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多處理器系統診斷能力之量測

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多處理機系統診斷能力之量測

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摘要

有關多處理機系統錯誤診斷問題已經在相當多的文獻被廣泛的討論，並且很多著名連結網路的診斷能力也已經被提出來了。在這篇論文當中，我們針對多處理機系統研究了一些不同的診斷問題。首先，我們介紹了一種新的診斷能力量測方法稱為*局部診斷能力量測*，並且提出了一些架構用來決定系統中一個處理機在PMC診斷模式下是否為局部t-可診斷的。針對超立方體網路(hypercube)和星狀網路(star graph)，我們證明了網路中每一個點的局部診斷能力等於它們自己的分支度。接著，我們針對系統診斷問題提出了一個新的觀念稱為強局部可診斷特性。一個系統我們說他具有強局部可診斷特性即表示此系統中每一個處理機的局部診斷能力等於它們自己的分支度。所以我們可以推得n維度超立方體網路Q_n和星狀網路S_n都有此很強的特性，當$n \geq 3$。下一步我們接著研究當多處理機系統具有一些壞掉的邊時，每一個點它們的局部診斷能力。對於具有一些壞掉的邊的n維度超立方體網路Q_n和星狀網S_n，我們證明了Q_n在壞n - 2條邊以內其仍然保有此很強的特性，而S_n在壞n - 3條邊以內也仍然保有此特性。假設網路在壞掉邊時每一個點具有至少兩條好的邊時，在這樣的條件下，我們證明了Q_n壞掉的邊數可以增加到$3(n-2)-1$條仍然保有這種強特性，而S_n在此條件下無論壞多少條邊仍然可保有此特性。更進一步地，我們考慮網路在壞掉邊時每一個點具有至少三條好的邊時，在這樣的條件下，我們證明了Q_n無論壞多少條邊仍可保有此很強的特性，並且我們所提出的這些壞邊數都是最佳值。除此之外，我們針對一般系統也提出了一個新的診斷演算法。此演算法的時間複雜度為$O(N \log N)$，此處*N代表系統中處理機的總數*。

條件式診斷能力量測是由賴等人所提出的，此量測方法在多處理機系統是另外一個有趣的議題。此量測方法在量測一個系統的診斷能力時給予一個條件，此條件為，在系統中

任一個錯誤點集合不能包含任一個點的所有鄰居。本篇論文當中，我們根據這個條件去計算一個n維度超立方體網路Q_n在比較式診斷模式下它的條件式診斷能力，並且得到的答案為$3(n - 2) + 1$，當$n \geq 5$。此條件式診斷能力約是傳統診斷能力的三倍之多。最後，我們延伸這個結果到BC(bijective connection)網路上，一個n維度BC網路記作X_n，此網路是一個n−正規圖具有2^n個點和$n2^{n-1}$條邊。一般常見的超立方體網路(hypercube)、交錯超立方體網路(crossed cube)、雙扭超立方體網路(twisted cube)和梅式超立方體網路(Mőbius cube)都是BC網路的一種。在這篇論文當中，我們也證明了一個n維度BC網路X_n在比較式診斷模式下它的條件式診斷能力為$3(n - 2) + 1$，當$n \geq 5$。根據這個結果，我們可以推得所有立方體網路的條件式診斷能力。

關鍵字：PMC 診斷模式、比較式診斷模式、t-可診斷的、診斷能力、局部診斷能力、強局部可診斷特性、條件式錯誤集合、條件式診斷能力、超立方體網路、星狀網路、BC網路、診斷演算法。

Diagnosability Measures for Multiprocessor Systems: A New Local Strategy

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Abstract

The problem of fault diagnosis has been discussed widely and the diagnosability of many well-known networks has been explored. In this thesis, we study some variants of diagnosis problems on multiprocessor systems. First of all, we introduce a new measure of diagnosability, called local diagnosability, and derive some structures for determining whether a vertex of a system is locally t-diagnosable under the PMC model. For hypercube network and star graph, we prove that the local diagnosability of each vertex is equal to its degree. Then, we propose a concept for system diagnosis, called strongly local-diagnosable property. A system $G(V,E)$ is said to have a strongly local-diagnosable property, if the local diagnosability of each vertex is equal to its degree. We show that both Q_n and S_n have this strong property for $n \geq 3$, where the two notations Q_n and S_n represent an n-dimensional hypercube and an n-dimensional star graph, respectively. Next, we study the local diagnosability of a faulty multiprocessor system. For a faulty hypercube Q_n and a faulty star graph S_n, we prove that both Q_n and S_n keep this strong property even if they have up to $n-2$ faulty edges and $n-3$ faulty edges, respectively. Assume that each vertex of a faulty hypercube Q_n and a faulty star graph S_n is incident with at least two fault-free edges, we prove that Q_n keeps this strong property even if it has up to $3(n-2)-1$ faulty edges and S_n will also keep this strong property no matter how many edges are faulty. Furthermore, we prove Q_n keeps this strong property no matter how many edges are faulty, provided that each vertex of a faulty hypercube Q_n is incident with at least three fault-free edges. Our bounds on the number of faulty edges are all tight. Besides, we propose a new diagnosis algorithm for general systems. The time complexity of our algorithm to diagnose all the faulty processors is bounded by $O(N \log N)$, where N is the total number of processors.

The conditional diagnosability measure, introduced by Lai et al., is another interesting issue for multiprocessor systems. They proposed this novel measure of diagnosability by adding an additional condition that any faulty set cannot contain all the neighbors of any vertex in a system. In this thesis, We make a contribution to the evaluation of diagnosability for hypercube networks under the comparison model and prove that the conditional diagnosability of n-dimensional hypercube Q_n is $3(n-2)+1$ for $n \geq 5$. The conditional diagnosability of Q_n is about three times larger than the classical diagnosability of Q_n. Furthermore, we extend the result to bijective connection network (in brief, BC network). An n-dimensional BC network, denoted by X_n, is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges. The n-dimensional hypercube, crossed cube, twisted cube, and Mőbius cube are some examples of the n-dimensional BC networks. In this thesis, we also prove that the conditional diagnosability of X_n is $3(n-2)+1$ under the comparison model, $n \geq 5$. As a corollary of this result, we obtain the conditional diagnosability of the cube family.

Keywords: PMC model, comparison model, *t*-diagnosable, diagnosability, local diagnosability, strongly local-diagnosable property, conditional faulty set, conditional diagnosability, hypercube network, star graph, BC network, diagnosis algorithm.

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Chapter 1

Introduction

With the continuous increase in the size of a multiprocessor system, the complexity of the system can adversely affect its reliability. In order to maintain reliability, the system should be able to identify faulty processors and replace them with fault-free ones. The process of identifying faulty processors is called the *diagnosis* of the system, and the *diagnosability* of the system refers to the maximum number of faulty processors that can be identified by the system. The problem of identifying faulty processors in a multiprocessor system has been widely studied in literatures [1, 3, 7, 9, 11, 12, 13, 14, 15, 16, 17, 18, 25, 28, 29, 30, 31, 32, 33, 38, 45, 48, 49, 53, 54, 55, 58]. There are two fundamental approaches to system-level diagnosis: *tested-based diagnosis* (PMC model) and *comparison-based diagnosis* (comparison model). In 1967, the Preparata, Metze, and Chien (PMC) model was proposed for system-level diagnosis in multiprocessor systems [45]. The PMC model uses tested-based diagnosis approach, under which a processor performs the diagnosis by testing on neighboring processors via the communication links between them. By analyzing the collection of all testing results, all of the faulty processors are identified. The PMC model was also used [4, 5, 6, 8, 26, 27, 34, 35, 36, 45]. In [26], Hakimi and Amin proved that a system is t-diagnosable if it is t-connected with at least $2t + 1$ vertices. They also gave a necessary and sufficient condition for verifying if a system is t-diagnosable under the PMC model.

The hypercube structure [46] and star graph [2] are two popular topologies for multiprocessor systems. An n-dimensional hypercube is denoted by Q_n, and the diagnosability of Q_n is shown to be n [35] under the PMC model, $n \geq 3$. An n-dimensional star graph is denoted by S_n, and the diagnosability of S_n is shown to be $n-1$ under the PMC model, $n \geq 3$ [37]. In [40], Lai et al. introduced a novel measure of diagnosability called *conditional diagnosability* by restricting that a faulty set cannot contain all the neighbors of any vertex. Based on this restriction, the conditional diagnosability of the n-dimensional hypercube is shown to be $4(n-2)+1$. Besides, Lai et al. introduced a concept called a *strongly t-diagnosable systems* and proved that the n-dimensional hypercube is strongly n-diagnosable. Essentially, it means that an n-dimensional hypercube is almost $(n+1)$-diagnosable except for the case where all the neighbors of some vertex are faulty simultaneously. In [50], Wang proved that the diagnosability of an incomplete hypercube under some conditions can be determined by simply checking the degree of each vertex under the PMC model. An incomplete hypercube is a hypercube with some missing edges. It is also called a faulty hypercube. There are some results concerning the diagnosability of several variations of the hypercube [4, 10, 21, 22, 26, 35, 50]. In classical measures of system-level diagnosability for multiprocessor systems, it has generally been assumed that any subset of processors can potentially fail at the same time. As a consequence, the diagnosability of a system is upper bounded by its minimum degree.

We observe that the diagnosability of a system discussed in previous literatures are all in a global sense, but ignored some local information. A system is t-diagnosable if, all the faulty processors can be uniquely identified, provided that the number of faulty processors does not exceed t. However, it is possible to correctly indicate all the faulty processors in a t-diagnosable system when the number of faulty processors is greater than t. For example, consider a multiprocessor system generated by integrating two arbitrary subsystems with a few communication links in some way, where the two subsystems are m-diagnosable and n-diagnosable, respectively, and $m >> n$. The diagnosability of this system is limited by n, but it is possible to correctly point out all the faulty processors even if the number of the faulty ones is between m and n. Therefore, if only considering the global faulty/fault-free status, we lose some local systematic details.

In this thesis, we propose a new measure of diagnosability, called *local diagnosability*, and study the local diagnosability of each processor of a system. We can identify the diagnosability of a system by computing the local diagnosability of each processor. This measure of the local diagnosability leads us to study the local diagnosability of each processor instead of the whole system. We propose a necessary and sufficient condition, Theorem 6, to determine the local diagnosability of a processor. We also provide two useful structures, called *Type I structure* and *Type II structure*, to determine the local diagnosability of a processor under the PMC model. Based on these structures, the local diagnosability of each vertex of hypercube and star graph is shown to be equal to its own degree. Then, we propose a concept for system diagnosis, called *strongly local-diagnosable property*. A system $G(V, E)$ is said to have a strongly local-diagnosable property, if the local diagnosability of each vertex is equal to its degree. We show that an n-dimensional hypercube Q_n and an n-dimensional star graph S_n all have this strong property. Then, we study the local diagnosability of an incomplete hypercube and an incomplete star graph. Firstly, we show that both Q_n and S_n keep this strong property even if it has up to $n-2$ faulty edges and $n-3$ faulty edges, respectively. Secondly, assume that each vertex of an incomplete hypercube Q_n and an incomplete star graph S_n is incident with at least two fault-free edges, we show Q_n keeps this strong property even if it has up to $3(n-2)-1$ faulty edges and S_n will also keep this strong property no matter how many edges are faulty. Furthermore, we show that Q_n keeps this strong property no matter how many edges are faulty, provided that each vertex of an incomplete hypercube Q_n is incident with at least three fault-free edges. Our bounds on the number of faulty edges are all tight. Besides, we propose a new diagnosis algorithm for general systems. The time complexity of our algorithm to diagnose all the faulty processors is bounded by $O(N \log N)$, where N is the total number of processors.

In 1980, Malek and Maeng introduced the comparison model using Comparison-based diagnosis approach, also known as the MM model [42, 43]. In this model, the number of faulty processors is limited and all faults are permanent. The MM model deals with the faulty diagnosis by sending the same input (or task) from a processor w to each pair of distinct neighbors, u and v, and then comparing their responses. The processor w is called the *comparator* of processors u and v. Different comparators may examine the same pair of processors. The result of the comparison is either the two responses agreed or two responses disagreed. Based on the results of all the comparisons, one need to decide the faulty or fault-free status of the processors in the system. Using a comparison diagnosis model, Sengupta and Dahbura described a diagnosable system and presented a polynomial algorithm to determine the set of all faulty processors [47].

Reviewing some previous literatures [4, 10, 21, 22, 23, 26, 35, 39, 41, 46, 51], Q_n, CQ_n, TQ_n and MQ_n, all have diagnosability n under the comparison model or the PMC model. The diagnosability of the Star S_n is shown to be $n-1$ under the comparison model [56]. In classical measures of system-level diagnosability for multiprocessor systems, if all the neighbors of some processor v are faulty simultaneously, it is not possible to determine whether processor v is fault-free or faulty. As a consequence, the diagnosability of a system is limited by its minimum degree. Hence, Lai et al. introduced a restricted diagnosability of multiprocessor systems called conditional diagnosability in [40]. Lai et al. considered a measure by restricting that, for each processor v in a system, all the processors which are directly connected to v do not fail at the same time. In this thesis, We make a contribution to the evaluation of diagnosability for hypercube networks under the comparison model and prove that the conditional diagnosability of n-dimensional hypercube Q_n is $3(n-2)+1$ for $n \geq 5$. The conditional diagnosability of Q_n is about three times larger than the classical diagnosability of Q_n. Furthermore, we extend the result to bijective connection network (in brief, BC network). An n-dimensional BC network, denoted by X_n, is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges. The n-dimensional hypercube, crossed cube, twisted cube, and Möbius cube are some examples of the n-dimensional BC networks. In this thesis, we also prove that the conditional diagnosability of X_n is $3(n-2)+1$ under the comparison model, $n \geq 5$. As a corollary of this result, we obtain the conditional diagnosability of the cube family.

1.1 Basic Terms and Notations

A multiprocessor system can be represented by a graph $G(V, E)$, where the set of vertices $V(G)$ represents processors and the set of edges $E(G)$ represents communication links between processors. Throughout this thesis, we focus on undirected graph without loops and follow [52] for graph theoretical definitions and notations.

Let $G(V, E)$ be a graph and $v \in V(G)$ be a vertex. We use the notation $E_G(v)$ to denote the set of edges incident with v. The cardinality $|E_G(v)|$ is called the degree of v, denoted by $deg_G(v)$ or simply $deg(v)$. The maximum degree is denoted by $\Delta(G)$, the minimum degree is $\delta(G)$, and G is regular if $\Delta(G) = \delta(G)$. G is d-regular if $deg(v) = d$ for every $v \in V(G)$. The neighborhood $N(v)$ of a vertex v in G is the set of all vertices that are adjacent to v in G. For a subset of vertices $V' \subset V(G)$, the neighborhood set of the vertex set V' is defined as $N(V') = \bigcup_{v \in V'} N(v) - V'$. For a set of edges(respectively, vertices) F, we use the notation $G - F$ to denote the graph obtained from G by removing all the edges(respectively, vertices) in F. The components of a graph G are its maximal connected subgraphs. A component is trivial if it has no edges; otherwise, it is nontrivial. The connectivity $\kappa(G)$ of a graph $G(V, E)$ is the minimum number of vertices whose removal results in a disconnected or a trivial graph. Let G_1 be a subgraph of G, we shall write the vertex set of G_1 as $V(G_1)$. The neighborhood set of $V(G_1)$ is defined as $N(V(G_1)) = \{u \in V(G) - V(G_1) \mid \text{there exists a vertex } v \in V(G_1) \text{ such that } (u, v) \in E(G)\}$. The following is an useful characterization for the distinguishability of two sets of vertices under the PMC model and the comparison model. Let $F_1, F_2 \subseteq V(G)$ be two distinct sets. The symmetric difference of the two sets F_1 and F_2 is defined as the set $F_1 \Delta F_2 = (F_1 - F_2) \bigcup (F_2 - F_1)$.

For studying the conditional diagnosability of a system, we also need some definitions for further discussion. Let $G(V, E)$ be a graph. For any set of vertices $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by the vertex subset U. Let H be a subgraph of G and v be a vertex in H. We use $V(H; 3) = \{v \in V(H) \mid deg_H(v) \geq 3\}$ to represent the set of vertices which has degree 3 or more in H. Let $F_1, F_2 \subseteq V(G)$ be two distinct sets and $S = F_1 \bigcap F_2$. We use $C_{F_1 \Delta F_2, S}$ to denote the subgraph induced by the vertex subset $(F_1 \Delta F_2) \bigcup \{u \mid \text{there exists a vertex } v \in F_1 \Delta F_2 \text{ such that } u \text{ and } v \text{ are connected in } G - S\}$.

1.2 Organization of the Thesis

The rest of this thesis is organized as follows. The details for the PMC model and the comparison model are described in Chapter 2, and the previous results for diagnosing a system are also provided in this chapter as well.

In Chapter 3, we introduce the concept of local diagnosability and propose a necessary and sufficient condition for verifying if it is locally t-diagnosable at a given processor in a system. Then, we define a strongly local-diagnosable property for a system and study the strong property in a faulty hypercube and a faulty star graph respectively. Next, we study the strong property in a conditional faulty hypercube and star graph. A diagnosis algorithm is proposed at the end of this chapter.

In Chapter 4, we focus on the measure of conditional diagnosability we study the conditional diagnosability of the hypercube Q_n under the comparison model. Finally, our conclusions and future works are given in Chapter 5.

Chapter 2

Diagnosis Model

The process of identifying faulty processors in a system is known as the system-level diagnosis. Several different approaches have been developed to diagnose faulty processors, among which there are two fundamental approaches on system-level diagnosis. One major approach is called PMC model established by Preparata, Metze and Chien [45]. Another major approach is the comparison model, proposed by Malek and Maeng [42, 43]. In the following, we describe the details of the two major models and give some previous results for diagnosing a system.

2.1 The PMC Model and Some Previous Results

The PMC diagnosis model is presented by Preparata, Metze and Chien [45]. In this model, a self-diagnosable system is often represented by a *directed graph* $T(V, E)$ in which an edge directed from vertex u to vertex v means that u can test v. In this situation, u is called the tester and v is called the tested vertex. The outcome of a test (u, v) is 1(respectively, 0) if u evaluates v as faulty (respectively, fault-free). We assume that the testing results of fault-free vertices are always reliable and the testing results of faulty vertices are unreliable. The collection of all testing results is called a *syndrome*. Formally, a syndrome is a function $\sigma : E \to \{0, 1\}$. The set of all faulty processors in the system is called a $faulty\ set$. This can be any subset of $V(T)$. For a given syndrome σ, a subset of vertices $F \subset V(T)$ is compatible with σ if the syndrome σ can be produced from the situation that all vertices in F are faulty and all vertices in $V - F$ are fault-free. Since faulty testers can give arbitrary testing results, any syndrome compatible with a faulty set F can occur when faulty processors in the system are exactly those in F. Let σ_F be the set of all syndromes which could be produced if F is the set of faulty vertices. Two distinct sets $F_1, F_2 \subseteq V(G)$ are said to be *distinguishable* if $\sigma_{F_1} \bigcap \sigma_{F_2} = \phi$; otherwise, F_1, F_2 are said to be *indistinguishable*. We say (F_1, F_2) is a *distinguishable pair* if $\sigma_{F_1} \bigcap \sigma_{F_2} = \phi$; otherwise, (F_1, F_2) is an *indistinguishable pair*. For PMC model, some known results about the definition of t-diagnosable system and related concepts are listed as follows. Some of these previous results are on directed graphs and others are on undirected.

Definition 1 *[45] A system G is called t-diagnosable if, given the test outcomes obtained by the testing link, all the faulty vertices can be uniquely identified without replacement, provided that the number of faulty vertices does not exceed t.*

Definition 2 *[45] The maximum number of faulty vertices that a system G can guarantee to identify is called the diagnosability of G, written as $t(G)$.*

Dahbura and Masson [19] proposed a polynomial time algorithm to check whether a system is t-diagnosable.

Lemma 1 *[19] A system $G(V, E)$ is t-diagnosable under the PMC model if and only if for each pair $F_1, F_2 \subset V$ with $|F_1|, |F_2| \leq t$ and $F_1 \neq F_2$, there is at least one test from $V - (F_1 \bigcup F_2)$ to $F_1 \Delta F_2$.*

The following two lemmas related to t-diagnosable systems are proposed by Preparata et al. [45] and Hakimi et al. [26], respectively.

Lemma 2 *[45] Let $G(V, E)$ be a graph and $|V| = N$. The following two conditions are necessary for G to be t-diagnosable;*

1. *$N \geq 2t + 1$, and*

2. each processor in G is tested by at least t other processors.

Lemma 3 *[26] Let $G(V,E)$ be a graph and $|V| = N$. G is t-diagnosable if*

1. $N \geq 2t+1$, and

2. $\kappa(G) \geq t$.

For a directed graph $G(V,E)$ and vertex $v \in V$, let $\Gamma(v) = \{v_i | (v, v_i) \in E\}$ and $\Gamma(X) = \bigcup_{v \in X} \Gamma(v) - X, X \subset V$. Hakimi and Amin presented a necessary and sufficient condition for a system G to be t-diagnosable as follows:

Theorem 1 *[26] Let $G(V,E)$ be the directed graph of a system G and $|V| = N$. Then G is t-diagnosable under the PMC model if and only if: (i) $N \geq 2t+1$, (ii) $d_{in}(v) \geq t$ for all $v \in V$, and (iii) for each integer p with $0 \leq p \leq t-1$, and each $X \subset V$ with $|X| = N - 2t + p$, $|\Gamma(X)| > p$.*

In this thesis, we propose some new concepts on diagnosis, and we focus on undirected graph. The following lemma follows directly from Lemma 1.

Lemma 4 *[19] Let $G(V,E)$ be a graph. For any two distinct sets $F_1, F_2 \subset V$, (F_1, F_2) is a distinguishable pair under the PMC model if and only if there exists a vertex $u \in V - (F_1 \bigcup F_2)$ and a vertex $v \in F_1 \Delta F_2$ such that $(u,v) \in E$ (see Figure 2.1).*

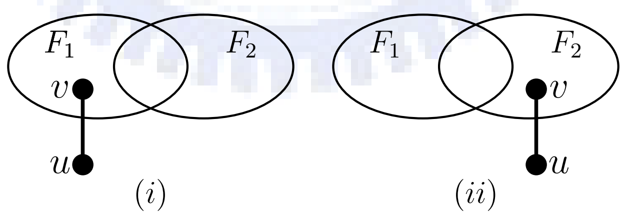


Figure 2.1: Illustration for a distinguishable pair (F_1, F_2)

It follows from Definition 1 that the following lemma holds.

Lemma 5 *[19] A system $G(V, E)$ is t-diagnosable under the PMC model if and only if, for any two distinct sets $F_1, F_2 \subset V$ with $|F_1| \leq t$ and $|F_2| \leq t$, (F_1, F_2) is a distinguishable pair.*

The following Lemma 6 is equivalent to Lemma 5.

Lemma 6 *[19] A system $G(V, E)$ is t-diagnosable if and only if, for each indistinguishable pair $F_1, F_2 \subset V$, it implies that $|F_1| > t$ or $|F_2| > t$.*

By Lemma 2, a similar result for undirected graph is stated as follows.

Corollary 1 *[45] Let $G(V, E)$ be an undirected graph and $|V| = N$. The following two conditions are necessary for G to be t-diagnosable under the PMC model:*

1. $N \geq 2t + 1$, *and*
2. $\delta(G) \geq t$.

For our discussion later, a useful result presented by Lai [40] is stated below.

Theorem 2 *[40] Let $G(V, E)$ be a graph. G is t-diagnosable if and only if, for each set of vertices $F \subset V$ with $|F| = p$, $0 \leq p \leq t - 1$, each connected component of $G - F$ has at least $2(t - p) + 1$ vertices.*

2.2 The Comparison Model and Some Previous Results

The comparison diagnosis model is proposed by Malek and Maeng [42, 43]. In this model, a self-diagnosable system is often represented by a multigraph $M(V, C)$, where V is the same vertex set defined in G and C is the labeled edge set. Let $(u, v)_w$ be a labeled edge. If (u, v) is an edge labeled by w, then $(u, v)_w$ is said to belong to C, which implies that the vertex u and v are being compared by vertex w. The same pair of vertices may be compared by different comparators, so M is a multigraph. For $(u, v)_w \in C$, we use $r((u, v)_w)$ to denote the result of comparing vertices u and v by w such that $r((u, v)_w) = 0$ if the outputs of u and v agree, and $r((u, v)_w) = 1$ if the outputs disagree. In this model, if $r((u, v)_w) = 0$ and w is fault-free, then both u and v are fault-free. If $r((u, v)_w) = 1$, then at least one of the three vertices u, v, w must be faulty. If the comparator w is faulty, then the result of the comparison is unreliable that means both $r((u, v)_w) = 0$ and $r((u, v)_w) = 1$ are possible outputs, and it outputs only one of these two possibilities. In this thesis, we consider a complete diagnosis that means each vertex diagnoses all pairs of distinct neighbors. For an n-dimensional hypercube Q_n, each vertex has degree n, and therefore, there are $\binom{n}{2}$ comparisons for each vertex acting as a comparator. Furthermore, there are 2^n vertices in Q_n so the total number of comparisons is $\binom{n}{2}2^n = O(n^2 2^n)$.

As the description for the PMC model, the collection of all comparison results defined as a function σ: $C \rightarrow \{0, 1\}$, is called the *syndrome* of the diagnosis. A subset $F \subset V$ is said to be *compatible* with a syndrome σ if σ can arise from the circumstance that all vertices in F are faulty and all vertices in $V - F$ are fault-free. A system is said to be *diagnosable* if, for every syndrome σ, there is a unique $F \subset V$ that is compatible with σ. In [47], a system is called a t-diagnosable system if the system is diagnosable as long as the number of faulty vertices does not exceed t. The maximum number of faulty vertices that the system G can guarantee to identify is called the *diagnosability* of G, written as $t(G)$. A faulty comparator can lead to unreliable results. So, a set of faulty vertices may produce different syndromes. Let $\sigma_F = \{\sigma \mid \sigma \text{ is compatible with } F\}$. Two distinct sets $F_1, F_2 \subset V$ are said to be *indistinguishable* if and only if $\sigma_{F_1} \bigcap \sigma_{F_2} \neq \emptyset$; otherwise, F_1, F_2 are said to be *distinguishable*. There are several different ways to verify a system to be t-diagnosable under the comparison approach. The following theorem given by Sengupta and Dahbura [47] is a necessary and sufficient condition for ensuring distinguishability.

Theorem 3 *[47] Let $G(V, E)$ be a graph. For any two distinct sets $F_1, F_2 \subset V$, (F_1, F_2) is a distinguishable pair under the comparison model if and only if at least one of the following conditions is satisfied (see Figure 2.2):*

1. $\exists u, w \in V - \{F_1 \bigcup F_2\}$ *and* $\exists v \in F_1 \triangle F_2$ *such that* $(u, v)_w \in C$,

2. $\exists u, v \in F_1 - F_2$ *and* $\exists w \in V - \{F_1 \bigcup F_2\}$ *such that* $(u, v)_w \in C$, *or*

3. $\exists u, v \in F_2 - F_1$ *and* $\exists w \in V - \{F_1 \bigcup F_2\}$ *such that* $(u, v)_w \in C$.

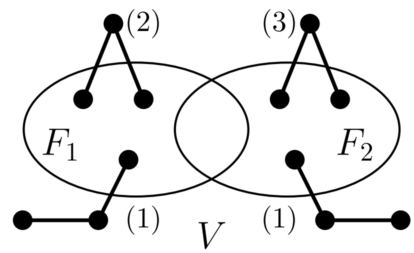


Figure 2.2: Description of distinguishability for Theorem 3.

The following result is a useful sufficient condition for checking whether (F_1, F_2) is a distinguishable pair.

Theorem 4 *Let* $G(V, E)$ *be a graph. For any two distinct sets* $F_1, F_2 \subset V$ *with* $|F_i| \leq t$, $i = 1, 2$, *and* $S = F_1 \bigcap F_2$. (F_1, F_2) *is distinguishable under the comparison model if, the subgraph* $C_{F_1 \Delta F_2, S}$ *of* $G - S$ *contains at least* $2(t - |S|) + 1$ *vertices having degree 3 or more.*

Proof.

Given any pair of distinct sets of vertices $F_1, F_2 \subset V$ with $|F_i| \leq t$, $i = 1, 2$. Let $S = F_1 \bigcap F_2$, then $0 \leq |S| \leq t - 1$, and $|F_1 \Delta F_2| \leq 2(t - |S|)$. Consider the subgraph $C_{F_1 \Delta F_2, S}$, the number of vertices having degree 3 or more is at least $2(t - |S|) + 1$ in $C_{F_1 \Delta F_2, S}$, the subgraph $C_{F_1 \Delta F_2, S}$ contains at least $2(t - |S|) + 1$ vertices. There is at least one vertex with degree 3 or more lying in $C_{F_1 \Delta F_2, S} - F_1 \Delta F_2$. Let u be one of such vertices with degree 3 or more. Let i, j, and k be three distinct vertices linked to u. If one of i, j, and k lies in $C_{F_1 \Delta F_2, S} - F_1 \Delta F_2$, condition 1 of Theorem 3 holds obviously. Suppose all these three vertices belong to $F_1 \Delta F_2$. Without loss of generality, assume i lies in $F_1 - F_2$, one of the two cases will happen: 1) if j lies in $F_1 - F_2$, condition 2 of Theorem 3 holds;

or, 2) if j lies in $F_2 - F_1$, wherever k lies in $F_1 - F_2$ or $F_2 - F_1$, condition 2 or 3 of Theorem 3 holds. So (F_1, F_2) is a distinguishable pair and the proof is complete. $\square$

By Theorem 4, we now propose a sufficient condition to verify whether a system is t-diagnosable under the comparison diagnosis model.

Corollary 2 *Let $G(V, E)$ be a graph. G is t-diagnosable under the comparison model if, for each set of vertices $S \subset V$ with $|S| = p$, $0 \leq p \leq t-1$, every connected component C of $G - S$ contains at least $2(t-p)+1$ vertices having degree at least three. More precisely, $|V(C; 3)| \geq 2(t-p)+1$.*

Chapter 3

Local Diagnosability

We first review some related results on system diagnosability of some well-known networks under the PMC model. In[35], Kavianpour et al. proved that the diagnosability of an n-dimensional hypercube Q_n is n. In [21] and [22], Fan proved that an n-dimensional Crossed cube and an n-dimensional Möbius cube have diagnosability n under the PMC model. In [50], Wang proved that the diagnosability of a faulty hypercube can be determined by checking the degree of each vertex under the PMC model, provided that the minimum degree of the faulty hypercube is at least three.

We observe that the traditional diagnosability discussed in most literatures describes the global status of a system. In this thesis, we study the local status of each processor instead of the global status of a system. For example, for any two positive integers m and n with $m >> n \geq 3$, the diagnosability of two hypercube systems Q_m and Q_n is m and n, respectively. Combining Q_m and Q_n with a few edges in some way may cause the diagnosability of the new system to become n. In this situation, the strong diagnosability of Q_m is disregarded. For this reason, we are motivated to study the local status of each processor. Given a single vertex, we require only identifying the status of this particular processor correctly. We now propose the following concept.

Definition 3 *Let $G(V, E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v if, given a syndrome σ_F produced by a set of faulty vertices $F \subseteq V$ containing vertex v with $|F| \leq t$, every set of faulty vertices F' compatible with σ_F and $|F'| \leq t$, must* *also contain vertex v.*

Definition 4 *Let $G(V, E)$ be a graph and $v \in V$ be a vertex. The local diagnosability of vertex v, written as $t_l(v)$, is defined to be the maximum value of t such that G is locally t-diagnosable at vertex v.*

The following result is another point of view for checking whether a vertex is locally t-diagnosable.

Lemma 7 *Let $G(V, E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v if and only if, for any two distinct sets of vertices $F_1, F_2 \subset V$, $|F_1| \leq t$, $|F_2| \leq t$ and $v \in F_1 \Delta F_2$, (F_1, F_2) is a distinguishable pair.*

In the following, we study some properties of a system being locally t-diagnosable at a given vertex, and its relationship between a system being t-diagnosable.

Proposition 1 *Let $G(V, E)$ be a graph and $v \in V(G)$ be a vertex. G is locally t-diagnosable at vertex v under the PMC model, then $|V(G)| \geq 2t + 1$.*

Proof.

We show this by contradiction. Assume that $|V(G)| \leq 2t$. We partition $V(G)$ into two disjoint subsets F_1, F_2 with $|F_1| \leq t$, $|F_2| \leq t$. The vertex v is either in F_1 or in F_2. Since $V - (F_1 \bigcup F_2) = \emptyset$, there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. By Lemma 4, (F_1, F_2) is an indistinguishable pair, this contradicts the assumption that G is locally t-diagnosable at vertex v. So the result follows. $\square$

Proposition 2 *Let $G(V, E)$ be a graph and $v \in V$ be a vertex with $deg(v) = n$. The local diagnosability of vertex v is at most n under the PMC model.*

Proof.

Let F_1 be the set of vertices adjacent to vertex v, $F_1 = N_G(v)$ and $|F_1| = n$. Let $F_2 = F_1 \bigcup \{v\}$ with $|F_2| = n + 1$. It is a simple matter to check that there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. By Lemma 4, (F_1, F_2) is an indistinguishable pair. Thus, G is not locally $(n+1)$-diagnosable at vertex v, so $t_l(v) \leq n = deg(v)$. We have the stated result. □

Proposition 3 *Let $G(V, E)$ be a graph. Under the PMC model, G is t-diagnosable if and only if G is locally t-diagnosable at every vertex.*

Proof.

To prove the necessity, we assume that G is t-diagnosable. If the result is not true, there exists a vertex $v \in V$ such that G is not locally t-diagnosable at vertex v. By Lemma 7, there exists a distinct pair of sets $F_1, F_2 \subset V$ with $|F_1| \leq t$, $|F_2| \leq t$ and $v \in F_1 \Delta F_2$, (F_1, F_2) is an indistinguishable pair. By Lemma 5, G is not t-diagnosable. This contradicts the assumption, hence the necessary condition follows.

To prove the sufficiency, suppose on the contrary that G is not t-diagnosable, there exists a distinct pair of sets $F_1, F_2 \subset V$ with $|F_1| \leq t$, $|F_2| \leq t$, (F_1, F_2) is an indistinguishable pair. Being distinct, the set $F_1 \Delta F_2 \neq \emptyset$, we can find a vertex $v \in F_1 \Delta F_2$. By Lemma 7, G is not locally t-diagnosable at vertex v, which is a contradiction. This completes the proof. □

By Definition 4 and Proposition 3, we know that the diagnosability of a multiprocessor system is equal to the minimum local diagnosability of all vertices of the system. Thus, we have the following theorem.

Theorem 5 *Let $G(V, E)$ be a multiprocessor system. Under the PMC model, the diagnosability of G is t if and only if*

$$min\{t_l(v) \mid for\ every\ v \in V\} = t.$$

From Theorem 5, we can identify the diagnosability of a system by computing the local diagnosability of each vertex. Because many well-known systems are vertex-symmetric, the diagnosability of these system can be easily identified by this effective method.

Before studying the local diagnosability of a vertex, we need some definitions for further discussion. Let F be a set of vertices and v be a vertex not in F. After deleting the vertices in F from G, we use C_v to denote the connected component which vertex v belongs to. Now, we propose a necessary and sufficient condition for verifying if a system is locally t-diagnosable at a given vertex v.

Theorem 6 *Let $G(V, E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v under the PMC model if and only if, for each set of vertices $F \subset V$ with $|F| = p$, $0 \leq p \leq t-1$ and $v \notin F$, the connected component, which v belongs to in $G - F$, has at least $2(t-p)+1$ vertices.*

Proof.

To prove the necessity, we assume that G is locally t-diagnosable at vertex v. If the result does not hold, there exists a set of vertices $F \subset V$ with $|F| = p, 0 \leq p \leq t-1$, $v \notin F$ such that the connected component C_v has strictly less than $2(t-p)+1$ vertices, $|V(C_v)| \leq 2(t-p)$. We then arbitrarily partition $V(C_v)$ into two disjoint subsets, $V(C_v) = F_1 \bigcup F_2$ with $|F_1| \leq t-p$, $|F_2| \leq t-p$. Let $A_1 = F_1 \bigcup F$ and $A_2 = F_2 \bigcup F$. It is clear that $|A_1| \leq (t-p)+p = t$, $|A_2| \leq (t-p)+p = t$, the vertex $v \in A_1 \Delta A_2$ and there is no edge between $V - (A_1 \bigcup A_2)$ and $A_1 \Delta A_2$. By Lemma 7, (A_1, A_2) is an indistinguishable pair. This contradicts the assumption that G is locally t-diagnosable at vertex v.

We now prove the sufficiency by contradiction. Suppose G is not locally t-diagnosable at vertex v, then, there exists an indistinguishable pair (F_1, F_2) with $|F_1| \leq t$, $|F_2| \leq t$ and $v \in F_1 \Delta F_2$. By Lemma 4, there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. Let $F = F_1 \bigcap F_2$ with $|F| = p$, $0 \leq p \leq t-1$ and $v \notin F$. $F_1 \Delta F_2$ is disconnected from other parts after removing all the vertices in F from G. We observe that $|F_1 \Delta F_2| \leq 2(t-p)$. Thus, the connected component C_v has at most $2(t-p)$ vertices and $|V(C_v)| \leq 2(t-p)$. This contradicts the assumption that the connected component C_v has to satisfy $|V(C_v)| \geq 2(t-p)+1$. Hence, the theorem holds. □

We now propose two special subgraphs called Type I structure and Type II structure. They provide us with an efficient and simple method to identify the local diagnosability of each vertex of a system under the PMC diagnosis model.

Definition 5 *Let $G(V, E)$ be a graph, $v \in V$ be a vertex and k be an integer, $k \geq 1$, a Type I structure $T_1(v; k)$ of order k at vertex v is defined to be the following graph,*

$$T_1(v; k) = [V(v; k), E(v; k)]$$

which is composed of $2k + 1$ vertices and of $2k$ edges as illustrated in Figure 3.1, where

- $V(v; k) = \{v\} \cup \{x_i, y_i \mid 1 \leq i \leq k\}$,
- $E(v; k) = \{(v, x_i), (x_i, y_i) | 1 \leq i \leq k\}$.

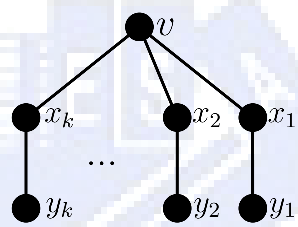


Figure 3.1: A Type I structure $T_1(v; k)$ consists of $2k + 1$ vertices and $2k$ edges.

Following Theorem 6 and Definition 5, we propose a sufficient condition for verifying if it is locally t-diagnosable at a given processor in a system.

Theorem 7 *Let $G(V, E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v under the PMC model if G contains a Type I structure $T_1(v; t)$ of order t at vertex v as a subgraph.*

Proof.

We use Theorem 6 to prove this result. Assume that G contains a subgraph $T_1(v; t)$ at vertex v. Let $e_i = (x_i, y_i)$ be the edge for each i, $1 \leq i \leq t$, with respect to $T_1(v; t)$. The number of vertices of the connected component including vertex v is at least $2t+1$. Let $F \subset V(G)$ be a set of vertices with $|F| = p, 0 \leq p \leq t-1$ and $v \notin F$. After deleting F from $V(G)$, there are at least $(t-p)$ complete e_i's still remain in $T_1(v;t)$. Therefore, the number of vertices of the connected component C_v is at least $2(t-p)+1$. By Theorem 6, G is locally t-diagnosable at vertex v. The proof is complete. □

A Type II structure $T_2(v;k,2)$ at a vertex v is defined as follows:

Definition 6 *Let $G(V,E)$ be a graph, $v \in V$ be a vertex and k be an integer, $k \geq 1$, a Type II structure $T_2(v;k,2)$ of order $k+2$ at vertex v is defined to be the following graph,*

$$T_2(v;k,2) = [V(v;k,2), E(v;k,2)]$$

which is composed of $2k+5$ vertices and of $2k+5$ edges as illustrated in Figure 3.2, where

- $V(v;k,2) = \{v\} \cup \{x_i, y_i \mid 1 \leq i \leq k\} \cup \{z_1, z_2, z_3, z_4\}$,
- $E(v;k,2) = \{(v,x_i),(x_i,y_i)|1 \leq i \leq k\} \cup \{(v,z_1),$ $(v,z_2),(z_1,z_3),(z_2,z_3),(z_3,z_4)\}$.

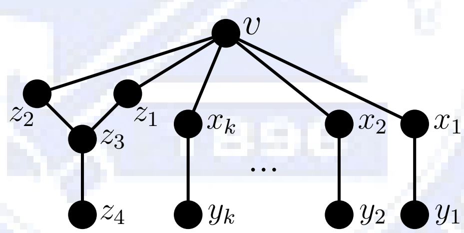


Figure 3.2: A Type II structure $T_2(v;k,2)$ consists of $2k+5$ vertices and $2k+5$ edges.

In the following, we propose another sufficient condition for verifying if it is locally t-diagnosable at a given processor in a system.

Theorem 8 *Let $G(V,E)$ be a graph and $v \in V$ be a vertex. G is locally t-diagnosable at vertex v under the PMC model if G contains a Type II structure $T_2(v;k,2)$ of order $k+2$ at vertex v as a subgraph, where $t = k+2$.*

Proof.

We use Theorem 6 to prove this result. Assume that G contains a subgraph $T_2(v;k,2)$ of order $t = k+2$ at vertex v. The number of vertices of the connected component including vertex v is at least $2k+5 = 2t+1$. Let $F \subset V$ be a set of vertices with $|F| = p$, $0 \le p \le t-1$ and $v \notin F$, the number of vertices of C_v is at least $(2k+5) - 2*1$ after removing one vertex in F, the number of vertices of C_v is at least $(2k+5) - 2*2$ after removing two vertices in F, and so on. Thus, the connected component C_v satisfies $|V(C_v)| \ge (2k+5) - 2p = 2(t-p)+1$. By Theorem 6, G is locally t-diagnosable at vertex v. This proves the theorem. □

In the following, we give some examples.

Example 1 *Let us consider a cycle of length four as shown in Figure 3.3(a). We can find a Type I structure $T_1(v;1)$ of order 1 at vertex v as shown in Figure 3.3(b), hence vertex v is locally 1-diagnosable.*

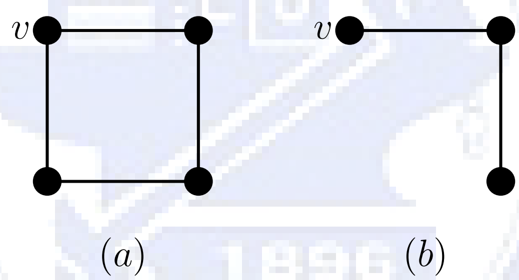
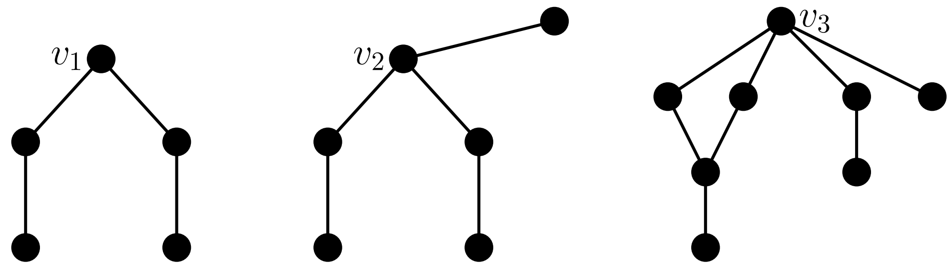


Figure 3.3: A cycle of length four and a Type I structure $T_1(v;1)$ of order 1 at v.

Example 2 *Consider examples as shown in Figure 3.4(a), 3.4(b) and 3.4(c). It is a routine work to check that there is a subgraph $T_1(v_1;2)$, $T_1(v_2;2)$ and $T_2(v_3;1,2)$ at vertex v_1, v_2 and v_3, respectively. Hence it is locally 2-diagnosable, 2-diagnosable and 3-diagnosable at vertex v_1, v_2 and v_3, respectively.*

By Theorem 7, Theorem 8 and Proposition 2, we have the following result.



(a) (b) (c)

Figure 3.4: Some examples of local diagnosability.

Theorem 9 *Let $G(V, E)$ be a graph and $v \in V$ be a vertex with $deg(v) = n$. The local diagnosability of vertex v is n under the PMC model if G contains a subgraph, which is either a Type I structure $T_1(v; n)$ of order n or a Type II structure $T_2(v; n-2, 2)$ of order n, at vertex v.*

3.1 The Local Diagnosability of Hypercube under the PMC Model

In this section, we study the local diagnosability of hypercube under the PMC model. An n-dimensional hypercube can be modeled as a graph Q_n, with the vertex set $V(Q_n)$ and the edge set $E(Q_n)$. There are 2^n vertices in Q_n, and each vertex has degree n. Each vertex v of Q_n can be distinctly labeled by a binary n-bit string, $v = v_{n-1}v_{n-2}...v_1v_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. Let u and v be two adjacent vertices. If the binary labels of u and v differ in ith position, then the edge between them is said to be in ith dimension and the edge (u, v) is called an ith dimensional edge. Let i be a fixed position, we use Q^0_{n-1} to denote the subgraph of Q_n induced by $\{v \in V(Q_n) \mid v_i = 0\}$ and Q^1_{n-1} to denote the subgraph of Q_n induced by $\{v \in V(Q_n) \mid v_i = 1\}$. Consequently, Q_n is decomposed to Q^0_{n-1} and Q^1_{n-1} by dimension i, and Q^0_{n-1} and Q^1_{n-1} are $(n-1)$-dimensional subcube of Q_n induced by the vertices with the ith bit position being 0 and 1 respectively. Q^0_{n-1} and Q^1_{n-1} are isomorphic to Q_{n-1}. For each vertex $v \in V(Q^0_{n-1})$, there is exactly one vertex in Q^1_{n-1}, denoted by $v^{(1)}$, such that $(v, v^{(1)}) \in E(Q_n)$. Conversely, for each vertex $v \in V(Q^1_{n-1})$, there is exactly one vertex in Q^0_{n-1}, denoted by $v^{(0)}$, such that $(v, v^{(0)}) \in E(Q_n)$. Let D_i be the set of all edges with one end in Q^0_{n-1} and the other in Q^1_{n-1}. These edges are called crossing edges in the ith dimension between Q^0_{n-1} and Q^1_{n-1}. We also call D_i the set of all ith dimensional edges.

Based on Theorem 9, we prove that the local diagnosability of each vertex in Q_n is equal to its degree.

Theorem 10 *Let Q_n be an n-dimensional hypercube. The local diagnosability of each vertex in Q_n is n under the PMC model, for $n \geq 3$.*

Proof.

We use Theorem 9 to prove this result, and we shall construct a Type I structure of order n at each vertex, for $n \geq 3$. We prove this by induction on n. Since an n-dimensional hypercube Q_n is vertex-symmetric, we can concentrate on the construction of Type I structure at a given vertex v. For $n = 3$, $deg(v) = 3$ and it is clear that Q_3 contains a Type I structure $T_1(v; 3)$ of order 3 at vertex v (see Figure 3.5). As the inductive hypothesis, we assume that Q_{n-1} contains a Type I structure $T_1(v; n-1)$ of order $n-1$ at each vertex, for some $n \geq 4$. Now we consider Q_n, Q_n can be decomposed into two subcubes Q^0_{n-1} and Q^1_{n-1} by some dimension. Without loss of generality, we may assume that the vertex $v \in Q^0_{n-1}$. By the inductive hypothesis, Q^0_{n-1} contains a Type I structure $T_1(v; n-1)$ of order $n-1$ at vertex v. Consider the vertex $v^{(1)}$ in Q^1_{n-1}. Vertex $v^{(1)}$ has an adjacent neighbor that is in Q^1_{n-1} due to $deg(v^{(1)}) = n$, where $n \geq 3$. Thus, Q_n contains a Type I structure $T_1(v; n)$ of order n at vertex v. By Theorem 9, the local diagnosability of each vertex in Q_n is n, for $n \geq 3$. □

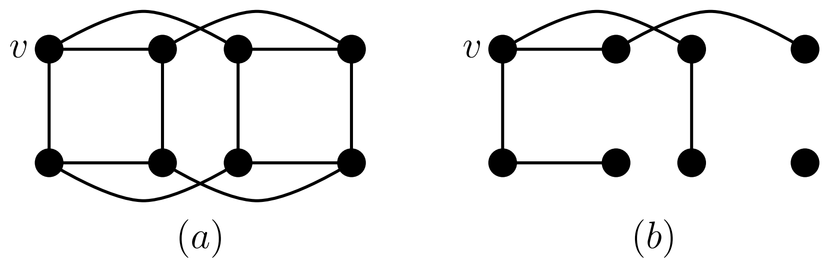


Figure 3.5: A Q_3 and a Type I structure $T_1(v;3)$ of order 3 at vertex v.

3.2 The Local Diagnosability of Star Graph under the PMC Model

In this section, we study the local diagnosability of star graph under the PMC model. An n-dimensional star graph S_n is an $(n-1)$-regular graph consisting of $n!$ vertices and $(n-1)n!/2$ edges. The set of vertices $V(S_n) = \{u_1u_2...u_n | u_i \in \langle n \rangle$ and $u_i \neq u_j$ for $i \neq j\}$, where $\langle n \rangle$ is the set $\{1, 2, ..., n\}$. The adjacency is defined as follows: $u_1u_2...u_i...u_n$ is adjacent to $v_1v_2...v_i...v_n$ through an edge of dimension i, if $v_1 = u_i$, $v_i = u_1$, and $v_j = u_j$ for $j \notin \{1, i\}$, where $2 \leq i \leq n$. Let $\mathbf{u} = u_1u_2...u_i...u_n$ be any vertex in S_n. We use $(\mathbf{u})_i$ to denote the ith coordinate u_i of $\mathbf{u}$ and $S_n^{\{i\}}$ to denote the ith subgraph of S_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Obviously, S_n can be decomposed into n vertex disjoint subgraphs $S_n^{\{i\}}$ for $1 \leq i \leq n$, such that each $S_n^{\{i\}}$ is isomorphic to S_{n-1}. Thus, the star graph can be constructed recursively. By the definition of S_n, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an edge of dimension i, for each $2 \leq i \leq n$. For example, S_4 contains 4! vertices in which two vertices $u_1u_2u_3u_4$ and $u_4u_2u_3u_1$ are neighbors and joined through an edge of dimensional 4. Let $(\mathbf{u})^i$ denote the unique i-neighbor of $\mathbf{u}$. We have $((\mathbf{u})^i)^i = \mathbf{u}$ and $(\mathbf{u})^n \in S_n^{\{(\mathbf{u})_1\}}$. For $1 \leq i, j \leq n$ and $i \neq j$, we use $E^{i,j}$ to denote the set of edges between $S_n^{\{i\}}$ and $S_n^{\{j\}}$. The star graph S_2, S_3 and S_4 are shown in Figure 3.6.

Based on Theorem 9, we prove that the local diagnosability of each vertex in S_n is equal to its degree.

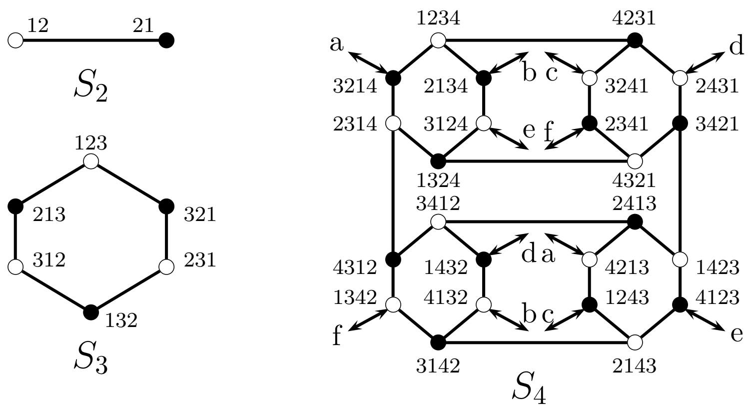


Figure 3.6: The star graph S_2, S_3 and S_4.

Theorem 11 *Let S_n be an n-dimensional star graph. The local diagnosability of each vertex in S_n is $n-1$ under the PMC model, for $n \geq 3$.*

Proof.

We shall construct a Type I structure of order $n-1$ at each vertex, for $n \geq 3$. We prove this by induction on n. Since an n-dimensional star graph S_n is vertex-symmetric, we can concentrate on an arbitrary vertex $\mathbf{v} = v_1v_2...v_n$. For $n = 3$, $deg(\mathbf{v}) = 2$ and it is clear that S_3 contains a Type I structure $T_1(\mathbf{v}; 2)$ of order 2 at vertex $\mathbf{v}$. As the inductive hypothesis, we assume that S_{n-1} contains a Type I structure $T_1(\mathbf{v}; n-2)$ of order $n-2$ at each vertex, for some $n \geq 4$. Now we consider S_n. By the definition of star graphs, S_n can be decomposed into n subgraphs $S_n^{\{v_1\}}$, $S_n^{\{v_2\}}$, ..., and $S_n^{\{v_n\}}$. So $\mathbf{v} \in S_n^{\{v_n\}}$. By the inductive hypothesis, $S_n^{\{v_n\}}$ contains a Type I structure $T_1(\mathbf{v}; n-2)$ of order $n-2$ at vertex $\mathbf{v}$. Consider the vertex $(\mathbf{v})^n$ in $S_n^{\{v_1\}}$. Vertex $(\mathbf{v})^n$ has at least one adjacent neighbor in $S_n^{\{v_1\}}$ due to $deg((\mathbf{v})^n) = n-1$, where $n \geq 3$. Thus, S_n contains a Type I structure $T_1(\mathbf{v}; n-1)$ of order $n-1$ at vertex $\mathbf{v}$. By Theorem 9, the local diagnosability of each vertex in S_n is $n-1$, for $n \geq 3$. □

3.3 Strongly Local-diagnosable Property

In this section, we use hypercube as an example to introduce our concept of the strongly local-diagnosable property. In previous section, we presented two sufficient conditions, Theorem 7 and Theorem 8, for identifying the local diagnosability of a vertex. It seems that identifying the local diagnosability of a vertex is the same as counting its degree. We give an example to show that this is not true in general. As shown in Figure 3.7, we take a vertex v in two-dimensional hypercube Q_2, let $F_1 = \{v, 1\}$ and $F_2 = \{2, 3\}$ with $|F_1| = 2$ and $|F_2| = 2$. It is a simple matter to check that (F_1, F_2) is an indistinguishable pair. Hence $t_l(v) \neq deg(v) = 2$. We then propose the following two concepts.

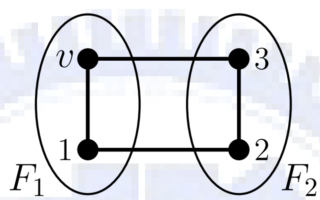


Figure 3.7: An indistinguishable pair (F_1, F_2) in Q_2.

Definition 7 *Let $G(V, E)$ be a graph and $v \in V$ be a vertex. Vertex v has the strongly local-diagnosable property if the local diagnosability of vertex v is equal to its degree.*

Definition 8 *Let $G(V, E)$ be a graph. G has the strongly local-diagnosable property if, every vertex in the graph G has the strongly local- diagnosable property.*

Following Definition 7, Definition 8, Theorem 10 and Theorem 11 imply the following two propositions.

Proposition 4 *Let Q_n be an n-dimensional hypercube, $n \geq 3$. Q_n has the strongly local-diagnosable property under the PMC model.*

Proposition 5 *Let S_n be an n-dimensional star graph, $n \geq 3$. S_n has the strongly local-diagnosable property under the PMC model.*

We now consider a system which is not vertex-symmetric. Let $G(V, E)$ be a graph and $F \subset E(G)$ be a set of edges. Removing the edges in F from G, the degree of each vertex in the resulting graph $G-F$ is called the remaining degree of v, and is denoted by $deg_{G-F}(v)$. We consider a faulty hypercube Q_n with a faulty set $F \subset E(Q_n)$, $n \geq 3$. We shall prove that Q_n has the strongly local-diagnosable property even if it has up to $(n-2)$ faulty edges. The number $n-2$ is optimal in the sense that a faulty hypercube Q_n cannot be guaranteed to have this strong property if there are $n-1$ faulty edges. As shown in Figure 3.8, we take a vertex $v \in V(Q_n)$ and a vertex x which is an adjacent neighbor of v. Let $F = \{(y,x) \in E(Q_n) \mid \text{vertex } y \text{ is directly adjacent to } x\} - \{(v,x)\}$, then $|F| = n-1$ and the remaining degree of v in $Q_n - F$ is n. Let $F_1 = (N_{Q_n-F}(v) - \{x\}) \bigcup \{v\}$ and $F_2 = N_{Q_n-F}(v)$, then $|F_1| = |F_2| = n$ and $v \in F_1 \Delta F_2$. It is clear that there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$. By Lemma 4, (F_1, F_2) is an indistinguishable pair, hence $t_l(v) \neq deg_{Q_n-F}(v) = n$. Therefore, $Q_n - F$ may not have this strong property, if $|F| \geq n-1$.

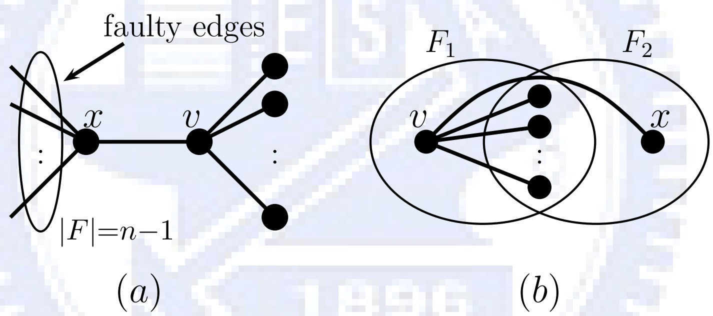


Figure 3.8: An indistinguishable pair (F_1, F_2), where $|F_1| = |F_2| = n$.

Theorem 12 *Let Q_n be an n-dimensional hypercube with $n \geq 3$, and $F \subset E(Q_n)$ be a set of edges, $0 \leq |F| \leq n-2$. Removing all the edges in F from Q_n, the local diagnosability of each vertex is still equal to its remaining degree under the PMC model.*

Proof.

We use Theorem 9 to prove this result, and we shall construct a Type I structure at each vertex. We prove this by induction on n. For $n = 3$, $0 \leq |F| \leq 1$, if $|F| = 0$, it is clear that Q_3 contains a Type I structure $T_1(v;3)$ of order 3 at every vertex. If $|F| = 1$, a three-dimensional hypercube Q_3 with one missing edge is shown in Figure 3.9. It is a routine work to see that every vertex has a Type I structure $T_1(v;k)$ of order k at it, where k is the remaining degree of the vertex. As the inductive hypothesis, we assume that the result is true for Q_{n-1}, $0 \leq |F| \leq (n-1)-2$, for some $n \geq 4$. Now we consider Q_n, $0 \leq |F| \leq n-2$. If $|F| = 0$, refer to the proof of Theorem 10, Q_n contains a Type I structure $T_1(v;n)$ of order n at every vertex. If $1 \leq |F| \leq n-2$, we choose an edge in F, the edge is in some dimension, decomposing Q_n into two subcubes Q^0_{n-1} and Q^1_{n-1} by this dimension, such that the edge is a crossing edge. Consider a vertex $v \in V(Q_n)$. Let $F_0 = F \bigcap E(Q^0_{n-1})$, $0 \leq |F_0| \leq (n-3)$ and $F_1 = F \bigcap E(Q^1_{n-1})$, $0 \leq |F_1| \leq (n-3)$. Without loss of generality, we may assume that the vertex v is in Q^0_{n-1} and $deg_{Q^0_{n-1}-F_0}(v) = k$. By the inductive hypothesis, $Q^0_{n-1} - F_0$ contains a Type I structure $T_1(v;k)$ at v. Consider the crossing edge $(v, v^{(1)})$. If $(v, v^{(1)}) \in F$, $Q_n - F$ contains a Type I structure $T_1(v;k)$ of order k at vertex v. If $(v, v^{(1)}) \notin F$, the remaining degree of v in $Q_n - F$ is $k+1$ and the vertex $v^{(1)}$ has at least an adjacent neighbor in Q^1_{n-1} due to $0 \leq |F_1| \leq (n-1)-2$. Therefore, $Q_n - F$ contains a Type I structure $T_1(v;k+1)$ of order $k+1$ at vertex v. By Theorem 9, removing all the edges in F from Q_n, the local diagnosability of each vertex is still equal to its remaining degree. $\square$

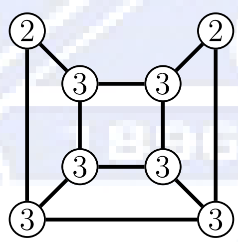


Figure 3.9: Q_3 with one missing edge. The number labeled on each vertex represents its local diagnosability.

We have the following corollary.

Corollary 3 *Let Q_n be an n-dimensional hypercube with $n \geq 3$, and $F \subset E(Q_n)$ be a set of edges, $0 \leq |F| \leq n-2$. Then, $Q_n - F$ has the strongly local-diagnosable property under the PMC model.*

We now consider a faulty star graph S_n with a faulty set $F \subset E(S_n)$, $n \geq 3$. Similarly, we shall prove that S_n has the strongly local-diagnosable property even if it has up to $(n-3)$ faulty edges and the number $(n-3)$ is also optimal.

Theorem 13 *Let S_n be an n-dimensional star graph with $n \geq 3$, and $F \subset E(S_n)$ be a set of edges, $0 \leq |F| \leq n-3$. Removing all the edges in F from S_n, the local diagnosability of each vertex is still equal to its remaining degree under the PMC model.*

Proof.

We prove this result by constructing a Type I structure T_1 at each vertex. We prove this by induction on n. For $n = 3$, $|F| = 0$, it is clear that S_3 contains a Type I structure $T_1(v; 2)$ of order 2 at every vertex. As the inductive hypothesis, we assume that the result is true for S_{n-1}, $0 \leq |F| \leq (n-1)-3$, for some $n \geq 4$. Now we consider S_n, $0 \leq |F| \leq n-3$. If $|F| = 0$, refer to the proof of Theorem 11, S_n contains a Type I structure $T_1(v; n-1)$ of order $n-1$ at every vertex. If $1 \leq |F| \leq n-3$, we choose an edge $e \in F$ in some dimension. The star graph can be decomposed into n subgraphs $S_n^{\{1\}}$, $S_n^{\{2\}}$, ..., and $S_n^{\{n\}}$. By the symmetric property of S_n, we may assume that e is a crossing edge between $S_n^{\{1\}}$ and $S_n^{\{2\}}$. Consider a vertex $\mathbf{v} \in V(S_n)$. Let $F_i = F \bigcap E(S_n^{\{i\}})$, $0 \leq |F_i| \leq (n-4)$ for all $1 \leq i \leq n$. Without loss of generality, we may assume that vertex $\mathbf{v}$ is in $S_n^{\{1\}}$ and $deg_{S_n^{\{1\}}-F_1}(\mathbf{v}) = k$. By the inductive hypothesis, $S_n^{\{1\}} - F_1$ contains a Type I structure $T_1(\mathbf{v}; k)$ at $\mathbf{v}$. Consider the crossing edge $(\mathbf{v}, (\mathbf{v})^n)$. If $(\mathbf{v}, (\mathbf{v})^n) \in F$, $S_n - F$ contains a Type I structure $T_1(\mathbf{v}; k)$ of order k at vertex $\mathbf{v}$. If $(\mathbf{v}, (\mathbf{v})^n) \notin F$, the remaining degree of $\mathbf{v}$ in $S_n - F$ is $k+1$ and the vertex $(\mathbf{v})^n$ has at least one adjacent neighbor in $S_n^{\{(\mathbf{v})_1\}}$ due to $0 \leq |F_{\{(\mathbf{v})_1\}}| \leq (n-1)-3$. Therefore, $S_n - F$ contains a Type I structure $T_1(\mathbf{v}; k+1)$ of order $k+1$ at vertex $\mathbf{v}$. By Theorem 9, removing all the edges in F from S_n, the local diagnosability of each vertex is still equal to its remaining degree. □

With Theorem 13, we have the following corollary.

Corollary 4 *Let S_n be an n-dimensional star graph with $n \geq 3$, and $F \subset E(S_n)$ be a set of edges, $0 \leq |F| \leq n-3$. Then, $S_n - F$ has the strongly local-diagnosable property under* *the PMC model.*

We now give an example to show that an n-regular graph $G(V, E)$ has the strong local diagnosability property, but it may not keep this strong property after removing $n-2$ edges from G. For example, a 3-regular graph is shown in Figure 3.10(a). The degree of each vertex is 3 and there exists a Type I structure $T_1(v; 3)$ of order 3 at each vertex. By Theorem 9, Definition 7 and Definition 8, this graph has the strong local diagnosability property. Let $F = \{(2, 3)\}$ be a set of one single edge, $G - F$ is shown in Figure 3.10 (b). The vertex u does not have the strong local diagnosability property. The reason is as follows. Let $F_1 = \{u, 1, 4\}$ and $F_2 = \{1, 2, 4\}$ with $|F_1| \leq 3$, $|F_2| \leq 3$. Since there is no edge between $V(G) - (F_1 \bigcup F_2)$ and $F_1 \Delta F_2$, by Lemma 4, (F_1, F_2) is an indistinguishable pair. Therefore, the local diagnosability of vertex u is at most 2 which is smaller than its degree.

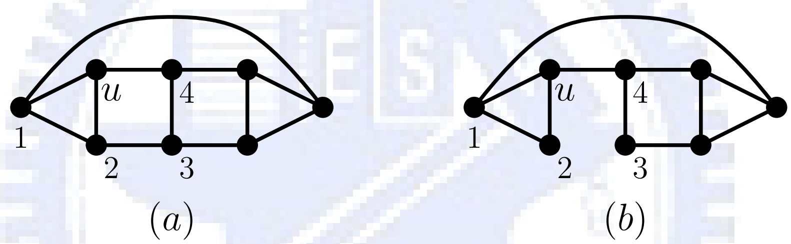


Figure 3.10: A 3-regular graph without the strong local diagnosability property after removing one edge.

3.4 Conditional Fault Local Diagnosability

In previous section, we know that Q_n does not have the strongly local-diagnosable property, if there are $n-1$ faulty edges, all these faulty edges are incident with a single vertex and this vertex is incident with only one fault-free edge. Therefore, we are led to the following question: How many edges can be removed from Q_n such that Q_n keeps the strongly local-diagnosable property under the condition that each vertex of the faulty hypercube Q_n is incident with at least two fault-free edges? Firstly, we give an example to show that a faulty hypercube Q_n with $3(n-2)$ faulty edges may not have the strongly local-diagnosable property, even if each vertex of the faulty hypercube Q_n is incident with at least two fault-free edges. As shown in Figure 3.11(a), we take a cycle of length four in Q_n, $n \geq 3$. Let $\{v, a, b, c\}$ be the four consecutive vertices on this cycle, and $F \subset E(Q_n)$ be a set of edges, $F = F_1 \bigcup F_2 \bigcup F_3$, where F_1 is the set of all edges incident with a except (v, a) and (b, a), F_2 is the set of all edges incident with b except (a, b) and (c, b), and F_3 is the set of all edges incident with c except (v, c) and (b, c), then $|F_1| = |F_2| = |F_3| = n-2$. The remaining degree of vertex v in $Q_n - F$ is n, $deg_{Q_n - F}(v) = n$. As shown in Figure 3.11(b), let $A_1 = (N_{Q_n-F}(v) - \{c\}) \bigcup \{v\}$ and $A_2 = (N_{Q_n-F}(v) - \{a\}) \bigcup \{b\}$, then $|A_1| = |A_2| = n$ and $v \in A_1 \Delta A_2$. It is clear that there is no edge between $V(Q_n) - (A_1 \bigcup A_2)$ and $A_1 \Delta A_2$. By Lemma 4, (A_1, A_2) is an indistinguishable pair, hence $t_l(v) \neq deg_{Q_n-F}(v) = n$. So some vertex of $Q_n - F$ may not have this strong property, if $|F| \geq 3(n-2)$. Then, we shall show that $Q_n - F$ has the strongly local-diagnosable property, if each vertex of $Q_n - F$ is incident with at least two fault-free edges and $|F| \leq 3(n-2)-1$. We need the following results to construct a Type I structure or a Type II structure at a vertex of a faulty hypercube.

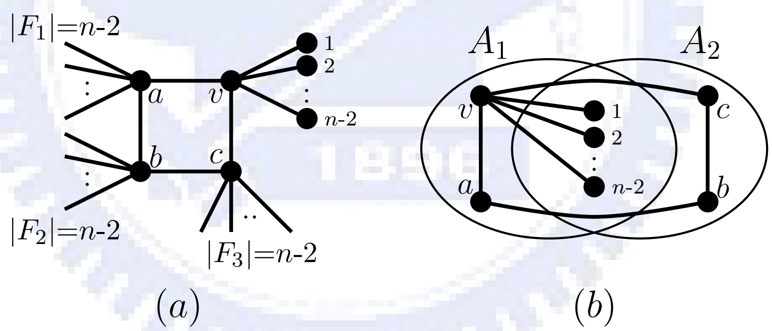


Figure 3.11: An indistinguishable pair (A_1, A_2), where $|A_1| = |A_2| = n$.

Theorem 14 *[52] Let $G(V, E)$ be a bipartite graph with bipartition (X, Y). Then G has a matching that saturates every vertex in X if and only if*

$$|N(S)| \geq |S|, \text{ for all } S \subseteq X.$$

Theorem 15 *[52] Let $G(V, E)$ be a bipartite graph. The maximum size of a matching in G equals the minimum size of a vertex cover of G.*

Lemma 8 *An n-dimensional hypercube Q_n has no cycle of length three and any two vertices have at most two common neighbors.*

For our discussion later, we need some definitions. Let Q_n be an n-dimensional hypercube and $F \subseteq E(Q_n)$ be a set of edges. Removing the edges in F from Q_n, for a vertex v in the resulting graph $Q_n - F$, we define $BG(v) = (L_1(v) \bigcup L_2(v), E)$ to be the bipartite graph under v with bipartition $(L_1(v), L_2(v))$, where $L_1(v) = \{x \in V(Q_n) \mid$ vertex x is adjacent to vertex v in $Q_n - F\}$, $L_2(v) = \{y \in V(Q_n) \mid$ there exists a vertex $x \in L_1(v)$ such that $(x, y) \in E(Q_n)$ in $Q_n - F\} - \{v\}$ and $E(BG(v)) = \{(x, y) \in E(Q_n) \mid$ vertex $x \in L_1(v)$ and vertex $y \in L_2(v)\}$. $L_1(v)$ ($L_2(v)$, respectively) is called the level one (level two, respectively) vertex under v (see Figure 3.12).

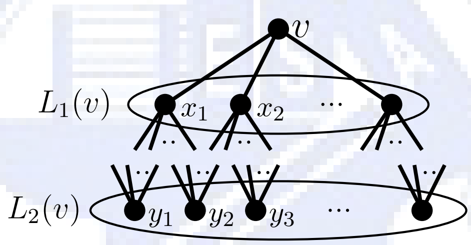


Figure 3.12: The bipartite graph $BG(v)$.

Theorem 16 *Let Q_n be an n-dimensional hypercube with $n \geq 3$, and $F \subset E(Q_n)$ be a set of edges, $0 \leq |F| \leq 3(n-2) - 1$. Assume that each vertex of $Q_n - F$ is incident with at least two fault-free edges. Removing all the edges in F from Q_n, the local diagnosability of each vertex is still equal to its remaining degree under the PMC model.*

Proof.

According to Theorem 9, we can concentrate on the construction of Type I structure or Type II structure at each vertex. Consider a vertex v in $Q_n - F$ with $deg_{Q_n - F}(v) = k$.

As shown in Figure 3.12, let $BG(v) = (L_1(v) \bigcup L_2(v), E)$ be the bipartite graph under v. Then, $|L_1(v)| = k$. Let $M \subset E(BG(v))$ be a maximum matching from $L_1(v)$ to $L_2(v)$. In the following proof, we consider three cases by the size of M: 1) $|M| = k$, 2) $|M| = k-1$ and 3) $|M| \leq k-2$.

Case 1: $|M| = k$

Since $|M| = k$ and $|L_1(v)| = k$, there exists a Type I structure $T_1(v;k)$ of order k at vertex v. By Theorem 9, the local diagnosability of vertex v is equal to k.

Case 2: $|M| = k-1$

We shall show that there is a Type II structure of order k at vertex v. As shown in Figure 3.13, let $L_1(v) = \{x_1, x_2, ..., x_k\}$ and let $ML_2(v) \subset L_2(v)$ be the set of vertices matched under M, $ML_2(v) = \{y \in L_2(v) \mid$ there exists a vertex $x \in L_1(v)$ such that $(x, y) \in M\}$. So $|ML_2(v)| = k-1$. Let $ML_2(v) = \{y_1, y_2, ..., y_{k-1}\}$ and assume vertex x_i is matched with vertex y_i for each i, $1 \leq i \leq k-1$. Then there exists a vertex $x_k \in L_1(v)$, x_k is unmatched by M. Since each vertex of $Q_n - F$ is incident with at least two fault-free edges, there exists a vertex $y_i \in ML_2(v)$, $i \in \{1, 2, ..., k-1\}$, such that $(x_k, y_i) \in E(BG(v))$. Without loss of generality, let $(x_k, y_1) \in E(BG(v))$. If the remaining degree of y_1 is at least three, as shown in Figure 3.14, there exists a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. By Theorem 9, the local diagnosability of vertex v is equal to k and the result follows. If the remaining degree of y_1 is two, the number of faulty edges incident with y_1 is $n-2$. Next, we divide the case into two subcases: 2.1), both x_k and x_1 have remaining degree two and 2.2), one of x_k and x_1 has remaining degree at least three and the other has at least two.

Subcase 2.1: Both x_k and x_1 have remaining degree two.

This is an impossible case. Since the number of faulty edges incident with x_k and x_1 is $2(n-2)$, the total number of faulty edges is at least $3(n-2)$ which is greater than $3(n-2)-1$, a contradiction.

Subcase 2.2: One of x_k and x_1 has remaining degree at least three and the other has at least two.

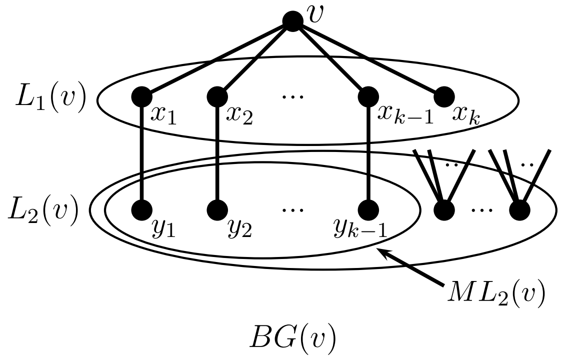


Figure 3.13: Illustration for the case 2 of Theorem 16 and Theorem 18.

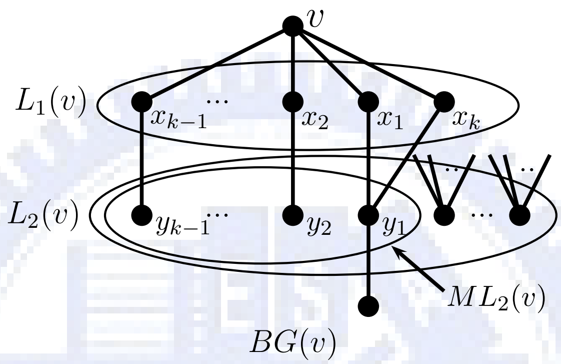


Figure 3.14: A Type II structure $T_2(v; k-2, 2)$ of order k at vertex v.

Without loss of generality, assume x_k has remaining degree at least three and x_1 has remaining degree at least two. Since $deg_{Q_n - F}(x_k) \geq 3$, there exist at least two vertices in $ML_2(v)$ that are the neighbors of vertex x_k. Then, we can find a vertex $y_i \in ML_2(v)$ and $y_i \neq y_1$, $i \in \{2, 3, ..., k-1\}$, such that $(x_k, y_i) \in E(BG(v))$. Without loss of generality, let $(x_k, y_2) \in E(BG(v))$. If the remaining degree of y_2 is at least three, there exists a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. By Theorem 9, the local diagnosability of vertex v is equal to k and the result follows. If the remaining degree of y_2 is two, the number of faulty edges incident with y_2 is $n-2$. We then consider two further cases:

Subcase 2.2.1: Vertex x_1 has remaining degree two.

This is an impossible case. Since the number of faulty edges incident with x_1 is $n-2$, the total number of faulty edges is at least $3(n-2)$ which is greater than $3(n-2)-1$, a contradiction.

Subcase 2.2.2: Vertex x_1 has remaining degree at least three.

Since $deg_{Q_n-F}(x_1) \geq 3$, there exist at least two vertices in $ML_2(v)$ that are the neighbors of vertex x_1. By Lemma 8, any two vertices of Q_n have at most two common neighbors. We can find a vertex $y_i \in ML_2(v)$, $y_i \neq y_1$ and $y_i \neq y_2$, $i \in \{3, 4, ..., k-1\}$, such that $(x_1, y_i) \in E(BG(v))$. Without loss of generality, let $(x_1, y_3) \in E(BG(v))$. If the remaining degree of y_3 is at least three, there exists a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. By Theorem 9, the local diagnosability of vertex v is equal to k and the result follows. If the remaining degree of y_3 is two, then the number of faulty edges incident with y_3 is $n-2$, and the total number of faulty edges is at least $3(n-2)$ which is greater than $3(n-2)-1$, a contradiction.

Case 3: $|M| \leq k-2$

We shall see that this is an impossible case. By Theorem 15, the minimum size of a vertex cover of the bipartite graph $BG(v)$ is no greater than $k-2$. We take a vertex cover with the minimum size, and let $VCL_1(v) \subset L_1(v)$, $VCL_2(v) \subset L_2(v)$ and $VCL_1(v) \bigcup VCL_2(v)$ be the vertex cover as shown in Figure 3.15. $VCL_1(v)$ and $VCL_2(v)$ can cover all the edges of $BG(v)$. Let $NVCL_1(v) = L_1(v) - VCL_1(v)$. We claim that the total number of faulty edges is at least $(n-1)|NVCL_1(v)| - 2|VCL_2(v)|$, and this number is greater than $3(n-2)$ which is a contradiction. With this claim, the case is impossible.

Now we prove the claim. First, for each vertex $x \in NVCL_1(v)$, the edges connecting x except (x, v) must be incident with the vertices in $VCL_2(v)$. For each vertex $y \in VCL_2(v)$, by Lemma 8, at most 2 edges connecting y are incident with the vertices in $NVCL_1(v)$. Then, the total number of faulty edges is at least $(n-1)|NVCL_1(v)| - 2|VCL_2(v)|$. Since $VCL_1(v) \bigcup VCL_2(v)$ is a minimum vertex cover, $|VCL_1(v)| + |VCL_2(v)| \leq k-2$. Since $|L_1(v)| = k$ and each vertex of $Q_n - F$ is incident with at least two fault-free edges, there exists a vertex in $L_1(v) - VCL_1(v)$ such that the vertex has at least one neighbor in $VCL_2(v)$. Thus, $|VCL_2(v)| \geq 1$. Now, we show that the number $(n-1)|NVCL_1(v)|-2|VCL_2(v)|$ is greater than $3(n-2)$. With $|VCL_1(v)|+|VCL_2(v)| \leq k-2$

and $|VCL_2(v)| \geq 1$, we have the following

$$\begin{aligned}
&[(n-1)|NVCL_1(v)| - 2|VCL_2(v)|] - [3(n-2)] \\
&= [(n-1)(k - |VCL_1(v)|) - 2|VCL_2(v)|] - [3(n-2)] \\
&\geq [(n-1)(|VCL_2(v)| + 2) - 2|VCL_2(v)|] - [3(n-2)] \\
&= (|VCL_2(v)| - 1)(n-3) + 1 \\
&> 0, \text{ for all } n \geq 3.
\end{aligned}$$

Thus, our claim holds.

In summary, aside from those impossible cases, we showed that $Q_n - F$ contains either a Type I structure $T_1(v;k)$ or a Type II structure $T_2(v;k-2,2)$ of order k at vertex v. By Theorem 9, removing all the edges in F from Q_n, the local diagnosability of each vertex is still equal to its remaining degree. □

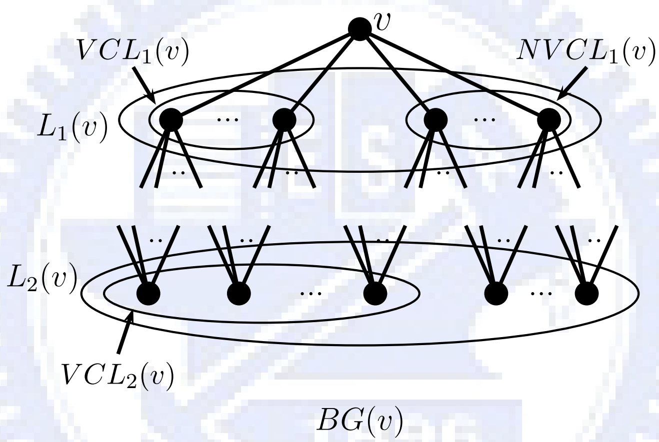


Figure 3.15: Illustration for the case 3 of Theorem 16 and Theorem 18.

By Theorem 16, we have the following corollary.

Corollary 5 *Let Q_n be an n-dimensional hypercube with $n \geq 3$, and $F \subset E(Q_n)$ be a set of edges, $0 \leq |F| \leq 3(n-2) - 1$. $Q_n - F$ has the strong local diagnosability property under the PMC model, provided that each vertex of $Q_n - F$ is incident with at least two fault-free edges.*

Based on the same requirement, we shall show that S_n keeps the strongly local-diagnosable property no matter how many edges are faulty.

Theorem 17 *Let S_n be an n-dimensional star graph with $n \geq 3$, and $F \subset E(S_n)$ be a set of edges. Assume that each vertex of $S_n - F$ is incident with at least two fault-free edges. Removing all the edges in F from S_n, the local diagnosability of each vertex is still equal to its remaining degree under the PMC model.*

Proof.

According to Theorem 9, we can concentrate on the construction of the Type I structure T_1 at each vertex. Consider a vertex v in $S_n - F$ with $deg_{S_n-F}(v) = k$. Let $N_{S_n-F}(v) = \{x_1, x_2, ..., x_k\}$ be the neighborhood of v. Let $L_2(v) = \{y \in V(S_n) \mid$ there exists a vertex $x \in N_{S_n-F}(v)$ such that $(x, y) \in E(S_n)\} - \{v\}$. Since each vertex of $S_n - F$ is incident with at least two fault-free edges and S_n has no cycle of length less than six, the maximum size of a matching from $N_{S_n-F}(v)$ to $L_2(v)$ is equal to k. As a result, there must exist a Type I structure $T_1(v; k)$ of order k at vertex v. By Theorem 9, removing all the edges in F from S_n, the local diagnosability of each vertex is still equal to its remaining degree. □

By Theorem 17, the following corollary holds.

Corollary 6 *Let S_n be an n-dimensional star graph with $n \geq 3$, and $F \subset E(S_n)$ be a set of edges. S_n keeps the strongly local- diagnosable property under the PMC model no matter how many edges are faulty, provided that each vertex of $S_n - F$ is incident with at least two fault-free edges.*

In the end of this section, we consider another condition: each vertex of a faulty hypercube Q_n is incident with at least three fault-free edges. Based on this condition, we prove that Q_n keeps the strong local diagnosability property no matter how many edges are faulty.

Theorem 18 *Let Q_n be an n-dimensional hypercube with $n \geq 3$, and $F \subset E(Q_n)$ be a set of edges. Assume that each vertex of $Q_n - F$ is incident with at least three fault-free edges. Removing all the edges in F from Q_n, the local diagnosability of each vertex is still equal to its remaining degree under the PMC model.*

Proof.

According to Theorem 9, we can concentrate on the construction of Type I structure or Type II structure at each vertex. Consider a vertex v in $Q_n - F$ with $deg_{Q_n-F}(v) = k$. Let $BG(v) = (L_1(v) \bigcup L_2(v), E)$ be the bipartite graph under v. Then, $|L_1(v)| = k$. Let $M \subset E(BG(v))$ be a maximum matching from $L_1(v)$ to $L_2(v)$. In the following proof, we consider three cases by the size of M: 1) $|M| = k$, 2) $|M| = k - 1$ and 3) $|M| \leq k - 2$.

Case 1: $|M| = k$

Since $|M| = k$ and $|L_1(v)| = k$, there exists a Type I structure $T_1(v; k)$ of order k at vertex v. By Theorem 9, the local diagnosability of vertex v is equal to k.

Case 2: $|M| = k - 1$

We will show that there is a Type II structure of order k at vertex v. As shown in Figure 3.13, let $L_1(v) = \{x_1, x_2, ..., x_k\}$ and let $ML_2(v) \subset L_2(v)$ be the set of vertices matched under M, $ML_2(v) = \{y \in L_2(v) \mid$ there exists a vertex $x \in L_1(v)$ such that $(x, y) \in M\}$. So $|ML_2(v)| = k - 1$. Let $ML_2(v) = \{y_1, y_2, ..., y_{k-1}\}$ and assume vertex x_i is matched with vertex y_i for each i, $1 \leq i \leq k - 1$. Then there exists a vertex $x_k \in L_1(v)$, x_k is unmatched by M. Since each vertex of $Q_n - F$ is incident with at least three fault-free edges, there exists a vertex $y_i \in ML_2(v)$, $i \in \{1, 2, ..., k - 1\}$, such that $(x_k, y_i) \in E(BG(v))$. Without loss of generality, let $(x_k, y_1) \in E(BG(v))$. Since the remaining degree of y_1 is at least three, as shown in Figure 3.14, there exists a Type II structure $T_2(v; k - 2, 2)$ of order k at vertex v. By Theorem 9, the local diagnosability of vertex v is equal to k and the result follows.

Case 3: $|M| \leq k - 2$

We will see that this is an impossible case. By Theorem 15, the minimum size of a vertex cover of the bipartite graph $BG(v)$ is no greater than $k-2$. However, we claim that any $k-2$ vertices of $BG(v)$ can not cover all the edges of $BG(v)$. With this claim, the case is impossible.

Now we prove this claim. Suppose not, we take a vertex cover with the minimum size, and let $VCL_1(v) \subset L_1(v)$, $VCL_2(v) \subset L_2(v)$ and $VCL_1(v) \bigcup VCL_2(v)$ be the vertex cover as shown in Figure 3.15. $VCL_1(v)$ and $VCL_2(v)$ can cover all the edges of $BG(v)$. Since $|VCL_1(v)| + |VCL_2(v)| \leq k-2$, we rewrite this inequality into the following equivalent form: $2(k - |VCL_1(v)|) \geq 2(|VCL_2(v)| + 2)$. Let $NVCL_1(v) = L_1(v) - VCL_1(v)$. Since each vertex of $Q_n - F$ is incident with at least three fault-free edges, for each vertex $x \in NVCL_1(v)$, aside from the edge (x, v), at least 2 edges connecting x must be incident with the vertices in $VCL_2(v)$. So the total number of edges incident with the vertices in $VCL_2(v)$ is at least $2|NVCL_1(v)|$. For each vertex $y \in VCL_2(v)$, by Lemma 8, at most 2 edges connecting y are incident with the vertices in $NVCL_1(v)$. So the total number of edges incident with the vertices in $NVCL_1(v)$ is at most $2|VCL_2(v)|$. Compare the lower bound $2|NVCL_1(v)|$ and the upper bound $2|VCL_2(v)|$. We have the following inequality

$$\begin{aligned} 2|NVCL_1(v)| &= 2(k - |VCL_1(v)|) \\ &\geq 2(|VCL_2(v)| + 2) > 2|VCL_2(v)|. \end{aligned}$$

The lower bound $2|NVCL_1(v)|$ is greater than the upper bound $2|VCL_2(v)|$. It means that some edges are not covered by $VCL_1(v)$ or $VCL_2(v)$ in $BG(v)$. Thus, our claim follows.

In Case 1, $Q_n - F$ contains a Type I structure $T_1(v; k)$ of order k at vertex v. In Case 2, $Q_n - F$ contains a Type II structure $T_2(v; k-2, 2)$ of order k at vertex v. We also proved that Case 3 is impossible. By Theorem 9, removing all the edges in F from Q_n, the local diagnosability of each vertex is still equal to its remaining degree. □

By Theorem 18, the following corollary holds.

Corollary 7 *Let Q_n be an n-dimensional hypercube with $n \geq 3$, and $F \subset E(Q_n)$ be a set of edges. Q_n keeps the strong local diagnosability property under the PMC model no* *matter how many edges are faulty, provided that each vertex of $Q_n - F$ is incident with at least three fault-free edges.*

3.5 A Diagnosis Algorithm

We now introduce a diagnosis algorithm to determine if a vertex is faulty or not for a given syndrome under the PMC model. Given a Type I structure $T_1(v;n)$ of order n at vertex v, there are communication links between v and x_i, x_i and y_i, for all $1 \le i \le n$, x_i and y_i can be the tester of the PMC model. After the test, each tester has a testing result denoted by 0 (1, respectively) representing the approval (disapproval, respectively). We define $r_i = (r^1, r^2)$, where r^1 is the result of x_i testing v and r^2 is the result of y_i testing x_i. Then, r_i can be in one of the four different states which are $r(0) = (0,0)$, $r(1) = (0,1)$, $r(2) = (1,0)$ and $r(3) = (1,1)$ (as illustrated in Figure 3.16). Let $R(k)$ be the collection of all $r(k)$, for all $0 \le k \le 3$. Obviously, $\sum_{k=0}^{3} |R(k)| = n$.

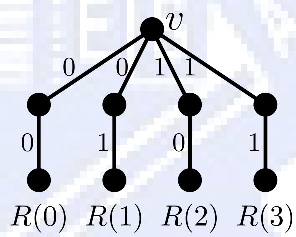


Figure 3.16: four different output states.

Suppose that there is a Type I structure $T_1(v;n)$ of order n at vertex v, where v has degree n. By Theorem 9, the local diagnosability of v is limited to n. Therefore, we may not be able to identify all the faulty vertices, if the number of faulty vertices in $T_1(v;n)$ is $n+1$ or more. Hence, we assume that the number of faulty vertices is at most n. Under this assumption, we propose the following algorithm to determine whether vertex v is faulty or not.

Theorem 19 *Let v be a vertex with degree n in $G(V,E)$. Suppose that there is a Type I structure $T_1(v;n)$ of order n at vertex v and the number of faulty vertices is at most n. The following two conditions are satisfied:*

1. *the vertex v is fault-free if $|R(0)| \geq |R(2)|$, and*

2. *the vertex v is faulty if $|R(0)| < |R(2)|$.*

Proof.

Let $l_i = (x_i, y_i)$ be an ordered double, $1 \leq i \leq n$, with respect to $T_1(v; n)$. First, we prove the condition 1 by contradiction. Assume that v is faulty, then the counting of all the other faulty vertices is as follows:

> For those l_i with result $r(0)$, there are at least two faulty vertices which are x_i, y_i.
>
> For those l_i with result $r(1)$, there is at least one faulty vertex which is x_i.
>
> For those l_i with result $r(2)$, the number of faulty vertices is uncertain.
>
> For those l_i with result $r(3)$, there is at least one faulty vertex which is either x_i or y_i.

Thus, the number of faulty vertices is at least $1+2|R(0)|+|R(1)|+|R(3)| = \sum_{k=0}^{3} |R(k)| + (1+|R(0)|-|R(2)|)$. By the assumption that $|R(0)| \geq |R(2)|$, the number of faulty vertices is strictly more than n. This contradicts to the assumption that the number of faulty vertices is at most n. Therefore, the vertex v is fault-free.

Next, we prove the condition 2 by contradiction again. Assume that v is fault-free, then the counting of all the other faulty vertices is as follows:

> For those l_i with result $r(0)$, the number of faulty vertices is uncertain.
>
> For those l_i with result $r(1)$, there is at least one faulty vertex which is either x_i or y_i.

For those l_i with result $r(2)$, there are at least two faulty vertices which are x_i and y_i.

For those l_i with result $r(3)$, there is at least one faulty vertex which is x_i.

Thus, the number of faulty vertices is at least $|R(1)| + 2|R(2)| + |R(3)| = \sum_{k=0}^{3} |R(k)| + (|R(2)| - |R(0)|)$. By the assumption that $|R(0)| < |R(2)|$, the number of faulty vertices is larger than n. This contradicts to the assumption that the number of faulty vertices is at most n. Therefore, the vertex v is faulty.

This completes the proof. $\square$

We now measure the time complexity of our algorithm to diagnose all the faulty vertices in a system. For many well-know general systems with N vertices, the degree of each vertex is in the order of $\log N$. For example, the n-dimensional Hypercube Q_n has $N = 2^n$ vertices and the degree of each vertex is n, $n = \log N$; the n-dimensional star graph S_n has $N = n!$ vertices and the degree of each vertex is $n - 1 = O(n) = O(\log N / \log n) = O(\log N / \log \log N)$. We assume that a testing result of each tester is directly stored in a syndrome table. Given a Type I structure $T_1(v; n)$ of order n at vertex v, assume the time for looking up the testing result of a tester in the syndrome table is constant c. Then, the time needed for determining the faulty or fault-free status of a vertex$v$ is $2c \log N = O(\log N)$. Consequently, the total time to diagnose all the faulty vertices is bounded by $O(N \log N)$.

Chapter 4

Conditionally Diagnosable Systems

In this section, we study the conditional diagnosis problem under the comparison model. In classical measures of diagnosability for multiprocessor systems, if all the neighbors of some processor v are faulty simultaneously, it is not possible to determine whether processor v is fault-free or faulty. For example, consider an n-dimensional hypercube Q_n and two faulty sets $F_1, F_2 \subset V(Q_n)$ as shown in Figure 4.1. As we observe the all neighbors of vertex v are included in F_1 and F_2. Let $F_1 = N(v) \bigcup \{v\}$ and $F_2 = N(v)$, then $|F_1| = n+1$ and $|F_2| = n$. By Theorem 3, F_1 and F_2 are indistinguishable under the comparison model. So the diagnosability of a system is limited by its minimum vertex degree.

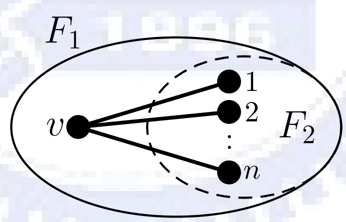


Figure 4.1: An indistinguishable pair (F_1, F_2).

In an n-dimensional hypercube Q_n, Q_n has $\binom{2^n}{n}$ vertex subsets of size n, among which there are only 2^n vertex subsets which contains all the neighbors of some vertex. Since the ratio $2^n/\binom{2^n}{n}$ is very small for large n, the probability of a faulty set containing all the neighbors of any vertex is very low. For this reason, Lai et al. introduced a new restricted diagnosability of multiprocessor systems called conditional diagnosability in [40]. They consider the situation that any faulty set cannot contain all the neighbors of any vertex in a system. In the following, we need some terms to define the conditional diagnosability formally. A faulty set $F \subset V$ is called a *conditional faulty set* if $N(v) \nsubseteq F$ for every vertex $v \in V$. A system $G(V, E)$ is said to be *conditionally t-diagnosable* if F_1 and F_2 are distinguishable, for each pair of conditional faulty sets $F_1, F_2 \subset V$, and $F_1 \neq F_2$, with $|F_1| \leq t$ and $|F_2| \leq t$. The maximum value of t such that G is conditionally t-diagnosable is called the *conditional diagnosability* of G, written as $t_c(G)$. It is trivial that $t_c(G) \geq t(G)$.

Lemma 9 *Let G be a multiprocessor system. Then, $t_c(G) \geq t(G)$.*

Let $G(V, E)$ be a graph and $F_1, F_2 \subset V$, $F_1 \neq F_2$. We say (F_1, F_2) is a distinguishable conditional-pair (an indistinguishable conditional-pair, respectively) if F_1 and F_2 are conditional faulty sets and are distinguishable (indistinguishable, respectively). Before discussing the conditional diagnosability, we have some observations as follows: Let $F_1, F_2 \subset V$ be an indistinguishable conditional-pair. Let $X = V - (F_1 \bigcup F_2)$. Since F_1 and F_2 are an indistinguishable conditional-pair, none of the three conditions of Theorem 3 holds and every vertex has at least one fault-free neighbor. Let vertex $u \in X$. If $N(u) \bigcap X \neq \emptyset$, then $N(u) \bigcap (F_1 \Delta F_2) = \emptyset$ (see Figure 4.2 (a)); otherwise $|N(u) \bigcap (F_1 - F_2)| = 1$ and $|N(u) \bigcap (F_2 - F_1)| = 1$(see Figure 4.2 (b)). Let vertex $v \in F_1 \Delta F_2$. If $N(v) \bigcap X = \emptyset$, then $|N(v) \bigcap (F_1 - F_2)| \geq 1$ and $|N(v) \bigcap (F_2 - F_1)| \geq 1$(see Figure 4.2 (c)). We state this fact in the following lemma.

Lemma 10 *Let $G(V, E)$ be a graph and $F_1, F_2 \subset V$ be an indistinguishable conditional-pair under the comparison model. Let $X = V - (F_1 \bigcup F_2)$. The following three conditions holds:*

1. $|N(u) \bigcap (F_1 \Delta F_2)| = 0$ *for* $u \in X$ *and* $N(u) \bigcap X \neq \emptyset$,
2. $|N(u) \bigcap (F_1 - F_2)| = 1$ *and* $|N(u) \bigcap (F_2 - F_1)| = 1$ *for* $u \in X$ *and* $N(u) \bigcap X = \emptyset$, *and*

3. $|N(v) \bigcap (F_1 - F_2)| \geq 1$ *and* $|N(v) \bigcap (F_2 - F_1)| \geq 1$ *for* $v \in F_1 \Delta F_2$ *and* $N(v) \bigcap X = \emptyset$.

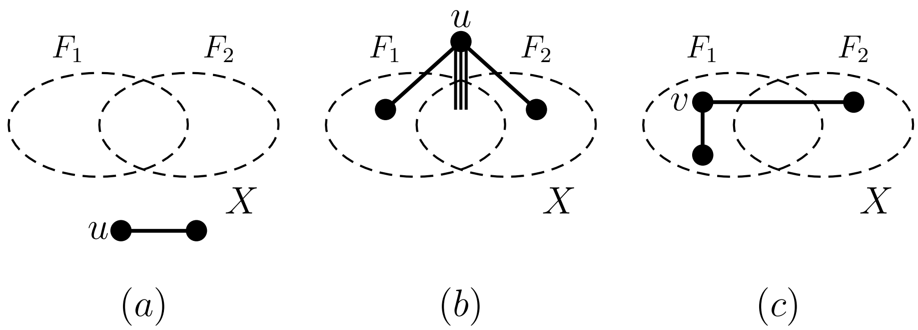


Figure 4.2: An indistinguishable conditional-pair (F_1, F_2).

In the following sections, we will first evaluate the conditional diagnosability for hypercube networks under the comparison model. Then, we extend the result to BC network.

4.1 Conditional Diagnosability of Hypercube under the Comparison Model

In this section, we study the conditional diagnosability of hypercube under the comparison model. First, we give an example to show that the conditional diagnosability of the hypercube Q_n is no greater than $3(n-2)+2$, $n \geq 5$. As shown in Figure 4.3, we take a cycle of length four in Q_n. Let $\{v_1, v_2, v_3, v_4\}$ be the four consecutive vertices on this cycle, and let $F_1 = N(\{v_1, v_3, v_4\}) \bigcup \{v_1\}$ and $F_2 = N(\{v_1, v_3, v_4\}) \bigcup \{v_3\}$, then $|F_1| = |F_2| = 3(n-2)+2$. It is straightforward to check that F_1 and F_2 are two conditional faulty sets, and F_1 and F_2 are indistinguishable by Theorem 3. Note that the hypercube Q_n has no cycle of length three and any two vertices have at most two common neighbors. As we can see, $|F_1 - F_2| = |F_2 - F_1| = 1$ and $|F_1 \bigcap F_2| = 3(n-2)+1$. Therefore, Q_n is not conditionally $(3(n-2)+2)$-diagnosable and $t_c(Q_n) \leq 3(n-2)+1$, $n \geq 3$. Then, we shall show that Q_n is conditionally t-diagnosable, where $t = 3(n-2)+1$.

Lemma 11 $t_c(Q_n) \leq 3(n-2)+1$ *under the comparison model, for* $n \geq 3$.

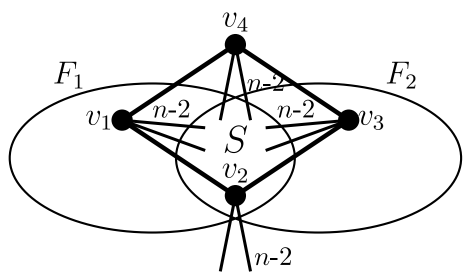


Figure 4.3: An indistinguishable conditional-pair (F_1, F_2), where $|F_1| = |F_2| = 3(n-2)+2$.

Let F be a set of vertices $F \subset V(Q_n)$ and C be a connected component of $Q_n - F$. We need some results on the cardinalities of F and $V(C)$ under some restricted conditions. The results are listed in Lemma 12 and 16. In Lemma 12, Lai et al. proved that deleting at most $2(n-1)-1$ vertices from Q_n, the incomplete hypercube Q_n has one connected component containing at least $2^n - |F| - 1$ vertices. We expand this result further. In Lemma 16, we show that deleting at most $3n-6$ vertices from Q_n, the incomplete hypercube Q_n has one connected component containing at least $2^n - |F| - 2$ vertices.

Lemma 12 *[40] Let Q_n be an n-dimensional hypercube, $n \geq 3$, and let F be a set of vertices $F \subset V(Q_n)$ with $n \leq |F| \leq 2(n-1)-1$. Suppose that $Q_n - F$ is disconnected. Then $Q_n - F$ has exactly two components, one is trivial and the other is nontrivial. The nontrivial component of $Q_n - F$ contains $2^n - |F| - 1$ vertices.*

In order to prove Lemma 16, we need some preliminary results as follows.

Lemma 13 *[46] Let Q_n be an n-dimensional hypercube. The connectivity of Q_n is $\kappa(Q_n) = n$.*

Lemma 14 *For any three vertices x, y, z in Q_4, $|N(\{x, y, z\})| \geq 7$.*

Proof.

A four-dimensional hypercube Q_4 can be divided into two Q_3's, denoted by Q_3^L and Q_3^R. Any two vertices in the Q_n have at most two common neighbors. If these three vertices x, y, z all fall in Q_3^L, then x, y, z have at least four neighboring vertices, all in Q_3^L. Besides, x, y, z have three more neighboring vertices in Q_3^R. Therefore, $|N(\{x,y,z\})| \geq 4+3=7$. Suppose now x, y fall in Q_3^L, z falls in Q_3^R. Vertex x and y have at least four neighboring vertices, all in Q_3^L. Vertex z will bring in at least three neighboring vertices in Q_3^R. Therefore, $|N(\{x,y,z\})| \geq 4+3=7$. □

We are going to prove Lemma 16 by induction on n, and we need a base case to start with. As we observed, for $n=4$, we found a counter example that the result of Lemma 16 does not hold. So we have to start with $n=5$.

Lemma 15 *Let Q_5 be a five-dimensional hypercube, and let F be a set of vertices $F \subset V(Q_5)$ with $|F| \leq 3n-6=9$. Then $Q_5 - F$ has a connected component containing at least $2^n - |F| - 2 = 30 - |F|$ vertices.*

Proof.

A five-dimensional hypercube Q_5 can be divided into two Q_4's, denoted by Q_4^L and Q_4^R. Let $F_L = F \bigcap V(Q_4^L), 0 \leq |F_L| \leq 9$ and $F_R = F \bigcap V(Q_4^R), 0 \leq |F_R| \leq 9$. Then $|F| = |F_L| + |F_R|$. Without loss of generality, we may assume that $|F_L| \geq |F_R|$. In the following proof, we consider three cases by the size of F_R: 1) $0 \leq |F_R| \leq 2$, 2) $|F_R| = 3$, and 3) $|F_R| = 4$.

Case 1: $0 \leq |F_R| \leq 2$**.**

Since $\kappa(Q_4) = 4$, $Q_4^R - F_R$ is connected and $|V(Q_4^R - F_R)| = 2^4 - |F_R|$. Let $F_R^{(L)} \subset V(Q_4^L)$ be the set of vertices which has neighboring vertices in F_R. For each vertex $v \in Q_4^L - F_L - F_R^{(L)}$, there is exactly one vertex $v^{(R)}$ in $Q_4^R - F_R$, such that $(v, v^{(R)}) \in E(Q_5)$. Besides, $|V(Q_4^L - F_L - F_R^{(L)})| \geq 2^4 - |F_L| - |F_R|$. Hence $Q_5 - F$ has a connected component that contains at least $[2^4 - |F_R|] + [2^4 - |F_L| - |F_R|] = 32 - |F| - |F_R| \geq 30 - |F|$ vertices.

Case 2: $|F_R| = 3$**.**

Since $\kappa(Q_4) = 4$, $Q_4^R - F_R$ is connected and $|V(Q_4^R - F_R)| = 2^4 - |F_R|$. Let $F_R = \{x, y, z\}$ and $F_R^{(L)} = \{x^{(L)}, y^{(L)}, z^{(L)}\} \subset V(Q_4^L)$, where $(x, x^{(L)}), (y, y^{(L)}), (z, z^{(L)}) \in E(Q_5)$. For each vertex $v \in Q_4^L - F_L - F_R^{(L)}$, there is exactly one vertex $v^{(R)}$ in $Q_4^R - F_R$, such that $(v, v^{(R)}) \in E(Q_5)$. If at least one of the three vertices $x^{(L)}, y^{(L)}, z^{(L)}$ belongs to F_L, then $|V(Q_4^L - F_L - F_R^{(L)})| \geq 2^4 - |F_L| - 2$. Hence $Q_5 - F$ has a connected component that contains at least $[2^4 - |F_R|] + [2^4 - |F_L| - 2] = 30 - |F|$ vertices; otherwise, $|V(Q_4^L - F_L - F_R^{(L)})| \geq 2^4 - |F_L| - 3$. Since $|F_L| \leq 6$, by Lemma 14, $x^{(L)}, y^{(L)}, z^{(L)}$ have at least one neighboring vertex in $Q_4^L - F_L - F_R^{(L)}$. Hence $Q_5 - F$ has a connected component that contains at least $[2^4 - |F_R|] + [2^4 - |F_L| - 3] + 1 = 30 - |F|$ vertices.

Case 3: $|F_R| = 4$.

Since $|F_R| = 4$ and $|F_L| \leq 5$, by Lemma 12, $Q_4^L - F_L$ ($Q_4^R - F_R$, respectively) has a connected component C_L (C_R, respectively) that contains at least $2^4 - |F_L| - 1$ ($2^4 - |F_R| - 1$, respectively) vertices. Since $|V(C_L)| \geq |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u, v) \in E(Q_5)$. Hence $Q_5 - F$ has a connected component that contains at least $[2^4 - |F_L| - 1] + [2^4 - |F_R| - 1] = 30 - |F|$ vertices.

Consequently, the lemma holds. □

We now prove Lemma 16.

Lemma 16 *Let Q_n be an n-dimensional hypercube, $n \geq 5$, and let F be a set of vertices $F \subset V(Q_n)$ with $|F| \leq 3n - 6$. Then $Q_n - F$ has a connected component containing at least $2^n - |F| - 2$ vertices.*

Proof.

We prove the lemma by induction on n. By Lemma 15, the lemma holds for $n = 5$. As the inductive hypothesis, we assume that the result is true for Q_{n-1}, for $|F| \leq 3(n-1) - 6$, and for some $n \geq 6$. Now we consider Q_n, $|F| \leq 3n - 6$. An n-dimensional hypercube Q_n can be divided into two Q_{n-1}'s, denoted by Q_{n-1}^L and Q_{n-1}^R. Let $F_L = F \bigcap V(Q_{n-1}^L), 0 \leq |F_L| \leq 3n - 6$ and $F_R = F \bigcap V(Q_{n-1}^R), 0 \leq |F_R| \leq 3n - 6$. Then $|F| = |F_L| + |F_R|$.

Without loss of generality, we may assume that $|F_L| \geq |F_R|$. In the following proof, we consider two cases by the size of F_R: 1) $0 \leq |F_R| \leq 2$ and 2) $|F_R| \geq 3$.

Case 1: $0 \leq |F_R| \leq 2$**.**

Since $0 \leq |F_R| \leq 2$, $Q^R_{n-1} - F_R$ is connected and $|V(Q^R_{n-1} - F_R)| = 2^{n-1} - |F_R|$. Let $F^{(L)}_R \subset V(Q^L_{n-1})$ be the set of vertices which has neighboring vertices in F_R. For each vertex $v \in Q^L_{n-1} - F_L - F^{(L)}_R$, there is exactly one vertex $v^{(R)}$ in $Q^R_{n-1} - F_R$, such that $(v, v^{(R)}) \in E(Q_n)$. Besides, $|V(Q^L_{n-1} - F_L - F^{(L)}_R)| \geq 2^{n-1} - |F_L| - |F_R|$. Hence $Q_n - F$ has a connected component that contains at least $[2^{n-1} - |F_R|] + [2^{n-1} - |F_L| - |F_R|] = 2^n - |F| - |F_R| \geq 2^n - |F| - 2$ vertices.

Case 2: $|F_R| \geq 3$**.**

Since $|F_R| \geq 3$, $3 \leq |F_L| \leq 3(n-1) - 6$ and $3 \leq |F_R| \leq 3(n-1) - 6$. By the inductive hypothesis, $Q^L_{n-1} - F_L$ ($Q^R_{n-1} - F_R$, respectively) has a connected component C_L (C_R, respectively) that contains at least $2^{n-1} - |F_L| - 2$ ($2^{n-1} - |F_R| - 2$, respectively) vertices. Next, we divide the case into three subcases: 2.1) $|V(C_L)| = 2^{n-1} - |F_L| - 2$ and $Q^R_{n-1} - F_R$ is disconnected, 2.2) $|V(C_L)| = 2^{n-1} - |F_L| - 2$ and $Q^R_{n-1} - F_R$ is connected, and 2.3) $|V(C_L)| \geq 2^{n-1} - |F_L| - 1$ and $|V(C_R)| \geq 2^{n-1} - |F_R| - 1$.

Case 2.1: $|V(C_L)| = 2^{n-1} - |F_L| - 2$ **and** $Q^R_{n-1} - F_R$ **is disconnected.**

This is an impossible case. Since $\kappa(Q_{n-1}) = n - 1$, $|F_R| \geq n - 1$. By Lemma 12, $|F_L| \geq 2((n-1)-1)$. Then the total number of faulty vertices is at least $(n-1)+2((n-1)-1) = 3n-5$ which is greater than $3n-6$, a contradiction.

Case 2.2: $|V(C_L)| = 2^{n-1} - |F_L| - 2$ **and** $Q^R_{n-1} - F_R$ **is connected.**

Since $Q^R_{n-1} - F_R$ is connected, $|V(Q^R_{n-1} - F_R)| = 2^{n-1} - |F_R|$. Since $|V(C_L)| \geq |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u, v) \in E(Q_n)$. Hence $Q_n - F$ has a connected component that contains at least $[2^{n-1} - |F_R|] + [2^{n-1} - |F_L| - 2] = 2^n - |F| - 2$ vertices.

Case 2.3: $|V(C_L)| \geq 2^{n-1} - |F_L| - 1$ **and** $|V(C_R)| \geq 2^{n-1} - |F_R| - 1$**.**

Since $|V(C_L)| \geq |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u, v) \in E(Q_n)$. Hence $Q_n - F$ has a connected component that contains at least $[2^{n-1} - |F_L| - 1] + [2^{n-1} - |F_R| - 1] = 2^n - |F| - 2$ vertices.

This completes the proof of the lemma. □

By Lemma 16, we have the following corollary.

Corollary 8 *Let Q_n be an n-dimensional hypercube, $n \geq 5$, and let F be a set of vertices $F \subset V(Q_n)$ with $|F| \leq 3n - 6$. Then $Q_n - F$ satisfies one of the following conditions:*

1. *$Q_n - F$ is connected.*
2. *$Q_n - F$ has two components, one of which is K_1, and the other one has $2^n - |F| - 1$ vertices.*
3. *$Q_n - F$ has two components, one of which is K_2, and the other one has $2^n - |F| - 2$ vertices.*
4. *$Q_n - F$ has three components, two of which are K_1, and the third one has $2^n - |F| - 2$ vertices*

Let $G(V, E)$ be a graph. A subset M of $E(G)$ is called a matching in G if its elements are links and no two are adjacent in G; the two ends of an edge in M are said to be matched under M. A vertex cover of G is a subset $\mathcal{K}$ of $V(G)$ such that every edge of G has at least one end in $\mathcal{K}$. A subset I of $V(G)$ is called an independent set of G if no two vertices of I are adjacent in G. As the description for Theorem 15, the maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover. To prove the conditional diagnosability of the hypercube, we need the following classical result.

Proposition 6 *[52] Let $G(V, E)$ be a bipartite graph. The set $I \subset V(G)$ is a maximum independent set of G if and only if $V - I$ is a minimum vertex cover of G.*

The hypercube can be described as follows: Let Q_n denote an n-dimensional hypercube. Q_1 is a complete graph with two vertices labeled with 0 and 1, respectively. For

$n \geq 2$, each Q_n consists of two Q_{n-1}'s, denoted by Q_{n-1}^0 and Q_{n-1}^1, with a perfect matching M between them. That is, M is a set of edges connecting the vertices of Q_{n-1}^0 and the vertices of Q_{n-1}^1 in a one-to-one manner. It is easy to see that there are 2^{n-1} edges between Q_{n-1}^0 and Q_{n-1}^1. The hypercube is a bipartite graph with 2^n vertices. Hence, we have the following Lemma.

Lemma 17 *Let Q_n be an n-dimensional hypercube. In hypercube Q_n, the maximum size of a matching, the minimum size of a vertex cover and the maximum size of an independent set are all 2^{n-1}.*

We are now ready to show that the conditional diagnosability of Q_n is $3(n-2)+1$ for $n \geq 5$. Let $F_1, F_2 \subset V(Q_n)$ be two conditional faulty sets with $F_1 \leq 3(n-2)+1$ and $F_2 \leq 3(n-2)+1$, $n \geq 5$. We shall show our result by proving that (F_1, F_2) is a distinguishable conditional-pair under the comparison diagnosis model.

Lemma 18 *Let Q_n be an n-dimensional hypercube with $n \geq 5$. For any two conditional faulty sets $F_1, F_2 \subset V(Q_n)$, and $F_1 \neq F_2$, with $F_1 \leq 3(n-2)+1$ and $F_2 \leq 3(n-2)+1$. Then (F_1, F_2) is a distinguishable conditional-pair under the comparison diagnosis model.*

Proof.

We use Theorem 4 to prove this result. Let $S = F_1 \bigcap F_2$, then $0 \leq |S| \leq 3(n-2)$. We will show that, deleting S from Q_n, the subgraph $C_{F_1 \Delta F_2, S}$ containing $F_1 \Delta F_2$ has "many" vertices having degree three or more. More precisely, we are going to prove that, in the subgraph $C_{F_1 \Delta F_2, S}$ the number of vertices having degree three or more is at least $2[3(n-2)+1-|S|]+1 = 6n-2|S|-9$. In the following proof, we consider three cases by the size of S: 1) $0 \leq |S| \leq n-1$, 2) $|S| = n$, and 3) $n+1 \leq |S| \leq 3(n-2)$.

Case 1: $0 \leq |S| \leq n-1$**.**

Since the connectivity of Q_n is n, $Q_n - S$ is connected, the subgraph $C_{F_1 \Delta F_2, S}$ is the only component in $Q_n - S$. Since the hypercube Q_n has no cycle of length three and any two vertices have at most two common neighbors, it is straightforward, though tedious, to check that the number of vertices which has degree two or one is at most two in $C_{F_1\Delta F_2,S}$. Hence, the number of vertices having degree three or more is at least $2^n - |S| - 2$ which is greater than $6n - 2|S| - 9$, for $n \geq 5$. By Theorem 4, (F_1, F_2) is a distinguishable conditional-pair under the comparison diagnosis model.

Case 2: $|S| = n$.

If $Q_n - S$ is disconnected, by Lemma 12, $Q_n - S$ has one trivial component $\{v\}$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. Since F_1 and F_2 are two conditional faulty sets, this is an impossible case. So $Q_n - S$ is connected, and the subgraph $C_{F_1\Delta F_2,S}$ is the only component in $Q_n - S$. Let $U = Q_n - (F_1 \bigcup F_2)$. If there exist two vertices u and v in $V(U)$ such that u is adjacent to v, then the condition 1 of Theorem 3 holds and therefore (F_1, F_2) is a distinguishable conditional-pair; otherwise $V(U)$ is an independent set. Since $|S| = n$ and $|F_1\Delta F_2| \leq 2(2n-5)$, $|V(U)| \geq 2^n - 2(2n-5) - n = 2^n - 5n + 10$. By Lemma 17, the maximum size of a independent set is 2^{n-1} in Q_n. Comparing the lower bound $2^n - 5n + 10$ and the upper bound 2^{n-1}, we have $2^n - 5n + 10 > 2^{n-1}$ for $n \geq 5$, a contradiction.

Case 3: $n + 1 \leq |S| \leq 3(n-2)$.

By Corollary 8, there are four cases in $Q_n - S$ we need to consider. For case 1 of Corollary 8, $Q_n - S$ is connected, the proof is exactly the same as that of Case 2, and hence the detail is omitted. For case 2 and 4 of Corollary 8, $Q_n - S$ has at least one trivial component $\{v\}$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. Since F_1 and F_2 are two conditional faulty sets, the two cases are disregarded. Therefore, we only need to consider that $Q_n - S$ has two components, one of which is K_2 and the other one has $2^n - |S| - 2$ vertices. Let (x, y) be the component with only one edge. Since $N(\{x, y\}) \subseteq S$ and F_1 and F_2 do not contain all the neighbors of any vertex, vertex x and y cannot belong to $F_1\Delta F_2$. So the subgraph $C_{F_1\Delta F_2,S}$ is the other large connected component of $Q_n - S$. Let $U = Q_n - (F_1 \bigcup F_2) - \{x, y\}$. If no two vertices of $V(U)$ are adjacent, then $V(U)$ is an independent set and $|V(U)| \geq 2^n - 6n + |S| + 8$. By Lemma 17, the maximum size of a matching is $2^{n-1} - 1$ in $Q_n - \{x, y\}$. By Theorem 15 and Proposition 6, the maximum size of a independent set is $2^{n-1} - 1$ in $Q_n - \{x, y\}$. Comparing the lower bound $2^n - 6n + |S| + 8$ and the upper bound $2^{n-1} - 1$, we have $2^n - 6n + |S| + 8 > 2^{n-1} - 1$

for $n \geq 5$, $n+1 \leq |S| \leq 3(n-2)$, a contradiction. Hence, there exist two vertices u and v in $V(U)$ such that u is adjacent to v, then condition 1 of Theorem 3 is satisfied and therefore (F_1, F_2) is a distinguishable conditional-pair.

In Case 1, we prove that at least one of the conditions of Theorem 3 is satisfied in subgraph $C_{F_1 \Delta F_2, S}$. In Case 2 and 3, the condition 1 of Theorem 3 holds in subgraph $C_{F_1 \Delta F_2, S}$. Therefore, (F_1, F_2) is a distinguishable conditional-pair under the comparison diagnosis model. □

By Lemma 11, $t_c(Q_n) \leq 3(n-2)+1$, and by Lemma 18, Q_n is conditionally $(3(n-2)+1)-$diagnosable for $n \geq 5$. Hence, $t_c(Q_n) = 3(n-2)+1$ for $n \geq 5$. For Q_3 and Q_4, we observe that Q_3 is not conditionally four-diagnosable and Q_4 is not conditionally six-diagnosable, as shown in Figure 4.4. So, $t_c(Q_3) \leq 3$ and $t_c(Q_4) \leq 5$. Hence, the conditional diagnosabilities of Q_3 and Q_4 are both strictly less than $3(n-2)+1$.

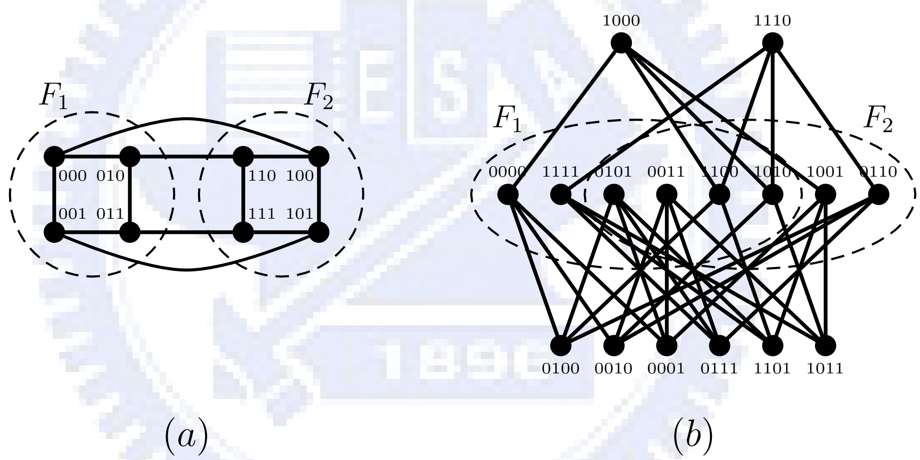


Figure 4.4: Two indistinguishable conditional-pairs for Q_3 and Q_4.

For the three-dimensional hypercube Q_3, Q_3 is three-diagnosable and it is not conditionally 4-diagnosable. It follows from Lemma 9 that $t_c(Q_3) = 3$. For the four-dimensional hypercube Q_4, we can use the similar technique used in proving Lemma 18 to prove that for any two conditional faulty sets $F_1, F_2 \subset V(Q_4)$, and $F_1 \neq F_2$, with $|F_1| \leq 5$ and $|F_2| \leq 5$, then (F_1, F_2) is a distinguishable conditional-pair under the comparison diagnosis model. Hence, the conditional diagnosability of Q_4 is 5. In summary, the conditional diagnosability of Q_n is stated as follows:

Theorem 20 *Under the comparison model, the conditional diagnosability of* Q_n *is* $3(n-2)+1$ *for* $n \geq 5$, $t_c(Q_3) = 3$ *and* $t_c(Q_4) = 5$.

4.2 Conditional Diagnosability of BC Networks under the Comparison Model

An n-dimensional bijective connection network (BC network), denoted by X_n, is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges. The set of all the n-dimensional BC networks is called the family of the n-dimensional BC networks, denoted by L_n. X_n and L_n may be recursively defined as below [24].

Definition 9 *The one-dimensional BC graph* X_1 *is a complete graph with two vertices. The family of the one-dimensional BC graph is defined as* $L_1 = \{X_1\}$. *Let* G *be a graph.* G *is an* n*-dimensional BC graph, denoted by* X_n*, if there exist* $V_0, V_1 \subset V(G)$ *such that the following two conditions hold:*

1. $V(G) = V_0 \bigcup V_1$, $V_0 \neq \emptyset$, $V_1 \neq \emptyset$, $V_0 \bigcap V_1 = \emptyset$; *and*
2. *There exists an edge set* $M \subset E(G)$ *such that* M *is a perfect matching between* V_0 *and* V_1, $G(V_0) \in L_{n-1}$ *and* $G(V_1) \in L_{n-1}$.

Now, we use again Figure 4.3 to show that the conditional diagnosability of BC graph X_n is no greater than $3(n-2)+2$, $n \geq 5$. As shown in Figure 4.3, we take a cycle of length four in X_n and it is easy to check the two conditional faulty sets F_1 and F_2 are indistinguishable, where $|F_1| = |F_2| = 3(n-2)+2$. Therefore, X_n is not conditionally $(3(n-2)+2)$-diagnosable and $t_c(X_n) \leq 3(n-2)+1$, $n \geq 3$. Next, we shall show that X_n is conditionally t-diagnosable, where $t = 3(n-2)+1$.

Lemma 19 $t_c(X_n) \leq 3(n-2)+1$ *under the comparison model, for* $n \geq 3$.

Let F be a set of vertices $F \subset V(X_n)$ and C be a connected component of $X_n - F$. We need some results on the cardinalities of F and $V(C)$ under some restricted conditions. The results are listed in Lemma 20 and 21. In Lemma 20, Zhu proved that deleting at most $2(n-1)-1$ vertices from X_n, the incomplete BC graph X_n has one connected component containing at least $2^n - |F| - 1$ vertices. We expand this result further. In Lemma 20, we show that deleting at most $3n-6$ vertices from X_n, the incomplete BC graph X_n has one connected component containing at least $2^n - |F| - 2$ vertices.

Lemma 20 *[57] $\forall X_n \in L_n (n \geq 3)$, let F be a set of vertices $F \subset V(X_n)$ with $n \leq |F| \leq 2(n-1)-1$. Suppose that $X_n - F$ is disconnected. Then $X_n - F$ has exactly two components, one is trivial and the other is nontrivial. The nontrivial component of $X_n - F$ contains $2^n - |F| - 1$ vertices.*

The BC graph can be described as follows: Let X_n denote an n-dimensional BC graph. X_1 is a complete graph with two vertices labeled with 0 and 1, respectively. For $n \geq 2$, each X_n consists of two X_{n-1}'s, denoted by X^L_{n-1} and X^R_{n-1}, with a perfect matching M between them. That is, M is a set of edges connecting the vertices of X^L_{n-1} and the vertices of X^R_{n-1} in a one-to-one manner. It is easy to see that there are 2^{n-1} edges between X^L_{n-1} and X^R_{n-1}. By using a simple induction, we can prove the following lemma.

Lemma 21 *$\forall X_n \in L_n (n \geq 5)$, let F be a set of vertices $F \subset V(X_n)$ with $|F| \leq 3n-6$. Then $X_n - F$ has a connected component containing at least $2^n - |F| - 2$ vertices.*

Proof.

We prove the lemma by induction on n. For $n = 5$, it is straightforward to verify that the lemma holds. As the inductive hypothesis, we assume that the result is true for X_{n-1}, for $|F| \leq 3(n-1)-6$, and for some $n \geq 6$. Now we consider X_n, $|F| \leq 3n-6$. An n-dimensional BC graph X_n can be divided into two X_{n-1}'s, denoted by X^L_{n-1} and X^R_{n-1}. Let $F_L = F \bigcap V(X^L_{n-1})$, $0 \leq |F_L| \leq 3n-6$ and $F_R = F \bigcap V(X^R_{n-1})$, $0 \leq |F_R| \leq 3n-6$. Then $|F| = |F_L| + |F_R|$. Without loss of generality, we may assume that $|F_L| \geq |F_R|$.

In the following proof, we consider two cases by the size of F_R: 1) $0 \leq |F_R| \leq 2$ and 2) $|F_R| \geq 3$.

Case 1: $0 \leq |F_R| \leq 2$**.**

Since $0 \leq |F_R| \leq 2$, $X^R_{n-1} - F_R$ is connected and $|V(X^R_{n-1} - F_R)| = 2^{n-1} - |F_R|$. Let $F^{(L)}_R \subset V(X^L_{n-1})$ be the set of vertices which has neighboring vertices in F_R. For each vertex $v \in X^L_{n-1} - F_L - F^{(L)}_R$, there is exactly one vertex $v^{(R)}$ in $X^R_{n-1} - F_R$, such that $(v, v^{(R)}) \in E(X_n)$. Besides, $|V(X^L_{n-1} - F_L - F^{(L)}_R)| \geq 2^{n-1} - |F_L| - |F_R|$. Hence $X_n - F$ has a connected component that contains at least $[2^{n-1} - |F_R|] + [2^{n-1} - |F_L| - |F_R|] = 2^n - |F| - |F_R| \geq 2^n - |F| - 2$ vertices.

Case 2: $|F_R| \geq 3$**.**

Since $|F_R| \geq 3$, $3 \leq |F_L| \leq 3(n-1) - 6$ and $3 \leq |F_R| \leq 3(n-1) - 6$. By the inductive hypothesis, $X^L_{n-1} - F_L$ ($X^R_{n-1} - F_R$, respectively) has a connected component C_L (C_R, respectively) that contains at least $2^{n-1} - |F_L| - 2$ ($2^{n-1} - |F_R| - 2$, respectively) vertices. Next, we divide the case into three subcases: 2.1) $|V(C_L)| = 2^{n-1} - |F_L| - 2$ and $X^R_{n-1} - F_R$ is disconnected, 2.2) $|V(C_L)| = 2^{n-1} - |F_L| - 2$ and $X^R_{n-1} - F_R$ is connected, and 2.3) $|V(C_L)| \geq 2^{n-1} - |F_L| - 1$ and $|V(C_R)| \geq 2^{n-1} - |F_R| - 1$.

Case 2.1: $|V(C_L)| = 2^{n-1} - |F_L| - 2$ **and** $X^R_{n-1} - F_R$ **is disconnected.**

This is an impossible case. Since $\kappa(X_{n-1}) = n - 1$, $|F_R| \geq n - 1$. By Lemma 20, $|F_L| \geq 2((n-1)-1)$. Then the total number of faulty vertices is at least $(n-1)+2((n-1)-1) = 3n - 5$ which is greater than $3n - 6$, a contradiction.

Case 2.2: $|V(C_L)| = 2^{n-1} - |F_L| - 2$ **and** $X^R_{n-1} - F_R$ **is connected.**

Since $X^R_{n-1} - F_R$ is connected, $|V(X^R_{n-1} - F_R)| = 2^{n-1} - |F_R|$. Since $|V(C_L)| \geq |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u, v) \in E(X_n)$. Hence $X_n - F$ has a connected component that contains at least $[2^{n-1} - |F_R|] + [2^{n-1} - |F_L| - 2] = 2^n - |F| - 2$ vertices.

Case 2.3: $|V(C_L)| \geq 2^{n-1} - |F_L| - 1$ **and** $|V(C_R)| \geq 2^{n-1} - |F_R| - 1$**.**

Since $|V(C_L)| \geq |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u, v) \in E(X_n)$. Hence $X_n - F$ has a connected component that contains at least $[2^{n-1} - |F_L| - 1] + [2^{n-1} - |F_R| - 1] = 2^n - |F| - 2$ vertices.

This completes the proof of the lemma. □

By Lemma 21, we have the following corollary.

Corollary 9 *$\forall X_n \in L_n(n \geq 5)$, let F be a set of vertices $F \subset V(X_n)$ with $|F| \leq 3n - 6$. Then $X_n - F$ satisfies one of the following conditions:*

1. *$X_n - F$ is connected.*
2. *$X_n - F$ has two components, one of which is K_1, and the other one has $2^n - |F| - 1$ vertices.*
3. *$X_n - F$ has two components, one of which is K_2, and the other one has $2^n - |F| - 2$ vertices.*
4. *$X_n - F$ has three components, two of which are K_1, and the third one has $2^n - |F| - 2$ vertices*

We are now ready to show that the conditional diagnosability of X_n is $3(n-2)+1$ for $n \geq 5$. Let $F_1, F_2 \subset V(X_n)$ be two conditional faulty sets with $|F_1| \leq 3(n-2)+1$ and $|F_2| \leq 3(n-2)+1$, $n \geq 5$. We shall show our result by proving that (F_1, F_2) is a distinguishable conditional-pair under the comparison model.

Lemma 22 *Let X_n be an n-dimensional BC graph with $n \geq 5$. For any two conditional faulty sets $F_1, F_2 \subset V(X_n)$, and $F_1 \neq F_2$, with $|F_1| \leq 3(n-2)+1$ and $|F_2| \leq 3(n-2)+1$. Then (F_1, F_2) is a distinguishable conditional-pair under the comparison model.*

Proof.

We use Theorem 4 to prove this result. Let $S = F_1 \bigcap F_2$, then $0 \leq |S| \leq 3(n-2)$. We will show that, deleting S from X_n, the subgraph $C_{F_1 \Delta F_2, S}$ containing $F_1 \Delta F_2$ has "many" vertices having degree three or more. More precisely, we are going to prove that, in the subgraph $C_{F_1\Delta F_2,S}$ the number of vertices having degree three or more is at least $2[3(n-2)+1-|S|]+1 = 6n-2|S|-9$. In the following proof, we consider three cases by the size of S: 1) $0 \le |S| \le n-1$, 2) $|S| = n$, and 3) $n+1 \le |S| \le 3(n-2)$.

Case 1: $0 \le |S| \le n-1$**.**

Since the connectivity of X_n is n [24], $X_n - S$ is connected, the subgraph $C_{F_1\Delta F_2,S}$ is the only component in $X_n - S$. Since the BC graph X_n has no cycle of length three and any two vertices have at most two common neighbors, it is straightforward, though tedious, to check that the number of vertices which has degree two or one is at most 2 in $C_{F_1\Delta F_2,S}$. Hence, the number of vertices having degree three or more is at least $2^n - |S| - 2$ which is greater than $6n-2|S|-9$, for $n \ge 5$. By Theorem 4, (F_1, F_2) is a distinguishable conditional-pair under the comparison diagnosis model.

Case 2: $|S| = n$**.**

If $X_n - S$ is disconnected, by Lemma 20, $X_n - S$ has one trivial component $\{v\}$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. Since F_1 and F_2 are two conditional faulty sets, this is an impossible case. So $X_n - S$ is connected, and the subgraph $C_{F_1\Delta F_2,S}$ is the only component in $X_n - S$. Let $U = X_n - (F_1 \bigcup F_2)$. If there exist two vertices u and v in $V(U)$ such that u is adjacent to v, then the condition 1 of Theorem 3 holds and therefore (F_1, F_2) is a distinguishable conditional-pair; otherwise $V(U)$ is an independent set. Hence, $N_{X_n-S}(v) \subset F_1\Delta F_2$, $\forall v \in U$, and we have the following inequality

$$\textstyle\sum_{v\in U} |deg_{X_n-S}(v)| \le \sum_{v\in F_1\Delta F_2} |deg_{X_n-S}(v)|.$$

To check the inequality, we have

$$\textstyle\sum_{v\in U} |deg_{X_n-S}(v)| \ge [2^n - 2(3(n-2)+1) + |S|]n - |S|n = n2^n - 6n^2 + 10n$$

and

$$\textstyle\sum_{v\in F_1\Delta F_2} |deg_{X_n-S}(v)| \le 2[3(n-2)+1-|S|]n = 4n^2 - 10n.$$

$n2^n - 6n^2 + 10n > 4n^2 - 10n$ for $n \ge 5$, a contradiction.

Case 3: $n+1 \le |S| \le 3(n-2)$**.**

By Corollary 9, there are four cases in $X_n - S$ we need to consider. For case 1 of Corollary 9, $X_n - S$ is connected, the proof is exactly the same as that of Case 2, and hence the detail is omitted. For case 2 and 4 of Corollary 9, $X_n - S$ has at least one trivial component $\{v\}$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. Since F_1 and F_2 are two conditional faulty sets, the two cases are disregarded. Therefore, we only need to consider that $X_n - S$ has two components, one of which is K_2 and the other one has $2^n - |S| - 2$ vertices. Let (x, y) be the component with only one edge. Since $N(\{x, y\}) \subseteq S$ and F_1 and F_2 do not contain all the neighbors of any vertex, vertex x and y cannot belong to $F_1 \Delta F_2$. So the subgraph $C_{F_1 \Delta F_2, S}$ is the other large connected component of $X_n - S$. Let $U = X_n - (F_1 \bigcup F_2) - \{x, y\}$. If there exist two vertices u and v in $V(U)$ such that u is adjacent to v, then the condition 1 of Theorem 3 holds and therefore (F_1, F_2) is a distinguishable conditional-pair; otherwise $V(U)$ is an independent set. Hence, $N_{X_n - S}(v) \subset F_1 \Delta F_2$, $\forall v \in U$, and we have the following inequality

$$\sum_{v \in U} |deg_{X_n - S}(v)| \le \sum_{v \in F_1 \Delta F_2} |deg_{X_n - S}(v)|.$$

To check the inequality, we have

$$\sum_{v \in U} |deg_{X_n - S}(v)| \ge [2^n - 2(3(n-2)+1) + |S| - 2]n - |S|n = n2^n - 6n^2 + 8n$$

and

$$\sum_{v \in F_1 \Delta F_2} |deg_{X_n - S}(v)| \le 2[3(n-2) + 1 - |S|]n \le 4n^2 - 12n.$$

$n2^n - 6n^2 + 8n > 4n^2 - 12n$ for $n \ge 5$, a contradiction.

In Case 1, we prove that at least one of the conditions of Theorem 3 is satisfied in subgraph $C_{F_1 \Delta F_2, S}$. In Case 2 and 3, the condition 1 of Theorem 3 holds in subgraph $C_{F_1 \Delta F_2, S}$. Therefore, (F_1, F_2) is a distinguishable conditional-pair under the comparison diagnosis model. □

By Lemma 19, $t_c(X_n) \le 3(n-2) + 1$, and by Lemma 22, X_n is conditionally $(3(n-2)+1)$-diagnosable for $n \ge 5$. We now have the following theorem.

Theorem 21 *Under the comparison model, the conditional diagnosability of X_n is* $3(n-$

2) + 1 *for* $n \geq 5$.

Since $Q_n, CQ_n, TQ_n, MQ_n \in L_n$, the following corollary holds.

Corollary 10 $t_c(Q_n) = t_c(CQ_n) = t_c(TQ_n) = t_c(MQ_n) = 3(n-2)+1$ *under the comparison model, for* $n \geq 5$.

Chapter 5

Conclusion, discussion, and future work

In this thesis, we propose a new concept called local diagnosability for a system and derive some structures for determining whether a system is locally t-diagnosable at a given vertex. Through this concept, the diagnosability of a system can be determined by computing the local diagnosability of each vertex. We also introduce a concept for system diagnosis, called strongly local-diagnosable property. A system has this strong property if the local diagnosability of every vertex is equal to its degree. We prove that both the hypercube network and the star graph have this strong property. Next, we study the local diagnosability of a faulty multiprocessor systems. For a faulty hypercube Q_n and a faulty star graph S_n, we prove that both Q_n and S_n keep this strong property even if they have up to $n-2$ faulty edges and $n-3$ faulty edges, respectively. According to Theorem 5, the global diagnosability of $Q_n - F$ is equal to the minimum local diagnosability of all vertices. A conditional local diagnosability measure for systems is also introduced in this thesis. Assume that each vertex of a faulty hypercube Q_n and a faulty star graph S_n is incident with at least two fault-free edges, we prove that Q_n keeps this strong property even if it has up to $3(n-2)-1$ faulty edges and S_n will also keep this strong property no matter how many edges are faulty. Furthermore, we prove Q_n keeps this strong property no matter how many edges are faulty, provided that each vertex of a faulty hypercube Q_n is incident with at least three fault-free edges. Our bounds on the number of faulty edges are all tight.

We use the hypercube and the star graph as two examples to introduce the concepts of the local diagnosability, the local structures and the strongly local-diagnosable property. In fact, many well-known systems also have these local structures and this strong property. Furthermore, there is a close relationship between its local structure and its local syndrome. So we propose a new diagnosis algorithm for general systems. The time complexity of our algorithm to diagnose all the faulty processors is bounded by $O(N \log N)$, where N is the total number of processors.

There are several different fault diagnosis models in the area of diagnosability. It is worth investigating, under various models, whether a system has this strongly local-diagnosable property after removing some edges. It is also an attractive work to develop more different measures of diagnosability based on network reliability, network topology, application environment and statistics related to fault patterns.

In the real world, processors fail independently and with different probabilities. The probability that any faulty set contains all the neighbors of some processor is very small [20, 44] so we are interested in the study of conditional diagnosability. A new diagnosis measure proposed by Lai et al. [40], it restricts that each processor of a system is incident with at least one fault-free processor. In this thesis, we first use the hypercube as an example and show that the conditional diagnosability of Q_n is $3(n-2)+1$ under the comparison model. This number $3(n-2)+1$ is about three times as large as the classical diagnosability. Furthermore, we extend the result to bijective connection network. Since the hypercube, crossed cube, twisted cube, and Möbius cube are some examples of BC networks, we can obtain the conditional diagnosability of the cube family.

In this thesis, we study the conditional diagnosability of Q_n under the comparison model. Under the PMC model, however, the conditional diagnosability of Q_n is shown to be $4(n-2)+1$ by Lai et al. [40]. In order to understand why the increase in diagnosability under the comparison model is lower than that under the PMC model, we take a look at Figure 4.3. As shown in Figure 4.3, there are two conditional faulty sets F_1 and F_2 with $|F_1| = |F_2| = 3(n-2)+2$. As shown, F_1 and F_2 are indistinguishable, and therefore the conditional diagnosability of Q_n is no greater than $3(n-2)+2$ under the comparison model. We now consider the same conditional faulty sets under the PMC model in Figure 4.3, either the edge (v_4, v_1) or the edge (v_4, v_3) provides "effective" test to distinguish the syndrome of F_1 and F_2 under the PMC model, namely v_4 tests v_1 or v_4 tests v_3. Therefore F_1 and F_2 are distinguishable. However, v_4 compares v_1 and v_3 is not an effective comparison to distinguish the syndrome of F_1 and F_2 under the comparison model. On the other hand, see Figure 2.2, every effective comparison under the comparison model provides effective test under the PMC model. So the conditional diagnosability of Q_n under the comparison model is intuitively lower than that under the PMC model. In this thesis, we give a complete proof to support our intuition and finally obtain the main result.

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