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碩 士 論 文

近似容忍區間

Approximate Tolerance Intervals

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摘 要

我們建立了近似容忍區間的方法，也利用例子和真實資料使用韋伯分配和極值分配的分析說明此一近似容忍區間是一個不錯的工具。最後我們也把這個技巧推廣到線性迴歸模型。

關鍵字：信賴區間；覆蓋區間；容忍區間

Approximate Tolerance Intervals

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Abstract

An technique of developing approximate tolerance intervals is established. Examples and real data analyses based on approximate tolerance intervals for Weibull distribution and extreme value distribution are provided. Furthermore, this technique is extended to linear regression model.

Key words: Confidence interval; coverage interval; tolerance interval.

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楊秀慧　　謹誌于

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Contents

List of tables and figures

Approximate Tolerance Intervals

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Abstract

A technique for developing approximate tolerance intervals is established. Examples and real data analyses based on approximate tolerance intervals for Weibull distribution and extreme value distributions are provided. Furthermore, this technique is extended to a linear regression model.

Key words: Confidence interval; coverage interval; tolerance interval;

1. Introduction

Statistical theory of interval estimation deals mostly with the confidence interval to contain a parameter θ. In many applications, we require an interval to contain the future random variable (r.v.), which is a prediction problem. Among the alternatives, intervals in the form of tolerance intervals are widely used in quality control and related prediction problems to monitor manufacturing processes, detect changes in such processes, ensure product compliance with specifications, etc.

In manufacturing industries, specification limits for a characteristic such as thickness or volume, say LSL and USL, define the boundaries of acceptable quality for a manufactured item (component). Consider the manufacturer of a mass-production item. The manufacturer is interested in an interval that contains a specified (usually large) percentage of the product and he knows that unless 90% of his production is acceptable in the sense that the item's characteristic falls in the limits, he will lose money in this production. For this purpose, having a random sample $X_1, ..., X_n$ from a distribution with probability density function (pdf) $f(x, \theta)$, Wilks (1941) established the

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX tolerance interval by saying that $(T_1, T_2) = (t_1(X_1, ..., X_n), t_2(X_1, ..., X_n))$ is a γ-content tolerance interval with confidence $1 - \alpha$ if it satisfies

$$P_\theta\{P_X^\theta[(T_1, T_2)] \geq \gamma\} \geq 1 - \alpha \text{ for } \theta \in \Theta \tag{1.1}$$

where Θ is the parameter space.

A vast literature on tolerance intervals has been developed (see for example Wilks (1941), Wald (1943), Paulson (1943), Guttman (1970) and a recent review in Patel (1986)). As noted by Bucchianico, Einmahl and Mushkudiani (2001), both the mathematically and the engineering oriented statistics textbooks hardly deal with this topic explicitly; and if they do, the treatment is often confined to tolerance intervals for the normal distribution. This is partly because tolerance intervals can be difficult to construct for particular distributions (although nonparametric tolerance intervals based on order statistics can be obtained for particular values of the content) and, perhaps, partly because as Carroll and Ruppert (1991) suggest, the idea of conditional coverage probability of (1.1) is considered to be too difficult for beginning students.

We consider a further problem that tolerance intervals for many distributions are still waiting to be developed. This is because when the tolerance intervals appear in the literature, they are generally constructed through some appropriate pivotal quantities. However, this design may not be available for many distributions useful in reliability and lifetime data models. Hence, it is desirable to solve this problem by approximation technique.

The idea for this research comes from an extension of the technique of confidence interval of coverage interval in Chen, Huang and Welsh (2005). We establish theories based on asymptotically normal quantile estimators to establish approximate tolerance intervals. An advantage of this techniques is that the explicit form of the interval may be formulated. For specific, we construct these approximate tolerance intervals for a Weibull distribution and extreme value distribution where large sample properties of the maximum likelihood estimators of the unknown parameters are emplyed in this setting. Real data analyses for these two approximate tolerance intervals are also provided. Finally, we extend this technique to the construction of nonparametric regression tolerance interval by using the regression quantile of Koenker and Bassett (1978).

2. Theoretical Basis for Approximate Tolerance Interval

Let $X_1, ..., X_n$ be a random sample from a distribution with distribution function F_θ where θ is an unknown parameter. We also denote $(T_1, T_2) = (t_1(X_1, ..., X_n), t_2(X_1, ..., X_n))$ as a random interval. We say that $(a(\theta), b(\theta))$ is a γ coverage interval for the underlying distribution if $\gamma = P_X^\theta[(a(\theta), b(\theta))]$ for $\theta \in \Theta$. It is interesting and desirable to compute the exact tolerance interval when possible. However, in general, the calculations are intractable and we have to resort to approximations. One way to construct an approximate tolerance interval is through the use of approximate confidence interval of a coverage interval.

Theorem 2.1. Let $(a(\theta), b(\theta))$ be a γ coverage interval. If (T_1, T_2) is an approximate $100(1-\alpha)\%$ confidence interval of $(a(\theta), b(\theta))$, then (T_1, T_2) is an approximate γ-content tolerance interval at confidence $1-\alpha$.
Proof.

$$\begin{aligned}
&P_\theta\{P_X^\theta[(T_1, T_2)] \geq \gamma\} = P_\theta\{F_\theta(T_2) - F_\theta(T_1) \geq \gamma\} \\
&= P_\theta\{F_\theta(T_2) - F_\theta(T_1) \geq F_\theta(b(\theta)) - F_\theta(a(\theta))\} \\
&\geq P_\theta\{F_\theta(T_1) \leq F_\theta(a(\theta)) < F_\theta(b(\theta)) \leq F_\theta(T_2)\} \\
&\geq P_\theta\{T_1 \leq a(\theta) < b(\theta) \leq T_2\}
\end{aligned}$$

as F_θ is nondecreasing. Then

$$\begin{aligned}
&\lim_{n\to\infty} P_\theta\{P_X^\theta[(T_1, T_2)] \geq \gamma\} \\
&\geq \lim_{n\to\infty} P_\theta\{T_1 \leq a(\theta) < b(\theta) \leq T_2\} \\
&= 1-\alpha. \quad \square
\end{aligned}$$

In all theorems introduced in this paper, approximate one sided tolerance intervals of the forms $(-\infty, T_2)$ and (T_1, ∞), respectively, corresponding with γ coverage one sided intervals $(-\infty, b(\theta))$ and $(a(\theta), \infty)$ are our special situations.

In the application of Theorem 2.1, to establish an approximate tolerance interval we need to decide a parameter γ coverage interval and a confidence interval for it. At this point, we presume that we have $(a(\theta), b(\theta))$ and then we consider asymptotic normal estimators of $a(\theta)$ and $b(\theta)$ to construct approximate confidence intervals for this γ coverage interval.

Theorem 2.2. Let $\hat{a}(\theta)$ with standard error $s_a(\theta)$ and $\hat{b}(\theta)$ with standard error $s_b(\theta)$ be asymptotically normal estimators of $a(\theta)$ and $b(\theta)$ respectively so that $n^{1/2}(\hat{a}(\theta) - a(\theta))/s_a \xrightarrow{d} N(0,1)$ and $n^{1/2}(\hat{b}(\theta) - b(\theta))/s_b \xrightarrow{d} N(0,1)$. Suppose we want to set a $100(1-\alpha)\%$ confidence interval for the interval $[a(\theta),\, b(\theta)]$. Then

$$\left[\hat{a}(\theta) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}},\ \hat{b}(\theta) + z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}}\right] \tag{2.1}$$

is an approximate $100(1-\alpha)\%$ confidence interval for $[a(\theta),\, b(\theta)]$.
Proof.

$$\begin{aligned}
&1 - P_\theta\left\{\frac{\hat{a}(\theta) - a(\theta)}{s_a/\sqrt{n}} \geq z_{1-\frac{\alpha}{2}}\right\} - P_\theta\left\{\frac{\hat{b}(\theta) - b(\theta)}{s_b/\sqrt{n}} \leq -z_{1-\frac{\alpha}{2}}\right\} \\
&= 1 - P_\theta\left\{\hat{a}(\theta) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}} \geq a(\theta)\right\} - P_\theta\left\{\hat{b}(\theta) + z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}} \leq b(\theta)\right\} \\
&\leq 1 - P_\theta\left\{\hat{a}(\theta) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}} \geq a(\theta) \text{ or } \hat{b}(\theta) + z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}} \leq b(\theta)\right\} \\
&= P_\theta\left\{\hat{a}(\theta) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}} \leq a(\theta) \text{ and } b(\theta) \leq \hat{b}(\theta) + z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}}\right\}
\end{aligned}$$

so

$$1-\alpha \leq \lim_{n\to\infty} P_\theta\left\{\hat{a}(\theta) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}} \leq a(\theta) \text{ and } b(\theta) \leq \hat{b}(\theta) + z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}}\right\}. \quad \square$$

When $(a(\theta), b(\theta))$ is a γ coverage interval, the random interval in (2.1) is an approximate γ-content tolerance interval with confidence $1-\alpha$. To choose coverage interval $(a(\theta), b(\theta))$, we restrict it on a quantile interval.

The general class of γ quantile coverage intervals is

$$\{[F_\theta^{-1}(\alpha), F_\theta^{-1}(\gamma+\alpha)] : 0 < \alpha < 1-\gamma\}. \tag{2.2}$$

The simplest general γ coverage intervals for the underlying distribution are the γ median coverage intervals given by $C_{med}(\gamma) = [F_\theta^{-1}\{(1-\gamma)/2\}, F_\theta^{-1}\{(1+\gamma)/2\}]$ and one sided ones $(-\infty, F_\theta^{-1}(\gamma)]$ and $(F_\theta^{-1}(1-\gamma), \infty)$. However, the shortest coverage interval

$$C_{mod}(\gamma) = [F_\theta^{-1}(\alpha^*), F_\theta^{-1}(\gamma+\alpha^*)],$$

is also very interesting where $\alpha^* = \arg_\alpha \min_{0<\alpha<1-\gamma}\{F_\theta^{-1}(\gamma+\alpha) - F_\theta^{-1}(\alpha)\}$. We tentatively consider the general coverage interval $C(\gamma) = [F_\theta^{-1}(\alpha_1), F_\theta^{-1}(\alpha_2)]$ where α_1 and α_2 are known. Given an estimator $\hat\theta$ of θ and by letting $\hat F_\theta^{-1}(\alpha_1) = F_{\hat\theta}^{-1}(\alpha_1)$ and $\hat F_\theta^{-1}(\alpha_2) = F_{\hat\theta}^{-1}(\alpha_2)$, then we can estimate the γ coverage interval by

$$\hat C(\gamma) = [\hat F_\theta^{-1}(\alpha_1), \quad \hat F_\theta^{-1}(\alpha_2)].$$

We restrict $\hat\theta$ as the maximum likelihood estimator for the convenience of using its large sample properties.

Theorem 2.3. Suppose that $\hat\theta$ is the maximum likelihood estimator of θ. Suppose that F is continuously differentiable and the regularity conditions hold. We also let $s_\alpha^2 = \left(\frac{\partial \hat F_\theta^{-1}(\alpha)}{\partial\hat\theta}\right)' I^{-1}(\hat\theta)\left(\frac{\partial \hat F_\theta^{-1}(\alpha)}{\partial\hat\theta}\right), 0 < \alpha < 1$. For any α_1, α_2 with $\alpha_2 - \alpha_1 = \gamma$,

$$[\hat F_\theta^{-1}(\alpha_1) - z_{1-\frac{\alpha}{2}}\frac{s_{\alpha_1}}{\sqrt{n}}, \hat F_\theta^{-1}(\alpha_2) + z_{1-\frac{\alpha}{2}}\frac{s_{\alpha_2}}{\sqrt{n}}] \tag{2.3}$$

is an approximate γ-content tolerance interval at confidence coefficient $1-\alpha$.
Proof. This is to use $\hat F_\theta^{-1}(\alpha_1)$ and $\hat F_\theta^{-1}(\alpha_2)$ to construct an approximate $100(1-\alpha)\%$ confidence interval for quantile interval $(F_\theta^{-1}(\alpha_1), F_\theta^{-1}(\alpha_2))$. We can write

$$\hat F_\theta^{-1}(\alpha_2) - F_\theta^{-1}(\alpha_2) = \left(\frac{\partial F_\theta^{-1}(\alpha_2)}{\partial\theta}\right)'(\hat\theta - \theta)$$

and

$$\hat{F}_\theta^{-1}(\alpha_1) - F_\theta^{-1}(\alpha_1) = \left(\frac{\partial F_\theta^{-1}(\alpha_1)}{\partial \theta}\right)' (\hat{\theta} - \theta).$$

By letting $\sigma_a^2 = \left(\frac{\partial F_\theta^{-1}(\alpha_2)}{\partial \theta}\right)' I^{-1}(\theta) \left(\frac{\partial F_\theta^{-1}(\alpha_2)}{\partial \theta}\right)$ and $\sigma_b^2 = \left(\frac{\partial F_\theta^{-1}(\alpha_2)}{\partial \theta}\right)' I^{-1}(\theta)$ $\left(\frac{\partial F_\theta^{-1}(\alpha_2)}{\partial \theta}\right)$, it follows that

$$\sqrt{n}(\hat{F}_\theta^{-1}(\alpha_1) - F_\theta^{-1}(\alpha_1)) \xrightarrow{d} N(0, \sigma_a^2) \text{ and } \sqrt{n}(\hat{F}_\theta^{-1}(\alpha_2) - F_\theta^{-1}(\alpha_2)) \xrightarrow{d} N(0, \sigma_b^2).$$

From the fact that $s_{\alpha_1} \xrightarrow{p} \sigma_a$ and $s_{\alpha_2} \xrightarrow{p} \sigma_b$, the interval in (2.3) is an approximate $100(1-\alpha)\%$ confidence interval for $(F_\theta^{-1}(\alpha_1), F_\theta^{-1}(\alpha_2))$ following from the Slutsky theorem and Theorem 2.2. $\square$

In this paper, we do not consider which type of the coverage interval is appropriate to use (see Chen, Huang and Welsh (2005) for an argument). However, the median (two sided) type and one sided lower covergae intervals will be used as examples.

Corollary 2.4. Let the regularity conditions hold. Then based on the median γ coverage interval,

$$[\hat{F}_\theta^{-1}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}} \frac{s_{(1-\gamma)/2}}{\sqrt{n}}, \hat{F}_\theta^{-1}(\frac{1+\gamma}{2}) + z_{1-\frac{\alpha}{2}} \frac{s_{(1+\gamma)/2}}{\sqrt{n}}] \tag{2.4}$$

is an approximate γ-content tolerance interval at confidence coefficient $1-\alpha$.

Owen (1964) argued that most tolerance intervals developed for normal distribution are set up for controlling the center part so that the percentage nondefective is controlled $100\gamma\%$, and hence the defectiveness could be all be in one tail. Then he consider a normal tolerance interval such that no more than the proportion $\frac{1-\gamma}{2}$ is below the lower tolerance limit and no more than the proportion $\frac{1-\gamma}{2}$ is above the upper tolerance limit. Extending from his idea, we may expect an approximate γ-content tolerance interval (T_1, T_2) with confidence $1-\alpha$ that satisfies

$$\lim_{n\to\infty} P_\theta\{P_X^\theta[(-\infty, T_1)] \le \frac{1-\gamma}{2} \text{ and } P_X^\theta[(T_2, \infty)] \le \frac{1-\gamma}{2}\} \ge 1-\alpha. \tag{2.5}$$

Theorem 2.5. Having chosen the median γ coverage interval, then the approximate γ-content tolerance interval of (2.4) at confidence $1-\alpha$ satisfies Owen's requirement (2.5).

Proof. First, we see that

$$P_\theta\{P_X^\theta[(-\infty, \hat{F}_\theta^{-1}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_{(1-\gamma)/2}}{\sqrt{n}})] \leq \frac{1-\gamma}{2} \text{ and } P_X^\theta[(\hat{F}_\theta^{-1}(\frac{1+\gamma}{2}) + z_{1-\frac{\alpha}{2}}\frac{s_{(1+\gamma)/2}}{\sqrt{n}}, \infty)] \leq \frac{1-\gamma}{2}\}$$

$$= P_\theta\{F_\theta(\hat{F}_\theta^{-1}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_{(1-\gamma)/2}}{\sqrt{n}}) \leq F_\theta(F_\theta^{-1}(\frac{1-\gamma}{2})) \text{ and } 1 - F_\theta(\hat{F}_\theta^{-1}(\frac{1+\gamma}{2}) + z_{1-\frac{\alpha}{2}}\frac{s_{(1+\gamma)/2}}{\sqrt{n}}) \leq 1 - F_\theta(F_\theta^{-1}(\frac{1+\gamma}{2}))\}$$

$$= P_\theta\{F_\theta(\hat{F}_\theta^{-1}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_{(1-\gamma)/2}}{\sqrt{n}}) \leq F_\theta(F_\theta^{-1}(\frac{1-\gamma}{2})) \text{ and } F_\theta(F_\theta^{-1}(\frac{1+\gamma}{2})) \leq F_\theta(\hat{F}_\theta^{-1}(\frac{1+\gamma}{2}) + z_{1-\frac{\alpha}{2}}\frac{s_{(1+\gamma)/2}}{\sqrt{n}})\}$$

$$\geq P_\theta\{\hat{F}_\theta^{-1}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_{(1-\gamma)/2}}{\sqrt{n}} \leq F_\theta^{-1}(\frac{1-\gamma}{2}) < F_\theta^{-1}(\frac{1+\gamma}{2}) \leq \hat{F}_\theta^{-1}(\frac{1+\gamma}{2}) + z_{1-\frac{\alpha}{2}}\frac{s_{(1+\gamma)/2}}{\sqrt{n}}\}.$$

Then, following from Theorem 2.2,

$$\lim_{n\to\infty} P_\theta\{P_X^\theta[(-\infty, \hat{F}_\theta^{-1}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_{(1-\gamma)/2}}{\sqrt{n}})] \leq \frac{1-\gamma}{2} \text{ and } P_X^\theta[(\hat{F}_\theta^{-1}(\frac{1+\gamma}{2}) + z_{1-\frac{\alpha}{2}}\frac{s_{(1+\gamma)/2}}{\sqrt{n}}, \infty)] \leq \frac{1-\gamma}{2}\} \geq 1-\alpha. \quad \square$$

Before introducing examples of approximate tolerance intervals, we should verify if the proposed approximation technique has an appropriate efficiency. We will simulate one example to compare an exact tolerance interval and an approximate one in terms of their difference in confidence.

Suppose that we have a random sample $X_1, ..., X_n$ drawn from the exponential distribution with pdf

$$f(x, \theta) = \theta e^{-\theta x}, x > 0.$$

The λth quantile function of this distribution is $F_\theta^{-1}(\lambda) = -\theta^{-1} log(1-\lambda)$ for $0 < \lambda < 1$. One sided tolerance intervals are derived by Goodman and Madansky (1962) and the two sided one with explicit form have been presented in Chen, Huang and Welsh (2005). Based on γ coverage interval

$$C(\gamma) = (-\theta^{-1}log(\frac{1+\gamma}{2}), -\theta^{-1}log(\frac{1-\gamma}{2})),$$

the following

$$(\frac{-2\sum_{i=1}^{n} X_i log((1+\gamma)/2))}{\chi^2_{1-\alpha/2}}, \frac{-2\sum_{i=1}^{n} X_i log((1-\gamma)/2)}{\chi^2_{\alpha/2}}) \tag{2.6}$$

is a γ-content tolerance interval with confidence $1-\alpha$. It is known that the maximum likelihood estimator of θ is $\bar{X}^{-1}$, the maximum likelihood estimator of the γ-content coverage interval is

$$\hat{C}(\gamma) = (-\bar{X}log(\frac{1+\gamma}{2}), -\bar{X}log(\frac{1-\gamma}{2}).$$

Because $\sqrt{n}(\bar{X} - \frac{1}{\theta})$ is approximately normal with mean zero and variance $\frac{1}{\theta^2}$, we have the following

$$\sqrt{n}(-\bar{X}log(\frac{1+\gamma}{2}) - (-\theta^{-1}log(\frac{1+\gamma}{2}))) \xrightarrow{d} N(0, \frac{(log((1+\gamma)/2))^2}{\theta^2})$$

$$\sqrt{n}(-\bar{X}log(\frac{1-\gamma}{2}) - (-\theta^{-1}log(\frac{1-\gamma}{2}))) \xrightarrow{d} N(0, \frac{(log((1-\gamma)/2))^2}{\theta^2})$$

Then we have that the random interval

$$(-\bar{X}log(\frac{1+\gamma}{2}) - z_{1-\frac{\alpha}{2}}\bar{X}\frac{-log((1+\gamma)/2)}{\sqrt{n}}, -\bar{X}log(\frac{1-\gamma}{2}) + z_{1-\frac{\alpha}{2}}\bar{X}\frac{-log((1-\gamma)/2)}{\sqrt{n}}) \tag{2.7}$$

is an approximate γ-content tolerance interval with confidence $1-\alpha$.

It is of interest to compare the exact tolerance interval of (2.6) and the approximate one of (2.7) in simulation. With replication $m = 100,000$, every time we select sample of size n randomly from the exponential distribution for a fixed θ. We let (t_1^j, t_2^j) represents the sample tolerance interval of (2.6) or (2.7) for the jth sample. The simulated confidence may be defined as

$$Conf = \frac{1}{m}\sum_{j=1}^{m} I(F_\theta(t_2^j) - F_\theta(t_1^j) \geq \gamma)$$

where F_θ is the true distribution function for the exponential distribution. We also consider that $\gamma = 0.9, 0.95, 0.99$, $1-\alpha = 0.9, 0.95, 0.99$ and $n = 10, 30, 50$. Let $Conf_{app}$ and $Conf_{exa}$ represent the simulated confidences for, respectively, the approximate and exact tolerance intervals. We then display the absolute values of confidence differences as

$$|Conf_{app} - Conf_{exa}|$$

in Table 1.

Table 1. Confidence differences for two tolerance interval when the distribution is exponential

	$1-\alpha = 0.9$	$1-\alpha = 0.95$	$1-\alpha = 0.99$
$\gamma = 0.9$			
$n = 10$	0.0012	0.0117	0.005
$n = 30$	0.004	0.0026	0.0016
$n = 50$	0.0059	0.0093	0.0055
$\gamma = 0.95$			
$n = 10$	0.0029	0.0033	0.0203
$n = 30$	0.0082	0.0037	0.0119
$n = 50$	0.0065	0.0071	0.0003
$\gamma = 0.99$			
$n = 10$	0.0015	0.0041	0.015
$n = 30$	0.0031	0.0094	0.0084
$n = 50$	0.0095	0.0085	0.0069

Since the confidences are all equal or above 0.9, the differences displayed in Table 1 are quite insignificant. This indicates that an approximate tolerance interval is appropriate as a choice when an exact one is not attainable.

3. Approximate Tolerance Intervals for Weibull Distributions

Location-scale and log-location-scale distributions are the most widely used for parametric reliability and lifetime data models. This section presents approximate tolerance intervals for a Weibull distribution. Consider the random variable with Weibull distribution that has a probablity density function of the form

$$f(x, \lambda, \beta) = \lambda\beta x^{\beta-1}e^{-\lambda x^\beta}, x > 0 \tag{3.1}$$

for some $\lambda > 0, \beta > 0$. The population quantile function is $F^{-1}(\alpha) = [-\lambda^{-1}log(1-\alpha)]^{1/\beta}$. With this, there are available explicit quantile intervals serving as coverage intervals. However, without appropriate pivotal quantities, there is difficulty developing exact tolerance intervals.

Let's denote

$$s^2_{wei}(m,\beta,\lambda) = \frac{(-log(m))^{2/\beta}}{\lambda^{2/\beta}\beta^2}\frac{\ell_1(m,\lambda)}{\ell_2(\lambda)}$$

where

$$\ell_1(m,\lambda) = 1 + \frac{5}{6}\pi - 10\delta + 5\delta^2 + 10log(\lambda)(-1+\delta) + 5log(\lambda)^2 - 2(1-\delta-log(\lambda))$$
$$log(-\frac{log(m)}{\lambda}) + [log(-\frac{log(m)}{\lambda})]^2$$

and

$$\ell_2(\lambda) = \frac{5}{6}\pi^2 + 8(-\delta - log(\lambda) + \delta log(\lambda)) + 4(\delta^2 + log(\lambda)^2)$$

and where δ is the Euler's constant with $\delta = -\int_0^\infty log(x)e^{-x}dx = 0.57722....$ We present an approximate tolerance interval for Weibull distribution where its proof is listed in the Appendix.

Theorem 3.1. Suppose that we have a random sample drawn from the Weibull distribution of (3.1). Then, with maximum likelihood estimators $\hat{\lambda}$ and $\hat{\beta}$,

$$([-\hat{\lambda}^{-1}log(\frac{1+\gamma}{2})]^{1/\hat{\beta}} - z_{1-\frac{\alpha}{2}}\frac{s_{wei}((1+\gamma)/2,\hat{\beta},\hat{\lambda})}{\sqrt{n}},$$
$$[-\hat{\lambda}^{-1}log(\frac{1-\gamma}{2})]^{1/\hat{\beta}} + z_{1-\frac{\alpha}{2}}\frac{s_{wei}((1-\gamma)/2,\hat{\beta},\hat{\lambda})}{\sqrt{n}}) \quad (3.2)$$

is an approximated γ-content tolerance interval with confidence $1-\alpha$ and

$$(0, [-\hat{\lambda}^{-1}log(1-\gamma)]^{1/\hat{\beta}} + z_{1-\alpha}\frac{s_{wei}(1-\gamma,\hat{\beta},\hat{\lambda})}{\sqrt{n}})$$

and

$$([-\hat{\lambda}^{-1}log(\gamma)]^{1/\hat{\beta}} - z_{1-\alpha}\frac{s_{wei}(\gamma,\hat{\beta},\hat{\lambda})}{\sqrt{n}}, \infty) \quad (3.3)$$

are respectively approximated one sided γ-content tolerance intervals with confidence $1-\alpha$.

We use real data analysis to explain these approximate tolerance intervals. The complete data set of $n = 23$ ball bearing failure times (originally discussed by Lieblein and Zelen (1956)) to test the endurance of deep groove ball bearings has been extensively studied. An ordered set of failure times measured in 10^6 revolutions is displayed in Leemis (1995, p190) with a data analysis. From an attempt to study the confidence interval for survival function, Leemis shows that the Weibull distribution is significantly better than the exponential distribution in fitting this data set. In fact, it is known that a Weibull distribution always provides a better fit than an exponential one since it is a generalization of the exponential distribution. We display the approximate one sided lower tolerance interval of (3.3) in the following table.

Table 2. Approximate Weibull tolerance interval for ball bearing data

	$1-\alpha=0.9$	0.95	0.99
$\gamma=0.9$	$(24.491,\infty)$	$(23.476,\infty)$	$(21.573,\infty)$
$\gamma=0.95$	$(17.369,\infty)$	$(16.644,\infty)$	$(15.286,\infty)$
$\gamma=0.99$	$(7.971,\infty)$	$(7.632,\infty)$	$(6.996,\infty)$

To evaluate the performance of the approximate Weibull tolerance interval in case of this data set, we compare it with other tolerance intervals. The exponential distribution has an exact lower γ-content tolerance interval with confidence $1-\alpha$ as

$$\left(\frac{-2nlog(\gamma)}{\chi^2_{1-\alpha}(2n)}\bar{X},\infty\right). \tag{3.4}$$

Basically the lower tolerance intervals are confidence intervals of the survival function $S(t) = P(X \geq t)$ and then they are expected to have the same pattern performed by the empirical survival function

$$S_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \geq t), t \geq 0. \tag{3.5}$$

Without knowing the true distribution in a real data analysis, this view of interval-like estimations being compared with an empirical distribution has been presented by Leemis (1995). Following Leemis's setting, we consider the survival function $\gamma = S_n(t)$ as a function of t and Weibull lower tolerance limits $t = [-\hat{\lambda}^{-1}log(\gamma)]^{1/\hat{\beta}} - z_{1-\alpha}\frac{s_{wei}(\gamma,\hat{\beta},\hat{\lambda})}{\sqrt{n}}$ and exponential lower tolerance limits $t = \frac{-2nlog(\gamma)}{\chi^2_{1-\alpha}(2n)}\bar{x}$ as functions of γ. In Figure 1, we present the pictures of empirical survival function and one sided lower exponential and Weibull tolerance intervals for the ball bearing data:

Figure 1 here

With confidence $1 - \alpha > 0.5$, a lower one-sided tolerance interval is expected to be under the sample survival function and also parallel to it. From this point, the tolerance interval based on Weibull distribution does achieves this aim appropriately. However, it is quite not appropriate for it based on exponential distribution since it is too low on its left part and too high on its right part.

Next, we compare the two sided tolerance interval for this ball bearing data with the log-normal tolerance interval. The log-normal distribution is another popularly means for monitoring lifetime data. Suppose that the logarithm of the ball bearing random variable follows the normal distribution $N(\mu, \sigma^2)$, using the technique of Chen, Huang and Welsh (2005), a γ-content log-normal tolerance interval with confidence $1 - \alpha$ is

$$(exp\{\bar{X}_T - t_{1-\frac{\alpha}{2}}(n-1, \sqrt{n}z_{\frac{1+\gamma}{2}})\frac{S_T}{\sqrt{n}}\}, exp\{\bar{X}_T - t_{1-\frac{\alpha}{2}}(n-1, -\sqrt{n}z_{\frac{1+\gamma}{2}})\frac{S_T}{\sqrt{n}}\}) \tag{3.6}$$

where $\bar{X}_T$ and S_T represent the sample mean and sample standard deviation of logarithms $log(X_1), log(X_2), ..., log(X_n)$. The two sided tolerance intervals are confidence intervals of two-sided quantile intervals $(F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2}))$ and so are expected to have the same pattern performed by its empirical function, when $n = 2k$,

$$S_n^*(x) = (\frac{1}{k}\sum_{i=1}^{k} I(X_i^* \geq x), \frac{1}{k}\sum_{i=k+1}^{2k} I(X_i^{**} \leq x)), \tag{3.7}$$

with $X_i^* = X_{(i)}, i = 1, ..., k$ and $X_i^{**} = X_{(i)}, = k+1, ..., 2k$ where $X_{(i)}, i =$

$1, ..., 2k$ are the order statistics of $X_1, ..., X_n$ and, when $n = 2k+1$,

$$S_n^*(x) = (\frac{1}{k+1}\sum_{i=1}^{k+1} I(X_i^* \geq x), \frac{1}{k+1}\sum_{i=k+1}^{2k+1} I(X_i^{**} \leq x)) \qquad (3.8)$$

with $X_i^* = X_{(i)}, i = 1, ..., k+1$ and $X_i^{**} = X_{(i)}, i = k+1, ..., 2k+1$. Now, we consider a two sided empirical function of (3.8), the two sided Weibull toleance limits functions in (3.2) and two sided log-normal toleance limits functions in (3.6) and plot them in Figure 2.

Figure 2 here

Basically two distributions both caught the right shape in interpreting the two sided sample cumulative distribution. However, the Weibull one fits the data better. Let's see this. Consider the 0.9-content tolerance interval with confidence 0.9. The Weibull and log-normal ones have sample tolerance intervals, respectively,

$$(15.7150, 163.3858) \text{ and } (18.1581, 221.8032)$$

with lengths 147.67 and 203.65. This indicates that in this example the approximate Weibull tolerance interval is significantly shorter than the exact log-normal tolerance interval.

4. Approximate Tolerance Intervals for Extreme Value Distribution

The next application of approximate tolerance interval is considered the extreme value distribution. This distribution is another example widely used in lifetime data and reliability analyses which has a pdf

$$f(x, \mu, b) = \frac{1}{b} e^{\frac{x-\mu}{b} - e^{\frac{x-\mu}{b}}}, x \in R \qquad (4.1)$$

for some $b > 0$ and $\mu \in R$. The αth quantile function for this distribution is $F^{-1}(\alpha) = \mu + b \ log(-log(1-\alpha))$.

Let's denote $s_{ext}^2(\gamma) = b^2(-\frac{\pi}{6}+\frac{\pi^2}{3})^{-1}[1-\frac{\pi}{6}+\frac{\pi^2}{3}-2\delta+\delta^2-2log(-\log(\gamma))+2\delta log(-\log(\gamma))+[log(-\log(\gamma))]^2]$. With the proof presented in the Appendix, we display its corresponding approximate tolerance interval in the following theorem.

Theorem 4.1. Suppose that we have a random sample drawn from the extreme value distribution of (4.1). Then, with maximum likelihood estimators $\hat{\mu}$ and $\hat{b}$,

$$(\hat{\mu} + \hat{b}\ log(-log(\frac{1+\gamma}{2})) - z_{1-\frac{\alpha}{2}}\frac{s_{ext}((1+\gamma)/2)}{\sqrt{n}},$$
$$\hat{\mu} + \hat{b}\ log(-log(\frac{1-\gamma}{2})) + z_{1-\frac{\alpha}{2}}\frac{s_{ext}((1-\gamma)/2)}{\sqrt{n}}) \quad (4.2)$$

is an approximate γ-content tolerance interval with coefficient $1-\alpha$ and

$$(0, \hat{\mu} + \hat{b}\ log(-log(1-\gamma)) + z_{1-\alpha}\frac{s_{ext}(1-\gamma)}{\sqrt{n}})$$

and

$$(\hat{\mu} + \hat{b}\ log(-log(\gamma)) - z_{1-\alpha}\frac{s_{ext}(\gamma)}{\sqrt{n}}, \infty) \quad (4.3)$$

are respectively approximate one sided γ-content tolerance intervals with coefficient $1-\alpha$.

Consider the analysis of a failure voltage for a type of cable. Data for voltage levels at which failures occurred in two types of electrical cable insulation when specimens were subjected to an increasing voltage stress in a laboratory test may be seen in Lawless (2003, p240). Engineering experience (Stone and Lawless (1979)) suggests that the log failure voltage of cable are adequately represented by extreme valued distribution.

We display the empirical survival function of (3.5), extreme value tolerance limit curve of (4.3) and exponential tolerance limit curve of (3.4) in Figure 3.

Figure 3 here

This data analysis reveals that an extreme value tolerance interval performs better in analyzing this data set than the exponential distribution since the exponential tolerance limit curve is too low on the left part and too high on the right part.

We now further dispaly the two sided empirical distribution function of (3.7), two sided extreme value tolerance limit functions of (4.2) and log-normal tolerance limit functions of (3.6) in Figure 4.

Figure 4 here

Although the log-normal is popularly used in analyze the lifetime data, however, in this case, the extreme value tolerance interval also performs better than the log-normal tolerance interval.

Two real data analyses in two sections confirm the effectiveness of using approximate tolerance intervals because that uses more general distributions.

5. Approximate Regression Tolerance Interval

A series of articles by Goodman and Madansky (1962), Liman and Thomas (1988) and Mee, Eberhardt and Reeve (1991) deals with tolerance intervals for regression with normal errors model. However, we can sometimes only make minimal assumptions on the shape of the family of distributions generating the regression data. Hence in this situation, we need to consider a nonparametric technique to develop regression tolerance intervals. In this situation, we have to resort to approximations. The quantile approach in regression will solve our problem.

Suppose that we have a linear regression model

$$y_i = x_i'\beta + \epsilon_i, i = 1, ..., n \tag{5.1}$$

where, for each i, x_i is a known design p-vector with value 1 in its first element and $\epsilon_i, i = 1, ..., n$ are independent and identically distributed error variables with distribution function F. Let x_0 be a known vector. The interest is to infer a random interval that includes at least a certain percentage of distribution of future response variable y_0 with confidence $1-\alpha$. The αth conditional quantile of the variable y_0 given x_0 is $x_0'\beta + F^{-1}(\alpha), 0 < \alpha < 1$ which can be expressed as $x_0'\beta(\alpha)$ with $\beta(\alpha) = \beta + \begin{pmatrix} F^{-1}(\alpha) \\ 0 \end{pmatrix}$, where 0 is the $(p-1)$-vector of zeros and $\beta(\alpha)$ is called the population regression quantile. A nonparametric method for developing regression tolerance interval is through a consistent estimator of population quantile.

Theorem 5.1. Let regression quantile estimators $\hat{\beta}(\frac{1-\gamma}{2})$ and $\hat{\beta}(\frac{1+\gamma}{2})$ be asymptotically normal. Suppose that there are standard errors s_a and s_b so that $n^{1/2}(x_0'\hat{\beta}(\frac{1-\gamma}{2}) - x_0'\beta(\frac{1-\gamma}{2}))/s_a \xrightarrow{d} N(0,1)$ and $n^{1/2}(x_0'\hat{\beta}(\frac{1+\gamma}{2}) -$

$x_0'\beta(\frac{1+\gamma}{2}))/s_b \xrightarrow{d} N(0,1)$. Then

$$\left[x_0'\hat{\beta}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}},\ x_0'\hat{\beta}(\frac{1+\gamma}{2}) + z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}}\right]$$

is an approximate two sided regression γ-content tolerance interval with confidence $1-\alpha$.

Proof.

$$1 - P\{\frac{x_0'\hat{\beta}(\frac{1-\gamma}{2}) - x_0'\beta(\frac{1-\gamma}{2})}{s_a/\sqrt{n}} \geq z_{1-\frac{\alpha}{2}}\} - P\{\frac{x_0'\hat{\beta}(\frac{1+\gamma}{2}) - x_0'\beta(\frac{1+\gamma}{2})}{s_b/\sqrt{n}} \leq -z_{1-\frac{\alpha}{2}}\}$$

$$= 1 - P\{x_0'\hat{\beta}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}} \geq x_0'\beta(\frac{1-\gamma}{2})\} - P\{x_0'\hat{\beta}(\frac{1+\gamma}{2})$$

$$+ z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}} \leq x_0'\beta(\frac{1+\gamma}{2})\}$$

$$\leq 1 - P\{x_0'\hat{\beta}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}} \geq x_0'\beta(\frac{1-\gamma}{2}) \text{ or } x_0'\hat{\beta}(\frac{1+\gamma}{2})$$

$$+ z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}} \leq x_0'\beta(\frac{1+\gamma}{2})\}$$

$$= P\{x_0'\hat{\beta}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}} \leq x_0'\beta(\frac{1-\gamma}{2}) \text{ and } x_0'\beta(\frac{1+\gamma}{2}) \leq$$

$$x_0'\hat{\beta}(\frac{1+\gamma}{2}) + z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}}\}$$

so the theorem is followed from Theorem 2.1 and the following

$$1-\alpha \leq \lim_{n\to\infty} P\{x_0'\hat{\beta}(\frac{1-\gamma}{2}) - z_{1-\frac{\alpha}{2}}\frac{s_a}{\sqrt{n}} \leq x_0'\beta(\frac{1-\gamma}{2}) \text{ and}$$

$$x_0'\beta(\frac{1+\gamma}{2}) \leq x_0'\hat{\beta}(\frac{1+\gamma}{2}) + z_{1-\frac{\alpha}{2}}\frac{s_b}{\sqrt{n}}\}. \quad \Box$$

Although there are several ways to construct consistent estimators of population regression quantile $\beta(\alpha)$ (see Koenker and Bassett (1978), Ruppert and Carroll (1980) and Chen and Chiang (1996)), however, the most popular method is that developed by Koenker and Bassett defining regression quantile $\hat{\beta}(\alpha)$ as the solution for the following minimization problem

$$\min_{b\in R^p}\sum_{i=1}^{n}\rho_\alpha(y_i - x_i'b),$$

where $\rho_\alpha(u) = u\psi_\alpha(u)$, $\psi_\alpha(u) = \alpha - I(u < 0)$ with $I(A)$ the indicator function of the event A.

This regression quantile, besides its popularity, it has been considered as the most natural extension of a sample quantile since it satisfies several properties of equivariance in location, scale and reparameterization of design. Furthermore, this regression quantile has been widely used to construct robust estimators; see, for example, Ruppert and Carroll (1980), Koenker and Portnoy (1987) and Chen and Portnoy (1996). Under some standard assumptions (see Ruppert and Carroll (1980) and Chen and Portnoy (1996)), the regression quantile $\sqrt{n}(\hat{\beta}(\alpha) - \beta(\alpha))$ has an asymptotic normal distribution with mean $\beta(\alpha)$ and covariance matrix $\alpha(1-\alpha)f^{-2}(F^{-1}(\alpha))Q^{-1}$ where $Q = \lim_{n\to\infty} n^{-1}X'X$. Further references such as Koenker and d'Orey (1987), Ruppert and Carroll (1980) and Lai, Chen and Chang (2004) may help in the estimation of regression quantiles and their standard errors for constructing regression tolerance interval. However, we leave it for further investigation to examine its performance.

6. Appendix

Proof of Theorem 3.1: The uth quantile of the Weibull distribution is $F^{-1}(u) = [-\lambda^{-1}log(1-u)]^{1/\beta}$. We have partial derivatives $\frac{\partial F^{-1}(u)}{\partial \begin{pmatrix}\lambda\\ \beta\end{pmatrix}'} = \frac{(-log(1-u))^{1/\beta}}{\beta\lambda^{1/\beta}}(\frac{-1}{\lambda}, -\frac{log(-log(1-u)/\lambda)}{\beta})$ and $\frac{\partial log\ f(t)}{\partial \begin{pmatrix}\lambda\\ \beta\end{pmatrix}} = \begin{bmatrix} \frac{1}{\lambda} - t^\beta \\ \frac{1}{\beta} + log(t) - \lambda(log(t))t^\beta \end{bmatrix}$.

With some calculations of integration, we may see that, with

$$c_{22} = \frac{1}{\beta^2}(1 + 5\delta^2 + \frac{5}{6}\pi^2 + 10\delta log(\lambda) + 5(log(\lambda))^2 - 10(\delta + log(\lambda)),$$

we have

$$I(\lambda, \beta) = E(\frac{\partial log\ f(t)}{\partial \begin{pmatrix}\lambda\\ \beta\end{pmatrix}}\frac{\partial log\ f(t)}{\partial \begin{pmatrix}\lambda\\ \beta\end{pmatrix}'}) = \begin{bmatrix} \frac{1}{\lambda^2} & \frac{1}{\lambda\beta}(1-\delta-log(\lambda)) \\ \frac{1}{\lambda\beta}(1-\delta-log(\lambda)) & c_{22} \end{bmatrix}.$$

This shows that, from (3.2),

$$\frac{\partial F^{-1}(\frac{1-\gamma}{2})}{\partial \begin{pmatrix} \lambda \\ \beta \end{pmatrix}'} I^{-1}(\lambda,\beta) \frac{\partial F^{-1}(\frac{1-\gamma}{2})}{\partial \begin{pmatrix} \lambda \\ \beta \end{pmatrix}} = s^2_{wei}(\frac{1+\gamma}{2})$$

$$\text{and } \frac{\partial F^{-1}(\frac{1+\gamma}{2})}{\partial \begin{pmatrix} \lambda \\ \beta \end{pmatrix}'} I^{-1}(\lambda,\beta) \frac{\partial F^{-1}(\frac{1+\gamma}{2})}{\partial \begin{pmatrix} \lambda \\ \beta \end{pmatrix}} = s^2_{wei}(\frac{1-\gamma}{2}).$$

The theorem is followed from Theorem 2.3.

Proof of Theorem 4.1:

We consider the γ-content coverage interval $C(\gamma) = (F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2}))$ with

$$F^{-1}(\frac{1-\gamma}{2}) = \mu + b\,log(-log(\frac{1+\gamma}{2})) \text{ and } F^{-1}(\frac{1+\gamma}{2}) = \mu + b\,log(-log(\frac{1-\gamma}{2})).$$

This indicates

$$\frac{\partial F^{-1}(\frac{1-\gamma}{2})}{\partial \begin{pmatrix} \mu \\ b \end{pmatrix}} = \begin{pmatrix} 1 \\ log(-log(\frac{1+\gamma}{2})) \end{pmatrix} \text{ and } \frac{\partial F^{-1}(\frac{1+\gamma}{2})}{\partial \begin{pmatrix} \mu \\ b \end{pmatrix}} = \begin{pmatrix} 1 \\ log(-log(\frac{1-\gamma}{2})) \end{pmatrix}.$$

We also have that the Fisher information matrix is

$$I(\mu,b) = \frac{1}{b^2}\begin{bmatrix} 1 & 1-\delta \\ 1-\delta & \frac{\pi^2}{3} - \frac{\pi}{6} + \delta^2 - 2\delta + 1 \end{bmatrix}.$$

We denote $s^2_{ext}(\gamma) = b^2(-\frac{\pi}{6} + \frac{\pi^2}{3})^{-1}[1 - \frac{\pi}{6} + \frac{\pi^2}{3} - 2\delta + \delta^2 - 2log(-\log(\gamma)) + 2\delta log(-\log(\gamma)) + [log(-\log(\gamma))]^2]$. We then see that

$$\frac{\partial F^{-1}(\frac{1-\gamma}{2})}{\partial \begin{pmatrix} \lambda \\ \beta \end{pmatrix}'} I^{-1}(\mu,b) \frac{\partial F^{-1}(\frac{1-\gamma}{2})}{\partial \begin{pmatrix} \lambda \\ \beta \end{pmatrix}} = s^2_{ext}(\frac{1+\gamma}{2})$$

$$\text{and } \frac{\partial F^{-1}(\frac{1+\gamma}{2})}{\partial \begin{pmatrix} \lambda \\ \beta \end{pmatrix}'} I^{-1}(\mu,b) \frac{\partial F^{-1}(\frac{1+\gamma}{2})}{\partial \begin{pmatrix} \lambda \\ \beta \end{pmatrix}} = s^2_{ext}(\frac{1-\gamma}{2}).$$

The theorem is followed from Theorem 2.3.

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Figure 1. Lower Weibull and exponential tolerance intervals for ball bearing data

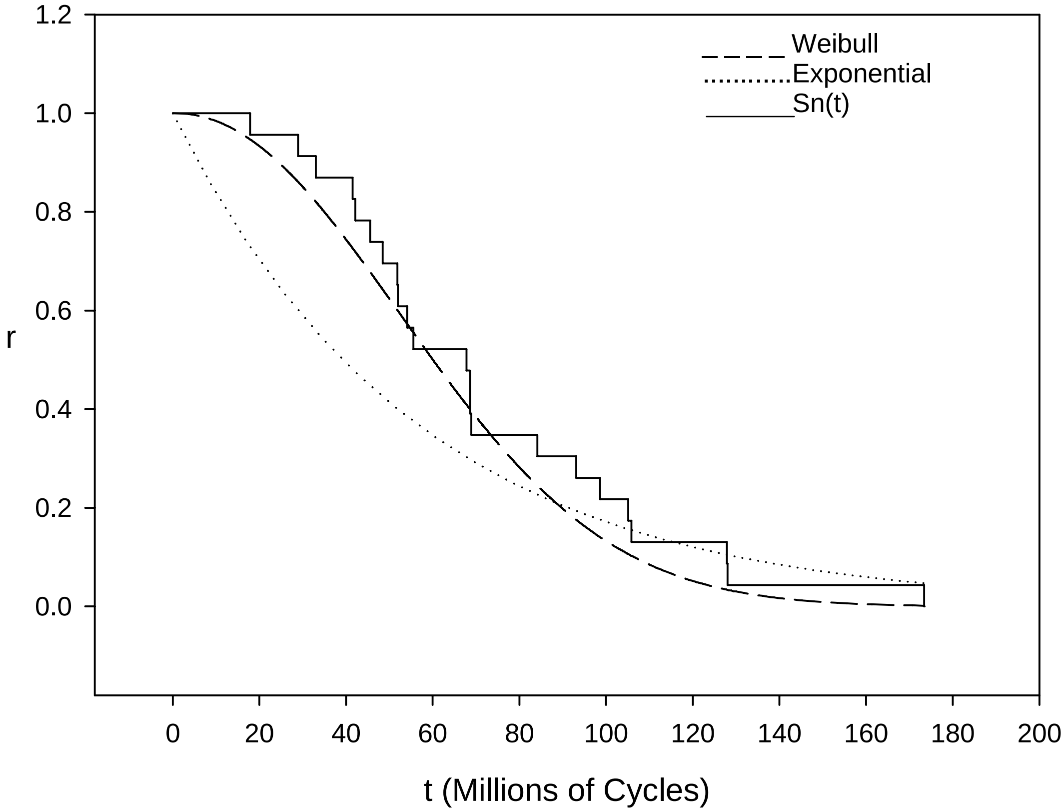

Figure 2. Two sided Weibull and Log-normal tolerance intervals for ball bearing data

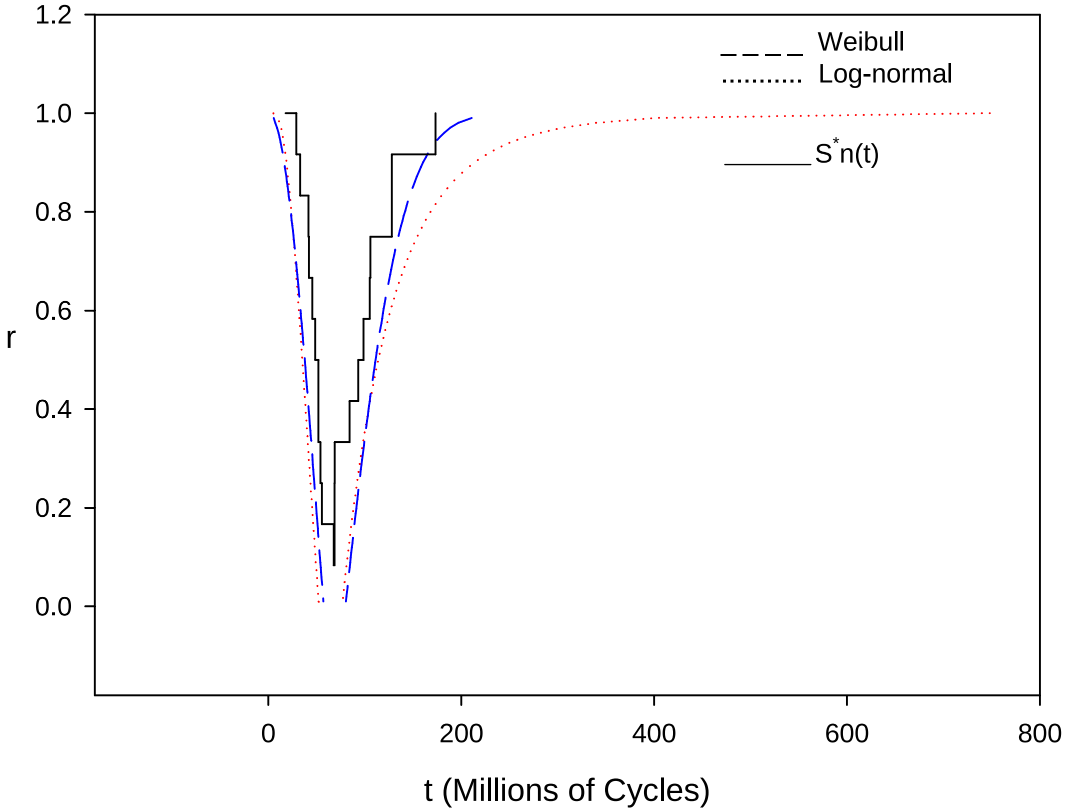

Figure 3. Lower extreme-value and exponential tolerance intervals for electrical cable insulation data

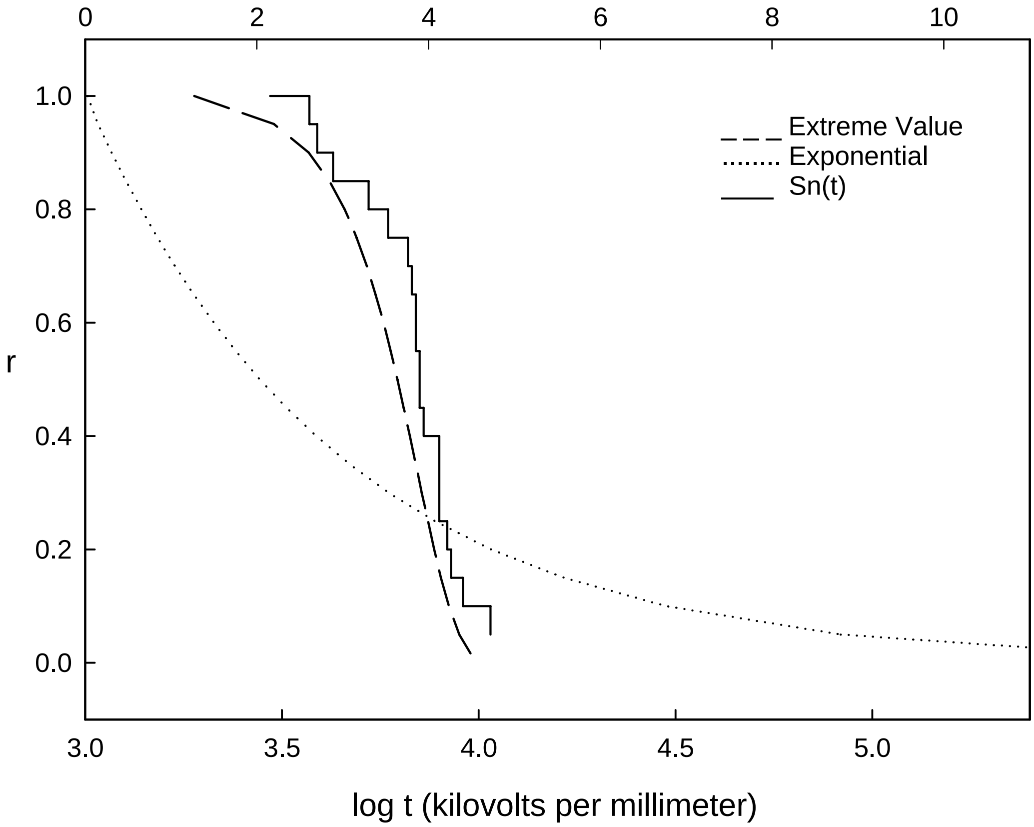

Figure 4. Two sided Extreme Value and Log-normal tolerance intervals for electrical cable insulation data