國立交通大學

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碩 士 論 文

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With Different Increment Shifts

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摘要

線性同餘法（linear congruential generator，LCG）與 k 階乘餘法（multiple recursive generator，MRG）至今仍為亂數產生器常使用的兩大線性方法；而近日，由於電腦處理器成本下降，平行化計算因而被廣泛研究與使用，以改善因分析與模擬資料量與日俱增，電腦運算效率降低之問題。除此之外，平行化處理亦可能增強許多模擬數據時所需具備的統計性質，如隨機性等。傳統平行化線性亂數產生器之方式，譬如使用不同初始值，多僅改善運算速度，對於統計性質的改變，則助益不大；但藉由改變線性亂數產生器的增量（increment），則可坐收同時改善速度與性質之效，且計算簡單，易於推廣，尤其輔以平行設計之跳蛙法（leapfrogging method），綜效較順序切割法（sequence splitting method）更佳。

Abstract

Two major linear random number generators (RNGs), the linear congruential generator (LCG) and the multiple recursive generator (MRG), have been widely studied and used for many decades. Nowadays, as the price decreasing of computer processors, parallelization of the generators is being concerned for, at least, the computational efficiency purpose. Besides, the proper design of parallel generator may also improve some statistical properties such as randomness. The Parallel linear random number generator with different increment shifts is efficient and feasible because the change of the increments only shifts the hyperplanes of the linear RNG. Additionally, parallelizing through the leapfrogging method can further improve than through the sequence splitting method.

Table of Contents

1. INTRODUCTION

It is very important to develop a random number generator (RNG) to generate pseudo-random numbers for simulations in computers (Knuth 1998). With the developed of parallel computing in modern computer technology, this study is aiming at the design of parallel linear random number generators with different increment shifts for simulations with parallel computing.

The linear congruential generator (LCG) has been widely used for its simplicity and theoretical properties since the first proposal in Lehmer (1951). With the increasing demand of high quality random numbers, the multiple recursive generator (MRG) and other methods were developed afterwards (L'Ecuyer, Blouin and Couture, 1993, Knuth 1998, Deng and Xu, 2003). The MRG greatly extends the period of the random sequence and improves properties of the LCG. It is challenging to determine the parameters in the MRG that improve the properties of a RNG. The DX generator has been proposed by Deng and Xu (2003) to select parameters in the MRG that prolongs the period of random sequences and improves the efficiency of simulation.

As the price decreasing of computer processors nowadays, it becomes very economic and efficient to generate numbers through parallelization by many computers. With a proper design of algorithms, it becomes feasible to shorten the computational speed and strengthen randomness of the output (Gentle, J. E. 2003). This study will propose methods to parallelize a linear generator by using different increment values. The theoretical and empirical properties will be explored in this study.

2. SERIAL LINEAR RANDOM NUMBER GENERATORS

2.1 Linear Congruential Generator (LCG)

The method of LCG has been proposed by Lehmer (1951). This recursive generator has the following form:

$$X_i = (aX_{i-1} + c) \bmod m, \tag{2.1}$$

$$U_i = X_i / m, \tag{2.2}$$

where a, c, m, and X_0 are integers; $m > 0$ is the modulus parameter; $0 < a < m$ is the multiplier parameter; $0 \le c < m$ is the increment parameter; X_0 is the starting value or the seed, $0 \le X_0 < m$ when $c \ne 0$ and $0 < X_0 < m$ otherwise.

The performances of linear congruential generators differ drastically considering the changes of a, c, and m. These parameters hence ought to be chosen very cautiously so that the resulting pseudorandom numbers possesses good properties, such as long period, uniformity, randomness, and so forth. Since the random number of X_i is obtained from the remainder after the modulo operation of m, the possible outcome X_i is in the set of $S = \{0, 1, 2, \ldots, m - 1\}$. An LCG is said to attain the full period m if and only if the following theorem is satisfied (Knuth 1998):

(1) c is relatively prime to m;

(2) q is a factor of $(a - 1)$, if q is a prime factor of m;

(3) $(a - 1)$ is a multiple of 4, if m is also a multiple of 4.

The LCGs can be classified into two types, mixed ($c > 0$) and multiplicative ($c = 0$). Apparently, the full period does not exist in the multiplicative case because $c = 0$ is a multiplier of any integer m. Also, 0 is an absorbing state in this case. Instead, its maximum period, $m - 1$, is obtained when a is a primitive element modulo m. Furthermore, it is widely used in 32 bit computers with the multiplicative LCGs, $m = 2^{31} - 1$ and $a = 7^k$ where k is a primitive root such as $a = 7^5 = 16{,}807$.

2.2 Multiple Recursive Generator (MRG)

The multiplicative LCG can be extended to the higher order recursively by adding up the previous k pseudorandom numbers. This is called as the multiple recursive generator (MRG), which is defined as follows:

$$X_i = (a_1 X_{i-1} + a_2 X_{i-2} + \cdots + a_k X_{i-k}) \bmod m, \tag{2.3}$$

$$U_i = X_i / m, \tag{2.4}$$

where k is the order. The absolute values of the multipliers a_1, a_2, …, a_k and the starting values X_0, X_1, …, X_{k-1} may be any numbers except for all zeros. In addition, a_k should be nonzero. If $k = 1$, the MRG degenerates to the multiplicative LCG.

The maximum period of an MRG is improved to $m^k - 1$ on condition that $f(x) = x^k - a_1 x^{k-1} - \cdots - a_{k-1} x - a_k$ is a primitive polynomial modulo m. Alanen and Knuth (1964) and Knuth (1998) showed the following criteria to find a_1, a_2, …, a_k:

(1) $(-1)^{k-1} a_k$ must be a primitive root modulo p;

(2) the polynomial x^r must be congruent to $(-1)^{k-1} a_k$, modulo $f(x)$ and m;

(3) the degree of $x^{r/q} \bmod f(x)$, using polynomial arithmetic modulo m, must be positive, for each prime factor q of r.

where $r = (m^k - 1)/(m - 1)$.

The MRG has improved the statistical properties for an RNG with slight increase of computational time because this generator requires more multiplication and calculation. Also, the criteria for selecting multiplier parameters requires the prime factorization of $r = (p^k - 1)/(p - 1)$, which is challenging in seaching when p and k are larger (Knuth 1998).

L'Ecuyer, Blouin, and Couture (1993) made an attempt to find some MRGs for $k \le 7$. With $m = 2^{31} - 1$, they suggested a sixth-order MRG:

$$X_i = (177{,}786 X_{i-1} + 64{,}654 X_{i-6}) \bmod (2^{31} - 1) \tag{2.5}$$

that attains the period of $(2^{31} - 1)^6 - 1 \approx 10^{55}$.

Other subgroups or special cases for the MRGs have been studied (Gentle, J. E. 2003). One typical instance is the generator,

$$X_i = (X_{i-j} + X_{i-k}) \bmod 2^m, \tag{2.6}$$

called the additive lagged-Fibonacci random number generator.

2.3 DX-*k*-*s* Random Number Generators

Research has been conducted to improve the computational speed of MRGs. L'Ecuyer (1990), for example, suggested considering the generator of the following form:

$$X_i = (a_j X_{i-j} + a_k X_{i-k}) \bmod m. \tag{2.7}$$

Thus, the primitive polynomial $f(x) = x^k - a_j x^{k-j} - a_k$ would be manipulated because of the accessibility of this primitive trinomial.

Deng and Lin (2000) proposed the fast multiple recursive random number generator (FMRG), which is defined as:

$$X_i = (-X_{i-1} + BX_{i-k}) \bmod m \tag{2.8}$$

The FMCG in (2.8) and the MRG in (2.7) improves the computational efficiency because it sets all multipliers to zeros except for a_1 in (2.8) or a_j in (2.7) and a_k. Deng and Lin (2000) also listed some FMCGs of orders up to four. L'Ecuyer (1997), however, suggested that the sum of squares of coefficients shall be large, which is a necessary condition for an MRG with a good lattice structure. FMCGs may fail to satisfy this condition.

Deng and Xu (2003) have further extended the idea of the FMRG to the so-called DX-*k*-*s* random number generator. To prolong the period and improve the efficiency, they place certain intermittent coefficients to the same value B. Here are some examples:

(1) DX-*k*-2

$$X_i = B(X_{i-1} + X_{i-k}) \bmod m; \tag{2.9}$$

(2) DX-*k*-3

$$X_i = B(X_{i-1} + X_{\left\lceil i-\frac{k}{2} \right\rceil} + X_{i-k}) \bmod m; \tag{2.10}$$

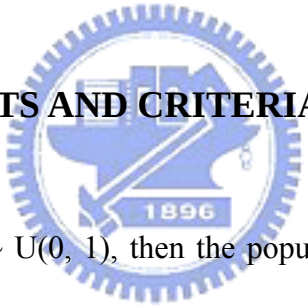
(3) DX-*k*-4

$$X_i = B(X_{i-1} + X_{\left\lceil i-\frac{k}{3} \right\rceil} + X_{\left\lceil i-\frac{2k}{3} \right\rceil} + X_{i-k}) \bmod m, \tag{2.11}$$

where $\lceil \cdot \rceil$ is the ceiling function. Deng and Xu (2003) and Deng (2005) recommended specific generators, and the following one has very long period and large coefficients:

$$X_i = 1{,}073{,}741{,}362(X_{i-1} + X_{i-533} + X_{i-1065} + X_{i-1597}) \bmod (2^{31} - 1). \tag{2.12}$$

The period greatly stretches to $10^{14,903}$ approximately.



3. STATISTICAL TESTS AND CRITERIA

3.1 Uniformity

If a random variable $U \sim$ U(0, 1), then the population mean is $E(U) = 1/2$, the variance is $Var(U) = 1/12$, and

$$P(a < U < b) = b - a, \tag{3.1}$$

for any $0 \le a < b \le 1$. A good uniform random number generator should generate similar results. Using the fact that $\sum_{0 \le X_i < m} X_i = \frac{m(m-1)}{2}$ and $\sum_{0 \le X_i < m} X_i^2 = \frac{m(m-1)(2m-1)}{6}$ for the RNG with maximum period, we know that $\bar{U} = \frac{m-1}{2m}$ and $S_U^2 = \frac{m+1}{12m}$. For the RNG with the period of $m - 1$, $\bar{U} = \frac{1}{2}$ and $S_U^2 = \frac{m-1}{12m}$. When m increases, both $\bar{U}$'s and S_U^2's approach to $E(U)$ and $Var(U)$.

Additionally, goodness-of-fit tests have been developed for testing uniformity, such as chi-squared test, Kolmogorov-Smirnov (K-S) test, and Anderson-Darling (A-D) test.

3.2 Randomness

Ideally, an RNG can generate random numbers with no correlation between X_i and X_{i+k}. For a LCG, the output X_{i+k} is a function of X_i as follows:

$$X_{i+k} = \left[a^k X_i + \frac{c(a^k - 1)}{a - 1} \right] \bmod m. \tag{3.2}$$

One method to evaluate this association is the Pearson correlation coefficient, R. When k = 1 or with lag one, it is called as the serial correlation coefficient. Making use of the formulas for $\sum_{0 \le X_i < m} X_i$ and $\sum_{0 \le X_i < m} X_i$ again, R can be induced from (3.3) to (3.5).

$$R = \frac{\sum_{i=1}^{m} X_i X_{i+1} - \frac{1}{m}\sum_{i=1}^{m} X_i \sum_{i=1}^{m} X_{i+1}}{\sqrt{\sum_{i=1}^{m} X_i^2 - \frac{1}{m}\left(\sum_{i=1}^{m} X_i\right)^2}\sqrt{\sum_{i=1}^{m} X_{i+1}^2 - \frac{1}{m}\left(\sum_{i=1}^{m} X_{i+1}\right)^2}} \tag{3.3}$$

$$= \frac{\sum_{i=1}^{m} X_i X_{i+1} - \frac{1}{m}\left[\frac{m(m-1)}{2}\right]^2}{\frac{m(m-1)(2m-1)}{6} - \frac{1}{m}\left[\frac{m(m-1)}{2}\right]^2} \tag{3.4}$$

$$= \frac{[\sigma(a,m,c)+6]m - (6X^* + 6c + 3)}{m^2 - 1}, \tag{3.5}$$

where $\sigma(a, m, c) = 12\sum_{i=0}^{m-1}\left(\left(\frac{i}{m}\right)\right)\left(\left(\frac{ai+c}{m}\right)\right)$, $((\cdot)) = \cdot - \frac{1}{2}(\lfloor \cdot \rfloor + \lceil \cdot \rceil)$, and X^* satisfies that $(aX^* + c) \bmod m = 0$.

When m is large in practical applications, the serial correlation can be approximated to (3.6) or (3.7).

$$R \approx \frac{\sigma(a,m,c)+6}{m} \tag{3.6}$$

$$\approx \frac{\sigma(a,m,c)}{m}. \tag{3.7}$$

The upper and lower bounds for the serial correlation can be obtained, which

only involved the parameters a and m (Knuth 1998). For $a = 16{,}807$ and $m = 2^{31} - 1$, the bounds are

$$\frac{-63918.5}{2^{31}-1} \le R \le \frac{127793}{2^{31}-1} \tag{3.8}$$

and

$$-2.9764 \times 10^{-5} \le R \le 5.9508 \times 10^{-5} \tag{3.9}$$

Hence, R will decrease with the increasing of m. More details are discussed in Knuth (1998).

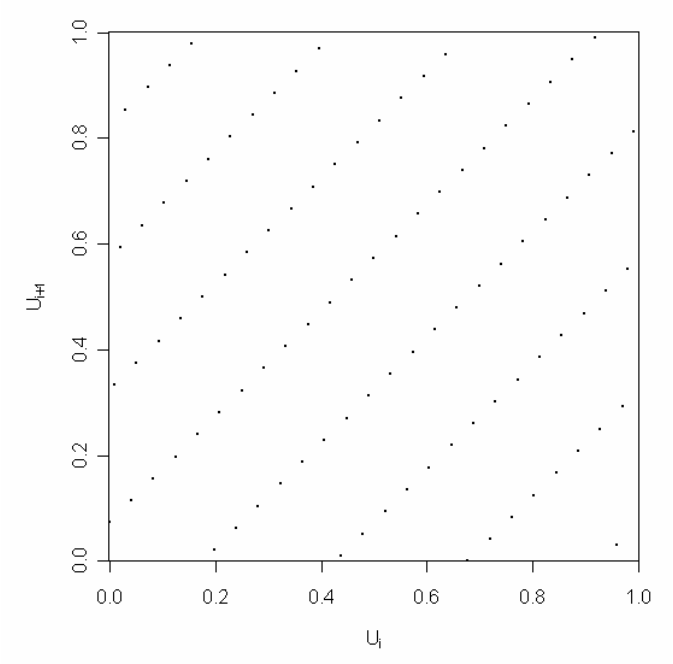
3.3 LATTICE STRUCTURE

The following two LCGs will be used for illustration:

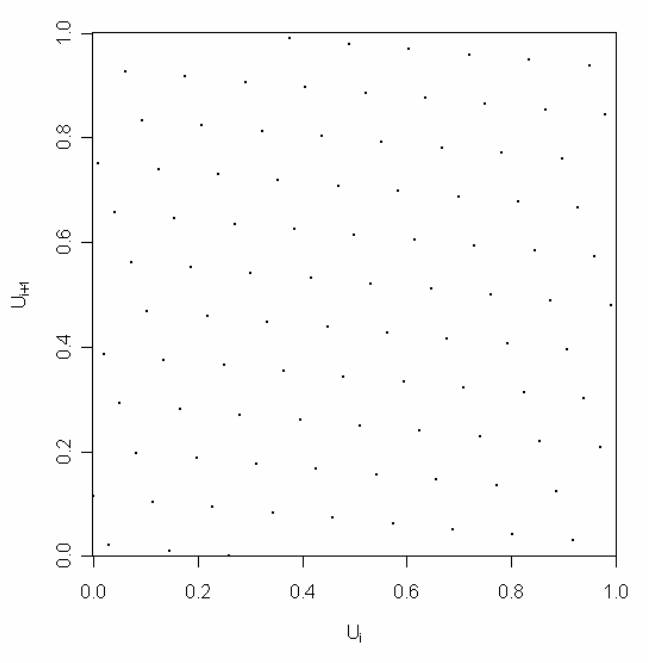
$$X_i = (25X_{i-1} + 7) \bmod 96, \tag{3.10}$$

$$X_i = (61X_{i-1} + 11) \bmod 96. \tag{3.11}$$

Both of them have the full period, 96. Their means and variances are the same consequently. How to make selection between (3.10) and (3.11)? The pairs of consecutive numbers of these two LCGs are plotted in Figure 3.1:



(a)



(b)

Figure 3.1: The pairs of consecutive numbers of these two LCGs in (3.10) and (3.11) are plotted in (a) and (b) respectively.

Ideally it is expected that these pairs of consecutive numbers should uniformly scatter in the unit square. In reality, these pairs fall on lattice structures as proved in Marsaglia (1968). Hence, one can examine the properties of lattice structures to select LCGs (Knuth 1998).

4. PARALLEL LINEAR RANDOM NUMBER GENERATORS

In order to accelerate the simulating time and improve efficiency, it is possible to parallelize the RNG with several computer processors. There are mainly two approaches to combine random numbers generated by different processors, sequence splitting and leapfrogging (Coddington, P. D. 1996). These two approaches will be discussed below.

4.1 Sequence Splitting

The generation of a total sequence of N random numbers is split by the random numbers generated by p processors. Suppose $N = np$ and n is the number of random number that one process generates. The processor j, $j = 1, 2, \ldots, p$, is responsible for producing the following random numbers in the total sequence:

$$X_{(j-1)n+1}, X_{(j-1)n+2}, \ldots, X_{jn} \tag{4.1}$$

4.2 Leapfrogging

The processor under the leapfrogging method generates random numbers in the total sequence with a lag of p. The system works like the allocation a deck of cards to p players. The processor j, $j = 1, 2, \ldots, p$, is responsible for producing the following random numbers in the total sequence:

$$X_j, X_{j+p}, X_{j+2p}, \ldots \tag{4.2}$$

5. PARALLELIZATION WITH DIFFERENT INCREMENT SHIFTS

For the LCG in (2.1), it is possible to consider the changing of a or c for different processor in parallel computing. This will be based on the lattice structures of LCGs as follows.

5.1 Minkowski Bases of the LCG

The Monkowski bases of the lattice structures for LCGs are shown in Marsaglia (1968). They are

$$L = \left\{ V_0 + \sum_{i=1}^{t} z_i V_i \mid z_1, z_2, \ldots, z_t \in \mathbb{Z} \right\} \tag{5.1}$$

where

$$V_0 = \frac{1}{m}\left(0,\ c,\ (1+a)c,\ (1+a+a^2)c,\ \ldots,\ \frac{c(a^{t-1}-1)}{a-1}\right),\ V_1 = \frac{1}{m}\left(1,\ a,\ a^2,\ \ldots,\ a^{t-1}\right),$$

$V_2 = (0, 1, 0, \ldots, 0)$, $V_3 = (0, 0, 1, \ldots, 0)$, …, $V_t = (0, 0, 0, \ldots, 1)$.

The increment c, simply showing up in V_0, accounts for nothing but the shift of lattice structure of an LCG. On the other hand, a does have a great effect on the shape of structure.

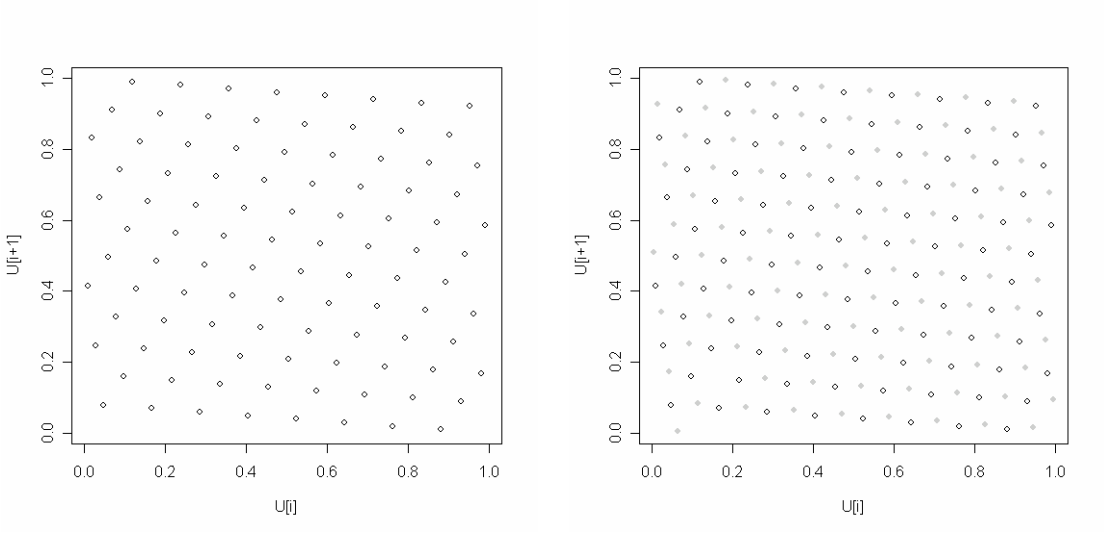
Moreover, (2.1) can be rewrite as composition of the equations below if c is smaller than m:

$$\begin{aligned} X_i &= aX_{i-1} + c, \\ X_i &= aX_{i-1} + c - m, \\ X_i &= aX_{i-1} + c - 2m, \\ &\vdots \\ X_i &= aX_{i-1} + c - (a-1)m; \end{aligned} \tag{5.2}$$

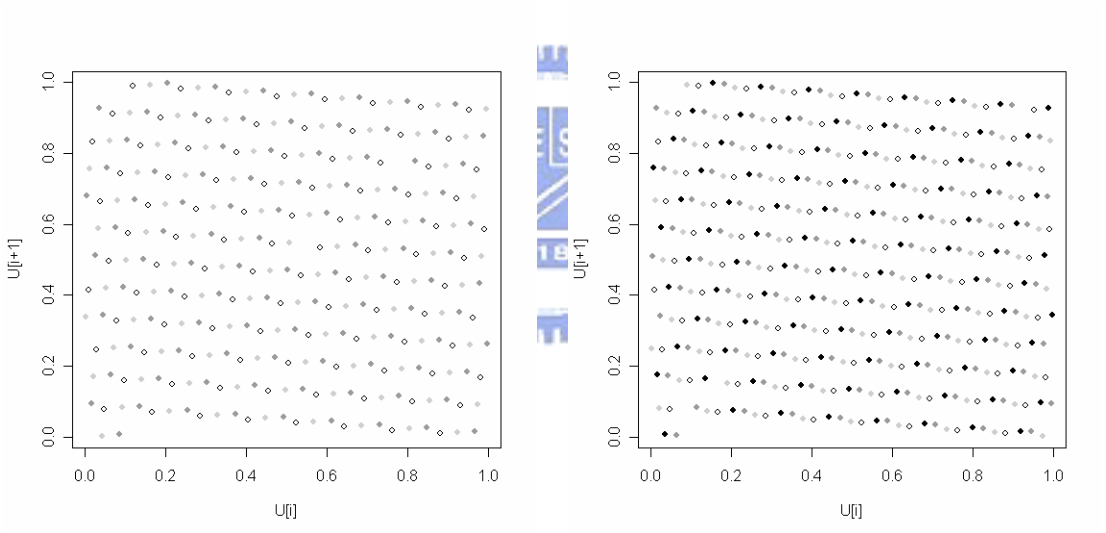
or the followings if c is equal to or greater than m (which is often avoided):

$$\begin{aligned} X_i &= aX_{i-1} + c, \\ X_i &= aX_{i-1} + c - m, \\ X_i &= aX_{i-1} + c - 2m, \\ &\vdots \\ X_i &= aX_{i-1} + c - am. \end{aligned} \tag{5.3}$$

Apparently, the pair of consecutive numbers from LCGs will fall on few parallel lines. The parameter c shifts the lines while the parameter a changes the slope of them. If we choose c appropriately in the parallelization of LCGs, we can fill in the space between parallel lines in the lattice structure efficiently as illustrated in Figure 5.1. However, the changing of a in parallelization is very challenging because the shapes of lattices are changing accordingly. Hence, we will consider the parallelization of LCGs with different shift parameters in this study.



(a) (b)



(c) (d)

Figure 5.1: The lattice structures of parallel LCGs with different shift increments for 1 to 4 processors are illustrated in part (a) to (d).

Let X_i be the random sequence from the processor 1 and Y_i be that from the processor 2. Suppose that these two processors employ LCGs with the same a but different increments, c and c'. Then

$$X_i - Y_i = [a(X_{i-1} - Y_{i-1}) + (c - c')] \bmod m \tag{5.4}$$

$$= \left[a^i (X_0 - Y_0) + \frac{(c - c')(a^i - 1)}{a - 1} \right] \bmod m. \tag{5.5}$$

That is, $X_i - Y_i$ has its special lattice structure. It also implies that the random sequence with leapfrogging method will follow a new lattice structure.

5.2 Parallel Design of Increment Shifts

Another reason for preferring c to a in parallelization is that given an LCG with a long period of m or $m - 1$, it is easy to find another c that yields the long period because it simply requires c be relatively prime to m. But the selection of a for parallelization for long period is very challenging.

To make the parallel design of increment shifts feasible for two processors, we consider:

$$\begin{aligned} &X_i^{(1)} = (aX_{i-1}^{(1)} + c) \bmod m, \quad U_i^{(1)} = X_i^{(1)} / m; \\ &X_i^{(2)} = (aX_{i-1}^{(2)} + \left(\left(\frac{c+m}{2} \right) \right)) \bmod m, \quad U_i^{(2)} = (X_i^{(2)} + 0.5)/(m + 0.5), \\ &\text{where } ((y)) = \begin{cases} \lfloor y \rfloor, & y \bmod 1 \geq 0.5; \\ \lceil y \rceil, & y \bmod 1 < 0.5. \end{cases} \end{aligned} \tag{5.6}$$

The $U_i^{(2)}$ is calculated with the extra addition of 0.5 in both numerator and denominator because it can expand the outcome space. Most important of all, this extra addition ensures the absence of zero which is undesired for most of the practical applications. Also, addition in both numerator and denominator changes neither the mean nor the variance. It doesn't draw different conclusions from $U_i^{(j)} = X_i^{(j)} / m$ under the empirical tests either since there is merely a subtle change for the value of $U_i^{(j)}$ and the difference between any two consecutive random numbers doesn't differ.

Besides, some empirical tests consider only $X_i^{(j)}$, which is not modified at all.

This idea can be extended to more than two processors. For three processors, we consider:

$$\begin{aligned} X_i^{(1)} &= (aX_{i-1}^{(1)} + c) \bmod m, & U_i^{(1)} &= X_i^{(1)} / m; \\ X_i^{(2)} &= (aX_{i-1}^{(2)} + [c + \left(\left(\frac{m-c}{3} \right) \right)]) \bmod m, & U_i^{(2)} &= (X_i^{(2)} + 1/3)/(m + 1/3); \\ X_i^{(3)} &= (aX_{i-1}^{(3)} + [c + \left(\left(\frac{2}{3}(m-c) \right) \right)]) \bmod m, & U_i^{(3)} &= (X_i^{(3)} + 2/3)/(m + 2/3) \end{aligned} \quad (5.7)$$

For p processors, we consider:

$$X_i^{(j)} = (aX_{i-1}^{(j)} + [c + \varepsilon]) \bmod m, \; U_i^{(j)} = (X_i^{(j)} + (j-1)/p)/(m + (j-1)/p), \quad (5.8)$$

where $\varepsilon = \left(\left(\frac{j-1}{p}(m-c) \right) \right)$, $j = 1, 2, \ldots, p$. If $c + \varepsilon$ is not relatively prime to m, set $\varepsilon = \varepsilon_0$, where ε_0 is the nearest integer to ε that makes $c + \varepsilon_0$ relatively prime to m. Moreover, the Euclidean algorithm is capable of helping to check if $c + \varepsilon_0$ is relatively prime to m (Cormem, T. H., Leiserson, C. E., Rivest, R. L., and Stein, C. 2001). In practice, this check can be omitted since m is often chosen from prime numbers.

5.3 Serial Correlation

The quality of this kind of parallel RNG (PRNG) will be investigated. The uniformity, the expectation, and the variance are surely satisfactory. It is necessary to check the serial correlation. Let p be the number of processors. Following the similar steps from (3.3) to (3.5), we get the correlation for the sequence splitting methods as follows:

$$R_{SS} \approx \frac{\left[\sum_{j=1}^{p} \sigma(a_j, m, c_j) + 6p \right] m - 3 \left[2 \sum_{j=1}^{p} (X^{*(j)} + c_j) + p \right]}{pm^2 - p} \quad (5.9)$$

$$R_{SS} \approx \frac{\sum_{j=1}^{p} \sigma(a_j, m, c_j) + 6p}{pm} \tag{5.10}$$

$$\approx \frac{\sum_{j=1}^{p} \sigma(a_j, m, c_j)}{pm} \tag{5.11}$$

Since the term of $\sigma(a_j, m, c_j)$ are only affected by a_j and m, this term does not change among processors with different c_j. As a result, we go back to check the primitive formula. If each processors generates the same amount of random numbers n, then:

$$R_{SS} \approx \frac{\sum_{j=1}^{p}\sum_{i=1}^{n}(X_i^{(j)}X_{i+1}^{(j)}) - \frac{1}{np}\sum_{j=1}^{p}\sum_{i=1}^{n}X_i^{(j)}\sum_{j=1}^{p}\sum_{i=1}^{n}X_{i+1}^{(j)}}{\sqrt{\sum_{j=1}^{p}\sum_{i=1}^{n}(X_i^{(j)})^2 - \frac{1}{np}\sum_{j=1}^{p}\left(\sum_{i=1}^{n}X_i^{(j)}\right)^2}\sqrt{\sum_{j=1}^{p}\sum_{i=1}^{n}(X_{i+1}^{(j)})^2 - \frac{1}{np}\sum_{j=1}^{p}\left(\sum_{i=1}^{n}X_{i+1}^{(j)}\right)^2}} \tag{5.12}$$

$$\approx \frac{12}{m^2 np}\sum_{j=1}^{p}\sum_{i=1}^{n}(X_i^{(j)}X_{i+1}^{(j)}) - 3 \tag{5.13}$$

The equation of (5.13) is derived by using the fact that the sample mean and the sample variance of $X^{(j)}$ are approximately to $0.5p$ and $\frac{p^2}{12}$. Then we get the bounds as following:

$$\min_{1\le j\le p} R_j \le R_{SS} \le \max_{1\le j\le p} R_j \tag{5.14}$$

Hence, the correlation of parallel LCGs by the sequence splitting method is between the best and the worst correlation of p serial LCGs.

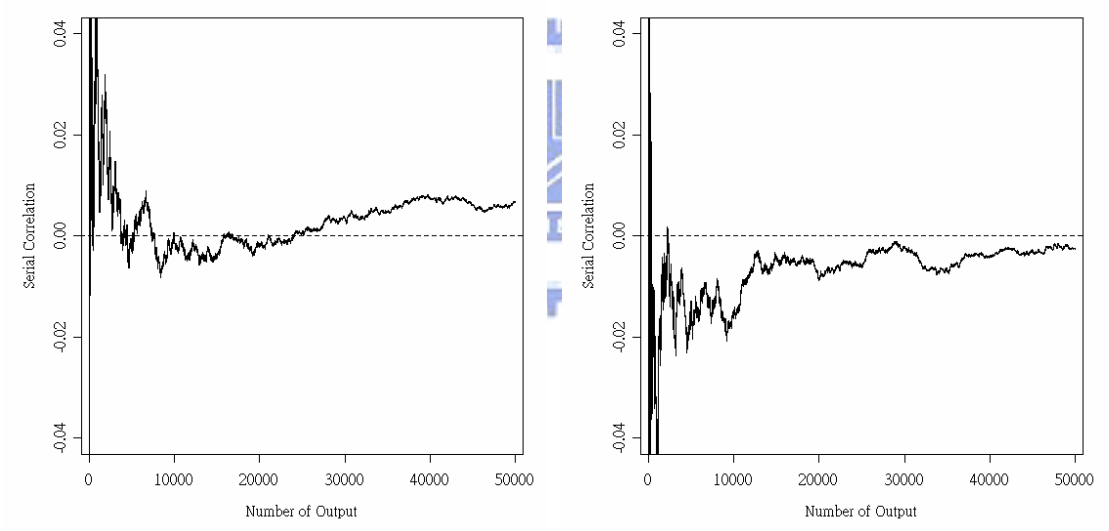
The serial correlation for the leapfrogging method R_{LF} is difficult to obtain by using the same idea from (3.3) to (3.5). We therefore directly check the primitive formula like (5.12):

$$R_{LF} = \frac{\left[\sum_{j=1}^{p-1}\sum_{i=1}^{n}(X_i^{(j)}X_i^{(j+1)}) + \sum_{i=1}^{n}(X_{i+1}^{(1)}X_i^{(p)})\right] - \frac{1}{np}\sum_{j=1}^{p}\sum_{i=1}^{n}X_i^{(j)}\sum_{j=1}^{p}\sum_{i=1}^{n}X_{i+1}^{(j)}}{\sqrt{\sum_{j=1}^{p}\sum_{i=1}^{n}(X_i^{(j)})^2 - \frac{1}{np}\sum_{j=1}^{p}\left(\sum_{i=1}^{n}X_i^{(j)}\right)^2}\sqrt{\sum_{j=1}^{p}\sum_{i=1}^{n}(X_{i+1}^{(j)})^2 - \frac{1}{np}\sum_{j=1}^{p}\left(\sum_{i=1}^{n}X_{i+1}^{(j)}\right)^2}} \tag{5.15}$$

$$R_{\mathrm{LF}} \approx \frac{12}{m^2 np}\left[\sum_{j=1}^{p-1}\sum_{i=1}^{n}(X_i^{(j)}X_i^{(j+1)}) + \sum_{i=1}^{n}(X_{i+1}^{(1)}X_i^{(p)})\right] - 3 \tag{5.16}$$

$$\min_{1\le j\le p} R_j \le R_{\mathrm{LF}} \le \max_{1\le j\le p} R_j \tag{5.17}$$

Thus, R_{LF} falls in the same bounds as R_{SS} does. What is more, the approximate serial correlation in (5.16) will decrease with the increase of $N = np$. The similar phenomenon can also be observed by the approximation of the serial correlation in (5.13). In practice, tremendous amount of random numbers is often generated in large simulation studies and the correlation will not be a major problem as illustrated in Figure 5.2.

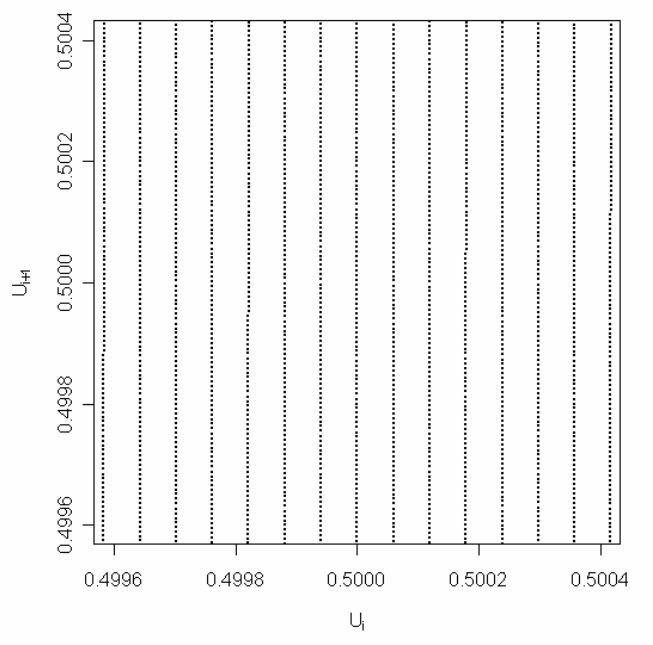


(a) (b)

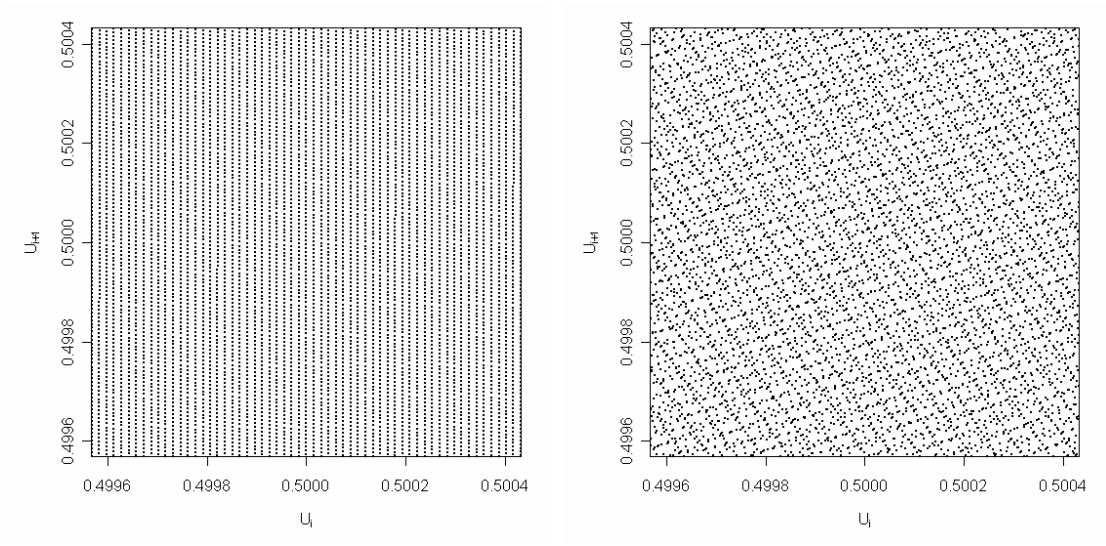
Figure 5.2: Serial correlations of the LCG with a = 16,807, c = 0, $m = 2^{31} - 1$, and p = 4 in both (a) sequence splitting method and (b) leapfrogging method with the increasing of the number of random sequence generated. The absolute value of both correlations have been smaller than 0.01 when the number of output is greater than 15,000.

5.4 Lattice Structures

From the perspectives of uniformity and serial correlation, sequence splitting method and leapfrogging methods have compatible performances. But there is a great difference when they are compared on the plots of the pair of consecutive numbers. Take four processors as an example. The lattice structure of one LCG is plotted in Figure 5.3 (a). The lattice structure of parallel LCG by sequence splitting is plotted in Figure 5.3 (b), which shows the adding of three more sets of parallel lines. The lattice structure of parallel LCG by leapfrogging is plotted in Figure 5.3 (c), which shows more changes of lattice structures that could be useful to create more randomness.



(a)



(b) (c)

Figure 5.3: The lattice structure of the LCG, $X_i = 16807X_{i-1} \bmod (2^{31} - 1)$ is shown in (a). Part (b) and (c) demonstrate the structures of the LCG mentioned-above with four processors and via sequence splitting method and leapfrogging method respectively.

5.5 Inter-processor Correlation

There may exist inter-processor correlation when applying the increment shift method for different processors. It might be expected that there exists highly positive correlation when the initial seeds are the same and the differences among the increments are very small.

For illustration, we simulate pseudo random numbers with the parallel LCG set, $X_i^{(1)} = 16807X_{i-1}^{(1)} \bmod (2^{31}-1)$ and $X_i^{(2)} = (16807X_{i-1}^{(2)}+1) \bmod (2^{31}-1)$ using leap-frogging method. Both initial seeds of these two LCGs are set to 1. The scatter plot is shown in Figure 5.4. The serial correlation is plotted in Figure 5.5. There does not seem to have high inter-correlation even when the initial seeds are the same and the differences among the increments are very small.

That is quite counter-intuition. But suppose the initial seeds are the same in (5.5),

then we can simplify the equation into

$$X_i - Y_i = \left[\frac{(c-c')(a^i - 1)}{a-1} \right] \bmod m. \tag{5.18}$$

$$Y_i = \{X_i - \left[\frac{(c-c')(a^i - 1)}{a-1} \right]\} \bmod m. \tag{5.19}$$

This shows the random sequence still possesses a new lattice structure even without the term of $a^i(X_0 - Y_0)$. Namely, the scatter plot of (X_i, Y_i) or that of $(X_i^{(1)}, X_i^{(2)})$ in Figure 5.3 does never appear with a highly correlated pattern, say, a line. The serial correlation

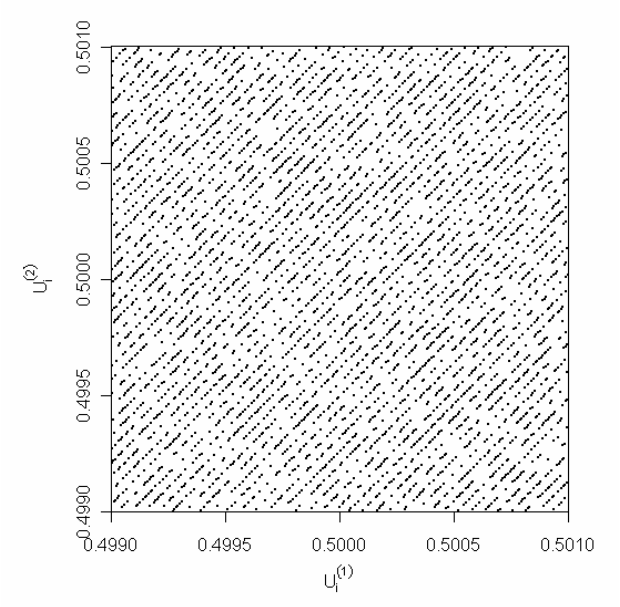


Figure 5.4: Scatter plot for the pair of $(X_i^{(1)}, X_i^{(2)})$ is shown for the parallel LCG set, $X_i^{(1)} = 16807 X_{i-1}^{(1)} \mod (2^{31} - 1)$ and $X_i^{(2)} = (16807 X_{i-1}^{(2)} + 1) \mod (2^{31} - 1)$.

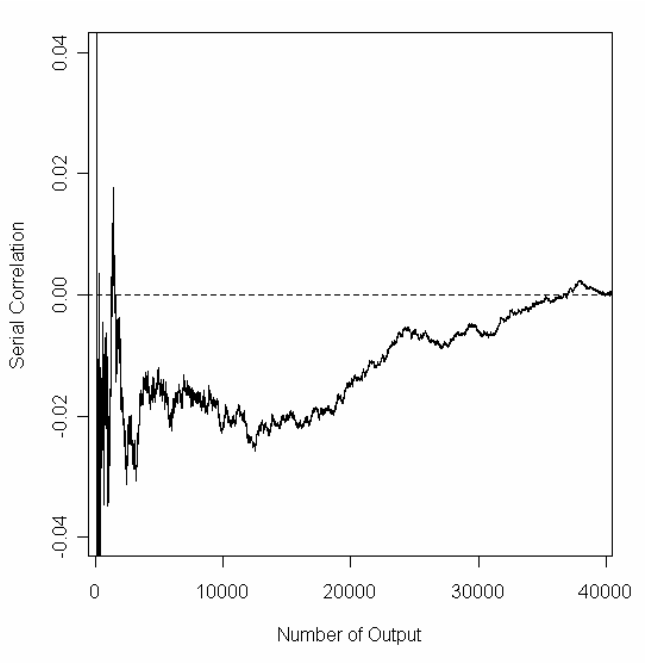


Figure 5.5: Serial correlations of this parallel LCG set with leapfrogging method. The correlation is desirably small especially when the number of pseudo-random number generated is larger than 25,000.

5.6 Absorbing States

It is aimed to parallelize the multiplicative LCGs that can attain the maximum periods. Each processor of the multiplicative LCG yields random numbers with a period of $m - 1$. It is evident that 0 is an absorbing state of the multiplicative LCG that is not desired. If the initial seed is the absorbing seed of 0, then the multiplicative LCG will enter the absorbing state of 0. The absorbing states and seeds shall be avoided in the parallelization of RNGs.

For the increment c in (2.1) $\neq 0$, we can figure out the absorbing states and seeds by means of the extended Euclidean algorithm in the followings. Let X_a denote as the absorbing seed, then we can rewrite (2.1) into (5.20) and (5.21):

$$X_a = (aX_a + c) \bmod m; \tag{5.20}$$

$$(a - 1)X_a \bmod m = -c \bmod m. \tag{5.21}$$

As $c < m$, we can simplify (5.21) as follows:

$$(a-1)X_a \bmod m = m - c. \tag{5.22}$$

Then it becomes the question of solving the linear congruence.

Firstly, we consider a linear congruence of $Ax \bmod M = 1$. We can find an integer k such that

$$Ax - kM = 1. \tag{5.23}$$

If $y = -k$,

$$Ax + My = 1 \tag{5.24}$$

Through the extended Euclidean algorithm (Cormem, T. H., Leiserson, C. E., Rivest, R. L., and Stein, C. 2001), we are able to obtain one of the solutions, say (x_0, y_0).

Next, we try to solve $Ax \bmod M = B$. In the beginning, we multiply B in the equation $Ax \bmod M = 1$ and substitute the solution x_0 into it:

$$B(Ax_0 \bmod M) = B; \tag{5.25}$$

$$(B \bmod M)(Ax_0 \bmod M) = B; \tag{5.26}$$

$$A(Bx_0) \bmod M = B. \tag{5.27}$$

Then, it is clear that Bx_0 satisfies the linear congruence $Ax \bmod M = B$. The same idea goes in (5.22). For the purpose of figuring out X_a, we can employ the extended Euclidean algorithm to find out α_0 in (5.28).

$$(a-1)\alpha_0 + m\beta_0 = 1. \tag{5.28}$$

In order to ensure that the absorbing seed is smaller than m, we let

$$X_a = [\alpha_0 \times (m-c)] \bmod m. \tag{5.29}$$

Since we can calculate the absorbing seeds, we ought to avoid them in advance. Obviously, the more processors are hired, the more starting values should be avoided.

Table 5.1: An example of $a = 16{,}807$ and $m = 2^{31} - 1$ to illustrate the implementation of the algorithm via the iterative method.

$$u_1 = m = 2^{31} - 1, \quad u_2 = a - 1 = 16806,$$

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \beta_1 = 1, \quad \beta_2 = 0,$$

$$q_i = \lfloor u_{i-1} / u_i \rfloor, \quad i = 2, 3, \ldots$$

$$u_j = u_{j-2} - u_{j-1} \times q_{j-1} \quad \alpha_j = \alpha_{j-2} - \alpha_{j-1} \times q_{j-1} \quad \beta_j = \beta_{j-2} - \beta_{j-1} \times q_{j-1}, \quad j = 3, 4, \ldots$$

Repeat the computation until $u_j = 1$. The last α_j is the solution, α_0 in (5.28).

For example, $q_2 = \lfloor (2^{31} - 1)/16806 \rfloor = 127{,}780$, $u_3 = (2^{31} - 1) - 16806 \times 127780 = 12{,}967$, $\alpha_3 = 0 - 1 \times 127780 = -127{,}780$, $\beta_3 = 1 - 0 \times 127780 = 1$, … Because $u_{14} = 3 - 2 \times 1 = 1$, the iteration is stopped, and $\alpha_{14} = 827{,}891{,}619$ is our solution.

u	α	β	q
2147483647	0	1	
16806	1	0	127780
12967	–127780	1	1
3839	127781	–1	3
1450	–511123	4	2
939	1150027	–9	1
511	–1661150	13	1
428	2811177	–22	1
83	–4472327	35	5
13	25172812	–197	6
5	–155509199	1217	2
3	336191210	–2631	1
2	–491700409	3848	1
1	827891619	–6479	2

From Table 5.1, we get $\alpha_0 = \alpha_{14} = 827{,}891{,}619$, and the absorbing seed is:

$$X_a = [827891619 \times (2^{31} - 1 - c)] \bmod (2^{31} - 1) \tag{5.30}$$

Table 5.2: The table shows the parallel LCG set with $a = 16{,}807$ and $m = 2^{31} - 1$ with processors varied from one to ten. For instance, for 3 processors, the parallel LCG set will be $X_i^{(1)} = 16807 X_{i-1}^{(1)} \bmod (2^{31} - 1)$, $X_i^{(2)} = (16807 X_{i-1}^{(2)} + 715827882) \bmod (2^{31} - 1)$, and $X_i^{(3)} = (16807 X_{i-1}^{(3)} + 1431655765) \bmod$. The absorbing state occurs at 0, 275,963,873, and 1,871,519,774, respectively.

No. of	Absorbing	j^{th} Processor				
Proc	State	1	2	3	4	5
1	Coefficient	0				
	Absorbing Seed	0				
2	Coefficient	0	1073741824			
	Absorbing Seed	0	659796014			
3	Coefficient	0	715827882	1431655765		
	Absorbing Seed	0	275963873	1871519774		
4	Coefficient	0	536870912	1073741824	1610612735	
	Absorbing Seed	0	329898007	659796014	1817585640	
5	Coefficient	0	429496729	858993459	1288490188	1717986918
	Absorbing Seed	0	760653377	693415135	1454068512	1386830270

Table 5.2 (Continued)

No. of Proc	Absorbing State	j^{th} Processor 1 (6)	2 (7)	3 (8)	4 (9)	5 (10)
6	Coefficient	0	357913941	715827882	1073741824	1431655765
	Absorbing Seed	0	1211723760	275963873	659796014	1871519774
	Coefficient	1789569706				
	Absorbing Seed	935759887				
7	Coefficient	0	306783378	613566756	920350134	1227133513
	Absorbing Seed	0	1652187122	1156890597	661594072	1485889575
	Coefficient	1533916891	1840700269			
	Absorbing Seed	990593050	495296525			
8	Coefficient	0	268435456	536870912	805306368	1073741824
	Absorbing Seed	0	1238690827	329898007	1568588834	659796014
	Coefficient	1342177279	1610612735	1879048191		
	Absorbing Seed	578894813	1817585640	908792820		
9	Coefficient	0	238609294	477218588	715827882	954437176
	Absorbing Seed	0	807815840	1615631680	275963873	1083779713
	Coefficient	1193046471	1431655765	1670265059	1908874353	
	Absorbing Seed	1063703934	1871519774	531851967	1339667807	
10	Coefficient	0	214748365	429496729	644245094	858993459
	Absorbing Seed	0	2113864526	760653377	727034256	693415135
	Coefficient	1073741824	1288490188	1503238553	1717986918	1932735282
	Absorbing Seed	659796014	1454068512	1420449391	1386830270	33619121

Table 5.3: List of α_0's for some other popular multipliers with the same modulus = $2^{31} - 1$. The absorbing seeds can be found by $X_a = [\alpha_0 \times (2^{31} - 1 - c)] \bmod (2^{31} - 1)$.

a	41358	48271	62089911	397204094	630360016	742938285
α_0	700423312	–179424105	751265057	–858674923	–895497791	–447186684

Table 5.2 shows different absorbing seeds for the parallel LCG set with a = 16,807 and $m = 2^{31} - 1$ under different processors. Via the same algorithm, absorbing states of other popular LCGs are listed in Table 5.3.

The increment shift method can also extend to the MRG case because (2.3) is able to be rewritten as composition of some parallel hyperplanes like (5.2) and (5.3). The parameter c only shifts this planes. Consequently, we now add the increment c in (2.3):

$$X_i = (a_1 X_{i-1} + a_2 X_{i-2} + \cdots + a_k X_{i-k} + c) \bmod m \tag{5.31}$$

If the generator enters the absorbing state, the output vector (X_{i-k}, X_{i-k+1}, …, X_{i-1}) and the next one (X_{i-k+1}, X_{i-k+2}, …, X_i) are suppose to be the same. It only occurs when $X_{i-k} = X_{i-k+1} = \cdots = X_{i-1} = X_i$. In other words, the absorbing state arises when the initial vector is (X_a, X_a, …, X_a). Therefore, we can solve out the absorbing seed by replacing all X_i's with X_a:

$$X_a = (a_1 X_a + a_2 X_a + \cdots + a_k X_a + c) \bmod m \tag{5.32}$$

$$(\sum_{i=1}^{k} a_k X_a - 1) \bmod m = m - c \tag{5.33}$$

Now it turns into the question concerning the linear congruence again. But the equation changes from (5.27) into (5.34).

$$(\sum_{i=1}^{k} a_k - 1)\alpha_0 + m\beta_0 = 1 \tag{5.34}$$

6. SIMULATION RESULT

First we study the parallel LCG with a = 16,807 and $m = 2^{31} - 1$. One billion pseudo-random numbers are generated with ten different seeds and processors by leapfrogging method. We employ DIEHARD tests which contains 17 test sets and 269 p-values (Marsaglia, G. 1995 and Gentle, J. E. 2003). Besides, α = 0.01 and 0.05 are chosen as the threshold of the p-values.

Figure 6.1 illustrates the average passing rate of 269 p-values from ten different initial seeds with different parameters. From only one processor to two, the performance of the parallel LCG greatly improves. It keeps improving until p = 4 and then falls back around 80% – but the performances are still better than that by only one processor.

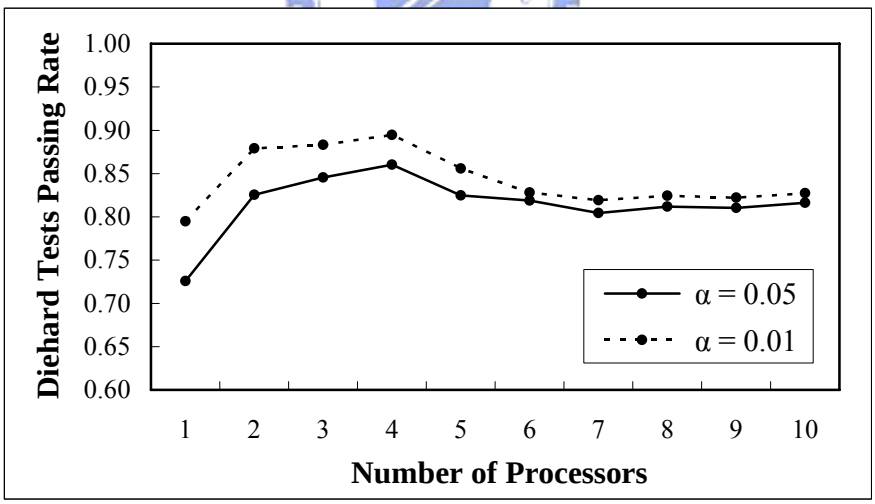


Figure 6.1: Passing rate of the DIEHARD test for the parallel LCG with a = 16,807 and $m = 2^{31} - 1$. Processors varied from one to ten are tested and the threshold α = 0.01 and 0.05 are used. Ten different initial seeds are make and averaged to calculate the passing rate.

Next, we check the MRG of DX-1597-4 in (2.12). For the purpose of avoiding the problem of overfloating, Jean-Louis (2005) suggest the change of (2.12) to (6.1) in programming:

$$X_i = 29{,}746[36{,}097(X_{i-1} + X_{i-533} + X_{i-1065} + X_{i-1597}) \bmod (2^{31} - 1)] \bmod (2^{31} - 1). \quad (6.1)$$

The same test package is employed to study this parallel generator. First, it is obvious that the MRG pass more tests than the LCG. The simulation result gives the similar results that the number of processors is suggested to be two or four. Additionally, Since the original MRG (with p = 1) has passed many tests, more processors don't do much improvement in this case.

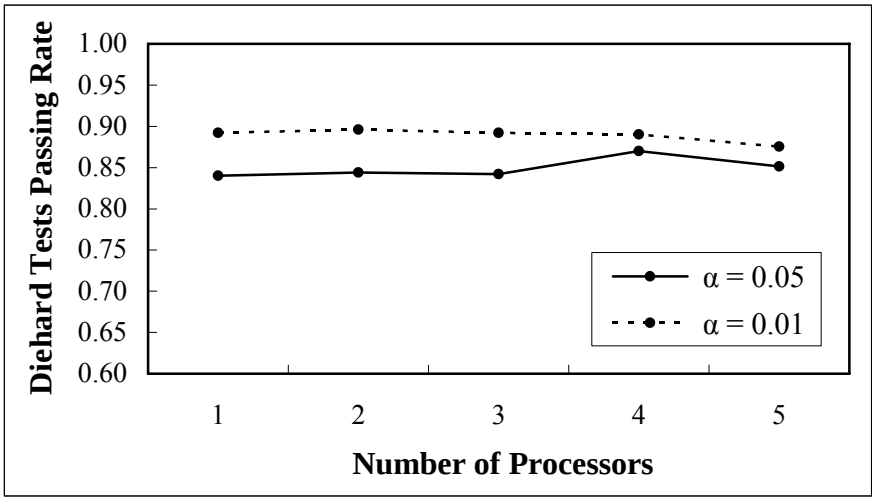


Figure 6.2: Passing rate of the DIEHARD test for the parallel MRG of DX-1597-4 in (2.12). Processors varied from one to five are tested and the threshold α = 0.01 and 0.05 are used. Two different initial seeds are make and averaged to calculate the passing rate.

7. CONCLUSION AND DISCUSSION

We have proposed a simple and effective approach to parallelize linear RNGs with different increment shifts, including LCGs and MRGs. The implementation of

this kind of parallel RNGs is easy and the improvements are as expected. Theoretic and simulation investigations are performed to confirm this new proposal.

There are two major approaches to connect random numbers generated by different processors, sequence splitting and leapfrogging. From the properties of lattice structure, the leapfrogging method dominates in the simulation studies. Also, theoretically, the leapfrogging method yields the much better lattice structure than the sequence splitting.

In particular, we examine the parallel LCG with a = 16,807 and $m = 2^{31} - 1$ through simulation tests. On the average, it performs best when p = 4 among 1 to 10 processors. It is interesting to perform more investigation to explore more about this phenomenon. More studies are of interest to pursuit in the future to understand the properties of this kind of parallel linear RNGs.

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APPENDIX

Table A1: Summary results of DIEHARD test for the parallel LCG with a = 16,807, $m = 2^{31} - 1$, and the number of processors p varied from one to ten. The threshold α = 0.01 and 0.05 are set to calculate the passing rate of the LCG. Ten different initial seeds X_0's are used and the results are averaged to plot the Figure 6.1.

Seed	1		214748365		429496729		644245094		858993458	
No of p	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01
1	0.7212	0.7918	0.7138	0.7955	0.7398	0.7993	0.7249	0.7955	0.7361	0.8030
2	0.8625	0.8959	0.8699	0.9071	0.8067	0.8848	0.7435	0.8290	0.8141	0.8587
3	0.8364	0.8810	0.8327	0.8699	0.8104	0.8736	0.8401	0.8699	0.8550	0.8848
4	0.8662	0.8996	0.8736	0.8996	0.8662	0.8922	0.8550	0.8959	0.8513	0.8959
5	0.8327	0.8625	0.8141	0.8476	0.8253	0.8587	0.8253	0.8550	0.8290	0.8587
6	0.8216	0.8253	0.8141	0.8290	0.8216	0.8290	0.8216	0.8327	0.8216	0.8290
7	0.8104	0.8216	0.7993	0.8141	0.7993	0.8141	0.8067	0.8178	0.8067	0.8216
8	0.8104	0.8253	0.8067	0.8253	0.8141	0.8253	0.8141	0.8216	0.8067	0.8253
9	0.8104	0.8216	0.8104	0.8253	0.7993	0.8141	0.8104	0.8216	0.8141	0.8178
10	0.8141	0.8253	0.8141	0.8290	0.8178	0.8290	0.8178	0.8290	0.8216	0.8290

Seed	1073741823		1288490188		1503238552		1717986917		1932735281	
No of p	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01
1	0.7100	0.7807	0.7323	0.7955	0.7249	0.7955	0.7286	0.7993	0.7249	0.7918
2	0.8141	0.8810	0.8513	0.8922	0.8253	0.8736	0.8439	0.8922	0.8253	0.8773
3	0.8699	0.8922	0.8699	0.8996	0.8178	0.8885	0.8550	0.8810	0.8662	0.8922
4	0.8587	0.8959	0.8550	0.8922	0.8625	0.8922	0.8699	0.8959	0.8439	0.8885
5	0.8290	0.8550	0.8178	0.8550	0.8178	0.8476	0.8253	0.8550	0.8327	0.8625
6	0.8216	0.8290	0.8253	0.8253	0.8104	0.8290	0.8178	0.8290	0.8141	0.8253
7	0.7993	0.8178	0.8030	0.8178	0.8067	0.8216	0.8104	0.8253	0.8030	0.8216
8	0.8141	0.8253	0.8104	0.8216	0.8141	0.8290	0.8104	0.8216	0.8178	0.8253
9	0.8178	0.8253	0.8104	0.8216	0.8067	0.8253	0.8141	0.8253	0.8104	0.8253
10	0.8178	0.8253	0.8178	0.8290	0.8141	0.8253	0.8141	0.8290	0.8141	0.8253

Table A2: Summary results of DIEHARD test for the parallel MRG of DX-1597-4 in (2.12), and the number of processors p varied from one to five. The threshold $\alpha = 0.01$ and 0.05 are set to calculate the passing rate of the MRG. Two initial seeds X_0's are used and the results are averaged to plot the Figure 6.2.

Seed	1		1073741823	
No of p	0.05	0.01	0.05	0.01
1	0.8364	0.8959	0.8439	0.8885
2	0.8476	0.8922	0.8401	0.8996
3	0.8364	0.8885	0.8476	0.8959
4	0.8699	0.8885	0.8699	0.8922
5	0.8476	0.8699	0.8550	0.8810