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碩 士 論 文

考慮流動性下之選擇權訂價模型：非線性拋物線偏微分方程式的數值方法應用

Pricing options under illiquidity: Numerical method applications in nonlinear parabolic PDE

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摘要

本篇論文提出一個在市場流動性不足情況發生時的選擇權訂價模型,並且發展一個新的數值方法來求解一個非線性拋物線型態的偏微分方程式,同時利用湯馬斯演算法來提升數值運算的效率。在實證研究的部份,我們使用美國的個股選擇權資料來進行分析,首先運用非線性最小平方法來估計標的物的市場流動性,並針對 Black-Scholes 與本文所運用的模型即 Frey 模型兩者之間對選擇權定價的損失函數分析。

關鍵字： 選擇權評價、非流動性的選擇權定價、非線性偏微分方程式、回饋效果、價格影響力、有限差分法、湯馬斯演算法、非線性最小平方法。

Pricing options under illiquidity: Numerical method applications in nonlinear parabolic PDE

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ABSTRACT

This paper considers the pricing model of options under illiquidity. A new numerical procedure for solving the nonlinear parabolic partial differential equation is explored and the Thomas algorithm is used to improving the efficiency of the numerical scheme. Using CBOE stock options, we employ the nonlinear least square method for obtaining the liquidity parameter of the underlying stock option in empirical work and then comparing the loss function between the Black-Scholes model and the model which is proposed by Frey and Patie (2001) and will be abbreviated as the Frey model in this paper.

Keyword: Option Valuation; Illiquidity Option Pricing; Nonlinear PDE; Feedback Effect; Price Impact; Finite Difference Method; Thomas Algorithm; NLS.

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在這兩年的求學歲月裡，很慶幸能和你們這些聰明絕頂的同學們在一起唸書、考 CFA、打球、唱歌、作研究共渡這些美好的時光。本篇論文之所以可以順利的完成，我想必需要感謝很多老師、學長和同學們的幫忙與協助，很多研究上的問題和一些新的想法都是在討論或是聊天中無意之間想出的。首先，很感謝鍾惠民老師在本篇論文上的細心指導，指點我相關寫作的觀念和重要的財務意涵，能讓我寫出一篇像樣的論文，並且有機會在三個研討會發表本篇論文[1]，讓我實在是倍感榮幸與受寵若驚，除了論文方面的指導，很感謝鍾老師讓我有機會擔任很多課程的助教以及參與一些短期的研究計畫，假如未來有唸博士班的可能，希望可以在跟老師多多學習與討教。除了鍾老師之外，在此必須要感謝王克陸、賴明治[2]、張傳章[3]與張焯然[4]等四位教授分別在研討會與在論文口試時給予很多有建設性的建議。

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[1] 2006 經濟金融會計國際研討會(台灣大學，四月十三日)、台灣財務工程學會年會季風險管理研討會(德明技術學院，六月二日)以及第十四屆亞太財務經濟及會計會議暨 2006 台灣財務工程會聯合研討會(台北圓山飯店，七月十五日) 。

[2] 交通大學應用數學系教授，其專長為數值分析、科學計算、數值偏微分方程、流體力學等等。

[3] 中央大學財務金融所教授。

[4] 清華大學計量財務金融學系副教授。

[5] 交通大學管理科學所博士生。

[6] 清華大學計量財務金融系助理教授，其專長為蒙地卡羅模擬、財務數學等等。

[7] 邱銘輝，專長為 MATLAB 與 C++等程式語言，程式功力出神入化，中央數學系畢業，曾經想報考資工所。

[8] 呂忠穎，專長為「當班長」以及閱讀各種財務相關論文，解決每個同學在論文上的疑難雜症，看論文的速度堪稱本所一絕，好比喝水一般，根本就是交大財金所地下所長，政大金融系畢業。

[9] 蔡呈偉，專長為英文和數學，英文說寫流利，口頭禪是「讚喔!」，政大金融系畢業。

[10] 交大統計所，與班長和偉哥為大學同學。

算法。不僅如此，很高興可以和儀貞[11]一起研究論文、討論程式以及分享生活上的點點滴滴；感謝電影團團長豪哥[12]在閒暇之餘帶領團員們去新竹風城看電影和品嘗一些美食，以抒發平日研究生苦悶乏味的生活。感謝慧妤、小素、小育、柏鈞、怡文、惠華、志揚、猛男、瑞娟、戰哥、波哥、小使、妃君等各位同學們，在我心情沮喪時給我的鼓舞與支持以及分享這兩年快樂且匆匆的時光;感謝美麗的財金所所辦小姐們文淇、碧鳳和佳芸平日的辛勞與付出；感謝克鈞、志勛、士榮，逸民、璨元、楷文等學弟妹，如果沒有你們各位，這兩年也不會過的如此快樂與充滿回憶，希望各位同學畢業後都能找到自己喜歡的工作以及幸福的歸宿，學弟妹們寫論文都可以順順利利，不要太常熬夜。

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楊家農 謹誌於
交通大學財務金融所
民國九十五年六月

[11] 鄭儀貞是大家的開心果，看到她總是很開心，此外，她兼了兩個高中數學的家教，靠著資優生的頭腦，賺了不少零用錢，成大統計系畢業。

[12] 邱柏豪，專長為經濟學與統計學，目前是陳偉補習班的助教，以後想當補習班老師。

Contents

1. Introduction

Market liquidity is one of the most critical factors in investment decisions and derivative pricing. There is a growing support for the proposition that liquidity affects the asset dynamics and trading strategies. This paper aims to investigate how the price process is affected by dynamic trading strategy and how dynamic trading strategy is influenced by stock price process. Market prices are determined by the supply and demand of traded assets. However, most of financial models assume that the supply and demand are perfect elasticity, which means the orders, including market order and limit order, do not affect the traded asset price. All investors who are considered as small traders have tiny impact in financial market individually. As a manner of fact, the volume of traded assets must influence on the price of traded assets in real market circumstance. The large traders are persons who have a market power and significant part of the shares. Consequently, we argue that their hedging strategy have great impact on stock price and volatility. Hence, the stock price is very likely affected by their hedging strategy and the influence of the large trader hedging strategy becoming a critical issue in financial market.

There are many theoretical studies and empirical studies which support the effect of liquidity of transaction cost on asset dynamics[13]. Furthermore, portfolio choice[14] is also determined by the liquidity assets. Liquidity of assets is one of the factors which play a major role in the formulation of optimal trading policy followed by traders.

This paper provides a comprehensive framework for the pricing of European option pricing and models the dynamic trading strategy in financial market due to illiquidity. According to many pervious research and related articles we know that it will have great influence on pricing and hedging strategy for traded asset such as

[13] See, for example, Pastor and Stambaugh (2001) and Lo, Mamaysky and Wang (2001).
[14] Koren and Szeidl (2002)

derivatives when market becomes illiquid or transaction cost becomes higher. Thus, the hedgers can hardly completely replicate their portfolio in illiquidity market and thus result in a lot of hedging error.

In our model analysis, we focus on the circumstance under market illiquidity and perfect liquidity. The classical Black-Scholes (BS) framework is based on many assumptions and the most important Black-Scholes formula assumption relative to this paper is “The stock and option price are not affected by placing orders.” We relax this assumption for the following further analysis and figure out the relationship between the option pricing and the market liquidity.

In market microstructure theory, we say that the feedback effect is based on the violation of this assumption. The large trader might be able to use his market power in order to manipulate market prices in his favor. Sometimes the large trader is called by informed trader who has more information than small trader in market.

We examine how price impact on the underlying asset market that affect the replication of a European contingent claim and find out the best hedging strategy. If the feedback effect exists, we need to develop a new financial model fitting the real market condition and the large trader or investors can use this model for the option pricing and hedging.

The standard market microstructure models of Kyle (1985) and Back (1992, 1993) use an equilibrium approach to investigate how informed traders reveal information and affect the market price through the trading. The equilibrium asset prices are directly influenced by the informed trader’s trades that shown by Kyle (1985) and Back (1992, 1993).

Jarrow (1992) investigates market manipulation trading strategy by large traders in the stock market and the large traders are defined as a person who has influence on prices by generalizing and extending Hart (1977) to a stochastic economy.

Furthermore, Jarrow make more generalization in model and distinguishing between the real wealth and the paper wealth while calculate the traders' position. He argues that asymmetry creates the manipulation opportunity and the large trader can use their market power to manipulate prices and generate profit without any risk.

Jarrow (1994) shows that the introduction of option markets might result market manipulation strategy. Simply speaking, he defines the manipulation strategy as arbitrage opportunity regarding the large trader. He shows that if the stock market and derivative market are perfectly aligned, so-called synchronous market condition, the large trader can hardly manipulate prices. Jarrow (1994) identifies this condition to be equivalent to the no arbitrage condition. But if the small trader or noise traders have only incomplete information about the large traders' behavior and reaction, the small trader could fail to synthetically replicate the call options.

Esser and Moench (2003) introduce a continuous-time model for an illiquid market and revise the market liquidity parameter from deterministic liquidity model to stochastic liquidity model (henceforth SL) which demonstrates that the market liquidity follows a stochastic process. Furthermore, they analyze positive feedback strategies and contrarian feedback strategies. They find the market volatility generally increasing compared to BS model when positive feedback strategies exist. Moreover, they derive a closed-form expression for the option pricing model and exploit a pragmatic method to calculate the price of liquidity from plain vanilla put options. However, the SL model is very sophisticated than the Frey model in numerical computation and empirical study. Thus, we do not consider the stochastic factor into the liquidity for Occam's razor purpose.

Cetin, Jarrow, Protter and Warachka (2006) use the stochastic supply curve modeling the liquidity risk and their empirical studies demonstrate that liquidity cost are a significant factor of option price. Furthermore, they find that in-the-money (ITM) options are subject to the lowest percentage impact of illiquidity component, even thought ITM options is expensive. On the contrary, the out-of-the-money (OTM) options are significant affected by the factor of illiquidity despite OTM options are cheaper than ITM options. They define liquidity cost of the discrete trading strategies and estimate the liquidity parameter of the stochastic supply curve. The empirical evidence shows that the liquidity cost increases quadratically with transaction sizes.

Recently research concentrates on the pricing and hedging aspects which are introduced by the market illiquidity and the presence of the price impact effects on stock prices regarding the large traders. Frey (1998, 2000), Schonbucher and Wilmott (2000), Frey and Patie (2001), Bank and Baum (2004), as well as Liu and Yong (2005).are some famous articles and they calibrate the nonlinear pricing PDE in the illiquidity of the option pricing. Cetin, Jarrow, Protter and Warachka (2006) is the latest paper which provide a "reduced form" illiquidity model for constructing a discrete trading strategy within temporary price impacts. They not only utilize a simple framework for estimating the parameter of the stochastic supply curve by regression but also build up an optimal discrete time hedging strategy rather than the nonlinear PDE pricing model.

There would be a tough problem as we introduce the large traders' trading and hedging actives into European option pricing model. In fact, the asset dynamics depend on many parameters such as the Delta hedging strategy, market liquidity, Gamma and so on. This characteristic renders the pricing problem nonlinear. Thus, we face the problem that the nonlinear PDE is more difficult than BS model for getting the exact solution. In section 3, we show a better way of numerical skill which can avoid solving the nonlinear PDE problem directly.

In practice, the traders often use the Black-Scholes model that the stock price is described by a lognormal random process. Nevertheless in BS model the traders' trading or hedging their position according to a misspecified model that could generate serious pricing and hedging error especially when liquidity becomes worse. That is the reason why we use the nonlinear PDE model rather than BS-PDE pricing model and the nonlinear PDE model is designated as the Frey model in the following research.

First of all, we provide the nonlinear parabolic partial differential equation (the Frey model) to modify the original Black-Scholes partial differential equation (BS-PDE) for option pricing under market illiquidity framework and the proof of the Frey model will present in Appendix. Secondly, we demonstrate the hedging error formula result from market illiquidity and claim the new volatility term for feedback effect trading strategies.

In general, PDE problem can be solved by certain numerical method including finite difference method (FDM), finite element method (FEM), and finite volume method and so on. In fact, obtaining the analytical solution of PDE is not easily even though there are many well-developed numerical methods. In our methodology, we utilize FDM which is the most fundamental and simplest framework in the computation of PDE.

The rest of this paper is organized as follows. In section 2, we introduce the model that is modification of BS-PDE and derive the Frey model (nonlinear PDE model). In section 3, we provide numerical results of the nonlinear PDE pricing model for European calls. Section 4 provides the empirical study and verifies the estimation loss function. Section 5 contains the concluding remark and further research. Appendix provides the concept of the Thomas algorithm and the comparison of the heat equation and the BS-PDE.

2. The Model

This paper considers the pricing model of options under illiquidity and the following several sections are the core of this paper. In this section, following Frey (2000) and Frey and Patie (2001), we assume that there are two traded assets: bond and stock in the market where bond is a risk-free asset (i.e. cash account) and stock is a risky asset which follows a stochastic process. Simultaneously, we consider the bond as a numeraire (i.e., sometimes called discount factor) and assume that bond market is perfect liquidity that there is no liquidity problem exist. Now we focus on liquidity problem in the stock market.

The BS model assumes that the underlying stock have perfect liquidity, meaning that investors can buy or sell a large amount of stock without affecting the stock price in market so that there is no feedback effect in the market. However, we take the market liquidity variable into account in the model due to the liquidity problem is an existent fact in the stock market. In this study, we do not assume the parameter of liquidity following a certain stochastic process, meaning that the liquidity is deterministic and it is not stochastic.[15]

The following sections will introduce the basic assumptions and asset dynamics firstly. Secondly, the Frey model will be conducted and then we introduce the tracking error of the model. After that we explore the numerical method applications in the model. Finally, we present the smooth version of the model which proposed by Frey and Patie (2001).

[15] In Esser and Moench (2003), the liquidity follows a certain stochastic process. Their framework generalizes the constant liquidity model of Frey (2000) and they impose a stochastic factor into the liquidity. Hence, the stochastic liquidity model of Esser and Moench (2003) is more sophisticated than the Frey model; it's becoming very complex in the modeling of option pricing and in constructing the hedging strategies when the market liquidity is considered as a random source. In this paper, we do not deal with this kind of complicated circumstance in the parameter of liquidity which follows a certain stochastic process.

2.1 Basic assumptions and asset dynamics

We now introduce the basic model setup proposed by Frey and Patie (2001). The risky asset (i.e. the stock) follows the stochastic process without drift term

$$dS_t = \sigma S_{t-} dW_t + \rho \lambda (S_{t-}) S_{t-} d\alpha_t^+ , \tag{1}$$

where α is the number of stock shares held by large investor, i.e. the trading strategy of the large trader. The variable α^+ denotes the right-continuous process, and ρ is a non-negative constant liquidity parameter. A large value of the parameter ρ means that the market becomes more illiquid. Moreover, we state that the parameter ρ is equal to zero as the market reduces to the BS world with perfect liquid. Recall that the drift term plays a role in stock dynamics in the assumption of the BS model. After the change of measure, however, the drift term is removed from the BS-PDE which is dominated by risk-free rate in risk neutral measure.

Frey (2000) and Frey and Patie (2001) discuss the influence of the trading strategy on the asset process with a smooth stock trading strategy α and suppose that the large trader utilize the strategy of the form $\alpha = \phi(t, S_t)$. Thus, the asset dynamic becomes a new dynamics and then we can obtain the new effective asset dynamics by Ito formula[16] with the following form

$$dS_t = v(t, S_t) S_t dW_t + b(t, S_t) S_t dt , \tag{2}$$

where

$$v(t, S_t) = \frac{\sigma}{[1 - \rho \lambda (S_t) S_t \phi_S (t, S_t)]} , \tag{3}$$

$$b(t, S_t) = \frac{\rho \lambda (S_t) \left(\phi_t (t, S_t) + \frac{1}{2} \phi_{SS} (t, S_t) v^2 (t, S_t) S_t^2 \right)}{[1 - \rho \lambda (S_t) S_t \phi_S (t, S_t)]} , \tag{4}$$

Derivation of the new asset dynamics

We suppose that the large trader utilize the strategy of the form $\alpha = \phi(t, S_t)$ for a

[16] See Shreve (2004) chapter 4.

function α and it is satisfying a mathematical assumption with two variables which are once continuously differentiable in time and twice continuously differentiable in stock[17]. The trading strategy of large trader expanded by Ito formula and thus we can get the form

$$d\alpha_t = \phi_S(t,S_t)dS_t + \left(\phi_t(t,S_t) + \frac{1}{2}\phi_{SS}(t,S_t)v^2(t,S_t)S_t^2\right)dt. \tag{5}$$

Firstly, we have already known the stock prices are controlled by the following stochastic process

$$dS_t = \sigma S_t dW_t + \rho\lambda(S_t)S_t d\alpha_t.$$

Secondly, we substitute the Equation (5) into the second term of the RHS of the Equation (1) and thus we obtain the Equation (6).

$$dS_t = \sigma S_t dW_t + \rho\lambda(S_t)S_t\left[\phi_S(t,S_t)dS_t + \left(\phi_t(t,S_t) + \frac{1}{2}\phi_{SS}(t,S_t)v^2(t,S_t)S_t^2\right)dt\right]. \tag{6}$$

By rearrangement,

$$\left[1-\rho\lambda(S_t)S_t\phi_S(t,S_t)\right]dS_t = \sigma S_t dW_t + \rho\lambda(S_t)S_t\left[\left(\phi_t(t,S_t) + \frac{1}{2}\phi_{SS}(t,S_t)v^2(t,S_t)S_t^2\right)dt\right] \tag{7}$$

Therefore, generates the following explicit form for asset dynamics

$$dS_t = \frac{\sigma S_t}{\left[1-\rho\lambda(S_t)S_t\phi_S(t,S_t)\right]}dW_t + \frac{\rho\lambda(S_t)S_t\left(\phi_t(t,S_t) + \frac{1}{2}\phi_{SS}(t,S_t)v^2(t,S_t)S_t^2\right)}{\left[1-\rho\lambda(S_t)S_t\phi_S(t,S_t)\right]}dt. \tag{8}$$

In this section, we provide a simple proof of the new effective asset dynamics. In next section, we interpret how the Frey model is controlled by the Equation (12) and clarify all of basic assumptions in the model.

2.2 The Frey model (nonlinear parabolic PDE)

The Frey model has two significant characteristics different from Black-Scholes PDE.

[17] The stock is often designated as the space in the FDM application

First, the risk-free rate does not play a role in Frey model. Second, the Frey model argues that the volatility is not a constant volatility. In the Frey model, the volatility term is dominated by three main parameters ρ, λ and u_{SS} in the Frey model. However, we can utilize the three main parameters to capture the volatility behavior in real markets. The parameter λ can be utilized to describe the asymmetry of liquidity[18]. Generally, markets tend to be more liquid in the bull market than in the bear market. Thus, Frey and Patie (2001) denote the parameter in the following form

$$\lambda(S) = 1 + (S - S_0)^2 \left\{ a_1 I_{(S \le S_0)} + a_2 I_{(S > S_0)} \right\}, \tag{9}$$

where the parameter a_1 is usually larger than a_2 in empirical study. The parameter λ plays an important role in this model. The asymmetry of liquidity can be explained by λ with financial sense. The third critical factor is u_{SS}. We are familiar with the Greeks in options pricing and hedging. In the Frey model, the parameter u_{SS} represents the value of gamma and it is also a crucial factor in the model.

First, u_{SS} plays a role in Equation (3) & (4) and thus it would affect the asset dynamic. If we want to simulate the sample path of the stock price afterward, the parameter u_{SS} must be calculated by FDM before the simulation. Second, it goes without saying that u_{SS} has a great influence on the size of hedging error and the large trader's trading strategy. In section 2.3, we will demonstrate the relationship between the parameter u_{SS} and tracking error with mathematical equation.

There are two major trading strategy which include positive feedback trading and contrarian feedback trading. When $u_{SS} > 0$ the large trader adopting the positive feedback trading strategy. On the other hand, the larger trader employs the contrarian feedback trading strategy as $u_{SS} < 0$. Moreover, u_{SS} is also vital when we discrete

[18] Kamara and Miller (1995) show that the relationship between moneyness and liquidity is asymmetric. Etling and Miller, Jr. (2000) also state that although the maximum value of liquidity is near the money, liquidity does not decrease symmetrically as strike price move away from at the money (ATM).

the Frey model in numerical computation and the detail of this part will be presented in section 2.4.

The Frey Model (nonlinear parabolic PDE)

We denote that there is a solution u of the Frey nonlinear PDE model. Specifically, the Frey model and terminal condition is given by

$$u_t(t,S)+\frac{1}{2}\frac{\sigma^2}{\left(1-\rho\lambda(S)Su_{SS}(t,S)\right)^2}S^2u_{SS}(t,S)=0, \tag{10}$$

$$u(T,S)=h(S), \tag{11}$$

where $u_S(t,S)=\dfrac{\partial u(t,S)}{\partial S}$ is the large traders' strategy and it must satisfies a critical assumption with $\rho\lambda(S)Su_{SS}(t,S)<1$. Obviously, we observe the Frey model setting risk-free rate equal to zero and illustrate the PDE formula under risk neutral measure without drift term (i.e. risk-free rate equal to zero). However, we can improve the Frey model by taking risk-free rate into account, making the pricing model more general than Frey's and we present the form of the Frey model with risk-free rate. For tractability, we still assume that the parameter, risk-free rate, equal to zero the same as Frey's model setting for "parsimonious principle" in numerical analysis.

2.3 Tracking error (hedging error)

Firstly, we realize the large traders following the trading strategy with $\alpha_t=u_S(t,S)$ and the volatility of asset price is $\sigma_u(t,S)=\dfrac{\sigma}{\left(1-\rho\lambda(S)Su_{SS}(t,S)\right)}$. Secondly, we employ the Ito formula to u so that we can obtain

$$du(t,S_t)=u_S(t,S_t)dS+\left(u_t(t,S_t)+\frac{1}{2}u_{SS}(t,S_t)\sigma_u^2S_t^2\right)dt, \tag{12}$$

equivalently,

$$u(T,S_T)=u(0,S_0)+\int_0^T u_S(t,S_t)dS_t+\int_0^T\left[u_t(t,S_t)+\frac{1}{2}u_{SS}(t,S_t)\sigma_u^2(t,S_t)S_t^2\right]dt, \tag{13}$$

where we denote $u(T,S_T)$ is the payoff of derivative at maturity day. Hence, $h(S_T)=u(T,S_T)$ and the payoff of derivative at maturity day can be represented in this form

$$h(S_T)=u(0,S_0)+\int_0^T u_S(t,S_t)dS_t+\int_0^T\left[u_t(t,S_t)+\frac{1}{2}u_{SS}(t,S_t)\sigma_u^2(t,S_t)S_t^2\right]dt \quad (14)$$

Assuming we have already known the Frey nonlinear PDE model. If the Frey model holds, we can eliminate the last term in the right hand side (abbreviated RHS) of the Equation (14). Now we denote the tracking error $e_T^M=h(S_T)-V_T^M$. The tracking error measures the difference between the terminal payoff of the European option (i.e. $h(S_T)$) and the replication of derivative (i.e. V_T^M) which duplicated by bond and stocks with the self-financing trading strategy. Using the tracking error can easily track and judge the performance of hedging strategy. In fact, we regard $h(S_T)$ and V_T^M as the total cost and the total revenue respectively in economic sense.

We conclude that a positive value of e_T^M displays the large trader who loss the money in the hedging strategy, meaning the payoff of the replication of derivative can not completely cover the payment of European style option at maturity date. Thus, the large traders suffer loss from under-hedging at maturity day.

According to self-financing trading strategy, we can obtain the option payoff at terminal time by the following representation

$$h(S_T(\rho,\alpha))=V_0+\int_0^T \alpha_t dS_t(\rho,\alpha), \quad (15)$$

where $\alpha_t=u_S^{BS}(t,S_t)$ and $V_0=u^{BS}(0,S_0)$. Next we demonstrate that the tracking error $e_T^M=0$ under BS world and as self-financing trading strategy holds.

$$
\begin{aligned}
e_T^M &= h(S_T(\rho,\alpha)) - V_T^M \\
&= h(S_T(\rho,\alpha)) - \left(V_0 + \int_0^T \alpha_t dS_t(\rho,\alpha)\right) \\
&= 0
\end{aligned}
\tag{16}
$$

Frey (2000) demonstrates that the Black-Scholes hedging is costly under imperfectly liquid market and the tracking error is always absolutely positive-value in BS world. We can display the tracking error in the following form[19]

$$
e_T^M = \int_0^T \left(\frac{1}{2}\sigma^2 u_{SS}^{BS} \left(\frac{1}{\left(1-\rho S_t u_{SS}^{BS}\right)^2} - 1 \right) S_t^2 \right) dt , \tag{17}
$$

If $u_{SS}^{BS} > 0$, then $\left(1-\rho S_t u_{SS}^{BS}\right) < 1$. Hence, we conclude $\frac{1}{\left(1-\rho S_t u_{SS}^{BS}\right)^2} > 1$ and the tracking error is positive. When $u_{SS}^{BS} < 0$, we can get the same result in this integral form. In a word, the value of the tracking error is always positive if the large trader uses the Black-Scholes option pricing model in illiquid market.

2.4 Numerical method applications in the Frey model

In this section, we are interested in how to discretize the Frey model which is a complex nonlinear PDE. Obviously, the coefficient of the Frey model is an unknown number which includes the solution that we want to solve it. In Frey and Patie (2001), they use the Newton method to solve the whole nonlinear system. However, using the Newton method might quite sophisticate and spending much more computational time in programming procedure. As a result, we provide an alternative approach that transfers the nonlinear problem into the linear system and this approach can reduce the computational costs. We will demonstrate the detail of methodology in the

[19] Frey (2000) shows the basic concept of tracking error. Theoretically, the tracking error can be treated as a "cumulative dividend stream with instantaneous dividend." By the way, we should notice that Frey (2000) does not consider the parameter λ in the model and in the tracking error.

following.

Firstly, we use the explicit method for the calculation of the coefficient. After we solve the coefficient in the first step, the nonlinear PDE becomes the linear PDE and therefore the coefficient of the model is known at this moment. Secondly, we use the implicit method[20] to solve every linear system at each time step. Recall the Frey model without the risk-free rate (zero drift)

$$u_t(t,S)+\frac{1}{2}\frac{\sigma^2}{\left(1-\rho\lambda(S)Su_{SS}(t,S)\right)^2}S^2u_{SS}(t,S)=0,$$

However, we impose the risk free rate term in the model which improves the model more general than the Frey model.

$$u_t(t,S)+\frac{1}{2}\frac{\sigma^2}{\left(1-\rho\lambda(S)Su_{SS}(t,S)\right)^2}S^2u_{SS}(t,S)+rSu_S(t,S)=ru(t,S), \tag{18}$$

Using the finite difference methods, the Frey model can be represented in this form

$$\frac{U_j^i-U_j^{i-1}}{\Delta t}+\frac{1}{2}\left(v_j^i\right)^2S_j^2\frac{U_{j+1}^{i-1}-2U_j^{i-1}+U_{j-1}^{i-1}}{(\Delta S)^2}+rS_j\frac{U_{j+1}^{i-1}-U_{j-1}^{i-1}}{2\Delta S}=rU_j^{i-1}, \tag{19}$$

where

$$v_j^i=\frac{\sigma}{\left(1-\rho\lambda(S_j)S_ju_{SS}^{(i)}\right)}=\frac{\sigma}{\left(1-\rho\lambda(S_j)S_j\frac{U_{j+1}^i-2U_j^i+U_{j-1}^i}{(\Delta S)^2}\right)}. \tag{20}$$

We denote that U_j^i is the numerical solution and u is the exact solution of the Frey model, i is the index of the time, j is the index of the space. We have already known the value of v_j^i at time i due to the explicit method as computing the PDE at time $i-1$. Therefore, all the gird of call option value can be obtained by FDM easily. Figure 1 displays the basic concept of the implicit method.

---

[20] See Appendix, the implicit method is unconditional stable. Thus, we purpose to solve the Frey model via the implicit method.

[Insert Figure 1 here]

2.5 The smooth version of nonlinear PDE

We find a serious problem in this term $(1-\rho\lambda(S)Su_{SS})$ as we check the numerical data and find that this term violates the basic assumption "$\rho\lambda(S)Su_{SS}(t,S)<1$" in programming process. If we do not address this numerical problem, the nonlinear PDE will display a non-smooth solution in option intrinsic value and generating bad numerical solution. Therefore, using some skill in nonlinear PDE, we revise the violation of the basic assumption by the following form:

$$u_t+\frac{1}{2}\sigma^2S^2\left\{\max\left[\sqrt{\alpha_0},\frac{1}{1-\min(\alpha_1,\rho\lambda(S)Su_{SS})}\right]\right\}^2u_{SS}+rSu_S=ru\,. \tag{21}$$

In Frey and Patie (2001), they choose α_0=0.02 and α_1=0.85[21] for the smooth version of nonlinear PDE. Frey and Patie (2001) do not explain how the parameter would be selected and the detail of methodology does not appear in their paper. If we do not impose artificial conditions in the PDE model, the denominator of second coefficient[22] in Equation (21) could be greater than one. Frey and Patie (2001) provide this approach to settle the non-smooth numerical solution problem in option pricing. General speaking, the more smooth PDE we have, the more precise solution we get.

The second coefficient of PDE in Equation (21) is controlled by some of the factors and the numerical boundary value are governed by α_0 and α_1. We can treat α_0 and α_1 as the artificial condition or the barrier. We state that α_1 is the maximum value of $\rho\lambda(S)Su_{SS}$ and use the parameter α_1 to control $(1-\rho\lambda(S)Su_{SS})$ and thus

[21] Two artificial conditions are imposed into the PDE model.

[22] The term is $\min(\alpha_1,\rho\lambda(S)Su_{SS})$.

this term will not be negative anymore. Hence, we obtain smooth solution in the Frey model. Secondly, the parameter α_0 control the volatility term in nonlinear PDE. The term $\sigma^2\alpha_0^2$ can be explained by the minimum value of the volatility under illiquidity market condition. The smooth version of PDE proposed by Frey and Patie (2001) sounds great but this approach might affect the stability of the numerical solution. Since the disadvantage of this revised PDE is too artificial that designedly limit the numerical value in certain boundary to avoid the violation of the basic assumption. We test the call option value, delta (i.e., first derivative) and gamma (i.e., second derivative) and finding the value of gamma will explode as $\rho > 0.4$. This phenomenon tells us that the smoothed version of PDE is absolutely not a unique approach as we calculate the second derivative. The second derivative represents the variation of the hedging strategy for large trader. Consequently, we should carefully use the smoothed version of PDE when we need to build up a hedging strategy in the certain period.

3. Numerical Result

Section 3 provides the numerical result of the Frey model. We compare the option pricing under different liquidity circumstances and compute the speed of programming with direct method and iterative method. Finally, we check the condition number of the linear system in each time step from “coarse grid” to “find grid.”

3.1 Option pricing under perfect liquidity market

In section 3.1, we set the parameter $\rho = 0$, which means that the Frey model is equivalent to the classical BS model. Our benchmark set of underlying parameters is reported in Table 1.

[Insert Table 1 here]

According to Table 1, we obtain that the exact solution of BS call option price is 7.9260. Moreover, we display the profile of the numerical solution in Figure 1 and verify the numerical solution of the Frey model with the BS model in perfect liquidity market. The terminal payoff and intrinsic value of call option are showed in Figure 1.

[Insert Figure 2 here]

3.2 Option pricing under imperfect liquidity market

In section 3.2, we implement the option pricing under illiquid market. The assumption and the parameter setting dose not change except the parameter ρ. We want to realize the call option value with different liquidity. In figure 2, we present the outcomes for a 1-year call option with other parameter setting for different values of the market liquidity ρ ranging from 0 to 0.5. Figure 2 shows that the relationship between ρ and call option value. However, we observe that the large trader will spend more money to hedge a call option under worse market liquidity condition. Thus, the hedge cost of the large trader is increasing in the parameter ρ.

[Insert Figure 3 here]

3.3 The computation speed of direct method and iterative method

One of the important issues is to explore the efficient method for solving the tridiagonal system. While there are several methods for solving the linear system, the iterative and direct methods are explored in our study. Jacobi, Gauss-Seidel (GS) and successive over-relaxation (henceforth, SOR) are most prevailing iterative method (indirect method). For direct method, the Gauss-eliminate type is the basic routines for solving linear system and usually based on some forms of Gaussian elimination with pivoting. The LU decomposition and the Thomas algorithm are the most popular approach in direct method for solving the linear system. It notes worthy that the Thomas algorithm is the most efficient way for solving "the tridiagonal linear system." If the linear system is not tridiagonal, the Thomas algorithm is not suitable and can not be used in non-tridiagonal type system. The setting of parameters is the same as Table 1 and the parameter ρ is assumed to be zero.

Table 2 demonstrates some of results from numerical method and compares the computational speed in three different methods. First, the Thomas algorithm is apparently the fastest way to address the tridiagonal system and it can save a lot of time for our procedure. According to the result of the Table 2 is not significant in the column of "elapsed time" because the mesh grid is "coarse." If we increase the partition of time and (stock) space, the Thomas algorithm will obviously exhibit its computational power. Second, the disadvantage of GS method is time-consuming and the accuracy of the solution depends on the number of iterations. The detail of the Thomas algorithm will show in Appendix. Hence, we abandon the iterative method because it is time-wasted. In our point of view, the Thomas algorithm is most efficient and fastest method for solving the tridiagonal system but a majority of textbook still uses LU-decomposition to solve the tridiagonal system in each time step. As a result, the Thomas algorithm provides a great improvement in numerical scheme and it decreases the computational costs.

[Insert Table 2 here]

3.4 The condition number of tridiagonal system

The condition number measures the sensitivity of the linear system. As the size of the coefficient matrix increases, the condition number will increase and the solution of the linear system becomes sensitive to the numerical methods. Using a different way to take an inverse in the coefficient matrix will result in a different solution. Table 3 shows that the relationship between the condition number and market liquidity.

[Insert Table 3 here]

Obviously, the condition number $\kappa(A)$ is increasing in the liquidity parameter ρ when ΔS holds constant. On the other side, we increase the partition of the space M and the parameter ρ holds constant at the same time. In Table 4, the condition number grows about quadruple as ΔS decreases a half and thus we claim that the solution is stable. Consequently, we conclude the numerical solution of the Frey model is not sensitive to the size of the coefficient matrix because the relative ratio of condition number is quite stable. The following table displays the relative ratio of condition number.

[Insert Table 4 here]

In this section, we not only check the condition number of the linear system at each time step but also list the maximum value of the condition number in each time step. We consider that the sensitivity of the numerical solution is highly correlated with the condition number[23]. At beginning, we conjecture that the coefficient of matrix could generate a great influence on numerical solution of the option pricing model as the market liquidity becomes worse. However, we demonstrate that the condition number increases quadruply as the partition of the space increases double and thus the sensitivity of the solution is conducted.[24]

[23] See Trefethen and Bau (1997).

[24] We should be careful as doing the numerical analysis especially for the application of the option pricing model in finance issue.

4. Empirical Study

Bakshi, Cao and Chen (1997) not only provide an empirical study for the performance of alternative option pricing models but also demonstrate the detail of the estimation procedure which is the least square type estimation method. They implement each model by adapting this procedure. Hence, the following study for obtaining the liquidity parameter is based on the spirit of the estimation procedure. We present a test of the performance of the option pricing model under illiquidity for individual stock option prices for the sample period from January 1, 2000 through December 31, 2004. Our empirical study is based on call option contract's close quote price taken to be the bid-ask mid-point price and the rollover effect is conducted. Moreover, we check the integrity of the quotes and remove unreasonable data in our empirical work. First, we compute the Black-Scholes implied volatility (BS-IV) for each exercise price everyday and then the BS-IV with higher than 100% or lower than 0% are excluded in our dataset. Let $C_{i,j}$ denote the daily close price of options i at day j. In addition, we denote $\tilde{C}$ as call option prices from the Frey nonlinear PDE model. The liquidity parameter is estimated numerically with two stages by the following equations[25].

Stage 1:

$$\rho_N^* = \arg \min_{\rho} \sum_{j=1}^{N} \sum_{i=1}^{M_j} \left(\underbrace{\tilde{C}\left(S_j, K_{i,j}, T_j, r_j, BSIV_j^{avg}; \rho_{initial}, a_1, a_2, \alpha_0, \alpha_1\right)}_{\text{numerical solution from the Frey nonlinear PDE Model}} - \underbrace{C_{i,j}^{obs}}_{\substack{\text{option} \\ \text{prices}}} \right)^2 , \quad (22)$$

Stage 2:

[25] The parameter ρ, a_1 and a_2 should not be estimated simultaneously since it does not easily obtain the accurate value of three parameters at the same time and it is time-consuming for getting three parameters. Therefore, we only focus on the estimation of the liquidity parameter ρ rather than the other two in our empirical study.

$$\rho_{S+1}^{*} = \arg \min_{\rho} \sum_{j=S-N+2}^{S+1} \sum_{i=1}^{M_j} \left(\underbrace{\tilde{C}\left(S_j, K_{i,j}, T_j, r_j, BSIV_j^{avg}; \rho_S, a_1, a_2, \alpha_0, \alpha_1\right)}_{\text{numerical solution from the Frey nonlinear PDE Model}} - \underbrace{C_{i,j}^{obs}}_{\substack{\text{option}\\ \text{prices}}} \right)^2 , \quad (23)$$

$$S = \{N, N+1, N+2, \cdots\cdots\cdots, n-1\} , \quad (24)$$

where the variable M_j presents the number of exercise quote price in each day and n presents the number of trading day. The above Equation (22) is a nonlinear least square (henceforth, NLS) estimator because the element $\tilde{C}$ is the numerical solution from the Frey nonlinear PDE model. On the other hand, we can also claim that the above Equation (22) is a type of estimation loss function and therefore the parameter ρ^* is a minimum (squared) distance estimator.

The following illustrates the detail procedure regarding the estimation of the liquidity parameter. First, we use the moving window method to do the whole estimation procedure. Second, we utilize the pattern search algorithm into the whole estimation procedure. Here the Pattern search algorithm can be employed for finding the minimum of objective function and obtaining the parameter ρ which correspond to the local minimum objective function. Third, we determine the number of day N in the moving window and then calculate BS-IV for each exercise price per day. After that we compute the arithmetic mean of BSIV after excluding the unreasonable value of the BS-IV. We use the data of the first N days to determine the parameter of the $N+1$-th day. i.e., if we select N =2 for the number of the moving windows which means that we use the first two days' observations to determine the parameter of the third day. Hence, the third parameter represents the first parameter that we estimated in our empirical study and then we use the same approach for getting the rest of parameters. As a result, we obtain a sequence of the parameter $\tilde{\rho}^*$:

$$\tilde{\rho}^* = (\rho_{N+1}^*, \rho_{N+2}^*, \cdots, \rho_n^*) , \quad (25)$$

where n is the length of date in the sample. We use the pattern search algorithm to determine the optimal parameter in the minimization of the square of the difference between the theoretical price and the observed price. We propose the pattern search algorithm for estimating liquidity parameter of the Frey model. The pattern search algorithm is a popular approach in optimization especially for solving bound constrained nonlinear programs, linear unconstrained problem and some kind of minimization problem[26].

However, we should give an initial-guess value, the lower bound and the upper bound of the parameter in this algorithm. Since the pattern search algorithm can not obtain the global minimum of the objective function. Hence, we set a vector of initial-guess which can be applied to find every optimal parameter for every different initial-guess value and then we pick up the most appropriate parameter[27] to be the first estimator. After we get the first estimator, the rest of estimation procedures are the same except we impose the former estimator ρ_S^* to be the initial-guess value as we estimate ρ_{S+1}^* in the rest of estimation procedure.

The empirical study can be divided into three parts. In the first part, we focus on the analysis of the bear market in short term period and then investigate the fitting ability of the Frey model from January 2002 to December 2002. Furthermore, we compare the pricing error ($MAE) with the BS and the Frey model. In the second part, we select a number of companies to be the sample of our empirical study and choice the period of time from 2000 to 2004. We want to figure out the practicability of the Frey model regarding the Top 20 of the average daily volume (ADV) of the underlying in stock option market and also make a comparison of the Top 20 with

[26] If the reader have more interest in the pattern search algorithm, you might search related article or take a look at the MATLAB help file where provide the concise programming code and the condensed introduction.

[27] We denote that the most appropriate parameter corresponds to the smallest objective function.

other companies which have worst liquidity than the Top 20. In the last part, we not only verify the suitability of the estimation loss function but also check the validation of the Frey model via a variety of loss function.

4.1 In bear market

The underlying asset IBM is arbitrarily selected by us. We observe the trajectory of the IBM stock price and find the underlying asset having the down trend phenomenon in 2002. According to the market microstructure theory, we argue that the bear market often occur the market illiquidity. Thus, we have more interest in the fitting ability of the Frey model especially when illiquid market happened. Figure 4 displays the trajectory of the underlying stock price of the IBM Company from 2000 to 2004 and only 2002 respectively and Table 5 reports the descriptive statistics of the sample from January 2002 to December 2002.

[Insert Figure 4 here]

[Insert Table 5 here]

The at-the-money (ATM) call options are only used in our empirical study because we consider that the OTM and ITM options are not suitable for the analysis of the pricing error. There are several empirical studies showing that the pricing error will generate more bias result from the volatility smile and skew pattern.

[Insert Table 6 here]

In Table 6, we compute the theoretical price of the BS model and the Frey model. First, we compare the mean of the ATM option pricing error of the IBM Company in

2002 and showing the numerical result with the different N and UB and given the fixed LB. According to the numerical result of the Table 6, the pricing of the Frey model is smaller than the BS model significantly and the result of pricing error is nothing to do with the upper bound of the parameter. Moreover, we find the number of the moving window does not have great impact on the pricing. Thus, the second part of the empirical study set $N=1$[28]. If the number of the moving window is given, no matter what the setting of UB, the pricing error of two option pricing model is not change almost surely. In Figure 5, we show that the profile of the liquidity parameter estimated from the Equation (23) & (24) by the pattern search algorithm. Obviously, we observe that the liquidity parameter is quite stable[29] so that we obtain accurate and reliable result from the pricing error of two option pricing model. Figure 6 shows the graph of the stock option price and the pricing error respectively.

[Insert Figure 5 here]

[Insert Figure 6 here]

4.2 The performance of the Top 20 & other companies

We select a number of companies which is listed on Chicago Board Options Exchange (CBOE). Moreover, we pick up the Top 20 active stock options in CBOE and also select less liquidity companies for our analysis. We want to figure out the fitting ability of the Frey model with respect to differ underlying stock options with different liquidity state. Table 7 reports the symbol and the name of the sample in the Top 20

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[28] The more N we set, the more computational time we need. However, the length of the moving window impact on the computational cost significantly but it does not affect the result of the pricing error of two option pricing model in our empirical study. As a result, we denote $N=1$ in the following analysis.

[29] The graph of liquidity parameter seems like unsmooth and volatile result from the scale of the vertical axis. In fact, the liquidity estimator is quite smooth as we readjust the range of the vertical axis.

active stock options and the less liquidity stock options. Table 8 provides the descriptive statistics of 27 stock options from Jan 2000 to Dec 2004 and the sample of the option prices are classified by moneyness and can be divided into three categories, respectively. We might notice that CE and XMSR are excluded in our empirical work due to their sample period are less than the length of the period from Jan 2000 to Dec 2004. However, YAHOO is also excluded in our dataset result from the first day of the implied volatility (IV) is not available and it can not be replaced by the former value of the IV. Hence, we ignore three improper samples for the following study.

[Insert Table 7 here]

[Insert Table 8 here]

The Frey model displays unexpected fitting ability and it can track the asset dynamics for every stock option. Moreover, all of the stock options have pass through the $MAE of the pair t-test except the AMR. Although the pricing error of the Frey model 0.3494 is smaller than the BS model's 0.4447, the outcome of the pair t-test is not significant result from the option price of the AMR have two jump phenomena happened on March 2000 and January 2001 respectively[30].

[Insert Table 9 here]

[Insert Figure 7 here]

If we eliminate the suspicious sample data which have jumps, the pricing error of the Frey model still significantly differs from the BS model in the AMR stock option

[30] The jump effect is not the consideration of the Frey model since we can not significantly distinguish the Frey from the BS when jump phenomenon happened. Therefore, we get a very reasonable consequent on the underlying asset of the AMR.

during the period from February 2001 to December 2004. Hence, we conclude that the Frey model exhibits a gorgeous practicability for the stock option and it obtains more precise solution than the BS model especially for illiquid market.

4.3 The loss function

Christoffersen and Jacobs (2004) emphasize the consistency in the choice of the loss function is important. If a theoretical model is implemented using an inappropriate estimation loss function, then the more mean squared error (MSE) we get. In the following analysis, we introduce many loss functions for investigating the accurate estimation of the liquidity parameter when evaluating the Frey model. We compare the value of MSE between the Frey model and the BS model since the loss function can be treated as the criteria of the model selection.

There are many loss functions are employed in literature and practice. First, the traditional loss function is composed of the dollar loss function and the percentage loss function and those can be divided into two categories respectively. Thus, mean squared dollar errors ($MSE), mean absolute dollar errors ($MAE), mean squared percentage errors (%MSE) and mean absolute percentage errors (%MAE) can be defined as

$$\$MSE = \frac{1}{n}\sum_{i=1}^{n}\left(\tilde{C}_i^{model} - C_i^{obs}\right)^2 , \tag{26}$$

$$\$MAE = \frac{1}{n}\sum_{i=1}^{n}\left|\tilde{C}_i^{model} - C_i^{obs}\right| , \tag{27}$$

$$\%MSE = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\tilde{C}_i^{model} - C_i^{obs}}{C_i^{obs}}\right)^2 , \tag{28}$$

$$\%MAE = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{\tilde{C}_i^{model} - C_i^{obs}}{C_i^{obs}}\right| , \tag{29}$$

where $\tilde{C}_i$ and C_i are the model call option prices and the observations respectively.

Secondly, we introduce the implied volatility loss function; that is, the implied volatility MSE and it also can be defined by

$$IVMSE \triangleq \frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\sigma}_i - \sigma_i\right)^2 , \tag{30}$$

where the implied volatility are

$$\tilde{\sigma}_i = BS^{-1}\left(\tilde{C}_i, S_i, K_i, T_i, r_i\right), \tag{31}$$

$$\sigma_i = BS^{-1}\left(C_i, S_i, K_i, T_i, r_i\right), \tag{32}$$

and BS^{-1} is the inverse of the BS formula, the rest of inputs in the BS model are also given as we calculate the implied volatility MSE.

[Insert Table 10 here]

Table 10 reports the results of the loss functions of two models with respect to 24 samples. The traditional loss functions show that the Frey model apparently performs better than the BS model and the IVMSE displays that the Frey model somewhat better than the BS model. Since jump phenomena are found by the trajectory of the implied volatility in the sample of AMR and WDC, the Frey model reduces the fitting ability per se. However, the rest of sample still exhibit a fabulous performance in the Frey model with smaller MSE or MAE and therefore we identify the performance of two theoretical option pricing models. Unquestionably, the Frey model shows that it can capture more the pattern of the market than the BS model by tracking the trajectory of the underlying asset. Furthermore, the estimation method of the liquidity is checked and it can obtain an accurate estimator by NLS method with respect to most of sample. Thus, we not only claim that the choice of the loss function is appropriate but also obtain the reliable results in our empirical work.

5. Conclusion

This paper considers the pricing model of options under illiquidity. According to the model proposed by Frey and Patie (2001), we explore a new numerical approach for solving the nonlinear PDE rather than the Newton method. Furthermore, we employ the Thomas algorithm for solving the tridiagonal system and propose the pattern search algorithm for getting the liquidity parameter, respectively. Therefore, the calculating process is not time-wasting by two algorithms. After a preliminary numerical study of the model, we apply it to stock call option prices for the sample period from January 1, 2000 to December 31, 2004. We demonstrate that the pricing error results from market illiquidity (the bear market) in the first part of empirical study. In the second part of empirical study, we enlarge the sample period and check the practicability of the Frey model for various companies which is listed on CBOE. The Frey model, for the most of sample, not only exhibits good outcomes regardless the length of the sample period but also presents excellent performance in illiquid market. The Frey model really represents a vital improvement with respect to the BS model in terms of pricing error and it provides a reasonable option pricing model for the pricing of a block order in terms of price impact. We argue that the serious pricing biases of the BS model can be explained by the nonlinear feedback effect and thus if the large trader uses the Frey model rather than the BS model, they could avoid unnecessary loss from the stock option market where illiquidity occurred.

In further research, the singularity separating method (SSM)[31] can be applied to the option pricing. Since the SSM is adopted, precise numerical solution can be obtained very quickly. The SSM method is proposed by You-Lan Zhu who improves

[31] Some of article says singularity removing transformation (SRT) method but they offer the same concept regarding the numerical method.

the terminal condition of the option[32]. There is a singularity point in the terminal payoff as the stock price equals to the exercise price and thus numerical solution will have a bad accuracy and reduced convergence rate around the singularity point even though the numerical solution will become smooth finally. However, we do not use the SSM method into our numerical scheme. If we want to get more rigorous solution of the PDE, we should adopt the SSM in our frameworks.

Recently, the field of the computational finance grows up quickly. We believe that the Frey model can be calculated by other numerical schemes but until now we consider that the new approach of mine is the fastest way for solving the Frey model. Accordingly, the large trader who uses the Frey model obtaining the more accurate theoretical price and the fastest way in a short term period so that they make the right strategy immediately and establish the optimal position in the market.

[32] See Zhu, Wu and Chern (2004)

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Appendix

A.1 The Thomas Algorithm

An efficient algorithm for the solution of tridiagonal system is based on Gaussian elimination with the coefficient of the diagonal elements scaled to one at each stage. The Thomas algorithm takes advantage of the zero elements that already present in the coefficient matrix and avoids unnecessary computational operations. Therefore, we need to store only the non-zero elements and save a lot of memory in programming process.

The Thomas Algorithm is an adaptation of the LU Decomposition (Gaussian elimination) idea to solution of a linear system with tridiagonal or band diagonal coefficient matrix. The fundamental linear system can be written in the form

$$Ax = b$$

Where x denote the unknown vector and b denote the right hand side (RHS) vector in this linear system. Here, the coefficient matrix A is full of zeros except for the diagonal, the super-diagonal and the sub-diagonal factors, as shown below

$$\begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & \ddots & \vdots \\ 0 & \gamma_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha_{n-1} & \beta_{n-1} \\ 0 & \cdots & 0 & \gamma_{n-1} & \alpha_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

The size of the coefficient matrix A is $n \times n$. When the matrix is extremely large (e.g., the dimension of A is 10000×10000), a considerable obviously reduction in memory requirements and increase in programming speed can be acquired by storing the nonzero elements in three vectors instead of saving the whole matrix A. By the way, this algorithm takes only $O(n)$ operations and the entire routine can be solved very fast, accurately and concisely. Finally, the Thomas algorithm works well when the coefficient matrix A is diagonally dominant. We only introduce the Thomas algorithm for solving the tridiagonal system in this section. However, the detail of the Thomas algorithm or others can refer to the related textbooks or papers[33].

A.2 The Comparison of the heat equation and the BS-PDE

Scheme	Heat Equation.	Heat Equation Stability Condition[34]	Local Truncation error	Black-Scholes PDE (BS-PDE)	BS-PDE Stability Condition[35]
FTCS	Explicit	$v \leq 0.5$	$O(\Delta t) + O(\Delta x^2)$	Implicit	unconditional stable
BTCS	Implicit	Unconditional stable	$O(\Delta t) + O(\Delta x^2)$	Explicit	$v \leq \frac{1}{\sigma^2 S^2}$

Note: The implicit method presents an unconditional stable property since we use it for solving the Frey model and obtaining the theoretical option prices. The forward difference in time and central difference in space is abbreviated as FTCS. Similarly, the backward difference in time and central difference in space is abbreviated as BTCS. The mesh ratio v is denoted as $\frac{\Delta t}{(\Delta x)^2}$.

33 See Fausett (2002).
34 See Higham (2004), chapter 23 and 24.
35 See Wilmott (2000), second volume, chapter 63.

Tables

Table 1: The basic parameter setting in option pricing

This table presents the basic parameter setting in the option pricing model and then we employ the finite difference method (FDM) for computing the theoretical option price which is based on the following parameter setting.

$S_0 = 50$	$r = 0$	$T = 1$	$\sigma = 0.4$
$X = 50$	$S_{\max} = 100$	$\Delta S = 0.5$	$\Delta t = 0.01$
$a_1 = 0$	$a_2 = 0$	$\alpha_0 = 0.02$	$\alpha_1 = 0.85$

Table 2: The comparison of the computational speed with different method

This table compares the different method for solving the linear system. Obviously, the Thomas algorithm displays the efficiency of the numerical scheme per se and therefore we use the Thomas algorithm in the following numerical analysis. However, the elapsed time is based on the following system requirement. The operation system of the computer is Windows XP SP2, CPU is AMD Athlon 1800+ 1.54 GHz and RAM is 768 MB. The speed of computation might rely on the different operation system, and it could be accelerating the computational speed and improving the efficiency as we using a better-level computer than our system requirement currently.

Method	Call Option Price	Elapsed Time (sec)	Category
X=inv(A)*b	7.9136	9.172	
X=A\b ("backslash operator")	7.9136	7.234	
Thomas algorithm	7.9136	6.281	direct method
LU-decomposition	7.9136	8.125	direct method
Gauss-Seidel (GS)	7.9134	79.344	iterative method

Note: the syntax is based on MATLAB.

Table 3: The condition number of tridiagonal system

This table shows the relationship between the condition number of the tridiagonal linear system and the market liquidity. Obviously, the condition number $\kappa(A)$ is increasing in the liquidity parameter ρ when ΔS holds constant.

		condition number $\kappa(A)$					
ΔS	M	$\rho = 0$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$
2	50	7.70	78.82	263.98	310.61	346.66	365.08
1	100	29.66	402.90	1155.90	1449.90	1613.50	1653.80
0.5	200	120.50	1536.10	5331.40	6204.70	6888.90	7543.30
0.25	400	491.48	6755.90	22713.00	26979.00	28570.00	30525.00

Table 4: The relative of the condition number

This table displays the relative of the condition number. As the size of the coefficient matrix increase twice, the ratio of the condition number is close to four. Thus, we consider that the solution of the linear system is not sensitive and is reliable.

$\rho = 0$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$
3.8512	5.1117	4.3787	4.6679	4.6544	4.5300
4.0624	3.8126	4.6123	4.2794	4.2695	4.5612
4.0787	4.3981	4.2602	4.3482	4.1473	4.0466

Table 5: The Descriptive Statistics of the IBM

This table provides the descriptive statistics of the underlying sample, IBM, and the sample of the option prices can be classified by moneyness and divided into three categories, respectively. In this table, we only focus on the analysis of the bear market circumstance and thus the sample period is selected from January 2002 to December 2002.

	IBM 2002Jan-2002Dec		
	Average call price	Standard error	Number of contracts
ITM	22.9692	0.3637	1691
ATM	2.4345	0.0710	256
OTM	0.1204	0.0057	2374

Note: moneyness denotes S/K , ATM denotes $0.97 < S/K < 1.03$, ITM denotes $1.03 \le S/K$ and OTM denotes $0.97 \ge S/K$.

Table 6: The pricing errors of the BS model and the Frey model

This table demonstrates the dollar absolute pricing errors ($MAE) with the different upper bound (UB) of the liquidity parameter and the different N. The Frey model shows smaller pricing error than the BS model significantly. However, we observe that the pricing error of two models are not sensitive with respect to UB and N. Thus, we let N =1 and UB=0.4 in the following empirical analysis and it can save a lot of time in the estimation of the liquidity as N =1. Furthermore, we relax the upper bound of the liquidity parameter in numerical scheme so that the liquidity parameter estimator is more closing to the real situation of the market.

Underlying stock: IBM 2002Jan-2002Dec					
N	BS	Frey	t-statistic	Std Dev	p-value
Panel A: UB = 0.1					
1	0.9361	0.2339	20.6333	0.3797	< 0.0001
2	0.9379	0.2618	19.8031	0.3802	< 0.0001
3	0.9340	0.2728	19.3945	0.3789	< 0.0001
4	0.9363	0.2807	19.1376	0.3800	< 0.0001
Panel B: UB = 0.2					
1	0.9361	0.2341	20.6183	0.3799	< 0.0001
2	0.9379	0.2618	19.8031	0.3802	< 0.0001
3	0.9340	0.2728	19.3945	0.3789	< 0.0001
4	0.9363	0.2807	19.1376	0.3800	< 0.0001
Panel C: UB = 0.3					
1	0.9361	0.2341	20.6183	0.3799	< 0.0001
2	0.9379	0.2618	19.8031	0.3802	< 0.0001
3	0.9340	0.2728	19.3945	0.3789	< 0.0001
4	0.9363	0.2807	19.1376	0.3800	< 0.0001
Panel D: UB = 0.4					
1	0.9361	0.2341	20.6183	0.3799	< 0.0001
2	0.9379	0.2618	19.8031	0.3802	< 0.0001
3	0.9340	0.2728	19.3945	0.3789	< 0.0001
4	0.9363	0.2807	19.1376	0.3800	< 0.0001

Table 7: The sample of the underlying stock options

This table displays the sample of the underlying stock options with two categories and the ranking order depends on the size of the total average daily volume (ADV) and the relative information of the ADV is listed on COBE official website.

Rank	Symbol	Name	Category
1	INTC	Intel Corporation	TOP 20
2	TWX	Time Warner, Inc.	TOP 20
3	GE	General Electric Company	TOP 20
4	MSFT	Microsoft Corporation	TOP 20
5	MO	Altria Group, Inc.	TOP 20
6	CE	Concord EFS, Inc.	TOP 20
7	CSCO	Cisco Systems, Inc.	TOP 20
8	JPM	J.P. Morgan Chase & Co.	TOP 20
9	XMSR	XM Satellite Radio Holdings, Inc	TOP 20
10	C	Citigroup, Inc.	TOP 20
11	HPQ	Hewlett-Packard Company	TOP 20
12	ORCL	Oracle Corporation	TOP 20
13	WMT	Wal-Mart Stores, Inc.	TOP 20
14	EP	El Paso Corporation	TOP 20
15	GM	General Motors Corporation	TOP 20
16	DELL	Dell Computer Corp.	TOP 20
17	IBM	International Business Machines Corporation	TOP 20
18	YHOO	Yahoo! Inc	TOP 20
19	BAC	Bank of America Corporation	TOP 20
20	QCOM	QUALCOMM, Inc.	TOP 20
21	BBY	Best Buy Co., Inc.	Other
41	AMR	AMR Corporation	Other
61	WFC	Wells Fargo & Company	Other
81	WDC	Western Digital Corporation	Other
151	SNPS	Synopsys, Inc.	Other
161	BAX	Baxter International, Inc.	Other
181	HDI	Harley-Davidson, Inc.	Other

Table 8: The Descriptive Statistics of the sample

This table provides the descriptive statistics of 27 stock options from Jan 2000 to Dec 2004 and the sample of the option prices are classified by moneyness and are divided into three categories, respectively.

Moneyness	Mean	Std Err	Number	Mean	Std Err	Number	Mean	Std Err	Number
		INTC			HPQ			QCOM	
ITM	15.4374	0.1688	8630	12.0888	0.1839	5290	20.3920	0.1618	9751
ATM	2.3064	0.0706	912	2.9236	0.1053	781	2.9065	0.0743	966
OTM	0.1891	0.0051	10283	0.3007	0.0091	7845	0.3622	0.0080	11862
		TWX			ORCL			BBY	
ITM	13.2076	0.1386	9147	11.5472	0.1682	6533	15.6534	0.1092	9161
ATM	1.5013	0.0421	952	1.8708	0.0971	643	2.1737	0.0412	945
OTM	0.1261	0.0029	13863	0.2229	0.0090	9549	0.258	0.0058	7943
		GE			WMT			AMR	
ITM	11.2768	0.1112	7579	11.5798	0.0837	7061	5.4963	0.065	4369
ATM	1.6954	0.0485	1114	1.5231	0.0263	894	1.0598	0.0778	544
OTM	0.1316	0.0041	7456	0.1261	0.0034	5730	0.5762	0.0297	6527
		MSFT			EP			WFC	
ITM	17.0212	0.1372	9875	7.8182	0.1142	4399	9.7272	0.0861	4916
ATM	1.9208	0.0463	974	1.3309	0.0375	606	1.1482	0.0210	793
OTM	0.1264	0.0029	13517	0.1321	0.0035	5039	0.107	0.0032	3342
		MO			GM			WDC	
ITM	12.7872	0.1006	8150	10.6015	0.0960	5486	3.1452	0.0542	1830
ATM	1.1598	0.0173	1105	1.5204	0.0281	1058	0.352	0.0136	178
OTM	0.1012	0.0023	6341	0.1478	0.0036	6347	0.0795	0.0012	3766
		CSCO			DELL			SNPS	
ITM	13.1432	0.1521	7673	9.7914	0.0816	7105	10.1935	0.1190	4089
ATM	2.2815	0.0798	653	1.2041	0.0227	904	1.9231	0.0376	729
OTM	0.1598	0.0045	10592	0.1369	0.0031	7032	0.3284	0.0082	3904
		JPM			IBM			BAX	
ITM	9.7623	0.0802	5790	23.5886	0.1395	9297	10.1935	0.119	4089
ATM	1.3339	0.0325	894	3.0174	0.0445	1435	1.9231	0.0376	729
OTM	0.1489	0.0040	5521	0.2510	0.0057	9879	0.3284	0.0082	3904
		C			BAC			HDI	
ITM	10.7200	0.0670	7988	13.0720	0.0945	6324	9.673	0.0833	4345
ATM	1.2983	0.0210	1185	1.5334	0.0236	1107	1.5253	0.0277	867
OTM	0.1190	0.0030	5728	0.1420	0.0036	4906	0.231	0.0063	3110

Table 9: The pricing errors of the BS model and the Frey model

This table provides the empirical results of the pricing errors with respect to two different option pricing models in $MAE sense. Undoubtedly, the Frey model produces smaller pricing error than the BS model significantly and there is only one sample, AMR, which can not distinguish the Frey model from the BS model via $MAE.

Jan 2000 - Dec 2004						
Rank	Symbol	BS $MAE	Frey $MAE	t-statistic	Std Dev	p-value
1	INTC	0.3895	0.1343	24.9263	0.2556	< 0.0001
2	TWX	0.3998	0.1264	29.1550	0.2341	< 0.0001
3	GE	0.0518	0.0369	3.2271	0.1157	0.0013
4	MSFT	0.9024	0.2437	50.1385	0.3279	< 0.0001
5	MO	0.6302	0.1855	38.9870	0.2847	< 0.0001
7	CSCO	0.3466	0.1547	16.9676	0.2823	< 0.0001
8	JPM	0.4573	0.1411	33.0852	0.2383	< 0.0001
10	C	0.6196	0.2030	35.8759	0.2895	< 0.0001
11	HPQ	0.3390	0.1449	17.3203	0.2795	< 0.0001
12	ORCL	0.2307	0.1178	11.7090	0.2405	< 0.0001
13	WMT	0.7026	0.2347	34.0733	0.3428	< 0.0001
14	EP	0.2557	0.0955	18.7780	0.2129	< 0.0001
15	GM	0.5643	0.1478	36.7269	0.2831	< 0.0001
16	DELL	0.3636	0.1230	30.3502	0.1979	< 0.0001
17	IBM	1.1160	0.3569	33.4107	0.5670	< 0.0001
19	BAC	0.7314	0.2228	33.2126	0.3823	< 0.0001
20	QCOM	0.5068	0.2443	17.3666	0.3773	< 0.0001
21	BBY	0.4812	0.2020	24.3921	0.2857	< 0.0001
41	AMR	0.4447	0.3494	0.9267	2.5678	0.3542
61	WFC	0.5254	0.1717	27.8892	0.3166	< 0.0001
81	WDC	0.1254	0.0921	6.4311	0.1293	< 0.0001
151	SNPS	0.3259	0.1577	19.6413	0.2139	< 0.0001
161	BAX	0.5640	0.3686	10.5374	0.4628	< 0.0001
181	HDI	0.3823	0.3518	2.4087	0.3157	0.0161

Note: the pricing error is calculated by the mean absolute dollar errors ($MAE).

Table 10: The comparison of the loss function

Model	$MAE	$MSE	%MAE	%MSE	IV_MSE	$MAE	$MSE	%MAE	%MSE	IV_MSE
	INTC					GM				
BS	0.3895	0.2430	0.4493	0.6340	0.0250	0.5643	0.4567	0.6179	1.0591	0.0259
Frey	0.1343	0.0573	0.1265	0.0594	0.0037	0.1478	0.0438	0.1782	0.1390	0.0027
	TWX					DELL				
BS	0.3998	0.2445	0.5739	0.9731	0.0318	0.3636	0.1950	0.4910	0.6063	0.0232
Frey	0.1264	0.0408	0.1776	0.1600	0.0051	0.1230	0.0306	0.1574	0.0837	0.0038
	GE					IBM				
BS	0.0518	0.0248	1.0508	1.7812	0.2079	1.1160	1.7377	0.6738	1.3286	0.0269
Frey	0.0369	0.0060	0.9417	0.9224	0.4560	0.3569	0.2776	0.2287	0.2419	0.0042
	MSFT					BAC				
BS	0.9024	0.9969	1.1259	5.8110	0.0602	0.7314	0.7738	0.9388	5.6094	0.0263
Frey	0.2437	0.0917	0.4297	1.3535	0.0093	0.2228	0.1028	0.3243	0.9943	0.0038
	MO					QCOM				
BS	0.6302	0.5269	0.9831	3.4754	0.0364	0.5068	0.4111	0.3454	0.3471	0.0198
Frey	0.1855	0.0666	0.3353	0.5740	0.0052	0.2443	0.1899	0.1274	0.0589	0.0142
	CSCO					BBY				
BS	0.3466	0.2109	0.6788	1.9420	0.0332	0.564	0.5571	0.3877	0.3863	0.0323
Frey	0.1547	0.0925	0.2358	0.2883	0.0071	0.3686	0.325	0.2057	0.0918	0.0163
	JPM					AMR				
BS	0.4573	0.3000	0.7047	1.9744	0.0264	0.4447	6.7816	0.372	0.3915	0.4365
Frey	0.1411	0.0426	0.2386	0.3501	0.0037	0.3494	6.7151	0.2059	0.1227	0.4525
	C					WFC				
BS	0.6196	0.5197	0.8433	2.2307	0.0323	0.5254	0.434	1.4964	15.6398	0.0247
Frey	0.2030	0.0729	0.3150	0.4883	0.0050	0.1717	0.0718	0.5514	2.7546	0.0042
	HPQ					WDC				
BS	0.3390	0.1952	0.5720	1.5650	0.0319	0.1254	0.0382	0.6473	1.0471	0.1025
Frey	0.1449	0.0968	0.1870	0.2403	0.0052	0.0921	0.0194	0.4417	0.3693	0.1908
	ORCL					SNPS				
BS	0.2307	0.1134	0.4485	0.5264	0.027	0.3259	0.1636	0.381	0.8984	0.0167
Frey	0.1178	0.0694	0.1559	0.0764	0.0061	0.1577	0.0588	0.1518	0.162	0.0042
	WMT					BAX				
BS	0.7026	0.6826	1.5098	14.8746	0.0345	0.564	0.5571	0.3877	0.3863	0.0323
Frey	0.2347	0.101	0.569	2.6561	0.0059	0.3686	0.325	0.2057	0.0918	0.0163
	EP					HDI				
BS	0.2557	0.1426	0.4010	0.4882	0.0313	0.3823	0.2624	0.5024	2.0671	0.0143
Frey	0.0955	0.0226	0.1489	0.0801	0.0113	0.1495	0.0481	0.1819	0.3581	0.0027

Figures

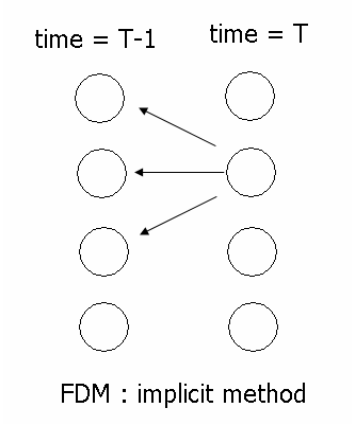

Figure 1: Implicit finite difference method

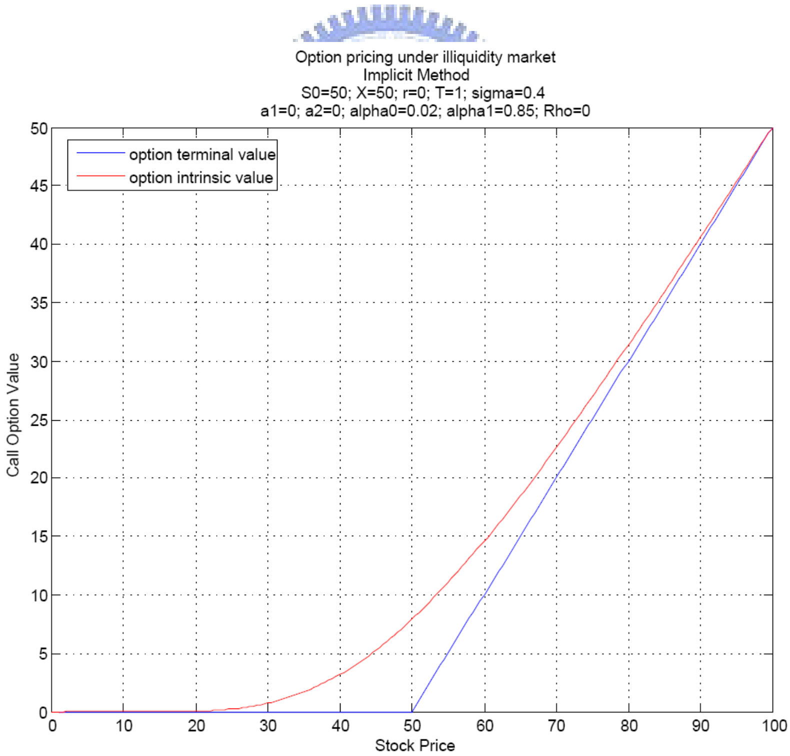

Figure 2: Option pricing under perfect liquidity market

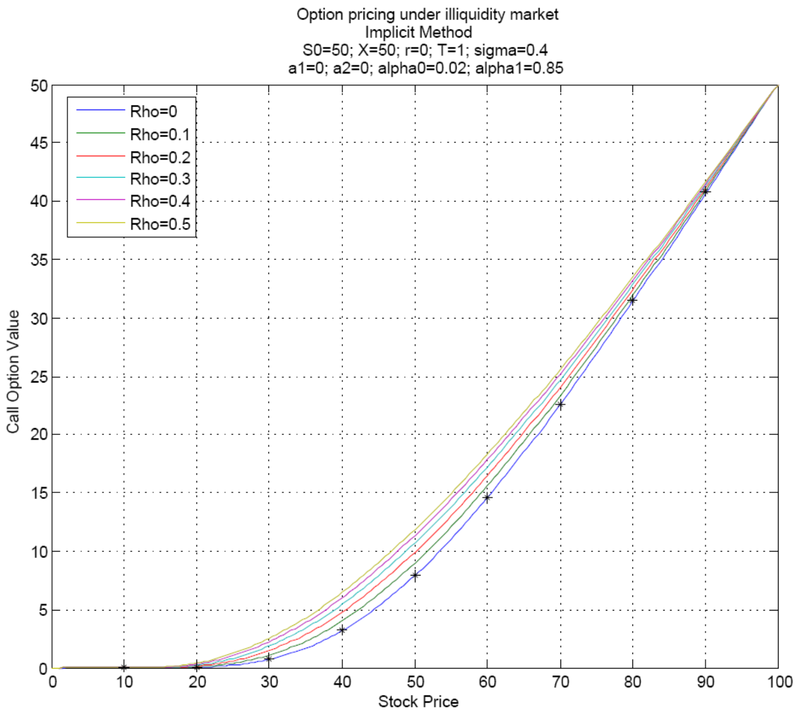

Figure 3: Option pricing under illiquidity market

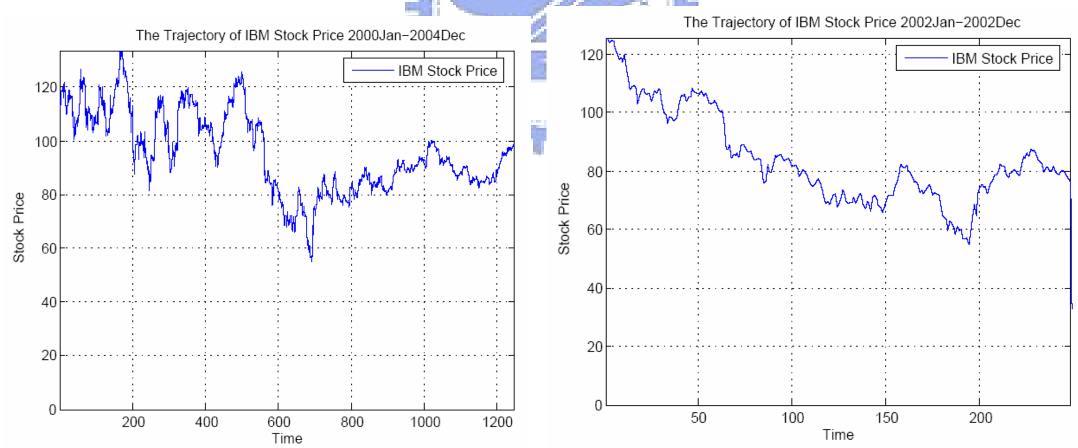

Figure 4: The trajectory of IBM stock price

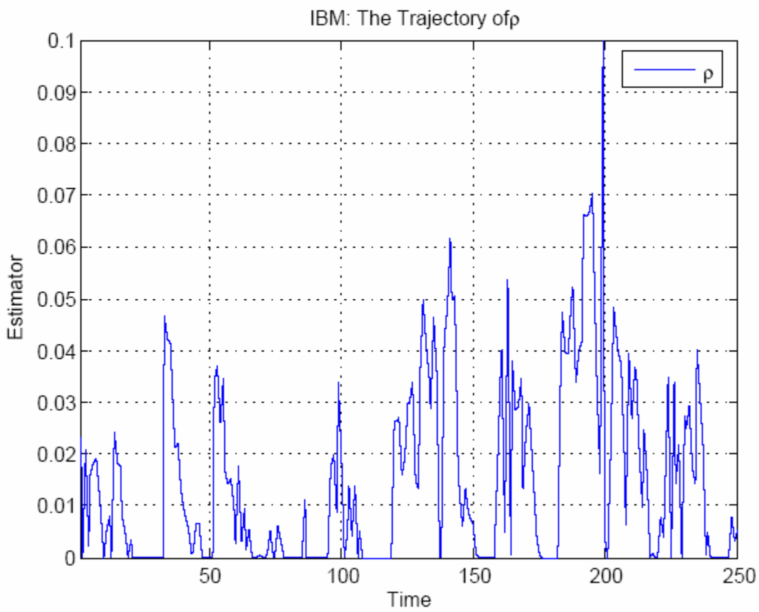

Figure 5: The trajectory of the liquidity parameter ρ

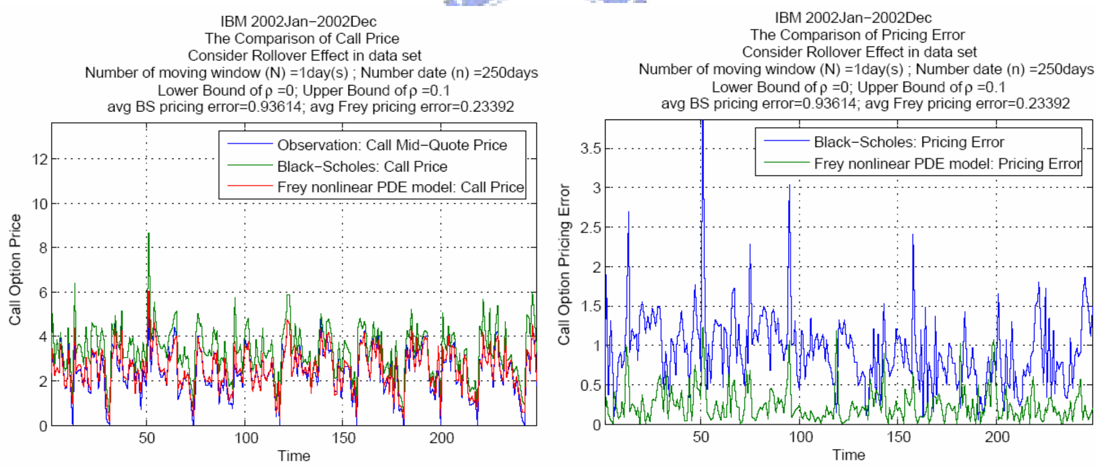

Figure 6: the option price and the pricing error of the IBM

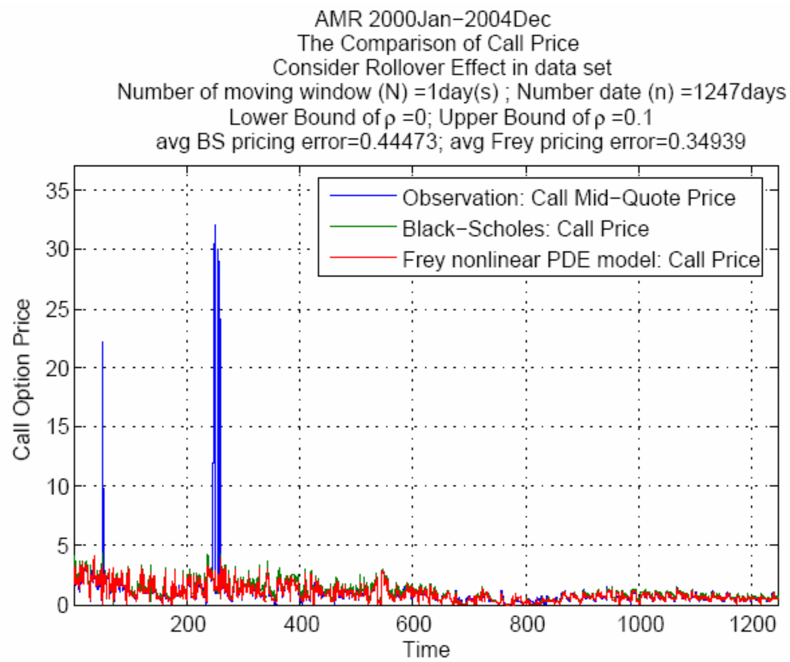

Figure 7: The stock option price of the AMR Company