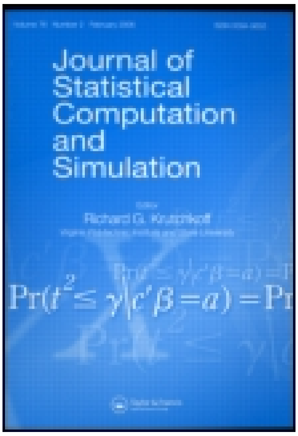

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A comparison of methods for estimating loss-based capability index

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The process capability index C_{pm}, sometimes called the loss-based index, has been proposed to the manufacturing industry for measuring process reproduction capability. This index incorporates the variation of production items with respect to the target value and the specification limits preset in the factory. To estimate the loss-based index properly and accurately, certain frequentist and Bayesian perspectives have been proposed to obtain lower confidence bounds (LCBs) for providing minimum process capability. The LCBs not only provide critical information regarding process performance but are also used to determine whether an improvement was made in a capability index and by extension in reducing the fraction of non-conforming items. In this paper, under the assumption of normality, based on frequentist and Bayesian senses, several existing approaches for constructing LCBs of C_{pm} are presented. Depending on the statistical methods used, we then classify these existing approaches into three categories and compared them in terms of the coverage rates and the mean values of the LCBs via simulations. The relative advantages and disadvantages of these approaches are summarized with some highlights of the relevant findings.

Keywords: coverage rate; loss-based capability index; lower confidence bound; performance comparison

1. Introduction

With the speedy advance of manufacturing technology, suppliers and manufacturers require their products to be of high quality, with a very low proportion of non-conformities (NC). This is true, particularly for today's high-technology products requiring a very low fraction of NC, often measured in parts per million (PPM). Traditional methods for measuring the fraction of NC become inapplicable for those high-quality processes because any manufacturing sample of a reasonable size likely contains no defective product items. For this reason, recently developed process capability indices (PCIs), including C_p, C_{pk}, and C_{pm}, have received substantial attention in the manufacturing industries, particularly for companies manufacturing microelectronics devices and accessories demanding strict quality requirements.

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The C_p and C_{pk} indices are appropriate measures of progress for quality improvement paradigms in which a reduction in variability is the guiding principle. These two indices are defined by Kane [1] as

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma}, \quad C_{pk} = \min\left\{\frac{\text{USL} - \mu}{3\sigma}, \frac{\mu - \text{LSL}}{3\sigma}\right\},$$

where USL and LSL refer to the upper and lower specification limits, respectively, μ stands for the process mean and σ stands for the process standard deviation of an in-control manufacturing process. Clearly, the C_p and C_{pk} indices are not related to the cost of failing to meet customers' requirement. Taguchi, on the other hand, emphasizes the loss in a product's worth when one of its characteristics departs from the customers' ideal value T. To help account for this, Hsiang and Taguchi [2] introduced the index C_{pm}, which was also proposed independently by Chan *et al.* [3]. As the index C_{pm} is related to the idea of squared error loss, this Taguchi index C_{pm} has been sometimes called the loss-based index. The index C_{pm} is defined as

$$C_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{1}$$

where $d = (\text{USL} = \text{LSL})/2$ refers to the half-length of the allowable tolerance of the process. From Equation (1), we can observe that the term $\sigma^2 + (\mu - T)^2$ incorporates two variation components: (i) variation in the process mean; and (ii) deviation of the process mean from the target; that is, the loss-based index C_{pm} emphasizes on measuring the ability of the process to cluster around the target, which therefore reflects the degrees of process targeting (centering). It is easy to see that if the process variance increases (decreases), then the denominator will increase (decrease) and C_{pm} will decrease (increase). Also, if process mean moves away from (closer to) the target value, then the denominator will increase (decrease) and C_{pm} will decrease (increase). Obviously, C_{pm} is sensitive to T, with an additional penalty of being off-target. Furthermore, Ruczinski [4] obtained a lower bound on the process yield as Yield $\geq 2\Phi(3C_{pm}) - 1$ or equivalently NC% $\leq 2\Phi(-3C_{pm})$ for $C_{pm} > \sqrt{3}/3$, where NC% is the fraction of nonconforming items.

In practice, in order to realize the manufacturing capability of a process based on the value of C_{pm}, the knowledge of the process mean μ and the process variance σ^2 are required. Thus, sample data are collected to calculate the estimated value of C_{pm}, and a great degree of uncertainty (sampling errors) is introduced into the capability analysis. In the case of the normal distribution conditions, the statistical properties of the estimator of C_{pm} have been investigated extensively, including Chan *et al.* [3], Boyles [5], Kushler and Hurley [6], Vännman and Kotz [7], Kotz and Lovelace [8], Wright [9], Zimmer *et al.* [10], Pearn and Shu [11], Perakis and Xekalaki [12], Daniels *et al.* [13], Wu and Pearn [14], and Pearn and Wu [15]. Kotz and Johnson [16] provided a compact survey with interpretations and comments of some 170 publications on process capability indices during 1992–2000. Among these studies, the main issue is to accurately estimate the loss-based index C_{pm} to properly measure the reproduction performance of an in-control manufacturing process. The lower confidence bound (LCB) conveys the minimum capability information of the manufacturing process, which is also the main criterion of quality assurance system. Although, many approaches based on the traditional frequentist and the Bayesian senses for constructing the LCB of C_{pm} have been carried out for measuring process performance, none of the research has been conducted on comparing the performance of these LCB approaches. Throughout this paper, it is assumed that the process measurements are independent, identically distributed as the normal distribution and the process is under statistical control.

2. Sampling distributions of the estimated C_{pm}

The definition of C_{pm} involves two unknown parameters μ and σ^2, which must be estimated from the collected sample. Chan *et al.* [3] and Boyles [5] proposed the following two estimators of C_{pm}, $\tilde{C}_{pm}$, and $\hat{C}_{pm}$, respectively,

$$\tilde{C}_{pm} = \frac{d}{3\sqrt{\sum_{i=1}^{n}(X_i - T)^2/(n-1)}} = \frac{d}{3\sqrt{S^2 + [n/(n-1)](\overline{X} - T)^2}}, \tag{2}$$

$$\hat{C}_{pm} = \frac{d}{3\sqrt{\sum_{i=1}^{n}(X_i - T)^2/n}} = \frac{d}{3\sqrt{S_n^2 + (\overline{X} - T)^2}}, \tag{3}$$

where $\overline{X} = \sum_{i=1}^{n} X_i/n$, $S^2 = \sum_{i=1}^{n}(X_t - \overline{X})^2/(n-1)$ and $S_n^2 = \sum_{i=1}^{n}(X_t - \overline{X})^2/n$.

The two estimators, $\tilde{C}_{pm}$ and $\hat{C}_{pm}$, are essentially the same. Note that $\overline{X}$ and S_n^2 are the maximum likelihood estimators (MLEs) of μ and σ^2, respectively. Hence, the estimated $\hat{C}_{pm}$ is also the MLE of C_{pm}. Boyles [5] considered that it would be more appropriate to replace the factor $n-1$ by n in the denominator since the terms $S_n^2 + (\overline{X} - T)^2 = \sum_{i=1}^{n}(X_i - T)^2/n$ and $E[S_n^2 + (\overline{X} - T)^2] = \sigma^2 + (\mu - T)^2$ in the denominator of $\hat{C}_{pm}$ are the uniformly minimum variance unbiased estimators (UMVUEs) of the term $\sigma^2 + (\mu - T)^2$. Therefore, it is reasonable, for reliability purposes, that we use the estimator $\hat{C}_{pm}$ to evaluate the process performance. However, the point estimate is a random variable which provides no confidence on the estimation accuracy. It would be more useful to construct LCBs that guarantee the minimal capability at some designated level of confidence. Kotz and Johnson [17] derived formulas for the rth moment of $\hat{C}_{pm}$. Note that the quantity $n[S_n^2 + (\overline{X} - T)^2]/\sigma^2$ has a non-central chi-square $\chi^2_{n,\lambda}$ distribution with n degrees of freedom and non-centrality parameter $\lambda = n\xi^2$, where $\xi = (\mu - T)/\sigma$. Boyles [5] and Pearn *et al.* [18] showed that $\hat{C}_{pm}$ is distributed as $C_p\sqrt{n/\chi^2_{n,\lambda}}$, which can be equivalently expressed as

$$\hat{C}_{pm} \sim C_{pm}\sqrt{1 + \frac{\lambda}{n}}\sqrt{\frac{n}{\chi^2_{n,\lambda}}}. \tag{4}$$

In fact, Equation (3) can be alternatively expressed as $\hat{C}_{pm} = D/(3\sqrt{K+Y})$, where $D = \sqrt{n}d/\sigma$, $K = nS_n^2/\sigma^2 \sim \chi^2_{n-1}$, $Y = n(\overline{X} - T)^2/\sigma^2$. Thus, the probability density function (PDF) and cumulative distribution function (CDF) of $\hat{C}_{pm}$ can be obtained as [19],

$$f_{\hat{C}_{pm}}(x) = \frac{2^{1-n/2}D^n}{3^n x^{n+1}} \exp\left(-\frac{\lambda}{2} - \frac{D^2}{18x^2}\right) \sum_{j=0}^{\infty} \left\{ \frac{\left(\lambda D^2/36x^2\right)^j}{j!\Gamma((n/2)+j)} \right\}, \tag{5}$$

$$F_{\hat{C}_{pm}}(x) = 1 - \int_0^x \frac{2^{1-n/2}D^n}{3^n x^{n+1}} \exp\left(-\frac{\lambda}{2} - \frac{D^2}{18x^2}\right) \sum_{j=0}^{\infty} \left\{ \frac{\left(\lambda D^2/36x^2\right)^j}{j!\Gamma((n/2)+j)} \right\} \mathrm{d}x \tag{6}$$

for $x > 0$, which is equivalent to the PDF and CDF of $\hat{C}_{pm}$ presented in [7]. It can be noted that expression (4) entailing a non-central χ^2 distribution and Equations (5) and (6) involved the infinite series. Applying the similar integration technique used in ref. [20], Pearn and Shu [11] obtained an alternatively explicit form of the PDF and CDF of $\hat{C}_{pm}$. The PDF and CDF of $\hat{C}_{pm}$

are expressed in terms of a mixture of the χ^2 distribution and the normal distribution,

$$f_{\hat{C}_{pm}} = \int_0^{b\sqrt{n}/(3x)} \frac{2b^2n}{9x^3} g\left(\frac{b^2n}{9x^2} - t^2\right)[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})]\mathrm{d}t, \tag{7}$$

$$F_{\hat{C}_{pm}} = 1 - \int_0^{b\sqrt{n}/(3x)} G\left(\frac{b^2n}{9x^2} - t^2\right)[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})]\mathrm{d}t \tag{8}$$

for $x > 0$, where $b = d/\sigma$, $\xi = (\mu - T)/\sigma$, $g(\cdot)$, and $G(\cdot)$ are the PDF and CDF of the χ^2 distribution with degrees of freedom $n-1$, and $\phi(\cdot)$ is the PDF of the standard normal distribution. It can obtain an identical equation if Equations (7) and (8) are substituted ξ by $-\xi$, for fixed values of x and n.

3. Existing approaches for estimating C_{pm}

In order to assess process performance and make decisions in manufacturing capability, the decision rules as (1) critical values, (2) p-values, and (3) LCBs are needed based on Equations (4)–(8). To keep the paper concise, the LCBs for the true value of C_{pm} are discussed here. Subsequently, several researchers devoted studies to constructing approximate or exact LCBs of C_{pm} are described and contrasted. From Equations (4)–(8), we know that the CDF and PDF of $\hat{C}_{pm}$ are quite complicated as the distribution parameter ξ is unknown.

3.1. *Marcucci and Beazley's approximation*

From expression (4), Marcucci and Beazley [21] set $\xi = 0$ to obtain the following approximate $100\gamma\%$ LCB of C_{pm}, denoted as $C_{pm}^{L(MB)}$,

$$C_{pm}^{L(MB)} \cong \sqrt{\frac{\chi_n^2(1-\gamma)}{n}}\hat{C}_{pm}, \tag{9}$$

where $\chi_n^2(1-\gamma)$ is the $100(1-\gamma)$th percentile of the ordinary χ^2 with n degrees of freedom.

3.2. *Boyles' approximation*

Boyles [5] used the ordinary χ^2 distribution with the correct low terminal and the first two moments to approximate the non-central χ^2 distribution, $\chi^2_{n,\lambda}$, in expression (4) (a technique used in [22]) to obtain an approximate LCB of C_{pm}, where a known parameter ξ is given. It has been shown that

$$\frac{S_n^2 + (\overline{X} - T)^2}{\sigma^2 + (\mu - T)^2} = \frac{\hat{\tau}^2}{\tau^2} \dot{\sim} \frac{\chi_v^2}{v}, \quad \text{where } v = \frac{n(1+\xi^2)^2}{(1+2\xi^2)}, \tag{10}$$

where the symbol $\dot{\sim}$ means 'is approximately distributed as'. As $C_{pm}/\hat{C}_{pm} = \hat{\tau}/\tau$, an approximate $100\gamma\%$ LCB for C_{pm}, with a known parameter ξ, $C_{pm}^{L(Bo)}$ can be expressed as

$$C_{pm}^{L(Bo)} \cong \hat{C}_{pm}\sqrt{\frac{\chi_v^2(1-\gamma)}{v}}. \tag{11}$$

In practice, v is unknown, which has to be estimated. Boyles [5] replaced the unknown parameter v by its MLE $\hat{v}$, naturally obtained by substituting μ and σ with the sample mean $\overline{X}$ and the sample standard deviation S_n. Consequently, the approximate 100γ% LCB of C_{pm} with an unknown parameter ξ, ${}^*C_{pm}^{L(Bo)}$, can be obtained as

$$ {}^*C_{pm}^{L(Bo)} \cong \hat{C}_{pm}\sqrt{\frac{\chi^2_{\hat{v}}(1-\gamma)}{\hat{v}}}, \quad \text{where } \hat{v} = \frac{n(1+\hat{\xi}^2)^2}{(1+2\hat{\xi}^2)}, \ \ \hat{\xi} = \frac{(\overline{X}-T)}{S_n}. \tag{12} $$

3.3. *Chan, Xiong and Zhang's approximation*

When the sample size is large ($\hat{v}$ is larger than 100 in Equation (12)), Boyles [5] used the standard normal distribution instead of the ordinary χ^2 distribution for the approximation, a similar approach is proposed by Chan *et al.* [23]. Thus, the approximate 100γ% LCB of C_{pm} with an unknown parameter ξ, $C_{pm}^{L(CXZ)}$, becomes

$$ C_{pm}^{L(CXZ)} \cong \hat{C}_{pm} - Z(\gamma)\frac{d}{3}\sqrt{\frac{S_n^2(\overline{X}-T)^2 + S_n^4/2}{n[S_n^2 + (\overline{X}-T)^2]^3}} $$

or, equivalently,

$$ C_{pm}^{L(CXZ)} \cong \hat{C}_{pm}\left(1 - Z(\gamma)\sqrt{\frac{1}{2\hat{v}}}\right), \tag{13} $$

where $Z(\gamma)$ denotes the 100γ% percentile of the standard normal distribution.

3.4. *Perakis and Xekalaki's approximation*

Perakis and Xekalaki [12] provided a different approximation of non-central χ^2 distribution improving Patnaik's moment approximation (a technique originally proposed by Pearson [24]) for constructing approximate LCB of C_{pm}. Perakis and Xekalaki [12] obtained the approximate LCB of C_{pm} with a known parameter ξ, $C_{pm}^{L(PX)}$, as

$$ C_{pm}^{L(PX)} \cong \hat{C}_{pm}\sqrt{\frac{c\chi^2_f(1-\gamma)+b}{n(1+\xi^2)}}, \tag{14} $$

where $c = (1+3\xi^2)/(1+2\xi^2)$, $f = n(1+2\xi^2)/c^2$, $b = -n\xi^4/(1+3\xi^2)$.

For an unknown parameter ξ, Perakis and Xekalaki [12] proposed Equation (15) to obtain the approximate LCB of C_{pm}, ${}^*C_{pm}^{L(PX)}$:

$$ {}^*C_{pm}^{L(PX)} \cong \hat{C}_{pm}\sqrt{\frac{\hat{c}\chi^2_{\hat{f}}(1-\gamma)+\hat{b}}{n(1+\hat{\xi}^2)}}, \tag{15} $$

where $\hat{c} = (1+3\hat{\xi}^2)/(1+2\hat{\xi}^2)$, $\hat{f} = (n(1+2\hat{\xi}^2)/\hat{c}^2)$, $\hat{b} = -n\hat{x}^4/(1+3\hat{\xi}^2)$.

Obviously, the unknown parameters ξ, c, f, and b are replaced by their MLEs $\hat{\xi}$, $\hat{c}$, $\hat{f}$, and $\hat{b}$, respectively.

3.5. *Zimmer and Hubele's approach*

On the other hand, based on expression (4), the $100\gamma\%$ LCB of C_{pm}, $C_{\mathrm{pm}}^{L(\mathrm{ZH})}$, can be found as [25]

$$C_{\mathrm{pm}}^{L(\mathrm{ZH})} = \hat{C}_{\mathrm{pm}}\sqrt{\frac{\chi^2_{n,\lambda}(1-\gamma)}{n+\lambda}}. \tag{16}$$

Using expression (4) and Equation (16), Zimmer and Hubele [25] and Zimmer *et al.* [10] presented some graphical procedures and tables of $\sqrt{n/\chi^2_{n,\lambda}(1-\gamma)}$ to obtain confidence intervals of C_{pm}, where the parameter ξ is assumed to be known.

In a real application with an unknown parameter ξ, the $100\gamma\%$ approximate LCB of C_{pm} based on Zimmer *et al.* [10], ${}^*C_{\mathrm{pm}}^{L(\mathrm{ZH})}$, can be presented as

$${}^*C_{\mathrm{pm}}^{L(\mathrm{ZH})} \cong \hat{C}_{\mathrm{pm}}\sqrt{\frac{\chi^2_{n,\hat{\lambda}}(1-\gamma)}{n+\hat{\lambda}}}, \quad \text{where } \hat{\lambda} = n\hat{\xi}^2. \tag{17}$$

3.6. *Pearn and Shu's approach*

By working with Equation (8), the $100\gamma\%$ LCBs of C_{pm}, $C_{\mathrm{pm}}^{L(\mathrm{PS})}$, can be obtained by solving Equation (18), where n and $\hat{C}_{\mathrm{pm}}$ are given with a known ξ and $b_L = 3C_{\mathrm{pm}}^{L(\mathrm{PS})}(1+\xi^2)^{1/2}$

$$\int_0^{b_L\sqrt{n}/(3\hat{C}_{\mathrm{pm}})} G\left(\frac{b_L^2 n}{9\hat{C}_{\mathrm{pm}}^2} - t^2\right)[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})]\mathrm{d}t = 1-\gamma. \tag{18}$$

In real application with an unknown parameter ξ, an approximate LCB, ${}^*C_{\mathrm{pm}}^{L(\mathrm{PS})}$, can be obtained by solving Equation (19) with ${}^*b_L = 3{}^*C_{\mathrm{pm}}^{L(\mathrm{PS})}(1+\hat{\xi}^2)^{1/2}$,

$$\int_0^{{}^*b_L\sqrt{n}/(3\hat{C}_{\mathrm{pm}})} G\left(\frac{{}^*b_L^2 n}{9\hat{C}_{\mathrm{pm}}^2}\right)[\phi(t+\hat{\xi}\sqrt{n}) + \phi(t-\hat{\xi}\sqrt{n})]\mathrm{d}t = 1-\gamma. \tag{19}$$

Pearn and Shu [11] commented that such approach introduces additional sampling errors from estimating ξ in finding the LCB, and certainly would make this approach (and of course other existing methods discussed above) less reliable. Pearn and Shu [11] investigated the analysis on LCB to find that it attains its minimal value at $\xi = 0$. Hence for quality assurance purpose, we may solve Equation (20) to obtain the required $100\gamma\%$ LCBs, ${}^{**}C_{\mathrm{pm}}^{L(\mathrm{PS})}$, for giving $\hat{C}_{\mathrm{pm}}$, n, and setting $\xi = 0$ without further estimating the parameter ξ.

$$2\int_0^{{}^{**}C_{\mathrm{pm}}^{L(\mathrm{PS})}\sqrt{n}/\hat{C}_{\mathrm{pm}}} G\left(\frac{n({}^{**}C_{\mathrm{pm}}^{L(\mathrm{PS})})^2}{\hat{C}_{\mathrm{pm}}^2} - t^2\right)\phi(t)\mathrm{d}t = 1-\gamma. \tag{20}$$

3.7. *Shiau, Chiang and Hung's Bayesian approach*

Shiau *et al.* [26] adopted a Bayesian approach to obtain the credible interval, a Bayesian analogue of classical confidence interval, by considering the posterior probability $p = Pr\{C_{\mathrm{pm}} > w|X\}$ that the process under investigation is capable.

Thus, the $100\gamma\%$ LCBs of $C_{\rm pm}$ based on Shiau *et al.'s* Bayesian approach, $C_{\rm pm}^{L({\rm SCH})}$, can be obtained from $\gamma = Pr\{C_{\rm pm} > C_{\rm pm}^{L({\rm SCH})}|X\}$, that is

$$\gamma = \int_0^{t'} \left[\frac{1}{\Gamma(k)\psi^k y^{k+1}}\right] \exp\left(-\frac{1}{\psi y}\right) [\Phi(b_1(y) + b_2(y)) - \Phi(b_1(y) - b_2(y))]\mathrm{d}y, \tag{21}$$

where $t' = (2/n)(\hat{C}_{\rm pm}/C_{\rm pm}^{L({\rm SCH})})^2$.

4. Performance comparisons

4.1. *Discussions*

Although Equations (16) and (18) proposed, respectively, by Zimmer and Hubele's approach (Section 3.5) and Pearn and Shu's approach (Section 3.6) have different mathematical expressions, given a known parameter $\xi = (\mu - T)/\sigma$ the exact LCBs of $C_{\rm pm}$ obtained from Equations (16) and (18) are identical i.e. $C_{\rm pm}^{L({\rm ZH})} = C_{\rm pm}^{L({\rm PS})}$. Note that the actual ξ is unobtainable in real industry application, hence the approaches expressed in Equations (16) and (18) are excluded in the subsequent performance comparisons.

When ξ is unknown, an exact confidence bound estimation of $C_{\rm pm}$ may be intractable due to the complicated statistical distribution. Existing approaches (Sections 3.1–3.7) based on the frequentist and Bayesian senses proposed in the preceding section are used to ease the difficulties. According to the statistical methods used for constructing the LCB of $C_{\rm pm}$, we then classify existing approaches (Sections 3.1–3.7) into three categories which are (1) the approach of exact sampling distribution, (2) the approaches of approximation, and (3) the Bayesian approach. Table 1 summarizes the frequentist and the Bayesian senses, the statistical methods, and three categories of approaches with corresponding equations.

4.2. *Simulation study*

The important consideration in choosing approaches for determining the LCB of $C_{\rm pm}$ is the performance of each approach. For LCBs, the natural performance characteristic is the coverage rate (CR) of the approach. If the desired confidence level is $100\gamma\%$, then the nominal CR is γ. For an exact approach, the actual CR equals the nominal rate, but when the approach involves an statistical methods of approximation, the actual rate can differ from the nominal rate. An approximate approach for which the difference is relatively small can be said to perform well.

Table 1. The frequentist and the Bayesian senses, the statistical methods, and three categories of approaches with corresponding equations.

	Statistical methods used for constructing LCBs of $C_{\rm pm}$	Category	Abbreviation of approach	Equations
Frequentist sense	Non-central χ^2 distribution	Approach of exact sampling distribution	ZH or PS	(17) or (19)
	Use ordinary χ^2 distribution by setting $\xi = 0$	Approaches of approximations	MB	(9) or (20)
	Patnaik's approximation		Bo	(12)
	Normal approximation		CXZ	(13)
	Pearson's approximation		PX	(15)
Bayesian sense	Non-informative prior	Bayesian approach	SCH	(21)

In order to compare the performance of those existing approaches for constructing the LCBs, a series of simulations were undertaken. Without loss of generality, the values USL $= 3$, $T = 0$, and LSL $= -3$ were used for all simulations. A class of normal processes with four different combinations of process mean and process standard deviation $(\mu, \sigma) = (1, 1)$, (0.5,0), (0,1), and (1,0) were considered. These values were chosen to represent processes that vary from 'not capable' (i.e. index value less than 1.00) to 'very capable' (index of 1.5 and larger). For each combination of (μ, σ), a sample of size $n = 25(5)150$ was drawn. The single simulation was then replicated 10,000 times. Thus, we are able to calculate the mean value of $\hat{C}_{\text{pm}}$ (ME), mean value of LCBs (MLCBs) of C_{pm} and its CR for each sample size n based on 10,000 trials. The ME and the MLCB are simply the average of 10,000 values of $\hat{C}_{\text{pm}}$ and the LCB of C_{pm}, respectively. The estimated CR is the proportion of times that the calculated LCB were actually smaller than the corresponding true value of C_{pm}. This actual CR could then be compared with the nominal confidence level.

Tables 2–5 display the values of ME, CR, and MLCB for each approach under various parameter (μ, σ) combinations and sample size n with a 95% confidence level at 10,000 replications. The ME seems to overestimate the actual value of C_{pm}, especially for small sample sizes. As n increases, the ME tends to close the actual value of C_{pm}. Obviously, the CR and MLCB are inverse ratios to each other. The lower CR, the closer to the actual value of the MLCB is. That is true because the lower CR means much more LCBs do not cover the actual value, and the MLCB is much closer to the actual value of C_{pm}.

From Tables 2–5, we observe that the performance of various approaches are affected by the value of ξ, thus we further perform extensive computations to calculate the LCBs of C_{pm} for $\xi = 0(0.1)2.0$, $C_{\text{pm}} = 0.8(0.1)2.0$, and $n = 25(5)150$, which covers a wide range of applications with $C_{\text{pm}} \geq 0.8$. Figure 1a–f plot CR *vs.* ξ for the 95% LCB of C_{pm} based on each approach with $\xi = 0(0.1)2.0$, $n = 25$, 50, 100, and 150, and $C_{\text{pm}} = 1.00$ at times $= 10{,}000$.

It can be noted that for small ξ (say $0 \leq \xi < 0.5$), the lower bounds of C_{pm} for the ZH(PS), Bo, PX, MB approaches perform well with the most accurate CRs for all the studied cases. Similarly, when $0.5 \leq \xi \leq 2.0$, the Bo, PX, ZH(PS) methods have the same acceptable performance measures for quality assurance with reasonable CRs for all the studied cases. As depicted in Figure 1e and f, the MB approach under $0.5 \leq \xi \leq 2.0$ and the SCH approach under $0 \leq \xi \leq 2.0$ keep type I error (α-risk) not greater than predetermined value (such as 0.05 or 0.01) to provide necessary protection to the customers. The conservative LCBs for the true value of C_{pm} can lead to higher level of type II error. In contrast, the CXZ approach keeps type I error (α-risk) greater than predetermined value, as shown in Figure 1c, to provide optimistic process capability for most of studies cases under $0 \leq \xi \leq 2.0$. The suggested approaches for constructing the LCBs of C_{pm} based on the value of ξ are displayed in Table 6.

4.3. *Some remarks*

Based on the simulation results and above discussion, we can conclude the following rule of thumb to practitioners for real-world factory applications:

(a) When $0 \leq \xi < 0.5$, the LCB of C_{pm} is in support of the use of the ZH(PS), Bo, PX, MB approaches.
(b) When $0.5 \leq \xi \leq 2.0$, the LCB of C_{pm} is in support of the use of the ZH(PS), Bo, PX approaches.
(c) If the parameter ξ is known, the exact LCB of C_{pm} is in support of the use of the Equations (16) and (18).

Table 2. Simulated results for 95% LCB of C_{pm} with $(\mu, \sigma) = (1, 1)$ and $C_{\mathrm{pm}} = 0.7071$.

Approach		ZH(PS)		MB		Bo		CXZ		PX		SCH	
n	ME	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB
25	0.7246	0.9516	0.5776	0.9692	0.5539	0.9494	0.5795	0.9468	0.5811	0.9516	0.5776	0.9587	0.5723
50	0.7158	0.9530	0.6128	0.9713	0.5968	0.9518	0.6136	0.9511	0.6146	0.9530	0.6128	0.9567	0.6107
100	0.7110	0.9510	0.6388	0.9707	0.6277	0.9498	0.6392	0.9492	0.6397	0.9510	0.6388	0.9538	0.6379
150	0.7094	0.9564	0.6506	0.9740	0.6416	0.9559	0.6509	0.9548	0.6512	0.9564	0.6506	0.9579	0.6501

Table 3. Simulated results for 95% LCB of C_{pm} with $(\mu, \sigma) = (1, 0.5)$ and $C_{\mathrm{pm}} = 0.8944$.

Approach		ZH(PS)		MB		Bo		CXZ		PX		SCH	
n	ME	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB
25	0.9039	0.9546	0.7775	0.9568	0.6910	0.9519	0.7793	0.9502	0.7804	0.9546	0.7752	0.9568	0.7752
50	0.8992	0.9529	0.8096	0.9972	0.7498	0.9516	0.8107	0.9501	0.8113	0.9529	0.9529	0.9538	0.8096
100	0.8973	0.9498	0.8343	0.9962	0.7921	0.9484	0.8347	0.9471	0.8350	0.9498	0.8343	0.9496	0.8345
150	0.8959	0.9555	0.8445	0.9971	0.8102	0.9545	0.8448	0.9540	0.8450	0.9555	0.8445	0.9547	0.8447

Table 4. Simulated results for 95% LCB of C_{pm} with $(\mu, \sigma) = (0, 1)$ and $C_{pm} = 1.00$.

Approach		ZH(PS)		MB		Bo		CXZ		PX		SCH	
n	ME	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB
25	1.0303	0.9501	0.7881	0.9503	0.7877	0.9501	0.7882	0.9482	0.7912	0.9501	0.7881	0.9718	0.7534
50	1.0150	0.9500	0.8465	0.9502	0.8464	0.9500	0.8465	0.9482	0.8482	0.9500	0.8465	0.9670	0.8279
100	1.0066	0.9518	0.8886	0.9518	0.8886	0.9518	0.8886	0.9495	0.8896	0.9518	0.8886	0.9653	0.8789
150	1.0051	0.9479	0.9090	0.9479	0.9090	0.9478	0.9090	0.9460	0.9097	0.9479	0.9090	0.9592	0.9025

Table 5. Simulated results for 95% LCB of C_{pm} with $(\mu, \sigma) = (0, 0.5)$ and $C_{pm} = 2.00$.

Approach		ZH(PS)		MB		Bo		CXZ		PX		SCH	
n	ME	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB	CR	MLCB
25	2.0575	0.9528	1.5739	0.9529	1.5730	0.9528	1.5741	0.9504	1.5800	0.9528	1.5738	0.9738	1.5046
50	2.0306	0.9516	1.6934	0.9518	1.6932	0.9516	1.6934	0.9504	1.6968	0.9516	1.6934	0.9666	1.6562
100	2.0171	0.9481	1.7807	0.9481	1.7806	0.9481	1.7807	0.9481	1.7825	0.9481	1.7807	0.9617	1.7613
150	2.0111	0.9479	1.8188	0.9479	1.8188	0.9479	1.8188	0.9466	1.8201	0.9479	1.8188	0.9599	1.8058

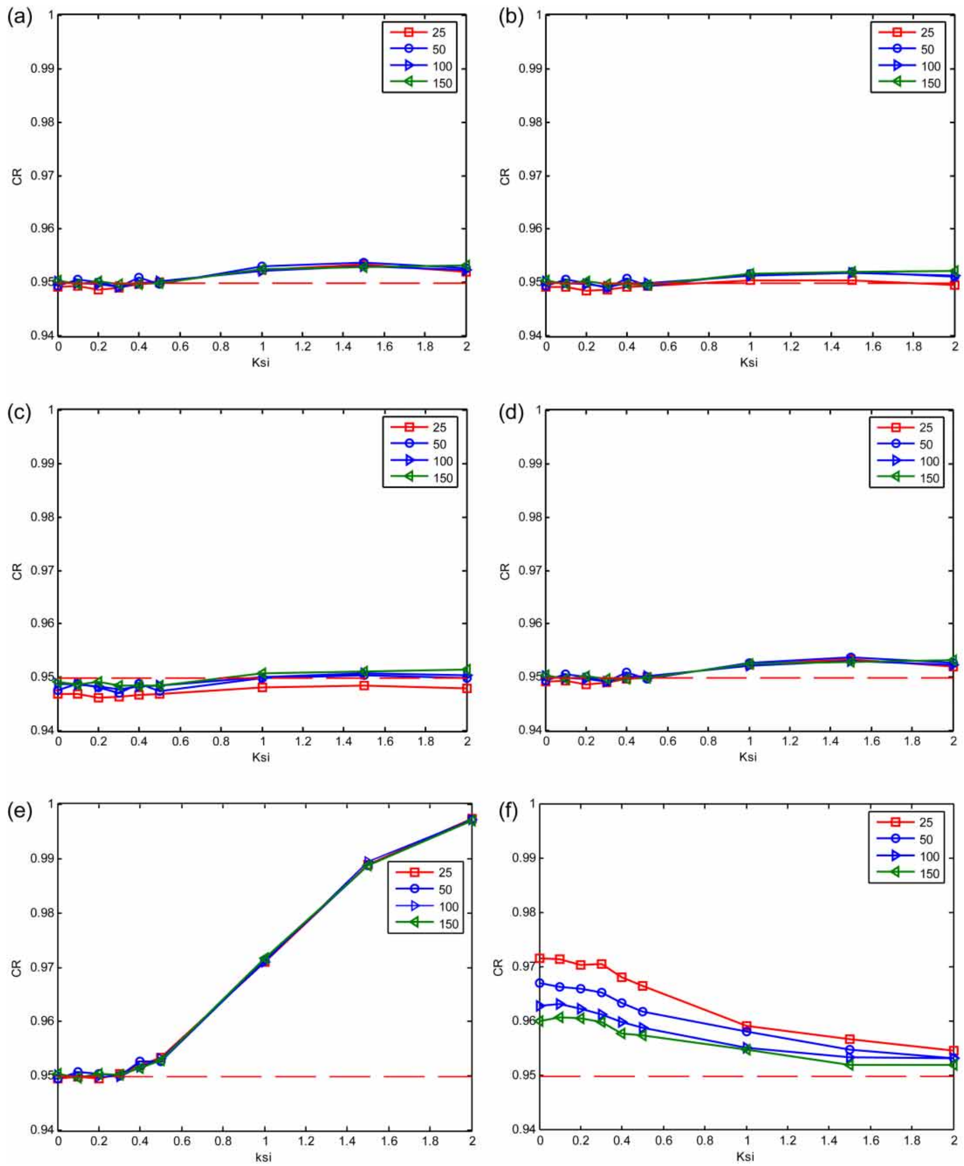

Figure 1. (a) Plots of CR *vs.* ξ for 95% LCB based on ZH(PS) approach; (b) plots of CR *vs.* ξ for 95% LCB based on Bo approach; (c) plots of CR *vs.* ξ for 95% LCB based on CXZ approach; (d) plots of CR *vs.* ξ for 95% LCB based on PX approach; (e) plots of CR *vs.* ξ for 95% LCB based on MB approach; (f) plots of CR *vs.* ξ for 95% LCB based on SCH approach.

Table 6. The suggested approaches for LCBs of C_{pm}.

ξ	Suggested approaches
$0.0 \leq \xi < 0.5$	ZH(PS), Bo, PX, MB
$0.5 \leq \xi < 2.0$	ZH(PS), Bo, PX

5. An application example

Printed circuit boards (PCBs) are widely used in the microelectronic manufacturing industry, making computers and peripherals, digital phones, fax machines, channel switch devices, remote controls, and many other things. Factories producing various PCBs and related products generally are classified as 'the PCB industry', because the core components inside those products are the PCBs. The PCB manufacturing process mainly consists of a series of chemical-related operations, and the chemical operations determine the functions of a PCB. PCBs are laminates. This means that they are made from two or more sheets of material stuck together; often copper and fiberglass.

Some parts of the side with copper tracks is coated with solder resist (usually green in colour) to prevent solder sticking to those areas where it is not required. This avoids unwanted solder bridges between tracks. The solder resist is an important operation in the post-process for the PCB manufacturing, which is chemically unrelated. The effects of the solder resist are protecting the metal ingredients inside the circuits from oxidizing, and also protecting the board itself from exterior damage, when embedding specific electronic components for various applications. The uniformly smooth surface of the PCB is an essential quality characteristic considered in all PCB quality control schemes. The operation of the solder resist is the key to surface coating in the PCB manufacturing industry. The simplest method to judge whether the PCBs satisfy the uniformity flat requirement after the solder resist, is to measure its thickness. It particularly checks the uneven parts including the caves and towers of a PCB. By measuring the thickness, one can obtain the degrees of the uniformity for the surface of a PCB which is used for PCBs capability measures on thickness.

The example investigated is taken from a company located on the Tao-Yuan Industrial Park in Taiwan, which has supplied manufacturing multi-layer PCBs for the company orders. The thickness of nominal-the-better characteristic is the key measurement for the PCBs. For a particular model of PCBs, USL, LSL and T of a PCB's thickness are 13.5 μm, 28.5 μm and 21.0 μm, respectively. According to today's modern quality assurance theory, reduction of the process loss is as important as increasing the process yield. The histogram and the normal probability plot of the 80 PCB data show no observations outside the upper specification limit, and both show that the sample data appears to be approximately normal; the Shapiro–Wilk test is also applied to verify the normality assumption.

5.1. *LCB applications*

The results obtained in the last discussion section show that we can suggest the LCB developed by ZH(PS), Bo, and PX approaches for providing the minimum process performance if one is unsure of the value of ξ. Thus, we calculate the LCB of C_{pm} based on ZH(PS), Bo, and PX approaches by executing the Matlab program. The program reads the sample data file; the sample size $n = 80$, $\text{LSL} = 13.5\,\mu\text{m}$, $\text{USL} = 28.5\,\mu\text{m}$, target value $T = 21.0\,\mu\text{m}$, and confidence level $\gamma = 0.95$ are taken as input, then it outputs the estimator $\hat{C}_{\text{pm}} = 1.405$, $\hat{\xi} = 1.3$, and the corresponding LCBs are $C_{\text{pm}}^{*(\text{ZH})} = 1.2608$, $C_{\text{pm}}^{L(\text{Bo})} = 1.2619$, and $C_{\text{pm}}^{L(\text{PX})} = 1.2521$. Table 7 shows these three LCBs of C_{pm} and the corresponding NC in PPM. We therefore conclude that the true value of the process

Table 7. Lower confidence bounds of C_{pm} and the corresponding NC.

Approaches	ZH (PS)	Bo	PX
LCB	1.2608	1.2619	1.2521
NC (in PPM)	156	154	173

capability C_{pm}, is not less than 1.2521 with 95% level of confidence. We can thus assure that the production yield is 99.9827% and the number of NC is less than 173 PPM.

6. Conclusions

The Taguchi capability index C_{pm} has been broadly used in the manufacturing industry dealing with problems of measuring reproduction capability of processes to enhance product development with a very low fraction of defectives. In this study, we exhaustively compare and contrast the several existing approaches based on the frequentist and the Bayesian senses for LCBs on C_{pm}. An intensive simulation study was conducted to compare the performance of approaches in terms of the attained CR and the MLCBs. The results recommend the appropriate approaches to practitioners for real-world factory applications.

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