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More on pooling spaces

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Abstract

A *pooling space* is a ranked poset P such that the subposet w^+ induced by the elements above w is atomic for each element w of P. Pooling spaces were introduced in [T. Huang, C. Weng, Pooling spaces and non-adaptive pooling designs, Discrete Math. 282 (2004) 163–169] for the purpose of giving a uniform way to construct pooling designs, which have applications to the screening of DNA sequences. Many examples of pooling spaces were given in that paper. In this paper, we clarify a few things about the definition of pooling spaces. Then we find that a *geometric lattice*, a well-studied structure in literature, is also a pooling space. This provides us many classes of pooling designs, some old and some new. We study the pooling designs constructed from affine geometries. We find that some of them meet the optimal bounds related to a conjecture of Erdös, Frankl and Füredi.

Keywords: Pooling spaces; Pooling designs; Ranked posets; Atomic; Geometric lattices; Affine geometries

1. Introduction

A binary matrix M is b^d*-disjunct* if for any $b+1$ columns $x, x_1, x_2, \ldots, x_b$ of M with x different to the others, there exist d rows such that x has values 1, and $x_1, x_2, \ldots, x_b$ all have values 0 at these d rows. b^1-disjunct matrices are also called *b-disjunct* matrices for simplicity. b-disjunct matrices were introduced in 1964 by Kautz and Singleton [9], and the concept of b^d-disjunct matrices were introduced by D'yachkov, Rykov and Rachad [4]. A b^d-disjunct matrix can be used to construct an error-tolerable design for non-adaptive group testing, which has applications to the screening of DNA sequences, and the corresponding decoding algorithm is efficient. See [2,7] for details. Hence a b^d-disjunct matrix is also called a *pooling design*.

The constructions of b^d-disjunct matrices were given by many authors, e.g. [11–13], [3]. These constructions use some properties of a ranked poset. In [8], the name *pooling spaces* was given to describe these ranked posets (formal definition in Section 3). Fix a pooling space P and positive integers $r < k$. Let M denote the incidence matrix between the rank r elements and the rank k elements in P. It was shown in [8] that M is b^d-disjunct for $b = r$ and $d = 1$. In some pooling spaces, e.g. the poset consisting of the subsets of a fixed set, the result is optimal in the sense that M is not b^d-disjunct for any $b > r$ or $d > 1$, but in some pooling spaces, e.g. a projective space, b can be sufficiently larger than r and d can be sufficiently larger than 1 [3]. We refer to the study of *optimal-disjunct properties* of a binary matrix M as the determination of b, d such that M is b^d-disjunct, but not b^{d+1}-disjunct. Note that the pair (b, d) may not be unique.

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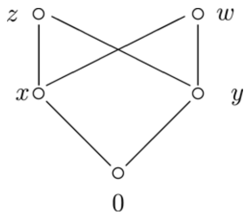

Fig. 1. A poset.

The main result of the paper appears in Section 5, in which we show that a geometric lattice (formal definition in Section 5) is a pooling space, and use this result to provide many classes of pooling spaces including the class of affine geometries. When we explore the optimal-disjunct properties of the binary matrices constructed from affine geometries in Section 6, we find a class of b-disjunct matrices that meets an optimal bound related to a conjecture of Erdös, Frankl and Füredi [5]. This optimal bound and its relation to the conjecture will be described in the end of Section 6.

The remaining sections are for background introduction and for the prior setting to realize the proof of our main theorem in Section 5. For example we review the basic definitions and properties of a partially ordered set in Section 2. In Section 3, we review the definition of pooling space which was first given in [8]. From the revisited we find a mistake appearing in the abstract of [8] and correct it in Proposition 3.5. To have a concrete realization of the proof of our main result, we do its special case in Section 4 by showing that the poset of contractions of a fixed graph is a pooling space. This class of posets is the first class of pooling spaces without modular intervals (formal definition in Section 2).

2. Preliminaries

In this section we give the basic definitions and properties of a partially ordered set. The expert may want to skip the remaining of this section and go to the next section.

Let P denote a finite set. By a *partial order* on P, we mean a binary relation $\leq$ on P such that

(i) $x \leq x$ for $x \in P$,
(ii) $x \leq y$ and $y \leq z \longrightarrow x \leq z$ for $x, y, z \in P$,
(iii) $x \leq y$ and $y \leq x \longrightarrow x = y$ for $x, y \in P$.

By a *partially ordered set* (or *poset*, for short), we mean a pair $(P, \leq)$, where P is a finite set, and where $\leq$ is a partial order on P. By abusing notation, we will suppress reference to $\leq$, and just write P instead of $(P, \leq)$.

Let P denote a poset with partial order $\leq$, and let x and y denote any elements in P. As usual, we write $x < y$ whenever $x \leq y$ and $x \neq y$, and write $x \nless y$ whenever $x < y$ is not true. We say that y *covers* x whenever $x < y$, and there is no $z \in P$ such that $x < z < y$. A sequence $x_0, x_1, \ldots, x_t$ of elements of P is said to be a *direct chain of length* t whenever x_i covers x_{i-1} for $1 \leq i \leq t$. A poset can be described by a diagram in the plane in which the elements are corresponding to dots, and y covers x whenever dot y is placed above dot x with an edge connecting them. See Fig. 1 for the diagram of the poset with five elements $\{0, x, y, z, w\}$, and x, y covers 0; z, w cover x, y respectively. Note that $0, x, z$ is a direct chain of length 2.

Let P denote any finite poset, and let S denote any subset of P. Then there is a unique partial order on S such that for all $x, y \in S$, $x \leq y$ in S if and only if $x \leq y$ in P. This partial order is said to be *induced* from P. By a *subposet* of P, we mean a subset of P, together with the partial order induced from P. An element $x \in S$ is said to be *minimal* (resp. *maximal*) in S whenever there is no $y \in S$ such that $y < x$ (resp. $x < y$). Let $\min(S)$ (resp. $\max(S)$) denote the set of all minimal (resp. maximal) elements in P. Whenever $\min(P)$ (resp. $\max(P)$) consists of a single element, we denote it by 0 (resp. 1), and we say that P has *the least element* 0 (resp. *the greatest element* 1).

Throughout the remaining of the paper we assume P is a poset with the least element 0. By an *atom* in P, we mean an element in P that covers 0. We let A_P denote the set of atoms in P. By the *interval* $[x, y]$, where $x, y \in P$ with $x \leq y$, we mean the subposet

$$[x, y] := \{z | z \in P, x \leq z \leq y\}$$

of P.

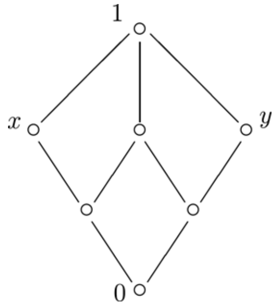

Fig. 2. An upper semi-modular lattice that is not lower semi-modular.

By a *rank function* on P, we mean a function rank from P to the set of nonnegative integers such that $\text{rank}(0) = 0$, and such that for all $x, y \in P$, y covers x implies $\text{rank}(y) - \text{rank}(x) = 1$. Observe that the rank function is unique if it exists. P is said to be *ranked* whenever P has a rank function. In this case, we set

$$\text{rank}(P) := \max\{\text{rank}(x) \mid x \in P\},$$
$$P_i := \{x \mid x \in P, \text{rank}(x) = i\},$$

and observe $P_0 = \{0\}$, $P_1 = A_P$. Observed P is ranked if and only if for any $x \in P$ every direct chain from 0 to x has the same length. Let P be a ranked poset of rank n and fix two integers $1 \leq r < k \leq n$. The *incidence matrix* M between P_r and P_k is a $|P_r| \times |P_k|$ binary matrix with rows indexed by P_r and columns indexed by P_k such that

$$M_{xy} := \begin{cases} 1, & x \leq y; \\ 0, & \text{else} \end{cases} \quad \text{for } x \in P_r, y \in P_k.$$

Let S be a subset of P. Fix $z \in P$. Then z is said to be an *upper bound* (resp. *lower bound*) of S, if $z \geq x$ (resp. $z \leq x$) for all $x \in S$. Suppose the subposet of upper bounds (resp. lower bounds) of S has a unique minimal (resp. maximal) element. In this case we call this element *the least upper bound* or *join* (resp. *the greatest lower bound* or *meet*) of S. If $S = \{x_1, x_2, \ldots, x_t\}$ we write $x_1 \vee x_2 \vee \cdots \vee x_t$ for the join of S and $x_1 \wedge x_2 \wedge \cdots \wedge x_t$ for the meet of S. P is said to be *atomic* whenever for each nonzero element x of P, x is the join of atoms in the interval $[0, x]$. Suppose P is atomic and $x < y$ are two elements in P. Observe that the atoms in the interval $[0, x]$ is a proper subset of the atoms in the interval $[0, y]$. P is said to be a *meet semi-lattice* (resp. *join semi-lattice*) whenever P is nonempty, and $x \wedge y$ (resp. $x \vee y$) exists for all $x, y \in P$. A meet semi-lattice (resp. *join semi-lattice*) has a 0 (resp. 1). A meet and join semi-lattice is called a *lattice*.

Suppose P is a lattice. Then P is said to be *upper semi-modular* (resp. *lower semi-modular*) whenever for all $x, y \in P$,

$$y \text{ covers } x \wedge y \longrightarrow x \vee y \text{ covers } x$$
$$(\text{resp. } x \vee y \text{ covers } x \longrightarrow y \text{ covers } x \wedge y).$$

P is said to be *modular* whenever P is upper semi-modular and lower semi-modular.

Fig. 2 is a diagram of an upper semi-modular lattice with 7 elements. This lattice is not lower semi-modular since $1 = x \vee y$ covers x but y does not cover $0 = x \wedge y$.

3. Pooling spaces

Definition 3.1. Let P be a ranked poset. For any $w \in P$, define

$$w^+ = \{y \geq w \mid y \in P\}.$$

P is said to be a *pooling space* whenever w^+ is atomic for each $w \in P$.

In particular a pooling space is atomic. It is immediate from the definition that if P is a pooling space, then so is w^+ for any $w \in P$.

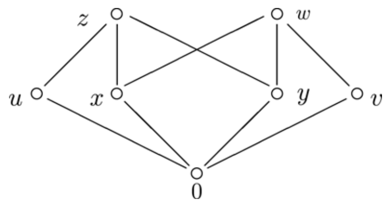

Fig. 3. A pooling space which is not a meet semi-lattice.

Lemma 3.2. *Let P be a pooling space. Then each interval in P is atomic.*

Proof. Let $[x, y]$ be an interval in P and $z \in [x, y]$ with $z \neq x$. Suppose $x \in P_i$. Note that the set of atoms contained in $[x, z]$, no matter considered in x^+ or in $[x, y]$, is the same set $[x, z] \cap P_{i+1}$. Since z is the join of $[x, z] \cap P_{i+1}$ in x^+ by assumption, z is also the join of $[x, z] \cap P_{i+1}$ in $[x, y]$. $\square$

Remark 3.3. The definition of pooling space was first given in [8]. However in the abstract of that paper, it was stated in an alternative way that a pooling space is a ranked poset with atomic intervals. The following example shows that this is not correct.

Example 3.4. Let $P = \{0, x, y, z, w\}$ and the partial order is defined as in Fig. 1 of Section 2. Then each interval in P is atomic. Since neither z nor w is the least upper bound of x and y, P is not atomic. Observe that P is not a meet semi-lattice.

We now give a revised version.

Proposition 3.5. *Let P be a ranked meet semi-lattice. Then P is a pooling space if and only if each interval in P is atomic.*

Proof. We have just proved the necessary condition in the previous lemma. To prove the sufficient condition we fix an element $w \in P$ and suppose $w \in P_s$ for some integer $0 \leq s \leq \text{rank}(P)$. We shall prove that w^+ is atomic. To do this fix $x \in w^+ \setminus \{w\}$ and we need to prove that x is the join of $[w, x] \cap P_{s+1}$ in w^+. By the assumption $[w, x]$ being atomic, x is the join of $[w, x] \cap P_{s+1}$ in $[w, x]$. In particular, x is an upper bound of $[w, x] \cap P_{s+1}$. Since P is a meet semi-lattice, the upper bounds of $[w, x] \cap P_{s+1}$ have a least element and denote it by y. Hence $y \leq x$ and clearly $w \leq y$, so equivalently $y \in [w, x]$. This forces $x \leq y$ and then obtains $x = y$. $\square$

We give a poling space which is not a meet semi-lattice.

Example 3.6. Let $P = \{0, u, x, y, v, z, w\}$ and let the partial order be defined as in the Fig. 3. Observe $z = u \vee x \vee y$ and $w = x \vee y \vee v$. The remaining properties of a pooling space hold trivially. Hence P is a pooling space. P is not a meet semi-lattice since $z \wedge w$ does not exist.

4. The contractions of a graph

Many examples of pooling spaces were given in [8]. They are related to the Hamming matroids, the attenuated spaces, and six classical polar spaces. Among these examples there is a common property: each interval is modular. In this section we will construct pooling spaces without modular intervals. The construction in this section also can be obtained as a consequence of our main theorem in the next section. We do it earlier and repeatedly here for the purpose to give the reader a concrete impression of a pooling space, and hope that one can find his own class of examples in the sequel.

Throughout the section let G denote a simple connected graph on n vertices.

Definition 4.1. Let $P = P(G)$ denote the set of partitions A of the vertex set $V(G)$ such that the subgraph induced by each block of A is connected. For $A, B \in P$, define

$$A \leq B \iff A \text{ is a refinement of } B.$$

The poset $(P(G), \leq)$ is called the poset of *contractions* of G.

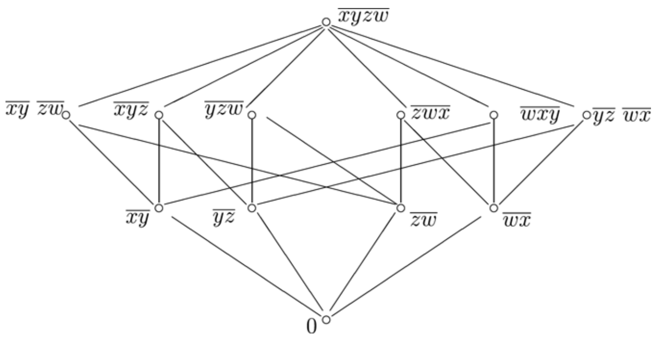

Fig. 4. The poset $P(C_4)$ of contractions of C_4. $P(C_4)$ is upper semi-modular, but not lower semi-modular.

Example 4.2. Let G denote a graph with the vertex set $\{x, y, z, w\}$ and edge set $\{\overline{xy}, \overline{yz}, \overline{zw}, \overline{wx}\}$, i.e. G is the 4-cycle C_4. Then the poset $P(G)$ is as in Fig. 4. We delete the single element blocks in the notation of a partition, e.g. the notation 0 is used to denote the partition with four blocks $\{x\}, \{y\}, \{x\}, \{w\}$ respectively, and $\overline{xy}$ is used to denote the partition with three blocks $\{x, y\}, \{z\}, \{w\}$ respectively. The poset is a lattice, but not a modular lattice. This is because the join of $\overline{xy}\,\overline{zw}$ and $\overline{yz}\,\overline{wx}$ is $\overline{xyzw}$, which covers $\overline{xy}\,\overline{zw}$, but $\overline{yz}\,\overline{wx}$ does not cover their meet 0. Observe the subposet induced on $\overline{xy}^+$ is $P(C_3)$, the poset of contractions of a triangle.

Lemma 4.3. *$P(G)$ is a ranked poset of rank $n - 1$. The rank i elements are those elements in $P(G)$ with $n - i$ blocks for $0 \le i \le n - 1$.*

Proof. For $D \in P(G)$ with $n - i$ blocks define the rank of D to be i, where $0 \le i \le n - 1$. We claim that this is a rank function. Suppose that B covers A and $\mathrm{rank}(A) = i$. Since A is a proper refinement of B, $\mathrm{rank}(B) \ge i + 1$ and there are two blocks in A that are contained in the same block of B. Let C be an element in $P(G)$ that glues these two blocks of A. Then $A < C \le B$ and $\mathrm{rank}(C) = \mathrm{rank}(A) + 1$. This shows $C = B$ and $\mathrm{rank}(B) = i + 1$. □

Proposition 4.4. *$P(G)$ is a pooling space of rank $n - 1$.*

Proof. $P(G)$ is ranked by the previous lemma. From the previous lemma and the definition, each atom in $P(G)$ contains $n - 1$ blocks, one block containing two adjacent vertices and each of the remaining $n - 2$ blocks containing a single vertex. By identifying the atoms with the edges of G we find that each element $A \in P(G)$ is the join of those edges contained in the induced subgraph of G corresponding to each block of A. This shows that $P(G)$ is atomic. More generally, for $B \in P(G)$, the subposet B^+ is also atomic. This is because the subposet B^+ is isomorphic to the poset $P(B_G)$ of contractions of B_G, where B_G is the graph with the vertex set B, and for two distinct blocks $x, y \in Bx$ is adjacent to y whenever some vertex in x is adjacent to some vertex in y. □

Remark 4.5. Let $G = K_n$ denote the complete graph on n vertices. Then the elements in $P = P(K_n)$ are all the partitions of the vertex set of K_n. $S(n, k) := |P_{n-k}|$ is called the *Stirling number of the second kind* where $k \ge 1$. It is well known that $S(n, k)$ can be solved by the recurrence relation

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k) \quad \text{for } 1 \le k \le n - 1$$

with initial condition $S(n, 0) := 0$ for $n \ge 1$, $S(0, 0) := 1$, and $S(n, n) = 1$ for $n \ge 1$. See [1, Section 8.2] for details.

By applying Proposition 4.4 and Remark 4.5 with the result in [8, Corollary 3.2] we immediately have the following corollary.

Corollary 4.6. *Let G denote a simple connected graph on n vertices and $P = P(G)$. Let $C(G, k, r)$ denote the incidence matrix between P_r and P_k where $1 \le r < k \le n - 1$. Then $C(G, k, r)$ is r-disjunct. In particular if $G = K_n$, then the matrix $C(G, k, r)$ has size $S(n, n - r) \times S(n, n - k)$.* □

5. Geometric lattice

The concept of geometric lattices can be described in very different ways. See [10, Chapter 23] for details. For the purpose to derive our main result easily, we adopt the definition that a *geometric lattice* is an upper semi-modular atomic lattice [10, Page 271]. We will show that a geometric lattice is a pooling space in this section. The following lemma is immediate from the definition.

Lemma 5.1. *Let P be an upper semi-modular lattice. Then the poset induced on every interval of P is an upper semi-modular lattice.* □

Lemma 5.2. *Let P be a geometry lattice. Then the poset induced on every interval of P is a geometric lattice.*

Proof. Let $[x, y]$ denote an interval in P where $x \in P_i$. By the previous lemma, it remains to show that $[x, y]$ is atomic. Fix $z \in [x, y]$ with $z \neq x$. Suppose that w is the join of $P_{i+1} \cap [x, z]$, the atoms in $[x, z]$. Then $w \leq z$. We are done if $w = z$, so assume $w < z$. Then there exists an atom in $a \in [0, z] \setminus [0, w]$. Note that $a \not\leq x$. By the upper semi-modularity, $a \vee x \in P_{i+1} \cap [x, z]$ is an atom in $[x, z]$, a contradiction to $a \vee x \not\leq w$. □

Lemma 5.3. *An upper semi-modular lattice is ranked. In particular, a geometric lattice is ranked.*

Proof. Let P be an upper semi-modular lattice and suppose that P is not ranked. Then there exists $x \in P$ such that $[x, 1]$ is not ranked, but for all atoms a of $[x, 1]$, $[a, 1]$ is ranked. Pick an atom $a \in [x, 1]$. Let f be a rank function on $[a, 1]$. We extend the function f to a function f' in $[x, 1]$ by defining

$$f'(y) := \begin{cases} f(y) + 1, & \text{if } y \in [a, 1]; \\ f(a \vee y), & \text{else.} \end{cases}$$

We shall prove that f' is a rank function in $[x, 1]$. Suppose that $u, v \in [x, 1]$ and u covers v. We need to show $f'(u) = f'(v) + 1$. This is clear if $v \in [a, 1]$. Assume $v \notin [a, 1]$. Suppose $u \in [a, 1]$. Then $u = a \vee v$ and $f'(u) = f(a \vee v) + 1 = f'(v) + 1$. Suppose $u \notin [a, 1]$. Since u covers $v = (a \vee v) \wedge u$, we have $a \vee u = (a \vee v) \vee u$ covers $a \vee v$. Then $f'(u) = f(a \vee u) = f(a \vee v) + 1 = f'(v) + 1$. This concludes that $[x, 1]$ is ranked, a contradiction. □

Theorem 5.4. *Let P be a geometric lattice. Then P is a pooling space.*

Proof. P is ranked by Lemma 5.3. Since each interval of P is a geometry lattice by Lemma 5.2, each interval is atomic. The theorem now follows from Proposition 3.5. □

By applying Theorem 5.4 with the result in [8, Corollary 3.2] we immediately have the following corollary.

Corollary 5.5. *Let P be a geometric lattice with rank n. Let $G(P, k, r)$ denote the incidence matrix between P_r and P_k where $1 \leq r < k \leq n$. Then $G(P, k, r)$ is r-disjunct.* □

Many examples of geometry lattices are listed in Chapter 23 of [10]. These are related to *linear system spaces*, *Steiner systems*, *affine geometries*, *projective geometries* and contractions of graphs. More examples are given in [6]. In some cases the corresponding results in Corollary 5.5 are not optimal. The optimal-disjunct properties on projective geometries were studied in [3].

6. Affine geometries

In this section we study the optimal-disjunct properties of the binary matrices constructed from affine geometries. The idea is exactly the same as the study of projective geometries in [3]. In fact this idea works for any geometric lattices with each interval isomorphic to a projective geometry. For completeness of the paper, we still provide the proof. Also there are some small computation mistakes in [3]. We will point out these mistakes after Corollary 6.8. In the beginning, we give the definition of affine geometries.

Definition 6.1. Let V denote an n-dimensional vector space over a finite field F_q, where q is the number of elements in the field. Let $P = P(V)$ denote the poset with element set

$$P = \{u + W \mid u \in V \text{ and } W \subseteq V \text{is a subspace}\} \cup \{\emptyset\},$$

where $\emptyset$ denote the empty set. The order is defined by inclusion. Note that P is a geometric lattice of rank $n + 1$. P is called the *affine geometry* and is denoted by $AG_n(F_q)$. The rank i elements in P_i are referred to as the *affine*

$(i-1)$-*subspaces* of V for $1 \le i \le n+1$. We say that the affine subspaces $u+W$ and $v+W$ are *parallel* for vectors $u, v \in V$ and subspace $W \subseteq V$.

We immediately have the following lemma.

Lemma 6.2. *Let V denote an n-dimensional vector space over a finite field F_q. Let $u_1, u_2 \in V$ be elements and let $W_1, W_2 \subseteq V$ be subspaces. Then $u_1 + W_1 = u_2 + W_2$ if and only if $W_1 = W_2$ and $u_1 - u_2 \in W_1$.* $\square$

Definition 6.3. Let $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denote the number of k-dimensional subspaces of an n-dimensional vector space over a finite field F_q.

Lemma 6.4. *Let V denote an n-dimensional vector space over a finite field F_q, and A denote an affine k-subspace of V. Then the number of affine r-subspaces contained in A is*

$$q^{k-r} \begin{bmatrix} k \\ r \end{bmatrix}_q,$$

where $r < k$. These affine r-subspaces in A are partitioned into

$$\begin{bmatrix} k \\ r \end{bmatrix}_q \tag{6.1}$$

classes, each class consisting of q^{k-r} parallel affine subspaces.

Proof. The parallel property defines an equivalent relation on the set of affine r-subspaces in A. The number of equivalent classes is as in (6.1) and each equivalent class consists of q^{k-r} elements by Lemma 6.2. $\square$

The following lemma is well known [10, p. 291].

Lemma 6.5. *For $1 \le k \le n$,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}. \quad \square$$

Theorem 6.6. *Let V denote an n-dimensional vector space over a finite field F_q. Fix integers $1 \le r < k \le n$ and a positive integer b. Let $A, A_1, A_2, \ldots, A_b$ denote affine k-subspaces of V with $A \ne A_i$ for $1 \le i \le b$. Then there are at least*

$$d := q^{k-r} \begin{bmatrix} k \\ r \end{bmatrix}_q - bq^{k-r-1} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \tag{6.2}$$

affine r-subspaces contained in A and not contained in any of A_i for $1 \le i \le b$.

Proof. There are

$$q^{k-r} \begin{bmatrix} k \\ r \end{bmatrix}_q$$

affine r-subspaces contained in A, some of them in some affine subspace $A \cap A_i$ for each $1 \le i \le b$ to be deducted. $A \cap A_i$ takes maximal coverage of these affine r-subspaces when $A \cap A_i$ is an affine $(k-1)$-subspace, and in this situation the number of these affine r-subspaces is

$$q^{(k-1)-r} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q. \quad \square$$

Remark 6.7. For positive integers $b \le q$ and $k < n$ the number in (6.2) is optimal. We choose A_i to an affine k-subspace with the meet with A corresponding to each of the q parallel affine $(k-1)$-subspaces in A. Then (6.2) is exactly the number of affine r-subspaces contained in A and not contained in any of A_i for $1 \le i \le b$.

From Lemma 6.4, Theorem 6.6 and Remark 6.7 we have the following corollary.

Corollary 6.8. *Let $P = AG_n(F_q)$ and let $E_q(n+1, k+1, r+1)$ denote the incidence matrix between P_{r+1} and P_{k+1} where $r < k$. Then $E_q(n+1, k+1, r+1)$ is b^d-disjunct with size*

$$q^{n-r}\begin{bmatrix} n \\ r \end{bmatrix}_q \times q^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_q,$$

where b is any positive integer less than $\frac{q(q^k-1)}{q^{k-r}-1}$ to ensure $d > 0$ in (6.2). Moreover, if $k < n$ and b is a positive integer such that

$$\begin{cases} b \leq q, & \text{if } r > 0; \\ b \leq q-1, & \text{if } r = 0, \end{cases} \tag{6.3}$$

then $E_q(n+1, k+1, r+1)$ is not b^{d+1}-disjunct. $\square$

The result in [3, Corollary 4.6] is similar to Corollary 6.8, but the former makes a mistake for not separating the case $r = 0$ in (6.3) from $r > 0$. This mistake inherits an earlier mistake in [3, Theorem 4.4], referring to the last line of its proof. The case $r = 0$ of (6.3) will be important in our following discussion. In view of (6.2) b increases if and only if d decreases. We set $r = 0$ and $b = q - 1$ to be the largest possible integer in Corollary 6.8 to obtain the following result.

Corollary 6.9. *$E_q(n+1, k+1, 1)$ is $(q-1)^{q^{k-1}}$-disjunct, but not $(q-1)^{q^{k-1}+1}$-disjunct, with size $q^n \times q^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_q$.* $\square$

We promised in the introduction to give some matrices that meet some optimal bound. These matrices are $E_q(3, 2, 1)$, where q is a power of a prime. We describe an optimal bound of an assumption below, and show the relation of this assumption and the conjecture of Erdös, Frankl and Füredi [5] later.

Assumption. Any b-disjunct matrix of size $s \times t$ with $s < t$ must have $s \geq (b+1)^2$.

We do not know if the above assumption is true, but $E_q(3, 2, 1)$ attains the equality $s = (b+1)^2$, since $E_q(3, 2, 1)$ is a $(q-1)$-disjunct matrix of size $q^2 \times (q^2 + q)$ by Corollary 6.9. In fact the above assumption is a consequence of the following conjecture of Erdös, Frankl and Füredi in [5]:

EFF Conjecture. *Any b-disjunct matrix of size $s \times (b+1)^2$ must have $s \geq (b+1)^2$.*

Also see [2, page 29] for the above conjecture. Suppose that EFF Conjecture is true and suppose that the above assumption fails. Let M be a b-disjunct matrix of size $s \times t$ with $s < t$, but $s < (b+1)^2$. If $t \geq (b+1)^2$ then we obtain a b-disjunct matrix of size $s \times (b+1)^2$ by deleting any $t - (b+1)^2$ columns of M. This contradicts the EFF Conjecture. Suppose $t < (b+1)^2$. Then we make a larger b-disjunct matrix by taking the direct sum of M and the $((b+1)^2 - t) \times ((b+1)^2 - t)$ identity matrix to become a matrix of size $((b+1)^2 - t + s) \times (b+1)^2$. We also have a contradiction to EFF Conjecture since $(b+1)^2 - t + s < (b+1)^2$.

To conclude the paper we stress again a special property of $E_q(3, 2, 1)$: $E_q(3, 2, 1)$ has more columns than rows. In the similar construction of disjunct matrices from a projective geometry of rank 3 [3], only square matrices can be obtained.

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