

Pricing American Asian options with higher moments in the underlying distribution

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Abstract

We develop a modified Edgeworth binomial model with higher moment consideration for pricing American Asian options. With lognormal underlying distribution for benchmark comparison, our algorithm is as precise as that of Chalasani et al. [P. Chalasani, S. Jha, F. Egriboyun, A. Varikooty, A refined binomial lattice for pricing American Asian options, *Rev. Derivatives Res.* 3 (1) (1999) 85–105] if the number of the time steps increases. If the underlying distribution displays negative skewness and leptokurtosis as often observed for stock index returns, our estimates can work better than those in Chalasani et al. [P. Chalasani, S. Jha, F. Egriboyun, A. Varikooty, A refined binomial lattice for pricing American Asian options, *Rev. Derivatives Res.* 3 (1) (1999) 85–105] and are very similar to the benchmarks in Hull and White [J. Hull, A. White, Efficient procedures for valuing European and American path-dependent options, *J. Derivatives* 1 (Fall) (1993) 21–31]. The numerical analysis shows that our modified Edgeworth binomial model can value American Asian options with greater accuracy and speed given higher moments in their underlying distribution.

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1. Introduction

Asian options, compared with vanilla options, are path-dependent and difficult for pricing. The holder obtains a payoff that depends on the average price of the underlying asset over some prescribed period. These options can be used to reduce the exposure of the risk by sudden movements in the underlying asset just before expiry. As a result, the Asian option is quite popular in risk management, and various methods have been developed to determine its value. Nielsen and Sandmann [45] and Boyle and Potapchik [7] have provided survey results of Monte Carlo simulation, the finite difference method and various quasi-analytical approaches for pricing European Asian options.

There are attempts to get closed-form pricing solutions for arithmetic Asian options. But the Black–Scholes [5] type of formula cannot be derived as it is difficult to transform its partial differential equation (PDE) into a standard

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heat equation. Turnbull and Wakeman [53], Lévy [40], and Zhang [56,57] have proposed analytical approximations. Curran [15], Rogers and Shi [47], and Thompson [52] also approximate analytically the lower and upper bounds. Several methods can be applied to price European Asian options: the Taylor expansion [34], the Laguerre series expansion [22,41], and the inversion technique of Laplace transform [28,50]. Recently Dhaene et al. [19,20] and Albrecher et al. [2] utilize the “comonotonic bounds” of the sum of random variables with given marginal distributions, a technique developed by Kaas et al. [35]. However, an analytical pricing formula is still a challenging problem. In recent years, numerical approaches have assumed increasing importance [29].

Three popular numerical methods, the Monte Carlo simulation, the finite difference method, and the binomial lattice, have been followed to value Asian options. Using geometric average, Kemna and Vorst [36] apply the Monte Carlo method to price the European Asian option. The simulation with a control variable estimate is further used to price Asian options [54]. Similarly, Grant et al. [29] value the arithmetic average American Asian option with a fixed exercise price. Recently, Das [17] also applies the Monte Carlo approach with a random lattice in derivative pricing. Although the Monte Carlo simulation is often used to compare with other pricing methods for its convenience and flexibility [39,6,8,25–27,51], its calculation is considered inefficient and very time-consuming.

Dewynne and Wilmott [18] use a modification of the explicit finite difference method, as suggested by Hull and White [31], to price Asian options. In addition, Večeř [55] develops techniques for pricing European Asian options on a traded account. D’Halluin et al. [21] demonstrate a semi-Lagrangian method to value American Asian options. The PDE can be transformed into a set of difference equations and solved quickly using numerical method. However, Barraquand and Pudet [4] point out that an augmentation of the state space is not a viable method to solve the path-dependent pricing problem.

Binomial tree models have been extensively applied in various option valuations [14,3,48,30,11,38]. In particular, Hull and White [32], Chalasani et al. [9,10], Neave and Ye [44], and Costabile et al. [13] all apply extended binomial models to value European or American path-dependent options. They claim that their algorithms are considerably faster and provide more accurate results compared with the analytical approximation approach used by Turnbull and Wakeman [53] which is discussed earlier. Also, Dai and Lyuu [16] have developed a new trinomial lattice to price Asian options. They find that their results are similar to those in [32].

In this study we intend to apply the binomial lattice model to price American Asian options, while considering the impacts of higher moments in the underlying distribution. The probability distributions of the financial variables constantly exhibit significant non-normal properties [33,23]. Many researchers have identified the asymmetric skewness and fat-tailed kurtosis in the financial data [24]. Corrado and Su [12], and Kim and White [37] both find significant negative skewness and positive excess kurtosis in the implied distribution of S&P 500 index options. Ahn et al. [1] consider non-normal distribution when using put options to reduce the cost of risk management. Moreover, the moment-matching analytical methods are applied to approximate the density function of the underlying average for the European Asian options [42,43,46].

Despite the fact that underlying distribution exhibits non-normal higher moments, it is interesting to note that few researchers have considered its impact on Asian option valuation. The purpose of this paper is to propose an algorithm to price American Asian options while considering the higher moments of underlying asset distribution. We apply the refined binomial lattice of [10] and use the Edgeworth expandible distribution [49] to include the parameters for higher moments.

We organize the paper as follows: The next section concisely introduces the pricing process of a binomial tree for an Asian option. The third section presents our Edgeworth binomial model for pricing American Asian options with higher moment consideration. In the fourth section, we discuss the lower and upper bounds of the prices of the American Asian options. The numerical results from our approach are then provided in Section 5. In particular, our estimates are compared with the benchmarks discussed in the literature. Finally, Section 6 offers our conclusions.

2. Pricing process of a binomial tree

Assume that the underlying asset of an American Asian option follows a geometric Brownian motion in a risk-neutral world. Its price at time t , $S(t)$, satisfies the following stochastic differential equation:

$$dS(t) = rS(t)dt + \sigma S(t)dB(t),$$

where the annualized risk-free interest rate, r , and the annualized risk-neutral volatility, σ , are constant, and $B(t)$ denotes a Brownian motion process.

We can use a binomial tree [14] to approximate the continuous time function $S(t)$. Assume that there are n time steps to the expiration date, T , of the option. In each time step, the underlying asset can move up by u percentage with probability p , or move down by d percentage, where $d = 1/u$ with probability $q = 1 - p$. Let

$$u = e^{\sigma\sqrt{\Delta t}},$$

and

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

A sample space Ω is the set of all possible sequences of n upticks or downticks. Its elements can be denoted as $\omega = \omega_1\omega_2 \dots \omega_n$, where ω_i denotes the i th uptick or downtick. Let random variable $H_k(\omega_k)$ be the number of upticks at time k and $H_0(\omega) = 0$ for all ω . A tree path ω is displayed to pass through or reach node (k, h) if and only if $H_k(\omega) = h$ for times $k = 0, 1, \dots, n$ and the number of possible upticks $h = 0, 1, \dots, k$. Then the underlying asset price at time k , S_k , is:

$$S_k = S_0 u^{H_k - (k - H_k)} = S_0 u^{2H_k - k}, \quad \text{with } k = 0, 1, \dots, n.$$

Thus the stock price at node (k, h) is $S_0 u^{2h - k}$. The average price at time k is denoted as $A_k = (S_0 + S_1 + \dots + S_k)/(k + 1)$, with $k \geq 0$. The payoff of an American Asian call with strike L at time n is $(A_n - L)^+$. The price of this option at time 0 is the maximum expected discounted payoff from all possible exercise strategy τ , i.e.,

$$C_0 = \max_{\tau} E[V_n^+]/(1 + r)^n,$$

where $E[V_n^+]$ is a probability-weighted average calculated by $\sum_k \varphi_k (A_k - L)^+$, and φ_k denotes the risk-neutral probability associated with A_k at the expiration date.

3. The modified Edgeworth binomial model

We first apply the Edgeworth binomial tree [49] in order to find the value of an American Asian option. Assume that an Edgeworth tree has n time steps. There are $n + 1$ nodes ($h = 0, \dots, n$) at step n . At each node h , the realization of the random variable is denoted by $y_h = [2h - n]/\sqrt{n}$ and the associated probability, $b(y_h)$, is the standardized binomial density, $[n!/(h!(n - h)!)](1/2)^n$. Given predetermined skewness and kurtosis, the binomial distribution is transformed by the Edgeworth expansion up to the fourth moment. The result is $f(y_h) = [1 + (1/6)\gamma_1(y_h^3 - 3y_h) + (1/24)(\gamma_2 - 3)(y_h^4 - 6y_h^2 + 3)] \times b(y_h)$ where $\gamma_1 = E^Q[y_h^3]$ is the skewness and $\gamma_2 = E^Q[y_h^4]$ is the kurtosis of the underlying distribution under risk-neutral measure. As the sum of $f(y_h)$ is not one, we normalize $f(y_h)$ with $f(y_h)/\sum_h f(y_h)$ and denote it P_h .¹ There is a wide range of higher moments (γ_1, γ_2) for which $f(y_h)$ is positive and unimodal.²

The variable y_h , which has probability P_h , can be standardized as $x_h = (y_h - M)/V$ with $M = \sum_h P_h y_h$ and $V^2 = \sum_h P_h (y_h - M)^2$. The variable x_h is used later in Eq. (1) to obtain the asset price and the corresponding risk-neutral probability, P_h , for a path to node h .

Subsequently, assume an n -step tree model with an underlying asset whose starting price is S_0 , and a realization of the random variable x_h under probability distribution P_h with mean 0 and variance 1. The asset price at the h th node ($h = 0, 1, \dots, n$, indicating the number of upticks) during the final step n , denoted as $\hat{S}_{n,h}$, is

$$\hat{S}_{n,h} = S_0 e^{\mu T + \sigma\sqrt{T}x_h} \tag{1}$$

with $\mu = r - \frac{1}{T} \ln \sum_{h=0}^n P_h e^{\sigma\sqrt{T}x_h}$,

¹ This modification may change the initial skewness and kurtosis. However, in practice, they are closely approximated. And the approximation tends to be improved by choosing ‘ n ’ larger so that $b(y_h)$ more closely resembles a normal distribution. See [49].

² Specifically, the skewness is between -0.8 and 0.8 , and the kurtosis is from 3 to 5.4.

where r is the continuously compounded annual risk-free rate, T is the time to expiration of the subject American Asian option (in years), and σ is the annualized volatility rate, assuming constant, of the cumulative asset return. μ is defined to ensure that the expected risk-neutral asset return equals r , and h indicates the number of upticks in the tree. Solving backwards recursively from the end of the tree, the nodal value, $S_{n-1,h}$, is

$$S_{n-1,h} = \left[\left(\frac{p_{n,h+1}}{p} \right) \hat{S}_{n,h+1} + \left(\frac{p_{n,h}}{p} \right) \hat{S}_{n,h} \right] \exp \left(-\frac{rT}{n} \right) \tag{2}$$

with $p = p_{n,h+1} + p_{n,h}$ where $p_{n,h}$ is $P_h/[n!/(h!(n-h)!)]$.

Given the underlying price process as described by Eq. (2), the value of the American Asian option depends on the arithmetic average of the asset prices over the life of the option. The average asset price at time k can be expressed as

$$A_k = (S_0 + S_{1,h} + \dots + S_{k,h})/(k + 1), \quad \text{where } k \geq 0 \text{ and } 0 \leq h \leq k. \tag{3}$$

Consider a new random variable $\chi_{k,h}(\omega)$ which denotes the area of any path ω reaching a node at time k and defines $\chi_0 = 0$. For any node (k, h) in the lattice, the lowest path reaching (k, h) is defined as the path with $k - h$ downticks followed by h upticks, and the highest path reaching (k, h) denotes the one with h upticks followed by $k - h$ downticks. The area $\chi_{k,h}(\omega)$ of any path ω reaching (k, h) is then defined as the number of boxes encompassed between this path ω and the lowest path reaching this node. The maximum area of any path reaching (k, h) is the number of boxes between the highest and the lowest paths reaching this node, i.e., $h(k - h)$. The minimum area of any path reaching (k, h) is zero. The set of possible areas of paths reaching node (k, h) is therefore $\{0, 1, \dots, h(k - h)\}$.

Each node (k, h) of a binomial lattice can be partitioned into nodelets representing the paths reaching (k, h) with the same geometric stock price average. As noted in [10], there is a one-to-one correspondence between the possible areas and the possible geometric underlying asset averages for the available paths reaching (k, h) . As a result, any path ω reaching or passing through a nodelet (k, h, a) has an area $\chi_{k,h}(\omega) = a$ with h upticks at time k .

Assume that the area of a path w reaching node (k, h) at time k is $\chi_{k,h}(w) = a$. When w has an uptick after this node, it reaches node $(k + 1, h + 1)$ at the next time step. The path w and the lowest path v reaching $(k + 1, h + 1)$ share the same edge connecting (k, h) and $(k + 1, h + 1)$ in the lattice. The number of boxes between w and v at time $k + 1$ is the same as the number a at time k , so the path w will reach nodelet $(k + 1, h + 1, a)$. On the other hand, when w has a downtick at time $k + 1$, it will reach node $(k + 1, h)$. Because the number of boxes between the path w and the lowest path at time $k + 1$ will increase by h , the path w then reaches nodelet $(k + 1, h, a + h)$.

All the tree paths that reach node (k, h) with the same area a will have the same geometric average asset price along the path. The arithmetic average of stock prices, \bar{A} , over all paths reaching nodelet (k, h, a) , as shown in Eq. (8) of [10], is as follows:

$$\bar{A}(k, h, a) = \frac{S''(k, h, a)}{(k + 1)M(k, h, a)}, \tag{4}$$

where $S''(k, h, a) = \sum_{m=1}^M S'_m(k, h, a)$, $S'_m(k, h, a) = \sum_{i=0}^k S_{i,h}$, with $k = 0, 1, \dots, n$; $h = 0, 1, \dots, k$; $a = 0, 1, \dots, h(k - h)$; $m = 1, 2, \dots, M(k, h, a)$; and $M(k, h, a)$ is the number of paths reaching (k, h, a) with $M(0, 0, 0) = 1$. Here $S''(k, h, a)$ is the sum of $S'_m(k, h, a)$ over all paths passing through (k, h, a) with $S''(0, 0, 0) = S_0$, while $S'_m(k, h, a)$ is the sum of the asset prices along any path in its m paths reaching (k, h, a) from time 0 to k .

A path passing through nodelet (k, h, a) and having an uptick will get to nodelet $(k + 1, h + 1, a)$ at time $k + 1$. Thus the number of paths reaching nodelet $(k + 1, h + 1, a)$, namely $M(k + 1, h + 1, a)$, should include $M(k, h, a)$ paths through (k, h, a) . Also, the sum of the prices from all the paths reaching $(k + 1, h + 1, a)$, namely $S''(k + 1, h + 1, a)$, would be $S''(k, h, a) + M(k, h, a)S_{k+1,h+1}$ for paths passing (k, h, a) . Likewise, all paths passing nodelet (k, h, a) with a downtick will reach nodelet $(k + 1, h, a + h)$ at time $k + 1$. By similar reasoning, $M(k + 1, h, a + h)$ should include $M(k, h, a)$ and $S''(k + 1, h, a + h)$ would contain $S''(k, h, a) + M(k, h, a)S_{k+1,h}$.

4. Upper bound and lower bound

Define $C_U(k, h, x)$ as the value of an American Asian option at time k , given that the number of upticks is h and the arithmetic asset price average A_k equals x . C_U can be expressed as follows:

$$C_U(n, h, x) = (x - L)^+, \quad h \leq n$$

$$C_U(k, h, x) = \max \left\{ (x - L)^+, \left(\left(\frac{p_{k+1,h+1}}{p} \right) C_U(k + 1, h + 1, x^U(k, h)) + \left(\frac{p_{k+1,h}}{p} \right) C_U(k + 1, h, x^L(k, h)) \right) \exp(-rt/n) \right\}, \quad k < n, h \leq [k], \tag{5}$$

where L is the exercise price, $p = p_{k+1,h+1} + p_{k+1,h}$ with $p_{k+1,h}$ being $P_h/[(k + 1)!/(h!((k + 1) - h)!)]$. Here $x^U(k, h) = [x(k + 1) + S_{k+1,h+1}]/(k + 2)$ is the arithmetic asset price average A_{k+1} , given that x is the arithmetic asset price average A_k and an uptick occurring at time $k + 1$. Similarly $x^L(k, h) = [x(k + 1) + S_{k+1,h}]/(k + 2)$ is the arithmetic asset price average A_{k+1} , given that x equals A_k and there is a downtick at time $k + 1$. Therefore, for any path ω such that $H_k(\omega) = h$ and $A_k(\omega) = x$, the price of an American Asian option at time k on ω is $C_U(k, h, x)$ whose value at time 0 is $C_U(0, 0, S_0)$.

However, we cannot directly calculate $C_U(0, 0, S_0)$. The quantity $x^U(k, h)$ may not equal any of the possible averages of asset prices over all paths reaching node $(k + 1, h + 1)$ with area a . And similarly, $x^L(k, h)$ may not equal any average at node $(k + 1, h)$. But we can use linear interpolation to derive these missing values because $\bar{A}(k, h, a)$ is a strictly increasing function in a .

Firstly, we compute the upper bound of the American Asian option using an idea similar to [32]. For a given $x = \bar{A}(k, h, a)$, we can find b such that for some $0 \leq \lambda \leq 1$,

$$x^U(k, h) = \lambda \bar{A}(k + 1, h + 1, b) + (1 - \lambda) \bar{A}(k + 1, h + 1, b + 1).$$

Therefore, on the right hand side of (5), $C_U(k + 1, h + 1, x^U(k, h))$ can be replaced by

$$W^U(k, h, a) = \lambda w_1 + (1 - \lambda)w_2,$$

where w_1 is $W(k + 1, h + 1, b) = [\bar{A}(k + 1, h + 1, b) - L]^+$ and w_2 is $W(k + 1, h + 1, b + 1) = [\bar{A}(k + 1, h + 1, b + 1) - L]^+$. Similarly $C_U(k + 1, h, x^L(k, h))$ can be substituted by $W^L(k, h, a) = \lambda w'_1 + (1 - \lambda) w'_2$, where w'_1 is $W(k + 1, h, b) = [\bar{A}(k + 1, h, b) - L]^+$ and w'_2 is $W(k + 1, h, b + 1) = [\bar{A}(k + 1, h, b + 1) - L]^+$. Using this procedure backward recursively, for $x = \bar{A}(k, h, a)$, $a = 0, 1, \dots, h(k - h)$, we have

$$W(k, h, a) = \max \left\{ (x - L)^+, \left(\left(\frac{p_{k+1,h+1}}{p} \right) W^U(k, h, a) + \left(\frac{p_{k+1,h}}{p} \right) W^L(k, h, a) \right) \exp \left(\frac{-rt}{n} \right) \right\} \tag{6}$$

for all (k, h, a) in the tree. It follows that the estimate $W(0, 0, 0)$ is an upper bound for the value of an American Asian option at time zero.

We now provide a lower bound of the American Asian option which can be obtained during the process for upper bound calculation applying a proper exercise rule. Such lower bound has been proposed for European Asian options in [47] and generalized in [19,20]. Let Z be a random variable with the property that all random variables $E[S_i|Z]$ are non-increasing or non-decreasing functions of Z . The notation S^l represents a comonotonic sum of n lognormal random variables.³ The cdf of this sum can be obtained by Theorem 5 from [19]. If we assume that the cdfs of the random variables $E[S_i|Z]$ are strictly increasing and continuous, then the cdf of S^l is also strictly increasing and continuous. From Eq. (48) of [19], we obtain that for all $nL \in (F_{E[S_i|Z]}^{-1+}(0), F_{E[S_i|Z]}^{-1}(1))$,

$$\sum_{i=0}^{n-1} F_{E[S_i|Z]}^{-1}(F_{S^l}(nL)) = nL,$$

where $F_X^{-1}(p)$ is the inverse of the cumulative distribution function of X .

Or equivalently,

$$\sum_{i=0}^{n-1} E[S_i|Z = F_Z^{-1+}(1 - F_{S^l}(nL))] = nL.$$

This determines the cdf of the convex order random variable $S^l = E[S'_m|Z]$ for $S'_m = \sum_{i=0}^{n-1} S_i$.

³ The detailed representation of comonotonic random variables is described in Section 4 in [19].

Under the same assumptions, the stop-loss premiums can be determined from Eq. (55) of [19]⁴:

$$E[(S^d - d)^+] = \sum_{i=0}^{n-1} E[(E[S_i|Z] - E[S_i|Z = F_Z^{-1+}(1 - F_{S^d}(d))])^+],$$

which holds for all retentions $d \in (F_{S^d}^{-1+}(0), F_{S^d}^{-1+}(1))$. Hence, the lower bound on the price of an American Asian call option with strike L , applying Jensen’s rule, is:

$$\begin{aligned} E\left[\frac{(A_\tau - L)^+}{R^\tau}\right] &= E\left[E\left(\frac{(A_\tau - L)^+}{R^\tau} \middle| Z_\tau\right)\right] \geq E\left[E\left(\frac{A_\tau - L}{R^\tau} \middle| Z_\tau\right)^+\right] \\ &= E\left[\frac{(E(A_\tau | Z_\tau) - L)^+}{R^\tau}\right] \\ &= \frac{e^{-rt/n}}{n} E[(E[S'_m(n-1, h, a) | Z_\tau] - nL)^+] \\ &= \frac{e^{-rt/n}}{n} E\left[\left(\sum_{i=0}^{n-1} E[S_{i,h} | Z_\tau] - nL\right)^+\right], \end{aligned} \tag{7}$$

where τ represents a stopping time for a fixed exercise rule, Z_k is \mathcal{F}_k -measurable and \mathcal{F}_k represents the information set at time k , and R is the discount factor, i.e. $R = e^{-rt/n}$. Note that each possible value of the random variable Z_k corresponds to a nodelet in the refined lattice. For an exercise rule τ , the calculation of the upper bound in inequality (7) will generate the estimate for the lower bound.

While calculating the upper bound, whenever $W(k, h, x) > [(p_{k+1,h+1}/p)W^U(k, h, a) + (p_{k+1,h}/p)W^L(k, h, a)] \exp(-rt/n)$ in Eq. (6), the option will be exercised at nodelet (k, h, a) . So $C_L(k, h, a) = [S'(k, h, a) - (k + 1)L]^+$ at each exercised nodelet (k, h, a) . For a nodelet at which the option is not exercised,

$$C_L(k, h, x) = [(p_{k+1,h+1}/p)C_L(k + 1, h + 1, a) + (p_{k+1,h}/p)C_L(k + 1, h, a + h)] \exp(-rt/n),$$

where $S'(k, h, a)/(k + 1) = E(A_k | Z_\tau = (k, h, a)) = \sum_{i=0}^k E[S_{i,h} | Z_\tau]/(k + 1)$, and $p = p_{k+1,h+1} + p_{k+1,h}$ with $p_{k+1,h}$ being $P_h/[(k + 1)!/(h!((k + 1) - h)!)]$. Thus $S'(k, h, a)$ is simply the average of $\Sigma S'_m(k, h, a)$ over all paths at nodelet (k, h, a) , while $S'_m(k, h, a)$ is the sum of the asset prices along any path with exercise point in its m paths reaching (k, h, a) from time 0 to k . We can easily see that $C_L(0, 0, 0)$ equals the expected value at time 0 in Eq. (7), which is the lower bound.

5. Numerical results

5.1. Comparisons of price estimates of American Asian options under lognormal distribution

In order to examine the possible errors from the Edgeworth approximations, we first compare our results with those of [10,32] for lognormal underlying distribution. Table 1 shows the estimated value of an at-the-money American Asian option using different time steps with yearly volatility at 0.3, the risk-free rate at 0.1, and the initial stock price at 50 dollars. Table 2 further provides the estimates of the values with 40 time steps but under various maturities and strike prices. Hull and White’s approximation for American Asian options is used as our benchmark.

In Table 1, we find that the lower and the upper price bounds of an American Asian option using our method are lower than those from [10,32]. However, when we gradually increase the number of the time steps our results are almost the same as theirs. Although the Edgeworth density $f(x)$ is not exactly a probability measure, the approximation errors tend to be very small if more time steps are used in the simulations.

Table 2 exhibits the results under different strikes and option lives using the same number of time steps. When the options are out-of-the-money, our estimates are slightly greater than those from [10,32]. However, for in-the-money

⁴ The stop-loss premium with retention d of a random variable X is shown by $E[(X - d)^+]$. See page 7 of [19] for a detailed discussion.

Table 1
Model comparisons for American Asian option valuations under lognormal distribution

Steps n	E-LB	E-UB	C-LB	C-UB	HW ($h = 0.005$)	HW ($h = 0.003$)
20	4.811	4.813	4.812	4.815	4.815	4.814
40	4.886	4.888	4.888	4.889	4.892	4.890
60	4.916	4.917	4.917	4.918	4.924	4.920
80	4.932	4.933	4.933	4.934	4.942	4.936

The American Asian options are valued with initial stock price $S_0 = 50$ dollars, strike $K = 50$ dollars, option life $T = 1.0$ year, volatility $\sigma = 0.3$ per year, and risk-free rate $r = 0.1$ per year. The estimated lower and upper bounds from our Edgeworth binomial model are indicated by E-LB and E-UB, respectively, whilst the estimates from [32] with different grid-size h are denoted by HW, and those from [10] are denoted by C-LB, C-UB. All simulations are conducted under different time steps with normal skewness $\gamma_1 = 0$ and kurtosis $\gamma_2 = 3$. Parameter values are selected so the results can be compared with those in the literature.

Table 2
Model comparisons for American Asian option valuations under lognormal distribution with different option lives and strikes

Option Life T	Strike L	E-LB	E-UB	C-LB	C-UB	HW
0.5	40	12.105	12.105	12.111	12.112	12.115
0.5	45	7.248	7.248	7.255	7.255	7.261
0.5	50	3.268	3.269	3.269	3.270	3.275
0.5	55	1.150	1.151	1.148	1.148	1.152
0.5	60	0.323	0.323	0.320	0.320	0.322
1.0	40	13.136	13.137	13.150	13.151	13.153
1.0	45	8.535	8.537	8.546	8.547	8.551
1.0	50	4.886	4.888	4.888	4.889	4.892
1.0	55	2.537	2.539	2.532	2.534	2.536
1.0	60	1.211	1.213	1.204	1.206	1.208
1.5	40	13.967	13.969	13.984	13.985	13.988
1.5	45	9.636	9.639	9.648	9.650	9.652
1.5	50	6.193	6.195	6.195	6.197	6.199
1.5	55	3.774	3.777	3.767	3.770	3.771
1.5	60	2.201	2.204	2.190	2.193	2.194
2.0	40	14.685	14.688	14.709	14.712	14.713
2.0	45	10.605	10.609	10.620	10.623	10.623
2.0	50	7.320	7.323	7.322	7.325	7.326
2.0	55	4.889	4.893	4.881	4.885	4.886
2.0	60	3.180	3.184	3.167	3.170	3.171

The American Asian options are valued with initial stock price $S_0 = 50$ dollars, time steps $n = 40$, volatility $\sigma = 0.3$ per year, and risk-free rate $r = 0.1$ per year. The estimated lower and upper bounds from our Edgeworth binomial model are indicated by E-LB and E-UB, respectively, whilst the estimates from [32] are denoted by HW, and those from [10] are denoted by C-LB, C-UB. All simulations are conducted under 40 time steps with normal skewness $\gamma_1 = 0$ and kurtosis $\gamma_2 = 3$, but with various option lives and strikes. Parameter values are selected so the results can be compared with those in the literature.

options, ours are slightly lower than theirs. The differences between our results and theirs are lowered if the options are closer to the expiration. Our simulations confirm the discussions in [34], that the Edgeworth expansion method works fine for shorter maturities, but not for long maturities, while pricing Asian options.

We also note when we increase the number of time steps to calculate the bounds of the option, the differences between our results and those of [10] become trivial. This suggests that the Edgeworth binomial model that we propose here can be properly used in pricing American Asian options under lognormal assumption.

5.2. Comparisons of price estimates of American Asian options under distributions with higher moments

We now compare the price estimates of American Asian options from our Edgeworth binomial model with those from [10,32] under distributions having higher moments. As indicated in Table 3, if the underlying distribution has negative skewness and positive excess kurtosis while other parameters are the same as in Table 1, we find that the results from our model (E-LB = 4.813, 4.890, 4.920, 4.937 and E-UB = 4.815, 4.891, 4.920, 4.937) are closer to

Table 3
Model comparisons for American Asian option valuations with various time steps, skewness and kurtosis

Steps n	γ_1	γ_2	E-LB	E-UB	C-LB	C-UB	HW ($h = 0.005$)	HW ($h = 0.003$)
20	-0.002	3.00	4.813	4.815	4.812	4.815	4.815	4.814
40	-0.046	3.06	4.890	4.891	4.888	4.889	4.892	4.890
60	-0.051	3.06	4.920	4.920	4.917	4.918	4.924	4.920
80	-0.050	3.05	4.937	4.937	4.933	4.934	4.942	4.936

The American Asian options are valued with initial stock price $S_0 = 50$ dollars, strike $K = 50$ dollars, option life $T = 1.0$ year, volatility $\sigma = 0.3$ per year, and risk-free rate $r = 0.1$ per year. The estimated lower and upper bounds from our Edgeworth binomial model are indicated by E-LB and E-UB, respectively, whilst the estimates from [32] with different grid-size h are denoted by HW, and those from [10] are denoted by C-LB, C-UB. All simulations are conducted under various time steps, skewness (γ_1) and kurtosis (γ_2). Parameter values are selected so the results can be compared with those in the literature.

Table 4
Model comparisons for American Asian option valuations with various option lives, strikes, skewness and kurtosis

Option Life T	Strike L	γ_1	γ_2	E-LB	E-UB	C-LB	C-UB	HW
0.5	40	-0.030	3.06	12.115	12.115	12.111	12.112	12.115
0.5	45	-0.030	3.06	7.260	7.261	7.255	7.255	7.261
0.5	50	-0.040	3.04	3.275	3.275	3.269	3.270	3.275
0.5	55	0.000	3.00	1.150	1.151	1.148	1.148	1.152
0.5	60	-0.020	3.25	0.321	0.322	0.320	0.320	0.322
1.0	40	-0.041	3.09	13.152	13.153	13.150	13.151	13.153
1.0	45	-0.040	3.10	8.551	8.552	8.546	8.547	8.551
1.0	50	-0.040	3.05	4.891	4.892	4.888	4.889	4.892
1.0	55	-0.003	3.00	2.536	2.538	2.532	2.534	2.536
1.0	60	-0.003	3.01	1.207	1.209	1.204	1.206	1.208
1.5	40	-0.017	3.00	13.987	13.989	13.984	13.985	13.988
1.5	45	-0.017	3.02	9.651	9.653	9.648	9.650	9.652
1.5	50	-0.007	3.00	6.199	6.201	6.195	6.197	6.199
1.5	55	-0.050	3.01	3.770	3.771	3.767	3.770	3.771
1.5	60	-0.007	3.01	2.192	2.194	2.190	2.193	2.194
2.0	40	-0.036	3.05	14.712	14.715	14.709	14.712	14.713
2.0	45	-0.029	3.05	10.623	10.625	10.620	10.623	10.623
2.0	50	-0.026	3.03	7.325	7.327	7.322	7.325	7.326
2.0	55	-0.028	3.01	4.886	4.889	4.881	4.885	4.886
2.0	60	-0.001	3.02	3.168	3.172	3.167	3.170	3.171

The American Asian options are valued with initial stock price $S_0 = 50$ dollars, time steps $n = 40$, volatility $\sigma = 0.3$ per year, and risk-free rate $r = 0.1$ per year. The estimated lower and upper bounds from our Edgeworth binomial model are indicated by E-LB and E-UB, respectively, whilst the estimates from [32] are denoted by HW, and those from [10] are denoted by C-LB, C-UB. All simulations are conducted under various option lives, strikes, skewness (γ_1) and kurtosis (γ_2). Parameter values are selected so the results can be compared with those in the literature.

the estimates from Hull–White (with grid-size $h = 0.003$ as used in [32]) than those from Chalasani et al. (C-LB = 4.812, 4.888, 4.917, 4.933 and C-UB = 4.815, 4.889, 4.918, 4.934). We also adopt the same parameter values used in Table 2 but assume an underlying distribution with left-skewness and leptokurtosis for calculating the option price. Our results, as shown in Table 4, are again closer to those from Hull–White (with $h = 0.005$ as used in [32]). The evidence in Tables 3 and 4 demonstrates that our modified Edgeworth binomial model performs better in simulating American Asian options.

6. Conclusion

In this study, we develop the modified Edgeworth binomial model to price American Asian options with higher moments in the underlying distribution. The model combines the refined binomial lattice with the Edgeworth expansible distribution to include the high-moment parameters.

Under lognormal distribution for the underlying asset, our results are slightly lower than those in [10] for in-the-money options but slightly higher than theirs for out-of-the-money options. The differences between the two models

become trivial when the time steps in the simulations are increased. Although our estimates are similar to those from Chalasani et al., both are still lower than those from Hull and White. However, after considering the higher moments of the underlying distribution, we find that our estimates of American Asian options are closer to those from Hull and White than from Chalasani et al.

Many studies in the literature have illustrated significant left-skewness and leptokurtosis for the distribution of the underlying asset. As the algorithm of the Edgeworth binomial tree can greatly enhance the computing efficiency, our modified model would be able to price American Asian options faster and with higher precision when the underlying distribution displays higher moments.

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References

- [1] D.-H. Ahn, J. Boudoukh, M. Richardson, R.F. Whitelaw, Optimal risk management using options, *J. Finance* 54 (1) (1999) 359–375.
- [2] H. Albrecher, J. Dhaene, M. Goovaerts, W. Schoutens, Static hedging of Asian options under Lévy models, *J. Derivatives* 12 (Spring) (2005) 63–72.
- [3] K.I. Amin, Jump diffusion option valuation in discrete time, *J. Finance* 48 (5) (1993) 1833–1863.
- [4] J. Barraquand, T. Pudet, Pricing of American path-dependent contingent claims, *Math. Finance* 6 (1996) 17–51.
- [5] F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Political Economy* 81 (1973) 637–654.
- [6] P. Boyle, M. Broadie, P. Glasserman, Monte Carlo method for security pricing, *J. Econ. Dyn. Control* 21 (1997) 1267–1321.
- [7] P. Boyle, A. Potapchik, Prices and sensitivities of Asian options: A survey, *Insurance Math. Econom.* 42 (2008) 189–211.
- [8] M. Broadie, P. Glasserman, Estimating security price derivatives using simulation, *Management Sci.* 42 (2) (1996) 269–285.
- [9] P. Chalasani, S. Jha, A. Varikooty, Accurate approximations for European Asian options, *J. Comput. Finance* 1 (3) (1998) 11–29.
- [10] P. Chalasani, S. Jha, F. Egriboyan, A. Varikooty, A refined binomial lattice for pricing American Asian options, *Rev. Derivatives Res.* 3 (1) (1999) 85–105.
- [11] C.C. Chang, H.C. Fu, A binomial options pricing model under stochastic volatility and jump, *CJAS* 18 (3) (2001) 192–203.
- [12] J.C. Corrado, T. Su, Implied volatility skews and stock index skewness and kurtosis implied by S&P 500 index option prices, *J. Derivatives* 4 (4) (1997) 8–19.
- [13] M. Costabile, I. Massabo, E. Russo, An adjusted binomial model for pricing Asian options, *Rev. Quan. Finance Account.* 27 (3) (2006) 285–296.
- [14] J.C. Cox, S.A. Ross, M. Rubinstein, Option pricing: A simplified approach, *J. Financial Econom.* 7 (1979) 229–263.
- [15] M. Curran, Valuing Asian and portfolio options by conditioning on the geometric mean, *Management Sci.* 40 (1994) 1705–1711.
- [16] T.-S. Dai, Y.-D. Lyuu, Efficient, exact algorithms for Asian options with multiresolution lattices, *Rev. Derivatives Res.* 5 (2002) 181–203.
- [17] S.R. Das, Random lattices for option pricing problems in finance, Working Paper, Leavey School of Business, Santa Clara University, 2002.
- [18] J. Dewynne, P. Wilmott, Partial to the exotic, *Risk* 6 (1993) 38–46.
- [19] J. Dhaene, M. Denuit, M.J. Goovaerts, R. Kaas, D. Vyncke, The concept of comonotonicity in actuarial science and finance: Theory, *Insurance Math. Econom.* 31 (2002) 3–33.
- [20] J. Dhaene, M. Denuit, M.J. Goovaerts, R. Kaas, D. Vyncke, The concept of comonotonicity in actuarial science and finance: Applications, *Insurance Math. Econom.* 31 (2002) 133–161.
- [21] Y. D'Halluin, P. Forsyth, P. Labahn, A semi-Lagrangian approach for American Asian options under jump diffusion, *SIAM J. Sci. Comput.* 27 (1) (2005) 315–345.
- [22] D. Dufresne, Laguerre series for Asian and other options, *Math. Finance* 10 (4) (2000) 407–428.
- [23] K.K. Dutta, D.F. Babbel, Extracting probabilistic information from the prices of interest rate options: Tests of distributional assumptions, *J. Business* 78 (2005) 841–870.
- [24] R.F. Engle, V.K. Ng, Measuring and testing the impact of news on volatility, *J. Finance* 48 (5) (1993) 1749–1778.
- [25] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, N. Touzi, Applications of Malliavin calculus to Monte Carlo methods in finance, *Finance Stochast.* 3 (1999) 391–412.
- [26] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, Applications of Malliavin calculus to Monte Carlo methods in finance II, *Finance Stochast.* 5 (2001) 201–236.
- [27] M.C. Fu, D.B. Madan, T. Wang, Pricing continuous Asian options: A comparison of Monte Carlo and Laplace transform inversion methods, *J. Comput. Finance* 2 (1999) 49–74.
- [28] H. Geman, M. Yor, Bessel processes, Asian options and perpetuities, *Math. Finance* 3 (1993) 349–375.
- [29] D. Grant, G. Vora, D. Weeks, Path-dependent options: Extending the Monte Carlo simulation approach, *Management Sci.* 43 (11) (1997) 1589–1602.
- [30] J.E. Hilliard, A.L. Schwartz, Binomial option pricing under stochastic volatility and correlated state variable, *J. Derivatives* (Fall) (1996) 23–39.
- [31] J. Hull, A. White, Pricing interest-rate derivative securities, *Rev. Financ. Stud.* 3 (4) (1990) 573–592.

- [32] J. Hull, A. White, Efficient procedures for valuing European and American path-dependent options, *J. Derivatives* 1 (Fall) (1993) 21–31.
- [33] J. Hull, A. White, Value at risk when daily changes in market variables are not normally distributed, *J. Derivatives* 5 (3) (1998) 9–19.
- [34] N. Ju, Pricing Asian and basket options via Taylor expansion, *J. Comput. Finance* 5 (3) (2002) 1–33.
- [35] R. Kaas, J. Dhaene, M.J. Goovaerts, Upper and lower bounds for sums of random variables, *Insurance Math. Econom.* 27 (2000) 151–168.
- [36] A. Kemna, A. Vorst, A pricing method for options based on average asset prices, *J. Banking Finance* 14 (1990) 113–129.
- [37] T.-H. Kim, H. White, On more robust estimation of skewness and kurtosis: Simulation and application to the S&P 500 Index, Working Paper, Department of Economics, UC San Diego, 2003.
- [38] T.R. Klassen, Simple, fast, and flexible pricing of Asian options, *J. Comput. Finance* 4 (2001) 89–124.
- [39] P. L'Ecuyer, G. Perron, On the convergence rates of IPA and FDC derivative estimators, *Operations Res.* 42 (4) (1994) 643–656.
- [40] E. Lévy, Pricing European average rate currency options, *J. Int. Money Fin.* 11 (1992) 474–491.
- [41] V. Linetsky, Spectral expansions for Asian (average price) options, *Operations Res.* 52 (6) (2004) 856–867.
- [42] M.A. Milevsky, S.E. Posner, Asian options, the sum of lognormals, and the reciprocal Gamma distribution, *J. Fin. Quan. Anal.* 33 (1998) 409–422.
- [43] M.A. Milevsky, S.E. Posner, A closed-form approximation for valuing basket options, *J. Derivatives* 5 (4) (1998) 54–61.
- [44] E.H. Neave, G.L. Ye, Pricing Asian options in the framework of the binomial model: A quick algorithm, *Derivative Use, Trading Regulation* 9 (3) (2003) 203–216.
- [45] J.A. Nielsen, K. Sandmann, Pricing bounds on Asian options, *J. Fin. Quan. Anal.* 38 (2) (2003) 449–473.
- [46] S.E. Posner, M.A. Milevsky, Valuing exotic options by approximating the SPD with higher moments, *J. Fin. Eng.* 7 (1998) 109–125.
- [47] L.C.G. Rogers, Z. Shi, The value of an asian option, *J. Appl. Probab.* 32 (1995) 1077–1088.
- [48] M. Rubinstein, Implied binomial trees, *J. Finance* 49 (3) (1994) 771–818.
- [49] M. Rubinstein, Edgeworth binomial trees, *J. Derivatives* 5 (3) (1998) 20–27.
- [50] W.T. Shaw, Pricing Asian options by contour integration, including asymptotic methods for low volatility, Working Paper, Math. Institute, 2002.
- [51] Y. Su, M.C. Fu, Importance sampling in derivative securities pricing, in: J.A. Joines, R.R. Barton, K. Kang, P.A. Fishwick (Eds.), *Proceedings of the 2000 Winter Simulation Conference*, 2000, pp. 587–596.
- [52] G.W.P. Thompson, Fast narrow bounds on the value of Asian options, Working Paper, Centre for Financial Research, Judge Institute of Management, University of Cambridge, 2000.
- [53] S. Turnbull, L. Wakeman, A quick algorithm for pricing European average options, *J. Fin. Quan. Anal.* 26 (September) (1991) 377–389.
- [54] F.J. Vázquez-Abad, D. Dufresne, Accelerated simulation for pricing Asian options, in: D.J. Medeiros, E.F. Watson, J.S. Carson, M.S. Manivannan (Eds.), *Proceedings of the 1998 Winter Simulation Conference*, 1998, pp. 1493–1500.
- [55] J. Večeř, A new PDE approach for pricing arithmetic average Asian options, *J. Comput. Finance* 4 (2001) 105–113.
- [56] J.E. Zhang, A semi-analytical method for pricing and hedging continuously sampled arithmetic average rate options, *J. Comput. Finance* 5 (2001) 59–79.
- [57] J.E. Zhang, Pricing continuously sampled Asian options with perturbation method, *J. Futures Markets* 23 (6) (2003) 535–560.