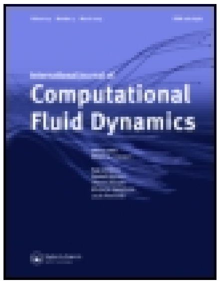

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Instability analysis of modulated Taylor vortices

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This study investigates the instability analysis of modulated Taylor vortices flow by utilising a numerical method. Based on the consideration that the outer cylinder is fixed and the inner cylinder rotates at a non-zero averaged speed under varying modulated amplitudes and frequencies, the flow is converted from one-dimension Couette flow to Taylor vortices. When the modulated amplitude is greater than 1 and the rotation speed of the inner cylinder exceeds the threshold value for one-dimensional flow, the flow will be more stable at intermediate and high frequencies. When the modulated amplitude is sufficiently large and the inner cylinder rotates at medium frequency, subharmonic flow arises.

Keywords: instability analysis; modulated effect; Taylor vortices; subharmonic flow; collocation

1. Introduction

Fluid motion between two concentric rotating cylinders is often investigated in the field of fluid dynamics. This study uses numerical methods to analyse and simulates flow patterns and relevant flow characteristics between two concentric rotating cylinders. Coles (1965) was the first researcher to definitively consider Taylor vortex flow to be non-linear, although several researchers had previously speculated that the Taylor vortex problem could be solved by considering non-linear flow. Donnelly (1964) experimentally analysed modulated flow stability. When the outer cylinder remains stationary and the inner cylinder rotates periodically, parameters such as interval, rotational frequency, and the modulated amplitude of the two cylinders can be varied to determine how the flow is affected by modulated rotation. Hall (1975) utilised linear theory to determine low and high frequencies and used non-linear theory to analyse the flow under a high frequency. Carmi and Tustaniwskyj (1981) examined modulated stability under a limited gap and the influence of axial symmetry and asymmetry on modulated flow. In a former study, it was shown that the critical Reynolds number exhibits an increased unstable offset under low frequency. Marques and Lopez (1997) and Lopez and Marques (2001) introduced and studied more cases of time-modulated Taylor–Couette problems in which the inner cylinder moves periodically along the axial direction. Youd *et al.* (2003, 2005), who analysed zero-equivalent modulated flow around concentric cylinders with a radius ratio of $\eta = 0.75$, defined as $\eta = R_1/R_2$ where R_1 and R_2, are the inner and outer radii of cylinders, identified the formation of reversing and non-reversing modulated Taylor–Couette flow. The main objectives of the present work investigate the instability of modulated Taylor vortices flow by utilising a numerical method. It is practical to focus attention on the transition based on varying modulated amplitudes and frequencies.

2. Numerical method

The flow is described by the incompressible, three-dimensional Navier–Stokes equations with cylindrical coordinates (R, θ, Z) in an absolute frame of reference according to the velocity–pressure formulation. The dimensionless factors r, z are the radial and axial coordinates, τ is the time and ω is the modulated frequency with corresponding dimensional quantities of t and ω'; Re is the Reynolds number of the inner cylinder and α is the axial wave number; the velocity components and pressure are $\bar{V}_r$, $\bar{V}_\theta$, $\bar{V}_z$, and $\bar{P}$, with the respective dimensional quantities denoted by lower-case symbols:

$$r = R/d, \quad z = Z/d, \quad \tau = v \cdot t/d^2, \quad \omega = d^2 \cdot \omega'/v,$$
$$Re = R_1 \cdot \Omega_1 \cdot d/v, \quad \alpha = 2\pi/\lambda, \quad \bar{P} = p \cdot d^2/(\rho \cdot v^2),$$
$$\bar{V}_r = v_r \cdot d/v, \quad \bar{V}_\theta = v_\theta \cdot d/v, \quad \bar{V}_z = v_z \cdot d/v$$

where $d = R_2 - R_1$, v is the kinematic viscosity and Ω_1 is the angular velocity of the inner cylinder; λ is the

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wavelength of the Taylor vortex and ρ is the density of the fluid, respectively.

The basic flow type is the one-dimensional Couette flow with the modulated amplitude and frequency of azimuthal velocity between two concentric cylinders. Then the inner cylinder rotates with the increasing Reynolds number *Re*, and the outer cylinder is considered to be at rest under all conditions. The dimensionless Navier–Stokes and continuity equations are as follows:

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\nabla\right)\vec{V} = -\nabla\bar{P} + \frac{1}{Re}\Delta\vec{V}, \quad \nabla\cdot\vec{V} = 0 \quad (1)$$

where $\vec{V} = (\bar{V}_r, \bar{V}_\theta, \bar{V}_z)$. The time scheme is semi-implicit and second-order accurate. It corresponds to a combination of the second-order backward implicit Euler scheme (for the time term) and an explicit Adam–Bashforth scheme (for the non-linear terms). The discretised form of the momentum equation is

$$\frac{3\vec{V}^{j+1} - 4\vec{V}^{j} + \vec{V}^{j-1}}{2\delta t} + 2\left(\vec{V}^{j}\cdot\nabla\right)\vec{V}^{j} - \left(\vec{V}^{j-1}\cdot\nabla\right)\vec{V}^{j-1} = -\nabla\bar{P}^{j+1} + \frac{1}{Re}\Delta\vec{V}^{j+1} \quad (2)$$

where $\vec{V}^{j}$ is the solution at time $t_j = j\delta t$, δt being the time step.

Cole (1976) investigated the effect of cylinder height on flow stability experimentally. Cole demonstrated that cylinder height does not influence the critical point for transformation from a Couette flow to a Taylor vortex flow unless the aspect ratio between cylinder height and interval is less than 8. Additionally, according to the study by Hall and Blennerhasset (1979), when the aspect ratio $L/d \geq 12$, the numerical and experimental results have no significant differences, indicating that the effect of cylinder height on flow stability can be neglected. We assume infinite cylinders and a periodic solution in the axial direction. The boundary conditions are

$$\bar{V}_r = \bar{V}_z = 0, \quad \bar{V}_\theta = Re(1 + \varepsilon\cos\ \omega\tau) \quad \text{at} \quad r = \frac{\eta}{1-\eta}, \quad \text{and} \quad \vec{V} = 0 \quad \text{at} \quad r = \frac{1}{1-\eta} \quad (3)$$

where ε and ω are the modulated amplitude and frequency, respectively.

The flow velocity and pressure profile of the Taylor vortices can be regarded as a one-dimensional flow with a perturbation and can be expressed as

$$\bar{V}_r = 0 + V_r'(r,z,\tau) \quad (4)$$

$$\bar{V}_\theta = \bar{\bar{V}}_\theta(r,\tau) + V_\theta'(r,z,\tau) \quad (5)$$

$$\bar{V}_z = 0 + V_z'(r,z,\tau) \quad (6)$$

$$\bar{P} = 0 + p'(r,z,\tau) \quad (7)$$

The perturbations are determined using a pseudo-spectral Fourier–Chebyshev collocation method, taking advantage of the orthogonality properties of Chebyshev polynomials and assuming exponential convergence (see Daoyi and Gerhard 1994, Speetjens and Clercx 2005).

$$V_r' = \sum_{m=0}^{M-1}\sum_{n=2}^{N+1} A_{mn}(\tau)\phi_n(\xi)\cos m\alpha z \quad (8)$$

$$V_\theta' = \sum_{m=0}^{M-1}\sum_{n=2}^{N+1} B_{mn}(\tau)\phi_n(\xi)\cos m\alpha z \quad (9)$$

$$V_z' = \sum_{m=1}^{M}\sum_{n=2}^{N+1} C_{mn}(\tau)\phi_n(\xi)\sin m\alpha z \quad (10)$$

$$p' = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} D_{mn}(\tau)T_n(\xi)\cos m\alpha z \quad (11)$$

Here, M and N are the number of terms in the Fourier series expansion and Chebyshev polynomial expansion, respectively, and A_{mn}, B_{mn}, $C_{\min}$, and D_{mn} are amplitude coefficients. The space can be defined as $T_n(\xi) = \cos(n\cdot\cos^{-1}\xi)$, and ϕ_n is a basis function that satisfies the boundary conditions. ϕ_n is expressed as

$$\phi_n(\xi) = T_n - [1-(-1)^n]\frac{T_1}{2} - [1+(-1)^n]\frac{T_0}{2}, \quad n = 2,3,4 \quad (12)$$

where $\xi \in [-1,1]$. The domain of r in the governing equation is transformed from $\eta/(1-\eta) \leq r \leq 1/(1-\eta)$ to $-1 \leq \xi \leq 1$ through the relational equation $\xi = 2r - (1+\eta)/(1-\eta)\cdot \bar{\bar{V}}_\theta(r,\tau)$ is the velocity of one-dimensional Couette flow and can be expressed as

$$\bar{\bar{V}}_\theta(r,\tau) = \frac{-\eta Re}{1+\eta}r + \frac{\eta Re}{(1-\eta)(1-\eta^2)}r^{-1} + \text{real}\left\{Re\cdot\varepsilon\left[\frac{K_1(sr_2)I_1(sr) - I_1(sr_2)K_1(sr)}{I_1(sr_1)K_1(sr_2) - I_1(sr_2)K_1(sr_1)}\right]\exp^{i\omega\tau}\right\} \quad (13)$$

where $s = \sqrt{i\omega}$ $r_1 = \frac{\eta}{1-\eta}$ and $r_2 = \frac{1}{1-\eta}$ are the position of inner and outer cylinders respectively. I_1 and K_1 are, respectively, the first and second kind first order modulated Bessel functions.

Substituting Equations (4)–(7) into the continuity Equation and Equation (2), the equations are transformed into an algebraic equation, which can be expressed as a matrix equation

$$AX^{j+1} = F^{j,j-1} \tag{14}$$

where A is a matrix of coefficients and vector $X = (A_{mn}, B_{mn}, C_{mn}, D_{mn})^T$. The unknown values of vector F, $F = (F_1, F_2, F_3, 0)^T$, are the summation of radial, azimuthal, and axial velocity components for linear term at time t_j and non-linear term at time t_{j-1}. The coefficients A_{mn}, B_{mn}, C_{mn}, and D_{mn} are determined iteratively until the convergence condition is satisfied. When the inner cylinder rotates with a fixed rotational speed, the convergence condition is

$$\left| \frac{X^{j+1} - X^j}{X^{j+1}} \right| < 10^{-4} \tag{15}$$

We adopt the tolerance (10^{-4}) to avoid the computation process to become time consuming and larger error ratio compared with those obtained by Jones (1985). When the pressure coefficient is converging to the tolerance (10^{-4}), the other coefficients in velocity components had been converging and lower to the tolerance (10^{-4}). The coefficient that satisfies the convergence condition is substituted in the appropriate equation among Equations (4)–(7); the speed and pressure in each time interval can then be determined. If the cylinder rotates periodically, the largest value of axial speed attained in a particular time interval at a selected observation point in the flow is compared with the axial speed in the preceding time interval, the convergence condition is

$$\left| \frac{\bar{V}_z{}^{i+1} - \bar{V}_z{}^{i}}{\bar{V}_z{}^{i+1}} \right| < 10^{-4} \tag{16}$$

where i is the periodic counter. If the difference is less than 10^{-4}, then the convergence condition is considered to be satisfied.

3. Results and discussion

Prior to the computation of the flow field, we analyse the degree of accuracy, which serves as the basis for the post computation. In theory, the greater the number of terms expanded, the higher is the accuracy; however, the limit to the increase of the number of terms will come from the round-off error and the computation process becomes time consuming. Therefore, the best option is to use the expansion with the least number of terms for which a certain degree of accuracy can be guaranteed. The results computed from the expanded number of terms, M and N in the computation mode of Taylor vortices, were compared with the results obtained by Jones (1985) when the inner cylinder rotated at a constant velocity. Jones (1985) used the Taylor number to obtain the rotating velocity of the inner cylinder, as shown in Table 1. For a low Re value ($Re = 72.5$), the radial velocity at the point of observation can be converged with the expanded terms 6×6. The difference ratio compared with the expanded terms 6×6 and 7×7 is converging to 0.19%. However, for a high Re value ($Re = 259.8$), the radial velocity at the point of observation can be converged with the expanded terms 10×10. The difference ratio compared with the expanded terms 10×10 and 11×11 is converging to 0.14%. These

Table 1. When the outer cylinder is fixed and the inner cylinder rotates at a constant velocity, the radial velocity at the point of observation is ($\xi = 0$, $z = 0$) when $\eta = 0.5$.

	Re (T_a)				
$M \times N$	72.5 (3,500)	106.1 (7,500)	150 (15,000)	212.1 (30,000)	259.8 (45,000)
4×4	4.6991	17.5775	–	–	–
5×5	3.9956	17.1988	30.5700	41.92323	–
6×6	4.2333	17.8566	32.8350	52.7493	67.0024
7×7	4.2253	17.9733	33.5851	52.8491	69.8342
8×8	4.2340	17.9840	33.5712	54.9975	70.7955
9×9	4.2376	17.9733	33.6452	55.5914	71.8347
10×10	4.2354	17.9840	33.6900	55.6550	72.2764
11×11	4.2347	17.9840	33.6754	55.6763	72.3803
12×12	4.2354	17.9840	33.6754	55.6975	72.4063
13×13	4.2354	17.9840	33.6754	55.6975	72.4063
Study of Jones (1985)	4.2336	17.9733	33.6768	55.7187	72.2764

The definition of T_a is represented by $T_a = 2(1-\eta)Re^2/(1+\eta)$.

results are in agreement with those obtained by Jones (1985). For the computation in this study, both M and N were expanded to 10 terms.

In Figures 1 and 2, the outer cylinder is fixed and amplitude of the inner cylinder is modulated at $\varepsilon = 1$ and $\varepsilon = 2$, respectively. The axial speed changes with time at different values of Reynolds number and modulated frequency when $\xi = 0.5$. For low frequency, the dimensionless time τ becomes time consuming to obtain the iterative convergence under different Reynolds number. But for high frequency, the computation process is rapidly converged. When the inner cylinder rotates at low frequency, the flow has sufficient time to change with the velocity. Once the rotation speed of the inner cylinder exceeds the threshold value for one-dimensional flow, the flow is transformed from one-dimensional circular Couette flow to axisymmetrical Taylor vortex flow. When the instantaneous Reynolds number reaches the maximum value, the Taylor vortex flow disappears in process of time; this phenomenon is referred to as transient stability, as shown in Figure 1a, b and Figure 2a, b. With an increase in the modulated frequency, transient stability disappears because a Stokes layer is produced near the wall. The flow beyond the Stokes layer cannot completely reflect the velocity change in the inner

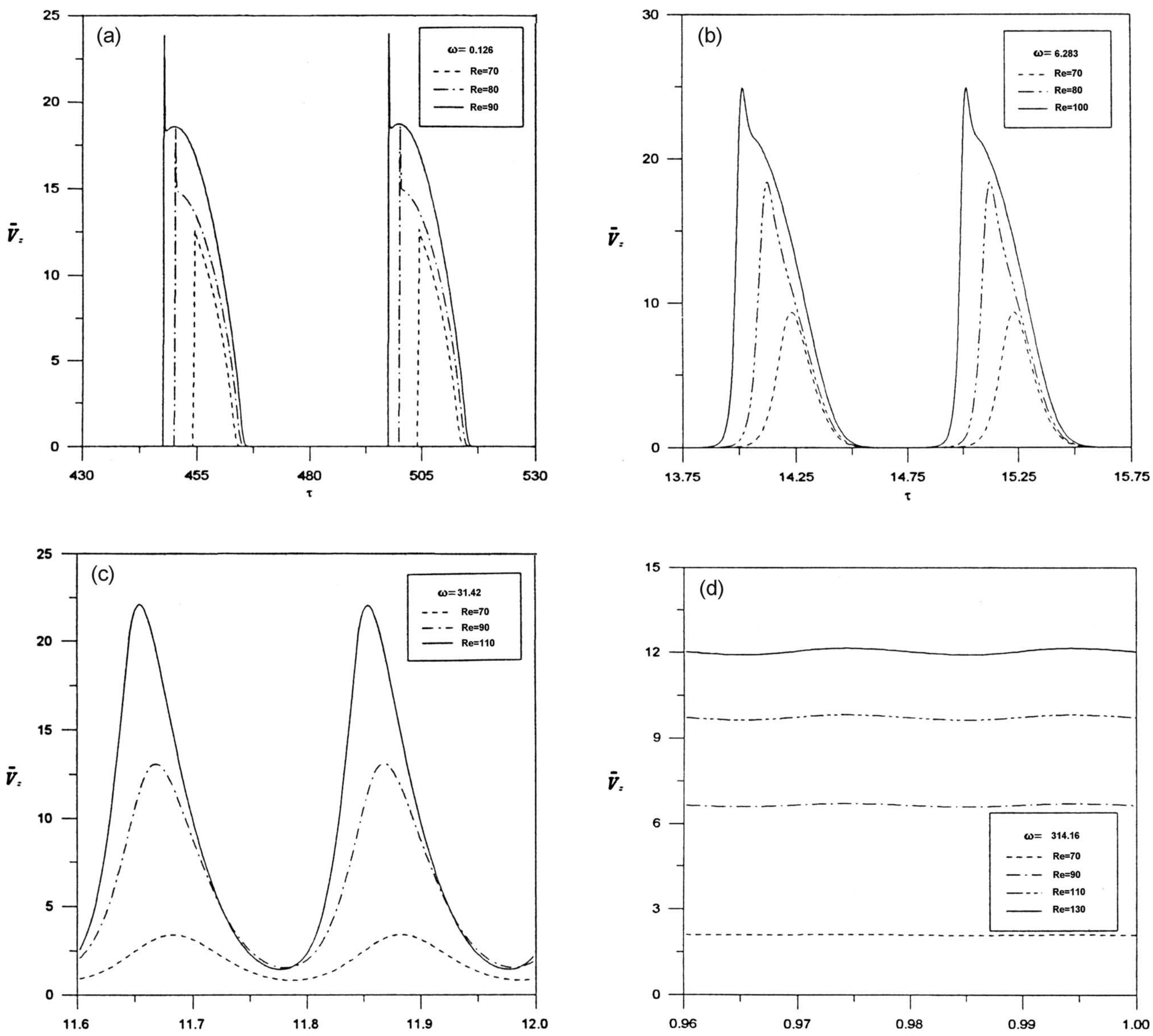

Figure 1. The outer cylinder is fixed and the inner cylinder rotates at different modulated frequencies: (a) $\omega = 0.126$, (b) $\omega = 6.283$, (c) $\omega = 31.42$, and (d) $\omega = 314.16$. The axial speed $\bar{V}_z$ changes with time τ at the point of observation ($\xi = 0.5$, $Z = \lambda/4$), ($\eta = 0.4833$, $\varepsilon = 1$).

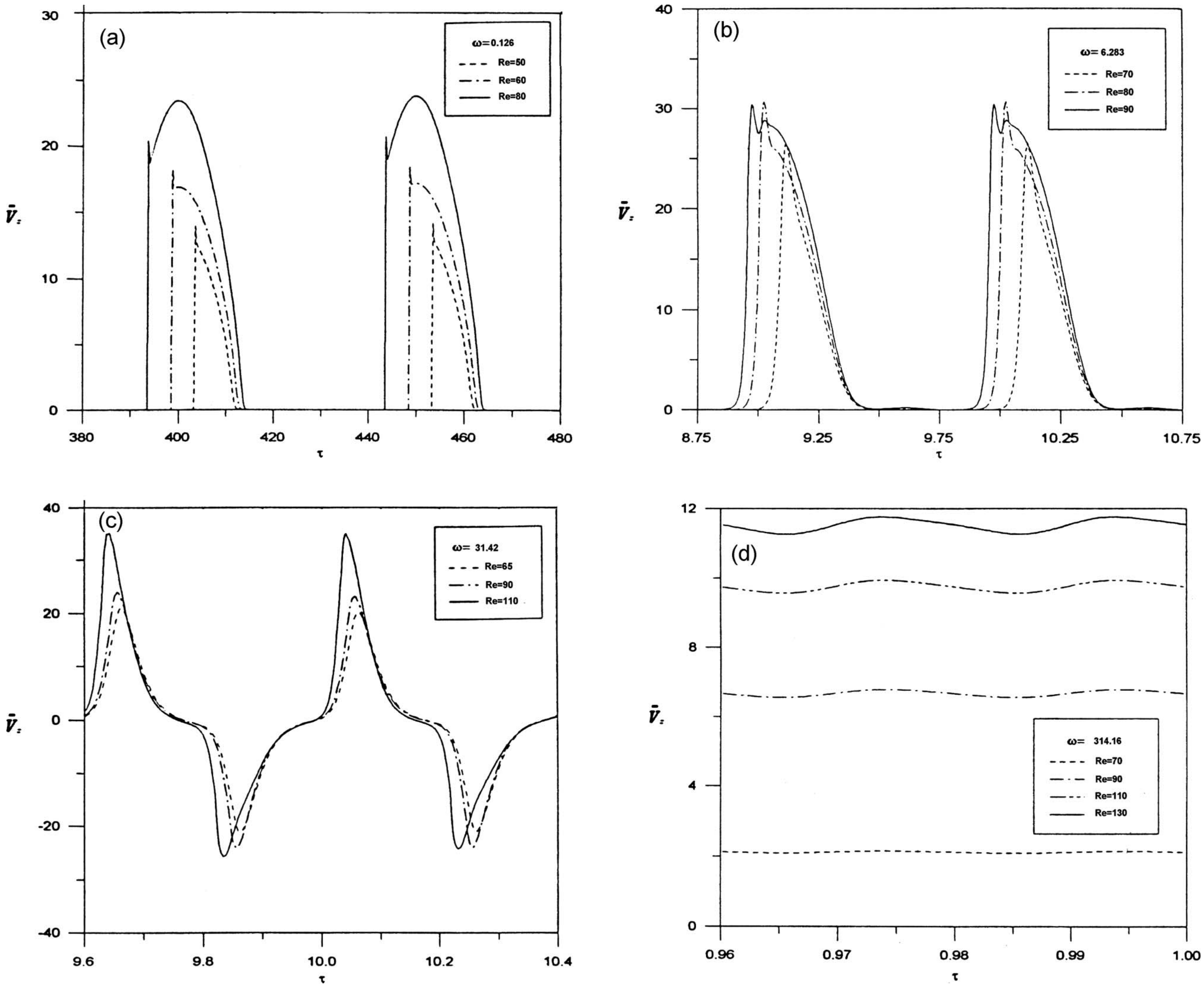

Figure 2. The outer cylinder is fixed and the inner cylinder rotates at different modulated frequencies: (a) $\omega = 0.126$, (b) $\omega = 6.283$, (c) $\omega = 31.42$, and (d) $\omega = 314.16$. The axial speed $\bar{V}_z$ changes with time τ at the point of observation ($\xi = 0.5$, $Z = \lambda/4$), ($\eta = 0.4833$, $\varepsilon = 2$).

cylinder (Figures 1c and 2c). Meanwhile, we can employ a larger Reynolds number and ensure that Taylor vortex flow occurs at an earlier time period. The plots in Figures 1d, 2d show the flow sustains the more stable Taylor vortex flow at high frequency. It is worth noting that subharmonic flow exists at intermediate and high frequencies when $\varepsilon = 2$, as shown in Figure 2c. However, no such flow is observed when $\varepsilon = 1$. The phenomenon is similar to Youd *et al.* (2003, 2005).

4. Conclusion

This study used numerical analysis to investigate the behaviour of flow between two concentric cylinders. The unstable Couette flow was transformed to Taylor vortex flow by considering the flow under non-zero averaged rotation speed at different modulated amplitudes and frequencies. In general, the flow generates larger instability at low-frequency modulation. At intermediate and high frequencies, the flow instability shows a gradually decreasing trend; when the modulated amplitude is sufficiently large, the period of flow at intermediate frequency is twice that of the rotational period of cylinders, which is $4\pi/\omega$ of subharmonic flow.

In addition to the modulated effects that affect the instability of Taylor vortices flow, most of the unstable state of supercritical Taylor vortices flow between concentric cylinders are the wavy form, so-called Taylor wavy vortices, at higher Reynolds numbers of the cylinders. The transition from Taylor vortices to wavy vortices takes place *via* a number of intermediate flow forms, and this article will be a milestone to investigate the phenomenon of Taylor wavy vortices.

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