Stochastic Analysis and Applications
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lsaa20

A Generalized Renewal Equation for Perturbed Compound Poisson Processes with Two-Sided Jumps

Yu-Ting Chen [a] & Yuan-Chung Sheu [b]
[a] Institute of Mathematics, Academia Sinica, Taipei, Taiwan
[b] Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan
Published online: 20 Aug 2009.

Stochastic Analysis and Applications, 27: 897–910, 2009

ISSN 0736-2994 print/1532-9356 online
DOI: 10.1080/07362990903136421

A Generalized Renewal Equation for Perturbed Compound Poisson Processes with Two-Sided Jumps

Yu-Ting Chen[1] and Yuan-Chung Sheu[2]
[1]Institute of Mathematics, Academia Sinica, Taipei, Taiwan
[2]Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan

Abstract: In this article, we study the discounted penalty at ruin in a perturbed compound Poisson model with two-sided jumps. We show that it satisfies a renewal equation under suitable conditions and consider an application of this renewal equation to study some perpetual American options. In particular, our renewal equation gives a generalization of the renewal equation in Gerber and Landry [2] where only downward jumps are allowed.

Keywords: Discounted penalty; Perpetual American option; Perturbed compound Poisson process; Renewal equation.

Mathematics Subject Classification (2000): 60J25; 60J75; 60G44.

1. INTRODUCTION

Consider a family of real-valued Markov processes $X = (X_t, \mathbb{P}_x)$ and let τ be the first-exit time of the process from the interval $(0, \infty)$ (i.e., $\tau = \inf\{t \geq 0, X_t \leq 0\}$). Given a bounded nonnegative Borel

Received October 25, 2007; Accepted March 21, 2009
This work was partially supported by National Center for Theoretical Science Mathematics Division and NSC 97-2628-M-009-014, Taiwan.
Address correspondence to Yuan-Chung Sheu, Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan; E-mail: sheu@math.nctu.edu.tw

function $g:\mathbb{R}_- \to \mathbb{R}_+$ and $r \geq 0$, the discounted penalty function is defined by

$$\Phi(x) \equiv \mathbb{E}_x\left[e^{-r\tau}g(X_\tau)\right], \tag{1.1}$$

with the convention that $e^{-r\cdot\infty} = 0$. Here, $\mathbb{E}_x[Z] = \int Z(\omega)d\mathbb{P}_x(\omega)$ for a random variable Z and we write $\mathbb{E}$ for $\mathbb{E}_0$ in the sequel.

The discounted penalty is considered in the insurance literature as a generalized notion of the ruin probability. It has been widely studied since the work of Dufresne and Gerber in [1]. In particular, Gerber and Landry gave a renewal equation for the discounted penalty in [2] when the Markov process X is a perturbed compound Poisson process which has no positive jumps and whose jump distribution is only subject to some mild technical conditions. They further showed that the optimal exercise boundary of an American option can be determined explicitly via this equation. This implies that the merit to derive such an equation is an advanced analysis of the discounted penalty function, given the fact that the explicit solution for the discounted penalty is unavailable in most cases.

In addition to the example given by Gerber and Landry [2], the discounted penalty function has been discussed widely in the literature of financial mathematics. See Mordecki [3], Hilberink and Rogers [4], Asmussen et al. [5], Alili and Kyprianou [6], Chen et al. [7], Lewis and Modecki [8], and many others. Except for Hilberink and Rogers [4] and Alili and Kyprianou [6], the authors worked under the classical framework of Merton [11] that the logarithm of the asset value process has two-sided jumps. In order to derive (semi-)explicit solutions for the discounted penalty function, one needs to restrict the jump distribution to some special class of probability distributions. Then, as shown by the authors, one can derive the desired solution for some large, but special, class of jump distributions via either differential equations or the fluctuation identities for Lévy processes.

The aim of this article is to study the discounted penalty in a perturbed compound Poisson model by another approach, *without the proviso that the jump distribution is either one-sided or follows a special distribution*. In Section 2, we follow Gerber and Landry [2] (see also Tsai and Wilmott [9]) and derive a generalized renewal equation for Φ (see Theorem 2.1 below). In particular, we have two byproducts of independent interests. First, we give an equation in which the first-order derivative of Φ at *zero* can be written as an integral of the function Φ; see Section 3 for an application of this equation to option pricing. (We refer to Chen et al. [7] for applications in credit risk modelling.) Second, by a suitable change of probability measures, we obtain an upper bound for the exponential decay rate for the function Φ. In Section 3, as an application of the main result(Theorem 2.1), we revisit the perpetual

American options considered in [2] and show that for some perpetual American options the smooth-pasting condition is equivalent to some first-order condition (see (3.10) below). By imposing the smooth-pasting condition, we derive an equation for the optimal levels for perpetual American options. When there are no upward jumps, we recover the explicit formula in Gerber and Landry [2] for the optimal level of a perpetual put option.

2. MAIN RESULT

From this point on, we assume that on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, there are a standard Brownian motion $W = (W_t; t \in \mathbb{R}_+)$ and a compound Poisson process $Z = (Z_t; t \in \mathbb{R}_+)$ with $Z_t = \sum_{n=1}^{N_t} Y_n$. Here $N = (N_t; t \in \mathbb{R}_+)$ is a Poisson process with parameter $\lambda > 0$ and $Y = (Y_n; n \in \mathbb{N})$ are independent and identically distributed with distribution F. We assume further that W, N, and Y are independent and the jump distribution F has a density. For every $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the law of the process

$$X_t = X_0 + ct + \sigma W_t - Z_t, \quad t \geq 0, \tag{2.1}$$

where $c \in \mathbb{R}$, $\sigma > 0$ and $X_0 = x$. Then the exponent of the Laplace transform of X is given by

$$\psi(\zeta) = \log \mathbb{E}[\exp \zeta X_1] = D\zeta^2 + c\zeta + \lambda \int e^{-\zeta y} dF(y) - \lambda, \tag{2.2}$$

where $D = \sigma^2/2$. Set

$$\Delta = \sup\left\{\zeta \in \mathbb{R}_+; \int e^{-\xi y} dF(y) < \infty, \forall \xi \in [0, \zeta]\right\}. \tag{2.3}$$

We assume throughout this paper that the **Lundberg's condition** holds for X:

$$\Delta > 0 \quad \text{and} \quad \lim_{s \uparrow\uparrow \Delta} \psi(s) > 0.$$

Under this condition, $\psi(0) - r = -r < 0$ given any $r > 0$ and ψ is strictly convex on $[0, \Delta)$, and hence there exists a *unique* number $\rho^* \in (0, \Delta)$ such that $\psi(\rho^*) - r = 0$. The number ρ^* is called the **Lundberg's constant** in literature. An implication of ρ^* is that $(e^{-rt+\rho^* X_t}; t \geq 0)$ is a martingale provided that $e^{\rho^* X_1}$ satisfies some integrability condition. (See Gerber and Landry [2].) Therefore, in risk-neutral pricing (that is, $\{V_t = e^{X_t}; t \geq 0\}$ is the price process of a security and $\{e^{-rt} V_t; t \geq 0\}$ is a martingale under a risk-neutral probability measure), we have $\rho^* = 1$.

In order to derive the renewal equation for Φ, we first give an explicit formula for the first order (right-hand) derivative of Φ at zero and sketch its proof. This formula is of independent interest and has many interesting applications. (For details, see Chen et al. [7] or Gerber and Landry [2]). In the sequel, we write $h_{\rho^*}(v) = e^{-\rho^* v}h(v)$ for any function h.

Proposition 2.1. *The derivative of* Φ *at* 0 *is given by*

$$\Phi'(0+) = \vartheta_0 - \frac{\lambda}{D}\int_{-\infty}^{0} dF(y)\int_0^{-y} dv\Phi_{\rho^*}(v)e^{-\rho^* y}, \tag{2.4}$$

where

$$\vartheta_0 = -\beta g(0) + \frac{\lambda}{D}\int_0^{\infty} dv\int_v^{\infty} dF(y)e^{-\rho^* y}g_{\rho^*}(v-y). \tag{2.5}$$

Sketch of Proof. Recall that the discounted penalty Φ in (1.1) satisfies the integro-differential equation

$$\frac{\sigma^2}{2}\Phi''(x) + c\Phi'(x) + \lambda\int_{-\infty}^{x}\Phi(x-y)dF(y)$$
$$+\lambda\int_x^{\infty} g(x-y)dF(y) - (\lambda + r)\Phi(x) = 0. \tag{2.6}$$

Moreover, if $r > 0$, both $\Phi(x)$ and $\Phi'(x)$ tend to zero as x goes to infinity. (For details and proofs, see Chen and Sheu [10].)

Multiplying both sides of the equation (2.6) by $e^{-\rho^* v}$ gives

$$e^{-\rho^* v}\left[D\frac{d^2}{dv^2} + c\frac{d}{dv} - (\lambda + r)\right]\Phi(v) + \lambda\int_{-\infty}^{\infty}\Phi(v-y)e^{-\rho^* v}dF(y) = 0. \tag{2.7}$$

Note that $\Phi(v) = e^{\rho^* v}\Phi_{\rho^*}(v)$ for $v \in \mathbb{R}$. Hence, (2.7) becomes

$$0 = D\Phi''_{\rho^*}(v) + \left(c + \rho^*\sigma^2\right)\Phi'_{\rho^*}(v) + \lambda\int_{-\infty}^{\infty}\Phi_{\rho^*}(v-y)e^{-\rho^* y}dF(y)$$
$$+\left[D(\rho^*)^2 + c\rho^* - (\lambda + r)\right]\Phi_{\rho^*}(v).$$

Recall that $\psi(\rho^*) = r$. Then

$$0 = D\Phi''_{\rho^*}(v) + \left(c + \rho^*\sigma^2\right)\Phi'_{\rho^*}(v) + \lambda\int_{-\infty}^{\infty}\Phi_{\rho^*}(v-y)e^{-\rho^* y}dF(y)$$
$$-\lambda\Phi_{\rho^*}(v)\int e^{-\rho^* y}dF(y). \tag{2.8}$$

Integrate (2.8) from $v = 0$ to $v = z$ gives

$$D[\Phi'_{\rho^*}(z) - \Phi'_{\rho^*}(0)] + (c + \rho^*\sigma^2)[\Phi_{\rho^*}(z) - \Phi_{\rho^*}(0)]$$
$$+\lambda\int_0^z dv\int_{-\infty}^{v} dF(y)\Phi_{\rho^*}(v-y)e^{-\rho^* y} - \lambda\int_0^z dv\int dF(y)\Phi_{\rho^*}(v)e^{-\rho^* y}$$
$$+\lambda\int_0^z dv\int_v^{\infty} dF(y)g_{\rho^*}(v-y)e^{-\rho^* y} = 0. \tag{2.9}$$

From this, we obtain

$$\begin{aligned}0 &= D[\Phi'_{\rho^*}(z) - \Phi'_{\rho^*}(0)] + (c + \rho^*\sigma^2)[\Phi_{\rho^*}(z) - \Phi_{\rho^*}(0)]\\ &\quad + \lambda\Bigg[\int_{-\infty}^{0} dF(y)\int_0^z dv\Phi_{\rho^*}(v-y)e^{-\rho^* y}\\ &\quad - \int_0^z dv\left(\int_{z-v}^{\infty} + \int_{-\infty}^{0}\right)dF(y)\Phi_{\rho^*}(v)e^{-\rho^* y}\\ &\quad + \int_0^z dv\int_v^{\infty} dF(y)e^{-\rho^* y}g_{\rho^*}(v-y)\Bigg]. \end{aligned} \tag{2.10}$$

Recall that $\Phi'(z)$ and $\Phi(z)$ tend to zero as $z \to \infty$. Letting $z \to \infty$ in the last equation gives

$$\begin{aligned}\Phi'_{\rho^*}(0) &= \frac{1}{D}\Bigg[-(c+\rho^*\sigma^2)g(0) - \lambda\int_{-\infty}^{0} dF(y)\int_0^{-y} dv\Phi_{\rho^*}(v)e^{-\rho^* y}\\ &\quad + \lambda\int_0^{\infty} dv\int_v^{\infty} dF(y)e^{-\rho^* y}g_{\rho^*}(v-y)\Bigg]. \end{aligned} \tag{2.11}$$

Since $\Phi'(0) = \rho^* g(0) + \Phi'_{\rho^*}(0)$, we get (2.4). □

To state our next result, we define a function G by:

$$G(v) = \begin{cases} \dfrac{\lambda}{D}\displaystyle\int_0^v dz e^{-\beta v}e^{\alpha z}\int_z^{\infty} dF(y)e^{-\rho^* y}, & v > 0,\\ \dfrac{\lambda}{D}\displaystyle\int_v^0 dz e^{-\beta v}e^{\alpha z}\int_{-\infty}^{z} dF(y)e^{-\rho^* y}, & v < 0, \end{cases} \tag{2.12}$$

where

$$\beta = \frac{c}{D} + \rho^* \quad\text{and}\quad \alpha = \beta + \rho^*. \tag{2.13}$$

Proposition 2.2. *The function Φ satisfies the equation*

$$\begin{aligned}\Phi(x) &= g(0)e^{-\beta x} + \lim_{n,m\to\infty}\left(\int_{-m}^{n}\Phi(x-v)G(v)dv\right.\\ &\quad \left. - e^{-\beta x}\int_{-m}^{n}\Phi(-v)G(v)dv\right), \quad x \in \mathbb{R}_+. \end{aligned} \tag{2.14}$$

Proof. We begin with (2.10). Based on the formula of $\Phi'_{\rho^*}(0)$ in (2.11), (2.10) is simplified to

$$D\Phi'_{\rho^*}(z) + (c+\rho^*\sigma^2)\Phi_{\rho^*}(z) + \lambda U(z) = 0, \tag{2.15}$$

where $U = U_1 + U_2 + U_3$ and

$$U_1(z) = \int_{-\infty}^{0} dF(y) \int_{0}^{-y} dv \Phi_{\rho^*}(v) e^{-\rho^* y} + \int_{-\infty}^{0} dF(y) \int_{0}^{z} dv \Phi_{\rho^*}(v-y) e^{-\rho^* y}$$
$$- \int_{0}^{z} dv \int_{-\infty}^{0} dF(y) \Phi_{\rho^*}(v) e^{-\rho^* y} = \int_{-\infty}^{0} dF(y) \int_{z}^{z-y} dv \Phi_{\rho^*}(v) e^{-\rho^* y},$$
$$U_2(z) = -\int_{0}^{\infty} dv \int_{v}^{\infty} dF(y) e^{-\rho^* y} g_{\rho^*}(v-y) + \int_{0}^{z} dv \int_{v}^{\infty} dF(y) e^{-\rho^* y} g_{\rho^*}(v-y)$$
$$= -\int_{z}^{\infty} dv \int_{v}^{\infty} dF(y) e^{-\rho^* y} g_{\rho^*}(v-y),$$
$$U_3(z) = -\int_{0}^{z} dv \int_{z-v}^{\infty} dF(y) \Phi_{\rho^*}(v) e^{-\rho^* y}.$$

We multiply both sides of (2.15) by the integration factor $e^{\alpha z}$, where α is defined in (2.13). Then

$$e^{\alpha z}\left[D\Phi'_{\rho^*}(z) + (c + \rho^* \sigma^2)\Phi_{\rho^*}(z)\right] = -\lambda e^{\alpha z} U(z). \tag{2.16}$$

On the other hand,

$$D(e^{\alpha z}\Phi_{\rho^*}(z))' = e^{\alpha z}[D\Phi'_{\rho^*}(z) + (c + \rho^* \sigma^2)\Phi_{\rho^*}(z)].$$

Then by (2.16), we obtain

$$D(e^{\alpha z}\Phi_{\rho^*}(z))' = -\lambda e^{\alpha z} U(z).$$

Integrating both sides of the last equation from $z = 0$ to $z = x$ yields

$$De^{\alpha x}\Phi_{\rho^*}(x) - Dg(0) = -\lambda \int_{0}^{x} e^{\alpha z} U(z) dz.$$

Divide both sides of the equation by $De^{\beta x}$, and we get

$$\Phi(x) = g(0)e^{-\beta x} - \frac{\lambda}{D} \int_{0}^{x} e^{-\beta x + \alpha z} U(z) dz.$$

Note by the definition of U,

$$\Phi(x) = g(0)e^{-\beta x} + \sum_{j=1}^{3} \frac{-\lambda}{D} \int_{0}^{x} e^{-\beta x + \alpha z} U_j(z) dz. \tag{2.17}$$

To complete the proof, we write each summand into the desired form. Using the results of Gerber and Landry [2, Equations (10) and (12)], we have

$$-\frac{\lambda}{D} \int_{0}^{x} e^{-\beta x + \alpha z} U_3(z) dz = \int_{0}^{x} \Phi(v) G(x-v) dv = \int_{0}^{x} \Phi(x-v) G(v) dv. \tag{2.18}$$

Next, we show that

$$
\begin{aligned}
&-\frac{\lambda}{D}\int_0^x e^{-\beta x+\alpha z}U_1(z)dz \\
&\quad = \lim_{m\to\infty}\left(\int_{-m}^0 \Phi(x-v)G(v)dv - e^{-\beta x}\int_{-m}^0 \Phi(-v)G(v)dv\right), \qquad (2.19)\\
&-\frac{\lambda}{D}\int_0^x e^{-\beta x+\alpha z}U_2(z)dz \\
&\quad = \lim_{n\to\infty}\left(\int_x^n g(x-v)G(v)dv - e^{-\beta x}\int_0^n g(-v)G(v)dv\right). \qquad (2.20)
\end{aligned}
$$

To prove (2.19), we have

$$
\begin{aligned}
&-\frac{\lambda}{D}\int_0^x dz e^{-\beta x+\alpha z}U_1(z) \\
&\quad = -\frac{\lambda}{D}\int_0^x dz e^{-\beta x+\alpha z}\int_{-\infty}^0 dF(y)\int_z^{z-y} dv\Phi(v)e^{-\rho^*(v+y)} \\
&\quad = -\frac{\lambda}{D}\int_0^x dz e^{-\beta x+\alpha z}\left(\int_z^x + \int_x^\infty\right)dv\Phi(v)e^{-\rho^* v}\int_{-\infty}^{z-v} dF(y)e^{-\rho^* y} \\
&\quad = -\frac{\lambda}{D}\int_0^x dv\Phi(v)\int_0^v dz e^{-\beta x+\alpha z-\rho^* v}\int_{-\infty}^{z-v} dF(y)e^{-\rho^* y} \\
&\qquad - \frac{\lambda}{D}\int_x^\infty dv\Phi(v)\int_0^x dz e^{-\beta x+\alpha z-\rho^* v}\int_{-\infty}^{z-v} dF(y)e^{-\rho^* y} \\
&\quad = -\frac{\lambda}{D}\int_0^x dv\Phi(v)e^{-\beta(x-v)}\int_{-v}^0 dz e^{\alpha z}\int_{-\infty}^z dF(y)e^{-\rho^* y} \\
&\qquad - \frac{\lambda}{D}\int_x^\infty dv\Phi(v)e^{-\beta(x-v)}\left(\int_{-v}^0 - \int_{x-v}^0\right)dz e^{\alpha z}\int_{-\infty}^z dF(y)e^{-\rho^* y} \\
&\quad = \lim_{m\to\infty}\left(-\frac{\lambda}{D}\int_0^m dv\Phi(v)e^{-\beta(x-v)}\int_{-v}^0 dz e^{\alpha z}\int_{-\infty}^z dF(y)e^{-\rho^* y}\right. \\
&\qquad \left. + \frac{\lambda}{D}\int_x^m dv\Phi(v)e^{-\beta(x-v)}\int_{x-v}^0 dz e^{\alpha z}\int_{-\infty}^z dF(y)e^{-\rho^* y}\right) \\
&\quad = \lim_{m\to\infty}\left(-e^{-\beta x}\int_{-m}^0 dv\Phi(-v)G(v) + \int_{-m}^0 dv\Phi(x-v)G(v)\right),
\end{aligned}
$$

where the last equation follows from the definition of G in $(-\infty, 0)$. So we see that (2.19) holds. To verify (2.20), we obtain

$$
\begin{aligned}
&-\frac{\lambda}{D}\int_0^x dz e^{-\beta x+\alpha z}U_2(z) \\
&\quad = \frac{\lambda}{D}\int_0^x e^{-\beta x+\alpha z}\int_z^\infty dF(y)\int_z^y dv e^{-\rho^* v}g(v-y)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lambda}{D}\int_0^x dze^{-\beta x+\alpha z}\int_z^\infty dF(y)\int_0^{y-z} dve^{\rho^* v-\rho^* y}g(-v)\\
&= \frac{\lambda}{D}\int_0^x dze^{-\beta x+\alpha z}\int_0^\infty dvg(-v)e^{\rho^* v}\int_{v+z}^\infty dF(y)e^{-\rho^* y}\\
&= \frac{\lambda}{D}\int_0^\infty dvg(-v)\int_0^x dze^{-\beta x+\alpha z+\rho^* v}\int_{v+z}^\infty dF(y)e^{-\rho^* y}\\
&= \frac{\lambda}{D}\int_0^\infty dvg(-v)\int_v^{x+v} dze^{-\beta x+\alpha z-\beta v}\int_z^\infty dF(y)e^{-\rho^* y}\\
&= \frac{\lambda}{D}\int_x^\infty dvg(x-v)\left(\int_0^v - \int_0^{v-x}\right)dze^{\alpha z-\beta v}\int_z^\infty dF(y)e^{-\rho^* y}\\
&= \lim_{n\to\infty}\left(\int_x^n dvg(x-v)G(v) - e^{-\beta x}\int_0^n dvg(-v)G(v)\right),
\end{aligned}
$$

where the last equality follows from the definition of G in $(0, \infty)$.

The desired result now follows from (2.18)–(2.20) and (2.17). □

Consider the two improper integrals in (2.14). We now turn to study the criterions under which both

$$\int_0^\infty g(-v)G(v)dv < \infty \tag{2.21}$$

and

$$\int_{-\infty}^0 \Phi(-v)G(v)dv < \infty. \tag{2.22}$$

We consider the criterion for (2.21) first.

Lemma 2.1. *Suppose that $\int_{(0,\infty)} dF(y) > 0$. Then $\beta > 0$ if and only if $\int_0^\infty h(v)G(v)dv < \infty$ for all nonnegative bounded Borel h on $[0, \infty)$.*

Proof. The proof of necessity is exactly given by Equations (13)–(16) in Gerber and Landry [2]. We only show the proof of sufficiency here.

Take $h \equiv 1$. By the definition of the function G in (2.12), the convergence of the integral $\int_0^\infty G(y)dy$ implies that

$$0 = \lim_{v\to\infty} e^{-\beta v}\int_0^v dze^{\alpha z}\int_z^\infty dF(y)e^{-\rho^* y}.$$

If $\beta \leq 0$ and $\int_{(0,\infty)} dF(y) > 0$, we get a contradiction. Hence, we have $\beta > 0$. □

Note that, by the definition of β, the condition that $\beta > 0$ is equivalent to the condition that $D(\rho^*)^2 + c\rho^* > 0$. The following example shows that we need not always have $\beta > 0$. (In [2], Gerber and Landry assumed implicitly that $\beta > 0$.)

Example 2.1. Fix $\sigma > 0$ and $c < 0$. Pick $\delta > 0$ such that $Dx^2 < -cx$ for all $x \in (0, \delta)$. Consider the double-exponential jump distribution

$$F(dy) = p\eta^{(+)}e^{-\eta^{(+)}y}\mathbf{1}_{y>0}dy + q\eta^{(-)}e^{\eta^{(-)}y}\mathbf{1}_{y<0}dy, \tag{2.23}$$

where $(p, q) \in (0, 1)$, $p + q = 1$, and $\eta^{\pm} > 0$. Then $\Delta = \eta^{(-)} > 0$ and $\psi(\Delta-) = +\infty$. If we take $\eta^{(-)} < \delta$, then $\rho^* \in (0, \eta^-) \subset (0, \delta)$ and $\beta = \frac{c}{D} + \rho^* < 0$ by the choice of δ.

Next, we consider the validity of (2.22) and assume that $\beta > 0$. Suppose $\int_{-\infty}^{0} dF(v) > 0$ and (2.22) holds. The convergence of the integral $\int_{-\infty}^{0} \Phi(-v)G(v)dv$ implies that

$$0 = \lim_{v\to\infty} \Phi(v)G(-v) = \lim_{v\to\infty} \Phi(v)e^{\beta v}\int_{-v}^{0} dze^{\alpha z}\int_{-\infty}^{z} dF(y)e^{-\rho^* y}. \tag{2.24}$$

Since $\beta > 0$, we have $\alpha = \beta + \rho^* > 0$ and $0 < \int_{-\infty}^{0} dze^{\alpha z}\int_{-\infty}^{z} dF(y)e^{-\rho^* y} < \infty$. By (2.24), we get

$$\lim_{v\to\infty} \Phi(v)e^{\beta v} = 0.$$

Conversely, suppose for some small $\delta > 0$,

$$\lim_{v\to\infty} \Phi(v)e^{(\beta+\delta)v} = 0. \tag{2.25}$$

Then

$$\int_{0}^{\infty} \Phi(v)G(-v)dv \leq \int_{0}^{\infty} e^{-\delta v}\left(\Phi(v)e^{(\beta+\delta)v}\int_{-\infty}^{0} dze^{\alpha z}\int_{-\infty}^{z} dF(y)e^{-\rho^* y}\right)dv < \infty,$$

which gives (2.22). We need to give a criterion under which (2.25) holds.

In order to obtain an estimate of the exponential decay rate for Φ, we use the change-of-measure technique. Let $\hat{\psi}(x)$ be the Laplace exponent of the dual process $\widehat{X} = -X$. Suppose that the Lundberg's condition for $\widehat{X}$ holds, that is with obvious notations, $\widehat{\Delta} > 0$ and $\hat{\psi}(\widehat{\Delta}-) > 0$. Then there exists a unique $\widehat{\rho^*} \in (0, \widehat{\Delta})$ such that $\hat{\psi}(\widehat{\rho^*}) = r$.

Lemma 2.2. *Given* $g \geq 0$, *the function* Φ *satisfies the estimate*:

$$\Phi(x) \leq \|g\|_\infty e^{-\widehat{\rho^*}x}, \quad x \geq 0.$$

Proof. As noted before, $\{e^{-rt-\widehat{\rho^*}X_t}; t \geq 0\}$ is a $(\mathcal{F}_t, \mathbb{P}_0)$-martingale. Fix $x > 0$. Define for each t a probability measure $\mathbb{P}_t^*$ on $\mathcal{F}_t$ by

$$\mathbb{P}_t^*(A) = \mathbb{E}_x\left[e^{-rt-\widehat{\rho^*}(X_t-x)}\mathbf{1}_A\right], \quad A \in \mathcal{F}_t.$$

By Rolski et al. [12, Corollary 10.2.1], there exists a unique probability measure $\mathbb{P}^*$ defined on $\bigvee_{t\geq 0}\mathscr{F}_t$ such that $\mathbb{P}^*$ agrees on $\mathscr{F}_t$ with $\mathbb{P}^*_t$.

Note the fact that $\widehat{\rho^*} \geq 0$ and $X_\tau \leq 0$ implies that $e^{\widehat{\rho^*}X_\tau} \leq 1$. Since τ is an $(\mathscr{F}_t)$-stopping time, we have by Rolski et al. [12] Lemma 10.2.2

$$\Phi(x) = \mathbb{E}_x\big[\big(e^{-r\tau-\widehat{\rho^*}(X_\tau - x)}\big)e^{\widehat{\rho^*}(X_\tau - x)}g(X_\tau);\, \tau < \infty\big]$$
$$= \mathbb{E}^*\big[g(X_\tau)e^{\widehat{\rho^*}X_\tau};\, \tau < \infty\big]e^{-\widehat{\rho^*}x} \leq \|g\|_\infty e^{-\widehat{\rho^*}x}.$$

This completes the proof. □

Theorem 2.1 (Main Result). *The discounted penalty* Φ *in* (1.1) *satisfies the renewal equation*

$$\Phi(x) = g(0)e^{-\beta x} + \int_{-\infty}^{\infty}\Phi(x-v)G(v)dv - e^{-\beta x}\int_{-\infty}^{\infty}\Phi(-v)G(v)dv, \quad x \geq 0 \tag{2.26}$$

in the following cases:

(1) $\int_{(0,\infty)} dF(y) = 1$ *and* $\beta > 0$;
(2) $\int_{(0,\infty)} dF(y) < 1$, $\beta > 0$, *and the following condition holds*:

$$\textit{for some } \delta > 0, \quad \Phi(x)e^{(\beta+\delta)} \longrightarrow 0 \textit{ as } x \longrightarrow \infty. \tag{2.27}$$

In particular, (2.27) *holds whenever the Lundberg's condition holds for the dual process* $\widehat{X}$ *and the constant* β *falls in* $(0, \widehat{\rho^*})$, *where* $\widehat{\rho^*}$ *is the Lundberg's constant for* $\widehat{X}$.

Proof. Firstly, assume (1) holds. Then $G = 0$ in $(-\infty, 0)$. Since $\beta > 0$ and Φ is bounded, by Lemma 2.1, both $\int_0^\infty \Phi(-v)G(v)dv$ and $\int_0^\infty \Phi(x-v)G(v)dv$ converge. Hence, Φ satisfies (2.26) under (1).

Secondly, assume (2) holds. Then by the discussion before Lemma 2.2, the assumption (2.27) implies that $\int_{-\infty}^0 \Phi(-v)G(v)dv < \infty$. Similarly, $\int_{-\infty}^0 \Phi(x-v)G(v)dv < \infty$. If $\int_{(0,\infty)} dF(y) = 0$, it follows that $G = 0$ in $(0,\infty)$ and hence $\int_0^\infty \Phi(x-v)G(v)dv = \int_0^\infty g(-v)G(v)dv = 0$. If $\int_{(0,\infty)} dF(y) > 0$, it follows from Lemma 2.1 that the assumption $\beta > 0$ implies $\int_0^\infty \Phi(x-v)G(v)dv < \infty$ and $\int_0^\infty g(-v)G(v)dv < \infty$. In either case, we see that (2.26) is satisfied under (2).

Finally, the last statement of the theorem follows from Lemma 2.2. The proof is complete. □

Remark. Our result thus generalizes Theorem 3 in Gerber and Landry [2] in which only (1) is considered. See also Tsai and Wilmott [9] for a more general penalty scheme.

We close this section by giving an example in which $\int_{(0,\infty)} dF(y) < 1$ and $0 < \beta < \widehat{\rho^*}$.

Example 2.2. Fix $\sigma > 0$ and let $c = 0$. We consider the jump distribution F in (2.23). In addition, we assume that F satisfies the following conditions: (i) $\eta^{(+)} > \eta^{(-)}$, (ii) $p\eta^{(+)} > q\eta^{(-)}$, (iii) $q\eta^{(+)} > p\eta^{(-)}$. Since $c = 0$, we have $\beta = \rho^*$. We show that $\rho^* < \widehat{\rho^*}$. Note that $\rho^* \in (0, \Delta) = (0, \eta^{(-)}) \subset (0, \eta^{(+)}) = (0, \widetilde{\Delta})$ and $\hat{\psi}$ is strictly convex on $(0, \widetilde{\Delta})$. Hence $\rho^* < \widehat{\rho^*}$ if and only if $\hat{\psi}(\rho^*) - r < 0 = \psi(\rho^*) - r$ (i.e., $\hat{\psi}(\rho^*) < \psi(\rho^*)$). Now, we have

$$\begin{aligned}\frac{1}{\lambda}[\psi(\rho^*) - \hat{\psi}(\rho^*)] &= \int e^{-\rho^* y} dF(y) - \int e^{\rho^* y} dF(y) \\ &= \left(\frac{p\eta^{(+)}}{\eta^{(+)} + \rho^*} + \frac{q\eta^{(-)}}{\eta^{(-)} - \rho^*}\right) - \left(\frac{p\eta^{(+)}}{\eta^{(+)} - \rho^*} + \frac{q\eta^{(-)}}{\eta^{(-)} + \rho^*}\right) \\ &= \frac{2\rho^*\{\eta^{(+)}\eta^{(-)}[(q\eta^{(+)} - p\eta^{(-)})] + (\rho^*)^2(p\eta^{(+)} - q\eta^{(-)})\}}{[(\eta^{(+)})^2 - (\rho^*)^2][(\eta^{(-)})^2 - (\rho^*)^2]},\end{aligned}$$

which is > 0 by (ii) and (iii). Therefore, $\beta = \rho^* < \widehat{\rho^*}$.

3. APPLICATIONS

Assume that there is a risk-free interest rate $r > 0$ in the market and the price process $(S_t; t \geq 0)$ of a stock (under a risk-neutral probability measure $\mathbb{Q}$) is given by:

$$S_t = s\exp\{ct + \sigma W_t - Z_t\}, \quad t \in \mathbb{R}_+,$$

where $s > 0$ is a constant. We consider a perpetual American option with a payoff function $g(s)$. In general, the price of this option is defined to be

$$v(s) = \max_T \mathbb{E}^{\mathbb{Q}}[e^{-rT} g(S_T) \mid S_0 = s]$$

where T ranges over all stopping times and $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation under the $\mathbb{Q}$ measure. For simplicity, we assume that T only ranges over all stopping times of the form:

$$\tau_L = \inf\{t \geq 0; S_t \leq L\}, \tag{3.1}$$

and that the supremum is attained. This implies that

$$v(s) = \max_L V(s, L) = V(s, L^*). \tag{3.2}$$

Here, $L^* > 0$ is called the **optimal level** in option pricing and the function $V(s, L)$ is defined by

$$V(s, L) = \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{-r\tau_L} g(S_{\tau_L}) \mid S_0 = s]. \tag{3.3}$$

Fix $L > 0$ and set

$$X_t = \log S_t - \log L = X_0 + ct + \sigma W_t - Z_t$$

where $X_0 = \log s - \log L$. Then (3.1) becomes

$$\tau_L = \inf\{t \geq 0; X_t \leq 0\} = \tau \tag{3.4}$$

and

$$V(s, L) = \mathbb{E}_x[e^{-r\tau} g_L(X_\tau)] \equiv \Phi(x, L) \tag{3.5}$$

where $g_L(y) = g(Le^y)$ and $x = \log s - \log L$. From these, we see that (3.2) is equivalent to

$$v(s) = \max_L V(s, L) = \max_L \Phi(x, L).$$

Hence, in option pricing and many financial applications, it is important to derive an explicit formula for Φ as well as the explicit solution for the optimal level L^*. For examples in which the (semi-)explicit solutions of Φ are available, we refer to Asmussen et al. [5], Chen et al. [7] and Lewis and Modecki [8].

In order to determine the optimal level, we impose the smooth-pasting condition which is conventionally assumed in literature of finance. (It is possible to show that the condition is necessary for the optimality.) We show that the smooth-pasting condition is equivalent to some condition on the first-order derivative of $V(s, L)$ with respect to L (see (3.10) below) and then derive an equation for the optimal level.

We start from the renewal equation

$$\begin{aligned} V(s, L) = \Phi(x, L) = g(L)e^{-\beta x} &+ \int_{-\infty}^{\infty} \Phi(x - v, L)G(v)dv \\ &- e^{-\beta x} \int_{-\infty}^{\infty} \Phi(-v, L)G(v)dv. \end{aligned} \tag{3.6}$$

Then we have

$$\begin{aligned} \frac{\partial}{\partial s} V(s, L) = -\frac{\beta}{s} g(L)e^{-\beta x} &+ \frac{1}{s} \int_{-\infty}^{\infty} \frac{\partial \Phi}{\partial x}(x - v, L)G(v)dv \\ &+ \frac{\beta}{s} e^{-\beta x} \int_{-\infty}^{\infty} \Phi(-v, L)G(v)dv \end{aligned} \tag{3.7}$$

and

$$\frac{\partial}{\partial L}V(s,L) = \left[g'(L) + \frac{\beta}{L}g(L)\right]e^{-\beta x} - \frac{1}{L}\int_{-\infty}^{\infty}\frac{\partial\Phi}{\partial x}(x-v,L)G(v)dv + \int_{-\infty}^{\infty}\frac{\partial\Phi}{\partial L}(x-v,L)G(v)dv - \frac{\beta}{L}e^{-\beta x}\int_{-\infty}^{\infty}\Phi(-v,L)G(v)dv - e^{-\beta x}\int_{-\infty}^{\infty}\frac{\partial\Phi}{\partial L}(-v,L)G(v)dv \tag{3.8}$$

(Here as in Gerber and Landry [2], we have to impose some suitable conditions on $\frac{\partial V}{\partial L}$.) The smooth pasting condition (i.e., $\frac{\partial}{\partial s}V(s,L^*)\,|_{s=L^*} = g'(L^*)$) and (3.7) imply that

$$g'(L^*) = -\frac{\beta}{L^*}g(L^*) + \frac{1}{sL^*}\int_{-\infty}^{\infty}\frac{\partial\Phi}{\partial x}(-v,L^*)G(v)dv + \frac{\beta}{L^*}\int_{-\infty}^{\infty}\Phi(-v,L^*)G(v)dv \tag{3.9}$$

By (3.8) and (3.9), we conclude that the smooth-pasting condition is equivalent to the first-order condition

$$\frac{\partial}{\partial L}V(L^*,L)\,|_{L=L^*} = 0. \tag{3.10}$$

Note that $\frac{\partial}{\partial s}V(s,L)\,|_{s=L} = \Phi'(0,L)\frac{1}{L}$. By Proposition 2.1 and the smooth-pasting condition, we obtain that the optimal level L^* satisfies the equation

$$Lg'(L) = -\beta g(L) + \frac{\lambda}{D}\int_0^{\infty}dv\int_v^{\infty}dF(y)e^{-\rho^* v}g(Le^{(v-y)}) - \frac{\lambda}{D}\int_{-\infty}^{0}dF(y)\int_0^{-y}dve^{-\rho^* v}\Phi(v,L). \tag{3.11}$$

In particular, if there are no upward jumps for X, then the equation (3.11) becomes

$$Lg'(L) = -\beta g(L) + \frac{\lambda}{D}\int_0^{\infty}dv\int_v^{\infty}dF(y)e^{-\rho^* v}g(Le^{(v-y)}). \tag{3.12}$$

Consider the special case that $g(s) = (K-s)^+$, where K is the strike price, and there are no upward jumps. Recall that $\rho^* = 1$ (see Section 2). We then get that, for $L < K$,

$$\begin{aligned}
-L = Lg'(L) &= \Phi'(0,L)\\
&= -\beta(K-L) + \frac{\lambda}{D}\int_0^{\infty}dv\int_v^{\infty}dF(y)e^{-v}(K-Le^{v-y})\\
&= -\beta(K-L) + \frac{\lambda}{D}\int_0^{\infty}dF(y)\int_0^{y}(e^{-v}K - Le^{-y})dv\\
&= -\beta(K-L) + \frac{\lambda}{D}\int_0^{\infty}dF(y)K(1-e^{-y}) - \frac{\lambda L}{D}\int_0^{\infty}dF(y)ye^{-y}.
\end{aligned}$$

The optimal level L^* is given by

$$L^* = \frac{\beta K - \frac{K}{D}\int_0^\infty \lambda(1-e^{-y})dF(y)}{\beta + 1 - \frac{\lambda}{D}\int_0^\infty ye^{-y}dF(y)}$$
$$= \frac{\beta K + \frac{K}{D}(r - D - c)}{\frac{c}{D} + 2 - \frac{\lambda}{D}\int_0^\infty ye^{-y}dF(y)}$$
$$= \frac{Kr}{c + 2D - \lambda\int_0^\infty ye^{-y}dF(y)}. \tag{3.13}$$

(In the last two equations we use respectively the fact $\psi(1) = r$ and $\beta = \frac{c}{D} + 1$.) The formula for the optimal level L^* coincides with (36) in Gerber and Landry [2].

REFERENCES

1. Dufresne, F., and Gerber, H.U. 1991. Risk theory for the compound Poisson process that is perturbed by diffusion. *Insurance: Mathematics and Economics* 10:51–59.
2. Gerber, H.U., and Landry, B. 1998. On the discounted penalty at ruin in a jump-diffusion and the perpetual put option. *Insurance: Mathematics and Economics* 22:263–276.
3. Mordecki, E. 2002. Optimal stopping and perpetual options for Lévy processes. *Finance and Stochastics* 6:473–493.
4. Hilberink, B., and Rogers L.C.G. 2002. Optimal capital structure and endogenous default. *Finance and Stochastics* 6:237–263.
5. Asmussen, S., Avram F., and Pistorius, M.R. 2004. Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and their Applications* 109:79–111.
6. Alili, L., and Kyprianou, A.E. 2005. Some remarks on first passage of Lévy processes, the American put and pasting principles. *Annals of Applied Probability* 15:2062–2080.
7. Chen, Y.T., Lee, C.F., and Sheu, Y.C. 2007. An ODE approach for the expected discounted penalty at ruin in jump diffusion model. *Finance and Stochastics* 11:323–355.
8. Lewis, A.L., and Mordecki, E. 2008. Wiener-Hopf factorization for Lévy processes having positive jumps with rational transforms. *Journal of Applied Probability* 45:118–134.
9. Tsai, C.L., and Wilmott, G.E. 2002. A generalized defective renewal equation for the surplus process perturbed by diffusion. *Insurance: Mathematics and Economics* 30:51–66.
10. Chen, Y.T., and Sheu, Y.C. to appear. On the discounted penalty at ruin in a jump-diffusion model. *Taiwanese Journal of Mathematics*.
11. Merton, R.C. 1976. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3:125–144.
12. Rolski, T., Schmidli, H., Schmidt, V., Teugels, J. 1999. *Stochastic Processes for Insurance and Finance*. John-Wiley, New York.