國 立 交 通 大 學

統計學研究所

碩 士 論 文

兩個逆高斯分配的平均值及尺度參數之廣義推論

Generalized inferences on the means and scales of two independent Inverse Gaussian populations

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摘要

逆高斯分配在分析全為正數且右偏的數據時是一個很好的模型，因此在統計應用上很受重視。過去的研究中，在不假設干擾參數相等的情況下比較兩個母體的平均值及尺度參數的推論還需要我們繼續研究。因此在本論文中，我們利用廣義p值法對於一個及兩個逆高斯分配母體參數尋求精確的檢定方法，並提出一個在使用上比過去更為便利的方法，而這個方法是建立在廣義方法的觀念上，解決了在使用過去文獻中檢定兩個母體平均值的比例和計算信賴區間時會遇到的困難，也就是統計量中包含了干擾參數的問題，而且我們也得到了確切的解。藉由實際數據的分析，我們發現我們的方法跟過去的方法比較起來可以得到長度最短或是很接近最短的信賴區間長度。並且在模擬的研究中，我們可以看出我們的方法所得出的覆蓋率與型I誤差都很接近我們所設定的水準。

關鍵字：覆蓋率；期望長度；廣義信賴區間；廣義p值；型I誤差

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ABSTRACT

The *IG* distribution has gotten intensive attentions in statistical application fields by reason of it is an ideal candidate for modeling positive, right-skewed data. The classical procedures have difficulties in analysis non-homogeneous *IG* data. Hence, the exact inferences on making inferences for two *IG* means and scales deserve further research. In this thesis, we present a convenient approach based on the generalized *p*-value and generalized confidence methods to perform the hypothesis testing and confidence intervals for mean and scale of one *IG* population as well as the ratio of means and scales of two independent *IG* populations. Illustrative examples show that the confidence lengths obtained by the generalized methods are the smallest or close to the smallest length. Furthermore, the simulation studies show that our coverage probabilities and type I error are very close to the nominal levels.

Keywords: Coverage probability; Expected length; Generalized confidence; Generalized *p*-value; Type I error

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Contents

Chapter 1 Introduction

For the last three decades, the inverse Gaussian (*IG*) distribution has gained tremendous attention in describing and analyzing right skewed data. *IG* distribution can accommodate data with a variety of shapes from highly skewed to almost normal and it is known that most of the data from applied fields are often positive and right-skewed, that is why *IG* distribution has gotten intensive attentions in statistical application fields. In 1915 Schrödinger introduced the probability distribution of the first passage time in Brownian motion, but we still are unaware of other references to the distribution until Tweedie (1945) proposed the name, *IG* distribution, for the first passage time distribution. Next, Wald (1947) derived the distribution as a limiting form for the distribution of sample size in a sequential probability ratio test. Because of this derivation, the distribution is also known as Wald's distribution, particularly in the Russian literature.

In many areas of statistical applications, handling of skewed data is by no means an exception but a fact of life. Hence if possible, it is desirable to analyze the data as observed using statistical methods based on skewed distributions. However, standard statistical methods for the normal distribution are commonly used for the data analysis. This is primarily due to lack of alternative methods that are easily available and also easy to understand. Although Gamma, Weibull, and lognormal distributions enjoy extensive use in certain special areas, none of them allow for a wide range of statistical methods comparable to those based on the normal distribution. Comparatively, *IG* can accommodate a variety of shapes from highly skewed to almost normal. See Chhikara and Folks (1989), Seshadri (1993, 1999) for more details of *IG* distribution analogies.

As the *IG* mean is inversely proportional to the drift of Brownian motion or the growth rate in Weiner process, it would be of some interest to compare two *IG* means if a comparison in the associated processes is desired. Chhikara (1975, 1989) derived UMP-unbiased tests for the equality of two inverse Gaussian population means, say $\mu_1 - \mu_2$, and constructed the confidence interval for the ratio of two means under the identical shape parameter λ assumption. However, the situation that two *IG* populations have the identical shape parameter does not always happen. Afterward, Tian and Wilding (2005) adopt the directed likelihood ratio and modified directed likelihood ratio method (Barndorff-Nielsen, 1986) to provide an approximate approach for constructing a confidence interval of μ_1/μ_2 of two *IG* populations. Even so, exact inferences on the ratio of two *IG* populations' means when the scale parameters λ_1 and λ_2 are unknown and possible unequal still need to explore. Therefore in this thesis, we would like to propose exact inferences on μ_1/μ_2 without making identical shape parameter assumption. We will develop significant tests and confidence intervals for the general cases without the assumption of equal scale parameters based on the concepts of generalized *p*-values and generalized confidence intervals. The concepts of the generalized *p*-value and generalized confidence interval were introduced by Tsui and Weerahandi (1989) and Weerahandi (1993), respectively, to solve many statistical problems involving nuisance parameters. Typically, the generalized *p*-value and the generalized confidence interval were found to be fruitful for problems where conventional frequentist procedures were non-existent or were difficult to obtain (see the book by Weerahandi (1995) for a detailed discussion). The lack of exact confidence intervals in many applications can be attributed to the statistical problems involving nuisance parameters. Therefore, for these reasons, we will use the idea of a generalized *p*-values approach to construct a pivotal variable, so it can be used for both hypothesis testing and confidence region.

The rest of the thesis is organized as follows. In chapter 2, the properties of inverse Gaussian distribution and the concept of generalized *p*-value and generalized confidence interval is reviewed. For one *IG* population, our procedures and Chhikara and Folks' (1976) methods for hypothesis testing and constructing the generalized confidence intervals about μ and λ are introduced in chapter 3. In chapter 4, we will present our procedures for hypothesis testing and constructing the generalized confidence intervals about $\frac{\mu_1}{\mu_2}$ and $\frac{\lambda_1}{\lambda_2}$ for two independent *IG* populations. The methods presented by Chhikara and Folks (1975) and Tian and Wilding (2005) will be addressed in this chapter as well. We apply these results to four sets of data, and compare our procedure with other methods with respect to their confidence intervals and confidence widths in chapter 5. Three sets of simulation studies are also presented in chapter 5 to compare the coverage probabilities, expected lengths, type I error and power performances of these methods. Concluding remarks are summarized in chapter 6.

Chapter 2 Properties of IG Distribution and the Generalized Methods

In this chapter we provide some of the properties that play a significant role in the development of statistical methods for the inverse Gaussian distribution and then briefly introduce the theories of the generalized *p*-value and the general confidence interval.

2.1 Properties of Inverse Gaussian distribution

The probability density function of a random variable X distributed as inverse Gaussian with parameters μ and λ, denoted by $X \sim IG(\mu,\lambda)$, is given by

$$f(x;\mu,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} x^{-\frac{3}{2}} \exp\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\},\quad x>0, \tag{2.1}$$

where $\mu>0$ and $\lambda>0$. And the inverse Gaussian distribution function $F(x)$ in terms of the normal distribution function, $\Phi(x)$, is given by

$$F(x)=\Phi\left[\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right]+e^{2\lambda/\mu}\Phi\left[-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}+1\right)\right]. \tag{2.2}$$

The parameter μ is the mean of $IG(\mu,\lambda)$ and λ is the scale parameter. When $\mu=1$, the distribution is often referred to as the standard Wald's distribution.

If $X \sim IG(\mu,\lambda)$, then the characteristic function, denoted by $C_X(t)$, is given by

$$C_X(t) = \exp\{\frac{\lambda}{\mu}[1-(1-\frac{2it\mu^2}{\lambda})^{\frac{1}{2}}]\}. \tag{2.3}$$

Suppose that all positive and negative moments exist, the moment generating function is

$$E(X^r) = \mu^r \sum_{s=0}^{r-1} \frac{(r-1+s)!}{s!(r-1-s)!} \left(2\frac{\lambda}{\mu}\right)^{-s}, \tag{2.4}$$

which can be obtained by taking the rth derivative of $C_X(t)$ and evaluating it at $t=0$. Thus the mean and variance of $IG(\mu,\lambda)$ can be derived as $E(X)=\mu$ and $Var(X)=\mu^3/\lambda$, respectively, through (2.4).

If $X_1, X_2, ..., X_n$ is a random sample from $IG(\mu,\lambda)$, then $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $V = \sum_{i=1}^{n}(X_i^{-1} - \bar{X}^{-1})/n$ are the maximum likelihood estimates of μ and λ^{-1}, respectively, with

$$\bar{X} \sim IG(\mu, n\lambda) \text{ and } n\lambda V \sim \chi^2_{n-1}, \tag{2.5}$$

where $IG(\mu,n\lambda)$ is the inverse Gaussian distribution and χ^2_{n-1} is the chi-square distribution with n-1 degrees of freedom. We can show that both of them are statistically independent. The density function (2.1) is seen to be a member of the exponential family, and $(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n}\frac{1}{X_i})$ is a complete sufficient statistic for inverse

Gaussian distribution.

By a simple characteristic function argument, it can be seen that if $X \sim IG(\mu, \lambda)$ then $cX \sim IG(c\mu, c\lambda)$ for $c > 0$. So the family of inverse Gaussian distributions is closed under a change of scale. Because of the normal analogy it would be natural to hope that any linear combination of inverse Gaussian variables would also be inverse Gaussian. Unfortunately, the property of reproducibility does not hold with respect to a change of location. Although this hope is not satisfied, Chhikara (1972) and Shuster and Miura (1972) have shown that under the necessary condition, $\lambda_i / \mu_i^2 = \xi$ for all i, inverse Gaussian variables do enjoy a certain additive property. That is if $X_i \sim IG(\mu_i, \lambda_i)$, $i = 1, 2, \ldots, n$, independently, such that $\lambda_i / \mu_i^2 = \xi$ for all i, then $\sum X_i \sim IG\left(\sum \mu_i, \xi\left(\sum \mu_i\right)^2\right)$.Therefore in order for the linear combination $\sum c_i X_i$ of independent inverse Gaussian variables to be inverse Gaussian, $\lambda_i / c_i \mu_i^2$ must be positive and constant, $i = 1, 2, \ldots, n$. Hence the additive property of the inverse Gaussian is restricted by a required relationship between the two parameters.

Furthermore, it is worth to notice that

$$\lambda (X - \mu)^2 / \mu^2 X \sim \chi_1^2 . \tag{2.6}$$

This useful property can be easily proved by finding the moment-generating function, and we will show how to use the statistic for our generalized method in next chapter.

2.2 Generalized *p*-value and generalized confidence interval

The concept of generalized *p*-value was first introduced by Tsui and Weerahandi (1989) to deal with the statistical testing problem in which nuisance parameters are present and it is difficult or impossible to obtain a nontrivial test with a fixed level of significance. The setup is as follows. Let $\mathbf{X}$ be a random quantity having a density function $f\left(\mathbf{X}\middle|\boldsymbol{\zeta}\right)$, where $\boldsymbol{\zeta}=(\theta,\boldsymbol{\eta})$ is a vector of unknown parameters, θ is the parameter of interest, and $\boldsymbol{\eta}$ is a vector of nuisance parameters. Suppose we are interested in testing the null hypothesis

$$H_0:\theta\le\theta_0 \text{ versus } H_1:\theta>\theta_0, \tag{2.7}$$

where θ_0 is a specified value.

Let $\boldsymbol{x}$ denote the observed value of $\mathbf{X}$ and consider the generalized test variable $R\left(\mathbf{X};\boldsymbol{x},\boldsymbol{\zeta}\right)$, which depends on the observed value $\boldsymbol{x}$ and the parameters $\boldsymbol{\zeta}$, and satisfies the following requirements:

(i) $r_{obs}=R\left(\boldsymbol{x};\boldsymbol{x},\theta,\boldsymbol{\eta}\right)$ does not depend on unknown parameters.

(ii) For fixed $\boldsymbol{x}$ and $\boldsymbol{\zeta}=(\theta,\boldsymbol{\eta})$, the distribution of $R\left(\mathbf{X};\boldsymbol{x},\boldsymbol{\zeta}\right)$ is independent of the nuisance parameters $\boldsymbol{\eta}$.

(iii)For fixed $\boldsymbol{x}$ and $\boldsymbol{\eta}$, $P\left(R\left(\mathbf{X};\boldsymbol{x},\zeta\right)\geq r\middle|\theta\right)$ is either increasing or decreasing in θ for any given r. (2.8)

Under the above conditions, if $R\left(\mathbf{X};\boldsymbol{x},\zeta\right)$ is stochastically increasing in θ, then the generalized p-values for testing the hypothesis in (2.8) can be defined as

$$p=\sup_{\theta\leq\theta_0}P\left\{R\left(\mathbf{X};\,\boldsymbol{x},\,\theta,\,\boldsymbol{\eta}\right)\geq r_{obs}\right\}=P\left\{R\left(\mathbf{X};\,\boldsymbol{x},\,\theta_0,\,\boldsymbol{\eta}\right)\geq r_{obs}\right\} \tag{2.9}$$

where $r_{obs}=R\left(\boldsymbol{x};\,\boldsymbol{x},\,\theta_0,\,\boldsymbol{\eta}\right)$.

In the same setup, suppose $R^*\left(\mathbf{X};\,\boldsymbol{x},\,\theta,\,\boldsymbol{\eta}\right)$ satisfies the following conditions:

(i) For any fixed $\boldsymbol{x}$, R^* has a probability distribution free of unknown parameters.

(ii) If $\mathbf{X}=\boldsymbol{x}$, then $r^*_{obs}=R^*\left(\boldsymbol{x};\,\boldsymbol{x},\,\theta,\,\boldsymbol{\eta}\right)$ does not depend on $\boldsymbol{\eta}$, the vector of nuisance parameters. (2.10)

Then, we say $R^*\left(\mathbf{X};\,\boldsymbol{x},\,\theta,\,\boldsymbol{\eta}\right)$ is a generalized pivotal quantity. If r_1 and r_2 are such that

$$P\left\{r_1\leq R^*\left(\mathbf{X};\,\boldsymbol{x},\,\theta,\,\boldsymbol{\eta}\right)\leq r_2\right\}=1-\alpha\,, \tag{2.11}$$

then, $\left\{\theta:r_1\leq R^*\left(\boldsymbol{x};\,\boldsymbol{x},\,\theta,\,\boldsymbol{\eta}\right)\leq r_2\right\}$ is a $100\left(1-\alpha\right)\%$ generalized confidence interval for θ. Following that, $\left\{R^*\left(\boldsymbol{x};\;\alpha/2\right),R^*\left(\boldsymbol{x};\,1\text{-}\alpha/2\right)\right\}$ is a $\left(1-\alpha\right)$ confidence interval for θ, where $R^*\left(\boldsymbol{x};\,\gamma\right)$ stands for the γth quantile of $R^*\left(\mathbf{X};\,\boldsymbol{x},\,\theta,\,\boldsymbol{\eta}\right)$.

For further derails and for several applications based on the generalized p-value, we refer to the book by Weerahandi (1995).

Chapter 3 Inferences on one population of Inverse Gaussian

In this chapter, we will provide inference on parameters μ and λ of inverse Gaussian distribution based on a generalized test variable and generalized pivotal quantity. In addition, Chhikara and Folks's (1989) method will be briefly introduced in this chapter as well.

3.1 Methods based on the generalized test variable and generalized pivotal quantity

3.1.1 Inferences on μ

Suppose $X_1, X_2, ..., X_n$ is a random sample from $IG(\mu, \lambda)$, where μ and λ are unknown. The sufficient statistics

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim IG(\mu, n\lambda) \quad \text{and} \quad V = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right) \sim \frac{\chi^2_{n-1}}{n\lambda} \tag{3.1}$$

are independent.

Consider the problem of significance testing of hypotheses

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0 \tag{3.2}$$

when λ is unknown. Since a generalized test variable can be a function of all unknown parameters, we can construct the random variable $R(\bar{X}, V; \bar{x}, v, \mu, \lambda)$ based on the random independent quantities

$$B = n\lambda V \sim \chi^2_{n-1} \text{ and } U = \frac{n\lambda(\bar{X}-\mu)^2}{\mu^2 \bar{X}} \sim \chi^2_1 \tag{3.3}$$

as mentioned in (2.5) and (2.6), respectively. For facilitation, we define U as

$$U = \frac{n\lambda(\bar{X}-\mu)^2}{\mu^2 \bar{X}} = \frac{n\lambda}{\bar{X}}\left(\frac{\bar{X}}{\mu}-1\right)^2, \tag{3.4}$$

which is chi-square distribution with 1 degree of freedom, then the generalized test variable for testing (3.2) can be deduced as following equation:

$$\begin{aligned} R &= \frac{n\lambda}{\bar{X}}\left(\frac{\bar{X}}{\mu}-1\right)^2 \times \frac{\bar{x}v}{n\lambda V} \\ &= U \times \frac{\bar{x}v}{B} \\ &= F \times \frac{\bar{x}v}{n-1}, \end{aligned} \tag{3.5}$$

where $\bar{x}$ and v are the observed values of $\bar{X}$ and V, respectively, and $U \sim \chi^2_1$, $B \sim \chi^2_{n-1}$ and $F \sim F_{1,n-1}$, the Snedecor's F distribution with 1 and n-1 degrees of freedom.

It is noted that the distribution of $R(\bar{X}, V; \bar{x}, v, \mu, \lambda)$ is free of the nuisance parameter λ, and the observed value $r_{obs} = R(\bar{x}, v; \bar{x}, v, \mu, \lambda) = \left(\frac{\bar{x}}{\mu}-1\right)^2$ is not dependent on λ. Besides, for fixed $\bar{x}, v$ and λ, $P\left[R\left(\bar{X}, V; \bar{x}, v, \mu, \lambda\right) \geq r\right]$ is increasing in μ. Therefore, R satisfies the three conditions in (2.8), R is a generalized test variable which can be applied for testing the null hypothesis $H_0: \mu = \mu_0$ versus $H_1: \mu \neq \mu_0$. The generalized *p*-value can be computed by

$$p = P\left[R\left(\bar{X}, V; \bar{x}, v, \mu_0, \lambda\right) \geq R\left(\bar{x}, v; \bar{x}, v, \mu_0, \lambda\right)\right]$$

$$= P\left[F_{1,n-1} \times \frac{\bar{x}v}{n-1} \geq \left(\frac{\bar{x}}{\mu_0} - 1\right)^2\right], \tag{3.6}$$

where R is defined as (3.5), $F_{1,n-1}$ is the Snedecor's F distribution with 1 and *n*-1 degrees of freedom, and H_0 is rejected when $p < \alpha$.

A generalized pivotal quantity in interval estimation can be treated as a counterpart of generalized test variable in significance testing of hypotheses. Because the distribution of $R\left(\bar{X}, V; \bar{x}, v, \mu, \lambda\right)$ does not depend on any unknown parameters and the observed value $r_{obs} = R\left(\bar{x}, v; \bar{x}, v, \mu, \lambda\right) = \left(\frac{\bar{x}}{\mu} - 1\right)^2$ does not depend on nuisance parameter λ, so R is indeed a generalized pivotal quantity satisfying the conditions in (2.10). Therefore, we can construct the $100(1-\alpha)\%$ confidence interval based on $R\left(\bar{X}, V; \bar{x}, v, \mu, \lambda\right)$.

Let $R(\bar{x}, v; \alpha)$ stand for the αth quantile of $R\left(\bar{X}, V; \bar{x}, v, \mu, \lambda\right)$ such that $P\left[R\left(\bar{X}, V; \bar{x}, v, \mu, \lambda\right) \leq R\left(\bar{x}, v; 1-\alpha\right)\right] = 1-\alpha$. Then

$$\left\{\mu : R\left(\bar{x}, v; \bar{x}, v, \mu, \lambda\right) \leq R\left(\bar{x}, v; 1-\alpha\right)\right\}$$

$$=\left\{\frac{\bar{x}}{1+\sqrt{\frac{\bar{x}v}{n-1}F_{1-\alpha}}}\le\mu\le\frac{\bar{x}}{1-\sqrt{\frac{\bar{x}v}{n-1}F_{1-\alpha}}}\right\} \tag{3.7}$$

is a $100(1-\alpha)\%$ generalized confidence interval of μ. For the fact that R is distributed as $\frac{\bar{x}v}{n-1}F$, where $F \sim F_{1,n-1}$ and F_α stands for the αth quantile of F distribution with 1, n-1 degrees of freedom.

Thus

$$\left(\frac{\bar{x}}{1+\sqrt{\frac{\bar{x}v}{n-1}F_{1-\alpha}}},\frac{\bar{x}}{1-\sqrt{\frac{\bar{x}v}{n-1}F_{1-\alpha}}}\right) \text{ if } 1-\sqrt{\frac{\bar{x}v}{n-1}F_{1-\alpha}}>0$$

$$\text{or } \left(\frac{\bar{x}}{1+\sqrt{\frac{\bar{x}v}{n-1}F_{1-\alpha}}},\infty\right) \text{ if } 1-\sqrt{\frac{\bar{x}v}{n-1}F_{1-\alpha}}<0 \tag{3.8}$$

is a $100(1-\alpha)\%$ confidence interval for μ.

3.1.2 Inferences on λ

Now consider the significance test of the hypothesis $H_0:\lambda=\lambda_0$ versus $H_1:\lambda\ne\lambda_0$ when μ is unknown. Since $\sum_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\bar{X}}\right)\sim\chi^2_{n-1}/\lambda$, we can construct the generalized test variable based on

$$W \equiv \lambda V \sim \chi^2_{n-1}, \tag{3.9}$$

with $V=\sum_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\bar{X}}\right)$ and χ^2_{n-1} is chi-square distribution with n-1 degrees of freedom. Therefore, the test variable, $T(V;v,\lambda)$ can be defined as

$$T(V;v,\lambda)=\frac{\lambda V}{v}=\frac{W}{v}. \tag{3.10}$$

For the fact that the distribution of random variable $T(V;v,\lambda)$ is free of nuisance parameter μ, the observed value $t_{obs}=T(v;v,\lambda)=\lambda$ is independent of μ, and $P[T\geq t]$ is non-increasing in λ for any given t, hence $T(V;v,\lambda)$ is a generalized test variable which satisfies the three conditions in (2.8). The generalized p-value for testing the null hypothesis $H_0:\lambda=\lambda_0$ versus $H_1:\lambda\neq\lambda_0$ can be obtained through

$$\begin{aligned} p&=2*\min\left\{P\left[T(V;v,\lambda_0)>T(v;v,\lambda_0)\right],P\left[T(V;v,\lambda_0)>T(v;v,\lambda_0)\right]\right\}\\ &=2*\min\left\{P\left[\frac{\chi^2_{n-1}}{v}>\lambda_0\right],P\left[\frac{\chi^2_{n-1}}{v}<\lambda_0\right]\right\}, \end{aligned} \tag{3.11}$$

where T is defined as (3.10), χ^2_{n-1} is chi-square distribution with n-1 degrees of freedom and H_0 is rejected when $p<\alpha$.

On the other hand, if we are interested in constructing confidence interval of λ, $T(V;v,\lambda)$ can be used as a generalized pivotal quantity. Because the observed value of $T(V;v,\lambda)$ is λ and $T(V;v,\lambda)$ satisfies the two conditions in (2.10), the $100(1-\alpha)\%$ equal tail confidence interval for λ is

$$\{T(v;\alpha/2),T(v;1-\alpha/2)\},$$

$$= \left\{ \frac{\chi^2_{\alpha/2}(n-1)}{v}, \frac{\chi^2_{1-\alpha/2}(n-1)}{v} \right\} \tag{3.12}$$

where $T(v;\gamma)$ stands for the γth quantile of $T(V;v,\lambda)$, and $\chi^2_\gamma(n-1)$ denotes the γth quantile of chi-square distribution with n-1 degrees of freedom.

3.2 Methods based on Chhikara and Folks (1989)

3.2.1 Inferences on μ

Suppose $\mathbf{X} = (X_1, X_2, ..., X_n)$ is from $IG(\mu,\lambda)$, the joint density function of $\mathbf{X}$ is

$$f(\mathbf{X};\mu,\lambda) = C(\theta,\psi)\left(\prod_{i=1}^n x_i^{-3/2}\right) \times \exp\left[\theta \sum_1^n x_i + \psi \sum_1^n \left(x_i + x_i^{-1}\right)\right],$$

where $\theta = \lambda\left(1-\mu^{-2}\right)/2$, $\psi = -\lambda/2$. Since $f(x;\mu,\lambda) = \mu^{-1} f(x/\mu;1,\lambda/\mu)$, without loss of generality, assume $\mu_0 = 1$, the hypothesis $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ when λ is unknown can equivalently be stated as follow:

$$H_0^{'} : \theta = 0 \quad \text{versus} \quad H_1^{'} : \theta \neq 0. \tag{3.13}$$

For a given level α and let $U = \sum_1^n X_i$, the UMP-unbiased critical region corresponds to $U < k_1$, or $U > k_2$, are determined by

$$\int_{k_1}^{k_2} h(u|s)du = 1-\alpha$$

and

$$\int_{k_1}^{k_2} uh(u|s)du = 1-\alpha\int_{-\infty}^{\infty} uh(u|s)du \tag{3.14}$$

where $h(u|s)$ denotes the conditional density function of U given s with

$$h(u|s) = \frac{n}{B[1/2,(n-1)/2]}\frac{1}{\sqrt{u^3(s-2n)}}\times\left[1-\frac{(u-n)^2}{u(s-2n)}\right]^{(n-3)/2},\ 0<\frac{(u-n)^2}{u(s-2n)}<1,$$

B is a Beta function and $s=\sum\left(x_i+x_i^{-1}\right)$.

Let

$$W = \frac{\sqrt{n-1}\left(\bar{X}-1\right)}{\sqrt{\bar{X}V}} \tag{3.15}$$

where $V=\sum_1^n\left(1/X_i-1/\bar{X}\right)/n$, the critical region, $U>k$, in (3.14) corresponds to $W>C$ where C is given by

$$H_{t,n-1}(-C)+\left(\frac{s+2n}{s-2n}\right)^{(n-2)/2} H_{t,n-1}\left(-\sqrt{4n+(s+2n)C^2}\right)=\alpha \tag{3.16}$$

and $H_{t,n-1}$ is the Student's *t* distribution function with *n*-1 degrees of freedom and $s=\sum\left(x_i+x_i^{-1}\right)$. In the two-sided case for testing $H_0:\mu=\mu_0$ versus $H_1:\mu\neq\mu_0$, λ unknown, a UMP-unbiased level α test is obtained by replacing X_i by X_i/μ_0, $i=1,2,...,n$, in (3.15), then the test statistics given by

$$\frac{\sqrt{n-1}\left(\bar{X}-\mu_0\right)}{\mu_0\sqrt{\bar{X}V}}. \tag{3.17}$$

Moreover, this critical region is

$$\left|\frac{\sqrt{n-1}\left(\bar{X}-\mu_0\right)}{\mu_0\sqrt{\bar{X}V}}\right| > t_{1-\alpha/2}, \tag{3.18}$$

where $t_{1-\alpha/2}$ is the $100\left(1-\alpha/2\right)$ percentage point of the Student's *t* distribution with *n*-1 degrees of freedom. (Chhikara and Folks, 1976)

It is interesting to note that the critical region in (3.18) is equivalent to $\left\{\frac{(n-1)}{\bar{X}V}\left(\frac{\bar{X}}{\mu_0}-1\right)^2 > F_{1,n-1,1-\alpha}\right\}$. The *p*-value is $p = P\left[F_{1,n-1} > \frac{(n-1)}{\bar{x}v}\left(\frac{\bar{x}}{\mu_0}-1\right)^2\right]$ which is the same as our result in (3.6). Thus we can conclude that our procedure is easily applicable.

On the other hand, according to Chhikara and Folks (1989), the confidence intervals for the parameter μ can be obtained by inverting the acceptance regions. Therefore, when λ is unknown, it follows from (3.18) that the $100\left(1-\alpha\right)$ percent confidence interval for μ is

$$\left(\bar{x}\left[1+\sqrt{\frac{\bar{x}v}{n-1}}t_{1-\alpha/2}\right]^{-1}, \bar{x}\left[1-\sqrt{\frac{\bar{x}v}{n-1}}t_{1-\alpha/2}\right]^{-1}\right), \text{ if } 1-\sqrt{\frac{\bar{x}v}{n-1}}t_{1-\alpha/2} > 0$$

$$\text{and } \left(\bar{x}\left[1+\sqrt{\frac{\bar{x}v}{n-1}}t_{1-\alpha/2}\right]^{-1}, \infty\right), \text{ otherwise.} \tag{3.19}$$

It can be also found that (3.19) is equivalent to our result in (3.8).

3.2.2 Inferences on λ

Roy and Wasan (1968) derived the UMP-unbiased test for $H_0 : \frac{1}{\lambda} = \frac{1}{\lambda_0}$ versus $H_1 : \frac{1}{\lambda} \neq \frac{1}{\lambda_0}$ when μ is unknown. The statistic $V = \sum_1^n \left(1/X_i - 1/\bar{X} \right)$ is distributed as χ^2_{n-1}/λ and the critical region given by $V \le k_1$ or $V \ge k_2$ corresponds to $\lambda V \le C_1$ or $\lambda V \ge C_2$, for a given level α of the test. C_1 and C_2 are determined by

$$\int_{C_1}^{C_2} g_{n-1}(t)dt = 1-\alpha \quad \text{and} \quad \int_{C_1}^{C_2} t g_{n-1}(t)dt = n(1-\alpha), \tag{3.20}$$

where $g_{n-1}(t)$ denotes the density function of χ^2_{n-1}. For the fact that $tg_{n-1}(t) = ng_{n+1}(t)$, (3.20) can be written as

$$F_{\chi^2_{n-1}}(C_2) - F_{\chi^2_{n-1}}(C_1) = F_{\chi^2_{n+1}}(C_2) - F_{\chi^2_{n+1}}(C_1) = 1-\alpha \tag{3.21}$$

where $F_{\chi^2_{n-1}}$ denotes the chi-square distribution function with n-1 degrees of freedom, and then C_1 and C_2 are uniquely determined from using tables of the chi-square distribution. Thus, for the equal tail test, C_1 and C_2 can be obtained by solving

$F_{\chi^2_{n-1}}(C_1) = 1 - F_{\chi^2_{n-1}}(C_2) = \frac{\alpha}{2}$. Hence $C_1 = \chi^2_{n-1,\alpha/2}$ and $C_2 = \chi^2_{n-1,1-\alpha/2}$, where $\chi^2_{n-1,\gamma}$ is the rth quantile of chi-square distribution with n-1 degrees of freedom. Therefore the p-value is $p = 2 * \min\left\{ P\left[\chi^2_{n-1} > \lambda_0 v\right], P\left[\chi^2_{n-1} < \lambda_0 v\right] \right\}$ and the $100(1-\alpha)\%$ confidence interval for λ is $\{\lambda : C_1 < \lambda v < C_2\}$ $= \left\{ \frac{\chi^2_{\alpha/2}(n-1)}{v} < \lambda < \frac{\chi^2_{1-\alpha/2}(n-1)}{v} \right\}$.

We note that these results are equivalent to our results in (3.11) and (3.12).

Chapter 4 Inferences on two populations of Inverse Gaussian

Although there has been a rapid growth in IG, the problem about making inference to the ratio of two IG means still need to be investigated. As the scale parameters λ_1 and λ_2 of two independent populations are the same, i.e. $\lambda_1 = \lambda_2$, the two-sided exact confidence interval of $\theta = \mu_1/\mu_2$ has been discussed by Chhikara and Folks (1989). However, it is not practical to expect two IG populations to have the identical scale parameter all the time. Recently, Tian and Wilding (2005) presented an approximate approach to construct the confidence interval of $\theta = \mu_1/\mu_2$ of two independent IG populations based on the modified directed likelihood ratio method (Barndorff-Nielsen, 1986). Nevertheless, the exact property of $\theta = \mu_1/\mu_2$ deserves further study. Therefore, in this chapter we will provide an exact and convenient method based on generalized p-value and generalized confidence interval to perform the hypothesis testing and then construct confidence intervals for $\theta = \mu_1/\mu_2$ and the ratio of two scale parameters $\delta = \lambda_1/\lambda_2$. In this chapter, we will also briefly introduce some methods in the literature which will be utilized to compare with our procedure in numerical examples and simulation studies.

4.1 Methods based on the generalized test variable and generalized pivotal quantity

4.1.1 Inferences on μ_1/μ_2

Let $X_{11}, X_{12}, \ldots, X_{1n_1}$ and $X_{21}, X_{22}, \ldots, X_{2n_2}$ be independent random samples from $IG(\mu_1, \lambda_1)$ and $IG(\mu_2, \lambda_2)$, respectively, where μ_i and λ_i are unknown and possible unequal with $i = 1, 2$. The independent sufficient statistics are given by

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij} \sim IG(\mu_i, n_i\lambda_i),\ V_i = \frac{1}{n_i}\sum_{j=1}^{n_i}\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i}\right) \sim \frac{\chi^2_{n_i-1}}{n_i\lambda_i},\ i = 1, 2. \tag{4.1}$$

Suppose we are interested in making inference in the parameter $\theta = \mu_1/\mu_2$, consider the following hypothesis testing:

$$H_0 : \frac{\mu_1}{\mu_2} = \theta_0 \text{ versus } H_1 : \frac{\mu_1}{\mu_2} \neq \theta_0,\ \theta_0 > 0, \tag{4.2}$$

when λ_i are unknown and possible unequal with $i = 1, 2$. Intuitively we may hope that the generalized test variable for two populations of inverse Gaussian is parallel that it for one population of IG in (3.5). In fact, the generalized test variable (3.5) and its observed value $r_{obs} = \left(\frac{\bar{x}}{\mu} - 1\right)^2$, can not be applied to the ratio μ_1/μ_2. Therefore we find another more flexible generalized test variable $G(\bar{X}_1, \bar{X}_2, V_1, V_2; \bar{x}_1, \bar{x}_2, v_1, v_2, \mu_1, \mu_2, \lambda_1, \lambda_2) \equiv G$ which is constructed by two independent statistics $G_1(\bar{X}_1, V_1; \bar{x}_1, v_1, \mu_1, \lambda_1) \equiv G_1$ and $G_2(\bar{X}_2, V_2; \bar{x}_2, v_2, \mu_2, \lambda_2) \equiv G_2$.

Above all we deliberate the statistic G_i based on the random independent quantities

$$B_i = n_i \lambda_i V_i \quad \text{and} \quad U_i = \frac{n_i \lambda_i \left(\bar{X}_i - \mu_i\right)^2}{\mu_i^2 \bar{X}_i} = \frac{n_i \lambda_i}{\bar{X}_i}\left(\frac{\bar{X}}{\mu_i} - 1\right)^2, \quad i = 1, 2, \tag{4.3}$$

which has been mentioned in (2.5) and (2.6), respectively. Since $B_i \sim \chi^2_{n_i - 1}$ and $U_i \sim \chi^2_1$, then one part of the generalized test variable for testing (4.2) can be deduced as following equation:

$$\begin{aligned}
G_i &= \bar{x}_i \left[\sqrt{\frac{n_i \lambda_i}{\bar{X}_i}} \left(\frac{\bar{X}_i}{\mu_i} - 1 \right) \times \sqrt{\frac{\bar{x}_i v_i}{n_i \lambda_i V_i}} + 1 \right]^{-1} \\
&= \bar{x}_i \left[\pm \sqrt{U_i} \times \sqrt{\frac{\bar{x}_i v_i}{B_i}} + 1 \right]^{-1} \\
&= \bar{x}_i \left[Z_i \times \sqrt{\frac{\bar{x}_i v_i}{B_i}} + 1 \right]^{-1} \quad (\text{ since } -Z_i = Z_i \text{ where } Z_i \sim N(0,1)) \\
&= \bar{x}_i \left[T_i \times \sqrt{\frac{\bar{x}_i v_i}{n_i - 1}} + 1 \right]^{-1} \qquad (4.4)
\end{aligned}$$

with $\bar{x}_i$ and v_i being the observed values of $\bar{X}_i$ and V_i, respectively, and $U_i \sim \chi^2_1$, $B_i \sim \chi^2_{n_i - 1}$ and $T_i \sim t_{n_i - 1}$, the Student's t distribution with $n_i - 1$ degrees of freedom for $i = 1, 2$. It is worthy to note that the observed value of G_i, $g_{i,obs}$, is μ_i which is the parameter we are interested in. There is no doubt that G_i can be also used as a generalized test variable in one population case and the result is equivalent to what we got in Chapter 3.

Eventually, since G_1 and G_2 are independent generalized test quantities, the generalized test variable G can be defined as follows:

$$G = G_1 / G_2$$

$$= \frac{\bar{x}_1 \left[\sqrt{\frac{n_1 \lambda_1}{\bar{X}_1}} \left(\frac{\bar{X}_1}{\mu_1} - 1 \right) \times \sqrt{\frac{\bar{x}_1 v_1}{n_1 \lambda_1 V_1}} + 1 \right]^{-1}}{\bar{x}_2 \left[\sqrt{\frac{n_2 \lambda_2}{\bar{X}_2}} \left(\frac{\bar{X}_2}{\mu_2} - 1 \right) \times \sqrt{\frac{\bar{x}_2 v_2}{n_2 \lambda_2 V_2}} + 1 \right]^{-1}}$$

$$= \frac{\bar{x}_1 \left[T_1 \times \sqrt{\frac{\bar{x}_1 v_1}{n_1 - 1}} + 1 \right]^{-1}}{\bar{x}_2 \left[T_2 \times \sqrt{\frac{\bar{x}_2 v_2}{n_2 - 1}} + 1 \right]^{-1}} \tag{4.5}$$

where $\bar{x}_1, \bar{x}_2, v_1, v_2$ are observed values of $\bar{X}_1, \bar{X}_2, V_1, V_2$, respectively, and $T_i \sim t_{n_i - 1}$, the Student's t distribution with $n_i - 1$ degrees of freedom for all *i*. It is noted that the distribution of G is independent of the nuisance parameters λ_1 or λ_2, and the observed value $g_{obs} = G(\bar{x}_1, \bar{x}_2, v_1, v_2; \bar{x}_1, \bar{x}_2, v_1, v_2, \mu_1, \mu_2, \lambda_1, \lambda_2) = \frac{\mu_1}{\mu_2}$ is free of λ_1 and λ_2. Besides, for fixed $\bar{x}_1, \bar{x}_2, v_1, v_2, \lambda_1, \lambda_2$ and given any g, $P[G \geq g]$ is monotonic in μ_1 / μ_2. Therefore, G satisfies the three conditions in (2.8), G is a generalized test variable which can be applied for testing the hypothesis $H_0 : \frac{\mu_1}{\mu_2} = \theta_0$ versus $H_1 : \frac{\mu_1}{\mu_2} \neq \theta_0$, $\theta_0 > 0$. The generalized *p*-value can be computed by

$$p = 2 * \min \left\{ P\left[G > g_{obs} \mid \theta = \theta_0 \right], P\left[G < g_{obs} \mid \theta = \theta_0 \right] \right\}$$

$$= 2 * \min\left\{ P\left[\frac{\bar{x}_1\left[T_1 \times \sqrt{\frac{\bar{x}_1 v_1}{n_1 - 1}} + 1\right]^{-1}}{\bar{x}_2\left[T_2 \times \sqrt{\frac{\bar{x}_2 v_2}{n_2 - 1}} + 1\right]^{-1}} > \theta_0 \right], P\left[\frac{\bar{x}_1\left[T_1 \times \sqrt{\frac{\bar{x}_1 v_1}{n_1 - 1}} + 1\right]^{-1}}{\bar{x}_2\left[T_2 \times \sqrt{\frac{\bar{x}_2 v_2}{n_2 - 1}} + 1\right]^{-1}} < \theta_0 \right] \right\} \quad (4.6)$$

where G is defined as (4.5), T_1 and T_2 are the Student's *t* distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom, respectively, and H_0 is rejected if $p < \alpha$.

We next consider the problem of interval estimation for μ_1/μ_2 based on generalized pivotal quantity. Since the observed value of G is μ_1/μ_2, the parameter of interest, and the properties of G fulfill the requirements in (2.10), thus G in (4.5) is indeed a generalized pivotal quantity which can be used to construct a generalized confidence interval. Therefore the $100(1-\alpha)\%$ equal tail confidence interval for μ_1/μ_2 can be computed by

$$\left\{ G_{\alpha/2} < \frac{\mu_1}{\mu_2} < G_{1-\alpha/2} \right\} \quad (4.7)$$

where G_γ stands for the γth quantile of G.

It is also noted that the statistics G_1 and G_2 can be utilized for testing the equality of μ_1 and μ_2 and constructing a confidence interval of $\mu_1 - \mu_2$ if it is necessary. Since the property of IG does not hold for the location change, it is hard to make inferences for the mean difference without any restriction. On the contrary, our procedure is readily applicable and easy to use to deal with mean difference problem without any restriction.

4.1.2 Inferences on λ_1/λ_2

It is also an interesting problem concerning the parameter $\delta = \lambda_1/\lambda_2$. Consider the hypothesis

$$H_0 : \frac{\lambda_1}{\lambda_2} = \delta_0 \quad \text{versus} \quad H_1 : \frac{\lambda_1}{\lambda_2} \neq \delta_0 , \tag{4.8}$$

when μ_1 and μ_2 are unknown and possible unequal. In this situation, the statistic $T(V;v,\lambda) = \frac{\lambda V}{v} = \frac{W}{v}$ with $V = \sum_{i=1}^{n}\left(X_i^{-1} - \bar{X}^{-1}\right)$ and $W \sim \chi^2_{n-1}$ in (3.10) which is employed in one population case can be applied to the two populations' case as well.

Similarly, since $V_i \equiv \sum_{j=1}^{n_i}\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i}\right)$ are sufficient statistics for λ_i and $V_i \sim \chi^2_{n_i-1}/\lambda_i$, $i = 1, 2$, we can construct the generalized test variable based on

$$W_i \equiv \lambda_i V_i \sim \chi^2_{n_i-1} \tag{4.9}$$

where $\chi^2_{n_i-1}$ is chi-square distribution with $n_i - 1$ degrees of freedom for $i = 1, 2$. Therefore the test variable $T^*(V_1, V_2; v_1, v_2, \delta)$ can be defined as $T^* = T_1/T_2$ with $T_i = \frac{\lambda_i V_i}{v_i} = \frac{W_i}{v_i}$, then

$$T^* = \frac{\lambda_1 V_1/v_1}{\lambda_2 V_2/v_2} = \frac{W_1/v_1}{W_2/v_2} = F\,\frac{(n_1-1)/v_1}{(n_2-1)/v_2} \tag{4.10}$$

where F denotes the F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom and v_1 and v_2 are the observed values of V_1 and V_2, respectively. For the fact that the distribution of random variable T^* is free of nuisance parameters, the observed value

$t_{obs}^* = T^*\left(v_1, v_2; v_1, v_2, \delta\right) = \delta$ is independent of nuisance parameters μ_1 and μ_2, and $P\left[T^* \ge t^*\right]$ is non-increasing in δ, hence $T^*\left(V_1, V_2; v_1, v_2, \delta\right)$ is a generalized test variable which satisfies the three conditions in (2.8). Therefore T^* is indeed a generalized test variable and can be used to test the hypothesis in (4.8). The generalized *p*-value for testing (4.8) can be computed by

$$p = 2*\min\left\{P\left[T^* > t_{obs}^* \,|\delta = \delta_0\right], P\left[T^* < t_{obs}^* \,|\delta = \delta_0\right]\right\}$$

$$= 2*\min\left\{P\left[\frac{W_1/v_1}{W_2/v_2} > \delta_0\right], P\left[\frac{W_1/v_1}{W_2/v_2} < \delta_0\right]\right\}$$

$$= 2*\min\left\{P\left[F\frac{(n_1-1)/v_1}{(n_2-1)/v_2} > \delta_0\right], P\left[F\frac{(n_1-1)/v_1}{(n_2-1)/v_2} < \delta_0\right]\right\}, \qquad (4.11)$$

where T^* is defined as (4.10), $W_1 \sim \chi^2_{n_1-1}$, $W_2 \sim \chi^2_{n_2-1}$ and H_0 is rejected when $p < \alpha$.

Furthermore, in order to construct confidence interval of λ_1/λ_2, $T^*\left(V_1, V_2; v_1, v_2, \delta\right)$ can be used as a generalized pivotal quantity as well. Because the observed value of T^* is λ_1/λ_2 and T^* satisfies the two conditions in (2.10), the $100\left(1-\alpha\right)\%$ equal tail confidence interval for λ_1/λ_2 is

$$\left\{T^*\left(v_1, v_2; \alpha/2\right), T^*\left(v_1, v_2; 1-\alpha/2\right)\right\}$$

$$= \left\{F_{\alpha/2}\frac{(n_1-1)/v_1}{(n_2-1)/v_2}, F_{1-\alpha/2}\frac{(n_1-1)/v_1}{(n_2-1)/v_2}\right\}, \qquad (4.12)$$

where $T^*(v_1, v_2; \gamma)$ stands for the γth quantile of $T^*(V_1, V_2; v_1, v_2, \psi)$ which is defined in (4.10).

4.2 Methods based on Chhikara and Folks (1989)

4.2.1 Inferences on μ_1/μ_2

Under the restriction of $\lambda_1 = \lambda_2 = \lambda$ and $\frac{\lambda_1}{\mu_1^2} = \frac{\lambda_2}{\mu_2^2} = \xi$, ξ is a constant, Chhikara and Folks (1989) derived a UMP-unbiased tests by constructing critical points of their rejection regions using percentage points of Student's t distribution.

For the significance size α test of $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$, $\lambda_1 = \lambda_2 = \lambda$ is unknown, the rejection region is given by

$$\left| \frac{\left[n_1 n_2 (n_1 + n_2 - 2)\right]^{1/2} (\bar{X}_1 - \bar{X}_2)}{\bar{X}_1 \bar{X}_2 (n_1 \bar{X}_1 + n_2 \bar{X}_2) \left[\sum_{j=1}^{n_1} (X_{1j}^{-1} - \bar{X}_1^{-1}) + \sum_{j=1}^{n_2} (X_{2j}^{-1} - \bar{X}_2^{-1}) \right]^{1/2}} \right| > t_{1-\alpha/2, n_1+n_2-2}, \quad (4.13)$$

where $t_{1-\alpha/2}$ is the $100(1-\alpha/2)$ percentage point of the Student's t distribution with $(n_1 + n_2 - 2)$ degrees of freedom. And thus the p-value is

$$p = \Pr\left[\left| \frac{\left[n_1 n_2 (n_1 + n_2 - 2)\right]^{1/2} (\bar{x}_1 - \bar{x}_2)}{\bar{x}_1 \bar{x}_2 (n_1 \bar{x}_1 + n_2 \bar{x}_2) \left[\sum_{j=1}^{n_1} (x_{1j}^{-1} - \bar{x}_1^{-1}) + \sum_{j=1}^{n_2} (x_{2j}^{-1} - \bar{x}_2^{-1}) \right]^{1/2}} \right| > t_{1-\alpha/2} \right]. \quad (4.14)$$

The test can be extended to compare the two inverse Gaussian means in terms of their ratio. This follows because of the property that density function $f(x;\mu,\lambda)=\mu^{-1}f(x;1,\lambda/\mu)$ for $X\sim IG(\mu,\lambda)$. UMP-unbiased test of the hypotheses $H_0:\frac{\mu_1}{\mu_2}=\theta_0$ versus $H_1:\frac{\mu_1}{\mu_2}\neq\theta_0$, $\theta_0>0$ can be derived from (4.13) by replacing X_{2j} by $\theta_0 X_{2j}$, $j=1,2,...,n_2$, provided the scale parameter of the distribution function of X_{2j} is assumed to be $\theta_0\lambda$.

It is straightforward to express the UMP-unbiased test procedures in terms of θ_0 obtained by inverting the acceptance regions of these tests at level α. When λ is unknown, the confidence interval for θ_0 is given by

$$\begin{cases}\left(A[B-C],A[B+C]\right) & ,\ 1-\bar{X}_1V_1d^2/n_1>0\\ \left(0,\quad A'[-B+C]\right) & ,\ \text{otherwise}\end{cases}, \tag{4.15}$$

where

$$A=\frac{\bar{X}_1}{\bar{X}_2}\left(1-\frac{\bar{X}_1V_1}{n_1}d^2\right)^{-1},\quad B=1+\frac{1}{2}\left(\frac{\bar{X}_1V_1}{n_2}+\frac{\bar{X}_2V_2}{n_1}\right)d^2,$$

$$C=d\left[\left(\frac{\bar{X}_1}{n_1}+\frac{\bar{X}_1}{n_2}\right)V_1+\left(\frac{\bar{X}_2}{n_1}+\frac{\bar{X}_2}{n_2}\right)V_2+\frac{1}{4}\left(\frac{\bar{X}_1V_1}{n_2}-\frac{\bar{X}_2V_2}{n_1}\right)^2d^2\right]^{1/2},$$

$$d=(n_1+n_2-2)^{-1/2}t_{1-\alpha/2},$$

$$A'=\frac{\bar{X}_1}{\bar{X}_2}\left(-1+\frac{\bar{X}_1V_1}{n_1}d^2\right)^{-1}$$

and $V_1 = \sum_{j=1}^{n_1} \left(X_{1j}^{-1} - \bar{X}_1^{-1} \right)$ and $V_2 = \sum_{j=1}^{n_2} \left(X_{2j}^{-1} - \bar{X}_2^{-1} \right)$.

For more details, we refer to the paper and the book by Chhikara and Folks (1975, 1989), respectively.

4.2.2 Inferences on λ_1 / λ_2

We now give a test for the ratio of two scale parameters λ_1 and λ_2 from two independent IG populations, say $IG\left(\mu_i, \lambda_i\right)$, $i = 1, 2$. To test the hypothesis $H_0 : \frac{\lambda_1}{\lambda_2} = \delta_0$ versus $H_1 : \frac{\lambda_1}{\lambda_2} \neq \delta_0$, take random samples of n_1 observations of $\mathbf{X}_1$ and n_2 observations of $\mathbf{X}_2$. For the fact that $\lambda_1 V_1$ and $\lambda_2 V_2$ have independent chi-square distributions $\chi^2_{n_1-1}$ and $\chi^2_{n_2-1}$, respectively, where $V_i = \sum_{j=1}^{n_i} \left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i} \right)$ for $i = 1, 2$. Thus the test statistic can be written as

$$Q = \frac{\lambda_1 V_1 / \left(n_1 - 1\right)}{\lambda_2 V_2 / \left(n_2 - 1\right)} = \frac{\chi^2_{n_1-1} / \left(n_1 - 1\right)}{\chi^2_{n_2-1} / \left(n_2 - 1\right)} \tag{4.16}$$

which is an F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom. Therefore the *p*-value for this size α test is computed by

$$p = 2 * \min \left\{ P\left[Q > \frac{\lambda_1 v_1 / \left(n_1 - 1\right)}{\lambda_2 v_2 / \left(n_2 - 1\right)} \middle| \frac{\lambda_1}{\lambda_2} = \delta_0 \right], P\left[Q < \frac{\lambda_1 v_1 / \left(n_1 - 1\right)}{\lambda_2 v_2 / \left(n_2 - 1\right)} \middle| \frac{\lambda_1}{\lambda_2} = \delta_0 \right] \right\}$$

$$= 2 * \min\left\{P\left[F_{n_1-1,n_2-1} > \delta_0 \cdot \frac{v_1/(n_1-1)}{v_2/(n_2-1)}\right], P\left[F_{n_1-1,n_2-1} < \delta_0 \cdot \frac{v_1/(n_1-1)}{v_2/(n_2-1)}\right]\right\}, \qquad (4.17)$$

where Q is defined in (4.20) and F_{n_1-1,n_2-1} denotes the F distribution with n_1-1 and n_2-1 degrees of freedom, and H_0 is rejected if $p < \alpha$.

It is straightforward to construct a confidence interval for $\frac{\lambda_1}{\lambda_2}$ based on the statistic Q given in (4.16). For the fact that $\Pr\left[F_{\alpha/2,n_1-1,n_2-1} < Q < F_{1-\alpha/2,n_1-1,n_2-1}\right]$ $= 1-\alpha$, where F_{γ,n_1-1,n_2-1} is the rth quantile of F distribution with n_1-1 and n_2-1 degrees of freedom. Let $q_1 = F_{\alpha/2,n_1-1,n_2-1}$ and $q_2 = F_{1-\alpha/2,n_1-1,n_2-1}$, a $100(1-\alpha)\%$ confidence interval for $\frac{\lambda_1}{\lambda_2}$ can be obtain through

$$\left\{\frac{\lambda_1}{\lambda_2} : q_1 < Q = \frac{\lambda_1 v_1/(n_1-1)}{\lambda_2 v_2/(n_2-1)} < q_2\right\}$$

$$= \left\{q_1 \frac{v_2/(n_2-1)}{v_1/(n_1-1)} < \frac{\lambda_1}{\lambda_2} < q_2 \frac{v_2/(n_2-1)}{v_1/(n_1-1)}\right\}. \qquad (4.18)$$

It is interesting to note that the result in (4.18) is the same as our result in (4.14). In our procedure, the pivotal quantity (4.10) of $\frac{\lambda_1}{\lambda_2}$ is $T^* = \frac{\chi^2_{n_1-1}/v_1}{\chi^2_{n_2-1}/v_2} = F_{n_1-1,n_2-1} \cdot \frac{v_2(n_2-1)}{v_1(n_1-1)}$, the quantile points which satisfys $\Pr\left[C_1 < T^* < C_2\right] = 1-\alpha$ are $C_1 = q_1 \frac{v_2/(n_2-1)}{v_1/(n_1-1)}$ and $C_2 = q_2 \frac{v_2/(n_2-1)}{v_1/(n_1-1)}$, where $q_1 = F_{\alpha/2,n_1-1,n_2-1}$ and $q_2 = F_{1-\alpha/2,n_1-1,n_2-1}$. Thus the $100(1-\alpha)\%$ confidence interval for $\frac{\lambda_1}{\lambda_2}$ is $\left\{\frac{\lambda_1}{\lambda_2} : C_1 < T^*_{obs} = \frac{\lambda_1}{\lambda_2} < C_2\right\}$ which is the same as (4.18).

4.3 Confidence intervals of $\boldsymbol{\mu}_1/\boldsymbol{\mu}_2$ based on the directed likelihood ratio statistic

The signed log likelihood ratio has been discussed by many authors, McCullagh (1982), Petersen (1981), Pierce and Schafer (1986), and Barndorff-Nielsen (1986) etc., to obtain a statistic which is asymptotically standard normally distributed with error of order $O(n^{-3/2})$ by repeated sampling. Tian and Wilding (2005) provided an estimating approach for constructing a confidence interval of μ_1/μ_2 based on the directed likelihood ratio method. The procedure is as follows.

Suppose the ratio of the two means is the parameter of interest, that is $\theta = \mu_1/\mu_2$ and the vector of nuisance parameters is $\boldsymbol{\eta} = (\mu_2, \lambda_1, \lambda_2)$ and $\boldsymbol{\zeta} = (\theta, \boldsymbol{\eta})$. Let $Y_{ij} = 1/\sqrt{X_{ij}}$, $j = 1, \ldots, n_i$; $i = 1, 2$, then Y_{1j} and Y_{2j} are two independent samples from $RRIG(\mu_1, \lambda_1)$ and $RRIG(\mu_2, \lambda_2)$, respectively, where $RRIG$ means the reciprocal root IG distribution. The log-likelihood function is

$$l(\zeta; x) = (n_1 + n_2)\log\left(\frac{2}{\sqrt{2\pi}}\right) + \frac{n_1}{2}\log\lambda_1 + \frac{n_2}{2}\log\lambda_2 + \frac{n_1\lambda_1}{\theta\mu_2}$$

$$+\frac{n_2\lambda_2}{\mu_2} - \frac{\lambda_1}{2}T_1 - \frac{\lambda_1}{2\psi^2\mu_2^2}S_1 - \frac{\lambda_2}{2}T_2 - \frac{\lambda_2}{2\mu_2^2}S_2, \tag{4.19}$$

where

$$S_i = \sum_{j=1}^{n_i} y_{ij}^{-2} = \sum_{j=1}^{n_i} x_{ij} \quad \text{and} \quad T_i = \sum_{j=1}^{n_i} y_{ij}^{2} = \sum_{j=1}^{n_i} 1/x_{ij}\ , \quad i = 1, 2\,. \tag{4.20}$$

The maximum likelihood estimates of the parameters of (4.19) are

$$\hat{\theta} = \left(S_1/n_1\right)/\left(S_2/n_2\right),\ \ \hat{\mu}_2 = S_2/n_2\ ,\ \ \hat{\lambda}_1 = 1/\left(T_1/n_1 - n_1/S_1\right) \ \text{ and } \ \hat{\lambda}_2 = 1/\left(T_2/n_2 - n_2/S_2\right).$$

For a given value of θ, the constrained maximum likelihood estimates of the nuisance parameters $\boldsymbol{\eta} = \left(\mu_2, \lambda_1, \lambda_2\right)$ can be obtained by solving

$$\hat{\mu}_{2\theta} = \left(\hat{\lambda}_{1\theta} S_1/\psi^2 + \hat{\lambda}_{2\theta} S_2\right)/\left(\hat{\lambda}_{1\theta} n_1/\psi + \hat{\lambda}_{2\theta} n_2\right),$$

$$\hat{\lambda}_{1\theta} = n_1/\left(T_1 + S_1/\theta^2 \hat{\mu}_{2\theta}^2 - 2n_1/\theta\hat{\mu}_{2\theta}\right),$$

$$\hat{\lambda}_{2\theta} = n_2/\left(T_2 + S_2/\hat{\mu}_{2\theta}^2 - 2n_2/\hat{\mu}_{2\theta}\right), \tag{4.21}$$

simultaneously. The approximate $100(1-\alpha)\%$ confidence intervals of μ_1/μ_2 based on the directed likelihood ratio statistic $r\left(\theta\right)$ is

$$\left\{\theta : \left|r\left(\theta\right)\right| \le Z_{\alpha/2}\right\} \tag{4.22}$$

with

$$r\left(\theta\right) = \operatorname{sgn}\left(\hat{\theta} - \theta\right)\left[2\left(l\left(\hat{\theta}, \hat{\boldsymbol{\eta}}\right) - l\left(\theta, \hat{\boldsymbol{\eta}}_\theta\right)\right)\right]^{1/2}, \tag{4.23}$$

where $l\left(\theta\right)$ is the log-likelihood function in (4.19).

Chapter 5 Numerical Examples and Simulation Studies

Some *IG* data are given to compare our procedure with other methods with respect to their confidence intervals and confidence lengths. Several simulation studies are also presented to compare the performances of three methods, (1) Chhikara and Folks (2) Tian and Wilding (3) the generalized approaches, in terms of their coverage probabilities, expected lengths and the Type I error.

5.1 Numerical examples

Example 1.

Gacula and Kubala (1975) reported certain sensory failure data for two refrigerated food products, M and K as these were called, and studied their shelf life which fit the *IG* distribution well. The summary data are given in Table 1 and the 95% confidence intervals for three methods are presented in Table 2.

Table 1. Summary data

Product	size	$\hat{\mu}$	$\hat{\lambda}$
M	26	42.885	18.622
K	17	56.941	14.881

Table 2. 95% confidence intervals and lengths for $\theta = \frac{\mu_1}{\mu_2}$

Method	$\hat{\theta}$	95% confidence interval	length
Chhikara	0.771	(0.635 , 0.905)	0.270
Directed	0.753	(0.562 , 0.962)	0.400
Generalized	0.755	(0.554 , 0.977)	0.422

Example 2.

Four sets of *IG* data presented in Folks and Chhikara (1978) who judged that the data are very well described by the Inverse Gaussian distribution. The first set, data (1), gives fracture toughnesses of MIG welds. The second set, data (2), gives data of precipitation (inches) from Jug Bridge, Maryland. The third set, data (3), gives runoff amounts at Jug Bridge, Maryland. Additionally, Gacula and Kubala (1975) gave data (4) on shelf-life of a food product. The summary data for four sets of *IG* data are shown in Table 3. For investigating the ratio of means of two independent populations when the scale parameters are more different than those in *Example 1*, we will compare the means of these four data sets mutually and show the results in Table 4.

Table 3. The summary data for four data sets

data	size	$\hat{\mu}$	$\hat{\lambda}$
(1)	19	74.300	4924.070
(2)	25	2.160	8.080
(3)	25	0.800	1.440
(4)	26	42.885	484.253

Table 4. 95% confidence intervals and lengths for $\theta = \frac{\mu_1}{\mu_2}$

(2)/(1)	$\hat{\theta}$	95% confidence interval	length
Chhikara	0.0303	(0.020 , 0.040)	0.020
Directed	0.0290	(0.024 , 0.036)	0.012
Generalized	0.0294	(0.024 , 0.037)	0.014

(3)/(1)	$\hat{\theta}$	95% confidence interval	length
Chhikara	0.0118	(0.006 , 0.017)	0.011
Directed	0.0108	(0.008 , 0.016)	0.008
Generalized	0.0111	(0.008 , 0.016)	0.008

(4)/(1)	$\hat{\theta}$	95% confidence interval	length
Chhikara	0.5857	(0.469 , 0.702)	0.233
Directed	0.5771	(0.509 , 0.661)	0.152
Generalized	0.5796	(0.505 , 0.667)	0.162

(3)/(2)	$\hat{\theta}$	95% confidence interval	length
Chhikara	0.4046	(0.154 , 0.655)	0.501
Directed	0.3726	(0.262 , 0.558)	0.296
Generalized	0.3812	(0.258 , 0.564)	0.306

(2)/(4)	$\hat{\theta}$	95% confidence interval	length
Chhikara	0.0521	(0.034 , 0.070)	0.036
Directed	0.0503	(0.040 , 0.066)	0.026
Generalized	0.0509	(0.040 , 0.065)	0.025

(3)/(4)	$\hat{\theta}$	95% confidence interval	length
Chhikara	0.0201	(0.011 , 0.029)	0.018
Directed	0.0187	(0.014 , 0.028)	0.014
Generalized	0.0193	(0.014 , 0.028)	0.014

From *Example 1* and *Example 2*, the results show that the confidence lengths obtained by the generalized methods are the smallest or close to the smallest confidence lengths no matter what the scale parameters perform when two IG populations are non-homogeneous. Some simulation studies are also worth to be inspected, and we will make discussion in next subsection.

5.2 Simulation studies

Some simulation studies are performed to compare the 95% coverage probabilities, expected lengths and type I errors of three procedures for the ratio of two means, $\theta = \mu_1/\mu_2$. We will choose different combinations of sample sizes $(n_1, n_2) = (5,10)$, $(10,5)$ and $(10,10)$, respectively, and various values of the ratio of scale parameters, λ_1/λ_2, with 1,000 replicates for each combination. The results appear in Tables 5-9. In addition, we will present powers of the tests obtained by the generalized method in Table 10.

Table 5. Coverage probabilities (CP) and expected lengths (length) of 95% confidence intervals of $\theta = \dfrac{\mu_1}{\mu_2}$

$\theta = 0.2$		Generalized①		Chhikara②		Directed③	
(n1 , n2)	$\dfrac{\lambda_1}{\lambda_2}$	CP	length	CP	length	CP	length
(5 , 10)	0.5	0.957	0.698	0.974	0.534	0.930	1.060
	1	0.954	0.472	0.974	0.466	0.917	0.787
	5	0.948	0.312	0.978	0.427	0.894	0.520
	10	0.949	0.294	0.982	0.422	0.907	0.280
(10 , 5)	0.5	0.946	0.565	0.906	0.483	0.895	0.671
	1	0.951	0.488	0.872	0.370	0.898	0.558
	5	0.950	0.427	0.811	0.290	0.876	0.379
	10	0.949	0.496	0.792	0.279	0.898	0.300
(10 , 10)	0.5	0.959	0.409	0.946	0.367	0.929	0.470
	1	0.951	0.335	0.941	0.317	0.912	0.373
	5	0.949	0.291	0.932	0.281	0.897	0.330
	10	0.953	0.285	0.930	0.276	0.886	0.259

①Generalized methods ②Chhikara and Folks (1989) ③Directed likelihood ratio statistic

Table 6. Coverage probabilities (CP) and expected lengths (length) of 95% confidence intervals of $\theta = \frac{\mu_1}{\mu_2}$

$\theta = 0.6$		Generalized①		Chhikara②		Directed③	
(n1 , n2)	$\frac{\lambda_1}{\lambda_2}$	CP	length	CP	length	CP	length
(5 , 10)	0.5	0.955	3.889	0.906	5.231	0.932	3.589
	1	0.957	2.651	0.962	2.023	0.902	2.602
	5	0.952	1.067	0.977	1.346	0.892	0.869
	10	0.950	0.974	0.976	1.295	0.895	0.721
(10 , 5)	0.5	0.956	2.891	0.934	6.967	0.938	2.536
	1	0.953	1.987	0.926	2.027	0.903	1.658
	5	0.955	1.370	0.846	0.989	0.893	0.866
	10	0.950	1.321	0.823	0.896	0.906	0.863
(10 , 10)	0.5	0.955	2.482	0.944	2.489	0.929	2.395
	1	0.952	1.536	0.948	1.281	0.933	1.563
	5	0.955	0.943	0.937	0.896	0.902	0.715
	10	0.949	0.881	0.931	0.856	0.902	0.661

①Generalized methods ②Chhikara and Folks (1989) ③Directed likelihood ratio statistic

Table 7. Coverage probabilities (CP) and expected lengths (length) of 95% confidence intervals of $\theta = \frac{\mu_1}{\mu_2}$

$\theta = 1$		Generalized①		Chhikara②		Directed③	
(n1 , n2)	$\frac{\lambda_1}{\lambda_2}$	CP	length	CP	length	CP	length
(5 , 10)	0.5	0.948	8.655	0.773	9.905	0.892	4.165
	1	0.958	6.370	0.930	6.323	0.909	3.501
	5	0.957	2.348	0.976	2.331	0.907	1.433
	10	0.956	1.872	0.978	2.218	0.898	1.233
(10 , 5)	0.5	0.954	7.571	0.801	22.061	0.919	3.813
	1	0.952	4.453	0.946	7.066	0.913	2.714
	5	0.948	2.440	0.868	1.873	0.899	1.480
	10	0.953	2.310	0.840	1.608	0.898	1.432
(10 , 10)	0.5	0.958	6.168	0.897	11.000	0.926	3.856
	1	0.955	3.450	0.953	3.080	0.916	2.872
	5	0.954	1.668	0.943	1.580	0.911	1.241
	10	0.952	1.550	0.937	1.472	0.909	1.411

①Generalized methods ②Chhikara and Folks (1989) ③Directed likelihood ratio statistic

Table 8. Type I error for testing $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$, $\theta = \frac{\mu_1}{\mu_2}$ (α =0.05)

$\theta_0 = 0.4$ (n1 , n2)	$\frac{\lambda_1}{\lambda_2}$	type I error Generalized①	Chhikara②	Directed③
(5 , 10)	0.5	0.04	0.05	0.08
	1	0.04	0.03	0.08
	5	0.05	0.02	0.08
	10	0.05	0.02	0.06
(10 , 10)	0.5	0.05	0.05	0.08
	1	0.04	0.05	0.07
	5	0.05	0.07	0.06
	10	0.05	0.07	0.07
(15 , 10)	0.5	0.05	0.06	0.06
	1	0.05	0.07	0.07
	5	0.04	0.11	0.08
	10	0.05	0.13	0.06

①Generalized methods ②Chhikara and Folks (1989) ③Directed likelihood ratio statistic

Table 9. Type I error for testing $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$, $\theta = \frac{\mu_1}{\mu_2}$ (α =0.05)

$\theta_0 = 0.8$ (n1 , n2)	$\frac{\lambda_1}{\lambda_2}$	type I error Generalized①	Chhikara②	Directed③
(5 , 10)	0.5	0.05	0.17	0.09
	1	0.04	0.05	0.08
	5	0.04	0.02	0.09
	10	0.05	0.02	0.10
(10 , 10)	0.5	0.04	0.07	0.08
	1	0.05	0.05	0.09
	5	0.05	0.06	0.09
	10	0.05	0.07	0.10
(15 , 10)	0.5	0.05	0.04	0.08
	1	0.05	0.06	0.08
	5	0.05	0.10	0.08
	10	0.05	0.11	0.08

①Generalized methods ②Chhikara and Folks (1989) ③Directed likelihood ratio statistic

Table 10. Simulated powers for testing $H_0: \theta = 1$ versus $H_1: \theta \neq 1$, $\theta = \frac{\mu_1}{\mu_2}$ (α =0.05)

		θ					
(n1 , n2)	$\frac{\lambda_1}{\lambda_2}$	0.2	0.4	0.6	0.8	1	1.2
(5 , 10)	0.5	0.763	0.245	0.128	0.061	0.053	0.051
	1	0.911	0.391	0.112	0.053	0.043	0.041
	5	0.998	0.726	0.210	0.077	0.045	0.047
	10	1.000	0.825	0.310	0.074	0.047	0.070
(10 , 5)	0.5	0.902	0.306	0.092	0.047	0.046	0.047
	1	0.957	0.381	0.120	0.050	0.048	0.051
	5	0.979	0.532	0.174	0.061	0.052	0.072
	10	0.983	0.556	0.201	0.072	0.054	0.079
(10 , 10)	0.5	0.971	0.397	0.136	0.065	0.042	0.042
	1	0.998	0.627	0.176	0.063	0.048	0.057
	5	1.000	0.865	0.335	0.080	0.045	0.069
	10	1.000	0.884	0.368	0.095	0.044	0.081
(15 , 10)	0.5	0.995	0.542	0.150	0.066	0.045	0.043
	1	1.000	0.722	0.349	0.057	0.044	0.071
	5	1.000	0.867	0.407	0.091	0.053	0.061
	10	1.000	0.889	0.386	0.099	0.046	0.080

From Table 5 to Table 10, we can conclude that the coverage probabilities obtained by generalized methods are very close to the nominal level 95% and the Type I error are exact or close to the nominal level 0.05. On the other hand, the coverage probabilities obtained by directed likelihood ratio are too small and the type I errors exceed the 5% in all cases. Besides, Chhikara and Folks (1989)'s procedure performs well under $\lambda_1 = \lambda_2$ and $\frac{\lambda_i}{\mu_i^2} = \xi$, a constant, $i = 1, 2$, but its performance becomes worse when the heteroscedasticity is increasing. In fact, the procedure based on generalized methods is readily applicable and easy to perform even under the sample sizes are quite small. The simulation results show that its results are better than the other two methods with respect to having almost exact coverage probabilities and the type I errors.

References

Barndorff-Nielsen, O. E. (1986). Inference on full and partial parameters based on the standardized signed log likelihood ratio. *Biometrika*, 73: 307-322.

Chhikara, R. S. (1972). Statistical inference related to the inverse Gaussian distribution. *Ph.D. Dissertation*, Oklahoma State University, Stillwater.

Chhikara, R. S. (1975). Optimum tests for the comparison of two inverse Gaussian distribution means. *Austral. J. Statist.*, 17: 77-83.

Chhikara, R. S., Folks, J. L. (1975). Statistical distributions related to the inverse Gaussian. *Commun. Statist.*, 4: 1081-1091.

Chhikara, R. S., Folks, J. L. (1976). Optimum tests procedures for the mean of first passage time in Brownian motion with positive drift (inverse Gaussian distribution). *Technometrics*, 18: 189-193.

Chhikara, R. S., Folks, J. L. (1989). The Inverse Gaussian distribution. *Marcel Dekker*, New York.

Folks, J. L., Chhikara, R. S. (1978). The Inverse Gaussian distribution and its statistical application-a review. *J. R. Statist. Soc. B*, 40: 263-289

Gacula, M. C., Kubala, J. J. (1975). Statistical models for shelf life failures. *J. Food Sci.*, 40: 404-409

McCullagh, P. (1982). Discussion of paper by A. C. Atkinson. *J. R. Statist. Soc. B*, 44: 29-30.

Petersen, B. V. (1981). A comparison of the Eforn-Hinkley ancillary and the likelihood ratio ancillary in a particular example. Ann. Statist. 9: 1328-1333.

Pierce, D. A., Schafer, D. W. (1986). Residuals in generalized linear models. *J. Am. Statist. Assoc.* 81: 977-986.

Roy, L. K., Wasan, M. T. (1968a). The first passage time distribution of Brownian motion with positive drift. Math. *Biosci.*, 3: 191-204.

Roy, L. K., Wasan, M. T. (1968b). Properties of the time distribution of standard Brownian motion. *Trabajos Estadist*., 19: 1-11.

Roy, L. K., Wasan, M. T. (1968c). Some characteristic properties of the time distribution of standard Brownian motion. *Recu en Mars*, 29-38.

Schrödinger, E. (1915). Zöur Theorie der Fall - und Steigversuche an Teilchenn mit Brownsche Bewegung. *Phys. Zeit.* 16: 289-295

Seshardi, V. (1993). The Inverse Gaussian Distribution: A Case Study in Exponential Families. *Clarendon Press*, Oxford.

Seshadri, V. (1999). The Inverse Gaussian Distribution: Statistical Theory and Applications. *Springer*, New York.

Shuster, J. J., Miura, C. (1972). Two way analysis of reciprocals. *Biometrika*, 59: 478-481.

Tian, Lili, Wilding, Gregory E. (2005). Confidence intervals of the ratio of means of two independent inverse Gaussian distributions. *Journal of Statistical Planing and Inference*, 133: 381-386.

Tsui, K. W., Weerahandi, S. (1989). Generalized *P*-values in significance testing of hypotheses in the presence of nuisance parameters. *J. Amer. Statist. Assoc*. 84: 602-607.

Tweedie, M. C. K. (1945). Inverse statistical variates. *Nature*, 155: 453.

Wald, A. (1947). Sequential Analysis, *Wiley*, New York.

Weerahandi, S. (1993). Generalized confidence intervals. *J. Amer. Statist. Assoc*. 88: 899-905.

Weerahandi, S. (1995). Exact Statistical Methods for Data Analysis. *Springer*, New York.