國 立 交 通 大 學

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博 士 論 文

M/G/1/K與G/M/1/K排隊含啟動時間的
*F*方策與*N*方策之相互關係

Interrelationships between *F* Policy and *N* Policy for
M/G/1/K and G/M/1/K Queues with Startup Time

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M/G/1/K 與 G/M/1/K 排隊含啟動時間的 *F* 方策與 *N* 方策之相互關係

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摘 要

此篇論文是在研究 F 方策 M/G/1/K 和 G/M/1/K 排隊含起動時間的問題，進而深入探討 F 方策和 N 方策的相互關係。在排隊問題中，F 方策主要是研究控制到達的問題。N 方策排隊問題主要是研究控制服務的問題。首先，我們探討對於在 F 方策 M/G/1/K 和 G/M/1/K 排隊含起動時間的問題。而 F 方策的定義如下：當顧客數目到達系統可承載數(例如系統容量)，系統不再允許任何到達顧客進入，直到有一定數目(足夠)的顧客已被服務，也就是顧客數目會降到一個門檻值 F $(0 \le F < K-1)$。同時，啟動系統開始讓顧客進入系統，其啟動時間的分配為指數分配，參數為 β。因此，系統會正常運作直到系統裡的顧客數目到達系統可承載的數目，所有的程序會再重新依序發生。我們分別針對在 F 方策 M/G/1/K 和 G/M/1/K 排隊中提出遞迴的方法與輔助變數技巧來推導。其方式如下，使用輔助變數代替剩餘服務時間(到達時間)再利用遞迴方法來計算在穩態下的機率。為了分析說明遞迴方法，此篇論文提出三種不同服務時間(到達時間)分配來解釋在 F 方策 M/G/1/K 和 G/M/1/K 排隊系統含指數分配的起動時間，其服務時間(到達時間)分配包括指數分配、3 階段 Erlang 和 deterministic 分配等。同時，針對最佳化的問題，建立其成本模型來決定最佳 F 值，使得成本最小。我們也使用 Maple 電腦程式來計算出最佳 F 值與系統參數的關係，並作敏感度的分析。為了進一步探討 F 方策和 N 方策的關係，我們同樣使用遞迴方法和輔助變數技巧來求取含啟動時間之 N 方策 M/G/1/K 排隊系統中穩態機率的演算法。而 N 方策的定義如下：當顧客數目增加到一個門檻值 N $(N \ge 1)$時，啟動系統開始服務，其啟動時間的分配為指數分配，參數為 γ。且直到系統內沒有顧客後關閉服務。透過一系列的演算法的比對與計算，驗證出 F 方策和 N 方策之間的互補關係：可由 F(或 N)方策排隊系統所得到的演算法去計算另一個方策的排隊問題的解。最後，我們提供兩個範例，分別為 3 階段 Erlang 與指數分配來作說明 F 方策和 N 方策之間的互補關係。

關鍵字：F 方策; N 方策; M/G/1/K 排隊; G/M/1/K 排隊; 遞歸方法; 啟動時間; 輔助變數; 成本; 敏感度分析.

Interrelationships between *F* Policy and *N* Policy for M/G/1/K and G/M/1/K Queues with Startup Time

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Abstract

This dissertation deals with the interrelationship between *F* policy and *N* policy. The *F* policy queuing problem investigates the most common issue of controlling arrival to a queuing system. The *N* policy queuing problem investigates the most common issue of controlling service. The optimal control arrival in M/G/1/K and G/M/1/K queues operating under the *F* policy and startup time is investigated in this dissertation. The definition of *F* policy is described as following: When the number of customers in the system reaches its capacity *K* (i.e. the system becomes full), no further arriving customers are allowed to enter the system until there are enough customers who have been served in the system. Consequently, the number of customers in the system decreases to a threshold value F ($0 \le F < K-1$). At that time, the server requires to take an exponential startup time with parameter β to start allowing customers in the system. Thus, the system operates normally until the number of customers in the system reaches its capacity at which time the above process is repeated all over again. A recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining service (or inter-arrival) time, is provided to develop the steady-state probability distributions of the number of customers in two finite queues. To illustrate analytically the two recursive methods, examples of different service (or interarrival) time distributions, such as exponential, 3-stage Erlang and deterministic distributions, in the *F* policy M/G/1/K queuing system and in the *F* policy G/M/1/K queuing system with exponential startup time distribution is present. In both queueing systems, a cost model is established to determine the optimal management *F* policy at minimum cost. An efficient Maple computer program is used to determine the optimal operating *F* policy and some system performance measures. Sensitivity analysis is also studied. To find the interrelationship between *F* policy and *N* policy, we have solved the solution algorithm of the *N* policy M/G/1/K queue with startup time. A recursive method and supplementary variable technique to obtain the solution algorithm is provided. The definition of *N* policy is described as following: The server needs a startup time when the number of customers in the system reaches the threshold N $(N \geq 1)$ for the first time until there are no customers present in the system. At that time, the server needs to take an exponential startup time with parameter γ to start servicing customers in the system. Through a series of the algorithm, the complementary interrelationship between the *F* policy and *N* policy queues is obtained. Therefore, the problem of *F* policy (*N* policy) queuing system with startup time gives the solution algorithm to the other problem. The two simple examples of 3-stage Erlang and exponential distribution to illustrate the interrelationship are provided.

Keywords: *F* policy, *N* policy, G/M/1/K queue, M/G/1/K queue, Recursive method, Startup times, Supplementary variable, Cost, Sensitivity analysis.

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Chapter 1

Introduction

In section 1.1, we describe the background of queueing theory that has been a continuously growing and interesting science. In section 1.2, an important technique that is supplementary variable technique will be introduced. In section 1.3, several researchers are shown in earlier works. In section 1.4, the description of the problems that are the interrelationships between F policy and N policy in M/G/1/K and G/M/1/K queues are presented. In section 1.5, we illustrate the scope of dissertation.

1.1 Background

The most of the optimization problems in queueing systems focused on design or static models in which the system characteristics did not vary with time in the last few years. It is evidence that this type of model does not meet the requirements of the majority of realworld queueing applications. For example, those models relative to the management of large-scale systems in several fields are: distribution, transportation, administration, production, informatics, etc. It is especially true in many computer and communication applications, in which the performance of the investigation system may be improved if some system parameters are adjusted as the system state changes. (see Kleinrock [24], Bolch et al. [6])

Hence, it is more concerned that the system characteristics are allowed to vary with time in the control or dynamic models. The aim of such models is to prescribe a certain behavior on the part of the decision maker. In these models, it is emphasized that the optimization is achieved over a class of operating policies, rather than over a set of parameters for a single operating policy, which is a fixed part of the model. A complete survey about the design and control of models was presented by Gross and Harris [15], and Crabill et al. [9].

Several researchers investigate five kinds of controllable queues as follows:

1. The N policy was first developed by Yadin and Naor [36] in 1963. When the number of customers in the system reaches the threshold N($N \geq 1$) for the first time until there are no customers present in the system, the server returns to provide

service.

Many researchers have worked on this subject such as Yadin and Naor [36], Hersh and Brosh [16], Kimura [23], Lee and Park [26], Medhi and Templeton [28], Takagi [29], Wang et al. [32], Wang and Ke [33], Ke and Wang [20] and others.

2. The *F* policy was developed by Gupta [12] in 1963. When the number of customers in the system reaches its capacity *K* (i.e. the system becomes full), no further arriving customers are allowed to enter the system until there are enough customers in the system have been served so that the number of customers in the system decreases to a threshold value F $(0 \le F < K-1)$. At that time, the server requires to take an exponential startup time to start allowing customers in the system. Thus, the system operates normally until the number of customers in the system reaches its capacity at which time the above process is repeated all over again.

3. The *T* policy was developed by Heyman [18], Levy and Yechiali [27], Tijm [31], Gakis et al. [11], and Ke[22]. Following the beginning of the idle period, the server returns to provide service immediately after *T* time units have elapsed from the epoch of server removal if there is at least one customer present in the waiting line, until there are no customers in the system. If after *T* time units have elapsed, there are no customers in the system to initiate service, the server waits another *T* time units, and so on, until at least one customer is present.

4. The *D* policy was developed by Balachandran [3], Tijm [31] and Gakis et al. [11]. The server returns to provide service if the accumulated backlog, which is the sum of service time of the new arriving customers, exceeds a given quantity *D*($D \ge 1$) for the first time.

5. The combined policies which are combined by any two of the *N*, *T*, and *D* policies. Doganata [10] first studied the *NT* policy M/G/1 queueing system that the vacation period is terminated if the time elapsed since the first arrival during the vacation period reaches the threshold *T*, or the number of customers in the system waiting for services reaches the threshod *N*. Alfa and Frigui [1] extended Doganata's model to the MAP/PH/1 case. Gakis, et al. [11] presented six dyadic policies for the M/G/1 queueing system.

1.2 Theoretical Analysis Techniques

Recall that a queue is characterized by the input process, the service mechanism, and the queue discipline. When arrivals have Poisson characteristics and service times are exponential, the resulting queueing process is Markovian. A Markovian model is in the framework of the birth-and-death process that completely specifies the state of the system at a given time. This information is sufficient to describe the future development of the process. These assumptions imply that the future evolution of the system from some time t depends only on the state of the system at time t, and is independent of the history of the system prior to time t. In these models, the "state" of the system could always be specified in terms of the number of customers present. (In a multidimensional case, the state is specified in terms of the number of customers of each type present at time t.)

Suppose that we are interested in a queue for which the number of customers present at any time t is not sufficient information to permit complete analysis of the model. Such a queue is impossibly solved by using birth-and-death process. Clearly, some methods are required.

In this section, we will introduce an important technique to study the F policy M/G/1/K and G/M/1/K queues with startup time and N policy M/G/1/K queue with startup time. It is supplementary variable technique, introduced by Cox [8].

Suppose that customers arrive at random with rate α in a single server queue. Let the service times of customers be independently distributed with p.d.f. $b(t)$ and hazard function $h_b(t)$. When $b(t)$ is not an exponential distribution, the probability of service being completed in $(t, t+\Delta t)$ depends on the length of time service has been in progress.

If the customer currently being served has been at the service point for a time u. Let $p_n(u,t)$ be the joint probability density of the state (n,u) at time t. More explicitly, if U is the random variable corresponding to u,

$$p_n(u,t) = \lim_{\Delta u \to 0^+} \frac{\Pr(u < U < u + \Delta u \text{ and } n \text{ customers present at time } t)}{\Delta u}.$$

Then we can have the forward equations for the process using the usual argument as in the birth and death model.

The following forward difference equations may be obtained in steady-state, for $n = 1, 2, \ldots$

$$p_0(t+dt) = p_0(t)(1-\alpha dt) + \int_0^\infty p_1(u;t)h_b(u)du + \Delta(dt), \tag{1.1}$$

$$p_n(u+dt, t+dt) = p_n(u,t)\{1-[\alpha + h_b(u)]dt\} + p_{n-1}(u,t)\alpha dt + \Delta(dt), \tag{1.2}$$

and

$$p_n(0,t) = \int_0^\infty p_{n+1}(u;t)h_b(u)du + \alpha p_0(t)\delta_{1,n} + \Delta(dt), \tag{1.3}$$

where $\delta_{1,j} = 1$ if $j = 1$ and $\delta_{1,j} = 0$ otherwise. ($\delta_{1,j}$ is the Kronecker delta function.)

By performing the following expansions to (1.2)

$$p_n(u+dt, t+dt) = p_n(u,t) + dt\frac{\partial p_n(u,t)}{\partial u} + dt\frac{\partial p_n(u,t)}{\partial t} + \Delta(dt),$$

we obtain the following partial differential equations

$$\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial t}\right) p_n(u,t) = -[\alpha + h_b(u)]p_n(u,t) - \alpha p_{n-1}(u,t). \tag{1.4}$$

Solutions to (1.1) and (1.3)-(1.4) can be obtained with the help of mathematical theory and techniques.

Let u denote the length of time when customer is being served at time t until his service completion. We have the following backward difference equations may be obtained in steady-state, for $n = 1, 2, \ldots$.

$$p_0(t+dt) = p_0(t)(1-\alpha dt) + p_1(0,t)dt + \Delta(dt), \tag{1.5}$$

$$p_n(u-dt, t+dt) = p_n(u,t)(1-\alpha dt) + p_{n-1}(u,t)\alpha dt + p_{n+1}(0,t)b(u)dt + \Delta(dt). \tag{1.6}$$

By performing the following expansions to (1.6)

$$p_n(u-dt, t+dt) = p_n(u,t) - dt\frac{\partial p_n(u,t)}{\partial u} + dt\frac{\partial p_n(u,t)}{\partial t} + \Delta(dt),$$

the following partial differential equations are obtained

$$\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial t}\right) p_n(u,t) = -\alpha p_n(u,t) + \alpha p_{n-1}(u,t) + p_{n+1}(0,t)b(u). \tag{1.7}$$

Solutions to (1.1)-(1.7) can be solved with the help of mathematical theory and techniques.

1.3 Literature Review

We use a supplementary variable technique to analyze the optimal control of the F policy M/G/1/K and F policy G/M/1/K queues where the server needs a startup time before start allowing customers in the system and $K < \infty$ denotes the maximum capacity of the system. The method of controlling arrivals focuses on reducing the number of customers in the system. The model proposed in this dissertation is very useful in real-life situations since the controlling of arriving customers is considered.

Steady-state analytical solutions of the F policy M/M/1/K queueing system with an exponential startup time were first developed by Gupta [12]. However, steady-state analytical solutions of the F policy queue with interarrival times or service times distribution of the general type have not been found. It is extremely difficult, if not possible, to obtain the explicit expressions for the steady-state probability distribution of the number of customers in the system. This becomes particularly helpful when the supplementary variable technique to the non-Markovian queueing system having general interarrival times or general service times is used. Cox [8] first introduced the supplementary variable technique. Based on this technique, Gupta and Rao [13-14] presented a recursive method to develop the steady-state probability distributions of the number of failed machines for the no-spare M/G/1 machine repair problem and the cold-standby M/G/1 machine repair problem, respectively.

Past work regarding queues may be divided into two parts according to whether the system is considered to control the service or the arrival. In the first category of controlling the service, the N policy M/M/1 queueing system without startup was first introduced by Yadin and Naor [36]. The extension of this model can be referred to Bell [4-5], Heyman [17], Kimura [23], Teghem [30], Wang and Ke [33], and others. Wang and Ke [33] provided a recursive method and used the supplementary variable technique to develop the steady-state probability distributions of the number of customers for the N policy M/G/1/L queueing system. Ke and Wang [20] presented a recursive method and applied the supplementary variable technique to obtain the steady-state probability distributions of the number of customers for the N policy G/M/1/L queueing system. The server startup corresponds to the preparatory work of the server before starting the service.

In some real-life situations, the server often needs a startup time before beginning to provide the service. Several authors research on queueing systems with startup time focus mainly on the *N* policy M/G/1 queues. Baker [2] first studied the *N* policy M/M/1 queueing system with an exponential startup time. Borthakur et al. [7] extended Baker's model to the general startup time. The *N* policy M/G/1 queueing system with startup time was investigated by several researchers such as Medhi and Templeton [28], Takagi [29], Lee and Park [26], Hur and Paik [19], Krishna et al. [25], Ke [21], Wang and Ke [33],Wang and Ke [34], Wang et al.[35], and so on. Ke [21] presented a recursive method and used the supplementary variable technique to compute the operating characteristics for the *N* policy G/M/1/L queueing system with an exponential startup time. In the second category of controlling the arrivals, the analytical developments for controlling the arrivals in queueing problems are rarely found in the literature, which are particularly for service time and interarrival time following general type. The work of related problems in the past mainly concentrates on Markovian system. The pioneering work in steady-state analytical solutions of the *F* policy M/M/1/K queueing system with an exponential startup time was first derived by Gupta [12]. Through a series of propositions, the interrelationship between the operating *N* policy and the operating *F* policy are established by Gupta [12].

Practically, the memoryless property of the arrival (input) process does not always meets the needs of applications because, for interarrival time, general distribution, rather than exponential distribution, appears to be more appropriate and reasonable. General distribution can include the special cases of exponential, Erlang, hyper-exponential, and deterministic, etc. However, aside from theoretical arguments, many real-life situations satisfy the assumptions of Markovian conditions for service time. Hence, we may consider inevitably to analyze the *F* policy M/G/1/K and *F* policy G/M/1/K queues.

1.4 Problem Statement

In this dissertation, we investigate the interrelationship between *F* policy and *N* policy in M/G/1/K and G/M/1/K queues. First, we study the optimal control in the *F* policy M/G/1/K and G/M/1/K queues. The definition of a *F* policy is described as follows: When the number of customers in the system reaches its capacity K (i.e. the system becomes full), no further arriving customers are allowed to enter the system until there are enough customers in the system have been served so that the number of customers in the system decreases to a threshold value F ($0 \le F < K - 1$). At that time, the server requires to take an exponential startup time to start allowing customers in the system. Thus, the system operates normally until the number of customers in the system reaches its capacity at which time the above process is repeated all over again. Gupta [12] first developed the concept of an F policy steady-state analytical solution of the F policy M/M/1/K queueing system with an exponential startup time.

A number of practical problems arise which may be formulated as one in which the server requires taking a startup time to start allowing customers in the system. Such models have potentially useful in practical real-life. For example, in computer process and service systems, massages are transmitted among the computers (processors). If the processor is free the message is accepted; otherwise the message is temporarily stored in a buffer to be served some time later. When the buffer is full, the arriving messages will be restricted entrance until the number of messages drops to some a threshold level. When system buffer reduces to the threshold level, the messages are immediately admitted to enter the system. This will help to prevent the system from becoming over-loaded. Another application of our model is transportation. In order to avoid traffic jams caused by motorists returning home for Chinese New Year, the entrance ramps along the highway will be controlled by a metering system. When traffic flow is congested, entrance ramps are closed to keep expressway traffic smooth. Vehicles are allowed to re-enter once the traffic is improved. The entrance ramps may need to maintain and the service may be temporarily shut down. The model is also applicable to controlling the amount of eco-tour visitors, e.g. in Kenting National Park (Taiwan). When applicants reach the limited numbers of the day, the application would be rejected.

We will study the interrelationship between F policy and N policy in M/G/1 and G/M/1 queues. We first consider the N policy M/G/1 queue with startup time. The decision-maker can turn a single server on at any arrival epoch or off at any service completion (departure) epoch. The term 'removable server' is just an abbreviation for the system of turning on and turning off the server, depending on the number of customers in the system. Yadin and Naor [36] first introduced the concept of an N policy which turns the server on when N $(N \geq 1)$ or more customers are present and turns the server off only when the system is empty. After the server is turned off, the server may not operate until N customers are present in the system.

Suppose that the time elapsing between two successive arrivals is independently and identically distributed (i.i.d.) random variable, having a general distribution $A(v)$ $(v \geq 0)$, a probability density function (p.d.f.) $a(v)$ $(v \geq 0)$ and mean interarrival time a_1. The service times of successive customers are independent and identically random variables having a common distribution $S(u)$ $(u \geq 0)$, a probability density function $s(u)$ $(u \geq 0)$ and mean service time s_1. The service process is independent of the arrival process. We assume that arriving customers form a single waiting line based on the order of their arrival; that is, the first-come, first-served discipline. The server can serve only one customer at a time. A customer, upon entry into the service facility, finding that the server is busy have to wait in the queue until the server is free.

1.5 Scope of Dissertation

In chapter 2, we provide a recursive method using the supplementary variable technique to derive the steady-state probability distributions in the F policy M/G/1/K queue. We illustrate the solution algorithm by presenting three simple examples for three different service time distributions: exponential (denoted M), 3-stage Erlang (denoted E_3), and deterministic (denoted D). Various system performance measures are also presented. The total expected cost function per unit time is developed. Numerical and comparative results are shown. In chapter 3, we follow the above method of the chapter 2 and treating the supplementary variable as the remaining interarrival time to develop the F policy G/M/1/K queue with startup time. In chapter 4, we study N policy M/G/1/K queue with startup time. In chapter 5, we examine the relationships between the F policy and N policy. Using the solution algorithm of N policy (F policy) M/G/1/K queue with startup time, we develop the steady state probabilities of F policy (N policy) G/M/1/K queue. Finally, chapter 6 consists of some concluding remarks.

Chapter 2

The F Policy M/G/1/K Queue with Startup Time

A supplementary variable technique is used to study the optimal management problem of the F policy M/G/1/K queue where the server needs a startup time before start allowing customers in the system and $K < \infty$ denotes the maximum number of customers in the system. The method of controlling arrivals focuses on reducing the number of customers in the system. The model presented in this dissertation is very useful in real-life situations since the controlling of arriving customers is considered.

The primary objective of this chapter is threefold. Firstly, we develop a recursive method using the supplementary variable technique and treating the supplementary variable as the remaining service time, to develop the steady-state probability distributions of the number of customers for the F policy M/G/1/K queue. The method can be utilized for any service time distribution, such as deterministic (denoted D), exponential (denoted M) and k-stage Erlang (denoted E_k), etc. Secondly, to illustrate a recursive method we present three simple examples for three different service time distributions such as exponential, 3-stage Erlang, and deterministic. Thirdly, we study various system performance measures, such as the average number of customers in the system, the probability that the server is busy, the blocking probability, etc. The total expected cost function per unit time for the F policy M/G/1/K queue with startup times is developed. Numerical and comparative results are also provided.

2.1 Assumptions and Notations

We consider the controlling arrivals to a finite capacity M/G/1 queue with combined F policy and exponential startup time. It is assumed that customers arrive according to a Poisson process with parameter λ, and the service times of the successive customers are independently and identically distributed (i.i.d.) random variables having a distribution $S(u)$ $(u \geq 0)$, a probability density function $s(u)$ $(u \geq 0)$ and mean service time s_1. The arrival process is independent of the service process. We assume that arriving customers form a single waiting line based on the order of their arrivals; that is, the first-come, first-served discipline. Suppose that the server can serve only one customer at a time. Customers entering into the service facility and finding that the server is busy have to wait in the queue until the server is available. Gupta [6] first introduced the concept of a F policy. The definition of a F policy is described as follows: When the number of customers in the system reaches its capacity K (i.e. the system becomes full), no further arriving customers are allowed to enter the system until there are enough customers in the system have been served so that the number of customers in the system decreases to a threshold value F $(0 \le F < K-1)$. At that time, the server requires to take an exponential startup time with parameter β to start allowing customers in the system. Thus, the system operates normally until the number of customers in the system reaches its capacity at which time the above process is repeated all over again.

The following notations and probabilities are used throughout this chapter.

F	threshold level
K	system capacity $(K > F+1)$
S	service time random variable
U	remaining service time random variable
$S(u)$	distribution function (d.f.) of S
$s(u)$	probability density function (p.d.f.) of S
$S^{*}(\theta)$	Laplace-Stieltjes transform (LST) of S
$S^{*(l)}(\theta)$	l th order derivative of $S^{*}(\theta)$ with respect to θ
$P_{0,0}(t)$	probability of no customers in the system at time t when the arrivals are not allowed to enter the system
$P_{0,n}(t)$	probability of n customers in the system at time t when the arrivals are not allowed to enter the system, where $n = 1, 2, \ldots, K$.
$P_{1,0}(t)$	probability of no customers in the system at time t when the arrivals are allowed to enter the system
$P_{1,n}(t)$	probability of n customers in the system at time t when the arrivals are allowed to enter the system, where $n = 1, 2, \ldots, K-1$.
$P_{0,0}$	steady state probability of no customers in the system when the arrivals are not allowed to enter the system
$P_{0,n}$	steady state probability of n customers in the system when the arrivals are not allowed to enter the system, where $n = 1, 2, \ldots, K$.
$P_{1,0}$	steady state probability of no customers in the system when the arrivals are allowed to enter the system
$P_{1,n}$	steady state probability of n customers in the system when the arrivals are allowed to enter the system, where $n = 1, 2, \ldots, K-1$.
s_1	mean service time

The special case with system capacity K=F+1 is presented in the appendix.

2.2 Development of the Equations and Solutions

We use the following supplementary variable: $U \equiv$remaining service time for the customer in service. The state of the system at time t is given by

$N(t) \equiv$number of customers in the system, and

$U(t) \equiv$remaining service time for the customer being served.

Let us define

$$P_{0,n}(u,t)du = \Pr\{N(t)=n, u<U(t)\le u+\mathrm{d}u\},\ u\ge 0,\quad n=0,1,\ldots,K.$$

$$P_{1,n}(u,t)du = \Pr\{N(t)=n, u<U(t)\le u+\mathrm{d}u\},\ u\ge 0,\quad n=0,1,\ldots,K-1.$$

$$P_{0,n}(t)=\int_0^\infty P_{0,n}(u,t)\mathrm{d}u,\quad n=0,1,\ldots,K.$$

$$P_{1,n}(t)=\int_0^\infty P_{1,n}(u,t)\mathrm{d}u,\quad n=0,1,\ldots,K-1.$$

Relating the state of the system at time t and $t+\mathrm{d}t$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,0}(t)=-\beta P_{0,0}(t)+P_{0,1}(0,t), \tag{2.1}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{0,n}(u,t)=-\beta P_{0,n}(u,t)+P_{0,n+1}(0,t)s(u),\quad 1\le n\le F, \tag{2.2}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{0,n}(u,t)=P_{0,n+1}(0,t)s(u),\quad F+1\le n\le K-1, \tag{2.3}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{0,K}(u,t)=\lambda P_{1,K-1}(u,t), \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{1,0}(t)=-\lambda P_{1,0}(t)+\beta P_{0,0}(t)+P_{1,1}(0,t), \tag{2.5}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{1,1}(u,t)=-\lambda P_{1,1}(u,t)+\beta P_{0,1}(u,t)+\lambda P_{1,0}(t)s(u)+ P_{1,2}(0,t)s(u), \tag{2.6}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{1,n}(u,t)=-\lambda P_{1,n}(u,t)+\beta P_{0,n}(u,t)+\lambda P_{1,n-1}(u,t)s(u) +P_{1,n+1}(0,t)s(u),\quad 2\le n\le F, \tag{2.7}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{1,n}(u,t)=-\lambda P_{1,n}(u,t)+\lambda P_{1,n-1}(u,t)s(u)+P_{1,n+1}(0,t)s(u), F+1\le n\le K-2, \tag{2.8}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{1,K-1}(u,t)=-\lambda P_{1,K-1}(u,t)+\lambda P_{1,K-2}(u,t). \tag{2.9}$$

2.3 Steady State Results

In steady state, let us define

$$P_{0,n} = \lim_{t\to\infty} P_{0,n}(t), \quad n = 0,1,\dots,K.$$

$$P_{1,n} = \lim_{t\to\infty} P_{0,n}(t), \quad n = 0,1,\dots,K-1.$$

$$P_{0,n}(u) = \lim_{t\to\infty} P_{0,n}(u,t), \quad n = 1,2,\dots,F.$$

$$P_{1,n}(u) = \lim_{t\to\infty} P_{1,n}(u,t), \quad n = 0,1,\dots,K-1.$$

and further define

$$P_{0,n}(u) = P_{0,n}s(u), \quad n = 1,2,\dots,F\ . \tag{2.10}$$

From (2.1)-(2.10), we can easily obtain the following steady state equations:

$$0 = -\beta P_{0,0} + P_{0,1}(0), \tag{2.11}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{0,n}(u) = -\beta P_{0,n}s(u) + P_{0,n+1}(0)s(u), \quad 1 \le n \le F, \tag{2.12}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{0,n}(u) = P_{0,n+1}(0)s(u), \quad F+1 \le n \le K-1, \tag{2.13}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{0,K}(u) = \lambda P_{1,K-1}(u), \tag{2.14}$$

$$0 = -\lambda P_{1,0} + \beta P_{0,0} + P_{1,1}(0), \tag{2.15}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{1,1}(u) = -\lambda P_{1,1}(u) + \beta P_{0,1}s(u) + \lambda P_{1,0}s(u) + P_{1,2}(0)s(u), \tag{2.16}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{1,n}(u) = -\lambda P_{1,n}(u) + \beta P_{0,n}s(u) + \lambda P_{1,n-1}(u) + P_{1,n+1}(0)s(u), \quad 2 \le n \le F, \tag{2.17}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{1,n}(u) = -\lambda P_{1,n}(u) + \lambda P_{1,n-1}(u) + P_{1,n+1}(0)s(u), \quad F+1 \le n \le K-2, \tag{2.18}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{1,K-1}(u) = -\lambda P_{1,K-1}(u) + \lambda P_{1,K-2}(u). \tag{2.19}$$

Further define

$$S^*(\theta) = \int_0^\infty e^{-\theta u}\mathrm{d}S(u) = \int_0^\infty e^{-\theta u}s(u)\mathrm{d}u,$$

$$P_{0,n}^*(\theta) = \int_0^\infty e^{-\theta u}P_{0,n}(u)\mathrm{d}u,$$

$$P_{1,n}^*(\theta) = \int_0^\infty e^{-\theta u}P_{1,n}(u)\mathrm{d}u,$$

$$P_{0,n} = P_{0,n}^*(0) = \int_0^\infty P_{0,n}(u)\mathrm{d}u,$$

$$P_{1,n} = P_{1,n}^*(0) = \int_0^\infty P_{1,n}(u)\mathrm{d}u,$$

$$\int_0^\infty e^{-\theta u}\frac{\partial}{\partial u}P_{0,n}(u)\mathrm{d}u = \theta P_{0,n}^*(\theta) - P_{0,n}(0),$$

and

$$\int_0^\infty e^{-\theta u} \frac{\partial}{\partial u} P_{1,n}(u)\,\mathrm{d}u = \theta P_{1,n}^*(\theta) - P_{1,n}(0).$$

Therefore, we take the LST on both sides of (2.12)-(2.14) and (2.16)-(2.19). It yields

$$-\theta P_{0,n}^*(\theta) = -\beta P_{0,n} S^*(\theta) + P_{0,n+1}(0) S^*(\theta) - P_{0,n}(0), \quad 1 \le n \le F, \tag{2.20}$$

$$-\theta P_{0,n}^*(\theta) = P_{0,n+1}(0) S^*(\theta) - P_{0,n}(0), \quad F+1 \le n \le K-1, \tag{2.21}$$

$$-\theta P_{0,K}^*(\theta) = \lambda P_{1,K}^*(\theta) - P_{0,K}(0), \tag{2.22}$$

$$(\lambda - \theta) P_{1,1}^*(\theta) = \beta P_{0,1} S^*(\theta) + \lambda P_{1,0} S^*(\theta) + P_{1,2}(0) S^*(\theta) - P_{1,1}(0), \tag{2.23}$$

$$(\lambda - \theta) P_{1,n}^*(\theta) = \beta P_{0,n} S^*(\theta) + \lambda P_{1,n-1}^*(\theta) + P_{1,n+1}(0) S^*(\theta) - P_{1,n}(0), \quad 2 \le n \le F, \tag{2.24}$$

$$(\lambda - \theta) P_{1,n}^*(\theta) = \lambda P_{1,n-1}^*(\theta) + P_{1,n+1}(0) S^*(\theta) - P_{1,n}(0), \quad F+1 \le n \le K-2, \tag{2.25}$$

$$(\lambda - \theta) P_{1,K-1}^*(\theta) = \lambda P_{1,K-2}^*(\theta) - P_{1,K-1}(0). \tag{2.26}$$

2.3.1 Recursive methods

The recursive method is developed to obtain $P_{0,n}^*(0)$ and $P_{1,n}^*(0)$. Our solution algorithm will first obtain $P_{0,n}(0)$ $(1 \le n \le K)$ which are then used for finding $P_{0,n}^*(0)$.

Using (2.11) and setting $\theta = 0$ in (2.20) and (2.21), we get

$$P_{0,n}(0) = \beta \sum_{i=0}^{\zeta_{n-1}} P_{0,i}, \quad 1 \le n \le K, \quad \text{where } \zeta_n = \begin{cases} n, & 0 \le n \le F-1, \\ F, & F \le n \le K, \end{cases} \tag{2.27}$$

and

$$P_{0,n+1}(0) = -\beta \varphi_{n,F} P_{0,n} + P_{0,n}(0), \quad 1 \le n \le K-1,$$

$$\text{where } \varphi_{n,F} = \begin{cases} 1, & 1 \le n \le F, \\ 0, & \text{otherwise}. \end{cases} \tag{2.28}$$

Using (2.28) in (2.20) and (2.21), we get

$$P_{0,n}^*(\theta) = \frac{1 - S^*(\theta)}{\theta} P_{0,n}(0), \quad 1 \le n \le K-1. \tag{2.29}$$

Taking $\lim_{\theta \to 0}$ in (2.29) and using L'Hôspital's rule once gives

$$P_{0,n}^*(0) = s_1 P_{0,n}(0), \quad 1 \le n \le K-1, \tag{2.30}$$

where $s_1 = -S^{*(1)}(0)$ is the mean service time.

Using (2.27) in (2.30), we have

$$P_{0,n}^*(0) = \phi_n P_{0,0}, \quad 1 \le n \le K-1, \tag{2.31}$$

where $\phi_n = \begin{cases} 1, & n=0, \\ s_1\beta(1+s_1\beta)^{\zeta_{n-1}}, & 1 \le n \le K. \end{cases}$ (2.32)

Thus, $P_{0,1}^*(0)$, $P_{0,2}^*(0)$,..., $P_{0,K-1}^*(0)$ can be obtained by using (2.31).

Next, we derive the expressions of $P_{1,n}(0)$ $(1 \le n \le K)$ in terms of $P_{1,0}$ and $P_{0,0}$. Using (2.31) in (2.23)-(2.24) and then setting $\theta = \lambda$ in (2.23)-(2.26), we finally obtain

$$P_{1,2}(0) = \frac{P_{1,1}(0) - \beta\phi_1 P_{0,0} S^*(\lambda) - \lambda P_{1,0} S^*(\lambda)}{S^*(\lambda)}, \tag{2.33}$$

$$P_{1,n+1}(0) = \frac{P_{1,n}(0) - \beta\varphi_{n,F}\phi_n P_{0,0} S^*(\lambda) - \lambda P_{1,n-1}^*(\lambda)}{S^*(\lambda)}, \quad 2 \le n \le K-2, \tag{2.34}$$

$$P_{1,K-1}(0) = \lambda P_{1,K-2}^*(\lambda). \tag{2.35}$$

To obtain $P_{1,n-1}^*(\lambda)$ $(1 \le n \le K-1)$ in (2.34)-(2.35), using (2.31) in (2.23)-(2.24) again, differentiating (2.23)-(2.26) $(l-1)$ times with respect to θ and setting $\theta = \lambda$, we finally get

$$P_{1,1}^{*(l-1)}(\lambda) = -\frac{S^{*(l)}(\lambda)}{l}\left[\lambda P_{1,0} + \beta\phi_1 P_{0,0} + \lambda P_{1,2}(0)\right], \quad l = 1,\ldots,K-2, \tag{2.36}$$

$$P_{1,n}^{*(l-1)}(\lambda) = -\frac{1}{l}\left[P_{1,n+1}(0) S^{*(l)}(\lambda) + \beta\varphi_{n,F}\phi_n P_{0,0} S^{*(l)}(\lambda) + \lambda P_{1,n-1}^{*(l)}(\lambda)\right], \quad 2 \le n \le K-2, \quad l = 1,\ldots,K-n-1, \tag{2.37}$$

$$P_{1,K-1}^*(\lambda) = -\lambda P_{1,K-2}^{*(1)}(\lambda), \tag{2.38}$$

where $P_{1,n}^{*(0)}(\lambda) = P_{1,n}^*(\lambda)$ and $S^{*(l)}(\theta) = \left[\left(\mathrm{d}^l/\mathrm{d}\theta^l\right) S^*(\theta)\right]$ denotes the lth derivative of $S^*(\theta)$.

Solving (2.36)-(2.38) recursively, we obtain

$$P_{1,n}^*(\lambda) = -\ell_n S^*(\lambda) P_{1,0} - \sum_{i=1}^{\zeta_n} \frac{\beta \ell_{n-i+1} \phi_i S^*(\lambda)}{\lambda} P_{0,0} - \sum_{i=1}^{n} \frac{\ell_{n-i+1} S^*(\lambda)}{\lambda} P_{1,i+1}(0), \quad 1 \le n \le K-1, \tag{2.39}$$

where

$$\ell_n = \begin{cases} -\dfrac{(-\lambda)^n S^{*(n)}(\lambda)}{n! S^*(\lambda)}, & 1 \le n \le K-1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.40}$$

Using (2.39) in (2.34), we can obtain

$$P_{1,n}(0) = \frac{1}{S^*(\lambda)} P_{1,n-1}(0) + \sum_{i=1}^{n-2} \ell_{n-i-1} P_{1,i+1}(0) + \beta\left[\sum_{i=1}^{\zeta_{n-2}} \ell_{n-i-1}\phi_i - \varphi_{n-1,F}\phi_{n-1}\right] P_{0,0} + \lambda \ell_{n-2} P_{1,0}, \quad 3 \le n \le K-1. \tag{2.41}$$

We further define

$$\Psi_n = \begin{cases} 1, & n = 0, \\ \sum_{1\le k\le n} \sum_{\substack{\tau_1+\tau_2+\cdots+\tau_k = n \\ \tau_1,\tau_2,\cdots,\tau_k \in \{1,2,\cdots,n\}}} \kappa_{\tau_1}\kappa_{\tau_2}\cdots\kappa_{\tau_k}, & n = 1,\ 2,\ldots,\ K-3, \\ 0, & \text{otherwise,} \end{cases} \tag{2.42}$$

where

$$\kappa_n = \begin{cases} \frac{1}{S^*(\lambda)} + \ell_1, & n = 1, \\ \ell_n, & n = 2,3,\cdots,K-3, \\ 0, & \text{otherwise.} \end{cases} \tag{2.43}$$

Remark: The representative meaning of the above formulation (2.42) is to sum up all possible products of k κs in which the total of subscript values of κ equals n. We give an easily understood example for n = 4:

$$\begin{aligned} \Psi_4 &= \kappa_4 + \kappa_3\kappa_1 + \kappa_2\kappa_2 + \kappa_1\kappa_3 + \kappa_1\kappa_1\kappa_2 + \kappa_1\kappa_2\kappa_1 + \kappa_2\kappa_1\kappa_1 + \kappa_1\kappa_1\kappa_1\kappa_1 \\ &= \kappa_4 + 2\kappa_3\kappa_1 + \kappa_2^2 + 3\kappa_1^2\kappa_2 + \kappa_1^4. \end{aligned}$$

Using (2.42) and (2.43) to solve (2.41) recursively, and including (2.15) and (2.33), we finally get

$$P_{1,1}(0) = A(1)P_{1,0} + B(1)P_{0,0}, \tag{2.44}$$

$$P_{1,n}(0) = \sum_{i=2}^{n} \Psi_{n-i}\left[A(i)P_{1,0} + B(i)P_{0,0}\right], \quad 2 \le n \le K-1, \tag{2.45}$$

where

$$A(n) = \begin{cases} \lambda, & n = 1, \\ \lambda\left[\frac{1-S^*(\lambda)}{S^*(\lambda)}\right], & n = 2, \\ \lambda\ell_{n-2}, & 3 \le n \le K-1, \end{cases} \tag{2.46}$$

$$B(n)=\begin{cases}-\beta, & n=1,\\ -\beta\left[\dfrac{1+\varphi_{1,F}\phi_1 S^*(\lambda)}{S^*(\lambda)}\right], & n=2,\\ \beta\displaystyle\sum_{i=1}^{\zeta_{n-2}}\ell_{n-i-1}\phi_i-\beta\varphi_{n-1,F}\phi_{n-1}, & 3\le n\le K-1.\end{cases} \tag{2.47}$$

Substituting (2.45), (2.44), and (2.35) into (2.39) finally yields

$$P_{1,0}=\frac{-\left[\displaystyle\sum_{i=1}^{K-2}\ell_{K-i-1}\sum_{j=2}^{i+1}\Psi(i-j+1)B(j)+\sum_{i=2}^{K-1}\frac{\Psi(K-i-1)B(i)}{S^*(\lambda)}+\sum_{i=1}^{\zeta_{K-2}}\beta\ell_{K-i-1}\phi_i\right]}{\left[\displaystyle\sum_{i=1}^{K-2}\ell_{K-i-1}\sum_{j=2}^{i+1}\Psi(i-j+1)A(j)+\sum_{i=2}^{K-1}\frac{\Psi(K-i-1)A(i)}{S^*(\lambda)}+\lambda\ell_{K-2}\right]}P_{0,0}. \tag{2.48}$$

Finally, we develop the steady-state probabilities $P_{1,n}^*(0)$ in terms of $P_{0,0}$. Setting $\theta=0$ in (2.23)-(2.26) we have

$$P_{1,n}^*(0)=\frac{1}{\lambda}\left[\beta\sum_{i=0}^{\zeta_n}\phi_i P_{0,0}+P_{1,n+1}(0)\right],\quad 0\le n\le K-2, \tag{2.49}$$

$$P_{1,K-1}^*(0)=\frac{\beta}{\lambda}\sum_{i=0}^{F}\phi_i P_{0,0}. \tag{2.50}$$

As $P_{1,1}(0)$, $P_{1,2}(0)$,..., $P_{1,K-1}(0)$ and $P_{1,0}$ are known, $P_{1,1}^*(0)$, $P_{1,2}^*(0)$, ..., $P_{1,K-1}^*(0)$ can be determined recursively using (2.49) and (2.50) in terms of $P_{0,0}$.

Now the only unknown quantity is $P_{0,K}^*(0)$ which can be obtained from (2.22). To find it, differentiating (2.22) with respect to θ and setting $\theta=0$, we have

$$P_{0,K}^*(0)=-\lambda P_{1,K-1}^{*(1)}(0). \tag{2.51}$$

To find $\lambda P_{1,K-1}^{*(1)}(0)$, differentiating (2.23)-(2.26) with respect to θ and setting $\theta=0$, we finally obtain

$$P_{1,1}^{*(1)}(0)=\frac{P_{1,1}+\beta\phi_1 P_{0,0}S^{*(1)}(0)+\lambda P_{1,0}S^{*(1)}(0)+P_{1,2}(0)S^{*(1)}(0)}{\lambda}, \tag{2.52}$$

$$P_{1,n}^{*(1)}(0)=\frac{P_{1,n}+\beta\varphi_{n,F}\phi_n P_{0,0}S^{*(1)}(0)+\lambda P_{1,n-1}^{*(1)}(0)+P_{1,n+1}(0)S^{*(1)}(0)}{\lambda},\quad 2\le n\le K-2, \tag{2.53}$$

$$P_{1,K-1}^{*(1)}(0)=\frac{P_{1,K-1}+\lambda P_{1,K-2}^{*(1)}(0)}{\lambda}. \tag{2.54}$$

As $P_{1,1}^{*(1)}(0)$ is known completely from (2.52), the values $P_{1,n}^{*(1)}(0)$ for $n = 2, 3, \ldots, K-1$ can be found recursively from (2.53) and (2.54). Therefore we obtain

$$P_{1,K-1}^{*(1)}(0) = \frac{1}{\lambda}\left[\sum_{i=1}^{K-1} P_{1,i} + \beta S^{*(1)}(0)\sum_{i=1}^{F}\phi_n P_{0,0} + S^{*(1)}(0)\sum_{i=2}^{K-1} P_{1,i}(0) + \lambda P_{1,0} S^{*(1)}(0)\right]. \tag{2.55}$$

Substituting (2.55) into (2.51), we have

$$P_{0,K}^{*}(0) = -\left[\sum_{i=1}^{K-1} P_{1,i} + \beta S^{*(1)}(0)\sum_{i=1}^{F}\phi_n P_{0,0} + S^{*(1)}(0)\sum_{i=2}^{K-1} P_{1,i}(0) + \lambda P_{1,0} S^{*(1)}(0)\right]. \tag{2.56}$$

So $P_{0,1}^{*}(0), P_{0,2}^{*}(0), \ldots, P_{0,K}^{*}(0)$ is known in terms of $P_{0,0}$, which can be determined using the normalizing condition

$$\sum_{i=0}^{K} P_{0,i} + \sum_{i=0}^{K-1} P_{1,i} = 1. \tag{2.57}$$

To demonstrate the working of the recursive method, we first describe the solution algorithm for calculating the steady state probabilities, $P_{0,n}^{*}(0)$ $(0 \le n \le K)$ and $P_{1,n}^{*}(0)$ $(0 \le n \le K-1)$. Next, to illustrate the solution algorithm, we provide three simple examples where the service time distributions are exponential, k-stage Erlang, and deterministic, respectively.

2.3.2 The solution algorithm

Let F be the threshold, K be the maximum capacity of the system, and let $S^{*(l)}(\theta)$ be the l-th derivative of $S^{*}(\theta)$, where $l = 1, 2, \ldots, K$. We set the values of F, K, and the LST expression of the service time distribution, namely $S^{*}(\theta)$. The steps of the solution algorithm are stated as follows:

Step 1. For each $n = 0, 1, \ldots, K$, compute ϕ_n using (2.32).

Step 2. For each $n = 1, 2, \ldots, K-1$, compute $P_{0,n}^{*}(0)$ using (2.31) in terms of $P_{0,0}$.

Step 3. Compute ℓ_n $(1 \le n \le K-2)$ and κ_n $(1 \le n \le K-3)$ using (2.40) and (2.43), respectively.

Step 4. For each $n = 0, 1, \ldots, K-3$, compute Ψ_n using (2.42).

Step 5. For each $n = 1, 2, \ldots, K-1$, compute $A(n)$ and $B(n)$ using (2.46) and (2.47).

Step 6. For each $n = 1, 2, \ldots, K-1$, compute $P_{1,n}(0)$ using (2.44) and (2.45) in terms of $P_{1,0}$ and $P_{0,0}$.

Step 7. Compute $P_{1,0}$ using (2.48) in terms of $P_{0,0}$. Thus $P_{1,n}(0)$ $(1 \le n \le K-1)$ are achieved from *Step* 6.

Step 8. For each $n = 1, 2, \ldots, K-1$, compute $P_{1,n}^{*}(0)$ using (2.49) and (2.50) in terms of $P_{0,0}$.

Step 9. For $n = K$, compute $P_{0,n}^{*}(0)$ using (2.56) in terms of $P_{0,0}$.

Step 10. Determine $P_{0,0}$ using (2.57). Thus $P_{0,n}^{*}(0)$ $(n = 1, 2, \ldots, K)$ are achieved from *Steps* 2 and 9, and $P_{1,n}^{*}(0)$ $(n = 0, 1, \ldots, K-1)$ are achieved from *Steps* 7 to 8.

2.4 Simple Examples

We use the solution algorithm to illustrate a recursive method. We provide three simple examples for three different service time distributions such as exponential, 3-stage Erlang, and deterministic, respectively.

Example 1 (For M/M/1 queue). We set the mean service time $s_1 = 1/\mu$, where μ is the service rate. Assume that $F = 1$ and $K = 4$. In this case, we have

$$S^{*}(\theta) = \frac{\mu}{\mu + \theta}.$$

Step 1. For each $n = 0, 1, \ldots, 4$, compute ϕ_n.

Using (2.32), we obtain

$$\phi_0 = 1,\ \phi_1 = (1-\alpha)/\alpha, \text{ and } \phi_2 = \phi_3 = \phi_4 = (1-\alpha)/\alpha^2, \text{ where } \alpha = \mu/(\mu + \beta).$$

Step 2. For each $n = 1, 2, 3$, compute $P_{0,n}^{*}(0)$ using (2.31) in terms of $P_{0,0}$.

From (2.31), we finally get

$$P_{0,1}^{*}(0) = \phi_1 P_{0,0} = \frac{1-\alpha}{\alpha} P_{0,0},$$

$$P_{0,2}^{*}(0) = P_{0,3}^{*}(0) = \phi_2 P_{0,0} = \frac{1-\alpha}{\alpha^2} P_{0,0}.$$

Step 3. For each $n = 1, 2$, compute ℓ_n and κ_n using (2.40) and (2.43), respectively.

For each $n = 1, 2$, using (2.40) yields $\ell_1 = -1/(1+\sigma)$ and $\ell_2 = -1/(1+\sigma)^2$, where

$\sigma = \mu/\lambda$.

For each $n = 1$, we find from (2.43) that $\kappa_1 = (1+\sigma+\sigma^2)/\sigma(1+\sigma)$.

Step 4. For each $n = 0,\ 1$, compute Ψ_n.

It implies from (2.42) that $\Psi_0 = 1$ and $\Psi_1 = (1+\sigma+\sigma^2)/\sigma(1+\sigma)$.

Step 5. For each $n = 1,\ 2,\ 3$, compute $A(n)$ and $B(n)$.

Using (2.46) and (2.47), it follows that

$$A(1) = \mu/\sigma,\quad A(2) = \mu/\sigma^2, \text{ and } A(3) = -\mu/\sigma(1+\sigma).$$

$$B(1) = -\frac{(1-\alpha)\mu}{\alpha},\quad B(2) = -\frac{(\alpha+\sigma)(1-\alpha)\mu}{\sigma\alpha^2}, \text{ and } B(3) = -\frac{(1-\alpha)^2\mu}{(1+\sigma)\alpha^2}.$$

Step 6. For each $n = 1,\ 2,\ 3$, compute $P_{1,n}(0)$ using (2.44) and (2.45) in terms of $P_{1,0}$ and $P_{0,0}$.

It yields from (2.44) and (2.45) that

$$P_{1,1}(0) = A(1)P_{1,0} + B(1)P_{0,0},$$
$$P_{1,2}(0) = \Psi_0\left[A(2)P_{1,0} + B(2)P_{0,0}\right],$$
$$P_{1,3}(0) = \Psi_1\left[A(2)P_{1,0} + B(2)P_{0,0}\right] + \Psi_0\left[A(3)P_{1,0} + B(3)P_{0,0}\right].$$

Step 7. Compute $P_{1,0}$ using (2.48) in terms of $P_{0,0}$. Thus $P_{1,n}(0)$ $(1 \le n \le 3)$ are achieved from *Step* 6.

From (2.48), we finally have

$$P_{1,0} = \frac{\sigma(1-\alpha)(\alpha+\sigma+\sigma^2+\sigma^3)}{\alpha^2}P_{0,0},\quad \left(P_{1,0}^*(0) = P_{1,0}\right),$$

$$P_{1,1}(0) = \frac{\sigma\mu(1-\alpha)(1+\sigma+\sigma^2)}{\alpha^2}P_{0,0},$$

$$P_{1,2}(0) = \frac{\sigma\mu(1-\alpha)(1+\sigma)}{\alpha^2}P_{0,0},$$

$$P_{1,3}(0) = \frac{\sigma\mu(1-\alpha)}{\alpha^2}P_{0,0}.$$

Step 8. For each $n = 1,\ 2,\ 3$, compute $P_{1,n}^*(0)$ using (2.49) and (2.50) in terms of $P_{0,0}$.

Using (2.49) and (2.50) yields

$$P_{1,1}^*(0) = \frac{\sigma(1-\alpha)(1+\sigma+\sigma^2)}{\alpha^2}P_{0,0},\quad P_{1,2}^*(0) = \frac{\sigma(1-\alpha)(1+\sigma)}{\alpha^2}P_{0,0},$$

and $P_{1,3}^{*}(0)=\dfrac{\sigma(1-\alpha)}{\alpha^2}P_{0,0}$.

Step 9. For $n=4$, compute $P_{0,n}^{*}(0)$ using (2.56) in terms of $P_{0,0}$.

Using (2.56), it follows that

$$P_{0,4}^{*}(0)=\frac{(1-\alpha)}{\alpha^2}P_{0,0}.$$

Step 10. Determine $P_{0,0}$ using (2.57). Thus $P_{0,n}^{*}(0)$ $(n=0,\ 1,\dots,4)$ are achieved from *Step*s 2 and 9, and $P_{1,n}^{*}(0)$ $(n=0,\ 1,\ 2,\ 3)$ are achieved from *Step*s 7 to 8.

$$P_{0,0}=\frac{\alpha^2}{\alpha^2+\alpha(1-\alpha)+3(1-\alpha)+\sigma(1-\alpha)\left(3+\alpha+3\sigma+2\sigma^2+\sigma^3\right)}.$$

It is to be noted that these results are the same as those given in Gupta [12, p1006].

Example 2 (For M/E_3/1 queue). The 3-stage Erlang distribution is made up of three independent and identical exponential stages, each with mean $1/3\mu$. We set the mean service time $s_1=1/\mu$, $F=1$, and $K=3$. In this case, we have

$$S^{*}(\theta)=\left(\frac{3\mu}{3\mu+\theta}\right)^3.$$

Step 1. For each $n=0,\ 1,\dots,3$, compute ϕ_n.

From (2.32), we finally obtain

$\phi_0=1$, $\phi_1=3(1-\gamma)/\gamma$, and $\phi_2=\phi_3=3(1-\gamma)(3-2\gamma)/\gamma^2$,

where $\gamma=3\mu/(3\mu+\beta)$.

Step 2. For each $n=1,\ 2$, compute $P_{0,n}^{*}(0)$ using (2.31) in terms of $P_{0,0}$.

From (2.31), it follows that

$$P_{0,1}^{*}(0)=\phi_1 P_{0,0}=3\frac{1-\gamma}{\gamma}P_{0,0},$$

$$P_{0,2}^{*}(0)=\phi_2 P_{0,0}=3\frac{(1-\gamma)(3-2\gamma)}{\gamma^2}P_{0,0}.$$

Step 3. For each $n=1$, compute ℓ_n.

Using (2.40) yields $\ell_1=-3/(1+\tau)$, where $\tau=3\mu/\lambda$.

Step 4. For each $n=0$, compute Ψ_n.

It implies from (2.42) that $\Psi_0=1$.

Step 5. For each $n=1,\ 2$, compute $A(n)$ and $B(n)$.

It yields from (2.46) and (2.47) that

$$A(1)=3\mu/\tau \quad \text{and} \quad A(2)=3\mu\left(1+3\tau+3\tau^2\right)\Big/\tau^4 .$$

$$B(1)=-3\frac{(1-\gamma)\mu}{\gamma} \quad \text{and} \quad B(2)=-3\frac{\left[\alpha\left(1+3\tau+3\tau^2\right)+\tau^3(3-2\gamma)\right](1-\gamma)\mu}{\tau^3\gamma^2}.$$

Step 6. For each $n=1,\ 2$, compute $P_{1,n}(0)$ using (2.44) and (2.45) in terms of $P_{1,0}$ and $P_{0,0}$.

From (2.44) and (2.45), we find that

$$P_{1,1}(0)=A(1)P_{1,0}+B(1)P_{0,0},$$
$$P_{1,2}(0)=\Psi_0\left[A(2)P_{1,0}+B(2)P_{0,0}\right],$$

Step 7. Compute $P_{1,0}$ using (2.48) in terms of $P_{0,0}$. Thus $P_{1,n}(0)$ $(1\le n\le 2)$ are achieved from *Step* 6.

It implies from (2.48) that

$$P_{1,0}=\frac{\tau(1-\gamma)\left[\tau^3(1+\tau)(3-2\gamma)+\gamma\left(1+4\tau+6\tau^2\right)\right]}{\left(1+4\tau+6\tau^2\right)\gamma^2}P_{0,0},\ \left(P_{1,0}^*(0)=P_{1,0}\right),$$

$$P_{1,1}(0)=3\frac{\tau^3\mu(1+\tau)(1-\gamma)(3-2\gamma)}{\left(1+4\tau+6\tau^2\right)\gamma^2}P_{0,0},$$

$$P_{1,2}(0)=9\frac{\tau^3\mu(1-\gamma)(3-2\gamma)}{\left(1+4\tau+6\tau^2\right)\gamma^2}P_{0,0}.$$

Step 8. For each $n=1,\ 2$, compute $P_{1,n}^*(0)$ using (2.49) and (2.50) in terms of $P_{0,0}$.

Using (2.49) and (2.50) yields

$$P_{1,1}^*(0)=\frac{\tau(1+\tau)(1-\gamma)(3-2\gamma)\left(1+3\tau+3\tau^2\right)}{\left(1+4\tau+6\tau^2\right)\gamma^2}P_{0,0},$$

and $$P_{1,2}^*(0)=\frac{\tau(1-\gamma)(3-2\gamma)\left(1+3\tau+3\tau^2\right)}{\gamma^2}P_{0,0}.$$

Step 9. For $n=3$, compute $P_{0,n}^{*}(0)$ using (2.56) in terms of $P_{0,0}$.

It follows from (2.56) that

$$P_{0,3}^{*}(0)=\frac{(1-\gamma)(3-2\gamma)\left(3+10\tau+10\tau^{2}\right)}{\left(1+4\tau+6\tau^{2}\right)\gamma^{2}}P_{0,0}.$$

Step 10. Determine $P_{0,0}$ using (2.57). Thus $P_{0,n}^{*}(0)$ $(n=0,\ 1,\ldots,3)$ are achieved from *Steps* 2 and 9, and $P_{1,n}^{*}(0)$ $(n=0,\ 1,\ 2)$ are achieved from *Steps* 7 to 8.

$$P_{0,0}=\left(1+4\tau+6\tau^{2}\right)\gamma^{2}\times$$
$$\left\{\left(1+4\tau+6\tau^{2}\right)\left[\gamma^{2}+(1-\gamma)\left(9-\gamma^{2}\right)\right]\right.$$
$$\left.+(1-\gamma)(3-2\gamma)\left(3+11\tau+14\tau^{2}+6\tau^{3}+4\tau^{4}+\tau^{5}\right)\right\}^{-1}.$$

Example 3 (For M/D/1 queue). We set the mean service time $s_1=1/\mu$, $F=1$, and $K=3$. In this case,

$$S^{*}(\theta)=e^{-\theta/\mu}.$$

Step 1. For each $n=0,\ 1,\ldots,3$, compute ϕ_n.

Using (2.32) yields

$$\phi_0=1,\ \phi_1=(1-\alpha)/\alpha,\ \text{and}\ \phi_2=\phi_3=(1-\alpha)/\alpha^{2},\ \text{where}\ \alpha=\mu/(\mu+\beta).$$

Step 2. For each $n=1,\ 2$, compute $P_{0,n}^{*}(0)$ using (2.31) in terms of $P_{0,0}$.

Using (2.31), we finally get

$$P_{0,1}^{*}(0)=\phi_1 P_{0,0}=\frac{1-\alpha}{\alpha}P_{0,0},$$
$$P_{0,2}^{*}(0)=\phi_2 P_{0,0}=\frac{1-\alpha}{\alpha^{2}}P_{0,0}.$$

Step 3. For each $n=1$, compute ℓ_n.

From (2.40), we find that $\ell_1=-\rho$, where $\rho=\lambda/\mu$.

Step 4. For each $n=0$, compute Ψ_n.

It implies from (2.42) that $\Psi_0=1$.

Step 5. For each $n=1,\ 2$, compute $A(n)$ and $B(n)$.

From (2.46) and (2.47), it follows that

$$A(1)=\rho\mu \quad \text{and} \quad A(2)=\rho\mu\left(1-e^{\rho}\right).$$

$$B(1)=-\frac{(1-\alpha)\mu}{\alpha} \quad \text{and} \quad B(2)=-\frac{\mu(1-\alpha)\left(1-\alpha+e^{\rho}\right)}{\alpha^2}.$$

Step 6. For each $n=1,\ 2$, compute $P_{1,n}(0)$ using (2.44) and (2.45) in terms of $P_{1,0}$ and $P_{0,0}$.

Using (2.44) and (2.45) yields

$$P_{1,1}(0)=A(1)P_{1,0}+B(1)P_{0,0},$$
$$P_{1,2}(0)=\Psi_0\left[A(2)P_{1,0}+B(2)P_{0,0}\right],$$

Step 7. Compute $P_{1,0}$ using (2.48) in terms of $P_{0,0}$. Thus $P_{1,n}(0)$ $(1\le n\le 2)$ are achieved from *Step* 6.

We find from (2.48) that

$$P_{1,0}=\frac{(1-\alpha)\left[\alpha(1+\rho)-1-\alpha e^{\rho}\right]}{\alpha^2\rho\left(1+\rho-e^{\rho}\right)}P_{0,0},\ \left(P_{1,0}^{*}(0)=P_{1,0}\right),$$

$$P_{1,1}(0)=-\frac{(1-\alpha)\mu}{\alpha^2\left(1+\rho-e^{\rho}\right)}P_{0,0},$$

$$P_{1,2}(0)=-\frac{(1-\alpha)\rho\mu}{\alpha^2\left(1+\rho-e^{\rho}\right)}P_{0,0}.$$

Step 8. For each $n=1,\ 2$, compute $P_{1,n}^{*}(0)$ using (2.49) and (2.50) in terms of $P_{0,0}$.

Using (2.49) and (2.50) yields

$$P_{1,1}^{*}(0)=-\frac{(1-\alpha)\left(1-e^{\rho}\right)}{\rho\alpha^2\left(1+\rho-e^{\rho}\right)}P_{0,0} \quad \text{and} \quad P_{1,2}^{*}(0)=-\frac{(1-\alpha)}{\rho\alpha^2}P_{0,0}.$$

Step 9. For $n=3$, compute $P_{0,n}^{*}(0)$ using (2.56) in terms of $P_{0,0}$.

It follows from (2.56) that

$$P_{0,3}^{*}(0)=\frac{(1-\alpha)\left[2\left(e^{\rho}-1\right)-\rho\left(1+e^{\rho}\right)\right]}{\rho\alpha^2\left(1+\rho-e^{\rho}\right)}P_{0,0}.$$

Step 10. Determine $P_{0,0}$ using (2.57). Thus $P_{0,n}^{*}(0)$ $(n=0,\ 1,\ldots,3)$ are achieved from *Steps* 2 and 9, and $P_{1,n}^{*}(0)$ $(n=0,\ 1,\ 2)$ are achieved from *Steps* 7 to 8.

$$P_{0,0}=\rho\alpha^2\left(1+\rho-e^{\rho}\right)\times$$
$$\left[\rho\left(1+\rho+\alpha-\alpha^2\right)+e^{\rho}\left(\alpha\rho+\alpha^2-2\rho^2-\alpha\right)-(1-\alpha)^2\right]^{-1}.$$

2.5 Optimal *F* policy

Our analysis is based on the following system performance measures of the F policy M/G/1/K queue with exponential startup time. Let

$L_s \equiv$ the average number of customers in the system;

$P_b \equiv$ the probability that the server is busy;

$P_s \equiv$ the probability that the server requires a startup time before starting the service;

$P_{bl} \equiv$ the probability that the server is blocked.

The expressions for L_s, P_b, P_s, and P_{bl} are give by

$$L_s=\sum_{n=1}^{K} nP_{0,n}+\sum_{n=1}^{K-1} nP_{1,n},$$

$$P_b=\sum_{n=0}^{K} P_{0,n}+\sum_{n=0}^{K-1} P_{1,n},$$

$$P_s=\sum_{n=0}^{F} P_{0,n},$$

$$P_{bl}=\sum_{n=0}^{K} P_{0,n}.$$

2.5.1 Cost function

We develop the total expected cost function per unit time for the F policy M/G/1/K queue with startup times, in which F is a management decision variable. The main purpose of this subsection is to determine the optimum management F policy so as to minimize this total expected cost function. Let

$C_h \equiv$ holding cost per unit time for each customer present in the system;

$C_b \equiv$ busy cost per unit time for a busy server;

$C_s \equiv$ startup cost per unit time for the preparatory work of the server before starting the service;

$C_{bl} \equiv$ fixed cost for every lost customer when the system is blocked.

Utilizing the definitions of each cost element listed above, the total expected cost function per unit time is given by

$$TC(F)=C_hL_s+C_bP_b+C_sP_s+C_{bl}\lambda P_{bl}. \qquad (2.58)$$

The optimal value of F, F^* is determined by satisfying the following inequality

$$TC(F^*-1) \geq TC(F^*) \text{ and } TC(F^*+1) \geq TC(F^*). \quad (2.59)$$

2.5.2 Numerical examples

We now perform a sensitivity analysis on the optimum value F^* based on changes in specific values of the system parameters and fix the system capacity K=15. We consider the three simple examples for three different service time distributions such as exponential, 3-stage Erlang, and deterministic and employ the following cost elements:

Case 1: $C_h = 5,\ C_b = 200,\ C_s = 250,\ C_{bl} = 300.$
Case 2: $C_h = 5,\ C_b = 200,\ C_s = 250,\ C_{bl} = 350.$
Case 3: $C_h = 5,\ C_b = 200,\ C_s = 300,\ C_{bl} = 350.$
Case 4: $C_h = 5,\ C_b = 225,\ C_s = 300,\ C_{bl} = 350.$
Case 5: $C_h = 10,\ C_b = 225,\ C_s = 300,\ C_{bl} = 350.$

In this section we provide the numerical results of the optimal value F^* and the minimum expected cost for three interarrival time distributions and specific values of λ, μ, β. We first fix (μ, β) = (1.0, 3.0) and choose different values of λ = 0.5, 0.6, 0.7. Next, we fix (λ, β) = (0.8, 3.0) and consider various values of μ = 1.0, 1.1, 1.2. Finally, we fix (λ, μ) = (0.8, 1.0) and select different values of β = 2.0, 4.0, 5.0.

The optimal value of F, F^*, and its minimum expected cost $TC\left(F^*\right)$ for the above five cases are shown in Tables 1-3. For fixed values of (μ, β) and various values of λ in Tables 1-3, we observe that (i) $TC\left(F^*\right)$ increases as λ increases for any case; and (ii) F^* decreases as λ increases for any case. For fixed values of (λ, β) and various values of μ in Tables 1-3, we find that (i) $TC\left(F^*\right)$ decreases as μ increases for any case; and (ii) F^* increases as μ increases for any case. Again, for fixed (λ, μ) and various values of β in Tables 1-3, we observe that (i) $TC\left(F^*\right)$ slightly decreases as β increases for any case; and (ii) F^* does not change at all when β changes from 2.0 to 5.0 for any case. Intuitively, F^* is insensitive to changes in β.

It can be easily see from Tables 1 through 3 that (i) F^* increases as C_h decreases or C_{bl} increases (see cases 4-5 and cases 1-2); and (ii) C_h and C_{bl} have a larger effect on F^* than C_b and C_s (see cases 3-4 and cases 2-3).

Table 1. The optimal value of F and its minimum expected cost for the service time distribution such as exponential.

		$\lambda\ (\mu,\beta)=(1.0,3.0)$			$\mu\ (\lambda,\beta)=(0.8,3.0)$			$\beta\ (\lambda,\mu)=(0.8,1.0)$		
		0.5	0.6	0.7	1.0	1.1	1.2	2.0	4.0	5.0
Case1	F^*	9	7	5	4	7	10	5	4	4
	$TC(F^*)$	105.000	127.486	151.420	177.597	158.454	143.314	177.680	177.561	177.540
Case2	F^*	12	11	9	6	10	12	6	6	6
	$TC(F^*)$	105.001	127.501	151.554	178.285	158.655	143.367	178.361	178.247	178.225
Case3	F^*	12	11	8	6	10	12	6	6	6
	$TC(F^*)$	105.001	127.502	151.562	178.314	158.669	143.374	178.404	178.269	178.242
Case4	F^*	11	9	7	4	8	11	5	5	5
	$TC(F^*)$	117.500	142.496	169.000	197.985	176.767	160.020	198.072	197.941	197.915
Case5	F^*	5	4	3	2	4	6	2	2	2
	$TC(F^*)$	122.470	149.933	180.049	213.873	189.095	169.773	213.960	213.830	213.804

Table 2. The optimal value of F and its minimum expected cost for the service time distribution such as 3-stage Erlang.

		$\lambda\ (\mu,\beta)=(1.0,3.0)$			$\mu\ (\lambda,\beta)=(0.8,3.0)$			$\beta\ (\lambda,\mu)=(0.8,1.0)$		
		0.5	0.6	0.7	1.0	1.1	1.2	2.0	4.0	5.0
Case1	F^*	9	7	6	4	7	10	4	4	4
	$TC(F^*)$	104.167	125.999	148.912	173.998	155.504	141.111	174.022	173.986	173.979
Case2	F^*	12	11	9	6	10	12	6	6	6
	$TC(F^*)$	104.167	126.000	148.932	174.216	155.541	141.116	174.241	174.204	174.197
Case3	F^*	12	11	9	6	10	12	6	6	6
	$TC(F^*)$	104.167	126.000	148.933	174.226	155.544	141.117	174.255	174.211	174.202
Case4	F^*	11	9	7	5	9	12	5	5	5
	$TC(F^*)$	116.667	141.000	166.424	194.121	173.710	157.781	194.150	194.107	194.099
Case5	F^*	6	4	3	2	4	6	2	2	2
	$TC(F^*)$	120.833	146.996	175.278	207.572	183.632	165.532	207.601	207.557	207.548

Table 3. The optimal value of F and its minimum expected cost for the service time distribution such as deterministic.

		$\lambda\ (\mu,\beta)=(1.0,3.0)$			$\mu\ (\lambda,\beta)=(0.8,3.0)$			$\beta\ (\lambda,\mu)=(0.8,1.0)$		
		0.5	0.6	0.7	1.0	1.1	1.2	2.0	4.0	5.0
Case1	F^*	10	8	6	4	7	10	4	4	4
	$TC(F^*)$	103.750	125.250	147.578	171.798	153.930	140.000	171.806	171.794	171.792
Case2	F^*	12	11	9	6	10	12	6	6	6
	$TC(F^*)$	103.750	125.250	147.582	171.869	153.938	140.001	171.877	171.865	171.863
Case3	F^*	12	11	9	6	10	12	6	6	6
	$TC(F^*)$	103.750	125.250	147.582	171.872	153.938	140.001	171.882	171.867	171.864
Case4	F^*	12	10	7	5	9	12	5	5	5
	$TC(F^*)$	116.250	140.250	165.080	191.839	172.117	156.667	191.848	191.834	191.831
Case5	F^*	7	5	3	2	4	6	2	2	2
	$TC(F^*)$	120.000	145.500	172.649	203.459	180.568	163.331	203.469	203.454	203.451

Chapter 3

The *F* Policy G/M/1/K Queue with Startup Time

We use a supplementary variable technique to analyze the optimal control of the *F* policy G/M/1/K queue where the server needs a startup time before start allowing customers in the system and $K < \infty$ denotes the maximum capacity of the system. The method of controlling arrivals focuses on reducing the number of customers in the system. The model proposed in this dissertation is very useful in real-life situations since the controlling of arriving customers is considered.

In section 3.1, the queue model is briefly described. Section 3.2 develops the equations and solutions. Section 3.3 provides a recursive method using the supplementary variable technique and treating the supplementary variable as the remaining interarrival time, to obtain the steady-state probability distributions of the number of customers in the *F* policy G/M/1/K queue. In section 3.4, we illustrate the solution algorithm by presenting three simple examples for three different interarrival time distributions: exponential (denoted M), 3-stage Erlang (denoted E_3), and deterministic (denoted D). In section 3.5, various system performance measures are presented. The total expected cost function per unit time for the *F* policy G/M/1/K queue with startup times is developed. Numerical and comparative results are also provided.

3.1 Assumptions and Notations

We consider the category of controlling the arrivals to the *F* policy G/M/1/K queue with exponential startup time. It is assumed that the times elapsing between successive arrivals are independently and identically distributed (i.i.d) random variables having general distribution $A(v)$ $(v \geq 0)$, a probability density function $a(v)$ $(v \geq 0)$ and mean interarrival time b_1. The service times of the customers are independently random variables having a common exponential distribution with mean $1/\mu$. Let us assume that customers arriving at the server form a single waiting line and are served in the order of their arrivals; that is, according to the first-come, first-served (FCFS) discipline. Suppose that the server can serve only one customer at a time, and that the service is independent of the arrival of the customers. Customers entering into the service facility and finding that the server is busy have to wait in the queue until the server is available. Gupta [6] first introduced the concept of a *F* policy. The definition of a *F* policy is described as follows: When the number of customers in the system reaches its capacity *K* (i.e. the system becomes full), no further arriving customers are allowed to enter the system until there are enough customers in the system have been served so that the number of customers in the system decreases to a threshold value F $(0 \le F < K-1)$. At that time, the server needs to take an exponential startup time with parameter β to start allowing customers in the system. Thus, the system operates normally until the number of customers in the system reaches its capacity at which time the above process is repeated all over again.

The following notations and probabilities are used throughout this chapter.

F	threshold level
K	system capacity $(K > F+1)$
A	Interarrival time random variable
V	remaining interarrival time random variable
$A(v)$	distribution function (d.f.) of A
$a(v)$	probability density function (p.d.f.) of A
$a^*(\theta)$	Laplace-Stieltjes transform (LST) of A
$a^{*(l)}(\theta)$	l th order derivative of $a^*(\theta)$ with respect to θ
$P_{0,0}(t)$	probability of no customers in the system at time t when the arrivals are not allowed to enter the system
$P_{0,n}(t)$	probability of n customers in the system at time t when the arrivals are not allowed to enter the system, where $n = 1, 2, \ldots, K$.
$P_{1,0}(t)$	probability of no customers in the system at time t when the arrivals are allowed to enter the system
$P_{1,n}(t)$	probability of n customers in the system at time t when the arrivals are allowed to enter the system, where $n = 1, 2, \ldots, K-1$.

The special case with system capacity *K*=*F*+1 is presented in the appendix.

3.2 Development of the Equations and Solutions

We use the following supplementary variable: $V \equiv$ remaining interarrival time for the customer in arrival process. The state of the system at time t is given by

$N(t) \equiv$ number of customers in the system, and

$V(t) \equiv$ remaining interarrival time for the customer who is arriving.

Let us define

$$P_{0,n}(v,t)dv = \Pr\{N(t)=n, v<V(t)\le v+\mathrm{d}v\}, \quad v\ge 0, \quad n=0,1,\dots,F.$$

$$P_{1,n}(v,t)dv = \Pr\{N(t)=n, v<V(t)\le v+\mathrm{d}v\}, \quad v\ge 0, \quad n=0,1,\dots,K-1.$$

$$P_{0,n}(t)=\int_0^\infty P_{0,n}(v,t)\mathrm{d}v, \quad n=0,1,\dots,F.$$

$$P_{1,n}(t)=\int_0^\infty P_{1,n}(v,t)\mathrm{d}v, \quad n=0,1,\dots,K-1.$$

Relating the state of the system at time t and $t+\mathrm{d}t$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,0}(t) = -\beta P_{0,0}(t)+\mu P_{0,1}(t), \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,n}(t) = -(\beta+\mu)P_{0,n}(t)+\mu P_{0,n+1}(t), \quad 1\le n\le F, \tag{3.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,n}(t) = -\mu P_{0,n}(t)+\mu P_{0,n+1}(t), \quad F+1\le n\le K-1, \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,K}(t) = -\mu P_{0,K}(t)+P_{1,K-1}(0,t), \tag{3.4}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial v}\right)P_{1,0}(v,t) = \beta P_{0,0}(v,t)+\mu P_{1,1}(v,t), \tag{3.5}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial v}\right)P_{1,n}(v,t) = {}& -\mu P_{1,n}(v,t)+\beta P_{0,n}(v,t)+P_{1,n-1}(0,t)a(v)+ \\ & \mu P_{1,n+1}(v,t), \quad 1\le n\le F,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial v}\right)P_{1,n}(v,t) = {}& -\mu P_{1,n}(v,t)+P_{1,n-1}(0,t)a(v)+\mu P_{1,n+1}(v,t), \\ & F+1\le n\le K-2,\end{aligned} \tag{3.7}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial v}\right)P_{1,K-1}(v,t) = -\mu P_{1,K-1}(v,t)+P_{1,K-2}(0,t)a(v). \tag{3.8}$$

3.3 Steady State Results

In steady state, let us define

$$P_{0,n} = \lim_{t\to\infty}P_{0,n}(t), \quad n=0,1,\dots,K.$$

$$P_{1,n} = \lim_{t\to\infty}P_{0,n}(t), \quad n=0,1,\dots,K-1.$$

$$P_{0,n}(v) = \lim_{t\to\infty}P_{0,n}(v,t), \quad n=0,1,\dots,F.$$

$$P_{1,n}(v) = \lim_{t\to\infty}P_{1,n}(v,t), \quad n=0,1,\dots,K-1.$$

and further define

$$P_{0,n}(v) = P_{0,n}a(v), \; n=0,1,\dots,F\,. \tag{3.9}$$

From (3.1)-(3.8) we can easily obtain the following steady state equations:

$$0=-\beta P_{0,0}+\mu P_{0,1}, \tag{3.10}$$
$$0=-(\beta+\mu)P_{0,n}+\mu P_{0,n+1}, \quad 1\le n\le F, \tag{3.11}$$
$$0=-\mu P_{0,n}+\mu P_{0,n+1}, \quad F+1\le n\le K-1, \tag{3.12}$$
$$0=-\mu P_{0,K}+P_{1,K-1}(0), \tag{3.13}$$
$$-\frac{\mathrm{d}}{\mathrm{d}v}P_{1,0}(v)=\beta P_{0,0}a(v)+\mu P_{1,1}(v), \tag{3.14}$$
$$-\frac{\mathrm{d}}{\mathrm{d}v}P_{1,n}(v)=-\mu P_{1,n}(v)+\beta P_{0,n}a(v)+P_{1,n-1}(0)a(v)+\mu P_{1,n+1}(v), \quad 1\le n\le F, \tag{3.15}$$
$$-\frac{\mathrm{d}}{\mathrm{d}v}P_{1,n}(v)=-\mu P_{1,n}(v)+P_{1,n-1}(0)a(v)+\mu P_{1,n+1}(v,t), \; F+1\le n\le K-2, \tag{3.16}$$
$$-\frac{\mathrm{d}}{\mathrm{d}v}P_{1,K-1}(v)=-\mu P_{1,K-1}(v)+P_{1,K-2}(0)a(v). \tag{3.17}$$

Further define

$$a^*(\theta)=\int_0^\infty e^{-\theta v}\mathrm{d}A(v)=\int_0^\infty e^{-\theta v}a(v)\mathrm{d}v,$$
$$P_{i,n}^*(\theta)=\int_0^\infty e^{-\theta v}P_{i,n}(v)\mathrm{d}v, \quad i=0,1,$$
$$P_{i,n}=P_{i,n}^*(0)=\int_0^\infty P_{i,n}(v)\mathrm{d}v, \quad i=0,1,$$
$$\int_0^\infty e^{-\theta v}\frac{\partial}{\partial u}P_{i,n}(v)\mathrm{d}v=\theta P_{i,n}^*(\theta)-P_{i,n}(0), \quad i=0,1.$$

Therefore if the LST is taken on both sides of (3.14)-(3.17), it is found that

$$-\theta P_{1,0}^*(\theta)=\beta P_{0,0}a^*(\theta)+\mu P_{1,1}^*(\theta)-P_{1,0}(0), \tag{3.18}$$
$$(\mu-\theta)P_{1,n}^*(\theta)=\beta P_{0,n}a^*(\theta)+\mu P_{1,n+1}^*(\theta)+P_{1,n-1}(0)a^*(\theta)-P_{1,n}(0), \quad 1\le n\le F, \tag{3.19}$$
$$(\mu-\theta)P_{1,n}^*(\theta)=\mu P_{1,n+1}^*(\theta)+P_{1,n-1}(0)a^*(\theta)-P_{1,n}(0), \; F+1\le n\le K-2, \tag{3.20}$$
$$(\mu-\theta)P_{1,K-1}^*(\theta)=P_{1,K-2}(0)a^*(\theta)-P_{1,K-1}(0). \tag{3.21}$$

3.3.1 Recursive methods

The recursive method is developed to obtain $P_{0,n}^*(0)$ and $P_{1,n}^*(0)$. Our solution algorithm will first obtain $P_{0,n}^*(0)$ $(1\le n\le K)$.

Using (3.10)-(3.13), we get

$$P_{0,n}^*(0)=\phi_n P_{0,0}, \quad 1\le n\le K, \tag{3.22}$$
$$P_{1,K-1}(0)=\mu\phi_{F+1}P_{0,0}, \tag{3.23}$$

where $\phi_n = \begin{cases} 1, & n = 0, \\ \dfrac{\beta}{\mu}\left(1+\dfrac{\beta}{\mu}\right)^{\zeta_{n-1}}, & 1 \le n \le K, \end{cases}$ (3.24)

where $\zeta_n = \begin{cases} n, & 0 \le n \le F-1, \\ F, & F \le n \le K. \end{cases}$

Thus, $P_{0,1}^*(0)$, $P_{0,2}^*(0)$,..., $P_{0,K}^*(0)$ can be obtained by using (3.22).

Next, we derive the expressions of $P_{1,n}(0)$ $(0 \le n \le K-2)$ in terms of $P_{0,0}$. Using (3.22)-(3.23) in (3.19)-(3.21) and then setting $\theta = \mu$ in (3.19)-(3.21), we finally obtain

$$P_{1,n}(0) = \frac{P_{1,n+1}(0) - \mu P_{1,n+2}^*(\mu) - \beta\varphi_{n+1,F}\phi_{n+1}P_{0,0}a^*(\mu)}{a^*(\mu)}, \quad 0 \le n \le K-3, \tag{3.25}$$

where $\varphi_{n,F} = \begin{cases} 1, & 1 \le n \le F, \\ 0, & \text{otherwise}, \end{cases}$

$$P_{1,K-2}(0) = \frac{\mu\phi_{F+1}}{a^*(\mu)}P_{0,0}. \tag{3.26}$$

To obtain $P_{1,n+2}^*(\mu)$ $(0 \le n \le K-3)$ in (3.25), use (3.22) and (3.26) in (3.19)-(3.21), differentiating (3.19)-(3.21) $(l-1)$ times with respect to θ and setting $\theta = \mu$, we finally get

$$P_{1,n}^{*(l-1)}(\mu) = -\frac{1}{l}\left[P_{1,n-1}(0)a^{*(l)}(\mu) + \beta\varphi_{n,F}\phi_n P_{0,0}a^{*(l)}(\mu) + \mu P_{1,n+1}^{*(l)}(\mu)\right], \tag{3.27}$$

where $2 \le n \le K-2, \quad l = 1,\ldots,K-n-1,$

$$P_{1,K-1}^*(\mu) = -\frac{\mu\phi_{F+1}a^{*(1)}(\mu)}{a^*(\mu)}P_{0,0}, \tag{3.28}$$

where $P_{1,n}^{*(0)}(\mu) = P_{1,n}^*(\mu)$ and $a^{*(l)}(\theta) = \left[\left(\mathrm{d}^l/\mathrm{d}\theta^l\right)a^*(\theta)\right]$ denotes the lth derivative of $a^*(\theta)$.

Solving (3.27)-(3.28) recursively, we obtain

$$P_{1,n}^*(\mu) = -\frac{\beta\varphi_{n+2,F}a^*(\mu)}{\mu}\sum_{i=\zeta_{n+2}}^{F}\ell_{n-i-1}\phi_i P_{0,0} - \frac{a^*(\mu)}{\mu}\sum_{i=n+2}^{K-1}\ell_{n-i-1}P_{1,i-1}(0), \tag{3.29}$$

$2 \le n \le K-2,$

where

$$\ell_n = \begin{cases} -\dfrac{(-\mu)^n a^{*(n)}(\mu)}{n!a^*(\mu)}, & 1 \le n \le K-1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.30}$$

Using (3.28)-(3.29) in (3.25), we can obtain

$$\begin{aligned} P_{1,n}(0) &= \frac{P_{1,n+1}(0)}{a^*(\mu)} + \sum_{i=n+2}^{K-1} \ell_{i-n-1} P_{1,i-1}(0) + \\ &\beta\left[\varphi_{n+2,F} \sum_{i=\zeta_{n+2}}^{F} \ell_{i-n-1}\phi_i - \varphi_{n+1,F}\phi_{n+1}\right] P_{0,0}, \quad 0 \le n \le K-3. \end{aligned} \tag{3.31}$$

We further define

$$\Psi_n = \begin{cases} 1, & n = 0, \\ \displaystyle\sum_{1\le k\le n} \sum_{\substack{\tau_1+\tau_2+\cdots+\tau_k=n \\ \tau_1,\tau_2,\cdots,\tau_k\in\{1,2,\cdots,n\}}} \kappa_{\tau_1}\kappa_{\tau_2}\cdots\kappa_{\tau_k}, & n = 1,\ 2,\ldots,\ K-2, \\ 0, & \text{otherwise,} \end{cases} \tag{3.32}$$

where

$$\kappa_n = \begin{cases} \dfrac{1}{a^*(\mu)} + \ell_1, & n = 1, \\ \ell_n, & n = 2,3,\cdots,K-2, \\ 0, & \text{otherwise.} \end{cases} \tag{3.33}$$

The representative meaning of the above formulation (3.32) is the same as (2.42). Using (3.32)-(3.33) to solve (3.31) recursively, and including (3.26), we finally get

$$P_{1,n}(0) = \sum_{i=1}^{K-n-1} \Psi_{K-n-i-1}\Lambda(K-i-1)P_{0,0}, \quad 0 \le n \le K-2, \tag{3.35}$$

where

$$\Lambda(n) = \begin{cases} \beta\varphi_{n+2,F} \displaystyle\sum_{i=\zeta_{n+2}}^{F} \ell_{i-n-1}\phi_i - \beta\varphi_{n+1,F}\phi_{n+1}, & 0 \le n \le K-3, \\ \dfrac{\mu\phi_{F+1}}{a^*(\mu)}, & n = K-2. \end{cases} \tag{3.36}$$

Finally, we develop the steady-state probabilities $P_{1,n}^*(0)$ in terms of $P_{0,0}$. Setting $\theta = 0$ in (3.18)-(3.21) we have

$$P_{1,n}^{*}(0)=\frac{1}{\mu}\left[P_{1,n-1}(0)-\beta\sum_{i=0}^{\zeta_{n-1}}\phi_i P_{0,0}\right],\quad 1\le n\le K-1. \tag{3.37}$$

As $P_{1,1}(0), P_{1,2}(0),\ldots, P_{1,K-1}(0)$ are known, $P_{1,1}^{*}(0), P_{1,2}^{*}(0),\ldots, P_{1,K-1}^{*}(0)$ can be determined recursively using (3.37) in terms of $P_{0,0}$.

Now the only unknown quantity is $P_{1,0}^{*}(0)$ which can be obtained from (3.18)-(3.21). To find it, differentiate (3.18)-(3.21) with respect to θ and set $\theta=0$, we have

$$P_{1,0}^{*}(0)=-\beta P_{0,0}a^{*(1)}(0)-\mu P_{1,1}^{*(1)}(0), \tag{3.38}$$

$$P_{1,n}^{*(1)}(0)=\frac{P_{1,n}+\beta\varphi_{n,F}\phi_n P_{0,0}a^{*(1)}(0)+\mu P_{1,n+1}^{*(1)}(0)+P_{1,n-1}(0)a^{*(1)}(0)}{\mu}, \tag{3.39}$$

$1\le n\le K-1.$

The values $P_{1,n}^{*(1)}(0)$ for $n=1,2,\ldots,K-1$ can be found recursively from (3.39). Therefore, we obtain

$$P_{1,0}^{*}(0)=-\left[\beta a^{*(1)}(0)\sum_{i=0}^{F}\phi_n P_{0,0}+\sum_{i=1}^{K-1}P_{1,i}+a^{*(1)}(0)\sum_{i=0}^{K-2}P_{1,i}(0)\right]. \tag{3.40}$$

So $P_{1,0}^{*}(0), P_{1,1}^{*}(0),\ldots, P_{1,K-1}^{*}(0)$ is known in terms of $P_{0,0}$, which can be determined using the normalizing condition

$$\sum_{i=0}^{K}P_{0,i}+\sum_{i=0}^{K-1}P_{1,i}=1. \tag{3.41}$$

To demonstrate the working of the recursive method, we first describe the solution algorithm for calculating the steady state probabilities, $P_{0,n}^{*}(0)$ $(0\le n\le K)$ and $P_{1,n}^{*}(0)$ $(0\le n\le K-1)$. Next, to illustrate the solution algorithm, we provide three simple examples where the interarrival time distributions are exponential, k-stage Erlang, and deterministic, respectively.

3.3.2 The solution algorithm

Let F be the threshold, K be the maximum capacity of the system, and let $a^{*(l)}(\theta)$ where $l=1, 2,\ldots,K$ be the lth derivative of $a^{*}(\theta)$. We are given the values of F, K, and the LST expression of the interarrival time distribution, namely $a^{*}(\theta)$. The steps of the solution algorithm are stated as follows:

Step 1. For each $n=0, 1,\ldots,K$, compute ϕ_n using (3.24).

Step 2. For each $n = 1, 2, \ldots, K$, compute $P_{0,n}^{*}(0)$ using (3.22) in terms of $P_{0,0}$.

Step 3. Compute $\ell_n \ (1 \le n \le K-2)$ and $\kappa_n \ (1 \le n \le K-2)$ using (3.30) and (3.33), respectively.

Step 4. For each $n = 0, 1, \ldots, K-2$, compute Ψ_n using (3.32).

Step 5. For each $n = 0, 1, \ldots, K-2$, compute $\Lambda(n)$ using (3.36).

Step 6. For each $n = 0, 1, \ldots, K-2$, compute $P_{1,n}(0)$ using (3.35) in terms of $P_{0,0}$.

Step 7. For each $n = 1, 2, \ldots, K-1$, compute $P_{1,n}^{*}(0)$ using (3.37) in terms of $P_{0,0}$.

Step 8. Compute $P_{1,0}^{*}(0)$ using (3.40) in terms of $P_{0,0}$.

Step 9. Determine $P_{0,0}$ using (3.41). Thus $P_{0,n}^{*}(0)$ $(n = 1, 2, \ldots, K)$ are achieved from *Step* 2, and $P_{1,n}^{*}(0)$ $(n = 0, 1, \ldots, K-1)$ are achieved from *Steps* 7 to 8.

3.4 Simple Examples

We use the solution algorithm to illustrate a recursive method. We provide three simple examples for three different interarrival time distributions such as exponential, 3-stage Erlang, and deterministic, respectively.

Example 1 (For M/M/1 queue). We set λ is the interarrival rate. Assume that $F = 2$ and $K = 5$. In this case, we have

$$a^{*}(\theta) = \frac{\lambda}{\lambda + \theta}.$$

Step 1. For each $n = 0, 1, \ldots, 5$, compute ϕ_n using (3.24).

Using (3.24), we obtain

$$\phi_0 = 1, \ \phi_1 = \frac{1-\alpha}{\alpha}, \ \phi_2 = \frac{1-\alpha}{\alpha^2}, \text{ and } \phi_3 = \phi_4 = \phi_5 = \frac{1-\alpha}{\alpha^3}, \text{ where } \alpha = \mu/(\mu+\beta).$$

Step 2. For each $n = 1, 2, \cdots, 5$, compute $P_{0,n}^{*}(0)$ using (3.22) in terms of $P_{0,0}$.

Using (3.22), we finally get

$$P_{0,1}^{*}(0) = \phi_1 P_{0,0} = \frac{1-\alpha}{\alpha} P_{0,0}, \ P_{0,2}^{*}(0) = \phi_2 P_{0,0} = \frac{1-\alpha}{\alpha^2} P_{0,0},$$

$$P_{0,3}^*(0) = P_{0,4}^*(0) = P_{0,5}^*(0) = \phi_3 P_{0,0} = \frac{1-\alpha}{\alpha^3} P_{0,0}.$$

Step 3. For each $n = 1, 2, 3$, compute ℓ_n and κ_n using (3.30) and (3.32), respectively.

For each $n = 1, 2, 3,$ using (3.30) yields $\ell_1 = -\sigma/(1+\sigma)$, $\ell_2 = -\sigma^2/(1+\sigma)^2$ and $\ell_3 = -\sigma^3/(1+\sigma)^3$, where $\sigma = \mu/\lambda$.

For each $n = 1, 2, 3$, from (3.32), we obtain

$$\kappa_1 = -\left(1+\sigma+\sigma^2\right)/(1+\sigma), \quad \kappa_2 = -\sigma^2/(1+\sigma)^2 \text{ and } \kappa_3 = -\sigma^3/(1+\sigma)^3.$$

Step 4. For each $n = 0, 1, \cdots, 3$, compute Ψ_n using (3.31).

It implies from (3.31) that

$$\Psi_0 = 1, \quad \Psi_1 = \left(1+\sigma+\sigma^2\right)/(1+\sigma), \quad \Psi_2 = 1+\sigma^2, \text{ and}$$
$$\Psi_3 = \left(1+\sigma+\sigma^2+\sigma^3+\sigma^4\right)/(1+\sigma).$$

Step 5. For each $n = 0, 1, \cdots, 3$, compute $\Lambda(n)$ using (3.36).

It follows from (3.36) that

$$\Lambda(0) = -\frac{\mu(1-\alpha)^2\left(\alpha+\alpha\sigma+\sigma^2\right)}{\alpha^3(1+\sigma)}, \quad \Lambda(1) = -\frac{\mu(1-\alpha)^2}{\alpha^3}, \quad \Lambda(2) = 0, \text{ and}$$
$$\Lambda(3) = \frac{\mu(1-\alpha)(1+\sigma)}{\alpha^3}.$$

Step 6. For each $n = 0, 1, \cdots, 3$, compute $P_{1,n}(0)$ using (3.35) in terms of $P_{0,0}$.

Using (3.35) yields

$$P_{1,0}(0) = \left[\Psi_3\Lambda(3)+\Psi_2\Lambda(2)+\Psi_1\Lambda(1)+\Psi_0\Lambda(0)\right]P_{0,0}$$
$$= \frac{\mu(1-\alpha)\left(\sigma^2+\sigma^3+\sigma^4+\alpha\sigma+\alpha^2\right)}{\alpha^3}P_{0,0},$$
$$P_{1,1}(0) = \left[\Psi_2\Lambda(3)+\Psi_1\Lambda(2)+\Psi_0\Lambda(1)\right]P_{0,0} = \frac{\mu(1-\alpha)\left(\sigma+\sigma^2+\sigma^3+\alpha\right)}{\alpha^3}P_{0,0},$$
$$P_{1,2}(0) = \left[\Psi_1\Lambda(3)+\Psi_0\Lambda(2)\right]P_{0,0} = \frac{\mu(1-\alpha)\left(1+\sigma+\sigma^2\right)}{\alpha^3}P_{0,0},$$
$$P_{1,3}(0) = \Psi_3\Lambda(3)P_{0,0} = \frac{\mu(1-\alpha)(1+\sigma)}{\alpha^3}P_{0,0}.$$

Step 7. For each $n = 1,\ 2,\ \cdots, 4$, compute $P_{1,n}^{*}(0)$ using (3.37) in terms of $P_{0,0}$.

It implies from (3.37) that

$$P_{1,1}^{*}(0) = \frac{\sigma(1-\alpha)\left(\sigma+\sigma^2+\sigma^3+\alpha\right)}{\alpha^3}P_{0,0}, \quad P_{1,2}^{*}(0) = \frac{\sigma(1-\alpha)\left(1+\sigma+\sigma^2\right)}{\alpha^3}P_{0,0},$$

$$P_{1,3}^{*}(0) = \frac{\sigma(1-\alpha)(1+\sigma)}{\alpha^3}P_{0,0}, \text{ and } P_{1,4}^{*}(0) = \frac{\sigma(1-\alpha)}{\alpha^3}P_{0,0}.$$

Step 8. Compute $P_{1,0}^{*}(0)$ using (3.40) in terms of $P_{0,0}$.

Using (3.40) yields $P_{1,0}^{*}(0) = \dfrac{\sigma(1-\alpha)\left(\sigma^2+\sigma^3+\sigma^4+\alpha\sigma+\alpha^2\right)}{\alpha^3}P_{0,0}$.

Step 9. Determine $P_{0,0}$ using (3.41). Thus $P_{0,n}^{*}(0)$ $(n = 1,\ 2,\ldots,5)$ are achieved from *Step* 2, and $P_{1,n}^{*}(0)$ $(n = 0,\ 1,\ldots,4)$ are achieved from *Steps* 7 to 8.

$$P_{0,0} = \frac{\alpha^3}{\alpha^3+(1-\alpha)\left(3+\alpha+\alpha^2\right)+\sigma(1-\alpha)\left(3+\alpha+\alpha^2+\alpha\sigma+3+3\sigma+3\sigma^2+2\sigma^3+\sigma^4\right)}.$$

It is to be noted that these results are the same as those given in Gupta [12, p1006].

Example 2 (For $E_3/M/1$ queue). The 3-stage Erlang distribution is made up of three independent and identical exponential stages, each with mean $1/3\lambda$. We set λ is the interarrival rate, $F = 1$, and $K = 3$. In this case, we have

$$a^{*}(\theta) = \left(\frac{3\lambda}{3\lambda+\theta}\right)^3.$$

Step 1. For each $n = 0,\ 1,\ldots,3$, compute ϕ_n using (3.24).

Using (3.24), we obtain

$\phi_0 = 1$, $\phi_1 = 3(1-\gamma)/\gamma$, and $\phi_2 = \phi_3 = 3(1-\gamma)(3-2\gamma)/\gamma^2$,
where $\gamma = 3\mu/(3\mu+\beta)$.

Step 2. For each $n = 1,\ 2,\ 3$, compute $P_{0,n}^{*}(0)$ using (3.22) in terms of $P_{0,0}$.

Using (3.22), we finally get

$$P_{0,1}^{*}(0) = \phi_1 P_{0,0} = 3\frac{1-\gamma}{\gamma}P_{0,0}, \quad P_{0,2}^{*}(0) = P_{0,3}^{*}(0) = \phi_2 P_{0,0} = 3\frac{(1-\gamma)(3-2\gamma)}{\gamma^2}P_{0,0}.$$

Step 3. Compute ℓ_1 and κ_1 using (3.30) and (3.33), respectively.

Using (3.30) yields $\ell_1 = -3\tau/(1+\tau)$, where $\tau = \mu/3\lambda$.

Form (3.33), we obtain $\kappa_1 = \left(1+\tau+6\tau^2+4\tau^3+\tau^4\right)/(1+\tau)$.

Step 4. For each $n = 0,\ 1$, compute Ψ_n using (3.31).

It implies from (3.31) that $\Psi_0 = 1$ and $\Psi_1 = \left(1+\tau+6\tau^2+4\tau^3+\tau^4\right)/(1+\tau)$.

Step 5. For each $n = 0,\ 1$, compute $\Lambda(n)$ using (3.36).

It follows from (3.36) that

$$\Lambda(0) = -9\frac{\mu(1-\gamma)^2}{\gamma^2} \quad \text{and} \quad \Lambda(1) = 3\frac{\mu(1-\gamma)(3-2\gamma)(1+\tau)^2}{\gamma^2}.$$

Step 6. For each $n = 0,\ 1$, compute $P_{1,n}(0)$ using (3.35) in terms of $P_{0,0}$.

Using (3.35) yields

$$\begin{aligned} P_{1,0}(0) &= \left[\Psi_1\Lambda(1) + \Psi_0\Lambda(0)\right]P_{0,0} \\ &= \frac{3\mu(1-\gamma)(3-2\gamma)(1+\tau)^2\left(1+\tau+6\tau^2+4\tau^3+\tau^4\right) - 9\mu(1-\gamma)^2}{\gamma^2}P_{0,0}, \end{aligned}$$

$$P_{1,1}(0) = \Psi_0\Lambda(1)P_{0,0} = \frac{3\mu(1-\gamma)(3-2\gamma)(1+\tau)^3}{\gamma^2}P_{0,0}.$$

Step 7. For each $n = 1,\ 2$, compute $P_{1,n}^*(0)$ using (3.37) in terms of $P_{0,0}$.

It implies from (3.37) that

$$P_{1,1}^*(0) = \frac{3(1-\gamma)(3-2\gamma)\tau\left(3+9\tau+17\tau^2+15\tau^3+6\tau^4+\tau^5\right)}{\gamma^2}P_{0,0},$$

$$P_{1,2}^*(0) = \frac{3(1-\gamma)(3-2\gamma)\tau\left(3+3\tau+\tau^2\right)}{\gamma^2}P_{0,0}.$$

Step 8. Compute $P_{1,0}^*(0)$ using (3.40) in terms of $P_{0,0}$.

Using (3.40) yields

$$\begin{aligned} P_{1,0}^*(0) = \frac{1}{\gamma^2}\Big\{3\tau(1-\gamma)\Big[\gamma(3-12\tau-36\tau^2-78\tau^3-78\tau^4-34\tau^5-6\tau^6) \\ +\tau(18+54\tau+117\tau^2+117\tau^3+51\tau^4+9\tau^5)\Big]\Big\}P_{0,0}. \end{aligned}$$

Step 9. Determine $P_{0,0}$ using (3.41). Thus $P_{0,n}^*(0)$ $(n=1,\ 2,\ 3)$ are achieved from *Step* 2, and $P_{1,n}^*(0)$ $(n=0,\ 1,\ 2)$ are achieved from *Steps* 7 to 8.

$$P_{0,0}=\gamma^2\Big[18-\gamma(27-10\gamma)+27\tau(2+6\tau+12\tau^2+18\tau^3+15\tau^4+6\tau^5+\tau^6)$$
$$-9\tau\gamma(27+30\tau+60\tau^2+270\tau^3+75\tau^4+30\tau^5+5\tau^6)$$
$$+9\tau\gamma^2(3+12\tau+24\tau^2+36\tau^3+30\tau^4+12\tau^5+2\tau^6)\Big]^{-1}.$$

Example 3 (For D/M/1 queue). We set λ is the interarrival rate, $F=1$, and $K=3$. In this case,

$$a^*(\theta)=e^{-\theta/\lambda}.$$

Step 1. For each $n=0,\ 1,\ldots,3$, compute ϕ_n using (3.24).

Using (3.24), we obtain

$\phi_0=1$, $\phi_1=(1-\alpha)/\alpha$, and $\phi_2=\phi_3=(1-\alpha)/\alpha^2$, where $\alpha=\mu/(\mu+\beta)$.

Step 2. For each $n=1,\ 2,\ 3$, compute $P_{0,n}^*(0)$ using (3.22) in terms of $P_{0,0}$.

Using (3.22), we finally get

$$P_{0,1}^*(0)=\phi_1 P_{0,0}=\frac{1-\alpha}{\alpha}P_{0,0},\quad P_{0,2}^*(0)=P_{0,3}^*(0)=\phi_2 P_{0,0}=\frac{1-\alpha}{\alpha^2}P_{0,0}.$$

Step 3. Compute ℓ_1 and κ_1 using (3.30) and (3.33), respectively.

Using (3.30) yields $\ell_1=-\sigma$, where $\sigma=\mu/\lambda$. Form (3.33), we obtain $\kappa_1=e^{\sigma}-\sigma$.

Step 4. For each $n=0,\ 1$, compute Ψ_n using (3.31).

It implies from (3.31) that $\Psi_0=1$ and $\Psi_1=e^{\sigma}-\sigma$.

Step 5. For each $n=0,\ 1$, compute $\Lambda(n)$ using (3.36).

It follows from (3.36) that $\Lambda(0)=-\dfrac{\mu(1-\alpha)^2}{\alpha^2}$ and $\Lambda(1)=\dfrac{\mu(1-\alpha)e^{\sigma}}{\alpha^2}$.

Step 6. For each $n=0,\ 1$, compute $P_{1,n}(0)$ using (3.35) in terms of $P_{0,0}$.

Using (3.35) yields

$$P_{1,0}(0)=\left[\Psi_1\Lambda(1)+\Psi_0\Lambda(0)\right]P_{0,0}=\frac{\mu(1-\alpha)\left(e^{2\sigma}-\sigma e^{\sigma}+\alpha-1\right)}{\alpha^2}P_{0,0},$$

$$P_{1,1}(0)=\Psi_0\Lambda(1)P_{0,0}=\frac{\mu(1-\alpha)e^{\sigma}}{\alpha^2}P_{0,0}.$$

Step 7. For each $n = 1, 2$, compute $P_{1,n}^*(0)$ using (3.37) in terms of $P_{0,0}$.

It implies from (3.37) that

$$P_{1,1}^*(0) = \frac{(1-\alpha)\left(e^{2\sigma} - \sigma e^{\sigma} - 1\right)}{\alpha^2} P_{0,0}, \text{ and } P_{1,2}^*(0) = \frac{(1-\alpha)\left(e^{\sigma} - 1\right)}{\alpha^2} P_{0,0}.$$

Step 8. Compute $P_{1,0}^*(0)$ using (3.40) in terms of $P_{0,0}$.

Using (3.40) yields $P_{1,0}^*(0) = \dfrac{(1-\alpha)\left[2 + \alpha\sigma - (1-\sigma)e^{\sigma}\left(e^{\sigma} + 1 - \sigma\right)\right]}{\alpha^2} P_{0,0}.$

Step 9. Determine $P_{0,0}$ using (3.41). Thus $P_{0,n}^*(0)$ $(n = 1, 2, 3)$ are achieved from *Step* 2, and $P_{1,n}^*(0)$ $(n = 0, 1, 2)$ are achieved from *Steps* 7 to 8.

$$P_{0,0} = \frac{\alpha^2}{\sigma e^{2\sigma}(1-\alpha) + \sigma e^{\sigma}(1-\alpha)(1-\sigma) + 2 - \alpha + \alpha\sigma - \alpha^2\sigma}.$$

3.5 Optimal *F* policy

Our analysis is based on the following system performance measures of the F policy G/M/1/K queue with exponential startup time. Let

$L_s \equiv$ the average number of customers in the system;

$P_b \equiv$ the probability that the server is busy;

$P_s \equiv$ the probability that the server requires a startup time before starting the service;

$P_{bl} \equiv$ the probability that the server is blocked.

The expressions for L_s, P_b, P_s, and P_{bl} are give by

$$L_s = \sum_{n=1}^{K} nP_{0,n} + \sum_{n=1}^{K-1} nP_{1,n},$$

$$P_b = \sum_{n=0}^{K} P_{0,n} + \sum_{n=0}^{K-1} P_{1,n},$$

$$P_s = \sum_{n=0}^{F} P_{0,n},$$

$$P_{bl} = \sum_{n=0}^{K} P_{0,n}.$$

3.5.1 Cost function

We derive the total expected cost function per unit time for the F policy

G/M/1/K queue with startup times, in which F is a decision variable. The main purpose of this subsection is to determine the optimum operating F policy so as to minimize this total expected cost function. Let

$C_h \equiv$ holding cost per unit time for each customer present in the system;
$C_b \equiv$ busy cost per unit time for a busy server;
$C_s \equiv$ startup cost per unit time for the preparatory work of the server before starting the service;
$C_{bl} \equiv$ fixed cost for every lost customer when the system is blocked.

Utilizing the definitions of each cost element listed above, the total expected cost function per unit time is given by

$$TC(F) = C_h L_s + C_b P_b + C_s P_s + C_{bl} \lambda P_{bl}. \tag{3.58}$$

The optimal value of F, F^* is determined by satisfying the following inequality

$$TC(F^* - 1) \geq TC(F^*) \text{ and } TC(F^* + 1) \geq TC(F^*). \tag{3.59}$$

3.5.2 Numerical examples

We set the system capacity *K*=15. We perform a sensitivity analysis for changes in the optimum value F^* along with changes in specific values of the system parameters. We consider three simple examples for three different interarrival time distributions such as exponential, 3-stage Erlang, and deterministic. The following cost elements are employed:

Case 1: $C_h = 10,\ C_b = 200,\ C_s = 250,\ C_{bl} = 350.$
Case 2: $C_h = 10,\ C_b = 200,\ C_s = 250,\ C_{bl} = 400.$
Case 3: $C_h = 10,\ C_b = 200,\ C_s = 300,\ C_{bl} = 400.$
Case 4: $C_h = 10,\ C_b = 225,\ C_s = 300,\ C_{bl} = 400.$
Case 5: $C_h = 15,\ C_b = 225,\ C_s = 300,\ C_{bl} = 400.$

In this subsection, we provide the numerical results of the optimal value F^* and the minimum expected cost for three interarrival time distributions and specific values of λ, μ, β. We first fix (μ, β) = (1.0, 3.0) and choose different values of λ = 0.55, 0.65, 0.75. Next, we fix (λ, β) = (0.7, 3.0) and consider various values of μ = 1.0, 1.1, 1.2. Finally, we fix (λ, μ) = (0.7, 1.0) and select different values of β = 2.0, 4.0, 5.0.

The optimal value of F, F^*, and its minimum expected cost $TC\left(F^*\right)$ for the above five cases are shown in Tables 4-6. For fixed values of (μ, β) and various values of λ in Tables 4-6, we observe that (i) $TC\left(F^*\right)$ increases as λ increases for any case; and (ii) F^* decreases as λ increases for any case. For fixed values of (λ, β) and various values of μ in Tables 4-6, we find that (i) $TC\left(F^*\right)$ decreases as μ increases for any case; and (ii) F^* increases as μ increases for any case. Again, for fixed (λ, μ) and various values of β in Tables 4-6, we observe that (i) $TC\left(F^*\right)$ slightly decreases as β increases for any case; and (ii) F^* does not change at all when β changes from 2.0 to 5.0 for any case. Intuitively, F^* is insensitive to changes in β.

It can be easily seen from Tables 4 through 6 that (i) F^* increases as C_h decreases (see cases 4-5); and (ii) C_h has a larger effect on F^* than C_b, C_s and C_{bl} (see cases 3-4, cases 2-3 and cases 1-2).

Table 4. The optimal value of F and its minimum expected cost for exponential interarrival time

		$\lambda\ (\mu,\beta)=(1.0,3.0)$			$\mu\ (\lambda,\beta)=(0.7,3.0)$			$\beta\ (\lambda,\mu)=(0.7,1.0)$		
		0.55	0.65	0.75	1.0	1.1	1.2	2.0	4.0	5.0
Case1	F^*	6	4	3	4	6	8	4	4	4
	$TC(F^*)$	122.209	148.361	177.914	162.635	144.660	130.652	162.654	162.626	162.620
Case2	F^*	8	6	5	5	8	11	5	5	5
	$TC(F^*)$	122.215	148.425	178.303	162.803	144.705	130.663	162.823	162.793	162.787
Case3	F^*	8	6	5	5	8	11	5	5	5
	$TC(F^*)$	122.216	148.428	178.318	162.810	144.708	130.664	162.834	162.798	162.791
Case4	F^*	7	5	4	4	7	10	5	4	4
	$TC(F^*)$	135.963	164.647	196.879	180.230	160.598	145.243	180.253	180.218	180.211
Case5	F^*	4	3	2	2	4	6	2	2	2
	$TC(F^*)$	142.056	173.713	210.174	191.254	169.196	152.207	191.276	191.242	191.235

Table 5. The optimal value of F and its minimum expected cost for 3-stage Erlang interarrival time

		$\lambda\ (\mu,\beta)=(1.0,3.0)$			$\mu\ (\lambda,\beta)=(0.7,3.0)$			$\beta\ (\lambda,\mu)=(0.7,1.0)$		
		0.55	0.65	0.75	1.0	1.1	1.2	2.0	4.0	5.0
Case1	F^*	7	5	4	4	7	9	4	4	4
	$TC(F^*)$	118.991	143.288	170.687	156.448	139.842	126.868	156.450	156.447	156.447
Case2	F^*	9	7	5	6	9	12	6	6	6
	$TC(F^*)$	118.991	143.291	170.747	156.463	139.844	126.868	156.465	156.462	156.462
Case3	F^*	9	7	5	6	9	11	6	6	6
	$TC(F^*)$	118.991	143.291	170.750	156.464	139.844	126.868	156.466	156.463	156.462
Case4	F^*	8	6	4	5	8	11	5	5	5
	$TC(F^*)$	132.741	159.539	189.471	173.957	155.752	141.451	173.959	173.956	173.955
Case5	F^*	5	4	3	3	5	7	3	3	3
	$TC(F^*)$	137.236	166.178	199.711	182.155	162.033	146.552	182.157	182.153	182.153

Table 6. The optimal value of F and its minimum expected cost for deterministic interarrival time

		$\lambda\ (\mu,\beta)=(1.0,3.0)$			$\mu\ (\lambda,\beta)=(0.7,3.0)$			$\beta\ (\lambda,\mu)=(0.7,1.0)$		
		0.55	0.65	0.75	1.0	1.1	1.2	2.0	4.0	5.0
Case1	F^*	10	6	4	5	7	10	5	5	5
	$TC(F^*)$	117.440	140.710	166.469	153.130	137.435	125.030	153.130	153.129	153.129
Case2	F^*	10	7	5	6	9	12	6	6	6
	$TC(F^*)$	117.440	140.710	166.477	153.131	137.435	125.030	153.131	153.130	153.130
Case3	F^*	12	7	5	6	9	12	6	6	6
	$TC(F^*)$	117.440	140.710	166.478	153.131	137.435	125.030	153.131	153.131	153.130
Case4	F^*	9	6	5	5	8	11	6	5	5
	$TC(F^*)$	131.190	156.960	185.224	170.630	153.344	139.613	170.630	170.630	170.630
Case5	F^*	6	4	3	4	6	8	4	4	4
	$TC(F^*)$	134.911	162.315	193.445	177.193	158.425	143.794	177.193	177.193	177.193

Chapter 4

The N policy M/G/1/K Queue with Startup Time

In this chapter, we introduce the N policy M/G/1/K queue with startup time. We use a supplementary variable technique to analyze the optimal control of the N policy M/G/1/K queue where the server needs a startup time when the number of customers in the system reaches the threshold N $(N \geq 1)$ for the first time until there are no customers present in the system. At that time, the server needs to take an exponential startup time with parameter γ to start servicing customers in the system.

The primary objective of this chapter is twofold. Firstly, we develop a recursive method using the supplementary variable technique and treating the supplementary variable as the remaining service time, to develop the steady-state probability distributions of the number of customers for the N policy M/G/1/K queue. The method can be utilized for any service time distribution, such as deterministic (denoted D), exponential (denoted M) and k-stage Erlang (denoted E_k), etc. Secondly, to illustrate a recursive method we present one simple example for the service time distribution such as 3-stage Erlang.

In section 4.1, we describe the queueing model briefly. Section 4.2 the equations and solutions is developed. Section 4.3 provides a recursive method using the supplementary variable technique and treating the supplementary variable as the remaining service time, to obtain the steady-state probability distributions of the number of customers in the N policy M/G/1/K queue. Finally, presenting one simple example for 3-stage Erlang (denoted E_3), in section 4.4.

4.1 Assumptions and Notations

It is assumed that customers arrive following a Poisson process with parameter λ and the service times of the customers are independently and identically distributed (i.i.d.) random variables having a distribution $S(u)$ $(u \geq 0)$, a probability density function $s(u)$ $(u \geq 0)$ and mean service time s_1. If one customer is in service, then arriving customers have to wait in the queue until the server is available. Let us assume that customers arrive at the server form a single waiting line and are served in the order of their arrivals; that is, the first-come, first-served (FCFS) discipline. Suppose that the server can serve only one customer at a time, and that the service is independent of the arrival of the customers.

The following notations and probabilities are used throughout the chapter.

N	threshold level
K	system capacity $(K > N)$
S	service time random variable
U	remaining service time random variable
$S(u)$	distribution function (d.f.) of S
$s(u)$	probability density function (p.d.f.) of S
$S^*(\theta)$	Laplace-Stieltjes transform (LST) of S
$S^{*(l)}(\theta)$	l th order derivative of $S^*(\theta)$ with respect to θ
$P_{0,0}(t)$	probability of no customers in the system at time t when the server is turned off
$P_{0,n}(t)$	probability of n customers in the system at time t when the server is turned off where $n=1,2,\ldots,K$.
$P_{1,n}(t)$	probability of n customers in the system at time t when the server is turned on where $n=1,2,\ldots,K$.
$P_{0,0}$	steady state probability of no customers in the system when the server is turned off
$P_{0,n}$	steady state probability of n customers in the system when the server is turned off where $n=1,2,\ldots,K$.
$P_{1,n}$	steady state probability of n customers in the system when the server is turned on where $n=1,2,\ldots,K$.
s_1	mean service time.

4.2 Development of the Equations and Solutions

We use the following supplementary variable: $U \equiv$ remaining service time for the customer in service. The state of the system at time t is given by

$N(t) \equiv$ number of customers in the system, and

$U(t) \equiv$ remaining service time for the customer being served.

Let us define

$$P_{1,n}(u,t)du = \Pr\{N(t)=n, u<U(t)\le u+\mathrm{d}u\},\quad u\ge 0,\quad n=1,2,\ldots,K.$$

$$P_{1,n}(t) = \int_0^\infty P_{1,n}(u,t)\mathrm{d}u,\quad n=1,2,\ldots,K.$$

Relating the state of the system at time t and $t+\mathrm{d}t$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,0}(t) = -\lambda P_{0,0}(t) + P_{1,1}(0,t), \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,n}(t) = -\lambda P_{0,n}(t) + \lambda P_{0,n-1}(t), \quad 1 \le n \le N-1, \tag{4.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,n}(t) = -(\lambda+\gamma)P_{0,n}(t) + \lambda P_{0,n-1}(t), \quad N \le n \le K-1, \tag{4.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,K}(t) = -\gamma P_{0,K}(t) + \lambda P_{0,K-1}(t), \quad n = K, \tag{4.4}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_{1,1}(u,t) = -\lambda P_{1,1}(u,t) + P_{1,2}(0,t)s(u), \quad n = 1, \tag{4.5}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_{1,n}(u,t) = -\lambda P_{1,n}(u,t) + \lambda P_{1,n-1}(u,t) + P_{1,n+1}(0,t)s(u), \quad 2 \le n \le N-1, \tag{4.6}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_{1,n}(u,t) = -\lambda P_{1,n}(u,t) + \lambda P_{1,n-1}(u,t) + \gamma P_{0,n}(u,t) + P_{1,n+1}(0,t)s(u), \quad N \le n \le K-1, \tag{4.7}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_{1,K}(u,t) = \lambda P_{1,K-1}(u,t) + \gamma P_{0,K}(u,t), \quad n = K. \tag{4.8}$$

4.3 Steady State Results

In steady state, let us define

$$P_{0,n} = \lim_{t\to\infty} P_{0,n}(t), \quad n = 1,2,\ldots,K.$$

$$P_{1,n} = \lim_{t\to\infty} P_{0,n}(t), \quad n = 1,2,\ldots,K.$$

$$P_{1,n}(u) = \lim_{t\to\infty} P_{1,n}(u,t), \quad n = 1,2,\ldots,K.$$

and further define

$$P_{0,n}(u) = P_{0,n}s(u), \quad n = N, N+1,\ldots,K. \tag{4.9}$$

From (4.1)-(4.9) we can easily obtain the following steady state equations:

$$0 = -\lambda P_{0,0} + P_{1,1}(0), \tag{4.10}$$

$$0 = -\lambda P_{0,n}(t) + \lambda P_{0,n-1}, \quad 1 \le n \le N-1, \tag{4.11}$$

$$0 = -(\lambda+\gamma)P_{0,n} + \lambda P_{0,n-1}, \quad N \le n \le K-1, \tag{4.12}$$

$$0 = -\gamma P_{0,K} + \lambda P_{0,K-1}, \quad n = K. \tag{4.13}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{1,1}(u) = -\lambda P_{1,1}(u) + P_{1,2}(0)s(u), \quad n = 1, \tag{4.14}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{1,n}(u,t) = -\lambda P_{1,n}(u) + \lambda P_{1,n-1}(u) + P_{1,n+1}(0)s(u), \quad 2 \le n \le N-1, \tag{4.15}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{1,n}(u) = -\lambda P_{1,n}(u) + \lambda P_{1,n-1}(u) + \gamma P_{0,n}s(u) + P_{1,n+1}(0)s(u), \quad N \le n \le K-1, \tag{4.16}$$

$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{1,K}(u)=\lambda P_{1,K-1}(u)+\gamma P_{0,K}s(u),\quad n=K. \tag{4.17}$$

Further define

$$S^*(\theta)=\int_0^\infty e^{-\theta u}\mathrm{d}S(u)=\int_0^\infty e^{-\theta u}s(u)\mathrm{d}u,$$

$$P_{0,n}^*(\theta)=\int_0^\infty e^{-\theta u}P_{0,n}(u)\mathrm{d}u,$$

$$P_{1,n}^*(\theta)=\int_0^\infty e^{-\theta u}P_{1,n}(u)\mathrm{d}u,$$

$$P_{0,n}=P_{0,n}^*(0)=\int_0^\infty P_{0,n}(u)\mathrm{d}u,$$

$$P_{1,n}=P_{1,n}^*(0)=\int_0^\infty P_{1,n}(u)\mathrm{d}u,$$

$$\int_0^\infty e^{-\theta u}\frac{\partial}{\partial u}P_{0,n}(u)\mathrm{d}u=\theta P_{0,n}^*(\theta)-P_{0,n}(0),$$

and

$$\int_0^\infty e^{-\theta u}\frac{\partial}{\partial u}P_{1,n}(u)\mathrm{d}u=\theta P_{1,n}^*(\theta)-P_{1,n}(0).$$

Therefore if the LST is taken of both sides of (4.14)-(4.17), it is found that

$$(\lambda-\theta)P_{1,1}^*(\theta)=P_{1,2}(0)S^*(\theta)-P_{1,1}(0), \tag{4.18}$$

$$(\lambda-\theta)P_{1,n}^*(\theta)=\lambda P_{1,n-1}^*(\theta)+P_{1,n+1}(0)S^*(\theta)-P_{1,n}(0),\quad 2\le n\le N-1, \tag{4.19}$$

$$(\lambda-\theta)P_{1,n}^*(\theta)=\lambda P_{1,n-1}^*(\theta)+\gamma P_{0,n}S^*(\theta)+P_{1,n+1}(0)S^*(\theta)-P_{1,n}(0),\quad F+1\le n\le K-2, \tag{4.20}$$

$$-\theta P_{1,K}^*(\theta)=\lambda P_{1,K-1}^*(\theta)+\gamma P_{0,K}S^*(\theta)-P_{1,K}(0). \tag{4.21}$$

4.3.1 Recursive methods

The recursive method is developed to obtain $P_{0,n}^*(0)$ and $P_{1,n}^*(0)$. Our solution algorithm will first find obtain $P_{0,n}$ $(1\le n\le K)$.

Using (4.10)-(4.13), we get

$$P_{0,n}=\phi_n P_{0,0},\quad 1\le n\le K. \tag{4.22}$$

where
$$\phi_n=\begin{cases}1, & 1\le n\le N-1,\\ \left(\dfrac{\lambda}{\lambda+\gamma}\right)^{n-N+1}, & N\le n\le K-1,\\ \dfrac{\lambda^{K-N+1}}{\gamma(\lambda+\gamma)^{K-N}}, & n=K.\end{cases} \tag{4.23}$$

$P_{0,1}$, $P_{0,2}$,..., $P_{0,K}$ can be using (4.22) in terms of $P_{0,0}$.

We derive the expressions of $P_{1,n}(0)$ $(1 \le n \le K)$ in terms of $P_{0,0}$. Using (4.22) in (4.20) and then setting $\theta = \lambda$ in (4.18)-(4.20), we finally obtain

$$P_{1,2}(0) = \frac{P_{1,1}(0)}{S^*(\lambda)}, \quad n = 1, \tag{4.24}$$

$$P_{1,n+1}(0) = \frac{P_{1,n}(0) - \gamma\varphi_{n,N}\phi_n P_{0,0} S^*(\lambda) - \lambda P^*_{1,n-1}(\lambda)}{S^*(\lambda)}, \quad 2 \le n \le K-1, \tag{4.25}$$

where $\varphi_{n,N} = 1, \quad N \le n \le K-1$, otherwise, $\varphi_{n,N} = 0$.

To obtain $P^*_{1,n-1}(\lambda)$ $(1 \le n \le K\text{-}1)$ in (4.25), using (4.22) in (4.20) again, differentiating (4.18)-(4.20) $(l-1)$ times with respect to θ and setting $\theta = \lambda$, we finally get

$$P^{*(l-1)}_{1,1}(\lambda) = -\frac{S^{*(l)}(\lambda)}{l}\left[P_{1,2}(0)\right], \quad n = 1, \quad l = 1,\ldots,K-1, \tag{4.26}$$

$$P^{*(l-1)}_{1,n}(\lambda) = -\frac{1}{l}\left[P_{1,n+1}(0) S^{*(l)}(\lambda) + \gamma\varphi_{n,N}\phi_n P_{0,0} S^{*(l)}(\lambda) + \lambda P^{*(l)}_{1,n-1}(\lambda)\right], \quad 2 \le n \le K-1, \quad l = 1,\ldots,K-n, \tag{4.27}$$

where $P^{*(0)}_{1,n}(\lambda) = P^*_{1,n}(\lambda)$ and $S^{*(l)}(\theta) = \left[\left(\mathrm{d}^l/\mathrm{d}\theta^l\right) S^*(\theta)\right]$ denotes the lth derivative of $S^*(\theta)$.

Solving (4.26)-(4.27) recursively, we obtain

$$P^*_{1,n}(\lambda) = -\varphi_{n,N} \sum_{i=N}^{n} \frac{\gamma \ell_{n-i+1}\phi_i S^*(\lambda)}{\lambda} P_{0,0} - \sum_{i=2}^{n+1} \frac{\ell_{n-i+2} S^*(\lambda)}{\lambda} P_{1,i}(0), \quad 1 \le n \le K-1, \tag{4.28}$$

where

$$\ell_n = \begin{cases} -\dfrac{(-\lambda)^n S^{*(n)}(\lambda)}{n! S^*(\lambda)}, & 1 \le n \le K-2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.29}$$

Using (4.28) in (4.25), we can obtain

$$P_{1,n}(0) = \frac{1}{S^*(\lambda)} P_{1,n-1}(0) + \sum_{i=2}^{n-1} \ell_{n-i} P_{1,i}(0) + \gamma\left(\varphi_{n-2,N} \sum_{i=N}^{n-2} \ell_{n-i-1}\phi_i - \varphi_{n-1,N}\phi_{n-1}\right) P_{0,0}, \quad 3 \le n \le K. \tag{4.30}$$

We further define

$$\Psi_n = \begin{cases} 1, & n=0, \\ \sum_{1\le k\le n} \sum_{\substack{\tau_1+\tau_2+\cdots+\tau_k=n \\ \tau_1,\tau_2,\cdots,\tau_k\in\{1,2,\cdots,n\}}} \kappa_{\tau_1}\kappa_{\tau_2}\cdots\kappa_{\tau_k}, & n=1,\ 2,\ldots,\ K-2, \\ 0, & \text{otherwise}, \end{cases} \tag{4.31}$$

where

$$\kappa_n = \begin{cases} \dfrac{1}{S^*(\lambda)} + \ell_1, & n=1, \\ \ell_n, & n=2,3,\cdots,K-2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.32}$$

The representative meaning of the above formulation (4.31) is the same as (2.42). Using (4.31) and (4.32) to solve (4.30) recursively, and including (4.10) and (4.24), we finally get

$$P_{1,1}(0) = A(1)P_{0,0}, \quad n=1, \tag{4.33}$$

$$P_{1,n}(0) = \sum_{i=2}^{n} \Psi_{n-i} A(i) P_{0,0}, \quad 2 \le n \le K, \tag{4.34}$$

where

$$A(n) = \begin{cases} \lambda, & n=1, \\ \dfrac{\lambda}{S^*(\lambda)}, & n=2, \\ \gamma\left(\varphi_{n-2,N}\sum_{i=N}^{n-2}\ell_{n-i-1}\phi_i - \varphi_{n-1,N}\phi_{n-1}\right), & 3\le n\le K. \end{cases} \tag{4.35}$$

Finally, we develop the steady-state probabilities $P_{1,n}^*(0)$ in terms of $P_{0,0}$. Setting $\theta = 0$ in (4.18)-(4.20) we have

$$P_{1,n}^*(0) = \frac{1}{\lambda}\left[P_{1,n+1}(0) - P_{1,1}(0) + \gamma\varphi_{n,N}\sum_{i=N}^{n}\phi_i P_{0,0}\right], \quad 1\le n\le K-1, \tag{4.36}$$

As $P_{1,1}(0)$, $P_{1,2}(0),\ldots,$ $P_{1,K}(0)$ are known, $P_{1,1}^*(0)$, $P_{1,2}^*(0),\ldots,$ $P_{1,K-1}^*(0)$ can be determined recursively using (4.36) in terms of $P_{0,0}$.

Now the only unknown quantity is $P_{1,K}^{*}(0)$ which can be obtained from (4.21). To find it, differentiate (4.21) with respect to θ and set $\theta = 0$, we have

$$P_{1,K}^{*}(0) = -\lambda P_{1,K-1}^{*(1)}(0) - \gamma P_{0,K} S^{*(1)}(0). \tag{4.37}$$

To find $\lambda P_{1,K-1}^{*(1)}(0)$, differentiating (4.18)-(4.20) with respect to θ and $\theta = 0$, we obtain

$$P_{1,1}^{*(1)}(0) = \frac{P_{1,1} + P_{1,2}(0) S^{*(1)}(0)}{\lambda}, \quad n = 1, \tag{4.38}$$

$$P_{1,n}^{*(1)}(0) = \frac{P_{1,n} + \gamma \varphi_{n,N} \phi_n P_{0,0} S^{*(1)}(0) + \lambda P_{1,n-1}^{*(1)}(0) + P_{1,n+1}(0) S^{*(1)}(0)}{\lambda}, \tag{4.39}$$
$$2 \le n \le K-1,$$

As $P_{1,1}^{*(1)}(0)$ is known completely from (4.38), the values $P_{1,n}^{*(1)}(0)$ for $n = 2,3,\ldots,K-1$ can be found recursively from (4.39). Therefore we obtain

$$P_{1,K}^{*}(0) = -\left[\sum_{i=1}^{K-1} P_{1,i} + \gamma S^{*(1)}(0) \sum_{i=N}^{K} \phi_n P_{0,0} + S^{*(1)}(0) \sum_{i=2}^{K} P_{1,i}(0) \right]. \tag{4.40}$$

So $P_{1,1}^{*}(0), P_{1,2}^{*}(0), \ldots, P_{1,K}^{*}(0)$ is known in terms of $P_{0,0}$, which can be determined using the normalizing condition

$$\sum_{i=0}^{K} P_{0,i} + \sum_{i=1}^{K} P_{1,i} = 1. \tag{4.41}$$

To demonstrate the working of the recursive method, we first describe the solution algorithm for calculating the steady state probabilities, $P_{0,n}^{*}(0)$ $(0 \le n \le K)$ and $P_{1,n}^{*}(0)$ $(1 \le n \le K)$. Next, to illustrate the solution algorithm, we provide one simple example where the service time distribution is k-stage Erlang.

4.3.2 The solution algorithm

Let N be the threshold, K be the maximum capacity of the system, and let $S^{*(l)}(\theta)$ where $l = 1, 2, \ldots, K$ be the *l*th derivative of $S^{*(l)}(\theta)$. Given the values of *N*, *K*, and the LST expression of the service time distribution, namely $S^{*}(\theta)$, the steps of the solution algorithm are stated as follows:

Step 1. For each $n = 1, 2, \ldots, K$, compute ϕ_n using (4.23).

Step 2. For each $n = 1, 2, \ldots, K$, compute $P_{0,n}^{*}(0)$ using (4.22) in terms of $P_{0,0}$.

Step 3. Compute $\ell_n\ (1 \le n \le K-2)$ and $\kappa_n\ (1 \le n \le K-2)$ using (4.29) and (4.32).

Step 4. For each $n = 0,\ 1,\ldots,K-2$, compute Ψ_n using (4.31).

Step 5. For each $n = 1,\ 2,\ldots,K$, compute $A(n)$ using (4.35).

Step 6. For each $n = 1,\ 2,\ldots,K$, compute $P_{1,n}(0)$ using (4.33) and (4.34) in terms of $P_{0,0}$.

Step 7. For each $n = 1,\ 2,\ldots,K-1$, compute $P_{1,n}^{*}(0)$ using (4.36) in terms of $P_{0,0}$.

Step 8. For $n = K$, compute $P_{1,n}^{*}(0)$ using (4.40) in terms of $P_{0,0}$.

Step 9. Determine $P_{0,0}$ using (4.41). Thus $P_{0,n}^{*}(0)\ (n = 1,\ 2,\ldots,K)$ are achieved from *Step* 2, and $P_{1,n}^{*}(0)\ (n = 1,\ 2,\ldots,K)$ are achieved from *Steps* 7 to 8.

4.4 Simple Example

We use the solution algorithm to illustrate a recursive method. We provide a simple example for the service time distribution such as 3-stage Erlang.

Example (For M/E$_3$/1 queue). The 3-stage Erlang distribution is made up of three independent and identical exponential stages, each with mean $1/3\lambda$. We set the mean service time $s_1 = 1/\mu$, $N = 2$, and $K = 3$. In this case, we have

$$S^{*}(\theta) = \left(\frac{3\mu}{3\mu + \theta}\right)^{3}$$

Step 1. For each $n = 1,\ 2,\ 3$, compute ϕ_n using (4.23), we obtain

$\phi_1 = 1$, $\phi_2 = \eta/(3-2\eta)$, and $\phi_3 = \eta^2/3(1-\eta)(3-2\eta)$, where $\eta = 3\lambda/(3\lambda + \beta)$.

Step 2. For each $n = 1,\ 2,\ 3$, compute $P_{0,n}^{*}(0)$ using (4.22) in terms of $P_{0,0}$.
From (4.22), we finally get

$$P_{0,1}^{*}(0) = \phi_1 P_{0,0} = P_{0,0},$$

$$P_{0,2}^{*}(0) = \phi_2 P_{0,0} = \frac{\eta}{(3-2\eta)} P_{0,0} \quad \text{and} \quad P_{0,3}^{*}(0) = \phi_3 P_{0,0} = \frac{\eta^2}{3(1-\eta)(3-2\eta)} P_{0,0}.$$

Step 3. For $n = 1$, compute ℓ_n and κ_n using (4.29) and (4.32), respectively.

For each $n=1$, using (4.29) yields $\ell_1 = -3\sigma/(1+\sigma)$, where $\sigma = \lambda/3\mu$.

For each $n=1$, we find from (4.32) that $\kappa_1 = \left(1+\sigma+6\sigma^2+4\sigma^3+\sigma^4\right)/(1+\sigma)$

Step 4. For each $n=0,\ 1$, compute Ψ_n.

It implies from (4.31) that $\Psi_0 = 1$ and $\Psi_1 = \left(1+\sigma+6\sigma^2+4\sigma^3+\sigma^4\right)/(1+\sigma)$.

Step 5. For each $n=1,\ 2,\ 3$, compute $A(n)$.

Using (4.35), it follows that

$$A(1)=\lambda,\quad A(2)=\lambda(1+\sigma)^3,\ \text{and}\ \ A(3)=-3\lambda(1-\eta)/(3-2\eta).$$

Step 6. For each $n=1,\ 2,\ 3$, compute $P_{1,n}(0)$ using (4.33) and (4.34) in terms of $P_{0,0}$.

It yields from (4.33) and (4.34) that

$$P_{1,1}(0) = A(1)P_{0,0} = \lambda P_{0,0},$$
$$P_{1,2}(0) = \Psi_0 A(2) P_{0,0} = \lambda(1+\sigma)^3 P_{0,0},$$
$$P_{1,3}(0) = \left[\Psi_1 A(2) + \Psi_0 A(3)\right] P_{0,0}$$
$$= \frac{\lambda(3-2\eta)(1+\sigma)^2\left(1+\sigma+6\sigma^2+4\sigma^3+\sigma^4\right)-3\lambda(1-\eta)}{3-2\eta} P_{0,0}.$$

Step 7. For each $n=1,\ 2$, compute $P^*_{1,n}(0)$ using (4.36) in terms of $P_{0,0}$.

From (4.36), we finally have

$$P^*_{1,1}(0) = \sigma\left(3+3\sigma+\sigma^2\right)P_{0,0}\ \ \text{and}$$
$$P^*_{1,2}(0) = \sigma\left(3+9\sigma+17\sigma^2+15\sigma^3+6\sigma^4+\sigma^5\right)P_{0,0}.$$

Step 8. For $n=3$, compute $P^*_{1,n}(0)$ using (4.40) in terms of $P_{0,0}$.

Using (4.40) yields

$$P^*_{1,3}(0) = \frac{1}{(3-2\eta)}\Big\{\sigma\Big[\eta(3-12\sigma-36\sigma^2-78\sigma^3-78\sigma^4-34\sigma^5-6\sigma^6)$$
$$+\sigma(18+54\sigma+117\sigma^2+117\sigma^3+51\sigma^4+9\sigma^5)\Big]\Big\}P_{0,0}.$$

Step 9. Determine $P_{0,0}$ using (4.41). Thus $P^*_{0,n}(0)$ $(n=1,\ 2,\ 3)$ are achieved from *Step* 2, and $P^*_{1,n}(0)$ $(n=1,\ 2,\ 3)$ are achieved from *Steps* 7 to 8.

$$
\begin{aligned}
P_{0,0} &= 3(1-\eta)(3-2\eta)\times \\
&\Big[18-\eta(27-10\eta)+27\sigma(2+6\sigma+12\sigma^2+18\sigma^3+15\sigma^4+6\sigma^5+\sigma^6) \\
&-9\sigma\eta(27+30\sigma+60\sigma^2+270\sigma^3+75\sigma^4+30\sigma^5+5\sigma^6) \\
&+9\sigma\eta^2(3+12\sigma+24\sigma^2+36\sigma^3+30\sigma^4+12\sigma^5+2\sigma^6)\Big]^{-1}.
\end{aligned}
$$

Chapter 5

Interrelationships between the *F* policy and the *N* policy for M/G/1/K and G/M/1/K Queues with Startup Time

In this chapter, we study the interrelationship between the *F* policy and *N* policy. In section 5.1, to consider the interrelationship between *N* policy M/G/1/K queue with exponential startup time and *F* policy G/M/1/K queue with exponential startup time, we use the solution algorithm of *N* policy M/G/1/K queue with exponential startup time to derive the solution algorithm of *F* policy G/M/1/K queue with exponential startup time. In section 5.2, the above interrelationship is shown in Table 7. To illustrate the interrelationship, we provide a simple example for *N* policy M/E_3/1/K queue. In this example, we set the service time distribution such as 3-stage Erlang and *K*=3 and *N*=2 to device the steady-state probability of *F* policy E_3/M/1/K queue with *K*=3 and *F*=1. Similarly, we use the solution algorithm of *F* policy M/G/1/K queue with exponential startup time to derive the solution algorithm of *N* policy G/M/1/K queue with exponential startup time in section 5.3. In section 5.4, the Table 8 shows the interrelationship between *F* policy M/G/1/K queue with exponential startup time and *N* policy G/M/1/K queue with exponential startup time. We provide a simple for *F* policy M/M/1/K queue. In this example, we set the service time distribution such as exponential and *K*=5 and *F*=2 to device the steady-state probability of *F* policy M/M/1/K queue with *K*=5 and *N*=3.

5.1 Development of the *F* policy G/M/1/K Queue

We follow the solution algorithm of the *N* policy M/G/1/K queue to modify the parameters. Change (i) threshold from *N* to *K-F*; (ii) the arrival rate from λ to μ; (iii) from the service time random variable S to the interarrival time random variable A. The other parameters are the same. The steps are stated as follows:

Step 1. For each $n = 1,\ 2, \ldots, K$, compute ϕ_n.

$$\phi_n = \begin{cases} 1, & 1 \le n \le K-F-1, \\ \left(\dfrac{\mu}{\mu+\gamma}\right)^{n-K+F+1}, & K-F \le n \le K-1, \\ \dfrac{\mu}{\gamma}\left(\dfrac{\mu}{\mu+\gamma}\right)^{F}, & n = K. \end{cases} \tag{5.1}$$

Step 2. For each $n = 1, 2, \ldots, K$, compute $P_{0,n}^{*}(0)$ using (5.2) in terms of $P_{0,0}$.

$$P_{0,n}^{*}(0) = \phi_n P_{0,0}, \quad 1 \le n \le K. \tag{5.2}$$

Step 3. Compute $\ell_n \left(1 \le n \le K-2\right)$ and $\kappa_n \left(1 \le n \le K-2\right)$ using (5.3) and (5.4).

$$\ell_n = \begin{cases} -\dfrac{(-\mu)^n a^{*(n)}(\mu)}{n! a^{*}(\mu)}, & 1 \le n \le K-2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

$$\kappa_n = \begin{cases} \dfrac{1}{a^{*}(\mu)} + \ell_1, & n = 1, \\ \ell_n, & 2 \le n \le K-2, \\ 0, & \text{otherwise,} \end{cases} \tag{5.4}$$

where $a^{*}(\theta)$ is the Laplace-Stieltjes transform (LST) of A.

Step 4. For each $n = 0, 1, \ldots, K-2$, compute Ψ_n using (5.5).

$$\Psi_n = \begin{cases} 1, & n = 0, \\ \displaystyle\sum_{1 \le k \le n} \sum_{\substack{\tau_1+\tau_2+\cdots+\tau_k = n \\ \tau_1,\tau_2,\cdots\tau_k \in \{1,2,\cdots,n\}}} \kappa_{\tau_1}\kappa_{\tau_2}\cdots\kappa_{\tau_k}, & n = 1, 2, \ldots, K-2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

Step 5. For each $n = 1, 2, \ldots, K$, compute $A(n)$ using (5.6).

$$A(n) = \begin{cases} \mu, & n = 1, \\ \dfrac{\mu}{a^{*}(\mu)}, & n = 2, \\ \gamma\left(\varphi_{n-2,K-F} \displaystyle\sum_{i=K-F}^{n-2} \ell_{n-i-1}\phi_i - \varphi_{n-1,K-F}\phi_{n-1}\right), & 3 \le n \le K, \end{cases} \tag{5.6}$$

where $\varphi_{n,K-F} = 1,\ K-F \le n \le K-1,$ otherwise, $\varphi_{n,K-F} = 0.$

Step 6. For each $n = 1, 2, \ldots, K$, compute $P_{1,n}(0)$ using (5.7) and (5.8) in terms of $P_{0,0}$.

$$P_{1,1}(0) = A(1) P_{0,0}, \tag{5.7}$$

$$P_{1,n}(0) = \sum_{i=2}^{n} \Psi_{n-i} A(i) P_{0,0}, \quad 2 \le n \le K. \tag{5.8}$$

Step 7. For each $n = 1, 2, \ldots, K-1$, compute $P_{1,n}^{*}(0)$ using (5.9) in terms of $P_{0,0}$.

$$P_{1,n}^{*}(0) = \frac{1}{\mu}\left[P_{1,n+1}(0) - P_{1,1}(0) + \gamma\varphi_{n,K-F}\sum_{i=K-F}^{n}\phi_i P_{0,0}\right],\quad 1 \le n \le K-1. \tag{5.9}$$

Step 8. For $n = K$, compute $P_{1,n}^{*}(0)$ using (5.10) in terms of $P_{0,0}$.

$$P_{1,K}^{*}(0) = -\left[\sum_{i=1}^{K-1} P_{1,i} + \gamma a^{*(1)}(0)\sum_{i=K-F}^{K}\phi_i P_{0,0} + a^{*(1)}(0)\sum_{i=2}^{K} P_{1,i}(0)\right]. \tag{5.10}$$

Step 9. Determine $P_{0,0}$ using (5.11). Thus $P_{0,n}^{*}(0)$ $(n = 1, 2, \ldots, K)$ are achieved from *Step* 2, and $P_{1,n}^{*}(0)$ $(n = 1, 2, \ldots, K)$ are achieved from *Steps* 7 to 8.

$$\sum_{i=0}^{K} P_{0,i} + \sum_{i=1}^{K} P_{1,i} = 1. \tag{5.11}$$

Using the solution algorithm for calculating the steady state probabilities, $P_{0,n}^{*}(0)$ $(0 \le n \le K)$ and $P_{1,n}^{*}(0)$ $(1 \le n \le K)$. Next, change the index of $P_{0,n}^{*}(0)$ and $P_{1,n}^{*}(0)$ from n to $K-n$ to obtain the steady state probabilities of F policy G/M/1/K queue. To illustrate the solution algorithm, we provide a table that N policy M/G/1/K queue corresponds to F policy G/M/1/K queue.

5.2 Interrelationship between the *N* policy M/G/1/K Queue and the *F* policy G/M/1/K Queue

To illustrate the interrelationship between the *N* policy M/G/1/K queue and the *F* policy G/M/1/K queue, we construct the Table 7 in this section and provide one simple example for the service time distribution such as 3-stage Erlang.

Table 7. N policy M/G/1/K queue corresponds to F policy G/M/1/K queue.

	N policy M/G/1/K				F policy G/M/1/K
Capacity	K	$\Rightarrow$	K		K
Threshold	N	$\Rightarrow$	$K-F$		F
Arrival time	λ	$\Rightarrow$	μ		A
Service time	S	$\Rightarrow$	A		μ
Startup time	γ	$\Rightarrow$	γ		γ
Probabilities	$P_{0,n}^{*}(0)$ $(0 \le n \le K)$	$\Rightarrow$	$P_{0,K-n}^{*}(0)$	$=$	$P_{0,n}^{*}(0)$ $(0 \le n \le K)$
	$P_{1,n}^{*}(0)$ $(1 \le n \le K)$	$\Rightarrow$	$P_{1,K-n}^{*}(0)$	$=$	$P_{1,n}^{*}(0)$ $(0 \le n \le K-1)$

S : the service time random variable ; A : the interarrival time random variable.

Next, we provide a simple example for the service time distribution such as

3-stage Erlang.

Example 1 (For M/E$_3$/1 queue). The 3-stage Erlang distribution is made up of three independent and identical exponential stages, each with mean $1/3\mu$. We set the mean service time $s_1 = 1/\mu$, $N = 2$, and $K = 3$. In this case, we use the solution algorithm to solve the steady state probabilities of F policy E$_3$/M/1 queue.

We change from $S^*(\theta) = \left(\frac{3\mu}{3\mu+\theta}\right)^3$ to $a^*(\theta) = \left(\frac{3\lambda}{3\lambda+\theta}\right)^3$.

The steps are stated as follows:

Step 1. For each $n = 1,\ 2,\ 3$, compute ϕ_n

Using (5.1), we obtain $\phi_1 = 1$, $\phi_2 = \gamma/(3-2\gamma)$, and $\phi_3 = \gamma^2/3(1-\gamma)(3-2\gamma)$, where $\gamma = 3\mu/(3\mu+\gamma)$.

Step 2. For each $n = 1,\ 2,\ 3$, compute $P_{0,n}^*(0)$ using (5.2) in terms of $P_{0,0}$.

Using (5.2), it follows that

$P_{0,1}^*(0) = \phi_1 P_{0,0} = P_{0,0}$,

$P_{0,2}^*(0) = \phi_2 P_{0,0} = \frac{\gamma}{(3-2\gamma)} P_{0,0}$ and $P_{0,3}^*(0) = \phi_3 P_{0,0} = \frac{\gamma^2}{3(1-\gamma)(3-2\gamma)} P_{0,0}$.

Step 3. For $n = 1$, compute ℓ_n and κ_n using (5.3) and (5.4), respectively.

For $n = 1$, using (5.3) yields $\ell_1 = -3\tau/(1+\tau)$, where $\tau = \mu/3\lambda$.

For $n = 1$, we find from (5.4) that $\kappa_1 = \left(1+\tau+6\tau^2+4\tau^3+\tau^4\right)/(1+\tau)$.

Step 4. For each $n = 0,\ 1$, compute Ψ_n.

It implies from (5.5) that $\Psi_0 = 1$ and $\Psi_1 = \left(1+\tau+6\tau^2+4\tau^3+\tau^4\right)/(1+\tau)$.

Step 5. For each $n = 1,\ 2,\ 3$, compute $A(n)$.

Using (5.6), it follows that

$A(1) = \mu$, $A(2) = \mu(1+\tau)^3$, and $A(3) = -3\mu(1-\gamma)/(3-2\gamma)$.

Step 6. For each $n = 1,\ 2,\ 3$, compute $P_{1,n}(0)$ using (5.7) and (5.8) in terms of $P_{0,0}$.

It yields from (5.7) and (5.8) that

$$P_{1,1}(0)=A(1)P_{0,0}=\mu P_{0,0},$$

$$P_{1,2}(0)=\Psi_0 A(2)P_{0,0}=\mu(1+\tau)^3 P_{0,0},$$

$$P_{1,3}(0)=\left[\Psi_1 A(2)+\Psi_0 A(3)\right]P_{0,0}$$

$$=\frac{\mu(3-2\gamma)(1+\tau)^2\left(1+\tau+6\tau^2+4\tau^3+\tau^4\right)-3\mu(1-\gamma)}{3-2\gamma}P_{0,0}.$$

Step 7. For each $n=1,\ 2$, compute $P_{1,n}^*(0)$ using (5.9) in terms of $P_{0,0}$.

From (5.9), we finally have

$$P_{1,1}^*(0)=\tau\left(3+3\tau+\tau^2\right)P_{0,0} \text{ and } P_{1,2}^*(0)=\tau\left(3+9\tau+17\tau^2+15\tau^3+6\tau^4+\tau^5\right)P_{0,0}.$$

Step 8. For $n=3$, compute $P_{1,n}^*(0)$ using (5.10) in terms of $P_{0,0}$.

Using (5.10) yields

$$P_{1,3}^*(0)=\frac{1}{(3-2\gamma)}\Big\{\tau\Big[\gamma(3-12\tau-36\tau^2-78\tau^3-78\tau^4-34\tau^5-6\tau^6)$$
$$+\tau(18+54\tau+117\tau^2+117\tau^3+51\tau^4+9\tau^5)\Big]\Big\}P_{0,0}.$$

Step 9. Determine $P_{0,0}$ using (5.11). Thus $P_{0,n}^*(0)$ $(n=1,\ 2,\ 3)$ are achieved from *Step* 2, and $P_{1,n}^*(0)$ $(n=1,\ 2,\ 3)$ are achieved from *Steps* 7 to 8.

$$P_{0,0}=3(1-\gamma)(3-2\gamma)\times$$
$$\Big[18-\gamma(27-10\gamma)+27\tau(2+6\tau+12\tau^2+18\tau^3+15\tau^4+6\tau^5+\tau^6)$$
$$-9\tau\gamma(27+30\tau+60\tau^2+270\tau^3+75\tau^4+30\tau^5+5\tau^6)$$
$$+9\tau\gamma^2(3+12\tau+24\tau^2+36\tau^3+30\tau^4+12\tau^5+2\tau^6)\Big]^{-1}.$$

$P_{0,3-n}^*(0)$ $(n=0,\ 1,\ 2,\ 3)$ and $P_{1,3-n}^*(0)$ $(n=1,\ 2,\ 3)$ are noted that these results are the same as those given in F policy $E_3/M/1/K$ queue.

5.3 Development of the N policy G/M/1/K Queue

We follow the solution algorithm of the F policy M/G/1/K queue to modify the parameters. Change (i) threshold from *F* to *K-N*; (ii) the arrival rate from λ to μ; (iii) from the service time random variable S to the interarrival time random variable A. The other parameters are the same. The steps are stated as follows:

Step 1. For each $n = 0, 1, \ldots, K$, compute ϕ_n.

$$\phi_n = \begin{cases} 1, & n = 0, \\ a_1\gamma(1 + a_1\gamma)^{\zeta_{n-1}}, & 1 \le n \le K, \end{cases} \tag{5.12}$$

where $\zeta_n = \begin{cases} n, & 0 \le n \le K - N - 1, \\ K - N, & K - N \le n \le K. \end{cases}$

Step 2. For each $n = 1, 2, \ldots, K - 1$, compute $P_{0,n}^*(0)$ using (5.13) in terms of $P_{0,0}$.

$$P_{0,n}^*(0) = \phi_n P_{0,0}, \quad 1 \le n \le K - 1. \tag{5.13}$$

Step 3. Compute $\ell_n \ (1 \le n \le K - 2)$ and $\kappa_n \ (1 \le n \le K - 3)$ using (5.14) and (5.15).

$$\ell_n = \begin{cases} -\dfrac{(-\mu)^n a^{*(n)}(\mu)}{n! a^*(\mu)}, & 1 \le n \le K - 2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.14}$$

$$\kappa_n = \begin{cases} \dfrac{1}{a^*(\mu)} + \ell_1, & n = 1, \\ \ell_n, & 2 \le n \le K - 3, \\ 0, & \text{otherwise,} \end{cases} \tag{5.15}$$

where $a^*(\theta)$ is the Laplace-Stieltjes transform (LST) of A.

Step 4. For each $n = 0, 1, \ldots, K - 3$, compute Ψ_n using (5.16).

$$\Psi_n = \begin{cases} 1, & n = 0, \\ \displaystyle\sum_{1 \le k \le n} \sum_{\substack{\tau_1 + \tau_2 + \cdots + \tau_k = n \\ \tau_1, \tau_2, \cdots \tau_k \in \{1,2,\cdots,n\}}} \kappa_{\tau_1}\kappa_{\tau_2}\cdots\kappa_{\tau_k}, & n = 1, 2, \ldots, K - 2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.16}$$

Step 5. For each $n = 1, 2, \ldots, K - 1$, compute $A(n)$ and $B(n)$ using (5.17) and (5.18).

$$A(n) = \begin{cases} \mu, & n = 1, \\ \mu\left[\dfrac{1 - a^*(\mu)}{a^*(\mu)}\right], & n = 2, \\ \mu\ell_{n-2}, & 3 \le n \le K - 1, \end{cases} \tag{5.17}$$

and

$$B(n)=\begin{cases} -\gamma, & n=1, \\ -\gamma\left[\dfrac{1+\varphi_{1,K-N}\phi_1 a^*(\mu)}{a^*(\mu)}\right], & n=2, \\ \gamma\left(\sum_{i=1}^{\zeta_{n-2}}\ell_{n-i-1}\phi_i-\varphi_{n-1,K-N}\phi_{n-1}\right), & 3\le n\le K-1, \end{cases} \tag{5.18}$$

where $\varphi_{n,K-N}=1,\quad 1\le n\le K-N$, otherwise, $\varphi_{n,K-N}=0$.

Step 6. For $n=0$, compute $P^*_{1,n}(0)$ using (5.19) in terms of $P_{0,0}$.

$$P^*_{1,0}(0)=-\frac{\sum_{i=1}^{K-2}\ell_{K-i-1}\sum_{j=2}^{i+1}\Psi_{i-j+1}B(j)+\sum_{i=2}^{K-1}\dfrac{\Psi_{K-j-1}B(i)}{a^*(\mu)}+\sum_{i=1}^{\zeta_{K-2}}\gamma\ell_{K-i-1}\phi_i}{\sum_{i=1}^{K-2}\ell_{K-i-1}\sum_{j=2}^{i+1}\Psi_{i-j+1}A(j)+\sum_{i=2}^{K-1}\dfrac{\Psi_{K-j-1}A(i)}{a^*(\mu)}+\mu\ell_{K-2}}p_{0,0.} \tag{5.19}$$

Step 7. For each $n=1,\ 2,\ldots,K-1$, compute $P_{1,n}(0)$ using (5.20) and (5.21) in terms of $P_{0,0}$.

$$P_{1,1}(0)=A(1)P_{1,0}+B(1)P_{0,0}, \tag{5.20}$$

$$P_{1,n}(0)=\sum_{i=2}^{n}\Psi_{n-i}\left[A(i)P_{1,0}+B(i)P_{0,0}\right],\quad 2\le n\le K-1. \tag{5.21}$$

Step 8. For each $n=1,\ 2,\ldots,K-1$, compute $P^*_{1,n}(0)$ using (5.22) and (5.23) in terms of $P_{0,0}$.

$$P^*_{1,n}(0)=\frac{1}{\mu}\left[\gamma\sum_{i=0}^{\zeta_n}\phi_i P_{0,0}+P_{1,n+1}(0)\right],\quad 1\le n\le K-2, \tag{5.22}$$

$$P^*_{1,K-1}(0)=\frac{\gamma}{\mu}\sum_{i=0}^{K-N}\phi_i P_{0,0}. \tag{5.23}$$

Step 9. For $n=K$, compute $P^*_{0,n}(0)$ using (5.24) in terms of $P_{0,0}$.

$$P^*_{0,K}(0)=-\left[\sum_{i=1}^{K-1}P_{1,i}+\gamma a^{*(1)}(0)\sum_{i=1}^{K-N}\phi_i P_{0,0}+a^{*(1)}(0)\sum_{i=2}^{K-1}P_{1,i}(0)+\mu a^{*(1)}(0)P_{1,0}\right]. \tag{5.24}$$

Step 10. Determine $P_{0,0}$ using (5.25). Thus $P^*_{0,n}(0)$ $(n=1,\ 2,\ldots,K)$ are achieved from *Step* 2 and *Step* 9, and $P^*_{1,n}(0)$ $(n=0,\ 1,\ldots,K-1)$ are achieved from *Step* 6 and *Step* 8.

$$\sum_{i=0}^{K} P_{0,i} + \sum_{i=0}^{K-1} P_{1,i} = 1. \tag{5.25}$$

Using the solution algorithm for calculating the steady state probabilities, $P_{0,n}^{*}(0)$ $(0 \le n \le K)$ and $P_{1,n}^{*}(0)$ $(0 \le n \le K-1)$. Next, change the index of $P_{0,n}^{*}(0)$ and $P_{1,n}^{*}(0)$ from n to $K-n$ to obtain the steady state probabilities of N policy G/M/1/K queue. To illustrate the solution algorithm, we provide a table that F policy M/G/1/K queue corresponds to N policy G/M/1/K queue.

5.4 Interrelationship between the *F* policy M/G/1/K Queue and the *N* policy G/M/1/K Queue

To illustrate the interrelationship between the F policy M/G/1/K queue and the N policy G/M/1/K queue, we construct the Table 8 in this section and provide one simple example for the service time distribution such as exponential.

Table 8. F policy M/G/1/K queue corresponds to N policy G/M/1/K queue.

	F policy M/G/1/K				N policy G/M/1/K
Capacity	K	$\Rightarrow$	K		K
Threshold	F	$\Rightarrow$	$K-N$		N
Arrival time	λ	$\Rightarrow$	μ		A
Service time	S	$\Rightarrow$	A		μ
Startup time	γ	$\Rightarrow$	γ		γ
Probabilities	$P_{0,n}^{*}(0)$ $(0 \le n \le K)$	$\Rightarrow$	$P_{0,K-n}^{*}(0)$	$=$	$P_{0,n}^{*}(0)$ $(0 \le n \le K)$
	$P_{1,n}^{*}(0)$ $(0 \le n \le K-1)$	$\Rightarrow$	$P_{1,K-n}^{*}(0)$	$=$	$P_{1,n}^{*}(0)$ $(1 \le n \le K)$

S : the service time random variable ; A : the interarrival time random variable.

Next, we provide a simple example for the service time distribution such as exponential.

Example 2 (For M/M/1 queue). We set the mean service time $s_1 = 1/\mu$, where μ is the service rate. Assume that $F = 2$, and $K = 5$. In this case, we use the solution algorithm to solve the steady state probabilities of N policy M/M/1 queue.

We change from $S^{*}(\theta) = \dfrac{\mu}{\mu+\theta}$ to $a^{*}(\theta) = \dfrac{\lambda}{\lambda+\theta}$.

The steps are stated as follows:

Step 1. For each $n = 0,\ 1, \ldots,\ 5$, compute ϕ_n

Using (5.12), we obtain $\phi_0 = 1$, $\phi_2 = \gamma(\rho\mu+\gamma)/(\rho\mu)^2$, and $\phi_3 = \phi_4 = \phi_5 = \gamma(\rho\mu+\gamma)^2/(\rho\mu)^3$, where $\rho = \lambda/\mu$.

Step 2. For each $n = 1, 2, \ldots, 5$, compute $P^*_{0,n}(0)$ using (5.13) in terms of $P_{0,0}$.

Using (5.13), it follows that

$$P^*_{0,1}(0) = \phi_1 P_{0,0} = \frac{\gamma}{\rho\mu} P_{0,0}, \quad P^*_{0,2}(0) = \phi_2 P_{0,0} = \frac{\gamma(\rho\mu+\gamma)}{(\rho\mu)^2} P_{0,0},$$

and $P^*_{0,3}(0) = P^*_{0,4}(0) = \phi_3 P_{0,0} = \phi_4 P_{0,0} = \dfrac{\gamma(\rho\mu+\gamma)^2}{(\rho\mu)^3} P_{0,0}$.

Step 3. Compute ℓ_n $(1 \le n \le 3)$ and κ_n $(1 \le n \le 2)$ using (5.14) and (5.15), respectively.

For $n = 1, 2, 3$, using (5.14) yields $\ell_1 = -1/(1+\rho)$, $\ell_2 = -1/(1+\rho)^2$, and $\ell_3 = -1/(1+\rho)^3$.

For $n = 1, 2$, we find from (5.15) that $\kappa_1 = (1+\rho+\rho^2)/\rho(1+\rho)$ and $\kappa_2 = -1/(1+\rho)^2$.

Step 4. For each $n = 0, 1, 2$, compute Ψ_n.

It implies from (5.16) that $\Psi_0 = 1$, $\Psi_1 = (1+\rho+\rho^2)/\rho(1+\rho)$, and $\Psi_2 = (1+\rho^2)/\rho^2$.

Step 5. For each $n = 1, 2, 3, 4$, compute $A(n)$ and $B(n)$.

Using (5.17), it follows that

$$A(1) = \mu, \quad A(2) = \mu/\rho, \quad A(3) = -\mu/(1+\rho), \text{ and } A(4) = -\mu/(1+\rho)^2.$$

Using (5.18), it follows that

$$B(1) = -\gamma, \quad B(2) = -\frac{\gamma(\mu+\gamma+\rho\mu)}{\rho\mu}, \quad B(3) = -\frac{\gamma(\gamma+\gamma\rho+2\rho\mu+\rho^2\mu)}{(1+\rho)\rho^2\mu^2},$$

and $B(4) = -\dfrac{\gamma(\gamma+\gamma\rho+2\rho\mu+\rho^2\mu)}{(1+\rho)^2\rho^2\mu^2}$.

Step 6. For $n = 0$, compute $P^*_{1,n}(0)$ using (5.19) in terms of $P_{0,0}$.

$$P_{1,0} = \frac{\gamma}{\mu^3}\left[\rho^4\mu^2 + \rho^3(\mu^2+2\mu\gamma) + (\rho^2+\rho)(\mu+\gamma)^2 + \mu\gamma + \mu^2 + \gamma^2\right] p_{0,0}.$$

Step 7. For each $n = 1, 2, 3, 4$, compute $P_{1,n}(0)$ using (5.20) and (5.21) in terms of $P_{0,0}$.

It yields from (5.20) and (5.21) that

$$P_{1,1}(0) = A(1)P_{1,0} + B(1)P_{0,0} = \frac{\gamma(\rho\mu+\gamma)}{\mu^2}\left[\rho^3\mu + \left(\rho^2+\rho+1\right)(\mu+\gamma)\right]P_{0,0},$$

$$P_{1,2}(0) = \frac{\gamma(\rho\mu+\gamma)^2\left(\rho^2+\rho+1\right)}{\rho\mu^2}P_{0,0}, \quad P_{1,3}(0) = \frac{\gamma(\rho\mu+\gamma)^2(\rho+1)}{\rho\mu^2}P_{0,0},$$

and $P_{1,4}(0) = \dfrac{\gamma(\rho\mu+\gamma)^2}{\rho\mu^2}P_{0,0}$

Step 8. For each $n = 1, 2, 3$, compute $P^*_{1,n}(0)$ using (5.22) in terms of $P_{0,0}$.
From (5.22), we finally have

$$P^*_{1,1}(0) = \frac{\gamma(\rho\mu+\gamma)}{\rho\mu^3}\left[\rho^3\mu + \left(\rho^2+\rho+1\right)(\mu+\gamma)\right]P_{0,0},$$

$$P^*_{1,2}(0) = \frac{\gamma(\rho\mu+\gamma)^2\left(\rho^2+\rho+1\right)}{\rho^2\mu^3}P_{0,0}, \text{ and } P^*_{1,3}(0) = \frac{\gamma(\rho\mu+\gamma)^2(\rho+1)}{\rho^2\mu^3}P_{0,0},$$

For $n = 4$, compute $P^*_{1,n}(0)$ using (5.23) in terms of $P_{0,0}$.
Using (5.23) yields

$$P^*_{1,4}(0) = \frac{\gamma(\rho\mu+\gamma)^2}{\rho^2\mu^3}P_{0,0}.$$

Step 9. For $n = 5$, compute $P^*_{0,n}(0)$ using (5.24) in terms of $P_{0,0}$.

$$P^*_{0,5}(0) = \frac{\gamma(\rho\mu+\gamma)^2}{\rho^3\mu^3}P_{0,0}.$$

Step 10. Determine $P_{0,0}$ using (5.25). Thus $P^*_{0,n}(0)$ $(n = 1, 2, \ldots, 5)$ are achieved from *Step* 2 and *Step* 9, and $P^*_{1,n}(0)$ $(n = 0, 1, \ldots, 4)$ are achieved from *Step* 6 and *Step* 8.

$$P_{0,0} = (\rho\mu)^3 \times \Big[\rho^7\gamma\mu^2 + 2\rho^6\gamma\mu(\gamma+\mu) + \rho^5\gamma(\gamma+3\mu)(\gamma+\mu) + 2\rho^4\gamma(\gamma+\mu)(\gamma+2\mu) + \rho^3(5\gamma\mu^2 + 7\gamma^2\mu + \mu^3 + 3\gamma^3) + \rho^2\gamma(3\gamma^2 + 7\gamma\mu + 5\mu^2) + \rho\gamma^2(3\gamma + 7\mu) + 3\gamma^3\Big]^{-1}.$$

$P^*_{0,5-n}(0)$ $(n = 0, 1, \ldots, 5)$ and $P^*_{1,5-n}(0)$ $(n = 0, 1, \ldots, 4)$ are noted that these results are the same as those given in F policy M/M/1/K queue.

Chapter 6

Conclusions and Future Researches

Steady-state results have been presented for (i) the *F* policy M/G/1/K queue with startup time; (ii) the *F* policy G/M/1/K queue with startup time; and (iii) the *N* policy M/G/1/K queue with startup time; These steady-state results, simple to use and provide system characteristics, are convenient for the corresponding queue applications. This is important that the interrelationships between the *F* policy and *N* policy for M/G/1/K and G/M/1/K queues have been structured. In this chapter, we make conclusions and provide possible extensions of the present work for further research.

6.1 Conclusions

In this thesis, we have shown that the supplementary variable technique can be successfully applied to analyse various controllable queues operating under *F* policy and *N* policy:

1. For the *F* policy M/G/1/K queue with startup time, we provided a recursive method for computing the steady state probability distribution of the number of customers in a finite system. We also illustrated a recursive method by a study of three different interarrival time distributions, exponential, 3-stage Erlang, and deterministic. In addition, we derived the optimum value of the control parameter *F* so as to minimize an expected cost function. We performed a sensitivity analysis among the optimal value of *F*, specific values of system parameters, and the cost elements. Based on the numerical results, we could make an intelligent effective based on exact solutions for practical and general queue with quantitative measurement.

2. For the *F* policy G/M/1/K queue with startup time, we have first provided a recursive method for obtaining the steady-state probability distributions of the number of customers in the system. Next, we have illustrated our recursive method by a study of three different interarrival time distributions: exponential, 3-stage Erlang, and deterministic. In addition, we provide a very efficient solution algorithm to calculate the optimal threshold *F* at minimum cost. Finally, we have performed a sensitivity analysis among the optimal value of *F*, specific values of

system parameters, and the cost elements. Further, the developed controlling arrival systems in this dissertation can be modeled many quality and service (Q & S) system in real-life.

3. For the *N* policy M/G/1/K queue with startup time, we have developed analytic steady-state results by using the supplementary variable technique. We have provided a recursive method for obtaining the steady-state probability distributions of the number of customers for such a system. To demonstrate the working schemes of this method, we have constructed a solution algorithm to study analytically one simple example where the service time distribution is 3-stage Erlang.

Finally, we have developed the complementarity interrelationships between the *F* policy and *N* policy for M/G/1/K and G/M/1/K queues with startup time. It is important that we can simply obtain the solution of one if the other solution is given.

6.2 Further Researches

The following problems of the controllable queues are worthy of further investigations:

1. Optimal control of the *F* policy M/G/1/K queue with a unreliable removable and startup server.

In chapter 2, we have studied the *F* policy M/G/1/K queue with startup time. Hur and Par [19] dealt the *N* policy M/G/1 queueing system with server startup. Wang and Ke [34] treated the dyadic policies M/G/1 queueing system with server breakdowns. In the future, we may study the *F* policy M/G/1 queue with a unreliable removable and startup server.

2. Optimal control of the *F* policy G/M/1/K queue with an unreliable removable server.

In chapter 3, we have studied the *F* policy G/M/1/K queue with startup time. In the future, we may study the *F* policy G/M/1/K queue with an unreliable removable server.

3. System characteristics of the *F* policy G/M/1/K queue with startup time

and system characteristics of the *F* policy M/G/1/K queue with startup time.

Ke [21] studied the operating characteristic of the *N* policy G/M/1 with startup time. In the future, we may study the system characteristics of the *F* policy G/M/1/K queue and *F* policy M/G/1/K queue with startup time.

4. Optimal control of the *F* policy G/G/1/K queue with startup time.

In this dissertation, we have studied the *F* policy M/G/1/K and G/M/1/K queues with startup time. In the future, we may study the *F* policy G/G/1/K queue with startup time.

5. Optimal control of the *N* policy M/G/1/K queue with an unreliable removable server.

In chapter 4, we have studied the *N* policy M/G/1/K queue with startup time. In the future, we may study the *N* policy M/G/1/K queue with a removable server.

6. Interrelationships between *F* policy and *N* policy for M/G/1/K and G/M/1/K queues with general startup time.

In chapter 5, we have studied the interrelationships between *F* policy and *N* policy for M/G/1/K queue and G/M/1/K queue with exponential startup time. The relationships between *F* policy and *N* policy M/M/1/K queuing system with exponential startup time was considered by Gupta [12]. In the future, we may study the interrelationships between *F* policy and *N* policy for M/G/1/K and G/M/1/K queues with general startup time.

7. Optimal control of the *F* policy $M^{[X]}$/G/1/K queue with startup time.

In this dissertation, we have studied the *F* policy M/G/1/K queue with startup time. In the future, we may study batch arrival for the *F* policy $M^{[X]}$/G/1/K queue with startup time.

Appendix

We discuss the two cases with F+1=K : the F policy M/G/1/K queue with startup time and the F policy G/M/1/K queue with startup time.

First, we modified (2.1)-(2.9) to (2.1a)-(2.6a) in the F policy M/G/1/K queue with startup time. The new equations (2.1a)-(2.6a) are as following:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,0}(t) = -\beta P_{0,0}(t) + P_{0,1}(0,t), \tag{2.1a}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_{0,n}(u,t) = -\beta P_{0,n}(u,t) + P_{0,n+1}(0,t)s(u), \quad 1 \le n \le F, \tag{2.2a}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_{0,K}(u,t) = \lambda P_{1,K-1}(u,t), \tag{2.3a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{1,0}(t) = -\lambda P_{1,0}(t) + \beta P_{0,0}(t) + P_{1,1}(0,t), \tag{2.4a}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_{1,1}(u,t) &= -\lambda P_{1,1}(u,t) + \beta P_{0,1}(u,t) + \lambda P_{1,0}(t)s(u) + \\ &P_{1,2}(0,t)s(u),\end{aligned} \tag{2.5a}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)P_{1,n}(u,t) &= -\lambda P_{1,n}(u,t) + \beta P_{0,n}(u,t) + \lambda P_{1,n-1}(u,t)s(u) \\ &+ P_{1,n+1}(0,t)s(u), \quad 2 \le n \le F,\end{aligned} \tag{2.6a}$$

Similarly, we can use (2.1a)-(2.6a) to obtain the algorithm of the steady-state probability with the supplementary variable technique and the recursive method.

Second, we changed (3.1)-(3.8) to (3.1a)-(3.5a) in the F policy G/M/1/K queue with startup time. The new equations (3.1a)-(3.5a) are as following:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,0}(t) = -\beta P_{0,0}(t) + \mu P_{0,1}(t), \tag{3.1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,n}(t) = -(\beta + \mu)P_{0,n}(t) + \mu P_{0,n+1}(t), \quad 1 \le n \le F, \tag{3.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{0,K}(t) = -\mu P_{0,K}(t) + P_{1,K-1}(0,t), \tag{3.3a}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial v}\right)P_{1,0}(v,t) = \beta P_{0,0}(v,t) + \mu P_{1,1}(v,t), \tag{3.4a}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial v}\right)P_{1,n}(v,t) &= -\mu P_{1,n}(v,t) + \beta P_{0,n}(v,t) + P_{1,n-1}(0,t)a(v) + \\ &\mu P_{1,n+1}(v,t), \quad 1 \le n \le F,\end{aligned} \tag{3.5a}$$

Likewise, we use (3.1a)-(3.5a) to yield the algorithm of the steady-state probability with the supplementary variable technique and the recursive method.

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