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Note

A note on triangle-free distance-regular graphs with $a_2 \neq 0$ ☆

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ABSTRACT

Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geqslant 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. We show that the intersection number c_2 is either 1 or 2, and if $c_2 = 1$, then $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3)$.

1. Introduction

Brouwer, Cohen, and Neumaier invented the term classical parameters (D, b, α, β) to describe the intersection numbers of most known families of distance-regular graphs [3, p. ix, p. 193]. All classical parameters (D, b, α, β) of distance-regular graphs with $b = 1$ are classified by Y. Egawa, A. Neumaier and P. Terwilliger in a sequence of papers (see [3, p. 195] for a detailed description). For $b < -1$, the classification is done in the case $c_2 = 1$, $a_2 > a_1 > 1$, $D \geqslant 4$ [14], and in the case $a_1 \neq 0$, $c_2 > 1$, $D \geqslant 4$ [15]. For the case $a_1 = 0$, Miklavič shows the graph is 1-*homogeneous* [6]. A. Jurišić, J. Koolen, and Š. Miklavič study distance-regular graphs in the cases $a_1 = 0$, $a_2 = 0$, without the assumption of classical parameters, but instead with an additional assumption that the graphs have an eigenvalue multiplicity equal to the valency, and they almost classify such graphs [5]. In this note, we study distance-regular graphs with classical parameters (D, b, α, β), $D \geqslant 3$, $a_1 = 0$, and $a_2 \neq 0$ (hence $b < -1$ by [7, Lemma 3.3]). We prove $c_2 \leqslant 2$, and if $c_2 = 1$ then $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3)$. Note that Witt graph M_{23} is the only known example of such graph with $c_2 = 1$ and the class of Hermitian forms graphs $\mathrm{Her}_2(D)$ is the only known family satisfying the conditions with $c_2 = 2$.

First we review some definitions and basic concepts concerning distance-regular graphs. See Bannai and Ito [1] or Terwilliger [10,11] for more background information.

Let $\Gamma = (X, R)$ denote a finite undirected, connected graph without loops or multiple edges with vertex set X, edge set R, distance function ∂, and diameter $D := \max\{\partial(x, y) \mid x, y \in X\}$.

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For a vertex $x \in X$ and an integer $0 \leqslant i \leqslant D$, set $\Gamma_i(x) := \{z \in X \mid \partial(x,z) = i\}$. The *valency* $k(x)$ of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$. The graph Γ is called *regular* (with *valency* k) if each vertex in X has valency k.

A graph Γ is said to be *distance-regular* whenever for all integers $0 \leqslant h, i, j \leqslant D$, and all vertices $x, y \in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij} = \left|\Gamma_i(x) \cap \Gamma_j(y)\right|$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ.

For convenience, set $c_i := p^i_{1\,i-1}$ for $1 \leqslant i \leqslant D$, $a_i := p^i_{1\,i}$ for $0 \leqslant i \leqslant D$, $b_i := p^i_{1\,i+1}$ for $0 \leqslant i \leqslant D-1$, $k_i := p^0_{i\,i}$ for $0 \leqslant i \leqslant D$, and set $b_D := 0$, $c_0 := 0$, $k := b_0$. Note that k is the valency of Γ. It follows immediately from the definition of p^h_{ij} that $b_i \neq 0$ for $0 \leqslant i \leqslant D-1$ and $c_i \neq 0$ for $1 \leqslant i \leqslant D$. Moreover

$$k = a_i + b_i + c_i \quad \text{for } 0 \leqslant i \leqslant D, \tag{1.1}$$

and

$$k_i = \frac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i} \quad \text{for } 1 \leqslant i \leqslant D. \tag{1.2}$$

A *strongly regular graph* is a distance-regular graph with diameter 2. We quote a lemma about strongly regular graphs which will be used in the next section.

Lemma 1.1. *(See [2, Theorem 19, p. 276].) Let Ω be a strongly regular graph with valency k which has k^2+1 vertices. Then $k \in \{2, 3, 7, 57\}$.*

Let Γ be a distance-regular graph with diameter D. Γ is said to have *classical parameters* (D, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \quad \text{for } 0 \leqslant i \leqslant D, \tag{1.3}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) \quad \text{for } 0 \leqslant i \leqslant D, \tag{1.4}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}. \tag{1.5}$$

Suppose Γ has classical parameters (D, b, α, β). Combining (1.3)–(1.5) with (1.1), we have

$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(\beta - 1 + \alpha\left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\right) \quad \text{for } 0 \leqslant i \leqslant D. \tag{1.6}$$

A subgraph Δ of Γ is called *weak-geodetically closed* whenever for any vertices $x, y \in \Delta$ with $\partial(x,y) = i$,

$$\Gamma_1(x) \cap \left(\Gamma_i(y) \cup \Gamma_{i-1}(y)\right) \subseteq \Delta.$$

Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in [9]. Let t be a positive integer. Γ is said *t-bounded*, whenever for any integer $0 \leqslant i \leqslant t$, and any two vertices $x, y \in X$ with $\partial(x,y) = i$, x, y are contained in a regular weak-geodetically closed subgraph $\Delta(x,y)$ of diameter i. Furthermore by [13, Theorem 4.6], $\Delta(x,y)$ is a distance-regular graph with intersection numbers

$$a_j\left(\Delta(x,y)\right) = a_j(\Gamma), \tag{1.7}$$

$$c_j\left(\Delta(x,y)\right) = c_j(\Gamma), \tag{1.8}$$

$$b_j\left(\Delta(x,y)\right) = b_j(\Gamma) - b_i(\Gamma) \tag{1.9}$$

for $0 \leqslant j \leqslant i$. We need the following three lemmas in the proof of our main theorem.

Lemma 1.2. *(See [15, Lemma 4.10].) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β). Let Δ be a regular weak-geodetically closed subgraph of Γ. Then Δ is distance-regular with classical parameters (t, b, α, β'), where t denotes the diameter of Δ, and $\beta' = \beta + \alpha(\left[{D \atop 1}\right] - \left[{t \atop 1}\right])$.*

Lemma 1.3. *(See [14, Lemma 4.1(iii)].) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geqslant 3$. Assume Γ is D-bounded. Then*

$$b^{3D-3i-4}(b+1-c_2)\big(\alpha\big(b^D+1\big)+\beta(b-1)\big) \geqslant 0 \tag{1.10}$$

for $1 \leqslant i \leqslant D-2$.

Lemma 1.4. *(See [7, Lemma 3.3] and [8, Theorem 1.3].) Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geqslant 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then the following* (i) *and* (ii) *hold.*

(i) *$\alpha < 0$ and $b < -1$.*
(ii) *Γ is 3-bounded.*

2. Main result

Let $\Gamma = (X, R)$ be a distance-regular graph which has classical parameters (D, b, α, β) with $D \geqslant 3$. Suppose the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $\alpha < 0$ and $b < -1$ by Lemma 1.4. Now we are ready to prove our main theorem.

Theorem 2.1. *Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geqslant 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $c_2 \leqslant 2$.*

Proof. Γ is 3-bounded by Lemma 1.4. Let Δ be a weak-geodetically closed subgraph of Γ with diameter 3. Then Δ has classical parameters $(3, b, \alpha, \beta')$, where $\beta' = \beta + \alpha(\left[{D \atop 1}\right] - \left[{3 \atop 1}\right])$ by Lemma 1.2. Applying $a_1 = 0$ to (1.6) and by (1.5) we have

$$\beta' = 1 + \alpha - \alpha\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = 1 - \alpha b - \alpha b^2. \tag{2.1}$$

Applying Lemma 1.3 to $\Gamma = \Delta$ with $D = 3$ and $i = 1$, we have

$$b^2(b+1-c_2)\big(\alpha\big(b^3+1\big)+\beta'(b-1)\big) \geqslant 0. \tag{2.2}$$

Note that $b + 1 - c_2 < 0$ since $b < -1$. Combining this with inequality (2.2) we find

$$\alpha\big(b^3+1\big)+\beta'(b-1) \leqslant 0. \tag{2.3}$$

Evaluating (2.3) using (2.1) we find

$$\alpha b + \alpha + b - 1 \leqslant 0. \tag{2.4}$$

Note that $\alpha b + \alpha + b - 1 = c_2 - 2$ by (1.3) and hence $c_2 \leqslant 2$. $\square$

For the case $c_2 = 1$, we have the following result.

Theorem 2.2. *Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geqslant 3$. Assume the intersection numbers $a_1 = 0$, $a_2 \neq 0$, and $c_2 = 1$. Then $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3)$.*

Proof. Combining $a_1 = 0$ and $c_2 = 1$ with (1.6), (1.3), and (1.5) we have

$$\alpha = \frac{-b}{1+b}, \tag{2.5}$$

$$\beta = \frac{b^{D+1}-1}{b^2-1}. \tag{2.6}$$

Let $\Omega \subset \Delta$ be two weak-geodetically closed subgraphs of Γ with diameter 2 and 3 respectively. Note that Ω is a strongly regular graph with $a_1(\Omega) = 0$, $c_2(\Omega) = 1$ by (1.7) and (1.8). Combining this with (1.1) and (1.2) we have

$$|\Omega| = 1 + k_1(\Omega) + k_2(\Omega) = 1 + \big(b_0(\Omega)\big)^2.$$

Hence

$$b_0(\Omega) = 2,\ 3,\ 7,\ 57 \tag{2.7}$$

by Lemma 1.1. Note that

$$b_0(\Omega) = b_0 - b_2 = 1 + b + b^2 \tag{2.8}$$

by (1.9), (1.4), (2.5), and (2.6). Solving (2.7) with (2.8) for integer $b < -1$ we have $b = -2$, -3, or -8. Note that

$$k_3(\Delta) = \frac{(b_0 - b_3)(b_1 - b_3)(b_2 - b_3)}{c_1 c_2 c_3} \tag{2.9}$$

by (1.2), (1.8), and (1.9). Evaluating (2.9) using (1.3)–(1.5), (2.5), and (2.6) we have

$$k_3(\Delta) = \frac{b^3(b^2+1)(b^2+b+1)(b^3+b^2+2b+1)}{1-b}. \tag{2.10}$$

The number $k_3(\Delta)$ is not an integer when $b = -3$ or -8. Hence $b = -2$ and $\alpha = -2$, $\beta = ((-2)^{D+1} - 1)/3$ by (2.5) and (2.6) respectively. □

Example 2.3. [4] Hermitian forms graph $\mathrm{Her}_2(D)$ is a distance-regular graph with classical parameters (D, b, α, β) with $b = -2$, $\alpha = -3$, and $\beta = -(-2)^D - 1$, which has $a_1 = 0$, $a_2 \neq 0$, and $c_2 = (1+\alpha)(b+1) = 2$. This is the only known class of examples that satisfies the assumptions of Theorem 2.1 with $c_2 = 2$.

Example 2.4. (See [12, p. 237].) Gewirtz graph is a distance-regular graph with diameter 2 and intersection numbers $a_1 = 0$, $a_2 = 8$, $k = 10$, which can be written as classical parameters (D, b, α, β) with $D = 2$, $b = -3$, $\alpha = -2$, and $\beta = -5$, so we have $c_2 = (b+1)^2/2 = 2$. It is still open if there exists a class of distance-regular graphs with classical parameters $(D, -3, -2, (-1-(-3)^D)/2)$ for $D \geqslant 3$.

Example 2.5. (See [3, Table 6.1].) Witt graph M_{23} is a distance-regular graph with classical parameters (D, b, α, β) with $D = 3$, $b = -2$, $\alpha = -2$, and $\beta = 5$, which has $a_1 = 0$, $a_2 = 2$, and $c_2 = 1$. This is the only known example that satisfies the assumptions of Theorem 2.1 with $c_2 = 1$.

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