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Vertex-bipancyclicity of the generalized honeycomb tori ☆

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ABSTRACT

Assume that m, n and s are integers with $m \geq 2, n \geq 4, 0 \leq s < n$ and s is of the same parity of m. The generalized honeycomb tori GHT(m, n, s) have been recognized as an attractive architecture to existing torus interconnection networks in parallel and distributed applications. A bipartite graph G is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. G is *vertex-bipancyclic* if for any vertex $v \in V(G)$, there exists a cycle of every even length from 4 to $|V(G)|$ that passes v. A bipartite graph G is called *k-vertex-bipancyclic* if every vertex lies on a cycle of every even length from k to $|V(G)|$. In this article, we prove that GHT(m, n, s) is 6-bipancyclic, and is bipancyclic for some special cases. Since GHT(m, n, s) is vertex-transitive, the result implies that any vertex of GHT(m, n, s) lies on a cycle of length l, where $l \geq 6$ and is even. Besides, GHT(m, n, s) is vertex-bipancyclic in some special cases. The result is optimal in the sense that the absence of cycles of certain lengths on some GHT(m, n, s)'s is inevitable due to their hexagonal structure.

1. Introduction

Network topology is a crucial factor for an interconnection network since it determines the performance of the network [1]. Stojmenovic [2] proposed variations of honeycomb tori and those honeycomb tori have been recognized as an attractive architecture to existing torus interconnection networks in parallel and distributed applications [2–6]. Cho and Hsu [7] proved that all these honeycomb torus networks can be characterized in a unified way, called the *generalized honeycomb torus*. Recently, there have been many studies about honeycomb networks [8–13].

The cycle-embedding problem is one of the most popular research problems [14]. From the applicational point of view, efficient algorithms and execution methods are required for communication patterns in networks. The study of certain topological structures on network designs provides a systematic and logical analysis for the desired performance. Since cycles in networks are useful in embedding linear arrays and rings, the existence of cycles with various lengths on networks has been largely investigated. (See [15–18] and their references.) A graph G is *pancyclic* if it contains cycles of all lengths from 3 to $|V(G)|$. A graph G is *k-pancyclic* if it contains cycles of all lengths from k to $|V(G)|$. A graph G is called *vertex-pancyclic* (resp. *edge-pancyclic*) if every vertex (resp. edge) lies on a cycle of every length from 3 to $|V(G)|$. Moreover, G is called *k-vertex-pancyclic* if every vertex lies on a cycle of every length from k to $|V(G)|$. These concepts are defined for bipartite graphs similarly. Let $H = (B \cup W, E)$ be a bipartite graph, where $B \cup W = V(H)$ and $E \subseteq \{(u, v) \mid u \in B \text{ and } v \in W\}$. Obviously, H has no odd cycles. We say that H is *bipancyclic* if it has cycles of all even lengths from 4 to $|V(G)|$. H is *k-bipancyclic* if it contains cycles of all even lengths from k to $|V(G)|$. H is called *vertex-bipancyclic* (resp. *edge-bipancyclic*) if every vertex

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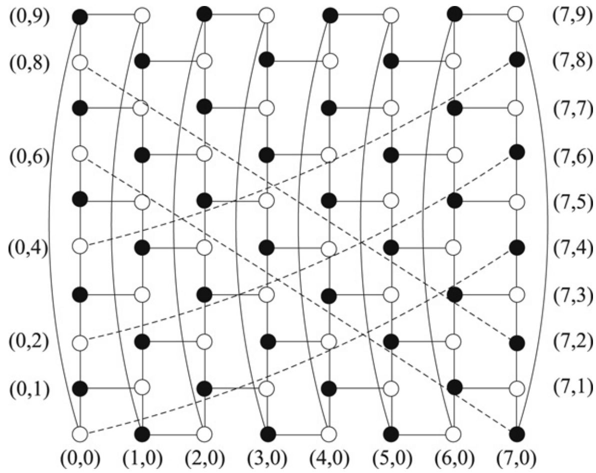

Fig. 1. The graph GHT(8, 10, 4). Notice that the dotted lines are crossing edges. For example, the edges ((1, 0), (2, 0)) and ((5, 0), (6, 0)) are the horizontal edges while the edges ((0, 0), (7, 4)) and ((0, 8), (7, 2)) are crossing edges.

(resp. edge) lies on a cycle of every even length from 4 to $|V(G)|$. Also, H is called *k-vertex-bipancyclic* if every vertex lies on a cycle of every even length from k to $|V(G)|$. There are numerous studies about the pancyclicity of hypercubes and their variants [19,20], products of graphs [21–23], and some classes of graphs [24,25]. Vertex-pancyclicity and edge-pancyclicity were discussed in [26–29], and many related studies were published recently [30–36].

In this article, we prove that the generalized honeycomb tori are vertex-bipancyclic in some special cases and contain cycles of length l, where $l \geq 6$ is an even integer, in most cases. More specifically, let GHT(m, n, s) be a generalized honeycomb torus, where $m \geq 3$ is an integer, $n \geq 6$ is an even integer, and $s \geq 0$ is an integer with $m + s$ even. We study the existence of cycles of different lengths in GHT(m, n, s) with various combinations of m, n, and s. The result is optimal in the sense that the absence of cycles of certain lengths on some GHT(m, n, s)'s is inevitable due to their hexagonal structure.

2. Preliminaries

An interconnection network is represented by a graph with vertices and edges symbolizing the processors and communication links between processors, respectively. In this paper, a network is represented as an undirected graph. For the graph definition and notation we follow [37]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* is represented by $\langle v_0, v_1, v_2, \ldots, v_k\rangle$. We also write the path $\langle v_0, v_1, v_2, \ldots, v_k\rangle$ as $\langle v_0, P_1, v_i, v_{i+1}, \ldots, v_j, P_2, v_t, \ldots, v_k\rangle$, where P_1 is the path $\langle v_0, v_1, \ldots, v_{i-1}, v_i\rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \ldots, v_{t-1}, v_t\rangle$. Hence, it is possible to write a path $\langle v_0, v_1, P, v_1, v_2, \ldots, v_k\rangle$ if the length of P is zero. If a path $Q = \langle v_0, v_1, v_2, \ldots, v_k\rangle$, then Q^{-1} denotes the path $\langle v_k, v_{k-1}, \ldots, v_1, v_0\rangle$. A *cycle* is a path of at least three vertices such that the first vertex and the last vertex are identical. Let C be a cycle and P a path. We use $|C|$ to denote the total number of distinct vertices/edges on C and $|P|$ the total number of distinct edges of P.

Throughout this paper, we use the following notations. For any two positive integers r and d, $[r]_d$ denotes $r \pmod d$. Let m, n and s be positive integers with $m \geq 2$, $n \geq 4$, n and $m + s$ are even. The *generalized honeycomb torus* GHT(m, n, s) is the graph with the vertex set $\{(i, j) \mid 0 \leq i < m, 0 \leq j < n\}$ such that (i, j) and (k, l) with $i \leq k$ are adjacent if they satisfy one of the following conditions:

1. $(k, l) = (i, [j \pm 1]_n)$;
2. $0 \leq i \leq m - 2$, $i + j$ is odd and $(k, l) = (i + 1, j)$;
3. $i = 0$, j is even, and $(k, l) = (m - 1, [j + s]_n)$.

Any edge satisfying the second condition is called a *horizontal edge*, and any edge satisfying the third condition is called a *crossing edge*. Fig. 1 gives an illustration of the graph GHT(8, 10, 4). For example, the edges ((1, 0), (2, 0)) and ((5, 0), (6, 0)) are the horizontal edges while the edges ((0, 0), (7, 4)) and ((0, 8), (7, 2)) are crossing edges. Fig. 2 shows that crossing edges can become horizontal edges in different layouts for the same graph. Obviously, any generalized honeycomb torus is a 3-regular bipartite graph. Moreover, any generalized honeycomb torus is vertex-transitive [7].

3. Bipancyclicity of GHT(m, n, s)

Let C_i denote a cycle in GHT(m, n, s) with $|C_i| = i$, where $i \in \{4, 6 + 2t \mid 0 \leq t \leq \frac{mn}{2} - 3\}$. Since GHT$(m, n, s)$ consists of hexagons, the existence of C_4 is missed in all cases except for $n = 4$. In addition, GHT(m, n, s) contains mn vertices, so

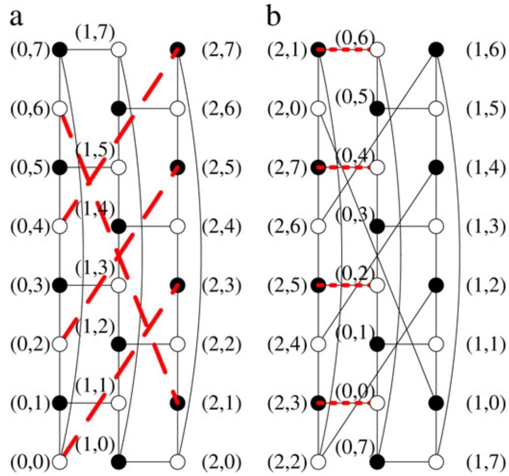

Fig. 2. The graph GHT(3, 8, 3). Notice that crossing edges in (a) can become horizontal edges in (b).

we should construct cycles of even lengths from 6 to mn. Yang et al. [13] proved that every generalized honeycomb torus is Hamiltonian. Therefore, we only need to construct cycles of even lengths from 6 to $mn-2$. That is, we should construct C_i in GHT(m, n, s), where $i = 6+2t$ for $0 \le t \le \frac{mn}{2}-4$. Obviously, $\{2t : 0 \le t \le \frac{mn}{2}-4\} = \{4t : 0 \le t \le \frac{mn}{4}-2\} \bigcup \{4t+2 : 0 \le t \le \frac{mn}{4}-3\}$. Thus, it suffices to construct C_{6+4t} for $0 \le t \le \frac{mn}{4}-2$ and C_{8+4t} for $0 \le t \le \frac{mn}{4}-3$ for the 6-bipancyclicity of GHT(m, n, s). In the following, some path patterns in generalized honeycomb tori are defined in order to construct C_i with various i.

$$I^t(i,j) = \langle (i,j), (i,[j+1]_n), (i,[j+2]_n), \ldots, (i,[j+t-1]_n), (i,[j+t]_n) \rangle, \quad t \in \mathbb{Z};$$
$$Q_{0,t}(i,j) = \langle (i,j), I^t(i,j), (i,[j+t]_n), (i+1,[j+t]_n), I^{-t}(i+1,[j+t]_n), (i+1,j) \rangle, \quad t \in \mathbb{Z};$$
$$Q_1(i,j) = \langle (i,j), (i,[j-1]_n), (i-1,[j-1]_n), (i-1,j), (i-2,j) \rangle;$$
$$Q_2(i,j) = \langle (i,j), (i,[j-1]_n), (i-1,[j-1]_n) \rangle;$$
$$P_{0,t}(i,j) = \langle (i,j), Q_{0,2t+1}(i,j), (i+1,j), Q_2(i+1,j), (i,j-1) \rangle, \quad t \in \mathbb{N};$$
$$P_{1,t}(i,j) = \langle (i,j), Q_{0,2t+1}(i,j), (i+1,j), (i+2,j) \rangle, \quad t \in \mathbb{Z};$$
$$P_{2,t}(i,j) = \langle (i,j), P_{1,t}(i,j), (i+2,j), (i+2,j-1) \rangle, \quad t \in \mathbb{N}.$$

3.1. *m is even*

Lemma 1. GHT$(4, 4, s)$ *is bipancyclic.*

Proof. By brute force, we construct cycles with different lengths in GHT$(4, 4, s)$ below.

$$C_4 = \langle (0,1), I^4(0,1), (0,1) \rangle;$$
$$C_6 = \langle (0,1), (0,2), P_{0,0}(0,2), (0,1) \rangle;$$
$$C_8 = \langle (0,0), P_{1,0}(0,0), (2,0), Q_{0,1}(2,0), (3,0), (0,0) \rangle, \quad \text{if } s = 0;$$
$$\text{or} \quad C_8 = \langle (0,1), (0,2), (3,0), (3,1), (2,1), Q_1(2,1), (0,1) \rangle, \quad \text{if } s = 2;$$
$$C_{10} = \langle (0,1), (0,2), P_{2,0}(0,2), (2,1), Q_1(2,1), (0,1) \rangle;$$
$$C_{12} = \langle (0,0), Q_{0,3}(0,0), (1,0), (Q_1(1,0))^{-1}, (3,0), (0,0) \rangle, \quad \text{if } s = 0;$$
$$\text{or} \quad C_{12} = \langle (0,0), P_{1,1}(0,0), (2,0), (2,1), (3,1), (3,2), (0,0) \rangle, \quad \text{if } s = 2;$$
$$C_{14} = \langle (0,1), (0,2), P_{1,0}(0,2), (2,2), P_{0,0}(2,2), (2,1), Q_1(2,1), (0,1) \rangle.$$

The lemma is proved. □

Lemma 2. GHT$(m, 4, s)$ *contains* C_l *for* $l \in \{4, 2m+4t | 0 \le t \le \frac{m}{2}-1\} \cup \{6+4t | 0 \le t \le m-2\}$, *if* $m \ge 6$.

Proof. It is obvious that GHT(m, n, s) consists of many hexagons and any two adjacent hexagons have two vertices in common. This structure implies the usage of a crossing edge in C_{4+4t} of GHT$(m, 4, s)$. The smallest size of C_l with $[l]_4 \equiv 0$ is $2m$. By brute force, we construct cycles with various lengths in GHT$(m, 4, s)$ as follows.

$$C_4 = \langle (0,0), I^4(0,0), (0,0) \rangle;$$

$$C_{2m+4t} = \left\langle (0,0), \left(P_{1,1}(0,0)\right)^t, (2t,0), \left(P_{1,0}(2t,0)\right)^{\frac{m}{2}-1-t}, (m-2,0), Q_{0,1}(m-2,0), (m-1,0), (0,0) \right\rangle \text{ for } s=0;$$

$$C_{2m+4t} = \left\langle (0,0), \left(P_{1,1}(0,0)\right)^t, (2t,0), \left(P_{1,0}(2t,0)\right)^{\frac{m}{2}-1-t}, (m-2,0), (m-2,1), (m-1,1), (m-1,2), (0,0) \right\rangle \text{ for } s=2;$$

$$C_{6+4t} = \left\langle (0,1), (0,2), \left(P_{1,0}(0,2)\right)^{\lfloor \frac{t}{2} \rfloor}, \left(2\left\lfloor \frac{t}{2}\right\rfloor, 2\right), P_{2[t]_2,0}\left(2\left\lfloor \frac{t}{2}\right\rfloor, 2\right), \left(2\left\lfloor \frac{t}{2}\right\rfloor + 2[t]_2, 1\right), Q_1\left(2\left\lfloor \frac{t}{2}\right\rfloor + 2[t]_2, 1\right)^{\lfloor \frac{t}{2}\rfloor + [t]_2}, (0,1) \right\rangle.$$

This proves the lemma. □

Lemma 3. GHT(m, n, s) *is* 6*-bipancyclic for* $m \geq 4$, $n \in \{6, 8\}$.

Proof. By brute force, we construct cycles with various lengths in GHT(m, n, s) for $n \in \{6, 8\}$ as follows.

Case 1. For GHT$(m, 6, s)$.

Case 1.1. C_{6+4t} for $0 \leq t \leq \frac{3}{2}m - 2$:

$C_{6+4t} = \langle (0,1), (0,2), \left(P_{1,1}(0,2)\right)^{\lfloor \frac{t+1}{3} \rfloor}, (2\lfloor \frac{t+1}{3} \rfloor, 2), \triangle, (2\lfloor \frac{t+1}{3} \rfloor, 1), \left(Q_1(2\lfloor \frac{t+1}{3} \rfloor, 1)\right)^{\lfloor \frac{t+1}{3} \rfloor}, (0,1) \rangle$, where $\triangle$ is $P_{0,[t]_3}(2\lfloor \frac{t+1}{3} \rfloor, 2)$ if $[t+1]_3 \not\equiv 0$ and is empty otherwise.

Case 1.2. C_{8+4t} for $0 \leq t \leq \frac{3}{2}m - 3$:

$t = 0$	$\langle (0,5), I^4(0,5), (0,3), (1,3), (1,4), (1,5), (0,5) \rangle$
$t = \{1, 2, 3\}$	$\langle (0,5), I^4(0,5), (0,3), (1,3), (1,2), (2,2), \triangle, (2,1), \left(Q_2(2,1)\right)^2, (0,5) \rangle$, where $\triangle$ is empty if $t = 1$ and is $P_{0,t-1}(2,2)$ otherwise.
$4 \leq t \leq \frac{3}{2}m - 3$	$\langle (0,5), I^4(0,5), (0,3), (1,3), (1,2), (2,2), \left(P_{1,1}(2,2)\right)^{\lfloor \frac{t-1}{3} \rfloor}, (2 + 2\lfloor \frac{t-1}{3} \rfloor, 2), \triangle, (2 + 2\lfloor \frac{t-1}{3} \rfloor, 1), \left(Q_1(2 + 2\lfloor \frac{t-1}{3} \rfloor, 1)\right)^{\lfloor \frac{t-1}{3} \rfloor}, (2,1), \left(Q_2(2,1)\right)^2, (0,5) \rangle$, where $\triangle = P_{0,[t-2]_3}(2 + 2\lfloor \frac{t-1}{3} \rfloor, 2)$ if $[t-1]_3 \not\equiv 0$ and is empty otherwise.

Case 2. For GHT$(m, 8, s)$.

Case 2.1. C_{6+4t} for $0 \leq t \leq 2m - 2$:

$C_{6+4t} = \langle (0,1), (0,2), \left(P_{1,2}(0,2)\right)^{\lfloor \frac{t+1}{4} \rfloor}, (2\lfloor \frac{t+1}{4} \rfloor, 2), \triangle, (2\lfloor \frac{t+1}{4} \rfloor, 1), \left(Q_1(2\lfloor \frac{t+1}{4} \rfloor, 1)\right)^{\lfloor \frac{t+1}{4} \rfloor}, (0,1) \rangle$, where $\triangle = P_{0,[t]_4}(2\lfloor \frac{t+1}{4} \rfloor, 2)$ if $[t+1]_4 \not\equiv 0$ and is empty otherwise.

Case 2.2. C_{8+4t} for $0 \leq t \leq 2m - 3$:

$t = 0$	$\langle (0,1), I^8, (0,1) \rangle$
$t = 1$	$\langle (0,7), I^4(0,7), (0,3), (1,3), (1,2), (2,2), (2,1), \left(Q_2(2,1)\right)^2, (0,7) \rangle$
$t = \{2, 3, 4, 5\}$	$\langle (0,7), I^6(0,7), (0,5), (1,5), I^{-3}(1,5), (1,2), (2,2), \triangle, (2,1), \left(Q_2(2,1)\right)^2, (0,7) \rangle$, where $\triangle$ is empty if $t = 2$ and is $P_{0,t-3}(2,2)$ otherwise.
$6 \leq t \leq 2m - 3$	$\langle (0,7), I^6(0,7), (0,5), (1,5), I^{-3}(1,5), (1,2), (2,2), \left(P_{1,2}(2,2)\right)^{\lfloor \frac{t-2}{4} \rfloor}, (2 + 2\lfloor \frac{t-2}{4} \rfloor, 2), \triangle, (2 + 2\lfloor \frac{t-2}{4} \rfloor, 1), \left(Q_1(2 + 2\lfloor \frac{t-2}{4} \rfloor, 1)\right)^{\lfloor \frac{t-2}{4} \rfloor}, (2,1), \left(Q_2(2,1)\right)^2, (0,7) \rangle$, where $\triangle$ is $P_{0,[t-3]_4}(2 + 2\lfloor \frac{t-2}{4} \rfloor, 2)$ if $[t-2]_4 \not\equiv 0$ and is empty otherwise.

□

Theorem 1. *Let* $n \geq 10$ *and* $s \geq 0$ *be even integers.* GHT$(4, n, s)$ *is* 6*-bipancyclic if* $s \in \{0, 2, 4\}$*. And* GHT$(4, n, s)$ *contains any cycle with length* l *for* $l \in \{6, 10 + 2t | 0 \leq t \leq 2n - 6\}$ *if* $s \geq 6$*. Moreover, there exists no* 8*-cycle in* GHT$(4, n, s)$ *for* $s \geq 6$*.*

Proof. The corresponding cycles are constructed below.

Case 1. $s \in \{0, 2, 4\}$.

Case 1.1. C_{6+4t} for $0 \leq t \leq n - 2$:

$0 \le t \le \frac{n}{2} - 2$	$\langle (0, 1), (0, 2), P_{0,t}(0, 2), (0, 1) \rangle$
$t = \frac{n}{2} - 1$	$\langle (0, 1), (0, 2), P_{2,\frac{n}{2}-2}(0, 2), (2, 1), Q_1(2, 1), (0, 1) \rangle$
$\frac{n}{2} \le t \le n - 2$	$\langle (0, 1), (0, 2), P_{1,\frac{n}{2}-2}(0, 2), (2, 2), P_{0,[t]_{\frac{n}{2}}}(2, 2), (2, 1), Q_1(2, 1), (0, 1) \rangle$

Case 1.2. C_{8+4t} for $0 \le t \le n - 3$:

$t = 0$ for $s = 0$	$\langle (0, 0), (3, 0), \left(Q_{0,1}(3, 0)\right)^{-1}, (2, 0), \left(P_{1,0}(2, 0)\right)^{-1}, (0, 0) \rangle$
$t = 0$ for $s = 2$	$\langle (0, 0), (3, 2), (Q_2(3, 2))^2, (1, 0), \left(Q_{0,1}(1, 0)\right)^{-1}, (0, 0) \rangle$
$t = 0$ for $s = 4$	$\langle (0, 0), (3, 4), (Q_2(3, 4))^3, (0, 1), (0, 0) \rangle$
$1 \le t \le \frac{n}{2} - 2$	$\langle (0, n-1), I^{2+2t}(0, n-1), (0, 1+2t), (1, 1+2t), I^{-(2t-1)}(1, 1+2t), (1, 2), (2, 2), (2, 1), (Q_2(2, 1))^2, (0, n-1) \rangle$
$\frac{n}{2} - 1 \le t \le n - 3$	$\langle (0, n-1), I^{n-2}(0, n-1), (0, n-3), (1, n-3), I^{-(n-5)}(1, n-3), (1, 2), (2, 2), P_{0,[t+1]_{\frac{n}{2}}}(2, 2), (2, 1), (Q_2(2, 1))^2, (0, n-1) \rangle$

Notice that the construction of all C_l's except for C_8 in Case 1 contains no crossing edge of GHT$(4, n, s)$ for $s \in \{0, 2, 4\}$.

Case 2. $s \ge 6$.

It is obvious that GHT(m, n, s) consists of many hexagons and any two adjacent hexagons have two vertices in common. Therefore, construction of cycles without crossing edges results in cycles with length $6 + 4t$, $0 \le t \le n - 2$, only. This implies that the usage of a crossing edge in C_8 is necessary. However, when $s \ge 6$, the smallest size of C_l with $[l]_4 \equiv 0$ is $s + 4 + [s]_4 \ge 12$, as shown in Fig. 3. Thus it is impossible to have C_8 in GHT$(4, n, s)$ for $s \ge 6$. On the other hand, we can construct C_l for $l \in \{6, 10 + 2t | 0 \le t \le 2n - 6\}$ the same as in Case 1. □

Definition 1. Let f_e be a function that maps $((m-1, 2), (m-1, 1))$ in C_{mn-4} of GHT(m, n, s) to $\langle (m-1, 2), (m, 2), (m, 1), (Q_{0,-1}(m, 1))^{-1}, (m-1, 1) \rangle$ in C_{mn} of GHT$(m+2, n, s)$ and maps $((m-1, 2), (m-1, 1))$ in C_{mn-2} of GHT(m, n, s) to $\langle (m-1, 2), (m, 2), (m, 1), (Q_{0,-1}(m, 1))^{-1}, (m-1, 1) \rangle$ in C_{mn+2} of GHT$(m+2, n, s)$. We give an illustration in Figs. 4 and 5.

Definition 2. Let $g_e(k)$ be a function that maps $((m-1, 2), (m-1, 1))$ in C_{mn-4} of GHT(m, n, s) to $\langle (m-1, 2), (m, 2), P_{0,k}(m, 2), (m, 1), \left(Q_{0,-1}(m, 1)\right)^{-1}, (m-1, 1) \rangle$ in $C_{mn+4+4k}$ of GHT$(m+2, n, s)$ and maps $((m-1, 2), (m-1, 1))$ in C_{mn-2} of GHT(m, n, s) to $\langle (m-1, 2), (m, 2), P_{0,k}(m, 2), (m, 1), \left(Q_{0,-1}(m, 1)\right)^{-1}, (m-1, 1) \rangle$ in $C_{mn+6+4k}$ of GHT$(m+2, n, s)$ for $0 \le k \le \frac{n}{2} - 2$. Examples are shown in Figs. 6–9.

Theorem 2. *Let $m \ge 6$, $n \ge 10$ and $s \ge 0$ be even integers.* GHT(m, n, s) *contains* 6*-cycle and all cycles with lengths l for $l \in \{10 + 2t | 0 \le t \le \frac{mn}{2} - 6\}$. Moreover, there exists no* 8*-cycle in* GHT(m, n, s).

Proof. We prove the theorem by the mathematical induction. For GHT$(6, n, s)$, we can construct C_l for $l \in \{6, 10 + 2t | 0 \le t \le 2n - 6\}$ the same as in Case 1 of Theorem 1 because there involves no crossing edge in those cycles. Then with Definitions 1 and 2, we construct C_k of GHT$(6, n, s)$ for $k \in \{10 + 2t | 2n - 5 \le t \le 3n - 6\}$ by using f_e and g_e. Using the induction hypothesis, we assume GHT(m, n, s) contains any cycle with length l for $l \in \{6, 10+2t | 0 \le t \le \frac{mn}{2} - 6\}$. Obviously, GHT$(m+2, n, s)$ contains the same $C_6, C_{10}, C_{12}, \dots, C_{mn-4}, C_{mn-2}$ as in GHT(m, n, s) since C_l, $l \in \{6, 10+2t | 0 \le t \le \frac{mn}{2} - 6\}$, contains no crossing edge in GHT(m, n, s). Then with Definitions 1 and 2, we construct C_{mn+2t} of GHT$(m+2, n, s)$ for $0 \le t \le n - 1$ by using f_e and g_e.

By induction, we know that GHT(m, n, s) contains any cycle with length l for $l \in \{6, 10 + 2t | 0 \le t \le \frac{mn}{2} - 6\}$. Moreover, for the same reason as in Case 2 of Theorem 1, there exists no 8-cycle in GHT(m, n, s) for even $m \ge 6$, even $n \ge 10$ and even $s \ge 0$. □

3.2. m is odd

Lemma 4. GHT$(3, 4, s)$ *is bipancyclic.*

Proof. By brute force, we construct cycles with different lengths in GHT$(3, 4, s)$ below.

$C_4 = \langle (0, 1), I^4(0, 1), (0, 1) \rangle$;

$C_6 = \langle (0, 1), (0, 2), P_{0,0}(0, 2), (0, 1) \rangle$;

$C_8 = \langle (0, 0), I^3(0, 0), (0, 3), (1, 3), (1, 2), (2, 2), (2, 1), (0, 0) \rangle$;

$C_{10} = \langle (0, 1), (0, 2), P_{2,0}(0, 2), (2, 1), Q_1(2, 1), (0, 1) \rangle$. □

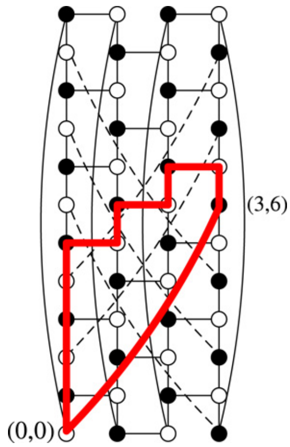

Fig. 3. The smallest cycle with a crossing edge in GHT(4, 12, 6) for $[l]_4 = 0$ is C_{12}.

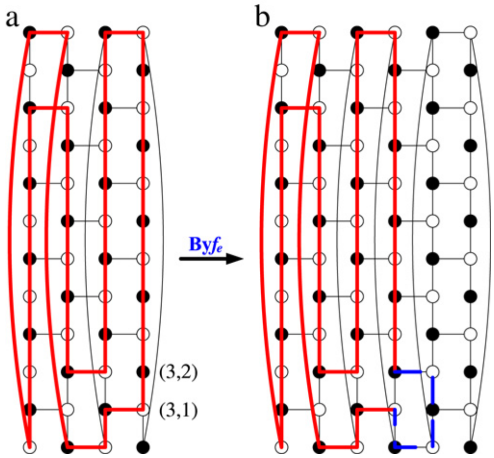

Fig. 4. C_{mn-4} of GHT(m, n, s) maps to C_{mn} of GHT($m+2$, n, s) in Definition 1 for m is even. Example: (a) C_{44} in GHT(4, 12, s); (b) C_{48} in GHT(6, 12, s). Note that the crossing edges are omitted in this figure, and the edges in C_{44} (C_{48}, resp.) are plotted by thick lines.

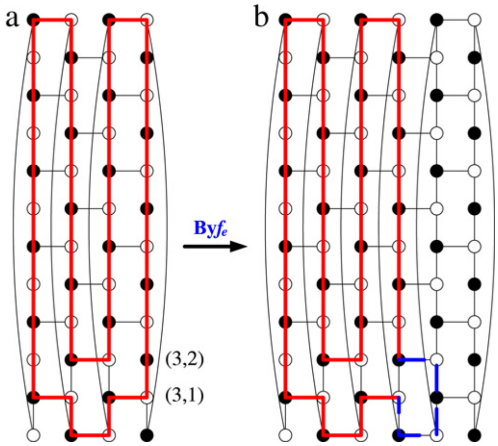

Fig. 5. C_{mn-2} of GHT(m, n, s) maps to C_{mn+2} of GHT($m+2$, n, s) in Definition 1 for m is even. Example: (a) C_{46} in GHT(4, 12, s); (b) C_{50} in GHT(6, 12, s). Note that the crossing edges are omitted in this figure, and the edges in C_{46} (C_{50}, resp.) are plotted by thick lines.

Lemma 5. GHT(m, 4, s) *contains* C_l *for* $l \in \{4, 2m+2+4t \mid 0 \le t \le \frac{m-3}{2}\} \cup \{6+4t \mid 0 \le t \le m-2\}$, *if* $m \ge 5$.

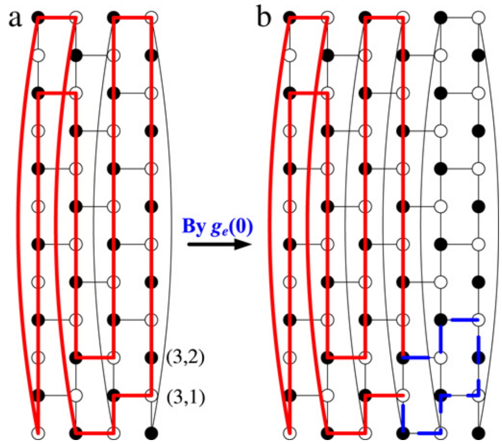

Fig. 6. C_{mn-4} of GHT(m, n, s) maps to C_{mn+4} of GHT($m + 2$, n, s) in Definition 2 for m is even. Example: (a) C_{44} in GHT(4, 12, s); (b) C_{52} in GHT(6, 12, s). Note that the crossing edges are omitted in this figure, and the edges in C_{44} (C_{52}, resp.) are plotted by thick lines.

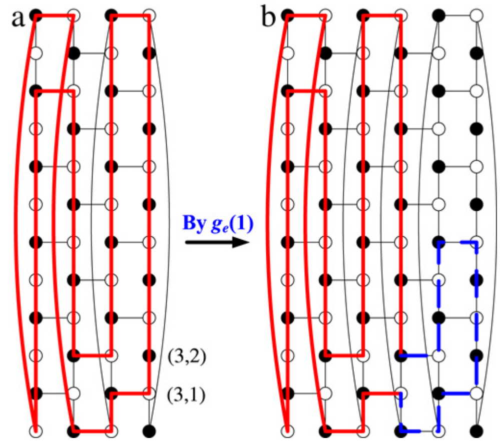

Fig. 7. C_{mn-4} of GHT(m, n, s) maps to C_{mn+8} of GHT($m + 2$, n, s) in Definition 2 for m is even. Example: (a) C_{44} in GHT(4, 12, s); (b) C_{56} in GHT(6, 12, s). Note that the crossing edges are omitted in this figure, and the edges in C_{44} (C_{56}, resp.) are plotted by thick lines.

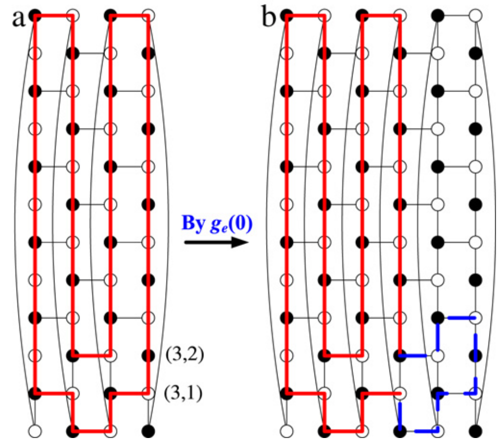

Fig. 8. C_{mn-2} of GHT(m, n, s) maps to C_{mn+6} of GHT($m + 2$, n, s) in Definition 2 for m is even. Example: (a) C_{46} in GHT(4, 12, s); (b) C_{54} in GHT(6, 12, s). Note that the crossing edges are omitted in this figure, and the edges in C_{46} (C_{54}, resp.) are plotted by thick lines.

Proof. It is obvious that GHT(m, n, s) consists of many hexagons and any two adjacent hexagons have two vertices in common. This structure implies the usage of a crossing edge in C_{4+4t} of GHT(m, 4, s). The smallest size of C_l with $[l]_4 \equiv 0$ is

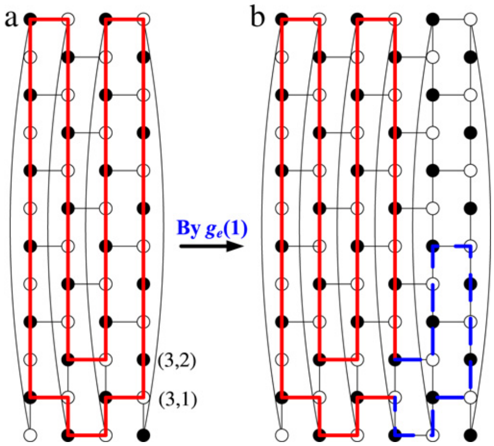

Fig. 9. C_{mn-2} of GHT(m, n, s) maps to C_{mn+10} of GHT$(m+2, n, s)$ in Definition 2 for m is even. Example: (a) C_{46} in GHT$(4, 12, s)$; (b) C_{58} in GHT$(6, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in C_{46} (C_{58}, resp.) are plotted by thick lines.

$2m+2$. By brute force, we construct cycles with various lengths in GHT$(m, 4, s)$ as follows.

$$C_4 = \langle (0,0), I^4(0,0), (0,0)\rangle;$$

$$C_{2m+2+4t} = \left\langle (0,0), \left(P_{1,1}(0,0)\right)^t, (2t,0), \left(P_{1,0}(2t,0)\right)^{\frac{m-1}{2}-t}, (m-1,0), I^{-3}(m-1,0), (m-1,1), (0,0)\right\rangle;$$

$$C_{6+4t} = \left\langle (0,1), (0,2), \left(P_{1,0}(0,2)\right)^{\lfloor \frac{t}{2}\rfloor}, \left(2\left\lfloor \frac{t}{2}\right\rfloor, 2\right), P_{2[t]_2,0}\left(2\left\lfloor \frac{t}{2}\right\rfloor, 2\right), \left(2\left\lfloor \frac{t}{2}\right\rfloor + 2[t]_2, 1\right), \right.$$
$$\left. Q_1\left(2\left\lfloor \frac{t}{2}\right\rfloor + 2[t]_2, 1\right)^{\lfloor \frac{t}{2}\rfloor + [t]_2}, (0,1)\right\rangle.$$

The lemma is proved. □

Lemma 6. *GHT(m, n, s) is 6-bipancyclic for $m \geq 3$, $n \in \{6, 8\}$.*

Proof. By brute force, we construct cycles with various lengths in GHT(m, n, s) for $n \in \{6, 8\}$ as follows.

Case 1. For GHT$(m, 6, s)$.

Case 1.1. C_{6+4t} for $0 \leq t \leq \frac{3}{2}m - \frac{5}{2}$:

$C_{6+4t} = \langle (0,1), (0,2), \left(P_{1,1}(0,2)\right)^{\lfloor \frac{t+1}{3}\rfloor}, (2\lfloor \frac{t+1}{3}\rfloor, 2), \triangle, (2\lfloor \frac{t+1}{3}\rfloor, 1), \left(Q_1(2\lfloor \frac{t+1}{3}\rfloor, 1)\right)^{\lfloor \frac{t+1}{3}\rfloor}, (0,1)\rangle$, where $\triangle$ is $P_{0,[t]_3}(2\lfloor \frac{t+1}{3}\rfloor, 2)$ if $[t+1]_3 \not\equiv 0$ and is empty otherwise.

Case 1.2. C_{8+4t} for $0 \leq t \leq \frac{3}{2}m - \frac{5}{2}$:

$t = 0$	$\langle (0,0), (0,5), (1,5), (1,4), (1,3), (0,3), I^{-3}(0,3), (0,0)\rangle$
$1 \leq t \leq \frac{3}{2}m - \frac{5}{2}$	$\langle (0,0), (0,5), (Q_1(0,5))^{-\lfloor \frac{t}{3}\rfloor}, (2\lfloor \frac{t}{3}\rfloor, 5), (2\lfloor \frac{t}{3}\rfloor + 1, 5), (2\lfloor \frac{t}{3}\rfloor + 1, 4), \triangle, (2\lfloor \frac{t}{3}\rfloor + 1, 3)$, $\left(P_{1,-2}(2\lfloor \frac{t}{3}\rfloor + 1, 3)\right)^{-\lfloor \frac{t}{3}\rfloor}, (1,3), (0,3), I^{-3}(0,3), (0,0)\rangle$, where $\triangle$ is $(2\lfloor \frac{t}{3}\rfloor + 2, 4), (2\lfloor \frac{t}{3}\rfloor + 2, 3), \left(Q_{0,-2[t]_3+1}(2\lfloor \frac{t}{3}\rfloor + 2, 3)\right)^{-1}$ if $[t]_3 \not\equiv 0$ and is empty otherwise.

Case 2. For GHT$(m, 8, s)$.

Case 2.1. C_{6+4t} for $0 \leq t \leq 2m-2$:

$t = \{0, 1\}$	$\langle (0,1), (0,2), P_{0,t}(0,2), (0,1)\rangle$
$2 \leq t \leq 2m-2$	$\langle (0,0), (0,7), (Q_1(0,7))^{-\lfloor \frac{t-1}{4}\rfloor}, (2\lfloor \frac{t-1}{4}\rfloor, 7), (2\lfloor \frac{t-1}{4}\rfloor + 1, 7), (2\lfloor \frac{t-1}{4}\rfloor + 1, 6), \triangle$, $(2\lfloor \frac{t-1}{4}\rfloor + 1, 5), \left(P_{1,-3}(2\lfloor \frac{t-1}{4}\rfloor + 1, 5)\right)^{-\lfloor \frac{t-1}{4}\rfloor}, (1,5), (0,5), I^{-5}(0,5), (0,0)\rangle$, where $\triangle$ is $(2\lfloor \frac{t-1}{4}\rfloor + 2, 6), (2\lfloor \frac{t-1}{4}\rfloor + 2, 5)$, $\left(Q_{0,-2[t-1]_4+1}(2\lfloor \frac{t-1}{4}\rfloor + 2, 5)\right)^{-1}$ if $[t-1]_4 \not\equiv 0$ and is empty otherwise.

Case 2.2. C_{8+4t} for $0 \le t \le 2m-3$:

$t=0$	$\langle (0,1), I^8(0,1), (0,1)\rangle$
$t=\{1,2\}$	$\langle (0,1), I^3(0,1), (0,4), P_{2,t-1}(0,4), (2,3), (Q_2(2,3))^2, (0,1)\rangle$
$t=3$	$\langle (0,1), (0,0), (0,7), (1,7), (1,6), (2,6), (2,5), Q_1(2,5), (0,5), (0,4), (0,3), (1,3), (1,2), (2,2), (2,1), Q_1(2,1), (0,1)\rangle$
$4 \le t \le 2m-3$	$\langle (0,1), (0,0), (0,7), (Q_1(0,7))^{-\lfloor \frac{t}{4}\rfloor}, (2\lfloor \frac{t}{4}\rfloor, 7), (1+2\lfloor \frac{t}{4}\rfloor, 7), (1+2\lfloor \frac{t}{4}\rfloor, 6), \triangle,$ $(1+2\lfloor \frac{t}{4}\rfloor, 5), \left(P_{1,-3}(1+2\lfloor \frac{t}{4}\rfloor, 5)\right)^{-(\lfloor \frac{k}{4}\rfloor -1)}, (3,5), (2,5), Q_1(2,5), (0,5), (0,4), (0,3),$ $(1,3), (1,2), (2,2), (2,1), Q_1(2,1), (0,1)\rangle$, where $\triangle$ is $(2+2\lfloor \frac{t}{4}\rfloor, 6), (2+2\lfloor \frac{t}{4}\rfloor, 5), \left(Q_{0,-2[t]_4+1}(2+2\lfloor \frac{t}{4}\rfloor, 5)\right)^{-1}$ if $[t]_4 \not\equiv 0$ and is empty otherwise.

□

Theorem 3. *Let $n \ge 10$ be an even integer and $s \ge 1$ be an odd integer.* GHT$(3,n,s)$ *is* 6*-bipancyclic if $s \in \{1,3,5\}$. And* GHT$(3,n,s)$ *contains any cycle with length l for $l \in \{6, 10+2t \mid 0 \le t \le \frac{3}{2}n-6\}$ if $s \ge 7$. Moreover, there exists no* 8*-cycle in* GHT$(3,n,s)$ *for $s \ge 7$.*

Proof. The corresponding cycles are constructed below. C_{l_1} of GHT$(3,n,s)$ for $l_1 \in \{6, 10+2t \mid 0 \le t \le \frac{n}{2}-5\}$ is the same as in GHT$(4,n,s)$. And C_{l_2} of GHT$(3,n,s)$ for $l_2 \in \{n+2+2t \mid 0 \le t \le n-2\}$ is constructed as follows.

Case 1. $s \in \{1,3,5\}$.

Case 1.1. C_8:

$$C_8 = \langle (0,0), I^3(0,0), (0,3), (1,3), (1,2), (2,2), (2,1), (0,0)\rangle, \quad \text{if } s=1;$$
$$C_8 = \left\langle (0,0), I^3(0,0), (0,3), \left(Q_{\frac{s-1}{2}}(0,3)\right)^{-\frac{s-1}{2}}, (2,s), (0,0)\right\rangle, \quad \text{if } s=\{3,5\}.$$

Case 1.2. C_{n+2+4t} for $0 \le t \le \frac{n}{2}-1$:

$t=0$	$\langle (0,0), (0,n-1), (1,n-1), (1,n-2), (1,n-3), (0,n-3), I^{-(n-3)}(0,n-3), (0,0)\rangle$
$1 \le t \le \frac{n}{2}-1$	$\langle (0,0), (0,n-1), (1,n-1), (1,n-2), (2,n-2), (2,n-3),$ $\left(Q_{0,-2[t]_{\frac{n}{2}}+1}(2,n-3)\right)^{-1}, (1,n-3), (0,n-3), I^{-(n-3)}(0,n-3), (0,0)\rangle$

Case 1.3. C_{n+4+4t} for $0 \le t \le \frac{n}{2}-2$:

$t=0$	$\langle (0,0), (0,n-1), (1,n-1), (1,n-2), (1,n-3), (0,n-3), (0,n-4), (0,n-5),$ $(1,n-5), (1,n-6), (1,n-7), (0,n-7), I^{-(n-7)}(0,n-7), (0,0)\rangle$
$1 \le t \le \frac{n}{2}-2$	$\langle (0,0), (0,n-1), (1,n-1), (1,n-2), (2,n-2), (2,n-3), Q_1(2,n-3), (0,n-3),$ $(0,n-4), (0,n-5), (1,n-5), (1,n-6), \triangle, (1,n-7), (0,n-7), I^{-(n-7)}(0,n-7), (0,0)\rangle$, where $\triangle$ is $(2,n-6), (2,n-7), \left(Q_{0,-2[t-1]_{\frac{n}{2}}+1}(2,n-7)\right)^{-1}$ if $t \ne 1$ and is empty otherwise.

Notice that the construction of all C_l's except for C_8 in Case 1 contains no crossing edge of GHT$(3,n,s)$ for $s \in \{1,3,5\}$.

Case 2. $s \ge 7$.

It is obvious that GHT(m,n,s) consists of many hexagons and any two adjacent hexagons have two vertices in common. Therefore, construction of cycles without crossing edges result in cycles with length $6+4t$, $0 \le t \le \frac{3}{2}n-2$, only. This implies that the usage of a crossing edge in C_8 is necessary. However, when $s \ge 7$, the smallest size of C_l with $[l]_4 \equiv 0$ is $s+3+[s+3]_4 \ge 12$, as shown in Fig. 10. Thus it is impossible to have C_8 in GHT$(3,n,s)$ for $s \ge 7$. On the other hand, we can construct C_l for $l \in \{6, 10+2t \mid 0 \le t \le \frac{3}{2}n-6\}$ the same as in Case 1. □

Definition 3. Let f_o be a function that maps $((m-1,n-2),(m-1,n-3))$ in C_{mn-4} of GHT(m,n,s) to $\langle (m-1,n-2), Q_{0,1}(m-1,n-2), (m,n-2), (m,n-3), (m-1,n-3)\rangle$ in C_{mn} of GHT$(m+2,n,s)$ and maps $((m-1,n-2),(m-1,n-3))$ in C_{mn-2} of GHT(m,n,s) to $\langle (m-1,n-2), Q_{0,1}(m-1,n-2), (m,n-2), (m,n-3), (m-1,n-3)\rangle$ in C_{mn+2} of GHT$(m+2,n,s)$. Examples are given in Figs. 11 and 12.

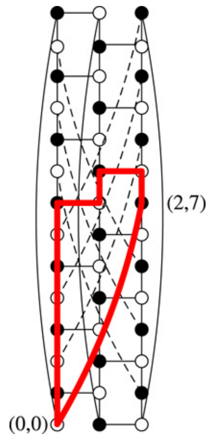

Fig. 10. The smallest cycle with a crossing edge in GHT(3, 14, 7) for $[l]_4 = 0$ is C_{12}.

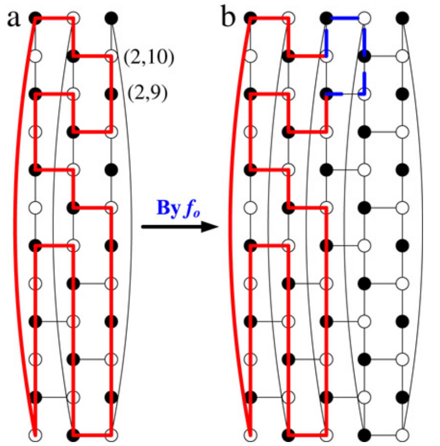

Fig. 11. C_{mn-4} of GHT(m, n, s) maps to C_{mn} of GHT$(m+2, n, s)$ in Definition 3 for m is odd. Example: (a) C_{32} in GHT$(3, 12, s)$; (b) C_{36} in GHT$(5, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in C_{32} (C_{36}, resp.) are plotted by thick lines.

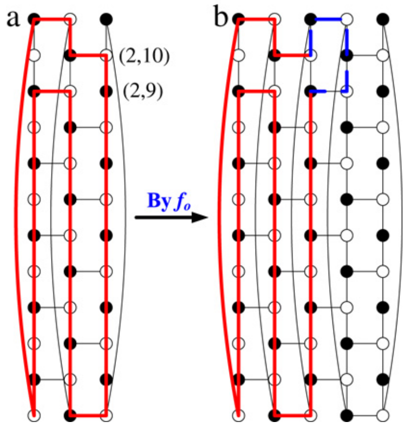

Fig. 12. C_{mn-2} of GHT(m, n, s) maps to C_{mn+2} of GHT$(m + 2, n, s)$ in Definition 3 for m is odd. Example: (a) C_{34} in GHT$(3, 12, s)$; (b) C_{38} in GHT$(5, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in C_{34} (C_{38}, resp.) are plotted by thick lines.

Definition 4. Let $g_o(k)$ be a function that maps $((m-1, n-2), (m-1, n-3))$ in C_{mn-4} of GHT(m, n, s) to $\langle(m-1, n-2), Q_{0,1}(m-1, n-2), (m, n-2), (m+1, n-2), (m+1, n-3), \left(Q_{0,-(2k+1)}(m+1, n-3)\right)^{-1}, (m, n-3), (m-1, n-3)\rangle$ in $C_{mn+4+4k}$ of GHT$(m+2, n, s)$ and maps $((m-1, n-2), (m-1, n-3))$ in C_{mn-2} of GHT(m, n, s) to $\langle(m-1, n-2), Q_{0,1}(m-$

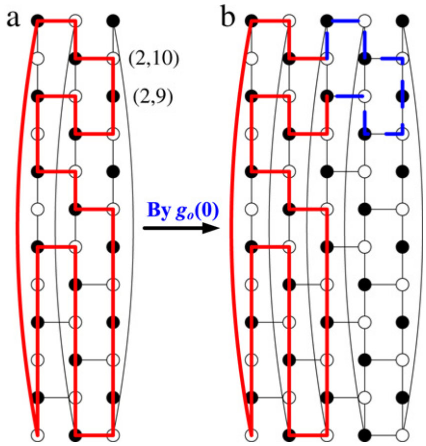

Fig. 13. C_{mn-4} of GHT(m, n, s) maps to C_{mn+4} of GHT$(m+2, n, s)$ in Definition 4 for m is odd. Example: (a) C_{32} in GHT$(3, 12, s)$; (b) C_{40} in GHT$(5, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in C_{32} (C_{40}, resp.) are plotted by thick lines.

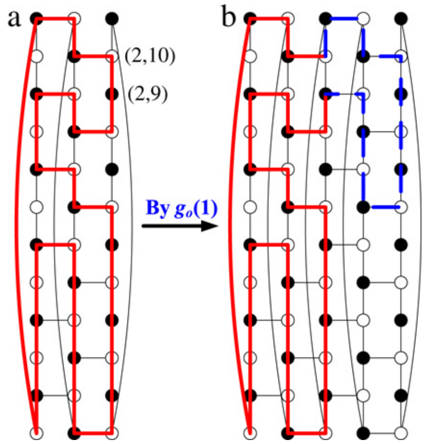

Fig. 14. C_{mn-4} of GHT(m, n, s) maps to C_{mn+8} of GHT$(m+2, n, s)$ in Definition 4 for m is odd. Example: (a) C_{32} in GHT$(3, 12, s)$; (b) C_{44} in GHT$(5, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in C_{32} (C_{44}, resp.) are plotted by thick lines.

$1, n-2), (m, n-2), (m+1, n-2), (m+1, n-3), \left(Q_{0,-(2k+1)}(m+1, n-3)\right)^{-1}, (m, n-3), (m-1, n-3)\rangle$ in $C_{mn+6+4k}$ of GHT$(m+2, n, s)$ for $0 \le k \le \frac{n}{2} - 2$. We give illustrations in Figs. 13–16.

Theorem 4. *Let $m \ge 5$ and $s \ge 1$ be odd integers and $n \ge 10$ be an even integer.* GHT(m, n, s) *contains* 6*-cycle and all cycles with lengths l where $l = 10 + 2t$ for $0 \le t \le \frac{mn}{2} - 6$. Moreover, there exists no* 8*-cycle in* GHT(m, n, s).

Proof. We prove the theorem by the mathematical induction. For GHT$(5, n, s)$ we can construct C_l for $l \in \{6, 10+2t | 0 \le t \le \frac{3}{2}n - 6\}$ the same as in Case 1 of Theorem 3 because there involves no crossing edge in those cycles. Then with Definitions 3 and 4, we construct C_k of GHT$(5, n, s)$ for $k \in \{10+2t | \frac{3}{2}n - 5 \le t \le \frac{5}{2}n - 6\}$ by using f_o and g_o. Using the induction hypothesis, we assume that GHT(m, n, s) contains any cycle with length l for $l \in \{6, 10+2t | 0 \le t \le \frac{mn}{2} - 6\}$. Obviously, GHT$(m+2, n, s)$ contains the same $C_6, C_{10}, C_{12}, \ldots, C_{mn-4}, C_{mn-2}$ as in GHT(m, n, s) since C_l, $l \in \{6, 10 + 2t | 0 \le t \le \frac{mn}{2} - 6\}$, contains no crossing edge in GHT(m, n, s). Then with Definitions 3 and 4, we construct C_{mn+2t} of GHT$(m+2, n, s)$ for $0 \le t \le n - 1$ by using f_o, g_o.

By induction, we know that GHT(m, n, s) contains 6-cycle and all cycles with lengths l for $l \in \{10 + 2t | 0 \le t \le \frac{mn}{2} - 6\}$. Moreover, for the same reason as in Case 2 of Theorem 3, there exists no 8-cycle in GHT(m, n, s) for odd $m \ge 5$, even $n \ge 10$ and odd $s \ge 1$. □

4. Conclusion

In this article, we study the vertex-bipancyclicity of the generalized honeycomb tori. In Section 3, we prove that GHT(m, n, s) is 6-bipancyclic, and is bipancyclic for some special cases. Moreover, some GHT(m, n, s) contains cycles with length l for any even integer $l \ge 6$ except 8 due to its hexagonal structure. Since GHT(m, n, s) is vertex-transitive, our

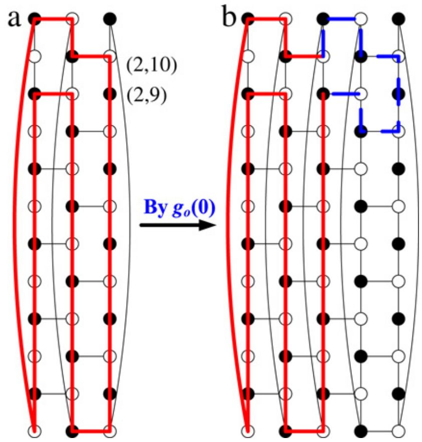

Fig. 15. C_{mn-2} of GHT(m, n, s) maps to C_{mn+6} of GHT$(m+2, n, s)$ in Definition 4 for m is odd. Example: (a) C_{34} in GHT$(3, 12, s)$; (b) C_{42} in GHT$(5, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in C_{34} (C_{42}, resp.) are plotted by thick lines.

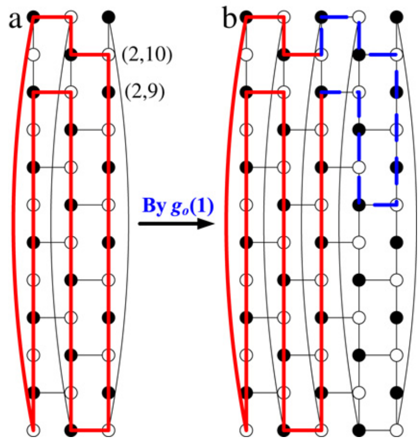

Fig. 16. C_{mn-2} of GHT(m, n, s) maps to C_{mn+10} of GHT$(m+2, n, s)$ in Definition 4 for m is odd. Example: (a) C_{34} in GHT$(3, 12, s)$; (b) C_{46} in GHT$(5, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in C_{34} (C_{46}, resp.) are plotted by thick lines.

theorems in Section 3 imply that given any vertex v of GHT(m, n, s), there exists a cycle with the required lengths that contains v. The results are summarized in the following tables and are shown to be optimal in the sense that the absence of cycles of certain lengths on some GHT(m, n, s)'s is inevitable due to their hexagonal structure. Let G be GHT(m, n, s).

When m is even:

$n \setminus m$	$m = 4$	$m \geq 6$
$n = 4$	Lemma 1. G is vertex-bipancyclic.	Lemma 2. G contains C_l for $l \in \{4, 2m + 4t, 6 + 4t \mid t \in \mathbb{N}\}$.
$n = \{6, 8\}$	Lemma 3. G is 6-vertex-bipancyclic.	Lemma 3. G is 6-vertex-bipancyclic.
$n \geq 10$	Theorem 1. G is 6-vertex-bipancyclic for $s \in \{0, 2, 4\}$; G contains a 6-cycle and is 10-vertex-bipancyclic for $s \geq 6$.	Theorem 2. G contains a 6-cycle and is 10-vertex-bipancyclic.

When m is odd:

$n \setminus m$	$m = 3$	$m \geq 5$
$n = 4$	Lemma 4. G is vertex-bipancyclic.	Lemma 5. G contains C_l for $l \in \{4, 2m + 2 + 4t, 6 + 4t \mid t \in \mathbb{N}\}$.
$n = \{6, 8\}$	Lemma 6. G is 6-vertex-bipancyclic.	Lemma 6. G is 6-vertex-bipancyclic.
$n \geq 10$	Theorem 3. G is 6-vertex-bipancyclic for $s \in \{1, 3, 5\}$; G contains a 6-cycle and is 10-vertex-bipancyclic for $s \geq 7$.	Theorem 4. G contains a 6-cycle and is 10-vertex-bipancyclic.

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