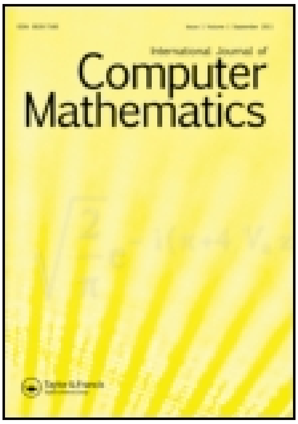

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Some research on the folded Petersen cube networks have been published for the past several years due to its favourite properties. In this paper, we consider the fault-tolerant hamiltonicity and the fault-tolerant hamiltonian connectivity of the folded Petersen cube networks. We use $FPQ_{n,k}$ to denote the folded Petersen cube networks of parameters n and k. In this paper, we show that $FPQ_{n,k} - F$ remains hamiltonian for any $F \subseteq V(FPQ_{n,k}) \cup E(FPQ_{n,k})$ with $|F| \leq n + 3k - 2$ and $FPQ_{n,k} - F$ remains hamiltonian connected for any $F \subseteq V(FPQ_{n,k}) \cup E(FPQ_{n,k})$ with $|F| \leq n + 3k - 3$ if $(n, k) \notin \{(0, 1)\} \cup \{(n, 0) \mid n$ is a positive integer$\}$. Moreover, this result is optimal.

Keywords: hamiltonian; hamiltonian connected; folded Petersen cube networks

2000 AMS Subject Classification: 05C45; 05C75; 05C85; 05C90; 68M10

1. Introduction

A large number of forms of topology have been proposed and studied for multicomputer interconnection networks. Such a form of topology is usually modelled as an undirected graph, where the set of vertices represents the processors and the set of edges represents the bidirectional communication links between the processors. The existing static interconnection networks include linear arrays, rings, meshes, complete binary trees, X-trees, full-ringed binary trees, tree machines, pyramids, fat trees [16], hypercubes [18], meshes of trees, cube-connected cycles [17], de Bruijn networks [19], and so on. For general surveys on multicomputer networks, refer to [13,14]. Among these networks, the hypercube family has been popular because of such properties as symmetry, regularity, high fault-tolerance, logarithmic degree and diameter, and selfrouting and simple broadcasting schemes. Also, several commercial multicomputer architectures (*e.g.*, Intel

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iPSC/2, NCUBE/10, and connection machine CM-2) use the hypercube topology for interconnecting processors. Nevertheless, new networks are being proposed and analysed with regard to their applicability and enhanced topological or performance properties.

It is almost impossible to design a network that is optimum from all aspects. One has to design a suitable network depending on the requirements of its properties. The hamiltonian property is one of the major requirements in designing the topology of a network. For example, 'Token Passing' approach is used in some distributed operation systems. Interconnection network requires the presence of hamiltonian cycles in the structure to meet this approach. Fault-tolerance is another major requirement in the designing of network topology. Thus, the fault-tolerant hamiltonicity is studied [9–12]. The fault-tolerant hamiltonicity of a network and the fault-tolerant hamiltonian connectivity for the same network are also studied [6–8,21].

In this paper, a network is represented as a loopless undirected graph. For graph definitions and notations, we follow [1]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. The *degree* of a vertex u of G, $\deg_G(u)$, is the number of edges incident with u. We use $\delta(G)$ to denote $\min\{\deg_G(x) \mid x \in V(G)\}$. A graph G is *k-regular* if $\deg_G(x) = k$ for any vertex in G. A *path*, $\langle v_0, v_1, v_2, \ldots, v_k\rangle$, is an ordered list of distinct vertices such that v_i and v_{i+1} are adjacent for $0 \le i \le k-1$. The *length* of a path P is the number of edges in P. The *distance* between two vertices u and v in G is the length of a shortest path joining them. The *diameter* of a graph G is the distance between the farthest points in G.

A path is a *hamiltonian path* if its vertices are distinct and span V. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last vertex. A cycle is a *hamiltonian cycle* if it traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle. We will use K_n to denote the complete graph with n vertices and use C_n to denote the cycle graph with n vertices.

A hamiltonian graph G is *k fault-tolerant hamiltonian* if $G - F$ remains hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \le k$. The *fault-tolerant hamiltonicity* $HF(G)$ is defined to be the maximum integer k such that G is k fault-tolerant hamiltonian if G is hamiltonian and is undefined otherwise. Clearly, $HF(G) \le \delta(G) - 2$ if $HF(G)$ is defined. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. All hamiltonian connected graphs except the complete graphs K_1 and K_2 are hamiltonian. A graph G is *k fault-tolerant hamiltonian connected* if $G - F$ remains hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \le k$. The *fault-tolerant hamiltonian connectivity* $HFK(G)$ is defined to be the maximum integer k such that G is k fault-tolerant hamiltonian connected if G is hamiltonian connected and is undefined otherwise. It can be checked that $HFK(G) \le \delta(G) - 3$ only if $HFK(G)$ is defined and $|V(G)| \ge 4$. There are a lot of studies on fault-tolerant hamiltonicity and fault-tolerant hamiltonian connectivity [6–12,21].

In [5], the performance of the hamiltonian property in faulty networks is discussed. Huang *et al.* [11] introduced the term, fault-tolerant hamiltonian connected. The Petersen graph [1] is a 3-regular graph with 10 vertices and diameter 2 as compared with the three-dimensional hypercube, which is a 3-regular graph with eight vertices and diameter 3. The folded Petersen cube networks are motivated by the Petersen graph. In this paper, we consider the fault-tolerant hamiltonicity and the fault-tolerant hamiltonian connectivity of the folded Petersen cube networks.

The k-dimensional folded Petersen network, FP_k, is constructed by an iterative Cartesian product on the Petersen graph. The folded Petersen network is then generalized into folded Petersen cube network, $FPQ_{n,k}$. The graph $FPQ_{n,k}$ is defined as a product of FP_k and the n-dimensional binary hypercube, Q_n. The number of vertices in FP_k is 10^k, whereas there are $2^n \times 10^k$ vertices in $FPQ_{n,k}$, which is therefore more scalable than FP_k. It turns out that $FPQ_{n,k}$ and even its special derivations $FPQ_{0,k} = FP_k$ and $FPQ_{n-3,1} = HP_n$, called the n-dimensional hyper Petersen network originally proposed by Das *et al.* [4], are better than the comparable-size hypercubes and several other networks with respect to the usual metrics (such as degree, diameter, connectivity, packing density, or cost) of a multicomputer architecture.

In this paper, we prove that $HF(FPQ_{n,k}) = n + 3k - 2$ and $HFK(FPQ_{n,k}) = n + 3k - 3$ if $(n, k) \notin \{(0, 1)\} \cup \{(n, 0) \mid n \text{ is a positive integer}\}$. Moreover, $FPQ_{0,1}$ is neither hamiltonian nor hamiltonian connected. Furthermore, $FPQ_{n,0}$ is hamiltonian but not hamiltonian connected if $n > 1$; $FPQ_{1,0}$ is hamiltonian connected but hamiltonian.

In the following section, we give the definition of the folded Petersen cube networks. In Section 3, we present some mathematical preliminary. In Section 4, we prove our main result. In the final section, we give a discussion of our work.

2. Folded Petersen cube networks

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *cartesian product* of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph with vertex set $V_1 \times V_2$ such that (u_1, v_1) is joined to (u_2, v_2) if and only if either $u_1 = u_2$ and v_1 is joined to v_2 in G_2 or $v_1 = v_2$ and u_1 is joined to u_2 in G_1. For any graph G and any positive integer k, we define $G^k = G$ if $k = 1$ and $G^k = G^{k-1} \times G$ if $k > 1$.

For any positive integer, the *n-dimensional hypercube* Q_n is defined as $Q_1 = K_2$ and $Q_n = Q_{n-1} \times Q_1$. Thus, $Q_n = K_2^n$. The *Petersen graph P*, shown in Figure 1(a), is a graph with 10 vertices having an outer 5-cycle, an inner 5-cycle, and five spokes joining them. The folded Petersen cube network $FPQ_{n,k}$ is defined as $Q_n \times P^k$. In particular, $FPQ_{0,k} = P^k$ and $FPQ_{n,0} = Q_n$. The graph $FPQ_{0,2} = P^2$ is shown in Figure 1(b).

The topology of an interconnection network plays an important role in the performance of a distributed system. Various network topologies are in use and have their respective advantages and disadvantages. A good network topology should be one with a small diameter. Regularity and symmetry are some other qualities, which a good network topology is expected to possess. Some of the well-known network topologies in use are the ring topology, the hypercube, the cube-connected cycles, and so on. The Petersen graph is a 3-regular graph with 10 vertices and of diameter 2. Compared with this graph, the three-dimensional hypercube is a 3-regular graph with

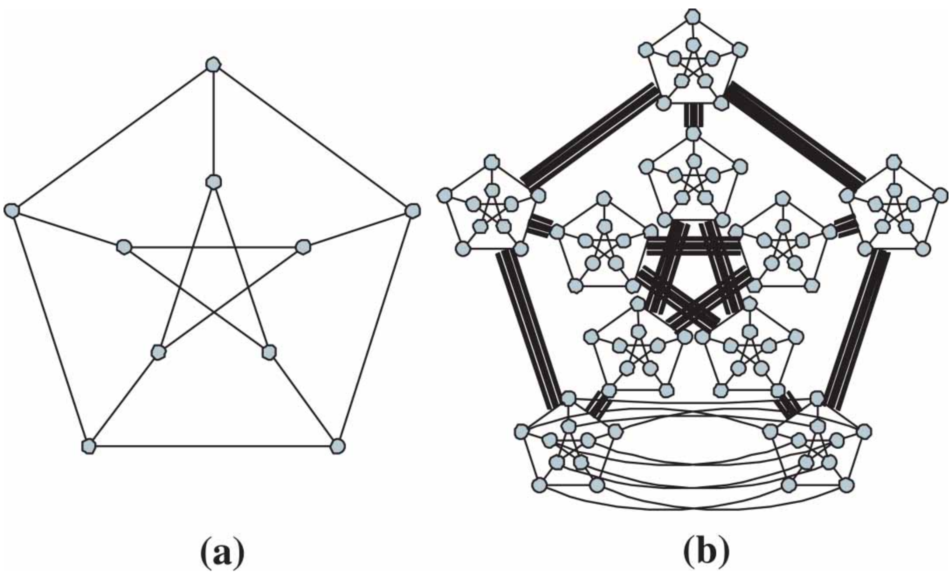

Figure 1. The Petersen graph P and (b) a schematic representation of P.

eight vertices and of diameter 3. It has more vertices as compared with the three-dimensional hypercube and of a smaller diameter. We call it the simple Petersen graph. As an extension, Öhring and Das [15] introduce the k-dimensional folded Petersen graph, FP_k, to be $FPQ_{0,k}$. It is observed that FP_k possesses qualities of a good network topology for distributed systems with large number of sites since it accommodates 10^k vertices and is a symmetric, $3k$-regular graph of diameter $2k$. Being an iterative Cartesian product on the Petersen graph, it is scalable. Moreover, Öhring and Das [15] define the folded Petersen cube networks $FPQ_{n,k}$ and show that a number of standard topologies like linear arrays, rings, meshes, hypercubes, and so on can be embedded into it. Recently, some research on the folded Petersen cube networks have been published for the past several years due to its favourite properties [4,15,20].

Yet, to our knowledge, there is no study on the hamiltonian property on folded Petersen cube networks. Perhaps, the difficulty is on the known result that the Petersen graph is not hamiltonian. With the result of this paper, there are only two non-hamiltonian graphs in the family of folded Petersen cube networks, namely $FPQ_{1,0}$ and $FPQ_{0,1}$.

3. Preliminary

We say a k-regular graph is *super fault-tolerant hamiltonian* if $HF(G) = k - 2$ and $HFK(G) = k - 3$. Some interesting families of interconnection networks are proved to be super fault-tolerant [9–11]. Chen *et al.* [2] observed the insight of the proofs of the aforementioned results and proposed the following construction scheme.

Let G_1 and G_2 be two graphs with the same number of vertices. Let M be an arbitrary *perfect matching* between the vertices of G_1 and G_2; *i.e.*, M is a set of edges connecting the vertices of G_1 and G_2 in a one to one fashion. For convenience, G_1 and G_2 are called *components*. Then, $G(G_1, G_2; M)$ is the graph with vertex set $V(G(G_1, G_2; M)) = V(G_1) \cup V(G_2)$ and edge set $E(G(G_1, G_2; M)) = E(G_1) \cup E(G_2) \cup M$ (see Figure 2 for an illustration).

Let H be a graph. Obviously, the Cartesian product $H \times K_2$ can be viewed as $G(H, H; M)$ for some matching M. The Petersen graph P can be viewed as $G(C_5, C_5; M)$ for some matching M as shown in Figure 3.

Furthermore, $P \times C_5$ can be viewed as $G(C_5 \times C_5, C_5 \times C_5; M)$ for some matching M (see Figure 4 for an illustration).

The following two theorems are proved in [2].

THEOREM 1 *Assume $k \geq 4$. Let G_1 and G_2 be two k-regular super fault-tolerant hamiltonian graphs and $|V(G_1)| = |V(G_2)|$. Then graph $G(G_1, G_2; M)$ is $(k - 1)$ fault-tolerant hamiltonian* [2].

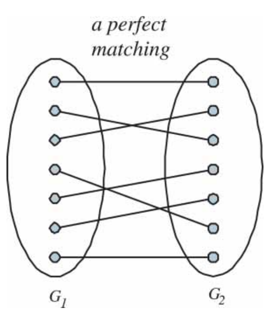

Figure 2. $G(G_1, G_2; M)$.

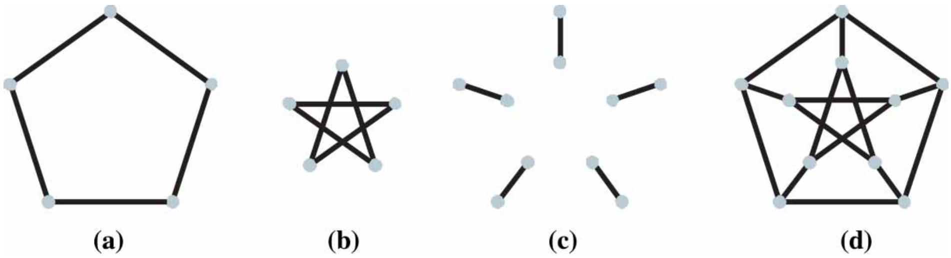

Figure 3. (a) A copy of C_5, (b) another copy of C_5, (c) the matching M, and (d) the Petersen graph P.

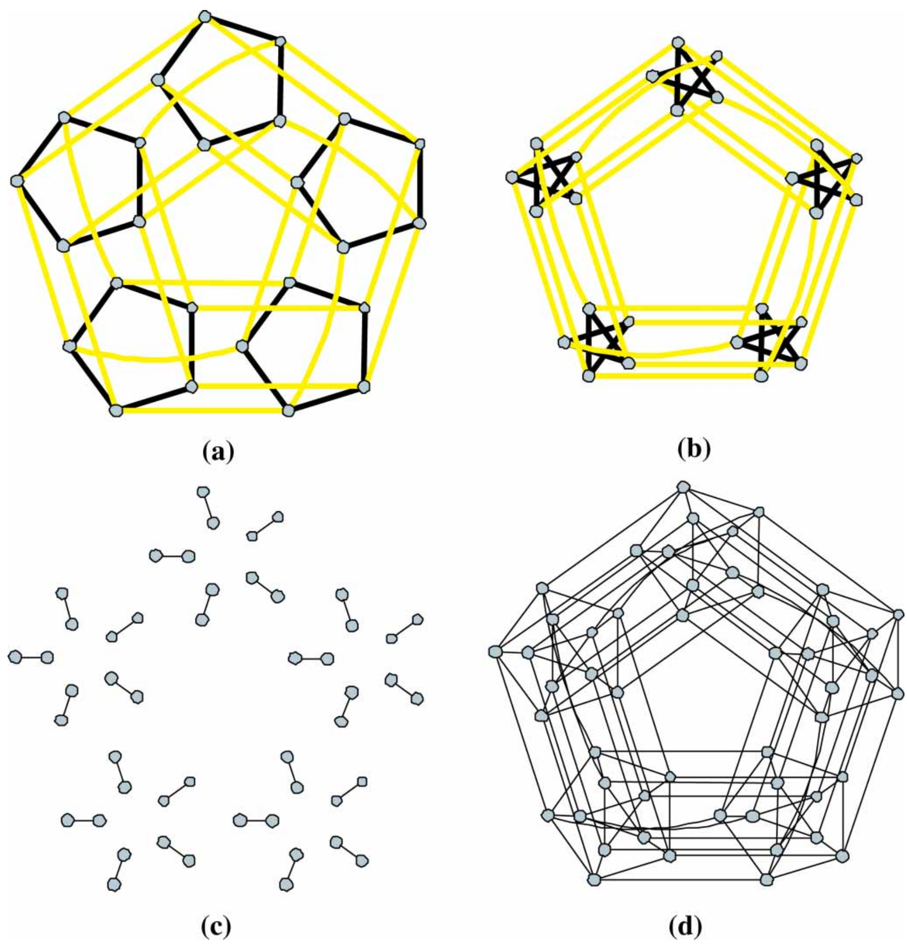

Figure 4. (a) A copy of $C_5 \times C_5$, (b) another copy of $C_5 \times C_5$, (c) the matching M, and (d) $P \times C_5$.

THEOREM 2 *Assume $k \geq 5$. Let G_1 and G_2 be two k-regular super fault-tolerant hamiltonian graphs and $|V(G_1)| = |V(G_2)|$. Then graph $G(G_1, G_2; M)$ is $(k-2)$ fault-tolerant hamiltonian connected* [2].

Combining Theorems 1 and 2, we have the following corollary.

COROLLARY 1 *Assume $k \geq 5$. Let G_1 and G_2 be two k-regular super fault-tolerant hamiltonian graphs and $|V(G_1)| = |V(G_2)|$. Then graph $G(G_1, G_2; M)$ is $(k+1)$-regular super fault-tolerant hamiltonian connected.*

Chen *et al.* [3] further extended their work by considering another construction scheme. Let r and t be positive integers with $r \geq 3$. Assume that $G_0, G_1, \ldots, G_{r-1}$ are graphs with $|V(G_i)| = t$ for $0 \leq i \leq r-1$. We define $H = G(G_0, G_1, \ldots, G_{r-1}; \mathcal{M})$ with $V(H) = \bigcup_{i=0}^{r-1} V(G_i)$ and $E(H) = \mathcal{M} \cup \bigcup_{i=0}^{r-1} E(G_i)$, where $\mathcal{M} = \bigcup_{i=0}^{r-1} M_{i,i+1 \pmod r}$ with $M_{i,i+1 \pmod r}$ is any arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1 \pmod r})$, (see Figure 5 for an illustration). Let H be any graph. Obviously, the Cartesian product $H \times C_4$ can be viewed as $G(H, H, H, H; \mathcal{M})$. The following theorem is proved in [3].

THEOREM 3 *Assume that $G_0, G_1, \ldots, G_{n-1}$ are k-regular super fault-tolerant hamiltonian with the same number of vertices where $n \geq 3$ and $k \geq 5$. Then $G(G_0, G_1, \ldots, G_{n-1}; \mathcal{M})$ is a $(k+2)$-regular super fault-tolerant hamiltonian* [3].

Theorems 1, 2, and 3 are useful to construct super fault-tolerant hamiltonian graphs. Yet, one drawback of these theorems is that $k = 4$ is excluded. However, we have some difficulty to improve Theorem 2 by including the case $k = 4$. For this reason, in this paper, we introduce the concept of extendable 4-regular super fault-tolerant hamiltonian graph. A 4-regular super fault-tolerant hamiltonian graph H is *extendable* if $H - \{x, y\}$ remains hamiltonian connected for any x and y such that $(x, y) \in E(H)$.

The following lemma is proved by brute force. Here, we just state the result to reduce the complexity.

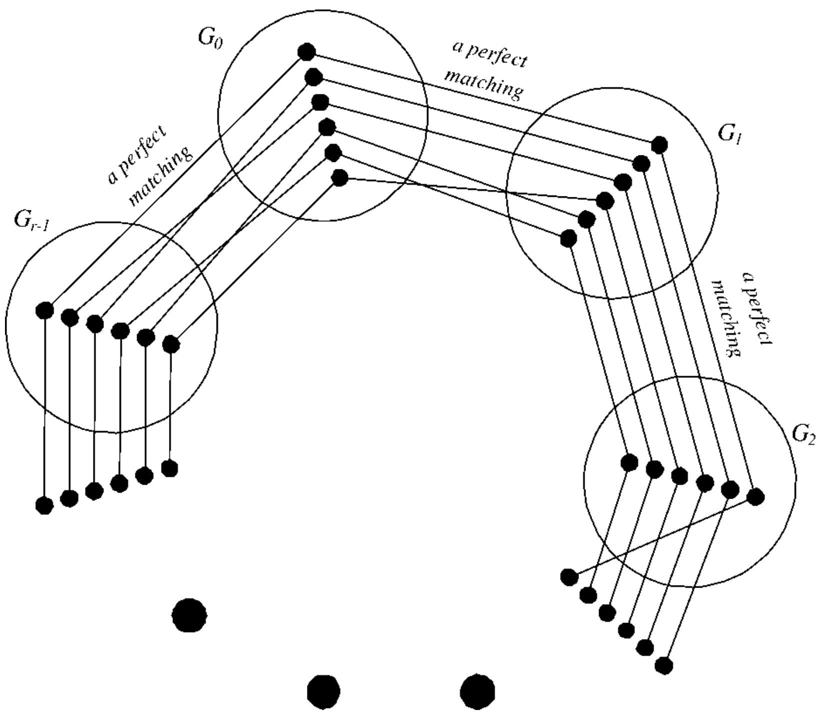

Figure 5. $H = G(G_0, G_1, \ldots, G_{r-1}; M)$.

LEMMA 1 *Both $P \times Q_1$ and $C_5 \times C_5$ are extendable 4-regular super fault-tolerant hamiltonian graphs. However, $C_5 \times C_4$ is a 4-regular super fault-tolerant hamiltonian but not extendable graph.*

THEOREM 4 *Assume that G_1 and G_2 are extendable 4-regular super fault-tolerant hamiltonian graphs with $|V(G_1)| = |V(G_2)|$. Then $G(G_1, G_2; M)$ is a 5-regular super fault-tolerant hamiltonian.*

Proof Since G_i is a 4-regular, $|V(G_i)| \geq 5$. Obviously, $G(G_1, G_2; M)$ is a 5-regular graph. By Theorem 1, $G(G_1, G_2; M)$ is a 3 fault-tolerant hamiltonian.

Now, we need to prove $G(G_1, G_2; M)$ is 2 fault-tolerant hamiltonian connected. Let F be any subset of $V(G(G_1, G_2; M)) \cup E(G(G_1, G_2; M))$ with $|F| \leq 2$. We need to find a hamiltonian path in $G(G_1, G_2; M) - F$ between all pairs of vertices in $V(G(G_1, G_2; M)) - F$.

We use F_i to denote $F \cap (V(G_i) \cup E(G_i))$ for $i = 1, 2$. Moreover, we use F_0 to denote $F \cap M$. Obviously, $|F| = |F_0| + |F_1| + |F_2|$. Let x be any vertex in $V(G(G_1, G_2; M))$. We use x' to denote the vertex of $V(G(G_1, G_2; M))$ matched under M. Thus, x' in $V(G_i)$ if and only if $x \in V(G_{3-i})$, for $i = 1, 2$. By considering the symmetric role of G_1 and G_2, we have the following cases.

Case 1 $|F_i| \leq 1$ for $i = 1, 2$.

(a) $u \in V(G_1)$ and $v \in V(G_2)$. Since $|V(G_i)| \geq 5$ and $|F| \leq 2$, we can find an edge $(x, x') \in M - F$ with $x \in V(G_1) - (F_1 \cup \{u\})$ and $x' \in V(G_2) - (F_2 \cup \{v\})$. Since G_1 and G_2 are 1 fault-tolerant hamiltonian connected, there exists a hamiltonian path R_1 of $G_1 - F_1$ joining u to x and there exists a hamiltonian path R_2 of $G_2 - F_2$ joining x' to v. Obviously, $\langle u, R_1, x, x', R_2, v\rangle$ forms a hamiltonian path of $G(G_1, G_2; M) - F$ joining u to v (see Figure 6(a) for an illustration).

(b) $\{u, v\} \subset V(G_1)$. Since G_1 is 1 fault-tolerant hamiltonian connected, there exists a hamiltonian path R_1 of $G_1 - F_1$ joining u to v. Thus, the length of R_1 is at least $|V(G_1 - F_1)| - 1$.

Suppose that we can find an edge (x, y) in R_1 such that $\{x', y', (x, x'), (y, y')\} \cap F = \emptyset$. We can rewrite R_1 as $\langle u, R_1^1, x, y, R_1^2, v\rangle$. Note the length of R_1^1 is 0 if $u = x$ and the length of R_1^2 is 0 if $y = v$. Since G_2 is 1 fault-tolerant hamiltonian connected, there exists a hamiltonian path R_2 of $G_2 - F_2$ joining x' to y'. Obviously, $\langle u, R_1^1, x, x', R_2, y', y, R_1^2, v\rangle$ forms a hamiltonian path of $G(G_1, G_2; M) - F$ joining u to v (see Figure 6(b) for an illustration).

Suppose that we cannot find an edge (x, y) in R_1 such that $\{x', y', (x, x'), (y, y')\} \cap F = \emptyset$. Then R_1 can be written as $\langle u = x_1, x_2, x_3, x_4, x_5 = v\rangle$ and F is one of the following cases: (1) $\{(x_2, x_2'), (x_4, x_4')\}$, (2) $\{x_2', (x_4, x_4')\}$, or (3) $\{(x_2, x_2'), x_4'\}$. In this case, $|V(G_1)| = 5$. Hence, G_1 and G_2 are isomorphic to the complete graph K_5. Moreover, $G(G_1, G_2; M)$ is isomorphic to the Cartesian product of K_5 and K_2. In this case, we can find another hamiltonian path R_1' joining u and v as $\langle u = x_1, x_2, x_4, x_3, x_5 = v\rangle$. Now, the edge (x_3, x_5) in R_1' satisfies $\{x_3', x_5', (x_3, x_3'), (x_5, x_5')\} \cap F = \emptyset$. As in the previous case, we can find a hamiltonian path of $G(G_1, G_2; M) - F$ joining u to v (see Figure 6(c) for an illustration).

Case 2 $|F_1| = 2$. Thus, $|F_0| = |F_2| = 0$.

(a) $\{u, v\} \subset V(G_1)$. Choose an element $f \in F$. Set $F' = F - \{f\}$. Thus, $|F'| = 1$. Since G_1 is 1 fault-tolerant hamiltonian connected, there exists a hamiltonian path R_1 of $G_1 - F'$ joining u to v. Note that f is in R_1 if f is a vertex. However, f may not be in R_1 if f is an edge. We can write R_1 as $\langle u, R_1^1, x, f, y, R_1^2, u\rangle$ if f is in R_1. Note that the length of R_1^1 is 0 if $u = x$ and the length of R_1^2 is 0 if $y = v$. We can also write R_1 as $\langle u, R_1^1, x, y, R_1^2, u\rangle$ by picking any

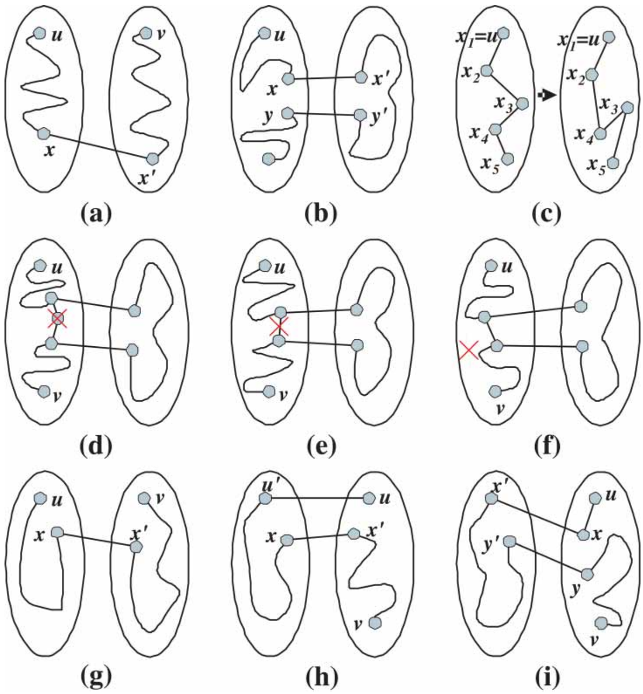

Figure 6. Illustration for Theorem 4.

edge $(x, y) \in R_1$ if f is not in R_1. Since G_2 is 1 fault-tolerant hamiltonian connected, there exists a hamiltonian path R_2 of G_2 joining x' to y'. Obviously, $\langle u, R_1^1, x, x', R_2, y', y, R_1^2, v\rangle$ forms a hamiltonian path of $G(G_1, G_2; M) - F$ joining u to v (see Figure 6(d),(e), and (f) for an illustration).

(b) $u \in V(G_1)$ and $v \in V(G_2)$. Since G_1 is a 2 fault-tolerant hamiltonian, there exists a hamiltonian cycle C of $G_1 - F$. Since C can be traversed backward and forward, we can write C as $\langle u, R_1, x, u\rangle$ such that $x' \neq v$. Since G_2 is 1 fault-tolerant hamiltonian connected, there exists a hamiltonian path R_2 of G_2 joining x' to v. Obviously, $\langle u, R_1, x, x', R_2, v\rangle$ forms a hamiltonian path of $G(G_1, G_2; M) - F$ joining u to v (see Figure 6(g) for an illustration).

(c) $\{u, v\} \subset V(G_2)$. Suppose $u' \notin F$. Since G_1 is a 2 fault-tolerant hamiltonian, there exists a hamiltonian cycle C of $G_1 - F$. Since C can be traversed backward and forward, we can write C as $\langle u', R_1, x, u'\rangle$ such that $x' \neq v$. Since G_2 is 1 fault-tolerant hamiltonian connected, there exists a hamiltonian path R_2 of $G_2 - \{u\}$ joining x' to v. Obviously, $\langle u, u', R_1, x, x', R_2, v\rangle$ forms a hamiltonian path of $G(G_1, G_2; M) - F$ joining u to v (see Figure 6(h) for an illustration).

Suppose $u' \in F$. Since G_2 is 4-regular, there exists a neighbour x of u in $G_2 - \{v\}$ such that $x' \notin F$. Since G_1 is a 2 fault-tolerant hamiltonian, there exists a hamiltonian cycle C of $G_1 - F$. Since C can be traversed backward and forward, we can write C as $\langle x', R_1, y', x'\rangle$ such that $y \neq v$. Since $u' \in F$, $y \neq u$. Since G_2 is extendable, there exists a hamiltonian path R_2 of $G_2 - \{u, x\}$ joining y to v. Obviously, $\langle u, x, x', R_1, y', y, R_2, v\rangle$ forms a hamiltonian path of $G(G_1, G_2; M) - F$ joining u to v (see Figure 6(i) for an illustration).

The theorem is proved. ■

4. Main result

LEMMA 2 *P^k is a $(3k)$-regular super fault-tolerant hamiltonian if and only if $k \geq 2$.*

Proof It is known that P is not hamiltonian. Hence, it is not hamiltonian connected. It is observed that P^2 can be viewed as $G(P \times C_5, P \times C_5; M)$. Moreover, $P \times C_5$ can be viewed as $G(C_5 \times C_5, C_5 \times C_5; M)$. By Lemma 1, $C_5 \times C_5$ is an extendable 4-regular super fault-tolerant hamiltonian. By Theorem 4, $P \times C_5$ is a 5-regular super fault-tolerant hamiltonian. By Corollary 1, P^2 is a 6-regular super fault-tolerant hamiltonian. Assume that k is a positive integer with $k \geq 3$. Again, P^k can be viewed as $G(P^{k-1} \times C_5, P^{k-1} \times C_5; M)$. By Theorem 3, $P^{k-1} \times C_5$ is a $(3k-1)$-regular super fault-tolerant hamiltonian. By Theorems 1 and 2, P^k is a $(3k)$-regular super fault-tolerant hamiltonian. Thus, the lemma is proved. ∎

THEOREM 5 *$HF(FPQ_{n,k}) = n + 3k - 2$ and $HFK(FPQ_{n,k}) = n + 3k - 3$ if $(n, k) \notin \{(0, 1)\} \cup \{(n, 0) \mid n$ is a positive integer$\}$. Moreover, $FPQ_{0,1}$ is neither hamiltonian nor hamiltonian connected. Furthermore, $FPQ_{n,0}$ is hamiltonian but not hamiltonian connected if $n > 1$; $FPQ_{1,0}$ is hamiltonian connected but not hamiltonian.*

Proof By Lemma 2, $HF(FPQ_{0,k}) = 3k - 2$ and $HFK(FPQ_{0,k}) = 3k - 3$ if $k \geq 2$. By Lemma 1, $HF(FPQ_{1,1}) = 2$ and $HFK(FPQ_{1,1}) = 1$. By Lemma 1 and Theorem 4, $HF(FPQ_{2,1}) = 3$ and $HFK(FPQ_{2,1}) = 2$. By Corollary 1, $HF(FPQ_{n,k}) = n + 3k - 2$ and $HFK(FPQ_{n,k}) = n + 3k - 3$ if $n \geq 3$ and $k \geq 1$. Thus, $HF(FPQ_{n,k}) = n + 3k - 2$ and $HFK(FPQ_{n,k}) = n + 3k - 3$ if $(n, k) \notin \{(0, 1)\} \cup \{(n, 0) \mid n \text{ is a positive integer}\}$. Obviously, $FPQ_{0,1}$ is neither hamiltonian nor hamiltonian connected. Note that $FPQ_{n,0}$ is isomorphic to Q_n. It is known that Q_1 is hamiltonian connected but not hamiltonian. Moreover, Q_n is hamiltonian and there is no hamiltonian path of Q_n joining any two vertices in the same partite set if $n \geq 2$. Thus, $FPQ_{n,0}$ is hamiltonian but not hamiltonian connected if $n > 1$. Since $FPQ_{1,0}$ is isomorphic to K_2, $FPQ_{1,0}$ is hamiltonian connected but not hamiltonian. ∎

We believe that our approach of this paper can be applied to the same problem on other interconnection networks. Definitely, we can repeatedly apply Theorems 1, 2, and 3 to obtain the result in this paper. However, we need a lot of efforts to check the base cases for the requirement $k \geq 5$ in Theorems 2 and 3. By introducing the concept of extendable 4-regular super fault-tolerant hamiltonian graph, all difficulties have been overcome. Thus, it would be a great improvement if Theorems 1, 2, and 3 remain true for smaller k.

Let H be the graph shown in Figure 7(a). Obviously, H is a 3-regular graph. By brute force, we can check that H is a 3-regular super fault-tolerant hamiltonian. Let G_1 and G_2 be two copies of H. Let G be the graph shown in Figure 7(b). Obviously, $G = (G_1, G_2; M)$ for some matching M.

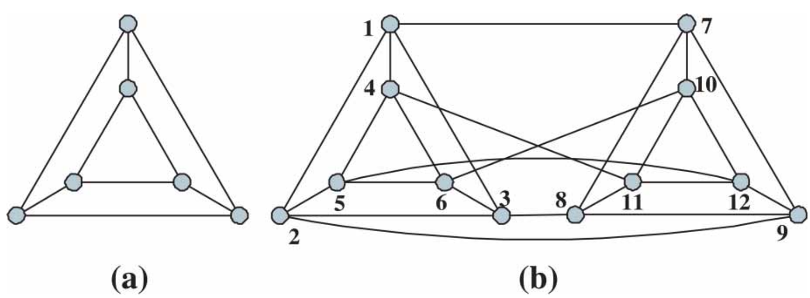

Figure 7. (a) The graph H and (b) the graph G.

By brute force, we can prove that $G - \{2, 12\}$ is not hamiltonian. Thus, G is not a 2 fault-tolerant hamiltonian.

Hence, the bound of k in Theorem 1 is optimal. Yet, we believe the bounds of k in Theorems 2 and 3 are optimal.

5. Conclusion

In this paper, we study the fault-tolerant hamiltonicity and fault-tolerant hamiltonian connectivity on the folded Petersen cube graph. We prove that, $HF(FPQ_{n,k}) = n + 3k - 2$ and $HFK(FPQ_{n,k}) = n + 3k - 3$ if $(n, k) \notin \{(0, 1)\} \cup \{(n, 0) \mid n \text{ is a positive integer}\}$. We believe that the approach of this paper can be used to obtain the fault-tolerant hamiltonicity of other interconnection networks. Moreover, it is interesting to prove that the bounds of k in Theorems 2 and 3 are optimal.

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