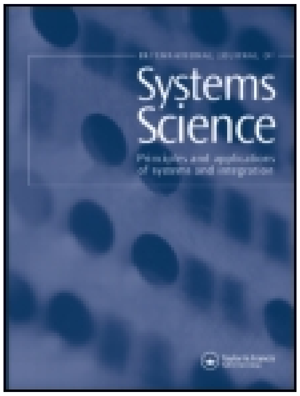

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Periodic review stochastic inventory models with service level constraint

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This article considers the periodic review stochastic inventory models with service level constraint to provide an improved solution procedure. The previous researchers assumed that the objective function is concave down in the lead time so that the minimum must occur on the boundary points of each sub-domain. In this article, we will show that their assumption is questionable since the minimum might not occur at the boundary points of each sub-domain. In a recent paper in *International Journal of Systems Science*, Ouyang and Chuang studied this problem. However, their solutions contained questionable results and their algorithm might not find the optimal solution due to flaws in their solution procedure. We develop some lemmas to reveal the parameter effects and then present our improved solution procedures for finding the optimal solution for periodic review stochastic inventory models in which the lead time demand is a normal distribution. The savings are illustrated by solving the same examples from Ouyang and Chuang's paper to demonstrate the improvement using our revised algorithm. In the direction of future research, we discuss the comparison between the reordered point being fixed and the reordered point as a new variable.

Keywords: inventory model; service level constraint; crashable lead time

Nomenclature

- A ordering cost per order
- D expected demand per year
- h holding cost per unit per year
- L length of lead time
- T the review period
- R target level
- X the lead time demand which has a distribution function F with finite mean μL and standard derivation $\sigma\sqrt{L}(>0)$
- α proportion of demands that are not met from stock so $1-\alpha$ is the service level
- β fraction of the demand during the stock-out period that will be backordered
- $C(L)$ lead time crashing cost
- x^+ maximum value of x and 0, i.e. $x^+ = \max\{x, 0\}$
- $T_i^{\#}$ the minimum solution under the constraint level, with $T_i^{\#} = \max\{T_i^*, (B^2 - L_i)^+\}$, for $\mathrm{EAC}_i(T, L_i)$ for $(B^2 - L_i)^+ \le T$
- T_i^{Ω} for those i, that satisfies that $B^2 \ge L_i$, the minimum solution under the constraint level, with $T_i^{\Omega} = \mathrm{MID}\{B^2 - L_i, (d_i/b)^{1/2}, (B^2 - L_{i-1})^+\}$ for $\mathrm{EAC}_i(T, B^2 - T)$ with $(B^2 - L_{i-1})^+ \le T \le B^2 - L_i$.

1. Introduction

In the traditional inventory model, the lead time is considered as a predetermined constant or a variable as in Silver and Peterson (1985) such that lead time is not controllable. Liao and Shyu (1991) developed a new inventory model to decompose the lead time into several components, each having a different piecewise linear crash cost function for lead time reduction. Therefore, lead time becomes a new decision variable. Gallego (1992) created a wonderful two-point distribution to serve as the most unfavourable case among the distributions with the same mean and variance to estimate the total expected cost of the lost sales such that the minmax distribution free approach of Scarf (1958) can apply to the stochastic inventory models. Moon and Gallego (1994) extended the minmax distribution free approach for stochastic inventory model with backorders and lost sales. Ouyang, Yeh, and Wu (1996) generalised Ben-Daya and Raouf's (1994) assumption allowing shortages. Ouyang and Wu (1997) extended it to inventory model with service level constraint. Moon and Choi (1998) and Lan, Chu, Chung, and Wan (1999) pointed out the problem in Ouyang et al.'s (1996) method. Ouyang and Wu (1998)

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extended the Ouyang et al. (1996) article to apply the minimax distribution free procedure. Ouyang and Chuang (1999) studied stochastic inventory models with service level constraint which are solved by the minimax distribution free procedure. Ouyang and Chuang (2000) developed periodic review stochastic inventory models involving variable lead time with a service level constraint for the lead time with normal distribution and distribution free. Wu and Tsai (2001) studied stochastic inventory models with a mixed normal distribution from different customers. Pan and Hsiao (2001) developed the model with backorder discount to ensure that customers would be willing to wait for backorders. Chu, Yang, and Chen (2005) improved the results of Ouyang and Wu (1997), first for lead time demand following a normal distribution, and then extending the minmax distribution free procedure to solve the problem.

From the stochastic inventory models with crashable lead time, there are many generalised extensions one can apply for more realistic inventory models. However, there is a questionable result in Ouyang and Chuang (2000) that deserves more detailed discussion. Ouyang et al. (1996) proved that the expected annual cost is a concave down function in lead time so that the minimum value will occur on the boundary points of each sub-domain. It is an excellent discovery that dramatically simplifies the solution procedure. To clearly indicate this property, we denote it as follows: the minimum values for concave down functions degenerate to the boundary points on the sub-domain of the crash cost. However, Ouyang et al. (1996) considered the stochastic inventory model without service level constraint. The researchers who followed them believed that this property also holds with the service level constraint.

In this article, we will point out that the degeneracy to the boundary points for the concave down function requires more detailed examination. We will construct a correct and efficient algorithm to find the optimum order quantity and lead time, develop lemmas to reveal the parameter effects and illustrate our improvement by solving the same numerical example in Ouyang and Chuang (2000) to indicate that sometimes their algorithm does not find the optimal solution.

2. Assumptions

We use the same assumptions as Ouyang and Chuang (2000) and several new expressions to clearly indicate our solution procedure.

(A1) The inventory level is reviewed every T units of time. A sufficient ordering quantity is ordered up to the target level R and the ordering quantity is arrived at after L units of time.

(A2) The length of the lead time L is less than the cycle length T so that there is never more than a single-order outstanding in any cycle.

(A3) The target level R = expected demand during protection interval + safety stock (SS), and $SS = k \times$ (SD of protection interval demand), that is, $R = D(T+L) + k\sigma(T+L)^{1/2}$ where k is the safety factor and satisfies $P(X > R) = q$, q representing the allowable stock-out probability during the protection interval and is provided.

(A4) The lead time L has n mutually independent components. The ith component has a minimum duration a_i, and normal duration b_i and a crash cost per unit time c_i. Further, we assume that $c_1 \le c_2 \le \cdots \le c_n$. The lead time components are crashed one at a time starting with the least c_i component and so on.

(A5) If we let $L_0 = \sum_{j=1}^{n} b_j$ and L_i the length of lead time with components $1, 2, \ldots, i$ crash to their minimum durations, then $L_i = \sum_{j=i+1}^{n} b_j + \sum_{j=1}^{i} a_j$. The lead time crash cost $C(L)$ per cycle for a given $L \in [L_i, L_{i-1}]$ is given by $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$.

(A6) If X has a normal distribution function $F(x)$, according to Ravindran, Phillips, and Solberg (1987), then $B(r) = \sigma L^{1/2}\psi(k)$, where $\psi(k) = \varphi(k) - k[1 - \Phi(k)] > 0$, and φ, Φ are the standard normal probability density function and distribution function, respectively.

(A7) In the beginning, the domain for (T, L) is $T > 0$ and $L_n \le L \le L_0$. However, the cost for crashed lead time is defined for each interval $L \in [L_i, L_{i-1}]$, where $i = 1, \ldots, n$. Hence, we need to consider $\mathrm{EAC}(T, L)$ for $L \in [L_i, L_{i-1}]$, where $i = 1, \ldots, n$. In the next section, we will only consider $L \in [L_i, L_{i-1}]$ for the time being.

(A8) $f_s(T)$ is defined as $4((T + L_s)/T^2)^{3/2} - c/2a_s$, for a family of auxiliary functions.

(A9) $g_s(T)$ is defined as $-a_s/T^2 + b + c/2(T + L_s)$, for another family of auxiliary functions.

(A10) $\mathrm{MID}(x, y, z)$ is the middle term in x, y and z.

3. Review of previous results

Ouyang and Chuang (2000) tried to solve the following problem:

$$\min \mathrm{EAC}_i(T, L) = \frac{A + C(L)}{T} + \frac{h}{2}DT + h\sigma(T+L)^{1/2}(k + (1-\beta)G(k)) \tag{1}$$

subject to

$$\sigma G(k) \leq D\alpha(T+L)^{1/2} \tag{2}$$

with $G(k) = \int_k^\infty (z-k) f_z(z)\mathrm{d}z$ and $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$ for $L \in [L_i, L_{i-1}]$.

The proper domain for the lead time should be $L \in [L_n, L_0]$, but they only considered the domain for the lead time for $L \in [L_i, L_{i-1}]$, with $i = 1, 2, \ldots, n$. The reason is that the crashing costs are different in each sub-domain $[L_i, L_{i-1}]$ with $i = 1, 2, \ldots, n$, so the minimum problem must be divided into sub-domain $[L_i, L_{i-1}]$ with $i = 1, 2, \ldots, n$. However, the crashing costs have similar expressions, hence the results for one sub-domain can be applied to others. Therefore, Ouyang and Chuang (2000) only studied one sub-domain.

They had derived that

$$\frac{\partial}{\partial T}\mathrm{EAC}_i(T,L) = -\frac{A + C(L)}{T^2} + \frac{h}{2}D + \frac{h\sigma}{2}(T+L)^{-1/2} \times (k + (1-\beta)G(k)), \tag{3}$$

$$\frac{\partial^2}{\partial T^2}\mathrm{EAC}_i(T,L) = \frac{hD}{T} + \frac{3T+4L}{4T}h\sigma(T+L)^{-3/2} \times (k + (1-\beta)G(k)), \tag{4}$$

and

$$\frac{\partial^2}{\partial L^2}\mathrm{EAC}_i(T,L) = \frac{-h\sigma}{4}(T+L)^{-3/2}[k + (1-\beta)G(k)]. \tag{5}$$

They claimed that $\mathrm{EAC}_i(T,L)$ is a convex function of T and a concave down function of L. Hence, they assumed that, for fixed T, the minimum total expected annual cost will occur at the end points of the interval $[L_i, L_{i-1}]$. Consequently, they only considered the minimum problem for $\mathrm{EAC}_i(T, L_i)$ and $\mathrm{EAC}_i(T, L_{i-1})$ for $L \in [L_i, L_{i-1}]$.

We will accept Equations (3) and (5), but Equation (4) is questionable (that will be explained by Equation (12)) and may require revision. We will point out that their assumption that the minimum cost will occur at L_i or L_{i-1} as the end points of the interval $[L_i, L_{i-1}]$ is questionable that will be demonstrated in Example 2. And finally, we will discuss why the minimum solution is existent and unique.

4. Our improvement

Fixing i in $\{1, \ldots, n\}$, we will find the minimum value for $\mathrm{EAC}_i(T,L)$ with $L \in [L_i, L_{i-1}]$. We consider the vector (T, L) in the first quadrant that satisfies the service level constraint $\sigma G(k) \leq D\alpha(T+L)^{1/2}$, which implies that the service level constraint being satisfied is equivalent to $B^2 \leq T + L$ where $B = (\sigma G(k)/D\alpha)$. Hence, we may divide the first quadrant into two sub-regions, where the vectors in the sub-region $\{(T,L)\colon B^2 \leq T+L\}$ satisfy the service level constraint and the vectors in the other sub-region $\{(T,L)\colon T+L < B^2\}$ do not satisfy the service level constraint. For a fixed i in $\{1, 2, \ldots, n\}$, we handle the minimum problem for $L \in [L_i, L_{i-1}]$ as L must reduce to boundaries and satisfies the constraint $B^2 - T \leq L$, then we divide the problem into three cases: (I) along the vertical line $L = L_i$, with $T \geq (B^2 - L_i)^+$, (II) along the vertical line $L = L_{i-1}$, with $T \geq (B^2 - L_{i-1})^+$ and (III) along the skew line $T + L = B^2$, with $(B^2 - L_i)^+ \geq T \geq (B^2 - L_{i-1})^+$.

Moreover, owing to $\mathrm{EAC}_i(T, L_i) = \mathrm{EAC}_{i+1}(T, L_i)$, we consider Cases (I) and (II) to reveal that we may merge them into one general case as follows: Case (1), along the vertical line $L = L_s$, with $T \geq (B^2 - L_s)^+$, for $s = i$ or $s = i-1$.

Here, we may point out that for Case (III), we only examine the minimum problem when $B^2 \geq L_i$, since if $B^2 < L_i$, then $(B^2 - L_i)^+ = 0$ and $(B^2 - L_{i-1})^+ = 0$, there is no need to consider the skew line. Therefore, we rewrite Case (III) to Case (2) as follows: Case (2), for those i that satisfies $B^2 \geq L_i$, along the skew line $T + L = B^2$, with $B^2 - L_i \geq T \geq (B^2 - L_{i-1})^+$.

For Case (1), the minimum problem becomes

$$\min \mathrm{EAC}_i(T, L_s) = \frac{A + C(L_s)}{T} + \frac{h}{2}DT + h\sigma(T + L_s)^{1/2} \times (k + (1-\beta)G(k)) \tag{6}$$

where $s = i$ or $s = i-1$, under the condition $T \geq (B^2 - L_s)^+$. To simplify the expression, we assume that $a_s = A + C(L_s)$, $b = (h/2)D$ and $c = h\sigma(k + (1-\beta)G(k))$, then rewrite Equation (6) in the following:

$$\min \mathrm{EAC}_i(T, L_s) = \frac{a_s}{T} + bT + c(T + L_s)^{1/2}. \tag{7}$$

It yields that

$$\frac{\mathrm{d}}{\mathrm{d}T}\mathrm{EAC}_i(T, L_s) = \frac{-a_s}{T^2} + b + \frac{c}{2}(T + L_s)^{-1/2} \tag{8}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}T^2}\mathrm{EAC}_i(T, L_s) = \frac{2a_s}{T^3} - \frac{c}{4}(T + L_s)^{-3/2}. \tag{9}$$

Rewriting Equation (9),

$$\frac{\mathrm{d}^2}{\mathrm{d}T^2}\mathrm{EAC}_i(T, L_s) = \frac{2a_s}{4}(T + L_s)^{-3/2} \times \left(\frac{4}{T^3}(T + L_s)^{3/2} - \frac{c}{2a_s}\right). \tag{10}$$

By Equation (10), we define a family of auxiliary functions, say $f_s(T)$, as follows:

$$f_s(T) = 4\left(\frac{T+L_s}{T^2}\right)^{3/2} - \frac{c}{2a_s}. \tag{11}$$

Since $(\mathrm{d}/\mathrm{d}T)f_s(T) = -6((T+L_s)/(T^2))^{1/2}((1/T^2)+(2L_s/T^3))<0$, $f_s(T)$ is a decreasing function of T, from $\lim_{T\to 0} f_s(T)=\infty$ to $\lim_{T\to\infty} f_s(T) = -(c/2a_s)$ such that there is a unique point, say $T_s^{\#}$ with $f_s(T_s^{\#})=0$. Moreover, $f_s(T)>0$ for $0<T<T_s^{\#}$ and $f_s(T)<0$ for $T_s^{\#}<T$ so we obtain that $(\mathrm{d}^2/\mathrm{d}T^2)\mathrm{EAC}_i(T,L_s)>0$ for $0<T<T_s^{\#}$ and $(\mathrm{d}^2/\mathrm{d}T^2)$ $\mathrm{EAC}_i(T,L_s)<0$ for $T_s^{\#}<T$. Hence, $\mathrm{EAC}_i(T,L_s)$ is a convex function for $0<T<T_s^{\#}$, and a concave down function for $T_s^{\#}<T$.

For completeness, here we point out that the corrected expression for Equation (4) should be

$$\frac{\partial^2}{\partial T^2}\mathrm{EAC}_i(T,L) = \frac{2[A+C(L)]}{T^3} - \frac{h\sigma}{4}(T+L)^{-3/2}(k+(1-\beta)G(k)) \tag{12}$$

so that Ouyang and Chuang claimed that $\mathrm{EAC}_i(T,L)$ being a convex function of T is questionable. This should be revised as follows: $\mathrm{EAC}_i(T,L_s)$ is a convex function for $0<T<T_s^{\#}$, and a concave down function for $T_s^{\#}<T$, where $s=i$ or $s=i-1$.

To simplify the expression, we assume another family of auxiliary functions, say $g_s(T)$, as follows:

$$g_s(T) = \frac{\mathrm{d}}{\mathrm{d}T}\mathrm{EAC}_i(T,L_s) = \frac{-a_s}{T^2} + b + \frac{c}{2}(T+L_s)^{-1/2}. \tag{13}$$

Next, we will prove that $g_s(T_s^{\#})>0$.

Since $(T_s^{\#})^2 = 4(a_s/c)^{2/3}(T_s^{\#}+L_s)$, we have that $(T_s^{\#})^2 > 4(a_s/c)^{2/3}T_s^{\#}$ and then

$$T_s^{\#} > 4(a_s/c)^{2/3}. \tag{14}$$

Using $(T_s^{\#}+L_s)^{-1/2} = 2(a_s/c)^{1/3}(T_s^{\#})^{-1}$, by Equation (14), we imply that

$$g\left(T_s^{\#}\right) = b + a_s^{1/3}c^{2/3}\left(T_s^{\#}\right)^{-2}\left(T_s^{\#} - (a_s/c)^{2/3}\right) > 0. \tag{15}$$

Moreover, we get that $\lim_{T\to 0} g_s(T) = -\infty$ and $\lim_{T\to\infty} g_s(T) = b>0$. By Equation (13), we know that $(\mathrm{d}/\mathrm{d}T)g_s(T) = (\mathrm{d}^2/\mathrm{d}T^2)\mathrm{EAC}(T,L_s)$, then $(\mathrm{d}/\mathrm{d}T)g_s(T)>0$ for $0<T<T_s^{\#}$ and $(\mathrm{d}/\mathrm{d}T)g_s(T)<0$ for $T_s^{\#}<T$. $g_s(T)$ increases from $\lim_{T\to 0} g_s(T)=-\infty$ to $g(T_s^{\#})>0$ and $g_s(T)$ decreases from $g(T_s^{\#})>0$ to $\lim_{T\to\infty} g_s(T)=b>0$. Therefore, there is a unique point, say T_s^* with $g_s(T_s^*)=0$. Consequently, $(\mathrm{d}/\mathrm{d}T)\mathrm{EAC}_i(T,L_s)$ is negative for $T\in(0,T_s^*)$ and positive for $T\in(T_s^*,\infty)$ so T_s^* is the minimum point for $\mathrm{EAC}_i(T,L_s)$, before we consider the constraint $(B^2-L_s)^+\le T$.

It yields that the minimum value of $\mathrm{EAC}_i(T,L_s)$ for $(B^2-L_s)^+\le T$ that occurs, say $T_s^{\#}$, with $T_s^{\#} = \max\{T_s^*, (B^2-L_s)^+\}$, for $s=i$ or $s=i-1$, and $i=1,2,\ldots,n$.

According to above discussions, we know that it yields for $1\le i\le n$, the minimum value of $\mathrm{EAC}_i(T,L_i)$ under the condition $(B^2-L_i)^+\le T$ that occurs at $T_i^{\#}$. We summarise our results in the next theorem.

Theorem 1: *For Case* (1), *with* $0\le i\le n$, *the minimum value occurs at* $T_i^{\#}$.

Next, we study the Case (2), *along the skew line* $T+L=B^2$ *to consider the minimum problem of* $\mathrm{EAC}_i(T,B^2-T)$ *under the condition* $B^2-L_i\ge T\ge(B^2-L_{i-1})^+$.

We know that

$$\mathrm{EAC}_i\left(T,B^2-T\right) = \frac{d_i}{T} + bT + e_i \tag{16}$$

with $d_i = A + c_i(L_{i-1}-B^2) + \sum_{j=1}^{i-1} c_j(b_j-a_j)$, $b=(hD/2)$ *and* $e_i = c_i + h\sigma B(k+(1-\beta)G(k))$. *We may rewrite Equation* (16) *as*

$$\mathrm{EAC}_i(T,B^2-T) = \left(\left(\frac{d_i}{T}\right)^{1/2} - (bT)^{1/2}\right)^2 + 2(d_ib)^{1/2} + e_i. \tag{17}$$

By Equation (17), *it shows that without considering the condition* $B^2-L_i\ge T\ge(B^2-L_{i-1})^+$, *the minimum point is* $T=(d_i/b)^{1/2}$. *To simplify the expression, we use a new expression, say* $\mathrm{MID}\{x,y,z\}$, *which stands for the middle term of* x, y *and* z. *For example*, $\mathrm{MID}\{7,1,5\}=5$ *and* $\mathrm{MID}\{6,2,6\}=6$. *We combine the above discussion in the next theorem. To simplify the expression, we assume that* $T_i^{\Omega} = \mathrm{MID}\{B^2-L_i, (d_i/b)^{1/2}, (B^2-L_{i-1})^+\}$.

Theorem 2: *For Case* (2), *the minimum solution for* T *of* $\mathrm{EAC}_i(T,B^2-T)$ *occurs at* T_i^{Ω}.

We summarise our findings in the last theorem.

Theorem 3: *The minimum for* $\mathrm{EAC}(T,L)$ *with* $L\in[L_n,L_0]$ *is expressed as* (T^*,L^*) *so that* $\mathrm{EAC}(T^*,L^*)$ *is the minimum value among* (a) $\mathrm{EAC}_i(T_i^{\#},L_i)$, *for* $1\le i\le n$, (b) $\mathrm{EAC}_1(T_0^{\#},L_0)$ *and* (c) *for those i that satisfies* $B^2\ge L_i$, $\mathrm{EAC}(T_i^{\Omega},L_i)$.

5. Numerical example

Example 1: To illustrate our improvement, we consider the same numerical example as Ouyang and Chuang (2000) with the following data: $D=625$ units per year, $A=\$350$ per order, $h=\$35$ unit per year,

$\mu = 11$ units per week, $\sigma = 7$ units per week, ($\sigma = 7\sqrt{52}$ units per year), $\beta = 1$ (backorders case), The lead time has three components with $c_1 = \$0.4$ per day, $a_1 = 6$ days, $b_1 = 20$ days, $c_2 = \$1.0$ per day, $a_2 = 6$ days, $b_2 = 20$ days, $c_3 = \$5.0$ per day, $a_3 = 9$ days, $b_3 = 16$ days. It yields $L_0 = (8/52) = 0.154$ years, $L_1 = (6/52) = 0.115$ years, $L_2 = (4/52) = 0.077$ years, and $L_3 = (3/52) = 0.058$ years. It is assumed that $q = 0.2$, so the safety factor k can be found directly from the standard normal table, and it equals 0.845.

Since $B = (\sigma G(k)/D\alpha)$, it shows that $B^2 = 0.201$. Here, we must point out that if we use $D = 625$ units per year, then our results cannot compare with those of Ouyang and Chuang (2000). We slightly modify the value of D to $D = 624$ units per years, then our results are consistent with their results. Hence, we modify D as $D = 624$ units per years.

For the first numerical example, we assume that the service level $1 - \alpha = 0.98$, i.e. the proportion of demand that is not met from stock is at most $\alpha = 0.02$. We compute the local minimum value for each interval $L \in [L_i, L_{i-1}]$ and list them in Table 1.

For easy comparison with the results of Ouyang and Chuang (2000), we quote their results in Table 2. They found that the optimal review period $T^* = 8.84$ weeks, that is, $T^* = 0.170$ years and the optimal lead time $L^* = L_1 = 6$, with $\mathrm{EAC}(T^*, L^*) = 4745.68$. We may say that Ouyang and Chuang (2000) and our improved method both have the same minimum solution for the first numerical example with $\alpha = 0.02$.

Example 2: We have the same value for parameters except that $\alpha = 0.015$. We obtain that $B^2 = 0.358$ and then based on Theorem 2, we find that

$$\mathrm{MID}\left\{B^2 - L_0 = 0.048, \sqrt{d_1/b} = 0.179,\ B^2 - L_1 = 0.086\right\} = 0.086 = B^2 - L_1,$$

$$\mathrm{MID}\left\{B^2 - L_1 = 0.086, \sqrt{d_2/b} = 0.180,\ B^2 - L_2 = 0.125\right\} = 0.125 = B^2 - L_2,$$

and

$$\mathrm{MID}\left\{B^2 - L_2 = 0.125, \sqrt{d_3/b} = 0.184,\ B^2 - L_3 = 0.144\right\} = 0.144 = B^2 - L_3,$$

such that there is no need to compute the minimum value along the skew line $T + L = B^2$, and then list the results in Table 3.

We find that the optimal review period, $T^* = B^2 - L_0 = 0.204$, the optimal lead time, $L^* = L_0 = 0.154$ and the minimum value $\mathrm{EAC}(B^2 - L_0, L_0) = 4837.378$. We compare the results with Ouyang and Chuang. From Table 2, the sixth column, according to Ouyang and Chuang's method, without considering the service level constraint in Table 2, the fourth row, they find the absolute minimum expected average cost $\mathrm{ETC}(T_1 = 8.84, L_1 = 6) = 4745.68$, then check $(E(X - R)^+/D(T_i + L_i)) = 0.0168 > \alpha = 0.015$, so it has not satisfied the service level constraint. Ouyang and Chuang will consider the next minimum value

Table 1. Local minimum values for each case.

Theorem	Minimum point	Minimum value
1	$\mathrm{Max}\{T_0^* = 0.169, B^2 - L_0 = 0.047\}$	4764.731
1	$\mathrm{Max}\{T_1^* = 0.170, B^2 - L_1 = 0.086\}$	4745.681
1	$\mathrm{Max}\{T_2^* = 0.173, B^2 - L_2 = 0.125\}$	4771.886
1	$\mathrm{Max}\{T_3^* = 0.180, B^2 - L_3 = 0.144\}$	4941.206
2	$\mathrm{MID}\{B^2 - L_0 = 0.048, \sqrt{(d_1/b)} = 0.179, B^2 - L_1 = 0.086\}$	5742.229
2	$\mathrm{MID}\{B^2 - L_1 = 0.086, \sqrt{(d_2/b)} = 0.180, B^2 - L_2 = 0.125\}$	4998.109
2	$\mathrm{MID}\{B^2 - L_2 = 0.125, \sqrt{(d_3/b)} = 0.184, B^2 - L_3 = 0.144\}$	5054.462

Table 2. Results from Ouyang and Chuang (2000).

i	L_i (weeks)	$C(L_i)$ (US \$)	T_i (weeks)	R_i (units)	$\mathrm{EAC}(T_i, L_i)$ (US \$ per year)	$E(X - R)^+/D(T_i + L_i)$
0	8	0	8.80	226	4764.73	0.0158
1	6	5.6	8.84	201	4745.68	0.0168
2	4	19.6	8.97	177	4771.89	0.0180
3	3	24.6	9.37	169	4941.21	0.0184

Table 3. Local minimum value for each case.

Theorem	Minimum point	Minimum value
1	$\text{Max}\{T_0^* = 0.169, B^2 - L_0 = 0.204\}$	4837.378
1	$\text{Max}\{T_1^* = 0.170, B^2 - L_1 = 0.243\}$	5008.922
1	$\text{Max}\{T_2^* = 0.173, B^2 - L_2 = 0.281\}$	5278.311
1	$\text{Max}\{T_3^* = 0.180, B^2 - L_3 = 0.300\}$	5520.673

in Table 2, as $\text{ETC}(T_1 = 8.80, L_1 = 8) = 4764.73$ and then check the service level constraint to find that failed again. Following this pattern, the ratios of $(E(X - R)^+/D(T_i + L_i))$ are all greater than 0.015.

This means that by Ouyang and Chuang's method all their solutions do not satisfy the service level constraint, so they concluded that the inventory system has no feasible solution. We discover that the solution procedure of Ouyang and Chuang sometimes cannot obtain the minimum solution.

6. Directions for future research

Here we want to discuss a possible direction for future research. Since Ouyang et al. (1996) developed a mixed inventory model with backorders and lost sales for variable lead time, there is a trend to treat the reordered point as a decision variable as pointed out by Moon and Choi (1998), Hariga and Ben-Daya (1999) and Yang, Ronald, and Chu (2005). Ouyang and Chuang (2000), for the normal distribution model, considered the reordered point being fixed, and for the distribution free model, they used the reordered point as a new decision variable.

In this article, we still take the fixed reordered point for the normal distribution model, such that one possible direction for the future research is to examine the normal distribution model where the reordered point is a new decision variable.

Taking the reordered point as a new decision variable is equivalent to saying that the safety factor, k, is a new variable such that the objective function becomes $\text{EAC}(T, L, k)$. On the other hand, if we treat the safety factor as fixed, then the objective function is $\text{EAC}(T, L)$. It is of little significance that the minimum value for $\text{EAC}(T, L, k)$ is less than or equal to the minimum value of $\text{EAC}(T, L)$. Generally speaking, if we are allowed to introduce another new parameter, say A (ordering cost per order), then $\text{EAC}(T, L, k, A)$ will have even an smaller minimum value. Here, we face a problem how to compare the results from $\text{EAC}(T, L)$, $\text{EAC}(T, L, k)$ and $\text{EAC}(T, L, k, A)$. If we directly use the minimum value then definitely $\text{EAC}(T, L, k, A)$ will be the best choice.

There are two methods, AIC or CAIC (Hartley and Helmbold 1995, p. 609–643), to test the difference between different methods with different numbers of parameters. The formulas for AIC and CAIC are given below, where p = the number of variables, n = number of constant data and Min = minimum value for each method: $\text{AIC}(p) = n \ln(\text{Min}/n) + 2p$ and $\text{CAIC}(p) = \text{AIC}(p) + p(\ln n - 1)$. In both statistics, the smaller the value, the better.

Up to now, we cannot solve the minimum problem for the $\text{EAC}(T, L, k)$ for the normal distribution model so that the comparison between $\text{EAC}(T, L, k)$ and $\text{EAC}(T, L)$ by AIC or CAIC may be a direction for the future research.

7. Conclusion

In the above discussions, we pointed out the questionable results in the paper of Ouyang and Chuang so that the minimum may not occur at the boundary points of sub-domain for the crash lead time cost. We offered the corrected algorithm to find the optimal solution. Our refined algorithms are easy to use and mathematically sound and provide the optimal replenishment solution for decision makers. In the second numerical example, we illustrated that sometimes Ouyang and Chuang cannot derive the optimal solution and so our improved method still provides the optimal solution for the decision makers.

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