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Mutually independent hamiltonian cycles of binary wrapped butterfly graphs

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ABSTRACT

Effective utilization of communication resources is crucial for improving performance in multiprocessor/communication systems. In this paper, the mutually independent hamiltonicity is addressed for its effective utilization of resources on the binary wrapped butterfly graph. Let G be a graph with N vertices. A hamiltonian cycle C of G is represented by $\langle u_1, u_2, \ldots, u_N, u_1\rangle$ to emphasize the order of vertices on C. Two hamiltonian cycles of G, namely $C_1 = \langle u_1, u_2, \ldots, u_N, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_N, v_1\rangle$, are said to be independent if $u_1 = v_1$ and $u_i \neq v_i$ for all $2 \leq i \leq N$. A collection of m hamiltonian cycles $C_1, \ldots, C_m$, starting from the same vertex, are m-mutually independent if any two different hamiltonian cycles are independent. The mutually independent hamiltonicity of a graph G, denoted by $\mathcal{IHC}(G)$, is defined to be the maximum integer m such that, for each vertex u of G, there exists a set of m-mutually independent hamiltonian cycles starting from u. Let $BF(n)$ denote the n-dimensional binary wrapped butterfly graph. Then we prove that $\mathcal{IHC}(BF(n)) = 4$ for all $n \geq 3$.

1. Introduction

A multiprocessor/communication *interconnection network* is usually modeled as a graph, in which the vertices correspond to processors/nodes, and the edges correspond to connections or communication links. In this paper, we use the terms, graphs and networks, interchangeably. Designing an interconnection network is multi-objected and complicated [1]. Hence, the topological properties of various interconnection networks have been widely addressed by many researchers [2–11]. Among various kinds of popular network topologies, butterfly networks are very suitable for VLSI implementation and parallel computing. In particular, the binary wrapped butterfly graph has gained many researchers' efforts for its nice topological properties. For example, it belongs to the family of constant degree-four Cayley graphs [12,13]. Therefore, it is vertex-transitive. Moreover, the hamiltonian properties were addressed in research by [2,3,11]. Until recently it is believed that the presence of such a constant-degree network topology, with both logarithmic diameter and optimal fault tolerance is critical to improve the performance of peer-to-peer architectures [14,15]. In practice, Malkhi et al. [16] build a peer-to-peer lookup network on the basis of butterfly graphs.

Network embedding [1] is an interesting subject, because the portability of the guest network onto the host network would permit executing the guest specified algorithms on the host with as little modification as possible. In the research of [4,7,9–11], embedding of various topologies, such as rings, linear arrays, and binary trees, etc., onto the butterfly networks had been addressed. In particular, the ring is a popular network topology, since many efficient communication algorithms have been designed based on a ring structure. For instance, the *token ring* [17] often serves as the underlying connection

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architecture of the local area network. In addition, the advantages of rings were discussed by Tel [18]. In this paper, we study the problem of embedding mutually independent hamiltonian cycles, as proposed by Sun et al. [19], onto a binary wrapped butterfly graph; that is, if some vertex is fixed as the start, any two of such hamiltonian cycles will traverse different vertices at every time step except the start-up and termination. Recently, Lin et al. [20] investigated how to embed the mutually independent hamiltonian cycles onto star networks and pancake networks. Moreover, Hsieh and Weng [21] further concerned the fault-tolerant embedding of pairwise independent hamiltonian paths on faulty hypercubes.

The concept of mutually independent hamiltonian cycles can be applied in many different areas. For example, communication applications on the interconnection network are often viewed as the interleaving of local computation and global communication stages. Such applications can be performed via a message routing protocol, by which information is transmitted along the communication links in packets of equal size. For the sake of simplification, the *store-and-forward all-port* communication model [8] has been widely adopted as one basic routing scheme, in which every processor is assumed to be capable of exchanging messages of fixed length, with all of its neighbors at each time step. Although routing messages over a spanning tree of the given network is intuitively the best strategy for message transmission, Baldi and Ofek [22] presented a systematic comparison between ring and tree embedding for group (many-to-many) multicast, and concluded that ring embedding remains a promising alternative. It is worth mentioning that there may be two potential shortcomings incurred by routing messages in a ring structured network [1]. One is that at least two message packets are likely to reside in the same processor, so as to provoke contention for the local computation resources. The other is that two or more message packets will contend for the use of some communication link (in the same direction). Clearly, mutually independent hamiltonian cycles can ease the effects of these two shortcomings.

As another example, a *Latin square* of order n is an $n \times n$ array containing the integers from 1 to n, arranged so that each integer appears exactly once in each row, and exactly once in each column. If we delete some rows from a Latin square, we will get a *Latin rectangle*. Obviously, a Latin square of order n can be thought of as the intermediate vertices of n mutually independent hamiltonian cycles on the complete graph with $n + 1$ vertices. Thus, the concept behind mutually independent hamiltonian cycles can be interpreted as a Latin square/rectangle for graphs. Furthermore, we consider the following scenario. A tour agency will organize a 10-day tour to Japan in the Christmas vacation. Suppose that there will be many people joining this tour. However, the maximum number of people staying in each local area is limited, say 100 people, for the sake of a hotel contract. One trivial solution is based on the First-Come-First-Served intuition. So, only 100 people can join this tour. Note that we cannot schedule the tour in a pipelined manner, because the holiday period is fixed. Fortunately, we observe that scheduling a tour is like a hamiltonian cycle of a graph, in which a vertex denotes a hotel and an edge denotes the connection between two hotels if they can be traveled in a reasonable time. Therefore, we can organize all the attendants into a number of subgroups; each subgroup has its own tour in such a way, that no two subgroups will stay in the same area during the same time period. So any two different tours are indeed independent hamiltonian cycles. If there exist five mutually independent hamiltonian cycles, then we may allow up to 500 attendees to visit Japan on a Christmas vacation. Obviously, if we can find the maximum number of mutually independent hamiltonian cycles, the number of tour attendants would be maximized.

The rest of this paper is organized as follows. In Section 2, the terminologies and notations are defined. In Section 3, the nearly recursive construction of the n-dimensional binary wrapped butterfly network, denoted by $BF(n)$, is introduced. The basic properties of $BF(n)$ are given in Section 4. In Section 5, we show that $BF(n)$ has four mutually independent hamiltonian cycles starting from any vertex. Finally, the concluding remarks are given in Section 6.

2. Definitions

In this paper, we concentrate on loopless undirected graphs. For the notations and graph-theoretic terminologies, we follow the ones given by Bondy and Murty [23]. A *graph* G is a two-tuple (V, E), where V is a nonempty set, and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that $V = V(G)$ is the *vertex set* and $E = E(G)$ is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. The number of vertices in a graph G is denoted by $|V(G)|$. The *degree* of any vertex u in a graph G, denoted by $deg_G(u)$, is the number of edges incident with u. The maximum and minimum degrees of graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph G is *k-regular* if $\Delta(G) = \delta(G) = k$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a nonempty subset of $V(G)$. The subgraph *induced* by S is the subgraph of G with the vertex set S and with the edge set consisting of those edges that join two vertices in S. Analogously, the subgraph *generated* by a nonempty set $F \subseteq E(G)$ is the subgraph of G with the edge set F and with the vertex set consisting of those vertices incident to at least one edge of F. Two graphs G_1 and G_2 are *isomorphic* if there is a bijection μ from $V(G_1)$ onto $V(G_2)$, such that $(u, v) \in E(G_1)$ if and only if $(\mu(u), \mu(v)) \in E(G_2)$. The bijection μ is called an *isomorphism*.

A *path* P of length k from vertex x to vertex y in a graph G is a sequence of distinct vertices $\langle v_1, v_2, \ldots, v_{k+1}\rangle$ such that $v_1 = x$, $v_{k+1} = y$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \le i \le k$. We also write P as $\langle x, P, y\rangle$ to emphasize its beginning and ending vertices. The i-th vertex of P is denoted by $P(i)$; i.e., $P(i) = v_i$. Both $P(1)$ and $P(k + 1)$ are *terminal* vertices of P. In particular, let $P^{-1} = \langle v_{k+1}, v_k, \ldots, v_1\rangle$ denote the reverse of P. For convenience, we use $V(P)$ to denote the set of vertices traversed by P. A *cycle* is a path with at least three vertices, such that the first vertex is adjacent to the last one. To emphasize the vertex order on a cycle, a cycle of length k is represented by $\langle v_1, v_2, \ldots, v_k, v_1\rangle$. A *hamiltonian cycle* (or *hamiltonian path*) of a graph G is a cycle (or path) that spans G. Two hamiltonian cycles starting from the same vertex s in a graph G, namely

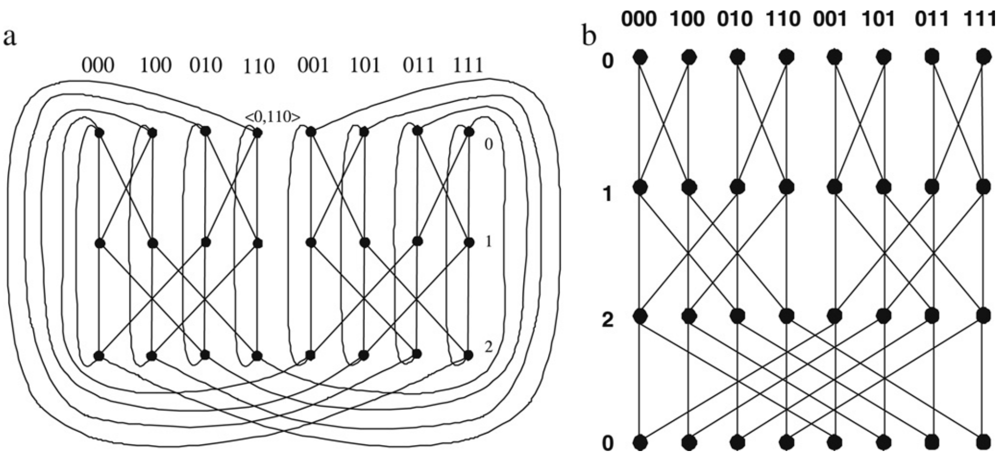

Fig. 1. (a) $BF(3)$; (b) $BF(3)$ with level-0 vertices replicated to ease visualization.

$C_1 = \langle v_1, v_2, \ldots, v_{|V(G)|}, v_1\rangle$ and $C_2 = \langle u_1, u_2, \ldots, u_{|V(G)|}, u_1\rangle$, are *independent* if $v_1 = u_1 = s$ and $v_i \neq u_i$ for $2 \leq i \leq |V(G)|$. A collection of m hamiltonian cycles $C_1, \ldots, C_m$, starting from the same vertex, are *m-mutually independent* if C_i and C_j are independent whenever $i \neq j$. Moreover, the *mutually independent hamiltonicity* of a graph G, denoted by $\mathcal{IHC}(G)$, is defined to be the maximum integer m, such that for any vertex u of G, there exists a set of m-mutually independent hamiltonian cycles starting from u. It is trivial that $\mathcal{IHC}(G) \leq \delta(G)$ for any graph G.

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ denote the set of integers modulo n. The n-dimensional *binary wrapped butterfly graph* (or *butterfly graph* for short) $BF(n)$ is a graph with the vertex set $\mathbb{Z}_n \times \mathbb{Z}_2^n$. Each vertex is labeled by a two-tuple $\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1}\rangle$ with a level $\ell \in \mathbb{Z}_n$ and an n-bit binary string $a_0a_1 \ldots a_{n-1} \in \mathbb{Z}_2^n$. A level-$\ell$ vertex $\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1}\rangle$ is adjacent to two vertices, $\langle (\ell+1)_{\text{mod } n}, a_0 \ldots a_\ell \ldots a_{n-1}\rangle$ and $\langle (\ell-1)_{\text{mod } n}, a_0 \ldots a_{\ell-1} \ldots a_{n-1}\rangle$, by *straight edges*, and is adjacent to another two vertices, $\langle (\ell+1)_{\text{mod } n}, a_0 \ldots a_{\ell-1}\overline{a}_\ell a_{\ell+1} \ldots a_{n-1}\rangle$ and $\langle (\ell-1)_{\text{mod } n}, a_0 \ldots a_{\ell-2}\overline{a}_{\ell-1}a_\ell \ldots a_{n-1}\rangle$, by *cross edges*. More formally, the edges of $BF(n)$ can be defined in terms of four generators g, g^{-1}, f, and f^{-1} as follows [13]:

$$g(\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1}\rangle) = \langle (\ell+1)_{\text{mod } n}, a_0 \ldots a_\ell \ldots a_{n-1}\rangle,$$
$$f(\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1}\rangle) = \langle (\ell+1)_{\text{mod } n}, a_0 \ldots a_{\ell-1}\overline{a}_\ell a_{\ell+1} \ldots a_{n-1}\rangle,$$
$$g^{-1}(\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1}\rangle) = \langle (\ell-1)_{\text{mod } n}, a_0 \ldots a_\ell \ldots a_{n-1}\rangle, \quad \text{and}$$
$$f^{-1}(\langle \ell, a_0 \ldots a_{\ell-1} \ldots a_{n-1}\rangle) = \langle (\ell-1)_{\text{mod } n}, a_0a_1 \ldots a_{\ell-2}\overline{a}_{\ell-1}a_\ell \ldots a_{n-1}\rangle,$$

where $\overline{a}_\ell \equiv a_\ell + 1 \pmod 2$. Throughout this paper, a level-ℓ edge of $BF(n)$ is an edge that joins a level-ℓ vertex and a level-$(\ell+1)_{\text{mod } n}$ vertex. To avoid the degenerate case, we only concern the case that $n \geq 3$. So, $BF(n)$ is 4-regular. Fig. 1(a) depicts the structure of $BF(3)$ and Fig. 1(b) is another layout of $BF(3)$ with the replication of level-0 vertices to ease visualization.

3. Nearly recursive construction of *BF*(*n*)

For any $\ell \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_2$, we use $BF_\ell^i(n)$ to denote the subgraph of $BF(n)$ induced by $\{\langle h, a_0 \ldots a_\ell \ldots a_{n-1}\rangle \in V(BF(n)) \mid a_\ell = i\}$. Obviously, $\{BF_\ell^0(n), BF_\ell^1(n)\}$ forms a partition of $BF(n)$. Moreover, $BF_{\ell_1}^i(n)$ is isomorphic to $BF_{\ell_2}^j(n)$ for any $i, j \in \mathbb{Z}_2$ and any $\ell_1, \ell_2 \in \mathbb{Z}_n$. With this observation, Wong [11] proposed a *stretching* operation to obtain $BF_\ell^i(n)$ from $BF(n-1)$. More precisely, the stretching operation can be described as follows.

Let $i \in \mathbb{Z}_2$ and $\ell \in \mathbb{Z}_n$ for $n \geq 3$. Furthermore, let $\Im_n$ denote the set of all subgraphs of $BF(n)$. Suppose that $G \in \Im_n$. We define the following subsets of $V(BF(n+1))$ and $E(BF(n+1))$:

$$V_1 = \{\langle h, a_0 \ldots a_{\ell-1}ia_\ell \ldots a_{n-1}\rangle \mid 0 \leq h < \ell, \langle h, a_0 \ldots a_{\ell-1}a_\ell \ldots a_{n-1}\rangle \in V(G)\},$$
$$V_2 = \{\langle h+1, a_0 \ldots a_{\ell-1}ia_\ell \ldots a_{n-1}\rangle \mid \ell < h \leq n-1, \langle h, a_0 \ldots a_{\ell-1}a_\ell \ldots a_{n-1}\rangle \in V(G)\},$$
$$V_3 = \{\langle \ell, a_0 \ldots a_{\ell-1}ia_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1}a_\ell \ldots a_{n-1}\rangle \text{ is incident to a level-}(\ell-1)_{\text{mod } n} \text{ edge in } G\},$$
$$V_4 = \{\langle \ell+1, a_0 \ldots a_{\ell-1}ia_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1}a_\ell \ldots a_{n-1}\rangle \text{ is incident to a level-}\ell \text{ edge in } G\},$$
$$E_1 = \{(\langle h, a_0 \ldots a_{\ell-1}ia_\ell \ldots a_{n-1}\rangle, \langle h+1, b_0 \ldots b_{\ell-1}ib_\ell \ldots b_{n-1}\rangle) \mid 0 \leq h < \ell, (\langle h, a_0 \ldots a_{\ell-1}a_\ell \ldots a_{n-1}\rangle, \langle h+1, b_0 \ldots b_{\ell-1}b_\ell \ldots b_{n-1}\rangle) \in E(G)\},$$
$$E_2 = \{(\langle h+1, a_0 \ldots a_{\ell-1}ia_\ell \ldots a_{n-1}\rangle, \langle (h+2)_{\text{mod }(n+1)}, b_0 \ldots b_{\ell-1}ib_\ell \ldots b_{n-1}\rangle) \mid \ell \leq h \leq n-1, (\langle h, a_0 \ldots a_{\ell-1}a_\ell \ldots a_{n-1}\rangle, \langle (h+1)_{\text{mod } n}, b_0 \ldots b_{\ell-1}b_\ell \ldots b_{n-1}\rangle) \in E(G)\},$$

and

$$E_3 = \{(\langle \ell, a_0 \ldots a_{\ell-1}ia_\ell \ldots a_{n-1}\rangle, \langle \ell+1, a_0 \ldots a_{\ell-1}ia_\ell \ldots a_{n-1}\rangle) \mid \langle \ell, a_0 \ldots a_{\ell-1}a_\ell \ldots a_{n-1}\rangle \text{ is incident to at least one level-}(\ell-1)_{\text{mod } n} \text{ edge and at least one level-}\ell \text{ edge in } G\}.$$

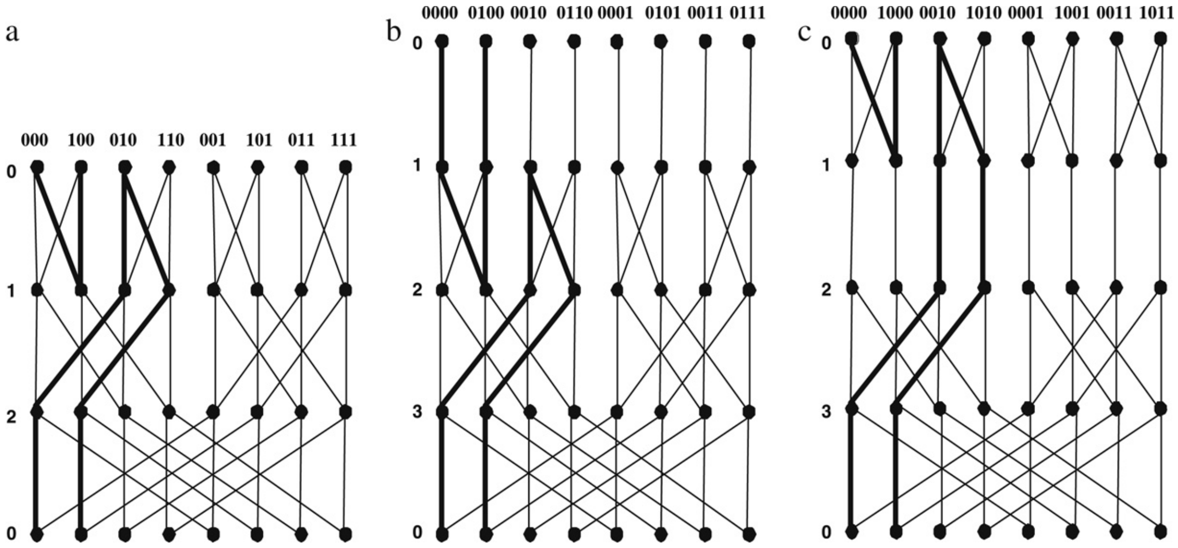

Fig. 2. (a) A subgraph G of $BF(3)$; (b) $\gamma_0^0(G)$ in $\gamma_0^0(BF(3))$; (c) $\gamma_1^0(G)$ in $\gamma_1^0(BF(3))$.

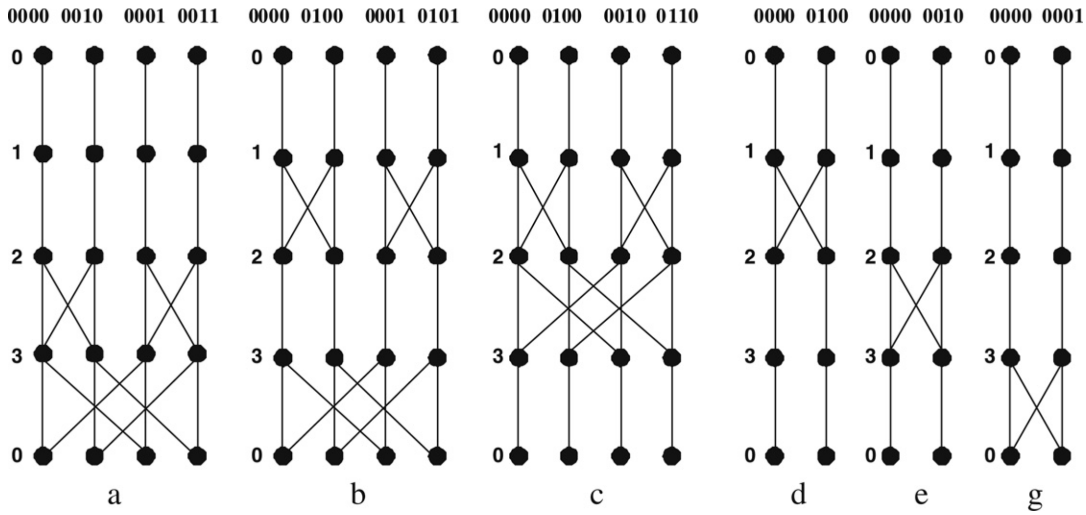

Fig. 3. (a) $BF_{0,1}^{0,0}(4)$; (b) $BF_{0,2}^{0,0}(4)$; (c) $BF_{0,3}^{0,0}(4)$; (d) $BF_{0,2,3}^{0,0,0}(4)$; (e) $BF_{0,1,3}^{0,0,0}(4)$; (g) $BF_{0,1,2}^{0,0,0}(4)$.

Then, the stretching function $\gamma_\ell^i : \bigcup_{n\geq 3} \mathfrak{I}_n \to \bigcup_{n\geq 4} \mathfrak{I}_n$ is defined by assigning $\gamma_\ell^i(G)$ as the graph with the vertex set $V_1 \cup V_2 \cup V_3 \cup V_4$ and the edge set $E_1 \cup E_2 \cup E_3$. Clearly γ_ℓ^i is well-defined and one-to-one. We have $\gamma_\ell^i(G) \in \mathfrak{I}_{n+1}$ if $G \in \mathfrak{I}_n$. In particular, $\gamma_\ell^i(BF(n)) = BF_\ell^i(n+1)$. In Fig. 2, we illustrate a subgraph G of $BF(3)$, $\gamma_0^0(G)$ in $\gamma_0^0(BF(3))$, and $\gamma_1^0(G)$ in $\gamma_1^0(BF(3))$. Obviously, $\gamma_{\ell_1}^i(BF(n))$ is isomorphic to $\gamma_{\ell_2}^j(BF(n))$ for any $\ell_1, \ell_2 \in \mathbb{Z}_n$ and $i, j \in \mathbb{Z}_2$. Moreover, $\gamma_\ell^i(P)$ is a path in $BF(n+1)$ if P is a path in $BF(n)$.

In fact, $BF(n)$ can be further partitioned. Let m be an integer with $1 \leq m \leq n$. Assume that $\ell_1, \ldots, \ell_m \in \mathbb{Z}_n$, such that $\ell_1 < \cdots < \ell_m$. For any $i_1, \ldots, i_m \in \mathbb{Z}_2$, we use $BF_{\ell_1,\ldots,\ell_m}^{i_1,\ldots,i_m}(n)$ to denote the subgraph of $BF(n)$ induced by $\{\langle h, a_0 \ldots a_{n-1}\rangle \in V(BF(n)) \mid a_{\ell_j} = i_j \text{ for } 1 \leq j \leq m\}$. In Fig. 3, we illustrate $BF_{0,1}^{0,0}(4)$, $BF_{0,2}^{0,0}(4)$, $BF_{0,3}^{0,0}(4)$, $BF_{0,2,3}^{0,0,0}(4)$, $BF_{0,1,3}^{0,0,0}(4)$, and $BF_{0,1,2}^{0,0,0}(4)$. Clearly $BF_{0,1}^{0,0}(4)$ is isomorphic with $BF_{0,3}^{0,0}(4)$. Moreover, $BF_{0,2,3}^{0,0,0}(4)$, $BF_{0,1,3}^{0,0,0}(4)$, and $BF_{0,1,2}^{0,0,0}(4)$ are also isomorphic. However, $BF_{0,1}^{0,0}(4)$ is not isomorphic to $BF_{0,2}^{0,0}(4)$.

Lemma 1. *Assume that $n \geq 3$ and $i, j, k \in \mathbb{Z}_2$. Then $BF_{0,1}^{i,j}(n)$ is isomorphic with $BF_{0,n-1}^{i,j}(n)$; $BF_{0,1,2}^{i,j,k}(n)$, $BF_{0,1,n-1}^{i,j,k}(n)$, and $BF_{0,n-2,n-1}^{i,j,k}(n)$ are isomorphic.*

Obviously, $\{BF^{i_1,\ldots,i_m}_{\ell_1,\ldots,\ell_m}(n) \mid i_1,\ldots,i_m \in \mathbb{Z}_2, \ell_1,\ldots,\ell_m \in \mathbb{Z}_n, \ell_1 < \cdots < \ell_m\}$ forms a partition of $BF(n)$ for any $1 \le m \le n$. To avoid the complication caused from modular arithmetic, we restrict our attention on the case that $1 \le m \le n-1$, $0 \le \ell_1 < \cdots < \ell_m$, and $\ell_j < n-m+j-1$ for each $1 \le j \le m$. The following two lemmas can be easily verified.

Lemma 2. *Let $1 \le m \le n-1$. Suppose that $i_1,\ldots,i_m \in \mathbb{Z}_2$ and $\ell_1,\ldots,\ell_m$ are integers such that $0 \le \ell_1 < \cdots < \ell_m$ and $\ell_j < n-m+j-1$ for each $1 \le j \le m$. Then*

$$BF^{i_1,\ldots,i_m}_{\ell_1,\ldots,\ell_m}(n) = \begin{cases} \gamma^{i_m}_{\ell_m} \circ \gamma^{i_{m-1}}_{\ell_{m-1}} \circ \cdots \circ \gamma^{i_3}_{\ell_3}(BF^{i_1,i_2}_{\ell_1,\ell_2}(3)) & \text{if } m = n-1, \\ \gamma^{i_m}_{\ell_m} \circ \gamma^{i_{m-1}}_{\ell_{m-1}} \circ \cdots \circ \gamma^{i_2}_{\ell_2}(BF^{i_1}_{\ell_1}(3)) & \text{if } m = n-2, \\ \gamma^{i_m}_{\ell_m} \circ \gamma^{i_{m-1}}_{\ell_{m-1}} \circ \cdots \circ \gamma^{i_1}_{\ell_1}(BF(n-m)) & \text{otherwise.} \end{cases}$$

Lemma 3. *Let G be a connected spanning subgraph of $BF^{i,j}_{0,1}(n)$, with $i,j \in \mathbb{Z}_2$ and $n \ge 3$. Assume that $2 \le \ell \le n-1$. Let*

$$F_0 = \{\langle \ell, a_0 \ldots a_{n-1}\rangle \in V(G) \mid \langle \ell, a_0 \ldots a_{n-1}\rangle \text{ is not incident to any level-}(\ell-1) \text{ edge in } G\},$$
$$F_1 = \{\langle \ell, a_0 \ldots a_{n-1}\rangle \in V(G) \mid \langle \ell, a_0 \ldots a_{n-1}\rangle \text{ is not incident to any level-}\ell \text{ edge in } G\}.$$

For any $p, q \in \mathbb{Z}_2$, let

$$\begin{aligned} \overline{F_0} &= \{\langle \ell, a_0 \ldots a_{\ell-1} pq a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in F_0\} \\ &\quad \cup \{\langle \ell+1, a_0 \ldots a_{\ell-1} pq a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in F_0\}, \\ \overline{F_1} &= \{\langle \ell+1, a_0 \ldots a_{\ell-1} pq a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in F_1\} \\ &\quad \cup \{\langle \ell+2, a_0 \ldots a_{\ell-1} pq a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in F_1\}, \\ M_0 &= \bigcup_{\langle \ell, a_0 \ldots a_{n-1}\rangle \notin F_0 \cup F_1} \{(\langle \ell, a_0 \ldots a_{\ell-1} pq a_\ell \ldots a_{n-1}\rangle, \langle \ell+1, a_0 \ldots a_{\ell-1} pq a_\ell \ldots a_{n-1}\rangle)\}, \quad \text{and} \\ M_1 &= \bigcup_{\langle \ell, a_0 \ldots a_{n-1}\rangle \notin F_0 \cup F_1} \{(\langle \ell+1, a_0 \ldots a_{\ell-1} pq a_\ell \ldots a_{n-1}\rangle, \langle \ell+2, a_0 \ldots a_{\ell-1} pq a_\ell \ldots a_{n-1}\rangle)\}. \end{aligned}$$

Then $F_0 \cap F_1 = \emptyset$, $\overline{F_0} \cap \overline{F_1} = \emptyset$, $\overline{F_0} \cup \overline{F_1} = V(BF^{i,j,p,q}_{0,1,\ell,\ell+1}(n+2)) - V(\gamma^q_{\ell+1} \circ \gamma^p_\ell(G))$, and $M_0 \cup M_1 \subseteq E(\gamma^q_{\ell+1} \circ \gamma^p_\ell(G))$.

Let G be a subgraph of $BF(n)$. A cycle C in G is called an *ℓ-scheduled* cycle of G if every level-ℓ vertex of G is incident to a level-$(\ell-1)_{\bmod n}$ edge and a level-ℓ edge on C [11]. Furthermore, a cycle C in G is a *totally scheduled* cycle of G if it is an ℓ-scheduled cycle of G for all $\ell \in \mathbb{Z}_n$ [11]. Obviously, $\gamma^i_\ell(C)$ with $i \in \{0,1\}$ is a totally scheduled cycle of $\gamma^i_\ell(G)$ if C is a totally scheduled cycle of G.

Lemma 4 (*[11]*). *Let $n \ge 3$. Then $BF(n)$ has a totally scheduled hamiltonian cycle.*

By stretching operation, we have the following two corollaries.

Corollary 1. *Assume that $n \ge 3$ and $i,j,k \in \mathbb{Z}_2$. Then there exists a totally scheduled hamiltonian cycle of $BF^{i,j,k}_{0,1,2}(n)$ including all straight edges of level 0, level 1, and level 2.*

Corollary 2. *Assume that $n \ge 4$ and $i,j,p,q \in \mathbb{Z}_2$. Then there exists a totally scheduled hamiltonian cycle of $BF^{i,j,p,q}_{0,1,2,3}(n)$ including all straight edges of level 0, level 1, level 2, and level 3 in $BF^{i,j,p,q}_{0,1,2,3}(n)$.*

4. Basic properties of *BF*(*n*)

Suppose that $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are either any two cross edges of $BF(n)$, or any two straight edges of $BF(n)$. Since $BF(n)$ is vertex-transitive, there exists an isomorphism μ over $V(BF(n))$, such that $u_2 = \mu(u_1)$ and $v_2 = \mu(v_1)$. Clearly, every hamiltonian cycle of $BF(n)$ includes at least one cross edge and at least one straight edge.

Lemma 5. *For any edge e of $BF(n)$ with $n \ge 3$, there exists a totally scheduled hamiltonian cycle of $BF(n)$ including e.*

Lemma 6. *Assume that $i,j,k \in \mathbb{Z}_2$. Let e be any edge of $BF^{i,j,k}_{0,1,2}(4)$ such that $e \notin \{(\langle 3, ijk0\rangle, \langle 0, ijk0\rangle), (\langle 3, ijk1\rangle, \langle 0, ijk1\rangle)\}$. Then there exists a totally scheduled hamiltonian cycle C of $BF^{i,j,k}_{0,1,2}(4)$ such that $e \in E(C)$.*

Proof. Obviously, $\langle \langle 0, ijk0\rangle, \langle 1, ijk0\rangle, \langle 2, ijk0\rangle, \langle 3, ijk0\rangle, \langle 0, ijk1\rangle, \langle 1, ijk1\rangle, \langle 2, ijk1\rangle, \langle 3, ijk1\rangle, \langle 0, ijk0\rangle\rangle$ is the unique hamiltonian cycle of $BF^{i,j,k}_{0,1,2}(4)$. Thus, this lemma is proved. □

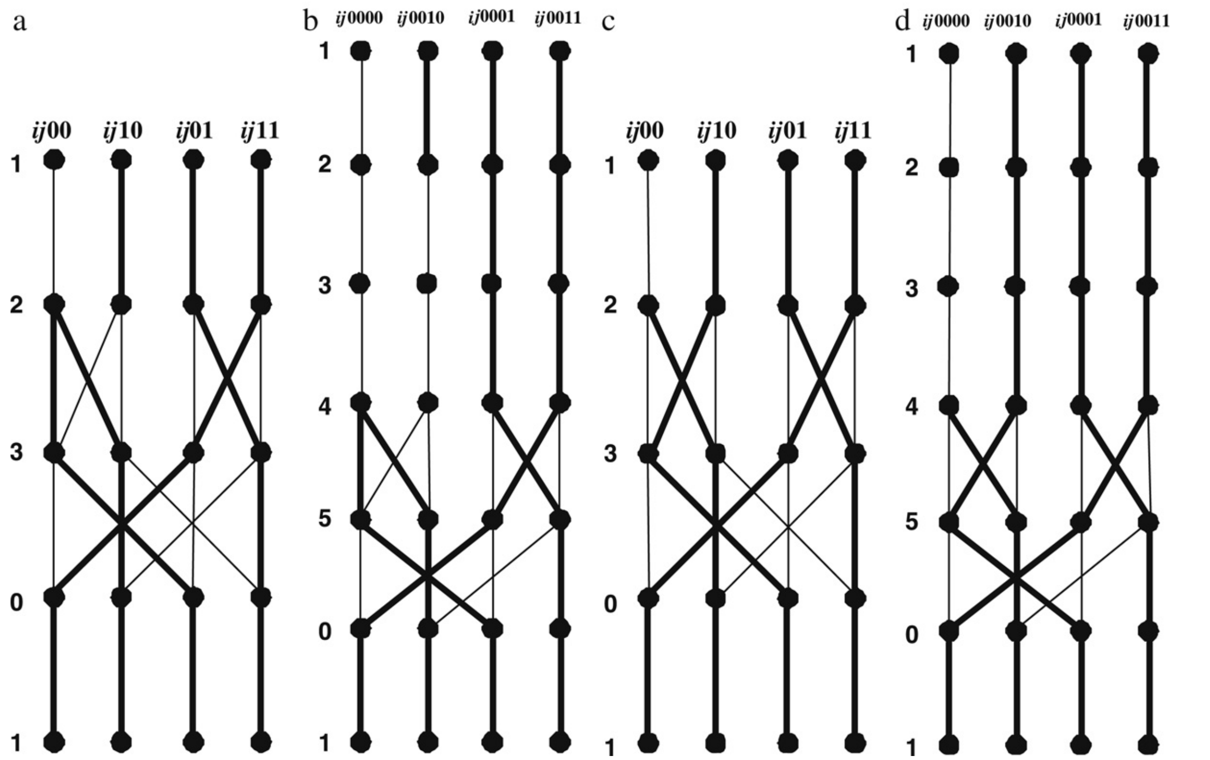

Fig. 4. (a) A weakly 2-scheduled hamiltonian path P_1 of $BF^{i,j}_{0,1}(4)$ joins $\langle 1, ij00\rangle$ to $\langle 2, ij10\rangle$; (b) $\gamma^0_3 \circ \gamma^0_2(P_1)$ in $BF^{i,j,0,0}_{0,1,2,3}(6) = \gamma^0_3 \circ \gamma^0_2(BF^{i,j}_{0,1}(4))$; (c) a weakly 2-scheduled hamiltonian path P_2 of $BF^{i,j}_{0,1}(4)$ joins $\langle 1, ij00\rangle$ to $\langle 2, ij00\rangle$; (d) $\gamma^0_3 \circ \gamma^0_2(P_2)$ in $BF^{i,j,0,0}_{0,1,2,3}(6)$.

Table 1
Hamiltonian paths of $BF^{i,j}_{0,1}(4)$ between $\langle 1, ij00\rangle$ and $\langle 2, ijpq\rangle$ for any $p, q \in \mathbb{Z}_2$

⟨⟨1, ij00⟩, ⟨0, ij00⟩, ⟨3, ij01⟩, ⟨2, ij11⟩, ⟨1, ij11⟩, ⟨0, ij11⟩, ⟨3, ij11⟩, ⟨2, ij01⟩, ⟨1, ij01⟩, ⟨0, ij01⟩, ⟨3, ij00⟩, ⟨2, ij10⟩, ⟨1, ij10⟩, ⟨0, ij10⟩, ⟨3, ij10⟩, ⟨2, ij00⟩⟩
⟨⟨1, ij00⟩, ⟨0, ij00⟩, ⟨3, ij00⟩, ⟨2, ij00⟩, ⟨3, ij10⟩, ⟨0, ij11⟩, ⟨1, ij11⟩, ⟨2, ij11⟩, ⟨3, ij01⟩, ⟨0, ij01⟩, ⟨1, ij01⟩, ⟨2, ij01⟩, ⟨3, ij11⟩, ⟨0, ij10⟩, ⟨1, ij10⟩, ⟨2, ij10⟩⟩
⟨⟨1, ij00⟩, ⟨0, ij00⟩, ⟨3, ij00⟩, ⟨2, ij00⟩, ⟨3, ij10⟩, ⟨2, ij10⟩, ⟨1, ij10⟩, ⟨0, ij10⟩, ⟨3, ij11⟩, ⟨0, ij11⟩, ⟨1, ij11⟩, ⟨2, ij11⟩, ⟨3, ij01⟩, ⟨0, ij01⟩, ⟨1, ij01⟩, ⟨2, ij01⟩⟩
⟨⟨1, ij00⟩, ⟨0, ij00⟩, ⟨3, ij01⟩, ⟨2, ij01⟩, ⟨1, ij01⟩, ⟨0, ij01⟩, ⟨3, ij00⟩, ⟨2, ij00⟩, ⟨3, ij10⟩, ⟨2, ij10⟩, ⟨1, ij10⟩, ⟨0, ij10⟩, ⟨3, ij11⟩, ⟨0, ij11⟩, ⟨1, ij11⟩, ⟨2, ij11⟩⟩

By stretching operation and Corollary 1, we have the following corollary.

Corollary 3. *Suppose that $n \geq 5$. Let e be any edge of $BF^{i,j,k}_{0,1,2}(n)$ with $i, j, k \in \mathbb{Z}_2$. Then there exists a totally scheduled hamiltonian cycle of $BF^{i,j,k}_{0,1,2}(n)$ including e.*

A path P of $BF(n)$ is *weakly ℓ-scheduled* if there is at least one non-terminal level-ℓ vertex v of P, such that v is incident to a level-$(\ell - 1) \bmod n$ edge and a level-ℓ edge on P. Fig. 4 illustrates two weakly 2-scheduled hamiltonian paths P_1 and P_2 of $BF^{i,j}_{0,1}(4)$ and their images $\gamma^0_3 \circ \gamma^0_2(P_1)$ and $\gamma^0_3 \circ \gamma^0_2(P_2)$ on $\gamma^0_3 \circ \gamma^0_2(BF^{i,j}_{0,1}(4)) = BF^{i,j,0,0}_{0,1,2,3}(6)$, respectively.

Lemma 7. *Let $n \geq 4$ and $i, j \in \mathbb{Z}_2$. Suppose that s is any level-1 vertex of $BF^{i,j}_{0,1}(n)$ and d is any level-2 vertex of $BF^{i,j}_{0,1}(n)$. Then there exists a weakly 2-scheduled hamiltonian path of $BF^{i,j}_{0,1}(n)$, joining s to d.*

Proof. Without loss of generality, we assume that $s = \langle 1, ij0^{n-2}\rangle$ and $d = \langle 2, ijpqx\rangle$ with $p, q \in \mathbb{Z}_2$ and $x \in \mathbb{Z}_2^{n-4}$. We prove this lemma by induction on n. The induction bases are listed in Tables 1 and 2.

As the inductive hypothesis, we assume that the statement holds for $BF^{i,j}_{0,1}(n-2)$ with $n \geq 6$. Now, we partition $BF^{i,j}_{0,1}(n)$ into $\{BF^{i,j,h,k}_{0,1,2,3}(n) \mid h, k \in \mathbb{Z}_2\}$. By the inductive hypothesis, there exists a weakly 2-scheduled hamiltonian path P^{00} of $BF^{i,j}_{0,1}(n-2)$ joining $\langle 1, ij0^{n-4}\rangle$ to $\langle 2, ijx\rangle$. Hence, there is at least one non-terminal level-2 vertex of P^{00}, say $v = \langle 2, ijy\rangle$ with $y \neq x$, such that v is incident to a level-1 edge and a level-2 edge on P^{00}. By Lemma 2, we have $BF^{i,j,0,0}_{0,1,2,3}(n) = \gamma^0_3 \circ \gamma^0_2 \circ \gamma^j_1(BF^i_0(n-3)) = \gamma^0_3 \circ \gamma^0_2(BF^{i,j}_{0,1}(n-2))$. Thus, $\gamma^0_3 \circ \gamma^0_2(P^{00})$ is a path on $BF^{i,j,0,0}_{0,1,2,3}(n)$ joining s to

Table 2
Hamiltonian paths of $BF^{i,j}_{0,1}(5)$ between $\langle 1, ij000\rangle$ and $\langle 2, ijpqx\rangle$ for any $p, q, x \in \mathbb{Z}_2$

⟨⟨1, ij000⟩, ⟨0, ij000⟩, ⟨4, ij001⟩, ⟨3, ij011⟩, ⟨2, ij111⟩, ⟨1, ij111⟩, ⟨0, ij111⟩, ⟨4, ij111⟩, ⟨3, ij111⟩, ⟨2, ij011⟩, ⟨1, ij011⟩, ⟨0, ij011⟩, ⟨4, ij011⟩, ⟨3, ij001⟩, ⟨2, ij101⟩, ⟨1, ij101⟩, ⟨0, ij101⟩, ⟨4, ij101⟩, ⟨3, ij101⟩, ⟨2, ij001⟩, ⟨1, ij001⟩, ⟨0, ij001⟩, ⟨4, ij000⟩, ⟨3, ij010⟩, ⟨2, ij110⟩, ⟨1, ij110⟩, ⟨0, ij110⟩, ⟨4, ij110⟩, ⟨3, ij110⟩, ⟨2, ij010⟩, ⟨1, ij010⟩, ⟨0, ij010⟩, ⟨4, ij010⟩, ⟨3, ij000⟩, ⟨2, ij100⟩, ⟨1, ij100⟩, ⟨0, ij100⟩, ⟨4, ij100⟩, ⟨3, ij100⟩, ⟨2, ij000⟩⟩
⟨⟨1, ij000⟩, ⟨0, ij000⟩, ⟨4, ij001⟩, ⟨3, ij011⟩, ⟨2, ij111⟩, ⟨1, ij111⟩, ⟨0, ij111⟩, ⟨4, ij111⟩, ⟨3, ij111⟩, ⟨2, ij011⟩, ⟨1, ij011⟩, ⟨0, ij011⟩, ⟨4, ij011⟩, ⟨3, ij001⟩, ⟨2, ij101⟩, ⟨1, ij101⟩, ⟨0, ij101⟩, ⟨4, ij101⟩, ⟨3, ij101⟩, ⟨2, ij001⟩, ⟨1, ij001⟩, ⟨0, ij001⟩, ⟨4, ij000⟩, ⟨3, ij010⟩, ⟨2, ij110⟩, ⟨1, ij110⟩, ⟨0, ij110⟩, ⟨4, ij110⟩, ⟨3, ij110⟩, ⟨2, ij010⟩, ⟨1, ij010⟩, ⟨0, ij010⟩, ⟨4, ij010⟩, ⟨3, ij000⟩, ⟨2, ij000⟩, ⟨3, ij100⟩, ⟨4, ij100⟩, ⟨0, ij100⟩, ⟨1, ij100⟩, ⟨2, ij100⟩⟩
⟨⟨1, ij000⟩, ⟨0, ij000⟩, ⟨4, ij001⟩, ⟨3, ij011⟩, ⟨2, ij111⟩, ⟨1, ij111⟩, ⟨0, ij111⟩, ⟨4, ij111⟩, ⟨3, ij111⟩, ⟨2, ij011⟩, ⟨1, ij011⟩, ⟨0, ij011⟩, ⟨4, ij011⟩, ⟨3, ij001⟩, ⟨2, ij101⟩, ⟨1, ij101⟩, ⟨0, ij101⟩, ⟨4, ij101⟩, ⟨3, ij101⟩, ⟨2, ij001⟩, ⟨1, ij001⟩, ⟨0, ij001⟩, ⟨4, ij000⟩, ⟨3, ij000⟩, ⟨2, ij000⟩, ⟨3, ij100⟩, ⟨2, ij100⟩, ⟨1, ij100⟩, ⟨0, ij100⟩, ⟨4, ij100⟩, ⟨3, ij110⟩, ⟨4, ij110⟩, ⟨0, ij110⟩, ⟨1, ij110⟩, ⟨2, ij110⟩, ⟨3, ij010⟩, ⟨4, ij010⟩, ⟨0, ij010⟩, ⟨1, ij010⟩, ⟨2, ij010⟩⟩
⟨⟨1, ij000⟩, ⟨0, ij000⟩, ⟨4, ij001⟩, ⟨3, ij011⟩, ⟨2, ij111⟩, ⟨1, ij111⟩, ⟨0, ij111⟩, ⟨4, ij111⟩, ⟨3, ij111⟩, ⟨2, ij011⟩, ⟨1, ij011⟩, ⟨0, ij011⟩, ⟨4, ij011⟩, ⟨3, ij001⟩, ⟨2, ij101⟩, ⟨1, ij101⟩, ⟨0, ij101⟩, ⟨4, ij101⟩, ⟨3, ij101⟩, ⟨2, ij001⟩, ⟨1, ij001⟩, ⟨0, ij001⟩, ⟨4, ij000⟩, ⟨3, ij010⟩, ⟨2, ij010⟩, ⟨1, ij010⟩, ⟨0, ij010⟩, ⟨4, ij010⟩, ⟨3, ij000⟩, ⟨2, ij000⟩, ⟨3, ij100⟩, ⟨2, ij100⟩, ⟨1, ij100⟩, ⟨0, ij100⟩, ⟨4, ij100⟩, ⟨3, ij110⟩, ⟨4, ij110⟩, ⟨0, ij110⟩, ⟨1, ij110⟩, ⟨2, ij110⟩⟩
⟨⟨1, ij000⟩, ⟨0, ij000⟩, ⟨4, ij000⟩, ⟨3, ij000⟩, ⟨2, ij000⟩, ⟨3, ij100⟩, ⟨2, ij100⟩, ⟨1, ij100⟩, ⟨0, ij100⟩, ⟨4, ij100⟩, ⟨3, ij110⟩, ⟨2, ij010⟩, ⟨1, ij010⟩, ⟨0, ij010⟩, ⟨4, ij010⟩, ⟨3, ij010⟩, ⟨2, ij110⟩, ⟨1, ij110⟩, ⟨0, ij110⟩, ⟨4, ij110⟩, ⟨0, ij111⟩, ⟨1, ij111⟩, ⟨2, ij111⟩, ⟨3, ij011⟩, ⟨4, ij011⟩, ⟨0, ij011⟩, ⟨1, ij011⟩, ⟨2, ij011⟩, ⟨3, ij111⟩, ⟨4, ij111⟩, ⟨3, ij101⟩, ⟨4, ij101⟩, ⟨0, ij101⟩, ⟨1, ij101⟩, ⟨2, ij101⟩, ⟨3, ij001⟩, ⟨4, ij001⟩, ⟨0, ij001⟩, ⟨1, ij001⟩, ⟨2, ij001⟩⟩
⟨⟨1, ij000⟩, ⟨0, ij000⟩, ⟨4, ij000⟩, ⟨3, ij000⟩, ⟨2, ij000⟩, ⟨3, ij100⟩, ⟨2, ij100⟩, ⟨1, ij100⟩, ⟨0, ij100⟩, ⟨4, ij100⟩, ⟨3, ij110⟩, ⟨2, ij010⟩, ⟨1, ij010⟩, ⟨0, ij010⟩, ⟨4, ij010⟩, ⟨3, ij010⟩, ⟨2, ij110⟩, ⟨1, ij110⟩, ⟨0, ij110⟩, ⟨4, ij110⟩, ⟨0, ij111⟩, ⟨1, ij111⟩, ⟨2, ij111⟩, ⟨3, ij011⟩, ⟨4, ij001⟩, ⟨0, ij001⟩, ⟨1, ij001⟩, ⟨2, ij001⟩, ⟨3, ij001⟩, ⟨4, ij011⟩, ⟨0, ij011⟩, ⟨1, ij011⟩, ⟨2, ij011⟩, ⟨3, ij111⟩, ⟨4, ij111⟩, ⟨3, ij101⟩, ⟨4, ij101⟩, ⟨0, ij101⟩, ⟨1, ij101⟩, ⟨2, ij101⟩⟩
⟨⟨1, ij000⟩, ⟨0, ij000⟩, ⟨4, ij000⟩, ⟨3, ij000⟩, ⟨2, ij000⟩, ⟨3, ij100⟩, ⟨2, ij100⟩, ⟨1, ij100⟩, ⟨0, ij100⟩, ⟨4, ij100⟩, ⟨3, ij110⟩, ⟨2, ij010⟩, ⟨1, ij010⟩, ⟨0, ij010⟩, ⟨4, ij010⟩, ⟨3, ij010⟩, ⟨2, ij110⟩, ⟨1, ij110⟩, ⟨0, ij110⟩, ⟨4, ij110⟩, ⟨0, ij111⟩, ⟨1, ij111⟩, ⟨2, ij111⟩, ⟨3, ij111⟩, ⟨4, ij111⟩, ⟨3, ij101⟩, ⟨4, ij101⟩, ⟨0, ij101⟩, ⟨1, ij101⟩, ⟨2, ij101⟩, ⟨3, ij001⟩, ⟨2, ij001⟩, ⟨1, ij001⟩, ⟨0, ij001⟩, ⟨4, ij001⟩, ⟨3, ij011⟩, ⟨4, ij011⟩, ⟨0, ij011⟩, ⟨1, ij011⟩, ⟨2, ij011⟩⟩
⟨⟨1, ij000⟩, ⟨0, ij000⟩, ⟨4, ij000⟩, ⟨3, ij010⟩, ⟨2, ij110⟩, ⟨1, ij110⟩, ⟨0, ij110⟩, ⟨4, ij110⟩, ⟨3, ij110⟩, ⟨2, ij010⟩, ⟨1, ij010⟩, ⟨0, ij010⟩, ⟨4, ij010⟩, ⟨3, ij000⟩, ⟨2, ij000⟩, ⟨3, ij100⟩, ⟨2, ij100⟩, ⟨1, ij100⟩, ⟨0, ij100⟩, ⟨4, ij100⟩, ⟨0, ij101⟩, ⟨1, ij101⟩, ⟨2, ij101⟩, ⟨3, ij001⟩, ⟨4, ij011⟩, ⟨0, ij011⟩, ⟨1, ij011⟩, ⟨2, ij011⟩, ⟨3, ij011⟩, ⟨4, ij001⟩, ⟨0, ij001⟩, ⟨1, ij001⟩, ⟨2, ij001⟩, ⟨3, ij101⟩, ⟨4, ij101⟩, ⟨3, ij111⟩, ⟨4, ij111⟩, ⟨0, ij111⟩, ⟨1, ij111⟩, ⟨2, ij111⟩⟩

$\langle 2, ij00x\rangle$ or joining s to $\langle 4, ij00x\rangle$. By Corollary 2, there is a totally scheduled hamiltonian cycle C^{hk} of $BF^{i,j,h,k}_{0,1,2,3}(n)$ including all straight edges of level 2 and level 3 for any $h, k \in \mathbb{Z}_2$.

Let $F_k = \{\langle 2, ijw\rangle \in V(P^{00}) \mid \langle 2, ijw\rangle$ is not incident to any level-$(k+1)$ edge on $P^{00}\}$ with $k \in \{0, 1\}$. Obviously, P^{00} is a connected spanning subgraph of $BF^{i,j}_{0,1}(n-2)$. By Lemma 3, we have $V(\gamma^0_3 \circ \gamma^0_2(P^{00})) = V(BF^{i,j,0,0}_{0,1,2,3}(n)) - (\overline{F_0} \cup \overline{F_1})$, where $\overline{F_0} = \{\langle 2, ij00w\rangle \mid \langle 2, ijw\rangle \in F_0\} \cup \{\langle 3, ij00w\rangle \mid \langle 2, ijw\rangle \in F_0\}$ and $\overline{F_1} = \{\langle 3, ij00w\rangle \mid \langle 2, ijw\rangle \in F_1\} \cup \{\langle 4, ij00w\rangle \mid \langle 2, ijw\rangle \in F_1\}$. In addition, we have $\overline{F_0} \cap \overline{F_1} = \emptyset$. If $\gamma^0_3 \circ \gamma^0_2(P^{00})$ joins s to $\langle 2, ij00x\rangle$, let $\overline{P^{00}} = \gamma^0_3 \circ \gamma^0_2(P^{00})$ and $\widetilde{F_0} = \overline{F_0}$. Otherwise, let $\overline{P^{00}} = \langle s, \gamma^0_3 \circ \gamma^0_2(P^{00}), \langle 4, ij00x\rangle, \langle 3, ij00x\rangle, \langle 2, ij00x\rangle\rangle$ and $\widetilde{F_0} = \overline{F_0} - \{\langle 2, ij00x\rangle, \langle 3, ij00x\rangle\}$. For any $h, k \in \mathbb{Z}_2$, let

$$X^{hk}_0 = \{(\langle 2, ijhkw\rangle, \langle 3, ijhkw\rangle) \mid \langle 2, ij00w\rangle \text{ and } \langle 3, ij00w\rangle \text{ are in } \widetilde{F_0}\},$$
$$Y^{hk}_0 = \{(\langle 2, ijhkw\rangle, \langle 3, ij\bar{h}kw\rangle) \mid \langle 2, ij00w\rangle \text{ and } \langle 3, ij00w\rangle \text{ are in } \widetilde{F_0}\},$$
$$X^{hk}_1 = \{(\langle 3, ijhkw\rangle, \langle 4, ijhkw\rangle) \mid \langle 3, ij00w\rangle \text{ and } \langle 4, ij00w\rangle \text{ are in } \overline{F_1}\}, \quad \text{and}$$
$$Y^{hk}_1 = \{(\langle 3, ijhkw\rangle, \langle 4, ijh\bar{k}w\rangle) \mid \langle 3, ij00w\rangle \text{ and } \langle 4, ij00w\rangle \text{ are in } \overline{F_1}\}.$$

Then we consider the following four cases.

Case 1: If $pq = 00$, then $d = \langle 2, ij00x\rangle$. It is noticed that $v \notin F_0 \cup F_1$. Let

$$A = \{(\langle 2, ij10y\rangle, \langle 3, ij00y\rangle), (\langle 2, ij00y\rangle, \langle 3, ij10y\rangle), (\langle 2, ij11y\rangle, \langle 3, ij01y\rangle), (\langle 2, ij01y\rangle, \langle 3, ij11y\rangle), (\langle 3, ij11y\rangle, \langle 4, ij10y\rangle), (\langle 3, ij10y\rangle, \langle 4, ij11y\rangle)\} \quad \text{and}$$
$$B = \{(\langle 2, ij00y\rangle, \langle 3, ij00y\rangle), (\langle 2, ij10y\rangle, \langle 3, ij10y\rangle), (\langle 2, ij01y\rangle, \langle 3, ij01y\rangle), (\langle 2, ij11y\rangle, \langle 3, ij11y\rangle), (\langle 3, ij10y\rangle, \langle 4, ij10y\rangle), (\langle 3, ij11y\rangle, \langle 4, ij11y\rangle)\}.$$

It follows from Lemma 3, that $(\langle 2, ij00y\rangle, \langle 3, ij00y\rangle) \in E(\overline{P^{00}})$. By Corollary 2, we have $(\langle 2, ij10y\rangle, \langle 3, ij10y\rangle) \in E(C^{10})$, $(\langle 2, ij01y\rangle, \langle 3, ij01y\rangle) \in E(C^{01})$, $(\langle 2, ij11y\rangle, \langle 3, ij11y\rangle) \in E(C^{11})$, $(\langle 3, ij10y\rangle, \langle 4, ij10y\rangle) \in E(C^{10})$, and $(\langle 3, ij11y\rangle, \langle 4, ij11y\rangle) \in E(C^{11})$. Then the subgraph P of $BF^{i,j}_{0,1}(n)$, generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$, forms a weakly 2-scheduled path of $BF^{i,j}_{0,1}(n)$ between s and d. Clearly, we have $V(P) = V(BF^{i,j}_{0,1}(n)) - (\widetilde{F_0} \cup \widetilde{F_1})$. Since

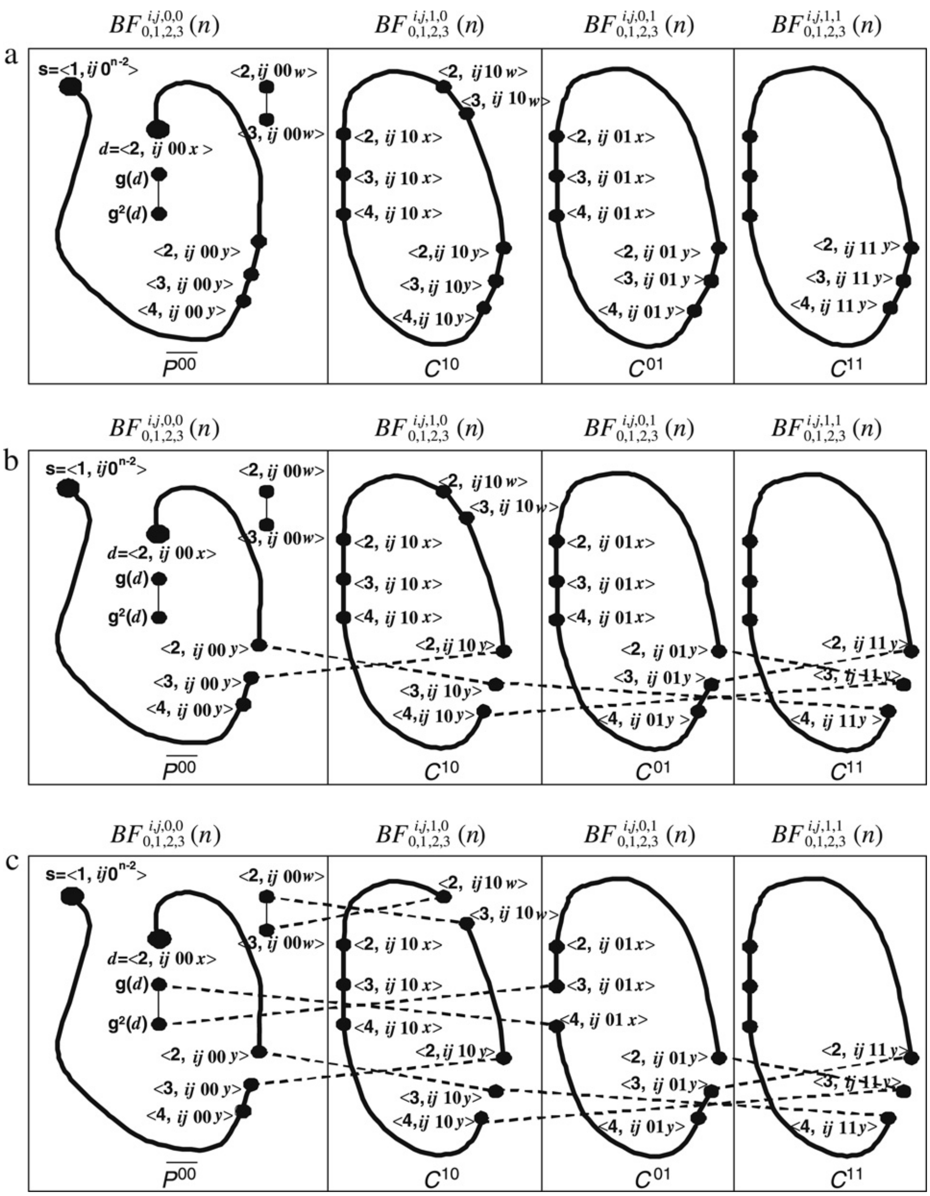

Fig. 5. (a) $\overline{P^{00}} = \gamma^0_3 \circ \gamma^0_2(P^{00})$, C^{10}, C^{01}, and C^{11}; (b) the path P generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$; (c) the path P' generated by $(E(P) \cup (X^{00}_0 \cup Y^{00}_0 \cup Y^{10}_0) \cup (X^{00}_1 \cup Y^{00}_1 \cup Y^{01}_1)) - (X^{10}_0 \cup X^{01}_1)$ to cover all vertices of $\widetilde{F_0} \cup \overline{F_1}$.

C^{hk} includes all straight edges of level 2 and level 3 in $BF^{i,j,h,k}_{0,1,2,3}(n)$, we have $X^{10}_0 \subset E(C^{10})$ and $X^{01}_1 \subset E(C^{01})$. Moreover, we have $(X^{10}_0 \cup X^{01}_1) \cap B = \emptyset$. Therefore, it follows that $(X^{10}_0 \cup X^{01}_1) \subset E(P)$. Let P' be the subgraph generated by $(E(P) \cup (X^{00}_0 \cup Y^{00}_0 \cup Y^{10}_0) \cup (X^{00}_1 \cup Y^{00}_1 \cup Y^{01}_1)) - (X^{10}_0 \cup X^{01}_1)$. Then P' is a weakly 2-scheduled hamiltonian path of $BF^{i,j}_{0,1}(n)$ joining s to d. See Fig. 5 for illustration, in which $\gamma^0_3 \circ \gamma^0_2(P^{00})$ is supposed to join s and $\langle 2, ij00x\rangle$.

Case 2: If $pq = 10$, then $d = \langle 2, ij10x\rangle$. Let

$$A = \{(\langle 2, ij00x\rangle, \langle 3, ij10x\rangle), (\langle 2, ij11y\rangle, \langle 3, ij01y\rangle), (\langle 2, ij01y\rangle, \langle 3, ij11y\rangle), (\langle 3, ij11y\rangle, \langle 4, ij10y\rangle), (\langle 3, ij10y\rangle, \langle 4, ij11y\rangle)\} \quad \text{and}$$

$$B = \{(\langle 2, ij10x\rangle, \langle 3, ij10x\rangle), (\langle 2, ij01y\rangle, \langle 3, ij01y\rangle), (\langle 2, ij11y\rangle, \langle 3, ij11y\rangle), (\langle 3, ij10y\rangle, \langle 4, ij10y\rangle), (\langle 3, ij11y\rangle, \langle 4, ij11y\rangle)\}.$$

Obviously, the subgraph P, generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$, forms a weakly 2-scheduled path of $BF^{i,j}_{0,1}(n)$ between s and d. Moreover, the subgraph P', generated by $(E(P) \cup (X^{00}_0 \cup Y^{00}_0 \cup Y^{10}_0) \cup (X^{00}_1 \cup Y^{00}_1 \cup Y^{01}_1)) - (X^{10}_0 \cup X^{01}_1)$, is a weakly 2-scheduled hamiltonian path of $BF^{i,j}_{0,1}(n)$ joining s to d.

Case 3: If $pq = 01$, then $d = \langle 2, ij01x\rangle$. Let

$$A = \{(\langle 2, ij00x\rangle, \langle 3, ij10x\rangle), (\langle 2, ij11x\rangle, \langle 3, ij01x\rangle), (\langle 3, ij11x\rangle, \langle 4, ij10x\rangle)\} \quad \text{and}$$
$$B = \{(\langle 2, ij01x\rangle, \langle 3, ij01x\rangle), (\langle 2, ij11x\rangle, \langle 3, ij11x\rangle), (\langle 3, ij10x\rangle, \langle 4, ij10x\rangle)\}.$$

Obviously, the subgraph P, generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$, forms a weakly 2-scheduled path of $BF^{i,j}_{0,1}(n)$, between s and d. Moreover, the subgraph P', generated by $(E(P) \cup (X^{00}_0 \cup Y^{00}_0 \cup Y^{10}_0) \cup (X^{00}_1 \cup Y^{00}_1 \cup Y^{01}_1)) - (X^{10}_0 \cup X^{01}_1)$, is a weakly 2-scheduled hamiltonian path of $BF^{i,j}_{0,1}(n)$ joining s to d.

Case 4: If $pq = 11$, then $d = \langle 2, ij11x\rangle$. Let

$$A = \{(\langle 2, ij00x\rangle, \langle 3, ij10x\rangle), (\langle 3, ij11x\rangle, \langle 4, ij10x\rangle), (\langle 3, ij01y\rangle, \langle 4, ij00y\rangle), (\langle 3, ij00y\rangle, \langle 4, ij01y\rangle)\} \quad \text{and}$$
$$B = \{(\langle 3, ij10x\rangle, \langle 4, ij10x\rangle), (\langle 3, ij00y\rangle, \langle 4, ij00y\rangle), (\langle 3, ij01y\rangle, \langle 4, ij01y\rangle), (\langle 2, ij11x\rangle, \langle 3, ij11x\rangle)\}.$$

The subgraph P, generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$, forms a weakly 2-scheduled path of $BF^{i,j}_{0,1}(n)$ between s and d. Moreover, the subgraph P', generated by $(E(P) \cup (X^{00}_0 \cup Y^{00}_0 \cup Y^{10}_0) \cup (X^{00}_1 \cup Y^{00}_1 \cup Y^{01}_1)) - (X^{10}_0 \cup X^{01}_1)$, is a weakly 2-scheduled hamiltonian path of $BF^{i,j}_{0,1}(n)$ joining s to d. □

By symmetry, the next corollary can be proved in the way similar to Lemma 7.

Corollary 4. *Assume that $n \geq 4$ and $i, j \in \mathbb{Z}_2$. Let s be any level-1 vertex of $BF^{i,j}_{0,1}(n)$ and d be any level-0 vertex of $BF^{i,j}_{0,1}(n)$. Then there exists a weakly 0-scheduled hamiltonian path of $BF^{i,j}_{0,1}(n)$ joining s to d.*

Lemma 8. *Assume that $n \geq 4$. Let $s = \langle 1, 0^n\rangle$, $d_1 = \langle 2, 0^2 10^{n-3}\rangle$, and $d_2 = \langle 0, 0^n\rangle$. Then there exist two hamiltonian paths H_1 and H_2 of $BF^{0,0}_{0,1}(n)$, such that the following conditions are all satisfied:* (i) *H_1 joins s to d_1,* (ii) *H_2 joins s to d_2, and* (iii) *$H_1(1) = H_2(1) = s$ and $H_1(t) \neq H_2(t)$ for each $2 \leq t \leq \left|V\left(BF^{0,0}_{0,1}(n)\right)\right| = n2^{n-2}$.*

Proof. Let $u_1 = g(s) = \langle 2, 0^n\rangle$, $u_2 = f(u_1) = g(d_1) = \langle 3, 0^2 10^{n-3}\rangle$, $u_3 = g^{-1}(d_1) = \langle 1, 0^2 10^{n-3}\rangle$, $u_4 = f(u_2)$, and $u_5 = g(u_1) = f(d_1) = \langle 3, 0^n\rangle$. Note that $u_4 = \langle 0, 0011\rangle$ if $n = 4$ and $u_4 = \langle 4, 0^2 1^2 0^{n-4}\rangle$ if $n \geq 5$. We partition $BF^{0,0}_{0,1}(n)$ into $\{BF^{0,0,0}_{0,1,2}(n), BF^{0,0,1}_{0,1,2}(n)\}$. By Corollary 1, there is a hamiltonian cycle C_0 of $BF^{0,0,0}_{0,1,2}(n)$ including all straight edges of level 2. Thus, we have $(u_1, u_5) \in E(C_0)$. By Lemma 6 and Corollary 3, there is a hamiltonian cycle C_1 of $BF^{0,0,1}_{0,1,2}(n)$, such that $(u_2, u_4) \in E(C_1)$. It is noticed that s and d_1 are vertices of degree two in $BF^{0,0,0}_{0,1,2}(n)$ and $BF^{0,0,1}_{0,1,2}(n)$, respectively. Therefore, we can write $C_0 = \langle s, u_1, u_5, P_0, d_2, s\rangle$ and $C_1 = \langle d_1, u_2, u_4, P_1, u_3, d_1\rangle$. As an illustrative example, Fig. 6(a) depicts C_0 and C_1 on $BF^{0,0}_{0,1}(4)$. Fig. 6(b) illustrates the abstraction of C_0 and C_1 for general n. Since $\{(u_1, u_2), (d_1, u_5)\} \subset E(BF^{0,0}_{0,1}(n))$, we set

$$H_1 = \langle s, d_2, P_0^{-1}, u_5, u_1, u_2, u_4, P_1, u_3, d_1\rangle \quad \text{and}$$
$$H_2 = \langle s, u_1, u_2, u_4, P_1, u_3, d_1, u_5, P_0, d_2\rangle.$$

Then it can be verified, as shown on Fig. 6(c), that H_1 and H_2 satisfy the conditions. □

Lemma 9. *Given any $k \in \{0, 1\}$ and $n \geq 4$, let (b_1, w_1) be a level-1 straight edge of $BF^{1,1,k}_{0,1,n-1}(n)$ and (b_2, w_2) be a level-0 straight edge of $BF^{1,1,k}_{0,1,n-1}(n)$ such that w_1 and w_2 are two distinct level-1 vertices. Then there exist two hamiltonian paths H_1 and H_2 of $BF^{1,1}_{0,1}(n)$, such that the following conditions are all satisfied:*

(i) *$H_1(1) = b_1$ and $H_1(n2^{n-2}) = w_1$,*
(ii) *$H_2(1) = b_2$ and $H_2(n2^{n-2}) = w_2$, and*
(iii) *$H_1(t) \neq H_2(t)$ for each $1 \leq t \leq n2^{n-2}$.*

Proof. Without loss of generality, we assume that $k = 0$. Let $u_1 = g^{n-3}(b_1)$, $u_2 = f(u_1)$, $u_3 = g(u_2)$, $u_4 = g(u_3)$, $u_5 = g^{n-3}(u_4) = g^{-1}(u_2)$, $u_6 = f(u_5) = g^{-1}(w_1)$, $v_1 = f^{-1}(b_2)$, $v_2 = g^{-n+3}(v_1)$, $v_3 = g^{-1}(v_2)$, $v_4 = g^{-1}(v_3) = g(v_1)$, $v_5 = f^{-1}(v_4) = g^{-1}(b_2)$, and $v_6 = g^{-n+3}(v_5) = g(w_2)$. By Corollary 1, $BF^{1,1,0}_{0,1,2}(n)$ has a totally scheduled hamiltonian cycle. By Lemma 1, $BF^{1,1,0}_{0,1,n-1}(n)$ is isomorphic with $BF^{1,1,0}_{0,1,2}(n)$. Hence, there also exists a totally scheduled hamiltonian cycle C_0 of $BF^{1,1,0}_{0,1,n-1}(n)$. It is noticed that w_1 is adjacent to u_6. Moreover, w_1, u_6, b_2, and w_2 are all vertices of degree two in $BF^{1,1,0}_{0,1,n-1}(n)$. Accordingly, C_0 can be written as $C_0 = \langle w_1, b_1, P_0, u_1, u_6, w_1\rangle$, where $P_0 = \langle b_1, P_{01}, v_5, b_2, w_2, v_6, P_{02}, u_1\rangle$.

By Lemma 6, $BF^{1,1,1}_{0,1,2}(4)$ has a totally scheduled hamiltonian cycle C such that $e \in E(C)$ if $e \in E(BF^{1,1,1}_{0,1,2}(4)) - \{(\langle 3, 1110\rangle, \langle 0, 1110\rangle), (\langle 3, 1111\rangle, \langle 0, 1111\rangle)\}$. By Lemma 1, $BF^{1,1,1}_{0,1,3}(4)$ is isomorphic with $BF^{1,1,1}_{0,1,2}(4)$. Hence, $BF^{1,1,1}_{0,1,3}(4)$ has a totally scheduled hamiltonian cycle C such that $e \in E(C)$ if $e \in E(BF^{1,1,1}_{0,1,3}(4)) - \{(\langle 2, 1101\rangle, \langle 3, 1101\rangle), (\langle 2, 1111\rangle, \langle 3, 1111\rangle)\}$. Obviously, (u_5, u_2) is a level-$(n-3)$ for any $n \geq 4$. Therefore, we have $(u_5, u_2) \in E(BF^{1,1,1}_{0,1,3}(4)) -$

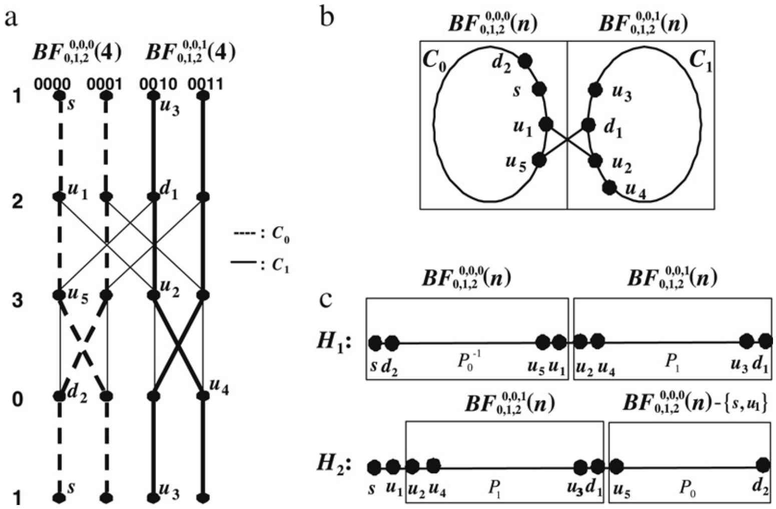

Fig. 6. Illustration for Lemma 8.

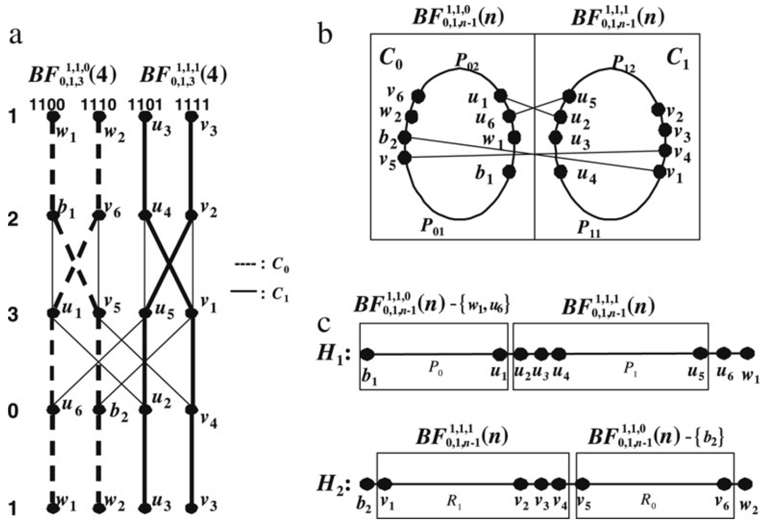

Fig. 7. Illustration for Lemma 9. In (a), $(b_1, w_1) = (\langle 2, 1100\rangle, \langle 1, 1100\rangle)$ and $(b_2, w_2) = (\langle 0, 1110\rangle, \langle 1, 1110\rangle)$ are assumed. In (c), we let $R_1 = \langle v_1, P_{11}^{-1}, u_4, u_3, u_2, , u_5, P_{12}^{-1}, v_2\rangle$ and $R_0 = \langle v_5, P_{01}^{-1}, b_1, w_1, u_6, u_1, P_{02}^{-1}, v_6\rangle$.

$\{(\langle 2, 1101\rangle, \langle 3, 1101\rangle), (\langle 2, 1111\rangle, \langle 3, 1111\rangle)\}$. It follows that $BF^{1,1,1}_{0,1,3}(4)$ has a totally scheduled hamiltonian cycle C_1 such that $(u_5, u_2) \in E(C_1)$. By Corollary 3, $BF^{1,1,1}_{0,1,2}(n)$, $n \geq 5$, has a totally scheduled hamiltonian cycle including any required edge. Since $BF^{1,1,1}_{0,1,n-1}(n)$ is isomorphic with $BF^{1,1,1}_{0,1,2}(n)$, it has a totally scheduled hamiltonian cycle C_1 such that $(u_5, u_2) \in E(C_1)$ if $n \geq 5$. In short, by Lemma 6 and Corollary 3, there is a totally scheduled hamiltonian cycle C_1 of $BF^{1,1,1}_{0,1,n-1}(n)$, such that $(u_5, u_2) \in E(C_1)$. Since u_2, u_3, v_3, and v_4 are vertices of degree two in $BF^{1,1,1}_{0,1,n-1}(n)$, we write $C_1 = \langle u_3, u_4, P_1, u_5, u_2, u_3\rangle$, where $P_1 = \langle u_4, P_{11}, v_1, v_4, v_3, v_2, P_{12}, u_5\rangle$. Fig. 7(a) depicts C_0 and C_1 on $BF^{1,1}_{0,1}(4)$. Fig. 7(b) illustrates the abstraction of C_0 and C_1 for general n. Then we set

$$H_1 = \langle b_1, P_{01}, v_5, b_2, w_2, v_6, P_{02}, u_1, u_2, u_3, u_4, P_{11}, v_1, v_4, v_3, v_2, P_{12}, u_5, u_6, w_1\rangle \quad \text{and}$$
$$H_2 = \langle b_2, v_1, P_{11}^{-1}, u_4, u_3, u_2, u_5, P_{12}^{-1}, v_2, v_3, v_4, v_5, P_{01}^{-1}, b_1, w_1, u_6, u_1, P_{02}^{-1}, v_6, w_2\rangle.$$

Since $w_1 \neq w_2$, $u_2 \neq v_2$, $u_3 \neq v_3$, $u_4 \neq v_4$, and $u_6 \neq v_6$, it can be checked that H_1 and H_2 satisfy the conditions. See Fig. 7(c) for illustration. □

Table 3
4-mutually independent hamiltonian cycles C_1, C_2, C_3, C_4 of $BF(3)$ starting from vertex $\langle 0, 000\rangle$

C_1	$\langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,011\rangle, \langle 0,111\rangle, \langle 2,111\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 1,101\rangle, \langle 0,101\rangle, \langle 2,100\rangle, \langle 0,100\rangle, \langle 1,100\rangle, \langle 2,110\rangle, \langle 0,110\rangle, \langle 1,110\rangle, \langle 0,010\rangle, \langle 2,010\rangle, \langle 1,010\rangle, \langle 2,000\rangle, \langle 1,000\rangle, \langle 0,000\rangle\rangle$
C_2	$\langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,000\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 0,101\rangle, \langle 1,101\rangle, \langle 2,111\rangle, \langle 0,110\rangle, \langle 1,010\rangle, \langle 2,010\rangle, \langle 0,010\rangle, \langle 1,110\rangle, \langle 2,100\rangle, \langle 0,100\rangle, \langle 1,100\rangle, \langle 2,110\rangle, \langle 0,111\rangle, \langle 1,011\rangle, \langle 2,001\rangle, \langle 0,000\rangle\rangle$
C_3	$\langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 2,100\rangle, \langle 0,100\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,010\rangle, \langle 1,110\rangle, \langle 2,110\rangle, \langle 0,111\rangle, \langle 1,111\rangle, \langle 0,011\rangle, \langle 2,011\rangle, \langle 1,011\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 0,101\rangle, \langle 2,101\rangle, \langle 1,101\rangle, \langle 2,111\rangle, \langle 0,110\rangle, \langle 1,010\rangle, \langle 2,000\rangle, \langle 0,000\rangle\rangle$
C_4	$\langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,010\rangle, \langle 1,010\rangle, \langle 0,110\rangle, \langle 2,110\rangle, \langle 1,110\rangle, \langle 2,100\rangle, \langle 0,101\rangle, \langle 1,001\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,101\rangle, \langle 2,111\rangle, \langle 0,111\rangle, \langle 1,011\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 0,100\rangle, \langle 1,100\rangle, \langle 0,000\rangle\rangle$

5. Mutually independent hamiltonian cycles of $BF(n)$

Theorem 1. *For all $n \geq 3$, $\mathcal{IHC}(BF(n)) = 4$.*

Proof. It is trivial that $\mathcal{IHC}(BF(n)) \leq \delta(BF(n)) = 4$. Suppose that $n = 3$. Since $BF(3)$ is vertex-transitive, we only find 4-mutually independent hamiltonian cycles starting from vertex $\langle 0, 000\rangle$. A set $\{C_1, C_2, C_3, C_4\}$ of four hamiltonian cycles is listed in Table 3. It is easy to check that they are mutually independent.

For $n \geq 4$, we partition $BF(n)$ into $\{BF^{i,j}_{0,1}(n) \mid i, j \in \mathbb{Z}_2\}$. Since $BF(n)$ is vertex-transitive, we assume that the beginning vertex is $s = \langle 1, 0^n\rangle$. Let $u_1 = \langle 2, 0^2 10^{n-3}\rangle$, $u_2 = f^{-1}(u_1) = \langle 1, 01^2 0^{n-3}\rangle$, $u_3 = g^{-1}(u_2) = \langle 0, 01^2 0^{n-3}\rangle$, $u_4 = f(u_3) = \langle 1, 1^3 0^{n-3}\rangle$, $u_5 = g(u_4) = \langle 2, 1^3 0^{n-3}\rangle$, $u_6 = f^{-1}(u_5) = \langle 1, 1010^{n-3}\rangle$, $u_7 = f^{-1}(s) = \langle 0, 10^{n-1}\rangle$, $v_1 = g^{-1}(s) = \langle 0, 0^n\rangle$, $v_2 = f(v_1) = \langle 1, 10^{n-1}\rangle$, $v_3 = g(v_2) = \langle 2, 10^{n-1}\rangle$, $v_4 = f^{-1}(v_3) = \langle 1, 1^2 0^{n-2}\rangle$, $v_5 = g^{-1}(v_4) = \langle 0, 1^2 0^{n-2}\rangle$, $v_6 = f(v_5) = \langle 1, 010^{n-2}\rangle$, and $v_7 = g(v_6) = f(s) = \langle 2, 010^{n-2}\rangle$. Obviously, $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ consists of 14 different vertices of $BF(n)$, such that all (u_1, u_2), (u_3, u_4), (u_5, u_6), (u_7, s), (v_1, v_2), (v_3, v_4), (v_5, v_6), and (v_7, s) are in $E(BF(n))$. By Lemma 8, there exist two hamiltonian paths P_1 and P_2 of $BF^{0,0}_{0,1}(n)$ such that (1) P_1 joins s to u_1, (2) P_2 joins s to v_1, and (3) $P_1(1) = P_2(1) = s$ and $P_1(t) \neq P_2(t)$ for each $2 \leq t \leq n2^{n-2}$. By Corollary 4, there is a hamiltonian path Q_1 of $BF^{0,1}_{0,1}(n)$ joining u_2 to u_3. Similarly, there is a hamiltonian path R_1 of $BF^{1,0}_{0,1}(n)$ joining u_6 to u_7. By Lemma 7, there is a hamiltonian path Q_2 of $BF^{1,0}_{0,1}(n)$ joining v_2 to v_3. Again, there is a hamiltonian path R_2 of $BF^{0,1}_{0,1}(n)$ joining v_6 to v_7. Applying Lemma 9, we can find two hamiltonian paths S_1 and S_2 of $BF^{1,1}_{0,1}(n)$, such that (1) S_1 joins u_4 to u_5, (2) S_2 joins v_4 to v_5, and (3) $S_1(t) \neq S_2(t)$ for each $1 \leq t \leq n2^{n-2}$. We set $C_1 = \langle s, P_1, u_1, u_2, Q_1, u_3, u_4, S_1, u_5, u_6, R_1, u_7, s\rangle$ and $C_2 = \langle s, P_2, v_1, v_2, Q_2, v_3, v_4, S_2, v_5, v_6, R_2, v_7, s\rangle$. Figs. 8(a) and (b) illustrate C_1 and C_2, respectively. Obviously, C_1 and C_2 are both hamiltonian cycles of $BF(n)$. In what follows, we claim that C_1 and C_2 are independent: first, Lemma 8 guarantees that $C_1(t) \neq C_2(t)$ for all $2 \leq t \leq n2^{n-2}$. Next, we have $C_1(t) \neq C_2(t)$ for $n2^{n-2} + 1 \leq t \leq n2^{n-1}$, because C_1 and C_2 pass through the vertices of $BF^{0,1}_{0,1}(n)$ and $BF^{1,0}_{0,1}(n)$, respectively. Moreover, Lemma 9 guarantees that $C_1(t) \neq C_2(t)$ for all $n2^{n-1} + 1 \leq t \leq 3 \times n2^{n-2}$. Finally, we have $C_1(t) \neq C_2(t)$ for $3 \times n2^{n-2} + 1 \leq t \leq n2^n$ since C_1 and C_2 pass through the vertices of $BF^{1,0}_{0,1}(n)$ and $BF^{0,1}_{0,1}(n)$, respectively. As a consequence, C_1 and C_2 are independent.

Let $u'_3 = \langle 0, 01^2 0^{n-4} 1\rangle$, $u'_4 = f(u'_3) = \langle 1, 1^3 0^{n-4} 1\rangle$, $u'_5 = g(u'_4) = \langle 2, 1^3 0^{n-4} 1\rangle$, $u'_6 = f^{-1}(u'_5) = \langle 1, 1010^{n-4} 1\rangle$, $v'_3 = \langle 2, 10^{n-2} 1\rangle$, $v'_4 = f^{-1}(v'_3) = \langle 1, 1^2 0^{n-3} 1\rangle$, $v'_5 = g^{-1}(v'_4) = \langle 0, 1^2 0^{n-3} 1\rangle$, and $v'_6 = f(v'_5) = \langle 1, 010^{n-3} 1\rangle$. Obviously, $u'_i \neq u_i$ and $v'_i \neq v_i$ for $3 \leq i \leq 6$. By Corollary 4, there is a hamiltonian path Q_3 of $BF^{0,1}_{0,1}(n)$ joining u_2 to u'_3. Similarly, there is a hamiltonian path R_3 of $BF^{1,0}_{0,1}(n)$ joining u'_6 to u_7. By Lemma 7, there is a hamiltonian path Q_4 of $BF^{1,0}_{0,1}(n)$ joining v_2 to v'_3. Similarly, there is a hamiltonian path R_4 of $BF^{0,1}_{0,1}(n)$ joining v'_6 to v_7. We apply Lemma 9 to construct two hamiltonian paths S_3 and S_4 of $BF^{1,1}_{0,1}(n)$, such that (1) S_3 joins u'_4 to u'_5, (2) S_4 joins v'_4 to v'_5, and (3) $S_3(t) \neq S_4(t)$ for all $1 \leq t \leq n2^{n-2}$. Then we set $O_1 = \langle s, P_1, u_1, u_2, Q_3, u'_3, u'_4, S_3, u'_5, u'_6, R_3, u_7, s\rangle$ and $O_2 = \langle s, P_2, v_1, v_2, Q_4, v'_3, v'_4, S_4, v'_5, v'_6, R_4, v_7, s\rangle$. Similar to C_1 and C_2, O_1 and O_2 are independent.

Let $C_3 = O_1^{-1}$ and $C_4 = O_2^{-1}$. For clarity, we list C_1, C_2, C_3, and C_4 as follows.

$$C_1 = \langle s, P_1, u_1, u_2, Q_1, u_3, u_4, S_1, u_5, u_6, R_1, u_7, s\rangle,$$
$$C_2 = \langle s, P_2, v_1, v_2, Q_2, v_3, v_4, S_2, v_5, v_6, R_2, v_7, s\rangle,$$
$$C_3 = \langle s, u_7, R_3^{-1}, u'_6, u'_5, S_3^{-1}, u'_4, u'_3, Q_3^{-1}, u_2, u_1, P_1^{-1}, s\rangle, \quad \text{and}$$
$$C_4 = \langle s, v_7, R_4^{-1}, v'_6, v'_5, S_4^{-1}, v'_4, v'_3, Q_4^{-1}, v_2, v_1, P_2^{-1}, s\rangle.$$

Then it is easy to check that C_1, C_2, C_3, and C_4 are 4-mutually independent hamiltonian cycles of $BF(n)$ starting from vertex s. See Fig. 8 for illustration. □

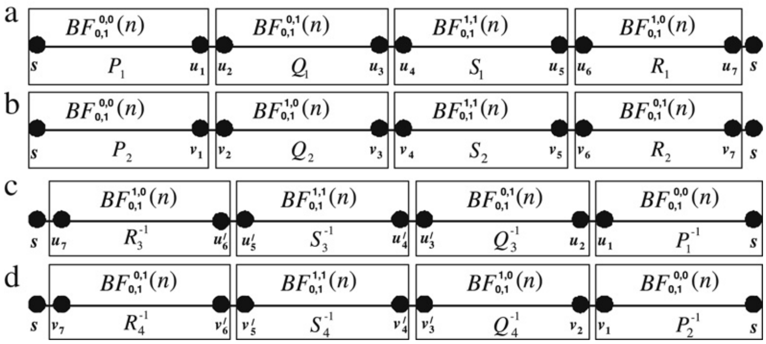

Fig. 8. Illustration for Theorem 1. (a) C_1; (b) C_2; (c) C_3; (d) C_4.

6. Conclusion

In this paper, we discuss the applications of mutually independent hamiltonian cycles, and prove that $\mathcal{IHC}(BF(n)) = 4$ for all $n \geq 3$. Wong [11] presented a recursive method to construct a hamiltonian cycle on the k-ary wrapped butterfly network, which is the generalization of the binary wrapped butterfly graph. Let $BF(k, n)$ denote the n-dimensional k-ary wrapped butterfly network. Then we have $BF(n) \cong BF(2, n)$. As an extension of our current research, it is intriguing to investigate the mutually independent hamiltonicity of $BF(k, n)$ for $k \geq 3$. By definition, $BF(k, n)$ is $2k$-regular. Therefore, we intuitively conjecture that $\mathcal{IHC}(BF(k, n)) = 2k$ for every $n \geq 3$. Since our current approach to proving that $\mathcal{IHC}(BF(2, n)) = 4$ depends upon Lemmas 7–9, it is inductive. For this reason, it is complicated to directly apply our approach to proving that $\mathcal{IHC}(BF(k, n)) = 2k$ for all $k \geq 3$. Perhaps it can be proved algebraically because $BF(k, n)$ is a Cayley graph.

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