0,0),(0,5),(0,4),(3,4),(3,5),(2,5),(2,4),(2,3),(3,3),(3,2),(0,2),(0,3),(1,3)\rangle$
$(1,3)$	$(2,2)$	$\langle(0,0),(0,5),(1,5),(1,4),(1,3)\rangle$ $\langle(0,0),(3,0),(3,5),(2,5),(2,4),(2,3),(3,3),(3,4),(0,4),(0,3),(1,3)\rangle$ $\langle(0,0),(0,1),(0,2),(3,2),(3,1),(2,1),(2,0),(1,0),Q_1(0,3),(1,3)\rangle$
$(1,3)$	$(2,4)$	$\langle(0,0),(0,5),(1,5),(1,4),(1,3)\rangle$ $\langle(0,0),(3,0),(3,5),(2,5),(2,0),(1,0),Q_1(0,3),(1,3)\rangle$ $\langle(0,0),(0,1),(0,2),(3,2),(3,1),(2,1),(2,2),(2,3),(3,3),(3,4),(0,4),(0,3),(1,3)\rangle$
$(2,0)$	$(0,2)$	$\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,4),(1,4),(1,3),(0,3),(0,4),(3,4),(3,5),(2,5),(2,0)\rangle$ $\langle(0,0),(0,5),(1,5),(1,0),(2,0)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,1),(2,0)\rangle$
$(2,2)$	$(0,2)$	$\langle(0,0),(0,1),(1,1),(1,0),(2,0),(2,1),(2,2)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,2)\rangle$ $\langle(0,0),(0,5),(1,5),(1,4),(2,4),(2,5),(3,5),(3,4),(0,4),(0,3),(1,3),(1,2),(2,2)\rangle$
$(2,2)$	$(0,4)$	$\langle(0,0),(0,5),(1,5),(1,0),(2,0),(2,5),(3,5),Q_3^{-1}(2,5),(3,2),(0,2),(0,3),(1,3),(1,4),(2,4),(2,3),(2,2)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,2)\rangle$
$(2,2)$	$(1,1)$	$\langle(0,0),(0,5),(1,5),(1,0),(2,0),(2,1),(2,2)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,2)\rangle$ $\langle(0,0),Q_0(0,4),(0,4),(3,4),(3,5),(2,5),(2,4),(1,4),(1,3),(1,2),(2,2)\rangle$

Hence $\text{HReT}(4,6)$ is hyper globally bi-3^*-connected. □

Lemma 17. *Assume that $n \geq 8$. Let x, y, and z be any three different vertices of $HReT(4,n) = (V_0 \cup V_1, E)$ in V_0. Then there exists a regular 3^*-container $C_{3^*}(x,y)$ of $HReT(4,n) - \{z\}$. Hence $HReT(4,n)$ is hyper globally bi-3^*-connected.*

Proof. Without loss of generality, we may assume that $x = (0,0)$, $y = (i,j)$, and $z = (k,l)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in $\text{HReT}(4,n) - \{z\}$. By the symmetric property of $\text{HReT}(4,n)$, we may assume that $i \in \{0,1,2\}$. We have the following cases:

Case 1: Suppose that $i \in \{0,1\}$ and $z \in \{0,1\}$. By Lemma 14, there exists a regular 3^*-container $C_{3^*}((0,0),(i,j))$ of $\mathrm{HReT}(2,n) - \{(k,l)\}$. By Lemma 12, $C'_{3^*,1}((0,0),(i,j))$ forms a 3^*-container of $\mathrm{HReT}(4,n) - \{(k,l)\}$.

Case 2: $i = 0$ and $k = 2$. Then j and l are even. By the symmetric property, we have the following subcases.

Case 2.1: Suppose that $j = 4$ and $l = 2$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0), Q_0(0,4), (0,4) \rangle; \\
P_2 &= \langle (0,0), (0,n-1), (0,n-2), (3,n-2), Q_3^{-1}(4,n-2), (3,4), (0,4) \rangle; \\
P_3 &= \langle (0,4), Q_0(4,n-3), (0,n-3), (1,n-3), Q_1^{-1}(0,n-3), (1,0), (1,n-1), \\
&\quad (1,n-2), (2,n-2), Q_2^{-1}(3,n-2), (2,3), (3,3), (3,2), (3,1), (2,1), (2,0), \\
&\quad (2,n-1), (3,n-1), (3,0), (0,0) \rangle.
\end{aligned}$$

Case 2.2: Suppose that $n-4 > j \geq 2$ and $l = j+2$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0), Q_0(0,j), (0,j) \rangle; \\
P_2 &= \langle (0,0), (3,0), Q_3(0,j), (3,j), (0,j) \rangle; \\
P_3 &= \langle (0,j), Q_0(j,j+4), (0,j+4), (3,j+4), (3,j+5), (2,j+5), (2,j+4), (2,j+3), \\
&\quad (3,j+3), (3,j+2), (3,j+1), (2,j+1), Q_2^{-1}(0,j+1), (2,0), (1,0), Q_1(0,j+5), \\
&\quad (1,j+5), (0,j+5), (0,j+6), (3,j+6), (3,j+7), (2,j+7), (2,j+6), \\
&\quad S_2^L(j+6,n-2), (2,n-2), (1,n-2), (1,n-1), (0,n-1), (0,0) \rangle.
\end{aligned}$$

Case 2.3: Suppose that $n-6 > j \geq 2$ and $n-4 > l > j+2$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0), Q_0(0,j), (0,j) \rangle; \\
P_2 &= \langle (0,0), (3,0), Q_3(0,j), (3,j), (0,j) \rangle; \\
P_3 &= \langle (1,j), (1,j+1), (1,j+2), (3,j+2), (3,j+1), (2,j+1), Q_2^{-1}(0,j+1), (2,0), \\
&\quad (1,0), Q_1(0,j+2), (1,j+2), (2,j+2), (2,j+3), (3,j+3), (3,j+4), (0,j+4), \\
&\quad (0,j+3), S_0^R(j+3,l-3), (0,l-3), (1,l-3), (1,l-2), (2,l-2), (2,l-1), \\
&\quad (3,l-1), (3,l), (3,l+1), (2,l+1), (2,l+2), (2,l+3), (3,l+3), (3,l+2), \\
&\quad (0,l+2), Q_0^{-1}(l-1,l+2), (0,l-1), (1,l-1), Q_1(l-1,l+3), (1,l+3), \\
&\quad (0,l+3), (0,l+4), (3,l+4), S_2^L(l+4,n-2), (2,n-2), (1,n-2), (1,n-1), \\
&\quad (0,n-1), (0,0) \rangle.
\end{aligned}$$

Case 2.4: Suppose that $n > 8$ and $j = l \geq 2$. The corresponding paths are:

$$P_1 = \langle (0,0), Q_0(0,j), (0,j) \rangle;$$

$$
\begin{aligned}
P_2 &= \langle (0,0),(3,0),Q_3(0,j-1),(3,j-1),(2,j-1),Q_2^{-1}(0,j-1),(2,0),(1,0),\\
&\quad Q_1(0,j+1),(1,j+1),(0,j+1),(0,j)\rangle;\\
P_3 &= \langle (0,j),(3,j),(3,j+1),(2,j+1),(2,j+2),(1,j+2),(1,j+3),(1,j+4),(2,j+4),\\
&\quad (2,j+3),(3,j+3),(3,j+2),(0,j+2),(0,j+3),(0,j+4),(3,j+4),(3,j+5),\\
&\quad (2,j+5),(2,j+6),(1,j+6),(1,j+5),S_1^L(j+5,n-5),(1,n-5),(0,n-5),\\
&\quad (0,n-4),(3,n-4),(3,n-3),(2,n-3),(2,n-2),(2,n-1),(3,n-1),(3,n-2),\\
&\quad (0,n-2),(0,n-3),(1,n-3),(1,n-2),(1,n-1),(0,n-1),(0,0)\rangle.
\end{aligned}
$$

Case 2.5: Suppose that $n = 8$, $j = 2$, and $l = 2$. The corresponding paths are:

$$
\begin{aligned}
P_1 &= \langle (0,0),(0,1),(0,2)\rangle;\\
P_2 &= \langle (0,2),(0,3),(0,4),(3,4),Q_3(4,7),(3,7),(2,7),(2,0),(2,1),(3,1),(3,0),(0,0)\rangle;\\
P_3 &= \langle (0,2),(3,2),(3,3),(2,3),Q_2(3,6),(2,6),(1,6),(1,7),(1,0),Q_1(0,5),\\
&\quad (1,5),(0,5),(0,6),(0,7),(0,0)\rangle.
\end{aligned}
$$

Case 2.6: Suppose that $n = 8$, $j = 4$, and $l = 4$. The corresponding paths are:

$$
\begin{aligned}
P_1 &= \langle (0,0),Q_0(0,4),(0,4)\rangle;\\
P_2 &= \langle (0,0),(0,7),(1,7),(1,0),Q_1(0,6),(1,6),(2,6),(2,5),(3,5),(3,4),(0,4)\rangle;\\
P_3 &= \langle (0,0),(3,0),Q_3(0,3),(3,3),(2,3),Q_2^{-1}(0,3),(2,0),(2,7),(3,7),(3,6),(0,6),\\
&\quad (0,5),(0,4)\rangle.
\end{aligned}
$$

Case 3: $i = 1$ and $k = 2$. Then j is odd and l is even. By the symmetric property, we have the following subcases.

Case 3.1: Suppose that $n-5 > j \geq 1$ and $n-4 > l > j+2$. The corresponding paths are:

$$
\begin{aligned}
P_1 &= \langle (0,0),Q_0(0,j),(0,j),(1,j)\rangle;\\
P_2 &= \langle (0,0),(3,0),Q_3(0,j),(3,j),(2,j),Q_2^{-1}(0,j),(2,0),(1,0),Q_1(0,j),(1,j)\rangle;\\
P_3 &= \langle (1,j),(1,j+1),(2,j+1),(2,j+2),(3,j+2),(3,j+1),S_3^L(j+1,l-2),(3,l-2),\\
&\quad (0,l-2),(0,l-1),(1,l-1),(1,l),(1,l+1),(1,l+2),(2,l+2),(2,l+1),(3,l+1),\\
&\quad (3,l),(0,l),(0,l+1),(0,l+2),(3,l+2),(3,l+3),(2,l+3),(2,l+4),(1,l+4),\\
&\quad (1,l+3),S_1^L(l+3,n-5),(1,n-5),(0,n-5),(0,n-4),(3,n-4),(3,n-3),\\
&\quad (2,n-3),(2,n-2),(2,n-1),(3,n-1),(3,n-2),(0,n-2),(0,n-3),(1,n-3),\\
&\quad (1,n-2),(1,n-1),(0,n-1),(0,0)\rangle.
\end{aligned}
$$

Case 3.2: Suppose that $n-5>j\geq 1$ and $l=j+1$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0), Q_0(0,j), (0,j), (1,j)\rangle;\\
P_2 &= \langle (0,0), (3,0), Q_3(0,j), (3,j), (2,j), Q_2^{-1}(0,j), (2,0), (1,0), Q_1(0,j), (1,j)\rangle;\\
P_3 &= \langle (1,j), Q_1(j,j+3), (1,j+3), (2,j+3), (2,j+2), (3,j+2), (3,j+1), (0,j+1),\\
&\quad (0,j+2), (0,j+3), (3,j+3), (3,j+4), (2,j+4), (2,j+5), (1,j+5), (1,j+4),\\
&\quad S_1^L(j+4,n-5), (1,n-5), (0,n-5), (0,n-4), (3,n-4), (3,n-3), (2,n-3),\\
&\quad (2,n-2), (2,n-1), (3,n-1), (3,n-2), (0,n-2), (0,n-3), (1,n-3), (1,n-2),\\
&\quad (1,n-1), (0,n-1), (0,0)\rangle.
\end{aligned}$$

Case 3.3: Suppose that $n-5>j\geq 1$ and $l=n-4$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0), Q_0(0,j), (0,j), (1,j)\rangle;\\
P_2 &= \langle (0,0), (0,n-1), (1,n-1), (1,0), Q_1(0,j), (1,j)\rangle;\\
P_3 &= \langle (1,j), (1,j+1), (2,j+1), (2,j+2), (3,j+2), (3,j+1), S_3^L(j+1,n-6), (0,n-6),\\
&\quad (0,n-5), (1,n-5), Q_1(n-5,n-2), (1,n-2), (2,n-2), (2,n-3), (3,n-3),\\
&\quad (3,n-4), (0,n-4), (0,n-3), (0,n-2), (3,n-2), (3,n-1), (2,n-1), (2,0),\\
&\quad (2,1), (3,1), (3,0), (0,0)\rangle.
\end{aligned}$$

Case 3.4: Suppose that $j=n-5$ and $l=n-4$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0), Q_0(0,n-5), (0,n-5), (1,n-5)\rangle;\\
P_2 &= \langle (0,0), (0,n-1), (1,n-1), (1,0), Q_1(0,n-5), (1,n-5)\rangle;\\
P_3 &= \langle (1,n-5), Q_1(n-5,n-2), (1,n-2), (2,n-2), (2,n-3), (3,n-3), (3,n-4),\\
&\quad (0,n-4), (0,n-3), (0,n-2), (3,n-2), (3,n-1), (2,n-1), (2,0), Q_2(0,n-5),\\
&\quad (2,n-5), (3,n-5), Q_3^{-1}(0,n-5), (3,0), (0,0)\rangle.
\end{aligned}$$

Case 3.5: Suppose that $n-5>j\geq 1$ and $l=n-2$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0), Q_0(0,j), (0,j), (1,j)\rangle;\\
P_2 &= \langle (0,0), (3,0), (3,n-1), (2,n-1), (2,0), (1,0), Q_1(0,j), (1,j)\rangle;\\
P_3 &= \langle (1,j), (1,j+1), (1,j+2), (0,j+2), (0,j+1), (3,j+1), Q_3^{-1}(1,j+1), (3,1), (2,1),\\
&\quad Q_2(1,j+2), (2,j+2), (3,j+2), (3,j+3), (0,j+3), (0,j+4), (1,j+4), (1,j+3),\\
&\quad S_1^R(j+3,n-6), (1,n-6), (2,n-6), (2,n-5), (3,n-5), (3,n-4), (0,n-4),\\
&\quad (0,n-3), (0,n-2), (3,n-2), (3,n-3), (2,n-3), (2,n-4), (1,n-4),\\
&\quad Q_1(n-4,n-1), (1,n-1), (0,n-1), (0,0)\rangle.
\end{aligned}$$

Case 4: $i=2$ and $k=0$. Then j and l are even. By the symmetric property, we have the following subcases.

Case 4.1: Suppose that $j = 0$ are $l > 0$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0),(0,n-1),(1,n-1),(1,0),(2,0)\rangle;\\
P_2 &= \langle (0,0),(0,1),(1,1),(1,2),(2,2),(2,1),(2,0)\rangle;\\
P_3 &= \langle (0,0),(3,0),(3,1),(3,2),(0,2),(0,3),(1,3),(1,4),(2,4),(2,3),\\
&\quad S_2^R(3,j-1),(2,j-1),(3,j-1),(3,j),(3,j+1),(2,j+1),(2,j+2),(1,j+2),\\
&\quad (1,j+1),S_1^L(j+1,n-3),(1,n-3),(0,n-3),(0,n-2),(3,n-2),(3,n-1),\\
&\quad (2,n-1),(2,0)\rangle.
\end{aligned}$$

Case 4.2: Suppose that $l > j > 0$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0),(0,1),(1,1),Q_1(1,j),(1,j),(2,j)\rangle;\\
P_2 &= \langle (0,0),(3,0),(3,1),(2,1),Q_2(1,j),(2,j)\rangle;\\
P_3 &= \langle (2,j),(2,j+1),(2,j+2),(1,j+2),(1,j+1),(0,j+1),Q_0^{-1}(2,j+1),\\
&\quad (0,2),(3,2),Q_3(2,j+2),(3,j+2),(0,j+2),(0,j+3),(1,j+3),(1,j+4),\\
&\quad (2,j+4),(2,j+3),S_2^R(j+3,l-1),(2,l-1),(3,l-1),(3,l),(3,l+1),\\
&\quad (2,l+1),(2,l+2),(1,l+2),(1,l+1),S_1^L(l+1,n-1),(1,n-1),(0,n-1),(0,0)\rangle.
\end{aligned}$$

Case 4.3: Suppose that $j = l > 0$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0),Q_0(0,j-1),(0,j-1),(1,j-1),Q_1^{-1}(0,j-1),(1,0),(2,0),Q_2(0,j),(2,j)\rangle;\\
P_2 &= \langle (0,0),(3,0),Q_3(0,j+1),(3,j+1),(2,j+1),(2,j)\rangle;\\
P_3 &= \langle (2,j),S_2^L(j,n-1),(2,n-2),(1,n-2),(1,n-1),(0,n-1),(0,0)\rangle.
\end{aligned}$$

Case 5: $i = 2$ and $k = 1$. Then j is even and l is odd. By the symmetric property, we have the following subcases.

Case 5.1: Suppose that $j = 0$ and $l = 1$. The corresponding paths are:

$$\begin{aligned}
P_1 &= \langle (0,0),(0,n-1),(1,n-1),(1,0),(2,0)\rangle;\\
P_2 &= \langle (0,0),(0,1),(0,2),(3,2),(3,1),(2,1),(2,0)\rangle;\\
P_3 &= \langle (0,0),(3,0),(3,n-1),(3,n-2),(0,n-2),Q_0^{-1}(3,n-2),(0,3),(1,3),\\
&\quad (1,2),(2,2),(2,3),(3,3),Q_3(3,n-3),(3,n-3),(2,n-3),Q_2^{-1}(4,n-3),\\
&\quad (2,4),(1,4),Q_1(4,n-2),(1,n-2),(2,n-2),(2,n-1),(2,0)\rangle.
\end{aligned}$$

Case 5.2: Suppose that $j = 0$ and $n-1 > l > 1$. The corresponding paths are:

$$P_1 = \langle (0,0),(0,n-1),(1,n-1),(1,0),(2,0)\rangle;$$

$$
\begin{aligned}
P_2 &= \langle (0,0),(3,0),(3,1),(2,1),(2,0)\rangle; \\
P_3 &= \langle (0,0),(0,1),(1,1),(1,2),(2,2),(2,3),(3,3),(3,2),S_3^L(2,j-3),(3,j-3),(0,j-3), \\
&\quad (0,j-2),(1,j-2),(1,j-1),(2,j-1),(2,j),(2,j+1),(1,j+1),(1,j+2), \\
&\quad (1,j+3),(2,j+3),(2,j+2),(3,j+2),Q_3^{-1}(j-1,j+2),(3,j-1),(0,j-1), \\
&\quad Q_0(j-1,j+3),(0,j+3),(3,j+3),(3,j+4),(2,j+4),(2,j+5),(1,j+5), \\
&\quad (1,j+4),S_1^L(j+4,n-3),(1,n-3),(0,n-3),(0,n-2),(3,n-2),(3,n-1), \\
&\quad (2,n-1),(2,0)\rangle.
\end{aligned}
$$

Case 5.3: Suppose that $n-1 > l > j+2$ and $j > 0$. The corresponding paths are:

$$
\begin{aligned}
P_1 &= \langle (0,0),(0,1),(1,1),Q_1(1,j),(1,j),(2,j)\rangle; \\
P_2 &= \langle (0,0),(3,0),(3,1),(2,1),Q_2(1,j),(2,j)\rangle; \\
P_3 &= \langle (2,j),(2,j+1),(3,j+1),(3,j),S_3^L(j,l-3),(3,l-3),(0,l-3),(0,l-2),(1,l-2), \\
&\quad (1,l-1),(2,l-1),(2,l),(2,l+1),(1,l+1),(1,l+2),(1,l+3),(2,l+3),(2,l+2), \\
&\quad (3,l+2),(3,l+1),(3,l),(3,l-1),(0,l-1),Q_0(l-1,l+3),(0,l+3),(3,l+3), \\
&\quad (3,l+4),(2,l+4),(2,l+5),(1,l+5),(1,l+4),S_1^L(l+4,n-1),(1,n-1), \\
&\quad (0,n-1),(0,0)\rangle.
\end{aligned}
$$

Case 5.4: Suppose that $n-2 > j$ and $l = n-1$. The corresponding paths are:

$$
\begin{aligned}
P_1 &= \langle (0,0),Q_0(0,j+1),(0,j+1),(1,j+1),(1,j+2),(2,j+2),(2,j+1),(2,j)\rangle; \\
P_2 &= \langle (0,0),(3,0),(3,n-1),(2,n-1),(2,0),(1,0),Q_1(0,j),(1,j),(2,j)\rangle; \\
P_3 &= \langle (2,j),Q_2^{-1}(1,j),(2,1),(3,1),Q_3(1,j+2),(3,j+2),(0,j+2),(0,j+3),(1,j+3), \\
&\quad (1,j+4),(2,j+4),(2,j+3),S_2^R(j+3,n-3),(2,n-3),(3,n-3),(3,n-2), \\
&\quad (0,n-2),(0,n-1),(0,0)\rangle.
\end{aligned}
$$

Case 5.5: Suppose that $j = n-2$ and $l = n-1$. The corresponding paths are:

$$
\begin{aligned}
P_1 &= \langle (0,0),(0,n-1),(0,n-2),(0,n-3),(1,n-3),(1,n-2),(2,n-2)\rangle; \\
P_2 &= \langle (0,0),Q_0(0,n-4),(3,n-4),Q_3(n-4,n-1),(2,n-1),(2,n-2)\rangle; \\
P_3 &= \langle (0,0),(3,0),Q_3(0,n-5),(3,n-5),(2,n-5),Q_2^{-1}(0,n-5),(2,0),Q_1(0,n-4), \\
&\quad (1,n-4),(2,n-4),(2,n-3),(2,n-2)\rangle.
\end{aligned}
$$

Hence HReT$(4,n)$ is hyper globally bi-3*-connected for $n \geq 8$. See Figure 4.7 for illustrations. □

Figure 4.7: Illustrations for Lemma 17.

4.5 The Globally Bi-3^*-Connected Property of Honeycomb Rectangular Torus HReT(m,n)

Lemma 18. *Assume that m and n are positive even integers with $m, n \geq 4$. Let x and y be any two vertices of HReT$(m,n) = (V_0 \cup V_1, E)$ with $x \in V_0$ and $y \in V_1$. Then there exists a regular 3^*-container $C_{3^*}(x,y)$ of HReT(m,n).*

Proof. Without loss of generality, we may assume that $x = (0,0)$ and $y = (i,j)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in HReT(m,n). We prove the lemma by induction on m. With Lemma 15, our theorem holds for $m = 4$. Now, we consider the case that $m \geq 6$.

Suppose that $i < m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in HReT$(m-2,n)$. By Lemma 12, $C'_{3^*,m-3}((0,0),(i,j))$ forms a 3^*-container of HReT(m,n). Suppose that $i \geq m-2$. By induction, there exists a regular $C_{3^*}(x,(i-2,j)) = \{P_1, P_2, P_3\}$ in HReT$(m-2,n)$. By Lemma 12, $C'_{3^*,1}((0,0),(i,j))$ forms a 3^*-container of HReT(m,n). □

Lemma 19. *Assume that m and n are positive even integers with $m \geq 4$ and $n \geq 6$. Let x, y, and z be any three different vertices of HReT$(m,n) = (V_0 \cup V_1, E)$ in V_0. Then there exists a regular 3^*-container $C_{3^*}(x,y)$ of HReT$(m,n) - \{z\}$.*

Proof. Without loss of generality, we may assume that $x = (0,0)$, $y = (i,j)$, and $z = (k,l)$. In order to prove this lemma, we will construct a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in HReT$(m,n) - \{z\}$. We prove the lemma by induction on m. With Lemmas 16 and 17, our theorem holds for $m = 4$. Now, we consider the case that $m \geq 6$.

Suppose that $i < m-2$ and $k < m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in HReT$(m-2,n) - \{z\}$. By Lemma 12, $C'_{3^*,m-3}((0,0),(i,j))$ forms a 3^*-container of HReT$(m,n) - \{z\}$. Suppose that $i < m-2$ and $k \geq m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ in HReT$(m-2,n) - (k-2,l)$. By Lemma 12, $C'_{3^*,i}((0,0),(i,j))$ forms a 3^*-container of HReT$(m,n) - \{z\}$. Suppose that $i \geq m-2$ and $k < m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,(i-2,j)) = \{P_1, P_2, P_3\}$ in HReT$(m-2,n) - \{z\}$. By Lemma 12, $C'_{3^*,k}((0,0),(i,j))$ forms a 3^*-container of HReT$(m,n) - \{z\}$. Suppose that $i \geq m-2$ and $k \geq m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,(i-2,j)) = \{P_1, P_2, P_3\}$ in HReT$(m-2,n) - (k-2,l)$. By Lemma 12, $C'_{3^*,1}((0,0),(i,j))$ forms a 3^*-container of HReT$(m,n) - \{z\}$. □

Theorem 13. *Assume that m and n are positive even integers with $n \geq 4$. Then*

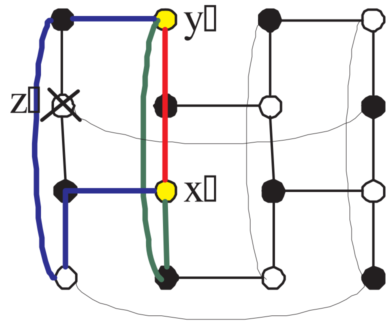

Figure 4.8: Illustration for Theorem 13.

HReT(m,n) is strongly globally bi-3*-connected. Moreover, HReT(m,n) is hyper globally bi-3*-connected if and only if $n \geq 6$ or $m = 2$.

Proof. With Lemmas 13 and 18, HReT(m,n) is globally bi-3*-connected if m, n are even integers with $n \geq 4$.

By Lemmas 14 and 19, HReT(m,n) is hyper globally bi-3*-connected if m, n are even integers with $n \geq 6$ or $m = 2$.

Now we consider the case HReT$(m,4)$ with m is an even integer and $m \geq 4$. We first prove that such HReT$(m,4)$ is not hyper globally bi-3*-connected.

To prove this fact, let $x = (1,1)$, $y = (1,3)$ and $z = (0,2)$. Suppose that there exists a 3*-container $C_{3^*}(x,y) = \{P_1, P_2, P_3\}$ of HReT$(m,4) - \{z\}$. Since $deg_{HReT(m,4)-z}(v) = 2$ for $v \in \{(0,1),(0,3),(3,2)\}$, $\langle(1,1),(1,2),(1,3)\rangle$ and $\langle(1,1),(0,1),(0,0),(0,3),(1,3)\rangle$ are two paths in $C_{3^*}(x,y)$. Without loss of generality, we assmue that $P_1 = \langle(1,1),(1,2),(1,3)\rangle$ and $P_2 = \langle(1,1),(0,1),(0,0),(0,3),(1,3)\rangle$. Since $deg_{HReT(m,4)-z}((1,1)) = deg_{HReT(m,4)-z}((1,3)) = 3$, $((1,3),(1,0))$ and $((1,0),(1,1))$ are edges in P_3. Thus $P_3 = \langle(1,1),(1,0),(1,3)\rangle$. Obviously, $\{P_1 \cup P_2 \cup P_3\}$ does not span HReT$(m,4) - \{z\}$. See Figure 4.8 for an illustration. Hence HReT$(m,4)$ is not hyper globally bi-3*-connected.

Although any HReT$(m,4)$ with m is an even integer and $m \geq 4$ is not hyper globally bi-3*-connected, we will prove that such HReT$(m,4)$ is strongly globally bi-3*-connected by induction.

We first prove that HReT(4,4) is strongly bi-3*-connected. Let x and y be any two different vertices in the same partite set of HReT(4,4). Without loss of generality, we may assume that x and y are vertices in V_0 and $x = (0,0)$. We need to find a vertex z in $V_0 - \{x, y\}$ such that there exists a 3^*-container $C_{3^*}(x, y) = \{P_1, P_2, P_3\}$ of HReT(4,4)$-\{z\}$. The corresponding vertex z and 3^*-container $C_{3^*}(x, y)$ are listed below.

y	z	$C_{3*}(x, y)$
$(0,2)$	$(1,3)$	$\langle(0,0),(0,1),(0,2)\rangle$ $\langle(0,0),(0,3),(0,2)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,0),(1,0),(1,1),(1,2),(2,2),(2,3),(3,3),(3,2),(0,2)\rangle$
$(1,1)$	$(1,3)$	$\langle(0,0),(0,1),(1,1)\rangle$ $\langle(0,0),(3,0),(3,1),(2,1),(2,0),(1,0),(1,1),\rangle$ $\langle(0,0),(0,3),(0,2),(3,2),(3,3),(2,3),(2,2),(1,2),(1,1)\rangle$
$(1,3)$	$(0,2)$	$\langle(0,0),(0,3),(1,3)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(1,3)\rangle$ $\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),Q_2^{-1}(0,3),(2,0),(1,0),(1,3)\rangle$
$(2,0)$	$(0,2)$	$\langle(0,0),(0,3),(1,3),(1,0),(2,0)\rangle$ $\langle(0,0),(3,0),(3,3),(3,2),(3,1),(2,1),(2,0)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,3),(2,0)\rangle$
$(2,2)$	$(0,2)$	$\langle(0,0),(3,0),Q_3(0,3),(3,3),(2,3),(2,2)\rangle$ $\langle(0,0),(0,3),(1,3),(1,0),(2,0),(2,1),(2,2)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2)\rangle$
$(3,1)$	$(0,2)$	$\langle(0,0),(3,0),(3,1)\rangle$ $\langle(0,0),(0,3),(1,3),(1,0),(2,0),(2,1),(3,1)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,3),(3,3),(3,2),(3,1)\rangle$
$(3,3)$	$(0,2)$	$\langle(0,0),(3,0),(3,3)\rangle$ $\langle(0,0),(0,3),(1,3),(1,0),(2,0),(2,1),(3,1),(3,2),(3,3)\rangle$ $\langle(0,0),(0,1),(1,1),(1,2),(2,2),(2,3),(3,3)\rangle$

Obviously, all these 3^*-containers of HReT(4,4)$-\{z\}$ are regular.

Now we consider the case HReT$(m, 4)$ with $m > 4$. Without loss of generality, we may assmue that $x = (0,0)$, $y = (i, j)$, and $z = (k, l)$. Suppose that $i < m-2$ and $k < m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x, y) = \{P_1, P_2, P_3\}$ in HReT$(m-2, 4) - \{z\}$. By Lemma 12, $C'_{3^*,m-3}((0,0),(i,j))$ forms a 3^*-container of HReT$(m, 4) - \{z\}$. Suppose that $i < m-2$ and $k \geq m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x, y) = \{P_1, P_2, P_3\}$ in HReT$(m-2, 4) - (k-2, l)$. By Lemma 12, $C'_{3^*,i}((0,0),(i,j))$ forms a 3^*-container of HReT$(m, 4) - \{z\}$. Suppose that $i \geq m-2$ and $k < m-2$. By induction, there exists a regular $C_{3^*}(x,(i-2, j)) = \{P_1, P_2, P_3\}$ in HReT$(m-2, 4) - \{z\}$. By Lemma 12, $C'_{3^*,k}((0,0),(i,j))$ forms a 3^*-container of HReT$(m, 4) - \{z\}$. Suppose that $i \geq m-2$ and $k \geq m-2$. By induction, there exists a regular 3^*-container $C_{3^*}(x,(i-2, j)) = \{P_1, P_2, P_3\}$ in HReT$(m-2, 4) - (k-2, l)$. By Lemma 12, $C'_{3^*,1}((0,0),(i,j))$ forms a 3^*-container of HReT$(m, 4) - \{z\}$.

Thus the theorem is proved. □

Chapter 5

Conclusion

There are a lot of studies on hamiltonian graphs. In this thesis, we are interested in some specific types of hamiltonian graphs. We introduce the concept of mutually independent hamiltonicity first. The concept of mutually independent hamiltonian arises from the following applications. If there are k pieces of data needed to be sent from u to v, and the data needed to be processed at every vertex, then we want mutually independent hamiltonian paths so that there will be no waiting time at a processor. Thus the mutually independent hamiltonian property is useful for communication algorithms. In chapter 2, we are interested in two families of graphs. The first family of graphs are those graphs with $\bar{e} \leq n-4$ and $n \geq 4$. It was proved [37] that such graphs are hamiltonian connected. In Theorem 6, we strengthen this classical result by proving that there are at least $n-2-\bar{e}$ mutually independent hamiltonian paths between every pair of distinct vertices of G. The second family of graphs are those graphs with the sum of the degree of any two non-adjacent vertices being at least $n+1$. Assume that G is a graph with the sum of any two non-adjacent vertices being at least $n+2$. Let u and v be any two distinct vertices of G. In Theorem 7, we show that there are $\deg_G(u)+\deg_G(v)-n$ mutually independent hamiltonian paths between u and v if $(u,v) \in E(G)$, and there are $\deg_G(u)+\deg_G(v)-n+2$ mutually independent hamiltonian paths between u and v if otherwise.

In chapter 3, we proposed a new concept called panpositionable hamiltonicity. We showed that the arrangement graph $A_{n,k}$ is panpositionable hamiltonian if $k \geq 1$ and $n-k \geq 2$ in Theorem 10. By applying this result, we can prove that $A_{n,k}$ is panconnected and pancyclic if $k \geq 1$ and $n-k \geq 2$. We also explained some relationship between the panpositionable hamiltonian property and the panconnected property by giving an example to show that a panconnected graph G is not necessarily panpositionable hamiltonian. Therefore, the panpositionable hamiltonian property is a stronger property for an interconnection network.

The honeycomb networks have been proposed as attractive alternatives to mesh and torus interconnection networks for computer architectures, interconnection topologies, parallel processes and distributed systems. In particular, the honeycomb rectangular torus $\text{HReT}(m,n)$ is a well-structured 3-connected cubic network. In chapter 4, we study the globally bi-3^*-connected property of the honeycomb rectangular torus $\text{HReT}(m,n)$. We have proved that any $\text{HReT}(m,n)$ is strongly globally bi-3^*-connected. We also proved that $\text{HReT}(m,n)$ is hyper globally bi-3^*-connected if and only if $n \geq 6$ or $m = 2$.

Future work will be directed to explore the mutually independent hamiltonicity and the panpositionable hamiltonicity of other interconnection networks. Moreover, we will try to find the globally 3^*-connected property of other cubic interconnection networks. It would be interesting to study some relationship between these specific properties, such as panpositionable hamiltonicity, panconnectivity and pancyclicity, and the other criteria for measuring the performance of a network.

Bibliography

[1] S.B. Akers, D. Harel and B. Krishnamurthy, "The Star Graph: An Attractive Alternative to the n-Cube," *Proc. Int'l Conf. Parallel Processing*, pp. 216-223, 1986.

[2] S.B. Akers and B. Krishnamurthy, "A Group-Theoretic Model for Symmetric Interconnection Networks," *IEEE Trans. Computers*, vol. 38, no. 4, pp. 555-566, 1989.

[3] Y. Alavi and J.E. Williamson, "Panconnected Graphs," *Studia Sci. Math. Hungar.* 10, pp. 19-22, 1975.

[4] M. Albert, R.E.L. Aldred and D. Holton, "On 3^*-Connected Graphs," *Australasian Journal of Combinatorics*, vol. 24, pp. 193-208, 2001.

[5] J.A. Bondy, "Pancyclic Graphs," *J. Combin. Theory Ser. B* 11, pp. 80-84, 1971.

[6] M.Y. Chan and S.J. Lee, "On the Existence of Hamiltonian Circuits in Faulty Hypercubes," *SIAM J. Discrete Math.,* vol. 4, no. 4, pp. 511-527, 1991.

[7] J.M. Chang, J.S. Yang, Y.L. Wang and Y. Cheng, "Panconnectivity, Fault-Tolerant Hamiltonicity and Hamiltonian-Connectivity in Alternating Group Graphs," *Networks,* vol. 44, pp. 302-310, 2004.

[8] W.K. Chiang and R.J. Chen, "On the Arrangement Graph," *Information Processing Letters,* vol. 66, no. 4, pp. 215-219, 1998.

[9] H.J. Cho and L.Y. Hsu, "Ring embedding in faulty honeycomb rectangular torus," *Information Processing Letters,* vol. 84, pp. 277-284, 2002.

[10] V. Chvátal, "On Hamilton's ideal," *J. Comb. Th. (B)*, 12, pp. 163-168, 1972.

[11] P. Cull and S. Larson, "The Möbius Cubes," *Distributed Memory Computing Conference, Proceedings., The Sixth*, pp. 699-702, 1991.

[12] K. Day and A. Tripathi, "Characterization of Node Disjoint Paths in Arrangement Graphs," Technical Report TR 91-43, Computer Science Dept., Univ. of Minnesota, 1991.

[13] K. Day and A. Tripathi, "Arrangement Graphs: A Class of Generalized Star Graphs," *Information Processing Letters,* vol. 42, no. 5, pp. 235-241, 1992.

[14] K. Day and A. Tripathi, "Embedding of Cycles in Arrangement Graphs," *IEEE Trans. Computers,* vol. 42, no. 8, pp. 1002-1006, Aug. 1993.

[15] K. Day and A. Tripathi, "Embedding Grids, Hypercubes, and Trees in Arrangement Graphs," *Proc. Int'l Conf. Parallel Processing,* pp. III-65-III-72, 1993.

[16] K. Efe, "The Crossed Cube Architecture for Parallel Computing," *IEEE Trans. Parallel and Distributed Systems,* vol. 3, no. 5, pp. 513-524, 1992.

[17] P. Erdös and T. Gallai, "On Maximal Paths and Circuits of Graphs," *Acta Math. Ac. Sc. Hung.,* 10, pp. 337-356, 1959.

[18] F. Harary, *Graph Theory,* Addison-Wesley, Reading MA 1994.

[19] P.A.J. Hilbers, M.R.J. Koopman and J.L.A. van de Snepscheut, "The Twisted Cube," *Parallel Architectures and Languages Europe, Lecture Notes in Computer Science,* pp. 152-159, 1987.

[20] S.Y. Hsieh, G.H. Chen and C.W. Ho, "Fault-Free Hamiltonian Cycles in Faulty Arrangement Graphs," *IEEE Trans. Parallel and Distributed Systems,* vol. 10, no. 3, pp. 223-237, 1999.

[21] S.Y. Hsieh and G.H. Chen, "Pancyclicity on Möbius Cubes with Maximal Edge Faults," *Parallel Computing,* vol. 30, pp. 407-421, 2004.

[22] D. Frank Hsu, "On Container Width and Length in Graphs, Groups and Networks," *IEICE Trans. Fundamentals,* vol. E77-A, no. 4, pp. 668-680, 1994.

[23] H.C. Hsu, L.C. Chiang, J.M. Tan and L.H. Hsu, "Fault Hamiltonicity of Augmented Cubes," *Parallel Computing,* Vol. 31, pp. 131-145, 2005.

[24] H.C. Hsu, T.K. Li, J.M. Tan and L.H. Hsu, "Fault Hamiltonicity and Fault Hamiltonian Connectivity of the Arrangement Graphs," *IEEE Trans. Computers,* vol. 53, no. 1, pp. 39-53, 2004.

[25] J.S. Jwo, S. Lakshmivarahan and S.K. Dhall, A New Class of Interconnection Networks Based on the Alternating Group, *Networks,* vol. 23, pp. 315-326, 1993.

[26] S.S. Kao, H.C. Hsu and L.H. Hsu, "Globally Bi-3*-Connected Graphs," submitted.

[27] S.S. Kao and L.H. Hsu, "Spider Web Networks: A Family of Optimal, Fault Tolerant, Hamiltonian Bipartite Graphs," *Applied Mathematics and Computation,* Vol. 160, pp. 269-282, 2005.

[28] S. Latifi and N. Bagherzadeh, "On Embedding Rings into a Star-Related Network," *Information Sciences,* vol. 99, pp. 21-35, 1997.

[29] S. Latifi, S.Q. Zheng and N. Bagherzadeh, "Optimal Ring Embedding in Hypercubes with Faulty Links," *Proc. IEEE Symp. Fault-Tolerant Computing,* vol. 42, pp. 178-184, 1992.

[30] F.T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes,* Morgan Kaufmann Publishers, San Mateo, CA, 1992.

[31] C.K. Lin, H.M. Huang, D.F. Hsu and L.H. Hsu, "On the Spanning w-Wide Diameter of the Star Graph," *Networks,* vol.48, pp. 235-249, 2006.

[32] C.K. Lin, H.M. Huang, L.H. Hsu and S. Bau, "Mutually Independent hamiltonian Paths in Star Networks," *Networks,* vol. 46, pp. 110-117, 2005.

[33] X. Lin and P.K. McKinley, "Deadlock-Free Multicast Wormhole Routing in 2-D Mesh Multicomputers," *IEEE Trans. Parallel and Distributed Systems,* vol. 5, pp. 793-804, 1994.

[34] R.S. Lo and G.H. Chen, "Embedding Hamiltonian Paths in Faulty Arrangement Graphs with the Backtracking Method," *IEEE Trans. Parrallel and Distributed Systems*, vol. 12, no. 2, pp. 209-221, 2001.

[35] G.M. Megson, X.F. Yang and X.P. Liu, "Honeycomb Tori are Hamiltonian," *Information Processing Letters* vol. 72, pp. 99-103, 1999.

[36] K. Menger, "Zur allgemeinen Kurventheorie," *Fund. Math.*, 10, pp. 95-115, 1927.

[37] O. Ore, "Hamiltonian Connected Graphs," *J. Math. PuresAppl.,* 42, pp. 21-27, 1963.

[38] R.A. Rowley and B. Bose, "Fault-Tolerant Ring Embedding in de Bruijn Networks," *IEEE Trans. Computers,* vol. 42, no. 12, pp. 1480-1486, 1993.

[39] Y. Saad and M.H. Schultz, "Topological Properties of Hypercubes," *IEEE Trans. Computers,* vol. 37, no. 7, pp. 867-872, 1988.

[40] I. Stojmenovic, "Honeycomb Networks: Topological Properties and Communication Algorithms," *IEEE Trans. Parallel and Distributed Systems,* vol. 8, no. 10, pp. 1036V1042, 1997.

[41] C.M. Sun, C.K. Lin, H.M. Huang and L.H. Hsu, "Mutually Independent Hamiltonian Paths and Cycles in Hypercubes," *Journal of Interconnection Networks,* Vol. 7, pp. 235-255, 2006.

[42] Y.H. Teng, J.M. Tan and L.H. Hsu, "Honeycomb Rectangular Disks," *Parallel Computing,* vol. 31, pp. 371-388, 2005.

[43] Y.H. Teng, J.M. Tan, T.Y. Ho and L.H. Hsu, "On Mutually Independent Hamiltonian Paths," *Applied Mathematics Letters,* Vol. 19, pp. 345-350, 2006.

[44] Y.H. Teng, J.M. Tan and L.H. Hsu, "Panpositionable Hamiltonicity of the Alternating Group Graphs," *Networks,* Vol. 50, pp. 146-156, 2007.

[45] Y.C. Tseng, S.H. Chang and J.P. Sheu, "Fault-Tolerant Ring Embedding in a Star Graph with Both Link and Node Failure," *IEEE Trans. Parallel and Distributed Systems,* vol. 8, pp. 1185-1195, 1997.

[46] Y.C. Tseng, M.H. Yang and T.Y. Juang, "Achieving Fault-Tolerant Multicast in Injured Wormhole-Routed Tori and Meshes Based on Euler Path Construction," *IEEE Trans. Computers,* vol. 48, pp. 1282-1296, 1999.

[47] D. Wang, "Embedding hamiltonian cycles into folded hypercubes with faulty links," *Journal of Parallel and Distributed Computing,* vol. 61, pp. 545-564, 2001.

[48] M.C. Yang, T.K. Li, Jimmy J.M. Tan and L.H. Hsu, "Fault-Tolerant Pancyclicity of the Möbius Cubes," *IEICE Trans. Fundamentals,* Vol. E88-A, pp. 346-352, 2005.

[49] M.C. Yang, T.K. Li, J.M. Tan and L.H. Hsu, "On Embedding Cycle in Faulty Twisted Cubes," *Information Sciences,* Vol. 176, pp. 676-690, 2006.

[50] X. Yang, D.J. Evans, H. Lai and G.M. Megson, "Generalized honeycomb torus is Hamiltonian," *Information Processing Letters,* Vol. 92, pp. 31-37, 2004.