國立交通大學

電機與控制工程系

博 士 論 文

T-S 模糊系統之最佳化時間控制

Time-Optimal Control of T-S Fuzzy Models

研 究 生：林保村

指導教授：王啟旭 博士

李祖添 博士

中 華 民 國 九 十 七 年 六 月

T-S 模糊系統的最佳化時間控制

Time-Optimal Control of T-S Fuzzy Models

研 究 生： 林保村 　　Student： Pao-Tsun Lin

指導教授： 王啓旭 　　Advisor(s)： Chi-Hsu Wang

李祖添 　　Tsu-Tian Lee

國 立 交 通 大 學

電機與控制工程系

博 士 論 文

A Dissertation

Submitted to Department of Electrical and Control Engineering

College of Electrical Engineering and Computer Science

National Chiao Tung University

in partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Electrical and Control Engineering

June 2008

Hsinchu, Taiwan, Republic of China

中華民國九十七年六月

T-S 模糊系統之最佳化時間控制

研究生： 林保村　　　　指導教授： 王啟旭 博士

李祖添 博士

國立交通大學電機與控制工程系博士班

摘　　要

本論文針對 Takagi-Sugeno (T-S) 模糊模型，設計一最佳化時間控制器，T-S 模糊模型可視為一多面體線性微分包圍(polytopic linear differential inclusion) 數學模型，利用李群論（Lie Algebra）的幾何特性推導最佳化時間之奇異性、存在性及切換次數。

本論文提出最佳化時間之控制器，首先針對控制器之存在性，推導 T-S 模糊模型之可控制性，在 T-S 模糊模型相關論文中，此為首次探討 T-S 模糊模型之控制性，並提出歸納式秩數(rank)條件式。透過最大值理論(maximum principle)，最佳化控制器為砰-砰(bang-bang)型式，在系統之奇異性之推導中，證明所提出之歸納式秩數(rank)條件式，可供設計非奇異之控制器使用;亦即，可控制之 T-S 模糊模型可設計最佳時間控制器，引用可解析李群論(solvable Lie algebra)於控制器之切換次數證明，此證明可提供於計算最佳演算解，讓演算法更容易找到最佳解。透過模擬本控制器設計法則均經在聯結車前進及倒車系統及多輸入系統之可控制性及最佳化時間控制器。

Time-Optimal Control of T-S Fuzzy Models

Student： Pao-Tsun Lin Advisors： Dr. Chi-Hsu Wang

Dr. Tsu-Tian Lee

Department of Electrical and Control Engineering
National Chiao Tung University

ABSTRACT

This dissertation investigates geometric property of time-optimal problem in Takagi-Sugeno (T-S) fuzzy model via Lie algebra. We will focus on the existence of time-optimal solution, singularity of switching function and number of switching. These inherent problems are considered because of their rich geometric properties. The necessary condition for the existence of time-optimal solution reveals the controllability of T-S fuzzy model which can be found by the generalized rank condition. The time-optimal controller can be found as the bang-bang type by applying maximum principle. In the study of singularity problem, we will focus on switching function whatever vanished on a finite time interval. The bounded number of switching can be found if the T-S model (also a nonlinear system) is solvable. This feature can be applied to solve the time-optimal problem by numerical approach. Fast response is always a considered property in this dissertation. A notion directly relate to the convergence rate of the state trajectories. A controller design of T-S fuzzy model on maximal convergence rate is introduced by the level set function. The result

of maximizing the convergence rate is characterized from the maximal invariant ellipsoid. The controller is also bang-bang within both the initial states and target states belong to level set.

Acknowledgements

I would like to thank all people who have helped and inspired me during my doctoral study.

I especially want to thank my advisors, Prof. Chi-Hsu Wang and Prof. Tsu-Tian Lee for their guidance during my research and study at National Chiao Tung University. Their perpetual energy and enthusiasm in research had motivated all advisees, including me. In addition, they were always accessible and willing to help me. As a result, research life became smooth and rewarding for me.

I would also like to thanks Prof. Hsin-Hsin Huang for listening to my sometimes very scattered thoughts and helping me channel them in the right direction. Our wonderful friendship is not only a turning point in my life, but also bringing me to the future. Without his encouragements, my ambition to study can hardly be realized.

Special thanks to Prof. Shun-Feng Su, Prof. Ching-Cheng Teng, Prof. Wei-Yen Wang and Prof. Yih-Guang Leu as my thesis committee members.

Fr. Visminlu Vicente L. Chua, S.J., have been my high school teacher. He offers advice and suggestions whenever I need them. Besides, they have set a role model of a typical teacher who cares and loves his students as if they were their own kids. Thank you.

Thanks to all members are at Intelligent Network Laboratory.

My deepest gratitude goes to my family for their unflagging love and support throughout my life; this dissertation is simply impossible without them.

Finally, I would like to thanks all my buddies at the swimming team of NTUST and D’Vine for making my life unforgettable.

Table of Content

List of Figures

Chapter 1
Introduction

This dissertation deals with the time-optimal control and maximal convergence rate for constrained T-S fuzzy system. In recent years, fuzzy logic control with human knowledge of the plant has witnessed an effective approach to the design of nonlinear control systems. Indeed, there have been many successful applications which are based on fuzzy control [1-8]. In [9], Takagi and Sugeno proposed an approach to model the nonlinear process. This type of models is the so-called T-S model with later further development in [10]. The T-S fuzzy model blends the dynamics of each fuzzy implication by a linear consequence part [11-13]. In this type of fuzzy model, lots of important issues are addressed such as stability [2, 8, 11], H_2/H_∞ performance [13-15] and robustness [16-18],…, etc. In [19], a fuzzy approach is used in the design of time-suboptimal feedback controllers.

1.1 Time-optimal Control

The maximum principle has been extensively applied in many time-optimal control problems [20-35]. A series of results have been published on the applications of maximum principle in time-optimal control of finite dimensional linear systems and certain low-order nonlinear systems [21-23]. It is well-known that Lie brackets play an essential role in the study of time-optimal control [31-35]. In general, the maximum principle can reduce the optimal control problem by Hamiltonian. However, the Hamiltonian formulation contains no information about the existence of time-optimal solution. It is better to convert the existence of time-optimal solution to the study of reachable sets [25, 26, 28]. While the existence of time-optimal solution is addressed as the compactness of researchable set, we still have to generalize the analytical process and this will lead us to the discussion of Lie algebra. An accessible Lie algebra spans a family of analytical vector fields which will imply the controllability of T-S fuzzy model. Time-optimal control for T-S fuzzy model is a new control problem with its rich geometric properties via Lie algebra.

Using the maximum principle, time-optimal trajectory combined with the corresponding control, is called an extremal. The bounded input is determined by the signs of the associated switching functions. The singularity of the system is a well-known problem in time-optimal control which is explored in [27, 31]. An optimal trajectory may be singular, i.e., switching functions may vanish along the trajectory. The characterization of such trajectories will be investigated in this dissertation. The existence of extremal will imply that the time-optimal controller of the T-S fuzzy model to have finite number of switching, which can be found by Lie algebra in this dissertation.

1.2 Controllability Revisit

Recently, the controllability of systems has also attracted many explorers, such as switched system [41-43], hybrid system [44, 45]. However, the controllability of T-S fuzzy model has not been found in the literature. The controllability of the fuzzy model is a pre-requisite of the proceeding controller design. The effort in this dissertation to design a time-optimal controller via controllable T-S fuzzy model is a new contribution. Since the control-affine system can be represented by a family of vector fields, this will have direct applications to control systems. Consider a T-S fuzzy model with a compact set of control input U, the Lie bracket taken at a point of an analytic family of vector fields form a complete set of its invariants. By formulating the T-S fuzzy model as a relaxed version, we can perform some algebraic operations on it, such as taking linear combinations and taking a product called Lie bracket.

1.3 On Maximal Convergence Rate

Fast response is always a considered property in this dissertation. A notion directly relate to fast response is the convergence rate of the state trajectories. For a linear system, the convergence rate is determined by the real part of the pole which is closest to the imaginary axis. We will give a controller design of T-S fuzzy model on maximal convergence rate by the introduced level set function. The result of maximizing the convergence rate is characterized from the maximal invariant ellipsoid. The controller is also bang-bang within both the initial states and target states are belong to level set.

1.4 Dissertation Overview

This dissertation is organized as follows. In section 2, we will formulate the time-optimal problem in T-S fuzzy model. In this section, the T-S fuzzy model is described as polytopic linear differential inclusion and Lie algebra is adopted to find the controllability of T-S fuzzy model. It can also be shown that if the T-S fuzzy model is controllable then the time-optimal does exist. Assuming the existence of time-optimal solution, we will investigate the singular structure in fuzzy model in section 3. The optimal trajectory is solved by the numerical illustrations are provided.

By introduced level set, the maximal convergence rate control discuss in section 4. Finally, conclusions are included in section 5.

Chapter 2

Controllability of T-S Fuzzy Models

Controllability properties of a control system are properties related to the following questions. Can the system be steered form a given initial state to a given final states? Can this be done for any pair of initial and final states? How large is the set of points to which the system can be steered from a given initial state? Which trajectories of the system are realizable and how do we find controls realizing them? Such questions can be motivated by practical problems and they are basic for any qualitative study of control systems.

Consider a nonlinear control system $\dot{x} = f(x, u)$, where $x \in X \subset \mathbb{R}^n$ and u is control in set U. This system can be viewed as collection of dynamical systems parameterized by control input. In study of controllability properties of systems, the set of available velocities $F(x) = \{ f(x, u) : u \in U \}$ by its convex hull, the trajectories of the convexified system can be approximated by the trajectories of the original system. In particular, if $0 \in \mathrm{int}\ \mathrm{co}\, F(x)$ for all $x \in X$, then the system is completely controllable.

2.1 Takagi-Sugeno (T-S) Fuzzy Models

Consider a nonlinear control-affine system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in X \subseteq \mathbb{R}^n$ is system state and u is control input is an arbitrary set U. The state space X is a smooth differential manifold of dimension n and U the control set. The vector fields f and g are assumed to be analytic.

In many situations, fuzzy model with the human knowledge can provide a linguistic description of the nonlinear system in terms of IF-THEN rules. The i-th rule of the T-S fuzzy model is described by the following form:

Rule i: IF $z_1(t)$ is $M_{i1} \cdots$ and $z_p(t)$ is M_{ip}, THEN

$$\dot{x} = A_i x + B_i u$$

where x is system states, taking values in an open subset X of $\mathbb{R}^n$, $u \in \mathbb{R}^m$ is a measurable bounded function on U, i is the number of IF-THEN rules, $z_i(t)$ are some fuzzy input variables, M_{ij} are fuzzy membership functions in the i-th rule, and $\dot{x} = A_i x + B_i u$ is the output from the i-th IF-THEN rule. The entire fuzzy model is formulated as follows:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z(t))(A_i x + B_i u) \tag{2}$$

where r is the total number of rules, $\mu_i(z(t))$ is the normalized membership function and $\mu_i(z(t)) = \alpha_i / \sum_{i=1}^{r} \alpha_i$ and α_i is the firing strength of i-th rule and $\alpha_i = \prod_{j=1}^{p} M_{ij}(z_j(t))$.

The T-S fuzzy model has strong connection with the polytopic linear differential inclusion (PLDI) [36, 37] which will lead to the relaxed version of T-S fuzzy model defined in this dissertation. The equivalence between the fuzzy model and the differential inclusion is revealed by the well-known Filippov's Selection Lemma [36, 37]. From Filippov's Selection Lemma, the set of solutions of T-S fuzzy model coincides with the set of solutions of the differential inclusion.

The relaxed version of T-S fuzzy model is described by

$$\dot{x} \in Co\left\{\left[A_i x + B_i u\right] \middle| i = 1,\ldots,r\right\} \tag{3}$$

where Co denotes as convex hull [36]. If the T-S fuzzy model is continuous and control input U is compact, the set of solutions of (2) coincides with the set of solutions of (3) [36, 37], i.e.,

$$Co\left\{\left[A_i x + B_i u\right] \middle| i = 1,\ldots,r\right\} \supseteq \sum_{i=1}^{r} \mu_i\left(z(t)\right)\left(A_i x + B_i u\right).$$

Therefore we represent the T-S fuzzy model by (3) as

$$\dot{x} = \sum_{i=1}^{r} \mu_i(t)\left(A_i x + B_i u\right) \tag{4}$$

where $\mu_i(t) \in [0,\ 1]$ and $\sum_{i=1}^{r} \mu_i(t) = 1$. To simplify the notion, we adopt $\sum A_i = \sum_{i=1}^{r} \mu_i(t) A_i$, $\sum B_i = \sum_{i=1}^{r} \mu_i(t) B_i$ and the j-th column vector of $\sum B_i$ are denoted as $\sum b_j = \sum_{i=1}^{r} \mu_i(t) B_{ij}$, $j = 1,\ldots,m$ and are assumed to be linearly independent. Throughout the rest of this dissertation, the T-S fuzzy model is denoted as

$$\dot{x} = \sum A_i x + \sum B_i u\,. \tag{5}$$

In general, the variable $z(t)$ in (2) sometimes is chosen as the state variables $x(t)$, thus de-fuzzification $\mu_i\left(z(t)\right)$ causes (2) to become a class of nonlinear systems. This lead to difficultly perform differential algebra on (2). To avoid this problem, such T-S fuzzy model (5) is introduced to allow us to perform differential algebraic on it.

2.2 Lie Algebras

The nonlinear control-affine system (1) can be viewed as a collection of dynamical system with control input. It is typical to expect that basic properties of such a system depend on interconnections between the different dynamical systems corresponding to different controls. The Lie bracket of two vector fields is another vector field which measures noncommutativeness of the flows of the vector fields.

Let f and g be vector fields on X, the corresponding *Lie bracket* of two smooth vector fields is denoted by $[f, g]$, and

$$[f,g](x)=\frac{\partial f}{\partial x}g(x)-\frac{\partial g}{\partial x}f(x),$$

where $\partial f/\partial x$ and $\partial g/\partial x$ denote the Jacobi matrices of their vector fields. The iterated Lie bracket of f and g is defined as

$$ad(f)^k(g)(x)=\left[f, ad(f)^{k-1}g\right](x) \tag{6}$$

where $ad(f)^0(g):=g$ and $k\geq 1$. The Lie algebra generated by the vector fields can be expressed as

$$\begin{aligned}\mathcal{L}&=\{f,g_1,\ldots,g_m\}_{LA}\\&=\operatorname{span}\left\{\left[g_{i_1},\cdots,\left[g_{i_{k-1}},g_{i_k}\right]\cdots\right]\middle| k\geq 1,\ 0\leq i_1,\ldots,i_k\leq m\right\}\end{aligned}$$

where $g_0=f$.

To study the coordinate change, consider a global diffeomorphism $\Phi: X\to X$ as tangent vectors are transformed through the Jacobian map. Consider a diffeomorphism is defined as

$$ad_\Phi(f)(p)=T\Phi(q)(p),\quad q=\Phi^{-1}(p),$$

where $T\Phi$ denotes the tangent map of Φ. Note that the coordinate change $p=\Phi(q)$ transforms the differential equation $\dot{p}=\overline{f}(q)$ where $\overline{f}=ad_\Phi(f)$. If the tangent map of Φ is a global diffeomorphism of X, then the operation ad_Φ is a linear operator on the vector fields X. For example, the additive of diffeomorphism is

$$ad_\Phi(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 ad_\Phi(f_1) + \alpha_2 ad_\Phi(f_2).$$

The global diffeomorphism of composition $\Phi \circ \Theta$ is

$$ad_{\Phi\circ\Theta}(f) = ad_\Phi(f) ad_\Theta(f).$$

From the definition of Lie bracket that $[f, g]$ transforms with coordinate changes like a vector field which is via the Jacobian map. If the tangent map of Φ is a diffeomorphism of X, the basic property of equivariance of Lie bracket with coordinate changes are as following:

$$[ad_\Phi(f), ad_\Phi(g)] = ad_\Phi[f, g].$$

2.3 Classical Controllability Results Revisited

In analyzing controllability properties of systems, the follow theorems are introduced. In the following, we will introduce notions and results which play a basic role in analyzing the structure of nonlinear control systems. They are directly related to controllability properties of nonlinear system. In the following, we denote X as a n dimensional C^∞ manifold.

Definition 1. *Let $T_x X$ be a subspace of the tangent space at any point $x \in X$. A distribution Δ on X is a map which is*

$$x \in X \to \Delta(x) \subset T_x X.$$

The distribution Δ is a smooth subspace of $\mathbb{R}^n$ to each point x. The dimension of Δ, in general, is not a constant. If the dimension is constant in a neighborhood of x, then x is said to be a regular point of the distribution.

Definition 2. *A distribution* $\Delta(x)$ *is called* ***involutive*** *if for any two vector fields* $f,\ g \in \Delta(x)$*, their Lie bracket* $[f,\ g] \in \Delta(x)$*.*

The involutive plays the basic role in following is well-know Forbenius theorem.

Theorem 1. (Frobenius' theorem) [35].

If distribution Δ *is involutive distribution of class* C^{∞} *and of dimension* k *on* X *then, locally around any points in* X*, there exists a smooth change of coordinates with transforms the distribution* Δ *to the following constant distribution*

$$span(e_1, \ldots, e_k),$$

where $e_1, \ldots, e_k$ *are the constant vector with 1 at the* i*-th place.*

In order to introduce a global version of Frobenius' theorem, we have following definitions.

Definition 3. A subset $S \subset X$ is called regular submanifold of X with dimension k if for any $x \subset S$ there exists a neighborhood U of x and a diffeomorphism $\Phi : U \to V \subset \mathbb{R}^n$ onto an open subset V such that

$$\Phi(U \cap V) = \left\{ x = (x_1, \ldots, x_n) \in V \middle| x_{k+1} = 0, \ldots, x_n = 0 \right\}.$$

If any point of the distribution is regular with dimension k, the distribution is said to be regular and the dimension of the distribution is k. In other words, a regular submanifolds of dimension k is a subset which locally looks like a piece of subspace of dimension k with changing of coordinates. A weaker version of a submanifold is introduced in the following definition.

Definition 4. A subset $S \subset X$ is called an immersed submanifold of X of dimension k if

$$S = \bigcup_{i=1}^{\infty} S_i \text{, where } S_1 \subset S_2 \cdots \subset S$$

and S_i are regular submanifolds of X of dimension k.

In fact, if subset S itself is regular submanifold, then $S_i = S$ and S is also an immersed submanifold. From geometric view, if two vectors field f and g are tangent to an immersed submanifold S then also their Lie bracket $[f, g]$ is tangent to this submanifold.

Remark 1. This is geometric definition of Lie bracket. If vectors field f is tangent to submanifold S, the fact that it is flow transforms points of S into points for any time t sufficiently small. With respect to t, the $[f, g]$ is gives a tangent vector to S.

Definition 5. A foliation $\{S_i\}_{i\in A}$ of X of dimension k is a partition

$$X = \bigcup_{i\in A} S_i$$

of X into arc-wise connected (immersed) submanifolds S_α. In here, S_α is called *leaves*.

Let g is a vector field of tangent to a foliation $\{S_i\}_{i\in A}$, that is, it is tangent to its leaves. The Lie bracket $[f, g]$ is tangent to this foliation, if the flow of f locally preserves this foliation. For any point $x \in S_i$, the f locally preserves the foliation $\{S_i\}_{i\in A}$ mean that there is a neighborhood U of x such that the image of a piece of a leaf is contained in a neighborhood of leaf of the foliation, for any time t sufficiently small.

Definition 6. Consider a set of vector fields $\mathcal{F}=\{f_u\}_{u\in U}$, the orbit of a point $x \in X$ is the set of points of X which and be reached by piecewisely by trajectories of vector fields,

$$Orb(x)=\left\{\gamma_{t_k}^{u_k}\circ\gamma_{t_{k-1}}^{u_{k-1}}\circ\cdots\circ\gamma_{t_1}^{u_1}\middle| k\geq 1,\ u_1,\ldots,u_k\in U, t_1,\ldots,t_k\in\mathbb{R}^+\right\},$$

where γ_t^u is denoted the flow of the vector field f_u.

Theorem 2. For all $x \in X$, the orbit $S = Orb(x)$ of a set of vector fields $\mathcal{F}=\{f_u\}_{u\in U}$ is an immersed submanifold. Further, the tangent space of this submanifold is given by the distribution $T_xS = \Delta(x)$.

Corollary 1. If the vector fields f_u are analytic, then the tangent space of the orbit can be obtained as

$$T_xS=\left\{g(x)\middle| g\in L\{f_u\}_{u\in U}\right\},$$

where $L\{f_u\}_{u\in U}$ denotes smallest set of vector fields which contains the set $\mathcal{F}$ and is closed under taking linear combinations and Lie bracket.

Denote X *be an open subset on* $\mathbb{R}^n$ or a differentiable manifold of dimension n. We have the following definition.

For convenience, the following Theorems 2 ~ 4 are listed here which are adapted from [35-37].

Theorem 3. (Chow's Theorem) [35]

Let $\mathcal{F}$ *be a set of* C^{∞} *vector fields on* X *and* $\mathcal{L}=\{\lambda_0,\lambda_1,\ldots,\lambda_k\}_{LA}$ *be the Lie algebra generated by* $\mathcal{F}$*. If* $\dim(\mathcal{L}(x))=n$ *for all* $x\in X$*, then any point of* X *is reachable by trajectory of the vector fields* $\mathcal{F}$*. Thus*

$$x_1 = e_{t_L}^{\lambda_L}\circ\cdots\circ e_{t_1}^{\lambda_1}(x_0)$$

for some $L\geq 1$, $\{\lambda_0,\lambda_1,\ldots,\lambda_k\}\in\mathcal{F}$ *and* $t_1,\ldots,t_L\in(0,\ \infty)$.

The following well-known theorem of Frobenius is characterized the integrable distribution [38].

Theorem 4. (Generalized Frobenius' theorem) [38]

If X *is a* C^{ω} *(regular) manifold of dimension* n *and* Δ *is an involutive distribution then around any point* $x\in X$*, there exists a largest integral manifold of* Δ *passing through* x.

Remark 2. A distribution Δ is said to be integrable if there exists a submanifold S on X such that for any $x\in X$

$$\Delta(x)=T_xS$$

where S is passing through x.

Remark 3. Any analytic involutive distribution Δ is integrable [39].

Theorem 5.[39]

Let $\mathcal{F}$ *be a set of* C^{ω} *vector fields on* X *and* $\mathcal{L}=\{\lambda_0,\lambda_1,\ldots,\lambda_k\}_{LA}$ *be the Lie algebra generated by* $\mathcal{F}$*. For all* $x\in X$*, there exists a largest integral manifold of* $\mathcal{F}$ *passing through* x.

The proof of Theorem 5 can be found by using the Campbell-Baker-Hausdorff formula and Theorem 4.

2.4 Lie Algebras of T-S Fuzzy Models

Since the control-affine system can be represented by a family of vector fields, this will have direct applications to control systems. Consider a T-S fuzzy model with a compact set of control input U, the Lie bracket taken at a point of an analytic family of vector fields form a complete set of its invariants. In particular, $\mathcal{L}(p_0)$ denotes the space of tangent vectors at p_0 defined by the Lie algebra. Due to the fact that $f = g_0 = \sum A_i x$, $g_1 = \sum b_1, \ldots, g_m = \sum b_m$, and that Lie bracket of constant vector fields is zero, the iterated Lie bracket can be found as

$$ad\left(\sum A_i x\right)^k \sum b_j = \left[\sum A_i x,\ ad\left(\sum A_i x\right)^{k-1} \sum b_j\right] \tag{7}$$

A Lie algebra $\mathcal{L}$ is recursively defined by

$$\mathcal{L}^{(1)} = [\mathcal{L},\ \mathcal{L}],\ \mathcal{L}^{(2)} = \left[\mathcal{L}^{(1)},\ \mathcal{L}^{(1)}\right], \ldots, \mathcal{L}^{(k)} = \left[\mathcal{L}^{(k-1)},\ \mathcal{L}^{(k-1)}\right], \ldots,$$

is called solvable if $\mathcal{L}^{(k)} = 0$ for large k, i.e., $\mathcal{L}^{(k)} \supset \mathcal{L}^{(k+1)}$. Furthermore, Lie algebra $\mathcal{L}$ is called nilpotent if the sequence of $\mathcal{L}$ is always decreasing with respect to

$$\mathcal{L}^1 = \mathcal{L},\ \mathcal{L}^2 = \left[\mathcal{L},\ \mathcal{L}^1\right], \ldots, \mathcal{L}^k = \left[\mathcal{L},\ \mathcal{L}^{k-1}\right], \ldots,$$

and $\mathcal{L}^k = 0$. Any nilpotent Lie algebra is solvable. More details can be found in [38].

2.5 Controllability of T-S Fuzzy Model

We begin with the formal definition of reachability and controllability. In this section, T-S fuzzy model (5) associated with Lie algebra is derived to show the controllability condition and imply the existence of optimal control.

Definition 7. *The* ***reachable set*** $\mathcal{R}(x)$ *of T-S fuzzy model (5) for time* $t \geq 0$, *subject to the initial condition* $x \in X$ *is the set*

$$\mathcal{R}_T(x) = \{x(t,u) : x \in X \text{ and } u : [0,\ T] \mapsto U\}.$$

Definition 8. *The T-S fuzzy model (5) is* ***accessible*** *if its reachable set* $\mathcal{R}_T(x)$, $x \in X$ *have non-empty interior. Similarly, We will call this T-S fuzzy model* ***strongly accessible*** *if the reachable set* $\mathcal{R}_T(x)$ *has nonempty interior for any* $T > 0$.

Definition 9. *The T-S fuzzy model (5) is* ***controllable*** *if* $\forall x_0$ *and* $\forall x_1$ *in the manifold of* X, *there exists a finite time* T *and admissible control function* $u:[0,\ T]$ *such that* $x(T; x_0, u) = x_1$.

Definition 10. *For T-S fuzzy model (5), the* ***accessibility Lie algebra*** *is defined as*

$$\mathcal{L}_a := \left\{\sum A_i x,\ \sum b_j \,\middle|\, \forall j = 1, \ldots, m\right\}_{LA}. \tag{8}$$

The $\mathcal{L}_a$ is a finite-dimensional Lie algebra of vector fields which contains the family $\{\sum A_i x,\ \sum b_j\}$. In fact, this ***accessibility Lie algebra*** plays basic role in the controllability of a T-S fuzzy model.

Theorem 6. *If the* ***accessibility Lie algebra*** *of the T-S fuzzy model in (5) is full rank at* x, *that is*

$$rank\left(\mathcal{L}_a(x)\right) = n,\ \ \forall x \in \mathbb{R}^n \tag{9}$$

then the reachable set up to any time $T > 0$ *has the nonempty interior and so the fuzzy model is strongly accessible.*

Proof:

According to Chow's theorem [35], the reachable set $\mathcal{R}(x)$ is the largest integral manifold of $\mathcal{L}_a$ for $\forall x \in \mathbb{R}^n$. From (9), it contains an open neighborhood Ω of x. This implies that for any x_0, its reachable set is an open set. We shall prove the theorem by contradiction. We claim that $\mathcal{R}(x_0)$ is closed and is denoted as $cl(\mathcal{R}(x_0))$. Therefore, there exists a $x_1 \in cl(\mathcal{R}(x_0)) \setminus \mathcal{R}(x_0)$. Hence $\mathcal{R}(x_1)$ contains an open neighborhood Ω of x_1, then $\Omega \cap \mathcal{R}(x_0) \neq \phi$. Let $\zeta \in \Omega \cap \mathcal{R}(x_0)$ then $x \in \mathcal{R}(x_1)$. By symmetry, $x_1 \in \mathcal{R}(\zeta)$, and $\zeta \in \mathcal{R}(x_0)$ then $x \in \mathcal{R}(x_0)$. Therefore $\Omega \subset \mathcal{R}(x_0)$, which is contradiction. We can conclude that the reachable set $\mathcal{R}(x)$ is arc-wise connected and span into $\mathbb{R}^n$ space. **Q.E.D.**

Remark 4. Since T-S fuzzy model (5) is analytic, using Chow's theorem [35] and Frobenius' theorem [38], the manifold X is maximal connected reachable manifolds. Each reachable manifold is the maximal integral manifold of $\mathcal{L}_a$.

Remark 5. By using Chow's theorem [35], the controllable manifolds can be spanned from $\left\{\sum A_i x,\ \sum b_j \,\middle|\, \forall j = 1, \ldots, m\right\}$.

Remark 6. The $\mathcal{L}_a$ implies that The T-S fuzzy model (5) is accessible form x_0 if the same collection of vectors together with $\sum A_i x_0 + \sum B_i u$ span the whole space. This condition means that no vector $\sum B_i u$ belongs to a proper invariant subspace of $\sum A_i x_0$.

Theorem 7. *If T-S fuzzy model is strongly accessible, then it's also controllable.*

Proof:

Using Remark 4, for a T-S fuzzy model the degree of largest integral manifold is related to rank of *accessibility Lie algebra* $\mathcal{L}_a$. Due to the fuzzy model is *strongly accessible*, there exists the n-th degree largest integral manifold. For a given point $x \in \mathbb{R}^n$, the fuzzy model is controllable. **Q.E.D.**

In the following, the generalized rank condition of accessible Lie algebra is derived to show the controllability of T-S fuzzy model.

Corollary 1. *The T-S fuzzy model* (5) *is controllable if and only if the following matrix*

$$(W_0,\ W_1,\dots,\ W_{n-1}) := \left(\sum b_j,\ \sum A_i \sum b_j,\dots,\ \left(\sum A_i\right)^{n-1} \sum b_j\right),\quad j = 1,\dots,m \quad (10)$$

is of rank n *for any* $t > 0$

Proof:

Firstly, we give the proof of sufficient part. Consider the T-S fuzzy model (5), let $f = g_0 = \sum A_i x$ and $g_1 = \sum b_j$ to be a vector filed. Then we have the following iterated Lie brackets,

$$\left[\sum A_i x,\ \sum b_j\right] = -\sum A_i \sum b_j,\ \left[\sum A_i x,\ \left[\sum A_i x,\ \sum b_j\right]\right] = \sum A_i^2 \sum b_j,\dots,.$$

From (7), the iterated Lie brackets are rewrote as

$$ad\left(\sum A_i x\right)^l \sum b_j = \left((-1)\sum A_i\right)^l \sum b_j\ .$$

Therefore, the *accessibility Lie algebra* $\mathcal{L}_a$ consists of constant vector fields only,

$$\mathcal{L}_a = Span\left\{\left(\sum A_i\right)^l \left(\sum b_j\right) \middle|\ l \geq 0,\ j = 1,\dots,m\right\}. \quad (11)$$

If (10) is satisfied, we can conclude that $\dim(\mathcal{L}_a)$ is of full rank n for any $t > 0$

then the fuzzy model is controllable.

From the Frobenius' theorem [38] and Remark 4, it follows that the T-S fuzzy model (5) is controllable, there exits the n-th degree largest integral manifold for $x \in X$. If (10) is satisfied, from Theorem 5 and Remark 2, there exists a largest integral submanifold S which is unique and contained in the largest integral manifold. **Q.E.D.**

Remark 7. In analyzing controllability properties of the fuzzy model (5) we can replace the set of $G(x) = \{A_i x + B_i u : u \in U,\ i = 1, \ldots, r\}$ by its convex hull, the trajectories of convexified system can be approximated by the trajectories of the original fuzzy model (2). In particular, if $0 \in \operatorname{int} Co\{G(x)\}$ for all $x \in X$, then the fuzzy model is controllable.

Remark 8. Obviously, for single rule T-S fuzzy model, Corollary 1 degenerates to the Kalman controllability matrix of linear system.

Remark 9. If all the subsystems are controllable, whereas the overall system can not concluded controllable, then the overall system can be called ***local controllable***.

The membership functions obviously play the critical roles in the controllability of system. In the following examples, the local controllability and controllability of T-S fuzzy model will be illustrated. The nonlinear system will be modeled with the distinct membership functions.

2.6 Existence of Optimal Control

In the following, we shall show that the existence of optimal solution of Problem 1 can be reduced to determine the accessibility of *reachable set*. The qualitative properties of the reachable sets can be established. One of the basic properties can be shown in the following context. The following theorem discusses the existence of the optimal solution for Problem 1.

Corollary 2. *If T-S fuzzy model in* (5) *is controllable, then there exists an optimal control for any bounded input.*

Proof:

Consider the T-S fuzzy model with bounded input $u(t) \in U \subseteq \mathbb{R}^m$. It is more convenient to consider the T-S fuzzy model in the form

$$\dot{x} = \sum A_i x + v,\ v \in V,$$

where V is the image of U under the map $\sum b : \mathbb{R}^m \to \mathbb{R}^n$ .Thus, the Lie brackets is

$$\left[\sum A_i x,\ v\right] = \sum A_i \cdot v,\ v \in V.$$

Let the set $W = \left\{v^{'} - v^{''} \middle| v^{'}, v^{''} \in V\right\}$. The Lie algebra of the T-S fuzzy model contains the vector fields

$$\sum A_i x + v^{'} - \left(\sum A_i x + v^{''}\right) = v^{'} - v^{''} \in W.$$

Consider all constant vector fields $f = w,\ w \in W$. Thus, it contains the Lie brackets $\left[w,\ \sum A_i x + v\right] = \sum A_i w$. Since the fuzzy model is controllable, the *accessibility Lie algebra* $\mathcal{L}_a$ consists of constant vector fields if

$$\mathcal{L}_a = \dim\operatorname{span}\left\{\left(\sum A_i\right)^l w \,\middle|\, 0 \le i \le n-1,\ w \in W\right\} = n \tag{12}$$

for $l = 0, \ldots, n-1$, $\forall t > 0$. This condition means that if the bounded input U is nonempty, then the controllability rank condition implies that the system can be spanned the whole space. **Q.E.D.**

The condition of Corollary 2 means that there exists no vector $v = v^{'} - v^{''} \in U$, $j \neq k$ such that, no image of U belongs to a invariant subspace of matrix $\sum A_i$. In the next section, we shall design the time-optimal controller for T-S fuzzy model with maximum principle.

2.7 Illustrative Examples

Example 1. Consider a nonlinear system:

$$
\begin{aligned}
\dot{x} &= \tan(u) \\
\dot{y} &= 10\sin(x)\cos(x).
\end{aligned}
$$

Assume that $x(t) \in [-\pi/2,\ \pi/2]$. Then the T-S fuzzy model of the nonlinear system can be formulated as:

Rule i: IF $x(t)$ is about "Positive" and "Negative", THEN

$$\dot{X}(t) = A_i X(t) + B_i u,\quad i = 1, 2 \tag{13}$$

where $X(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T$,

$$
A_1 = \begin{bmatrix} 0 & 0 \\ 10\beta & 0 \end{bmatrix},\quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$
$$
A_2 = \begin{bmatrix} 0 & 0 \\ -10\beta & 0 \end{bmatrix},\quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

and $\beta = \cos(88^{\circ})$. The membership functions are shown in Fig. 1.

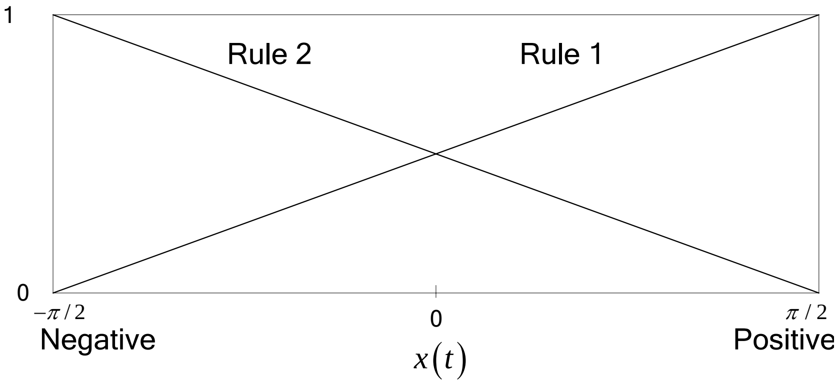

Fig. 1 The membership functions in Example 1.

According to Corollary 1, the corresponding rank of controllability matrix of the fuzzy model is,

$$Rank\left(\sum b_j,\ \sum A_i \sum b_j\right)$$

where

$$W_0 = \sum b_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$W_1 = \sum A_i \sum b_j = \left(\mu_1 \begin{bmatrix} 0 & 0 \\ 0.349 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 \\ -0.349 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.349(\mu_1 - \mu_2) \end{bmatrix} \right).$$

The fuzzy model is controllable if $Rank([W_0,\ W_1]) = 2$. We can check the controllability by the following determinant:

$$\left\| \begin{bmatrix} 1 & 0 \\ 0 & 0.349(\mu_1 - \mu_2) \end{bmatrix} \right\| = 0.349(\mu_1 - \mu_2).$$

Unfortunately, the rank of $[W_0,\ W_1]$ for $\mu_1 = \mu_2 = 0.5$ is 1. From the membership functions, we can observe that the fuzzy model is uncontrollable if $x(t) = 0$. Although $x(t) = 0$ is one of equilibrium points however the fuzzy model is concluded to be uncontrollable when $x(t) = 0$ and $y(t) \neq 0$.

In following example, we redesign the nonlinear system with different membership functions.

Example 2. Consider the nonlinear system in Example 1. If the membership functions are chosen as Fig. 2. Then the consequence parts of fuzzy model can be formulated as:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 10 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 0 \\ 10\beta & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

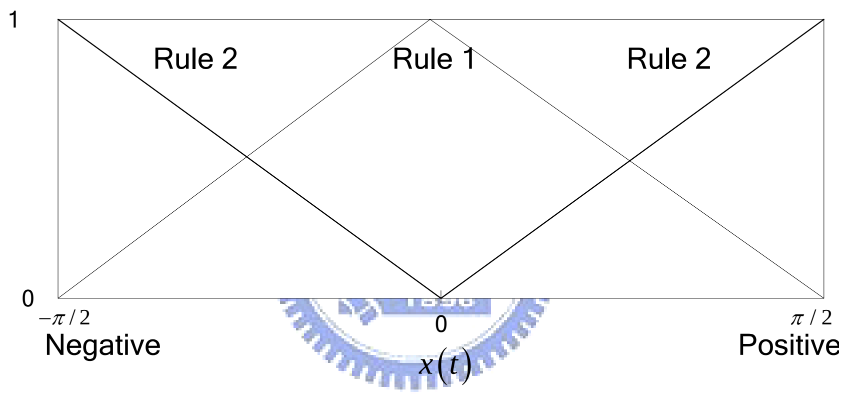

Fig. 2 The membership functions of Example 2.

By Corollary 1, the controllability matrix contains the vector fields

$$W_0 = \sum b_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$W_1 = \sum A_i \sum b_j = \left(\mu_1 \begin{bmatrix} 0 & 0 \\ 10 & 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 \\ 0.349 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10(\mu_1 + 0.0349\mu_2) \end{bmatrix}.$$

If the fuzzy model is controllable then the following condition is satisfied:

$$\left\| \begin{bmatrix} 1 & 0 \\ 0 & 10(\mu_1 + 0.0349\mu_2) \end{bmatrix} \right\| = 10(\mu_1 + 0.0349\mu_2) \neq 0 .$$

Since the firing strengths $\mu_i \in [0,\ 1]$ and $\mu_1 + \mu_2 = 1$, then $10(\mu_1 + 0.0349\mu_2) \neq 0$

for $\forall t$. Then we can conclude that the overall T-S fuzzy model is controllable. Since the Example 2 is controllable, the reachable set for $t = [0 \quad 2]$ is plotted in Fig. 3.

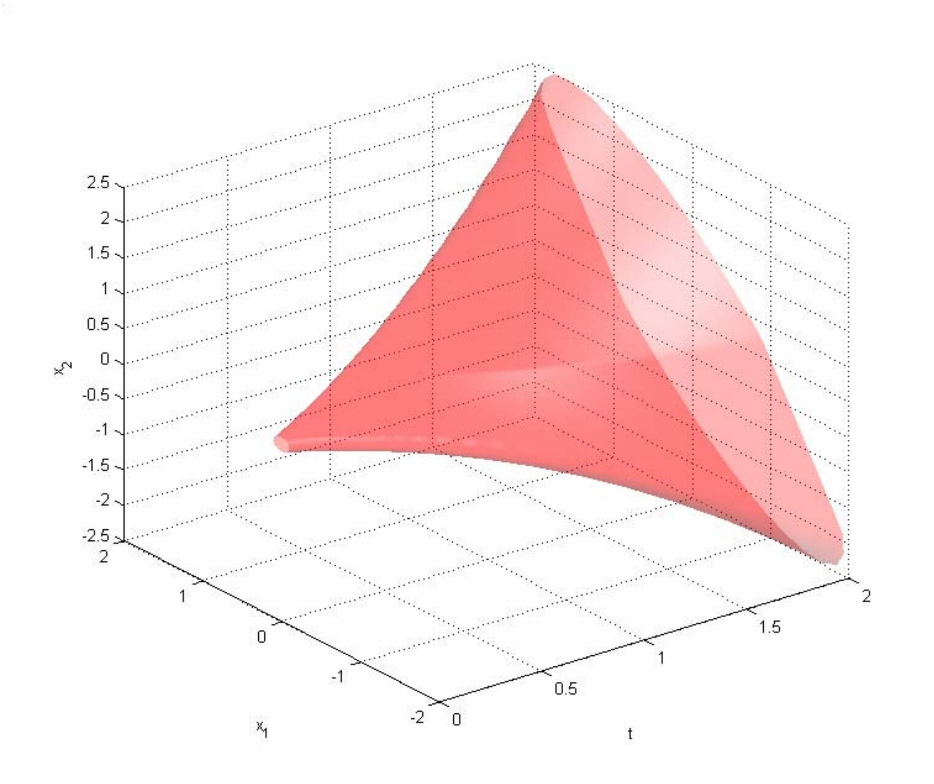

Fig. 3 Reachable tube in Example 2.

Example 3. Choose the closed-loop eigenvalues $[-1 \quad -1]$ for Example 1 and 2. The stabilizable controller is designed by Parallel Distributed Compensation (PDC) [2]. Fig. 4 shows the response of the controllable and uncontrollable system. The dotted lines show the responses of locally controllable case (Example 1). The solid lines indicate the responses of controllable case (Example 2). The controllable case is no surprising to stable the system. From Example 1, we know that the system is not controllable in $x(t) = 0$. The dotted lines show that the system can not converge to zero. This is due to the controllability of system is disappeared.

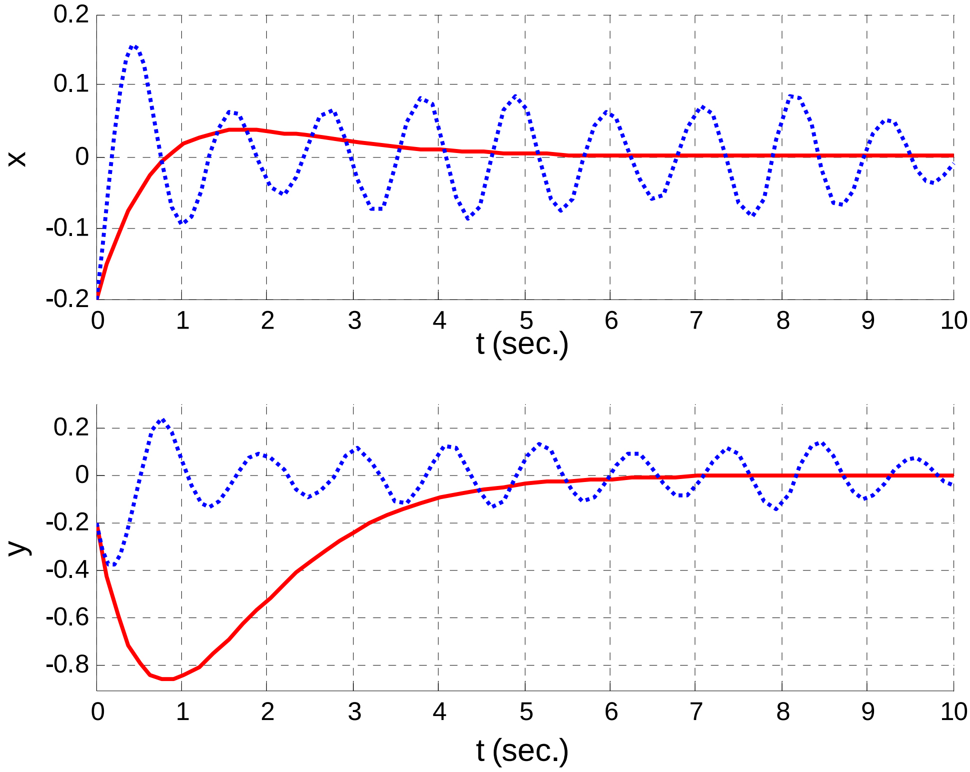

Fig. 4 Trajectories of controllable and uncontrollable case.

Remark 10. An important and natural question arises in the design of feedback controller using local controllability. The controllability of a physical system is a pre-requisite of the proceeding controller design.

Chapter 3

Time-optimal Control Design

Now, we give the Time-optimal control via Pontryagin's Minimum Principle. The controller is derived as bang-bang and the number of switching will be shown as below section.

3.1 Problem Formulation

We will make the following assumption on the control input.

Assumption 1. The control input is given by

$$U = \left\{ u \in \mathbb{R}^m \middle| a_j \le u_j \le b_j,\ j = 1,\dots,m \right\}.$$

For a given control $u(t) \subset U$ on a time interval $[0,\ t_1]$ and any initial point $x(t_0) = x_0 \in X$, let $x(.,x_0,u)$ denote the solution of the nonlinear control-affine (5) with an measurable control u defined on a interval of $[0,\ t_1]$. For performing optimality on a segment $[0,\ t_1]$, we introduce a cost functional

$$J(u) = \int_0^{t_1} \varphi\big(x(t), u(t)\big)\,dt \tag{14}$$

Let $x_0 \in X$ be an initial point and $x_1 \in X$ be a final point. We propose the following optimal control problem in terms of the cost functional J.

Problem 1. *Find a control* $u(t) \in U$ *that minimizes (14) along the solution of* (5) *and satisfies the boundary condition*

$$x(t_1, x_0, u) = x_1 . \tag{15}$$

We note that this problem is well posed, i.e., an optimal control does exist. The intuitive interpretation of Problem 1 is clear: find a control that will push the initial state to a given final condition in a given amount of time.

3.2 Introduction of Pontryagin's Minimum Principle

The system (1) under bounded controls $|u(t)| \le U$ can be formulated by using the Pontryagin's Minimum Principle. The minimization problem for (1) becomes

$$H\left[x^*(t),\ \lambda^T(t),\ u^*(t)\right] = \min_{u(t)\in U} H\left[x^*(t),\ \lambda^T(t),\ u(t)\right] \tag{16}$$

for $t \in [0 \quad t_1]$, or, equivalently,

$$H\left[x^*(t),\ \lambda^T(t),\ u^*(t)\right] \le H\left[x^*(t),\ \lambda^T(t),\ u(t)\right] \tag{17}$$

where H is called Hamiltonian, $x^*(t)$ is optimal trajectories and λ^T is a vector of costates. The superscript (*) denotes the optimal results. The Hamiltonian for system (1) can be written as

$$\begin{aligned} H\left[x(t),\ \lambda^T(t),\ u(t)\right] &= 1 + \lambda^T \cdot \dot{x} \\ &= 1 + \left\langle \lambda^T(t), f(x) + g(x)u \right\rangle \\ &= 1 + \left\langle \lambda^T(t), f(x) \right\rangle + \left\langle \lambda^T(t), g(x)u(t) \right\rangle \end{aligned} \tag{18}$$

Suppose that $u^*(t)$ is a time-optimal control and $x^*(t)$ is the resultant of time-optimal trajectory in minimum time, t^*. Substituting the equation (18) into the inequality (17), we can obtain

$$1+\left\langle \lambda^T,\ f\left(x^*\right)\right\rangle+\left\langle \lambda^T,\ g\left(x^*\right)u^*\right\rangle$$
$$\le 1+\left\langle \lambda^T(t),\ f\left(x^*\right)\right\rangle+\left\langle \lambda^T(t),\ \ g\left(x^*\right)u\right\rangle. \tag{19}$$

Since the first two terms are the same on both side of the inequality, therefore the above inequality equation can be simplified as follows

$$u^*(t)\lambda^T(t)\ g\left(x^*\right)\le u(t)\lambda^T(t)\ g\left(x^*\right). \tag{20}$$

By defining $\psi_j:[0,\ t_1]\to\mathbb{R},\quad \psi_j(t):=\lambda^T\ g\left(x^*\right)$, we can conclude that

$$u^*(t)\psi_j\le u(t)\psi_j. \tag{21}$$

From Assumption 1, $\left|u(t)\right|\le U$, therefore, time-optimal controller can be generalized as

$$u^*(t)=-SGN\left\{\psi_j\right\}\left|U\right|. \tag{22}$$

In (14), it is obvious that if time-optimal control, $u^*(t)$, exists then there is a unique bang-band control. After applying Pontryagin's Minimum Principle, we have the following necessary conditions,

Optimal state trajectory:

$$\dot{x}=\frac{\partial H\left[x^*(t),\ \lambda(t),\ u^*(t)\right]}{\partial\lambda(t)} \tag{23}$$

Costate equation:

$$\dot{\lambda}(t)=-\frac{\partial H\left[x(t),\ \lambda(t),\ u(t)\right]}{\partial x(t)} \tag{24}$$

, and stationary condition

$$H\left[x(t),\ \lambda(t),\ u(t)\right]=0 \tag{25}$$

for $k=1,2,\ldots,n$.

3.2.1 Shooting Method

The shooting method [40] is used to solve this problem. The shooting method can be used to determine the time-optimal control problem as described in what follows. In T-S fuzzy model, equations (5), (35), (38) and (40) can be rewritten as

$$\dot{X} = F\left[X(t),\ u(t)\right] \tag{26}$$

$$X(t_0) = [x_0,\ p_0]^T \tag{27}$$

$$e\left[X\left(p,\ t_f\right),\ t_f\right] = 0 \tag{28}$$

$$u(t) = -SGN\left\{\psi_j\right\}|U| \tag{29}$$

where $X = [x,\ p]^T$ is a vector of $2n$ variables, which are the states, x, and costates, p. $F[X,\ u]$ is combined with a vector of fuzzy system states and costates. $p_0 = p(t_0)$ is an n-dimensional vector of unknown initial costates, $X(t_0)$ is $2n$-dimensional vector of initial states and unknown initial costates, p_0. $e\left[x\left(p_0,\ t_f\right),\ t_f\right]$ is an l-dimensional vector, where $l \geq n$, representing the error at the target point. This vector includes the final conditions of states, and the extra condition for Hamiltonian (25) to be met at the target point. $q(X)$ is a switch function. In order to reduce $e\left[x\left(p_0,\ t_f\right),\ t_f\right]$ to zero, the values $p^k,\ t_f^k$ in the kth iteration have to be corrected in the next iteration using the following formula

$$\begin{bmatrix} p^{k+1} \\ t_f^{k+1} \end{bmatrix} = \begin{bmatrix} p^k \\ t_f^k \end{bmatrix} + \begin{bmatrix} \Delta p^k \\ \Delta t_f^k \end{bmatrix}. \tag{30}$$

The correction terms Δp^k and Δt_f^k can be computed by minimizing a norm of e given by

$$\|e\| = \left(\sum_{i=1}^{l} e_i^2 \right)^{1/2}. \tag{31}$$

Obtaining an analytical expression may not always be possible. However, the stationary condition (25) offers the gradient along which the decision variables can be corrected. In [40], Newton's method is adopted. The vector corrections is defined as

$$\begin{bmatrix} \Delta p^k \\ \Delta t_f^k \end{bmatrix} = -\alpha_k \begin{bmatrix} \delta p^k \\ \delta t_f^k \end{bmatrix} \tag{32}$$

where δp^k and δt_f^k can be calculated using following expression:

$$\begin{bmatrix} \dfrac{\partial e\left[p^k, t_f^k\right]}{\partial p} & \dfrac{\partial e\left[p^k, t_f^k\right]}{\partial t_f} \end{bmatrix} \begin{bmatrix} \delta p^k \\ \delta t_f^k \end{bmatrix} = e\left[p^k, t_f^k\right]. \tag{33}$$

The scalar α_k is chosen in the range $0 \le \alpha_k \le 1$. The initial gauss of the set of the values $\left[p^0, t_f^0\right]$ are required. Due to the fact that the costates don't have a physical meaning, initial gauss are difficult to obtain. For more complex problems, forward-backward method (FBM) was proposed in [25], which offered a good guess of the initial costates.

Determination of optimal control sequence of (29) is related to the trajectory of costates. This introduces other problems in that the initial costates and finial time are unknown. This kind of problem is called Two-Point Boundary Value Problems (TPBVP). The shooting method [40], however, has been used to solve this problem. The optimal solution can be obtain by solving equations (5), (35), (38) and (40) simultaneously. For TPBVP, no practical method has been developed yet for computing the time-optimal feedback control. The main reason is that it is generally impossible to characterize the switching surface. Suppose that in the time interval $[0,\ t_1]$ there exists one nontrivial (or more) subinterval, $[t_a,\ t_b] \subset [0,\ t_1]$, such that

$\psi_j(t)$ is identically zero, then the shooting method is fail. We will give more details for this case in following section.

3.3 Time-Optimal Controller of T-S Fuzzy Model

In this section, we will study the properties of time-optimal control using the maximum principle [20], [27]. The Time-optimal controller is designed via a controllable T-S fuzzy model. In general, Problem 1 can be formulated as a Hamiltonian by maximum principle. The Hamiltonian for Problem 1 can be described as

$$H(x,\lambda,u) := \lambda^T \sum A_i x + \lambda^T \sum B_i u \tag{34}$$

where $\lambda : [0,\ t_1]$ is a *costate* satisfying the *adjoint equation* associated with (5):

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\lambda^T \sum A_i . \tag{35}$$

By using the maximum principle [20], the Problem 1 becomes

$$H(x,\lambda,u) = \max_{v\in U} H(x,\lambda,v) . \tag{36}$$

Definition 11. *Trajectories of (5), (34) and (35) that satisfy the maximum principle is called* ***extremal*** $(x,\lambda,u):[0,\ t_1] \mapsto \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \times U$ *. When the constant* λ_0 *is zero, the extremal is said to be* ***abnormal*** *[31].*

Definition 12. *For* $j = 1,\ldots,m$ *, the* ***switching functions*** $\psi_j(\cdot)$ *, along an* ***extremal*** (x,λ,u) *are defined by*

$$\psi_j : [0,\ t_1] \to \mathbb{R}, \quad \psi_j(t) := \lambda^T \sum b_j \ . \tag{37}$$

They are absolutely continuous functions [31].

The necessary condition for optimality provided by the maximum principle states that $u:[0,\ t_1]$ must pointwise maximize $H\left(x(t),\lambda(t),\ \cdot\ \right)$ for the costate λ associated with the optimal trajectory. Moreover, the Hamiltonian is constant along the solutions of (34) and must satisfy

$$H(x,\lambda,u)=\lambda_0,\ \lambda_0\geq 0. \tag{38}$$

The maximum condition (36) is equivalent to the following:

$$u_j(t)\psi_j(t)=\max_{v_j\in U} v_j(t)\psi_j(t),\quad j=1,\ldots,m. \tag{39}$$

Obviously, the functions $\psi_j(t)$ play a crucial role in the study of time-optimal trajectories. Under Assumption 1, the time-optimal control must satisfy the following conditions almost everywhere,

$$\begin{aligned} u_j&=b_j && \text{if } \psi_j(t)>0 \\ u_j&=a_j && \text{if } \psi_j(t)<0 \end{aligned} \tag{40}$$

for $j=1,\ldots,m$. In case, switching functions having zeros have to be carefully analyzed.

Suppose that in the time interval $[0,\ t_1]$ there exists one nontrivial (or more) subinterval, $[t_a,\ t_b]\subset[0,\ t_1]$, such that $\psi_j(t)$ is identically zero, then the corresponding extremal is called *singular*. If $\psi_j(t)\neq 0$ for almost all $t\in[0,\ t_1]$, the maximum principle implies that the control u_j corresponds to piecewise constant controls taking values in the set of m vertices of U, is called *bang-bang*. An extremal is said to be *normal* if control u_j is *bang-bang* with at most a finite number of switching.

If T-S fuzzy model is smooth and (x,λ,u) is an *extremal*, then the time derivative of the absolutely continuous function $\psi_j(t)$ is given by

$$\begin{aligned}\dot{\psi}_j(t) &= \lambda^T\left[-\sum A_i x(t),\ \sum b_j\right] + \lambda^T\left[\sum b_k,\ \sum b_j\right]u_j(t)\\ &= \lambda^T\left[-\sum A_i x(t),\ \sum b_j\right].\end{aligned} \tag{41}$$

Since $\sum b_j,\ j=1,\ldots,m$ and $j \neq k$ are constant terms, therefore $\left[\sum b_k,\ \sum b_j\right]=0$.

It is obvious that the derivatives of the switching functions $\psi_j(t)$ are themselves absolutely continuous function, and therefore we can perform further derivatives of it. In the next theorem, Lie brackets will be crucial in establishing a bound on the number of switches for *bang-bang* controls will be derived.

Theorem 8. *If the T-S fuzzy model is controllable, then the* ***extremal*** *is* ***normal****.*

Proof:

Let (x,λ,u) be *extremal* in $t \in [0,\ t_1]$. We shall prove the theorem by contradiction. Suppose there exists a sequence of infinite distinct singular set

$$S = \{s_0,\ldots,s_i,\ldots\},$$

where s_i is the i-th time interval $[t_a,\ t_b]_i$ such that $\psi_j(t)=0,\ \forall t \in [t_a,\ t_b]_i$, $j=1,\ldots,m$. Assume $t_0 \in s_i$. Then we have the following relation:

$$\psi_j(t) = \lambda^T(t_0)\ \sum b_j = 0,\quad j=1,\ldots,m \tag{42}$$

From (42), we have the first derivation of $\psi_j(t)$:

$$\dot{\psi}_j(t) = \lambda^T(t_0)\left[\sum A_i x(t),\ \sum b_j\right] = 0. \tag{43}$$

Indeed, l-th derivative of $\psi_j(t)$ can be expressed as:

$$\psi_j^l(t) = \lambda^T(t_0)\, ad\left(\sum A_i x(t)\right)^l\left(\sum b_j\right) = 0,\quad l=1,\ldots,n-1. \tag{44}$$

By Corollary 2, we have

$$span\left\{ad\left(\sum A_i x(t)\right)^l \left(\sum b_j\right)\right\} \in \mathbb{R}^n, \quad l = 1, \ldots, n-1.$$

Hence, we have $\lambda(t_0) = 0$, which contradicts to the necessary condition of maximum principle. So we can conclude that the set S is finite. Outside the set S, the switching function $\lambda^T(t)\sum b_j$ attains the maximum on U at one vertex, thus the optimal control $u(t)$ is *bang-bang* on $[0,\ t_1] \setminus t_0$. **Q.E.D.**

If the T-S fuzzy model is extremal, then the system will also simultaneously establish a bounded number of switching for bang-bang optimal controls. Further, consider the trajectories for which m control vectors are simultaneously singular. From the proof of Corollary 2, we also know the set of all vector fields $\left\{\left[\sum A_i x,\ \sum b_j\right]\right\}$ are linear independent, so we have the following result.

Theorem 9. *If an extremal of the T-S fuzzy model in* (5) *is normal, then the switching function* $\psi_j(t)$, $j = 1, \ldots, m$ *will not be vanished for any* t.

Proof:

Assume that k is a fixed element of $\{1, \ldots, m\}$ and (x, λ, u) is extremal with a common accumulation point of zeros at $t = t_0$. From (42) and (43) we have

$$\psi_j(t) = \lambda^T(t_0)\sum b_j = 0$$

and its first derivative is

$$\dot{\psi}_j(t) = \lambda^T(t_0)\left[\sum A_i x(t),\ \sum b_j\right] = 0$$

for all $j = 1, \ldots, m$, $j \neq k$. If ψ_k and $\dot{\psi}_k$ vanish at $t = t_0$, Since The vector field $\sum b_k$, $\left[\sum A_i x,\ \sum b_j\right]$ for $j = 1,\ldots,m$, are linear independent. This yields a contradiction with the non-vanishing condition for costate in the maximum principle.

Q.E.D.

The solvable Lie algebra is defined for the T-S fuzzy model (5) as following.

Definition 13. *For T-S fuzzy model (5), the* ***solvable Lie algebra*** *is defined as*

$$\mathcal{L}^{(k)} := \left\{\sum A_i x,\ \sum b_j \,\middle|\, \forall j = 1,\ldots,m\right\}_{LA}. \tag{45}$$

if derived series $\mathcal{L}^{(k)}$ *is vanished for larger* k. *Then the T-S fuzzy model is called* ***solvable***.

In the next theorem, *solvable Lie algebra* will be crucial in establishing a bound on the number of switching for bang-bang control will be derived.

Theorem 10. *If the controllable T-S fuzzy model (5) is solvable, then the total number of switching is bounded.*

Proof:

The controllable *T-S fuzzy model* (5) will imply

$$\mathcal{L} = \mathrm{span}\left\{ad\left(\sum A_i x\right)^k \sum b_j\right\}, \text{ for } k = 1,\ldots,n-1.$$

If $\mathcal{L}$ is *solvable lie algebra*, .i.e., $\mathcal{L}^{(k)} = ad\left(\sum A_i x\right)^k \sum b_j = 0$ for $k \geq p \geq n-1$.

Form (44), we have

$$\psi_j^k(t) = \lambda^T(t_0)\, ad\left(\sum A_i x\right)^k \left(\sum b_j\right), \text{ for } k \geq p, \tag{46}$$

is identically zero due to the T-S fuzzy model is solvable. In (46), $\psi_j^k(t)$ is vanished for $k \geq p$, then the polynomial degree of switching function $\psi_j(t)$ do not exceed p. **Q.E.D.**

Remark 11. For $\sum b_j \neq 0$, the *solvable* condition (46) can be generalized as $\mathcal{L}^{(k)} = ad\left(\sum A_i x\right)^k = 0$.

For the single input case, Theorem 10 provides the condition that the number of switching is at most p. Similarly, For multiple m vertices of U, the number of switching will not exceed $m \cdot p$.

3.4 Illustrative Examples

To utilize the time-optimal design techniques, two systems with single input and two inputs respectively will be illustrated.

Example 4.

Consider an articulated vehicle [1] in Fig. 5. The kinematic model of the vehicle is the starting point to model the dynamics of the lateral and orientation motions.

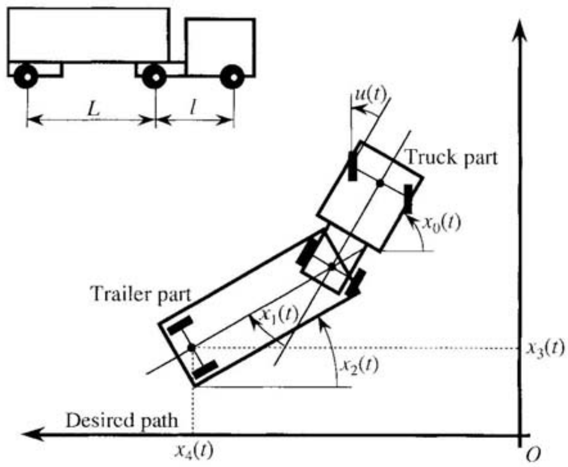

Fig. 5 Articulated vehicle model [1].

The dynamics of articulated vehicle can be formulated as

$$\dot{x}_0 = \frac{v}{l}\tan\left(u\left(t\right)\right)$$
$$x_1 = x_0 - x_2$$
$$\dot{x}_1 = \dot{x}_0 - \dot{x}_2 = \frac{v}{l}\tan\left(u\left(t\right)\right) - \frac{v}{L}\sin\left(x_1\left(t\right)\right)$$
$$\dot{x}_2 = \frac{v}{L}\sin\left(x_1\left(t\right)\right)$$
$$\dot{x}_3 = v\cos\left(x_1\left(t\right)\right)\cdot\sin\left(x_2\left(t\right)\right)$$
$$\dot{x}_4 = -v\cos\left(x_1\left(t\right)\right)\cdot\cos\left(x_2\left(t\right)\right)$$

where

$x_0\left(t\right)$ angle of truck;

$x_1\left(t\right)$ angle difference between truck and trailer;

$x_2\left(t\right)$ angle of trailer;

$x_3\left(t\right)$ vertical position of rear end of trailer;

$x_4\left(t\right)$ horizontal position of rear end of trailer;

$u\left(t\right)$ steering angle,

l is the length of truck, L is the length of trailer, and v is the constant speed. In this example, let $l = 1m$, $L = 2.5m$, $v = -5m/s$. The control purpose is to find the steering angle with constant backward speed so that the articulated vehicle will reach the straight line $x_3 = 0$, i.e.,

$$x_1\left(t\right) \to 0,\ x_2\left(t\right) \to 0,\ x_3\left(t\right) \to 0\,.$$

If the angle difference between the truck and trailer expands to 90°, i.e. $\left|x_1\right| = 90^\circ$, this phenomenon is called “jackknife”. When a jackknife phenomenon happens, an articulated vehicle becomes uncontrollable and the backward motion can not continue any more. To avoid this problem, the analysis of researchable set will be discussed in the following.

For constructing the T-S fuzzy model, assuming that $u(t)$, $x_2(t)$ are small and $x_1(t) \in (-\pi/2,\ \pi/2)$. Let $X(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T$. The dynamics of articulated vehicle can be formulated as:

Rule i: IF $x_1(t)$ is "Positive" and "Negative", THEN

$$\dot{X}(t) = A_i X(t) + B_i U(t), \quad i = 1, 2 \tag{47}$$

where the membership functions are given in Fig. 6 and the consequent parts are chosen as

$$A_1 = \begin{bmatrix} -v/L & 0 & 0 \\ v/L & 0 & 0 \\ 0 & v & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -v/L & 0 & 0 \\ v/L & 0 & 0 \\ 0 & \beta \cdot v & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix}.$$

and $\beta = \cos(88^\circ)$.

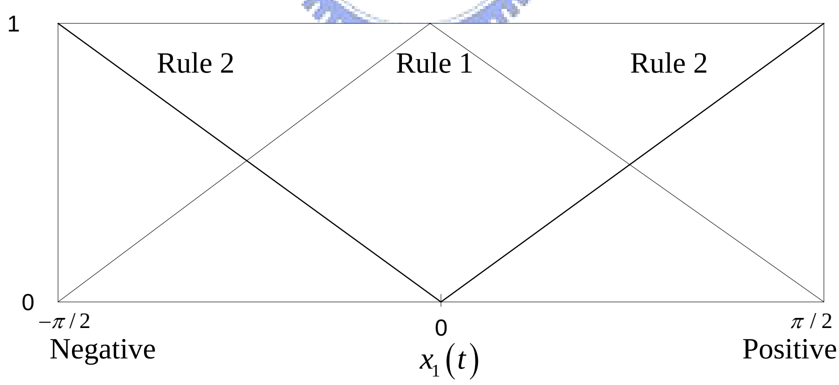

Fig. 6 The membership functions of Example 4.

From Corollary 1, we have

$$W_0 = \sum b_j = \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix}.$$

The matrix $\sum A_i \sum b_j$ is

$$W_1 = \left(\mu_1 \begin{bmatrix} -v/L & 0 & 0 \\ v/L & 0 & 0 \\ 0 & v & 0 \end{bmatrix} \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} -v/L & 0 & 0 \\ v/L & 0 & 0 \\ 0 & \beta v & 0 \end{bmatrix} \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix} \right)$$
$$= \begin{bmatrix} -v^2/(lL)(\mu_1+\mu_2) \\ v^2/(lL)(\mu_1+\mu_2) \\ 0 \end{bmatrix}.$$

The matrix $\sum A_i^2 \sum b_j$ is

$$W_2 = \begin{bmatrix} -v^3/(lL^2)(\mu_1+\mu_2) \\ v^3/(lL^2)(\mu_1+\mu_2) \\ -v^3/(lL)(\mu_1+\beta\mu_2) \end{bmatrix}$$

The controllability of the fuzzy model can be reformulated by finding the determinant of $[W_0,\ W_1,\ W_2]$:

$$\left\| \begin{matrix} v/l & -v^2/(lL) & -v^3/(lL^2)(\mu_1+\mu_2) \\ 0 & -v^2/(lL) & v^3/(lL^2)(\mu_1+\mu_2) \\ 0 & 0 & -v^3/(lL)(\mu_1+\beta\mu_2) \end{matrix} \right\|. \tag{48}$$

The determinant of (48) can be found as $(v/l)\cdot\left[-v^2/(lL)\right]\cdot\left\{-v^3/\left[(lL)(\mu_1+\beta\mu_2)\right]\right\}$.

Since the determinant of (48) can not be zero for $\forall \mu_i \in [0,\ 1]$ with $\sum \mu_i = 1$ ($i=1,2$), therefore we may conclude that the fuzzy model is controllable and time-optimal solution does exist. To realize time-optimal control, we consider a control as $U = \tilde{u} + u^*$ where control input $\tilde{u} = -kx$ can be designed by the pole assignment and time-optimal control u^* (steering angle) is constrained in $\left[5^\circ,\ -5^\circ\right]$. Choose the closed-loop eigenvalues as $[0 \quad 0 \quad 0]$ and we have $k = [\text{-}0.4 \quad 0 \quad 0]$. By closed-loop feedback, the consequent parts of the fuzzy model (47) can be reformulated as

$$A_1=\begin{bmatrix}0&0&0\\-2&0&0\\0&-5&0\end{bmatrix},\quad B_1=\begin{bmatrix}-5\\0\\0\end{bmatrix}$$
$$A_2=\begin{bmatrix}0&0&0\\-2&0&0\\0&\text{-0.1745}&0\end{bmatrix},\quad B_2=\begin{bmatrix}-5\\0\\0\end{bmatrix}.$$

Due to $\sum B_i \neq 0$ for $\forall t \geq 0$, by using Remark 11, we have

$$\mathcal{L}^{(0)}=\mu_1A_1+\mu_2A_2=\mu_1\begin{bmatrix}0&0&0\\-2&0&0\\0&-5&0\end{bmatrix}+\mu_2\begin{bmatrix}0&0&0\\-2&0&0\\0&\text{-0.1745}&0\end{bmatrix}$$
$$\mathcal{L}^{(1)}=\mu_1A_1A_1+\mu_2A_2A_2=\mu_1\begin{bmatrix}0&0&0\\0&0&0\\10&0&0\end{bmatrix}+\mu_2\begin{bmatrix}0&0&0\\0&0&0\\0.3490&0&0\end{bmatrix}.$$

$$\mathcal{L}^{(2)}=\mu_1A_1A_1A_1+\mu_2A_2A_2A_2=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$
$$\mathcal{L}^{(3)}=\mathcal{L}^{(4)}=\cdots=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

For $\forall\mu_i \in [0,\ 1]$ with $\sum \mu_i = 1$ ($i = 1,2$), and $k \geq 2$, $\mathcal{L}^{(k)}$ is identically zero therefore the fuzzy model is concluded to be solvable and the number of switching is at most 2. Let $u = 5^{\circ}$, the bang-bang control does exist and the possible control sequence can be concluded as:

$$\{u\},\ \{-u\},\ \{u,\ -u\},\ \{-u,\ u\},\{u,\ -u,\ u\},\ \{-u,\ u,\ -u\}.$$

The switching curves V are shown in Figures 7 and 8. The dotted line is the set V^{-} which is the trajectory by control input $\{-u\}$ and the solid line shows the set V^{+} which is the trajectory by control input $\{u\}$. Let V_1 denote the set of states which can be forced to the origin by the control sequence $\{u,\ -u\}$ or $\{-u,\ u\}$. The transition from the control input u to $-u$ must occur on the set V^-. If the control sequence from $-u$ to u, the transition must occur on the set V^+. The set V_1 are shown in Figures 9 and 10. The dotted line is the set V_1^- which is forced by the control sequence $\{-u,\ u\}$ and the solid line shows the set V_1^+ which is forced by the control sequence $\{u,\ -u\}$. The set V_2 is the trajectory which can be forced to the origin by the control sequence $\{u,\ -u,\ u\}$ or $\{-u,\ u,\ -u\}$. To prevent the jackknife phenomenon, the state x_1 should be constrained to be less than 90°. In Figures 11 and 12, the ellipses show the reachable set for $|x_1| \le 90^\circ$ where the solid ellipses are the set V_1 and the dotted ellipses are the set V_2. In fact, $V \subseteq V_1 \subseteq V_2$.The maximal reasonable range of initial positions will be restricted on the reachable set V_2.

Case I

For the initial position, $x_0 = 240^\circ$, $x_1 = 200^\circ$, $x_2 = 40^\circ$, $x_3 = 20m$ and $x_4 = 0m$, the time-optimal trajectory of x_3 vs. x_4 is depicted in Fig. 13. The corresponding time-optimal control $u^*(t)$ is shown in Fig. 14. The shortest time from initial position to the origin is 2.4115 (sec.).

Case II

For the initial position, $x_0 = 320^\circ$, $x_1 = 20^\circ$, $x_2 = 300^\circ$, $x_3 = 20m$ and $x_4 = 0m$, the time-optimal trajectory of x_3 vs. x_4 is depicted in Fig. 15. The corresponding time-optimal control $u^*(t)$ is shown in Fig. 16. The shortest time from initial position to the origin is 13.6715 (sec.).

Case III

In this case, the control purpose is to realize the forward movement the articulated vehicle along the straight line. For forward speed $v = 5m/s$, the consequence parts of the system are

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0.1745 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

The set V is depicted in Fig. 17 and Fig. 18, the solid line is the set of V^- and the dotted line is the set of V^+. The set V_1 are shown in Fig. 19 and Fig. 20. The solid line is the set of V_1^-. The dotted line is the set of V_1^+. For the initial position, $x_0 = -160^\circ$, $x_1 = -20^\circ$, $x_2 = -140^\circ$, $x_3 = 35m$ and $x_4 = 0m$, the time-optimal trajectories are depict in Fig. 21. The corresponding time-optimal control $u^*(t)$ are shown in Fig. 22.

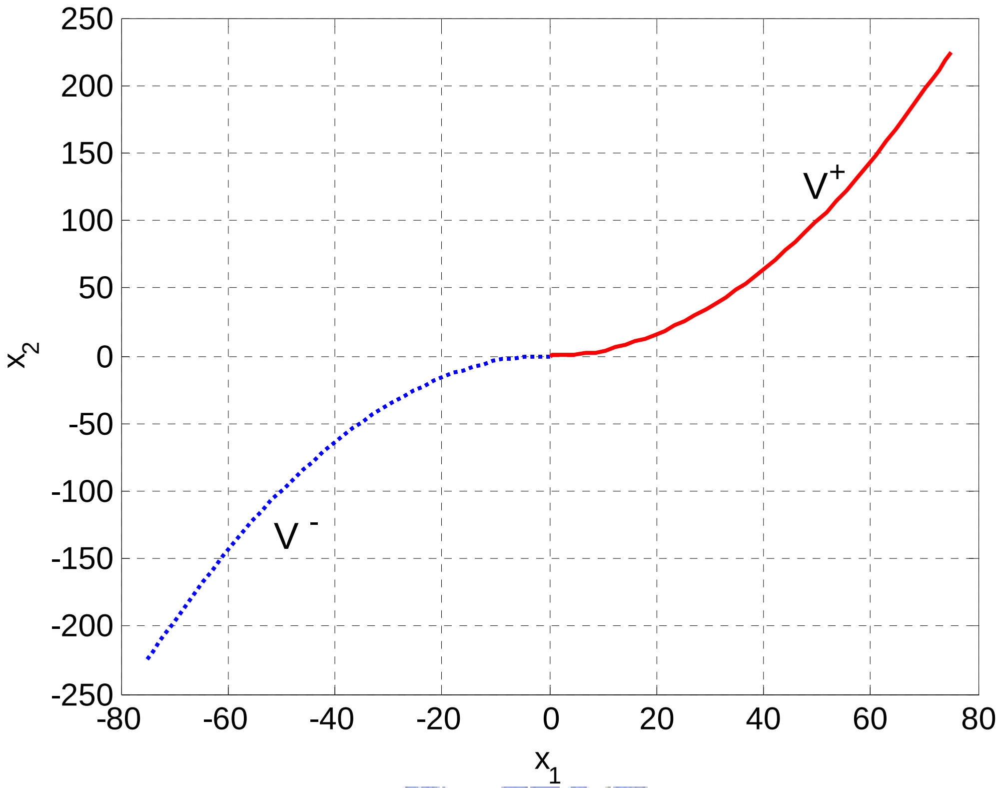

Fig. 7 The projection of the set V on the $x_1 - x_2$ plane.

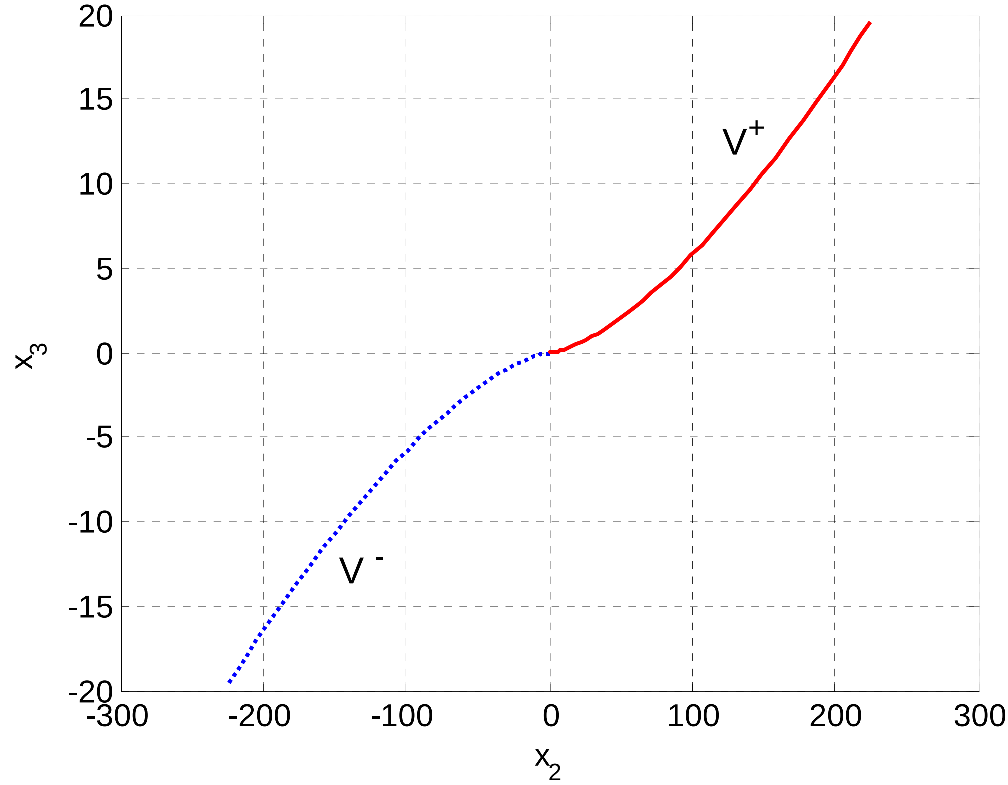

Fig. 8 The projection of the set V on the $x_2 - x_3$ plane.

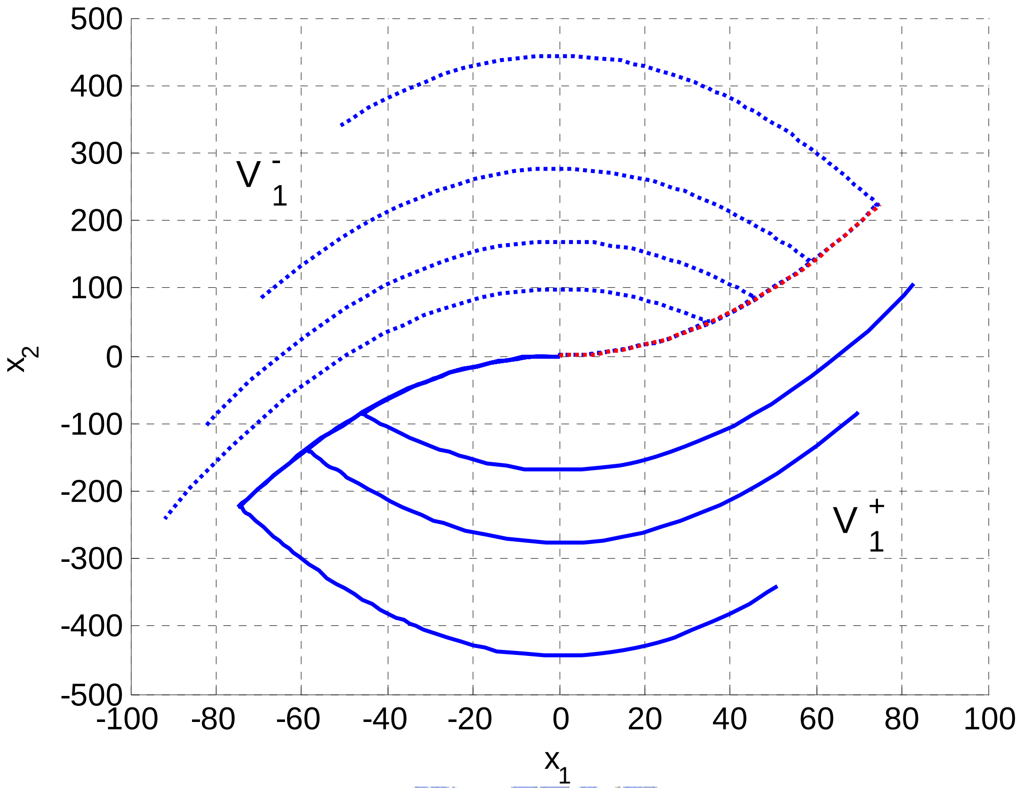

Fig. 9 The projection of the set V_1 on the $x_1 - x_2$ plane.

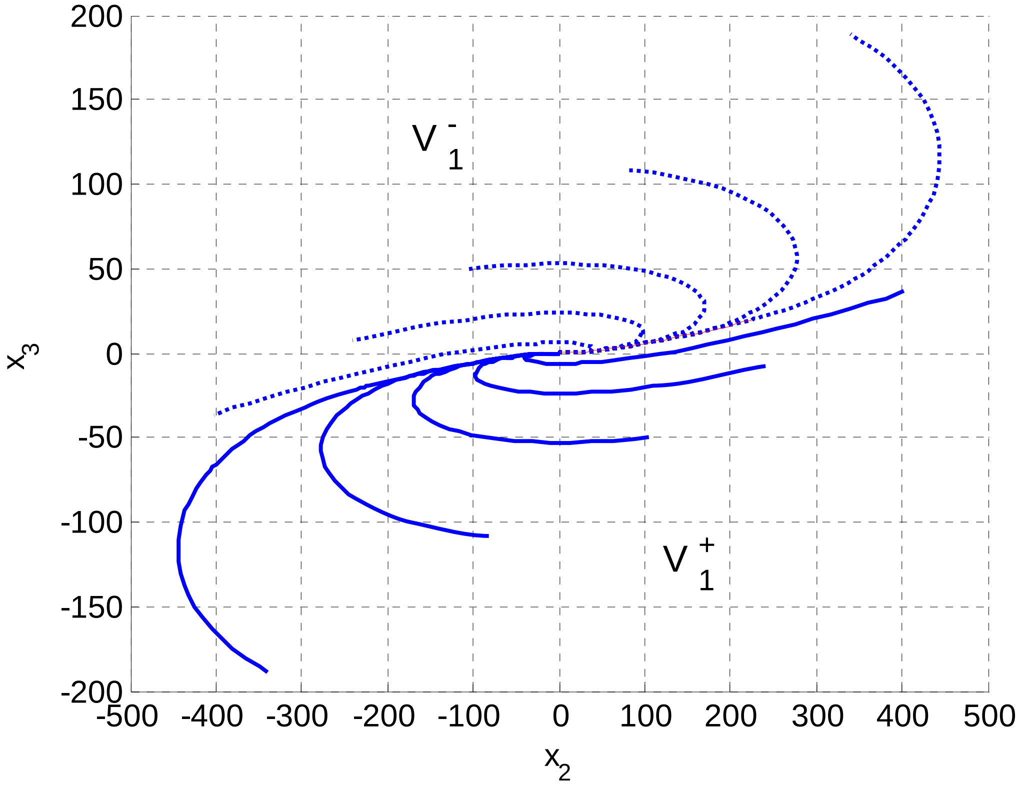

Fig. 10 The projection of the set V_1 on the $x_2 - x_3$ plane.

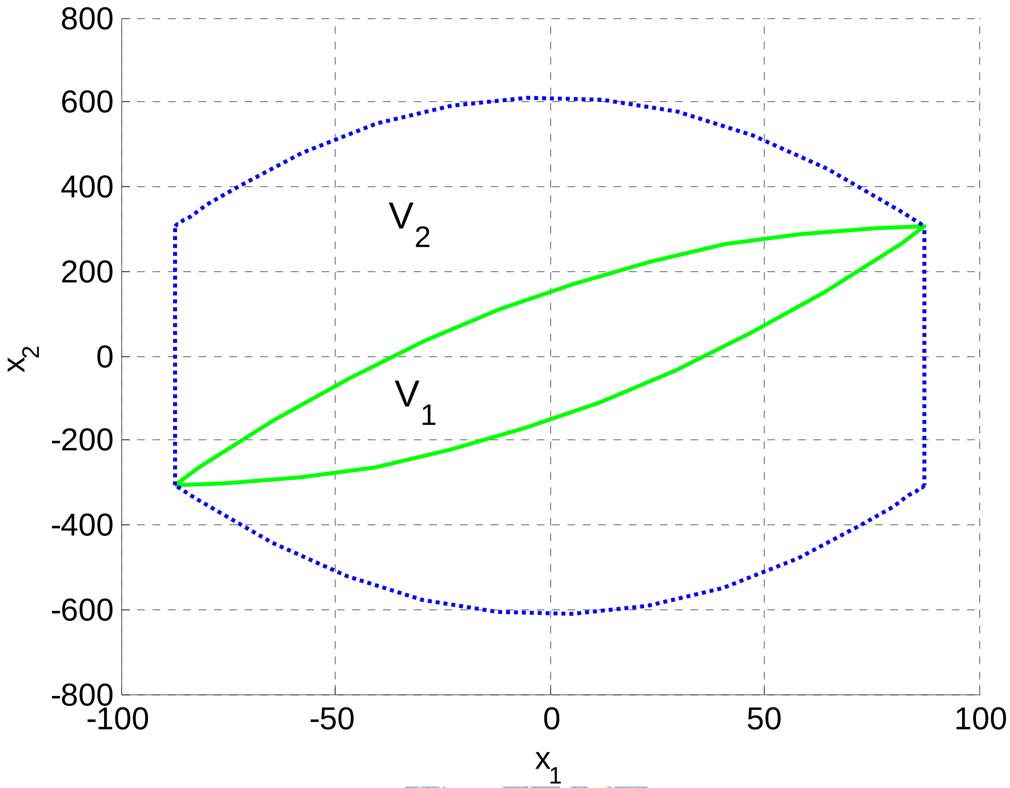

Fig. 11 The reachable set of V_1 and V_2 on the $x_1 - x_2$ plane.

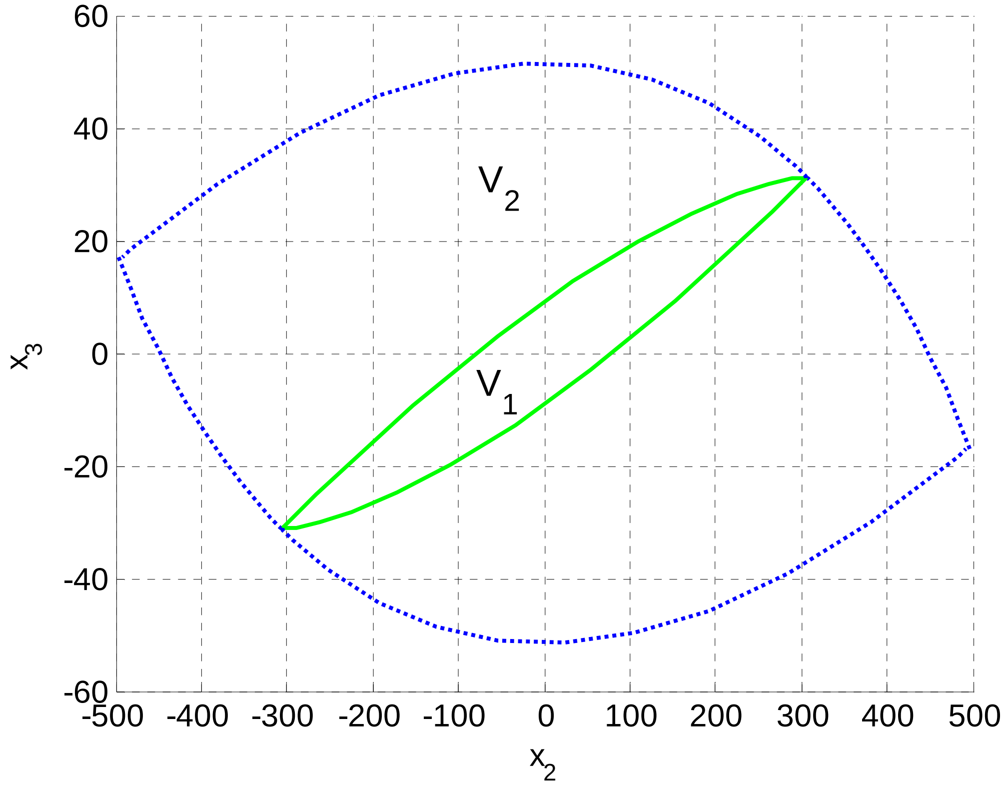

Fig. 12 The reachable set of V_1 and V_2 on the $x_2 - x_3$ plane.

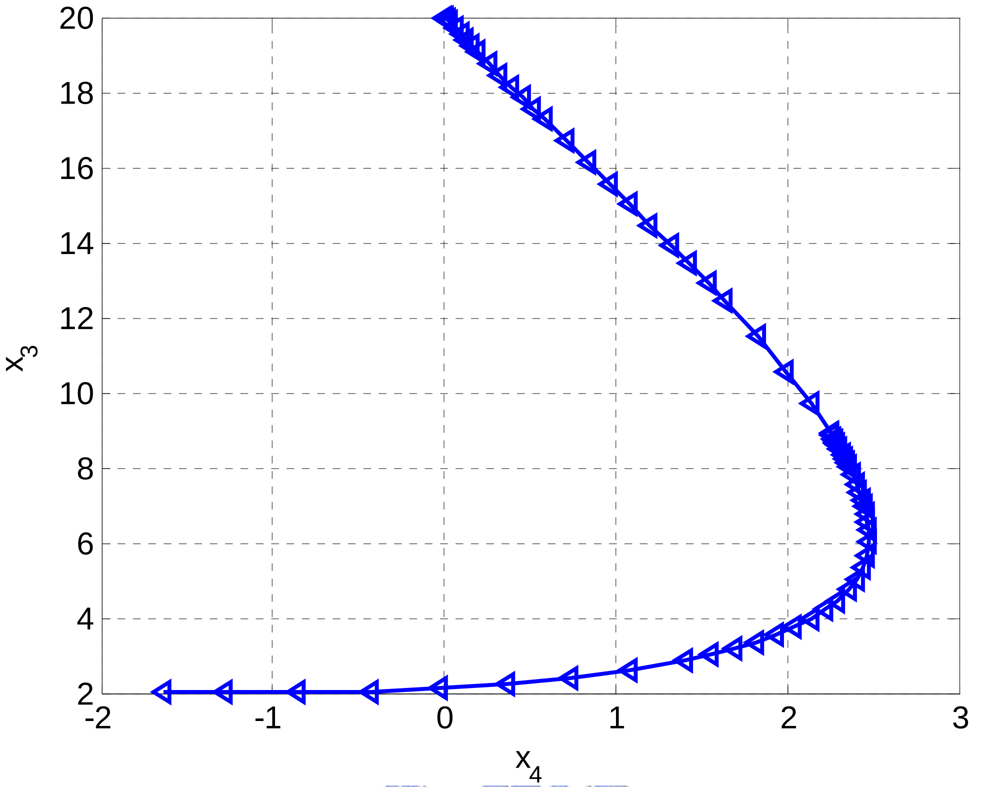

Fig. 13 Time-optimal trajectory in phase plane (CaseI)

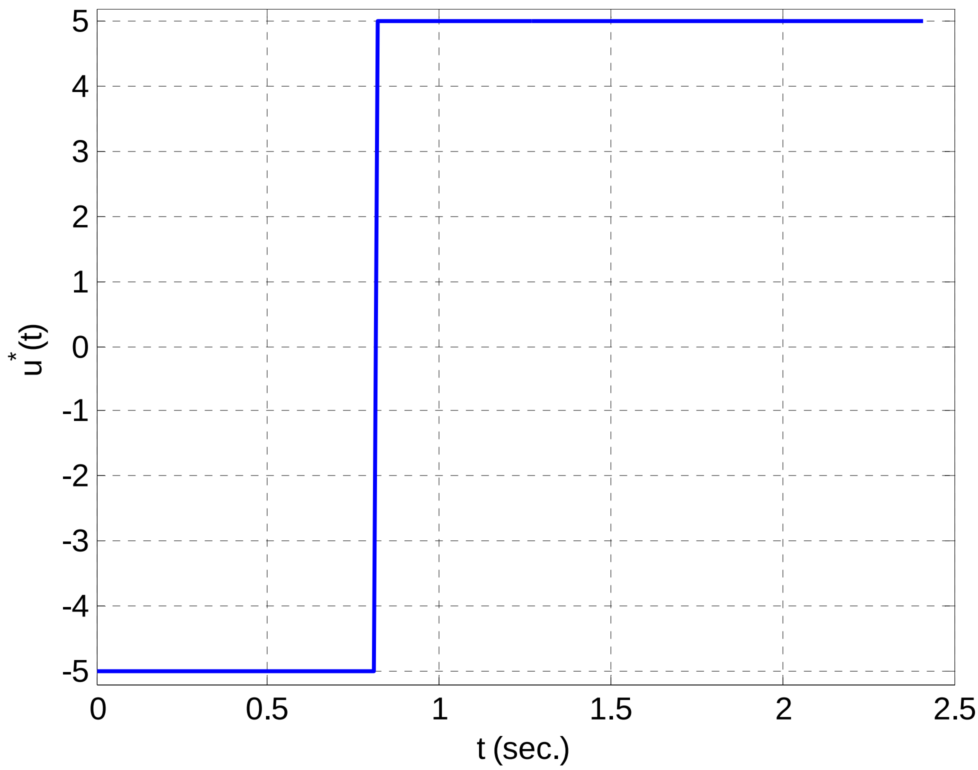

Fig. 14 The corresponded time-optimal control input (Case I).

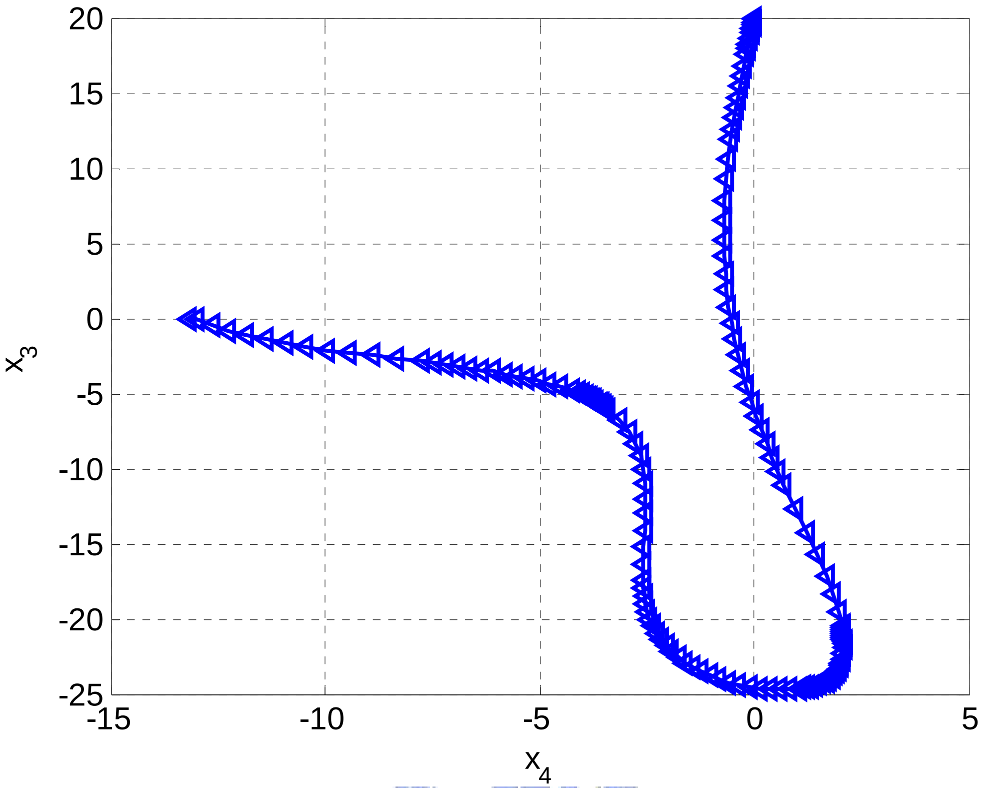

Fig. 15　Time-optimal trajectory in phase plane (Case II).

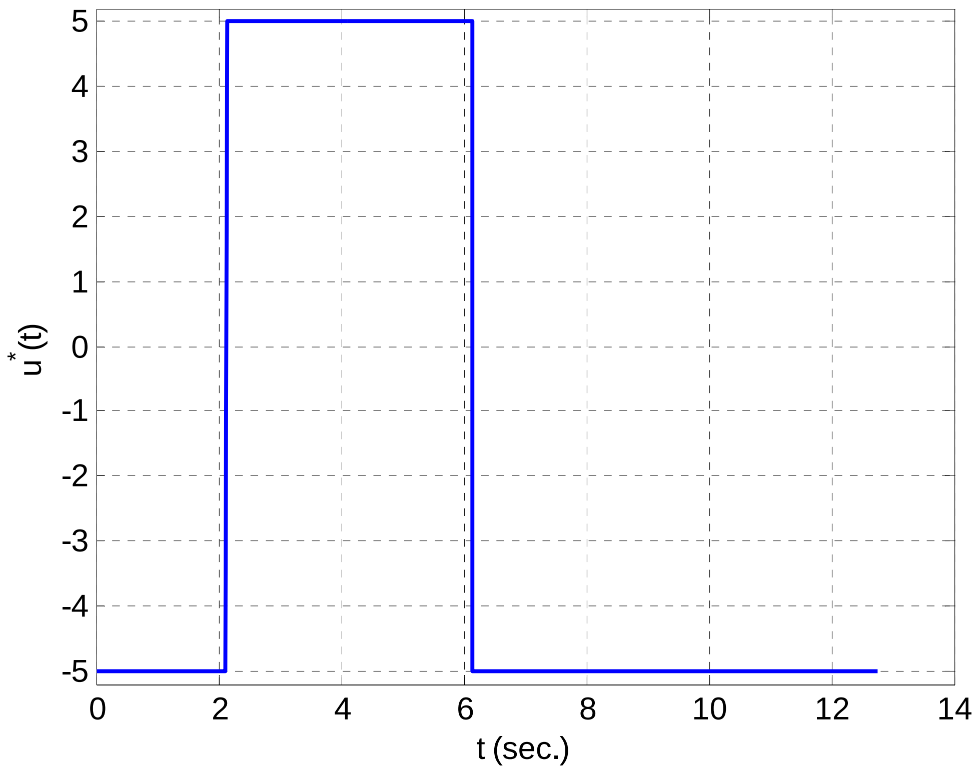

Fig. 16　The corresponded time-optimal control input (Case II).

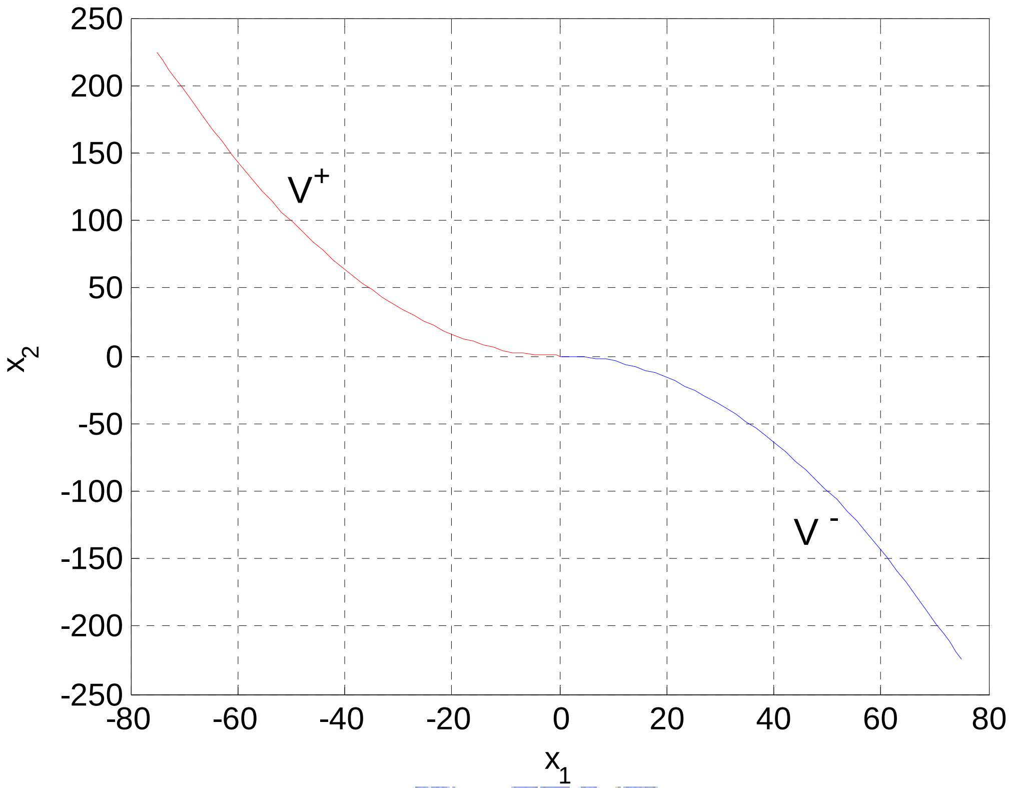

Fig. 17 The projection of the set V on the $x_1 - x_2$ plane.

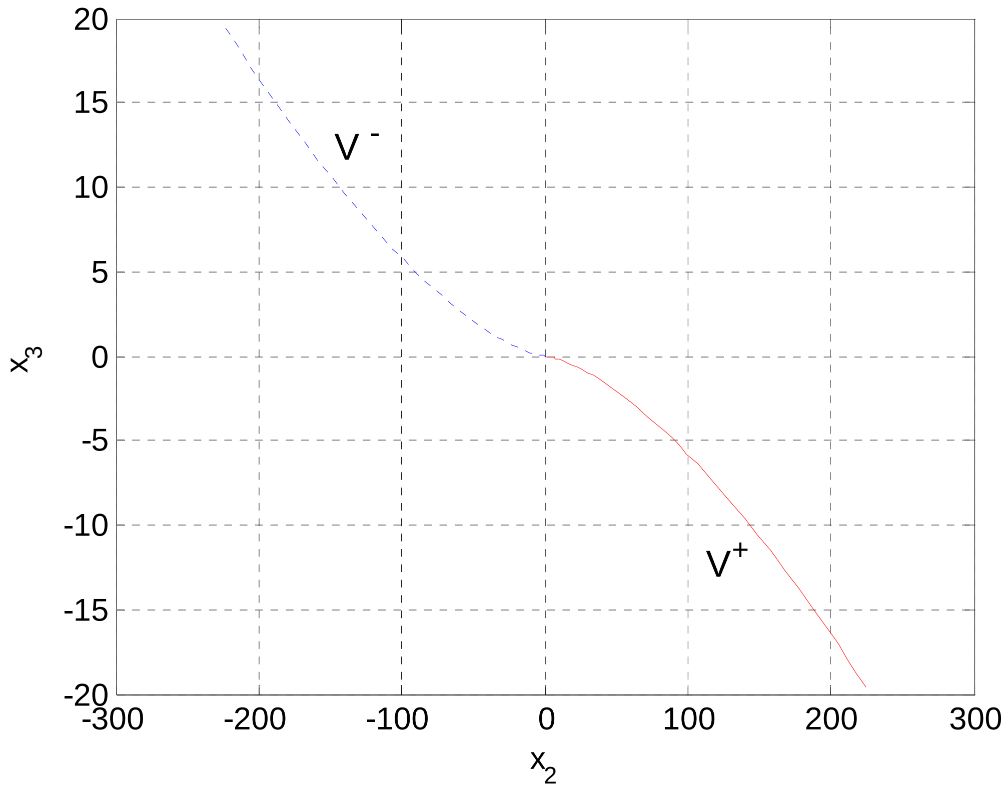

Fig. 18 The projection of the set V on the $x_2 - x_3$ plane.

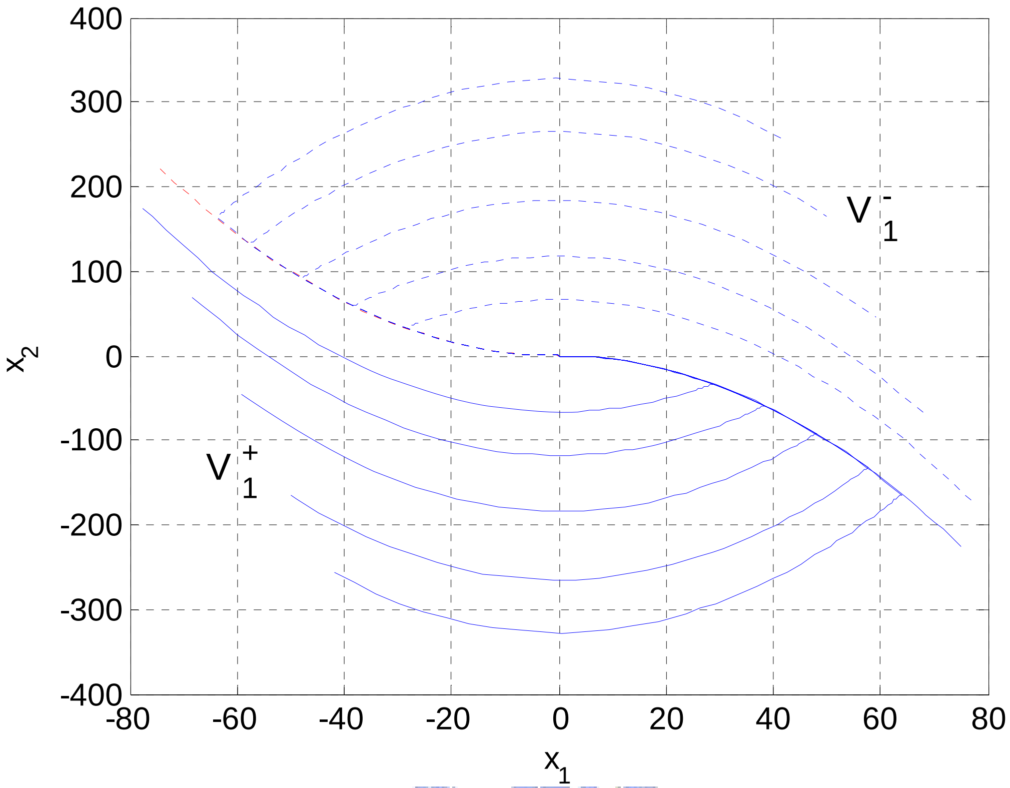

Fig. 19 The projection of the set V_1 on the $x_1 - x_2$ plane.

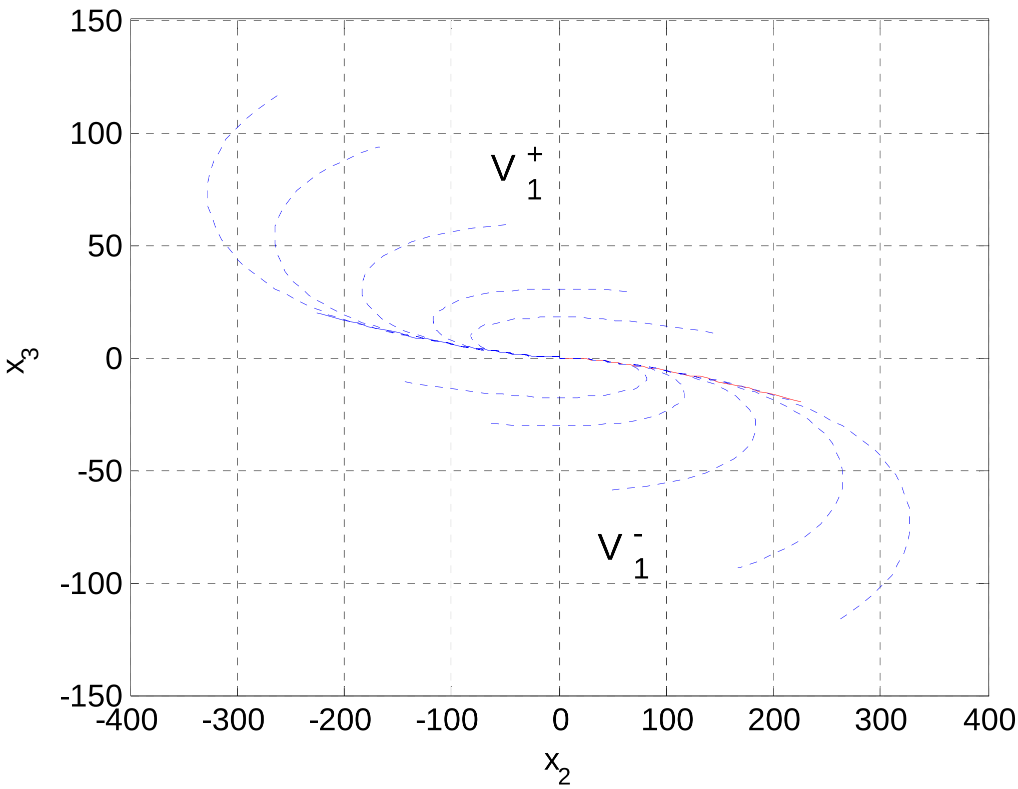

Fig. 20 The projection of the set V_1 on the $x_2 - x_3$ plane.

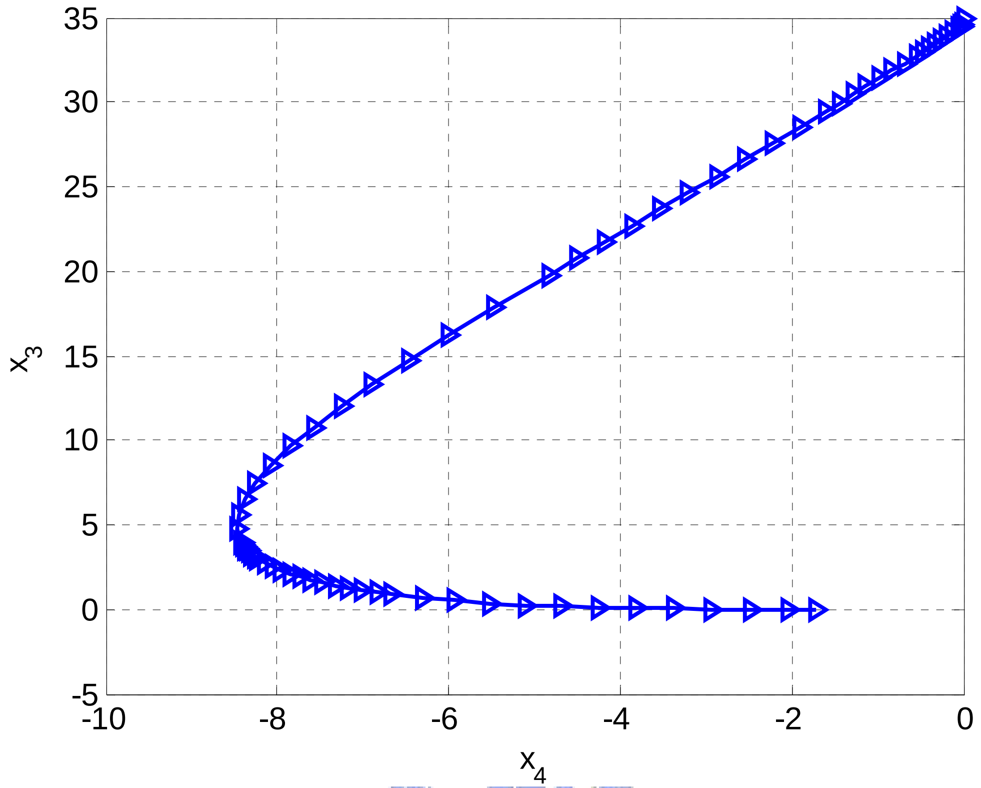

Fig. 21 Time-optimal trajectory in phase plane (Case III).

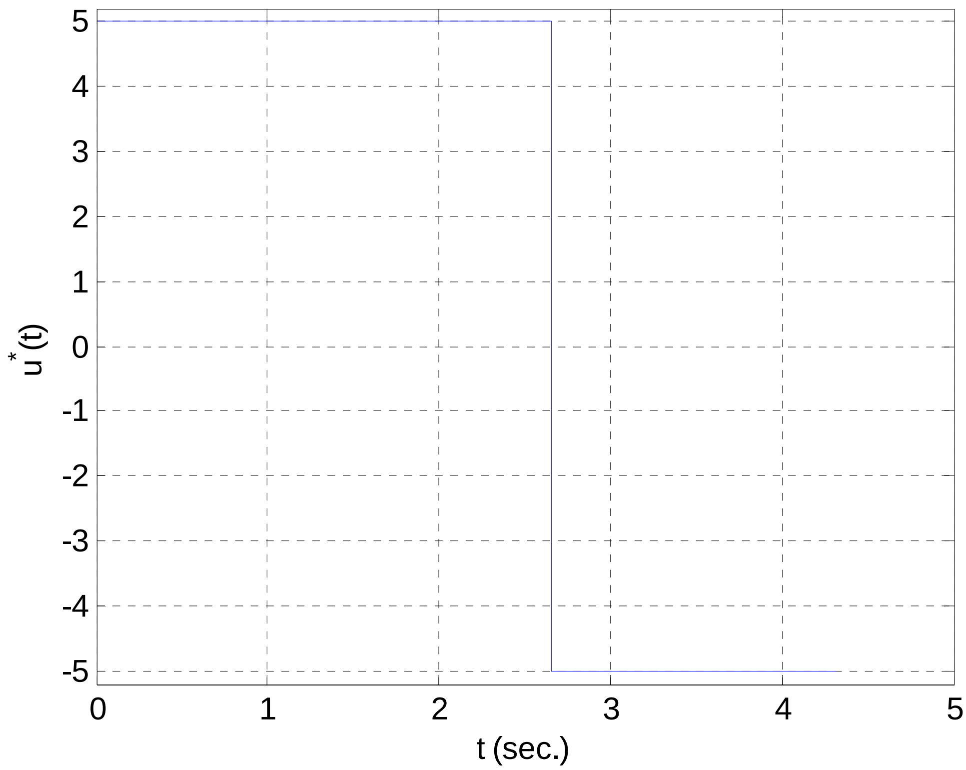

Fig. 22 The correspond time-optimal control input (Case III).

Example 5.

The multiple inputs system is considered here. Consider the following T-S fuzzy model:

Rule i: IF $x_1(t)$ is "Positive" and "Negative", THEN

$$\dot{X}(t) = A_i X(t) + B_i U(t), \quad i = 1, 2 \tag{49}$$

where $X(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$, $U(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T$, $|u_1(t)| \le 1$, $|u_2(t)| \le 1$, and the consequent parts are chosen as

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0.18 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 4 & 0.5 \\ 0.5 & -4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 & 0.5 \\ 0.5 & -4 \end{bmatrix}.$$

The membership functions of the fuzzy model are given in Fig. 23.

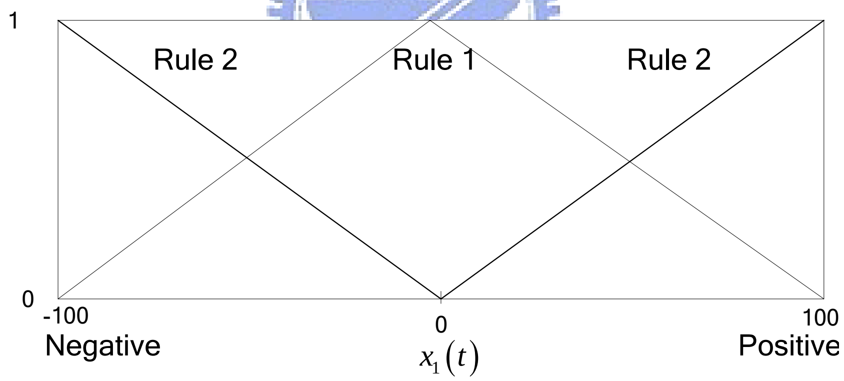

Fig. 23 The membership functions for Example 5.

The fuzzy model is found to be controllable by Corollary 1. The switching number is at most 2 which is obtained by using Remark 11. Therefore the time-optimal sequences are

$$\{1,\ 1\},\ \{-1,\ -1\},\ \{1,\ -1\},\ \{-1,\ 1\}.$$

Follow the same analysis in Example 4, the switching curves are explained in the followings. There are two possible switching curves in this example. Let the set of states V be forced by input $\{1,\ 1\}$ or $\{-1,\ -1\}$ and V_1 be forced by input $\{1,\ -1\}$ or $\{-1,\ 1\}$ to the origin. The switching curve V is depicted as solid line in Fig. 24 , the dotted line depicts switching curve V_1 and the time-optimal control inputs are also shown in Fig. 24. Assume $\mathcal{R}(T)$ and $\mathcal{R}_1(T)$ are reachable sets for V and V_1 respectively that can reach the origin at time T. Fig. 25 depicts reachable set which is sampled from $T=5$ to $T=20$ in every 5 seconds. The dotted line is the reachable set $\mathcal{R}_1(T)$ and the solid line is the reachable set $\mathcal{R}(T)$.

Case I

For the initial state $X_0=[40,\ -50]$, the time-optimal trajectory is shown in Fig. 26. The corresponding time-optimal control $u^*(t)$ is shown in Fig. 27. The shortest time from initial state to the origin is 9.350 (sec).

Case II

For the initial state $X_0=[40,\ 100]$, the time-optimal trajectory is depicted in Fig. 28. The corresponding time-optimal control $u^*(t)$ is shown in Fig. 29. The shortest time from initial state to the origin is 25.249 (sec).

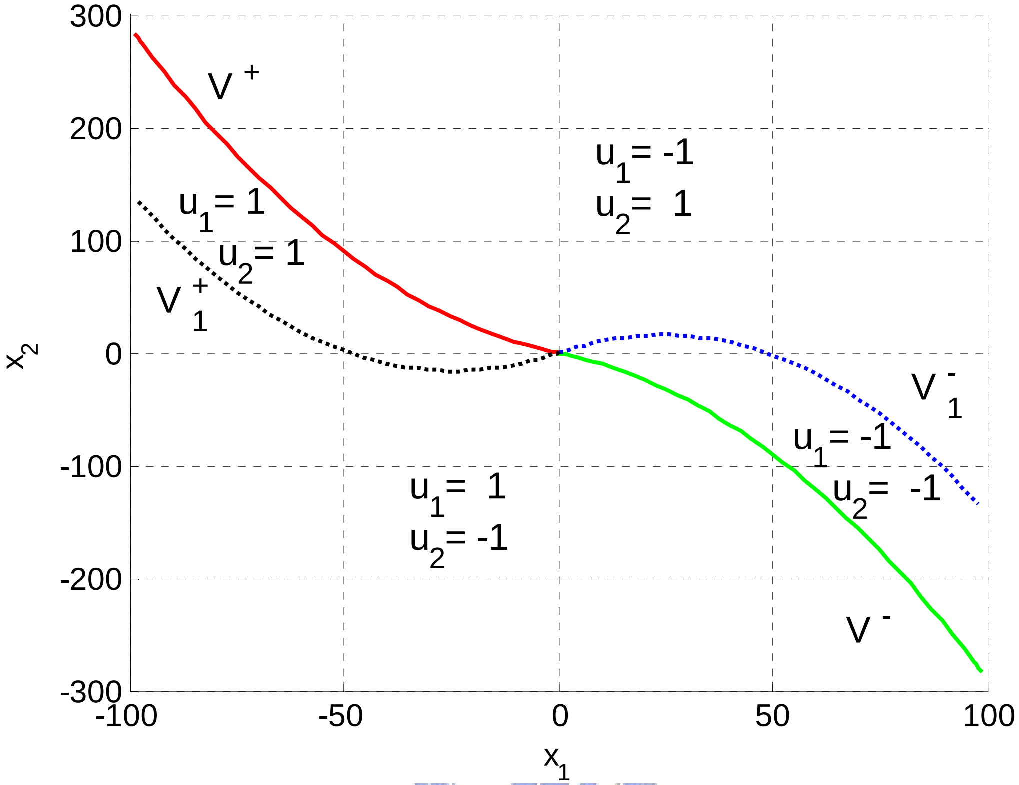

Fig. 24 The switching curve and time-optimal control input.

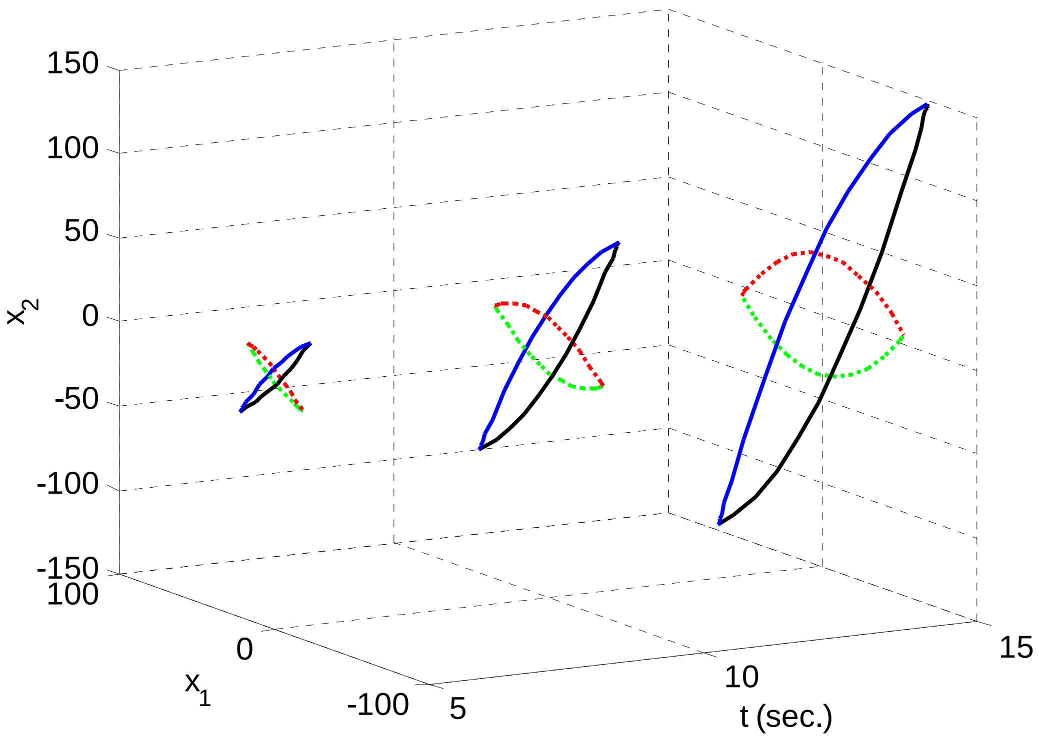

Fig. 25 The reachable sets of Example 5.

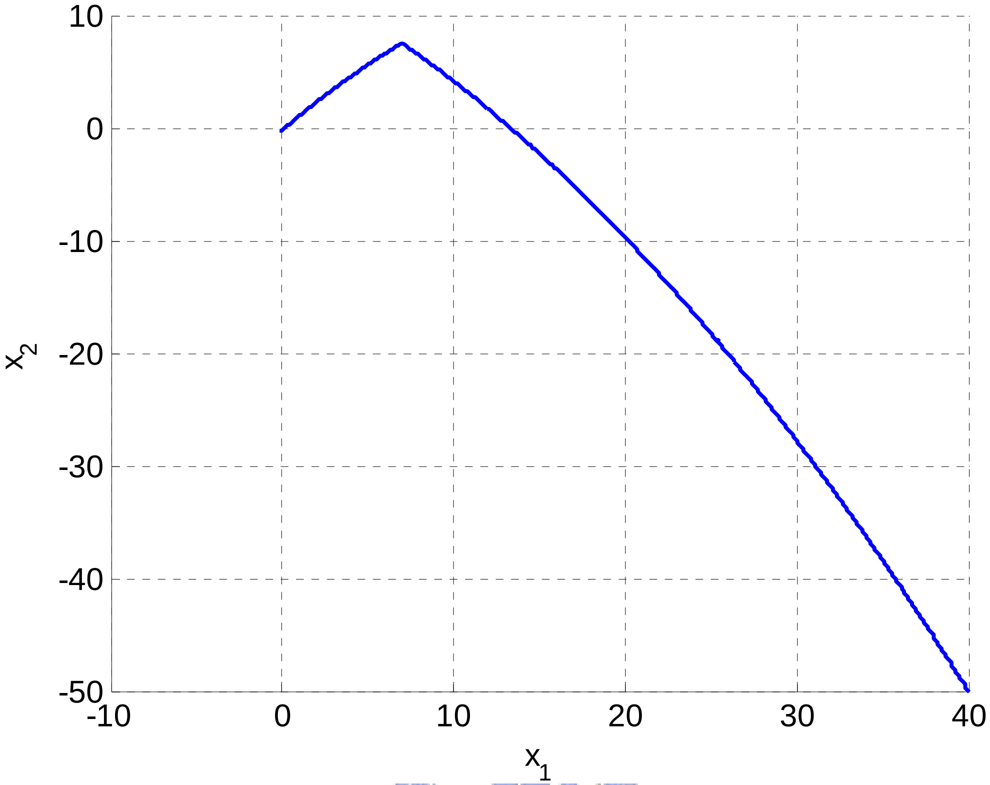

Fig. 26 Time-optimal trajectory in phase plane (Case I).

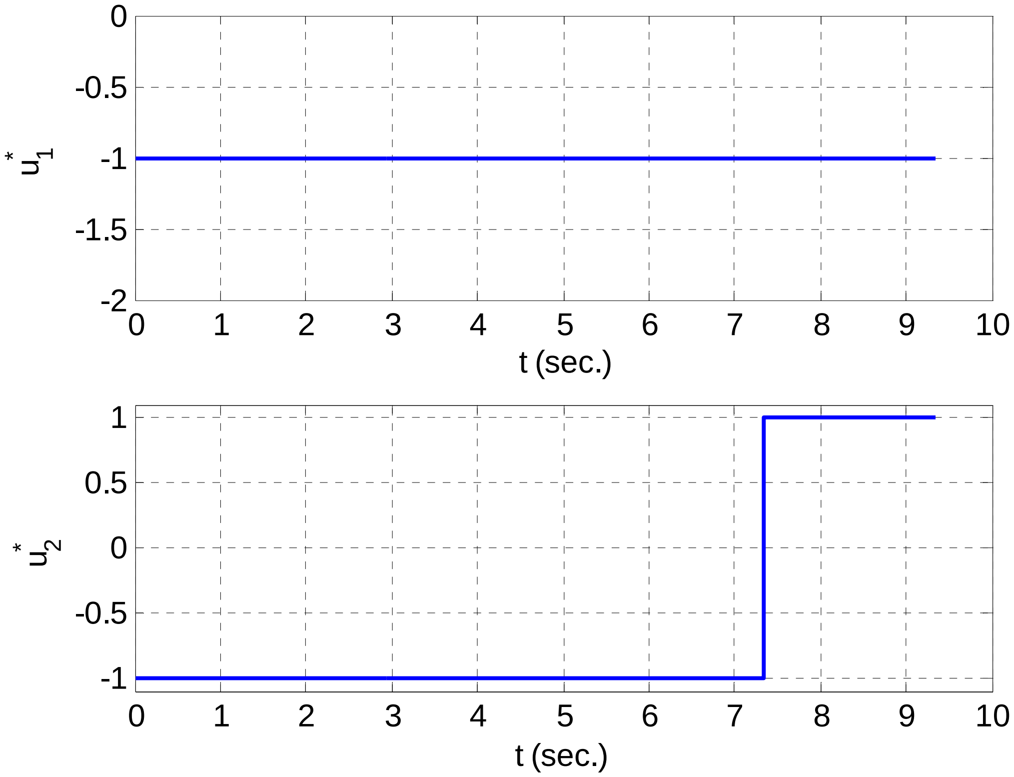

Fig. 27 The corresponded time-optimal control input (Case I).

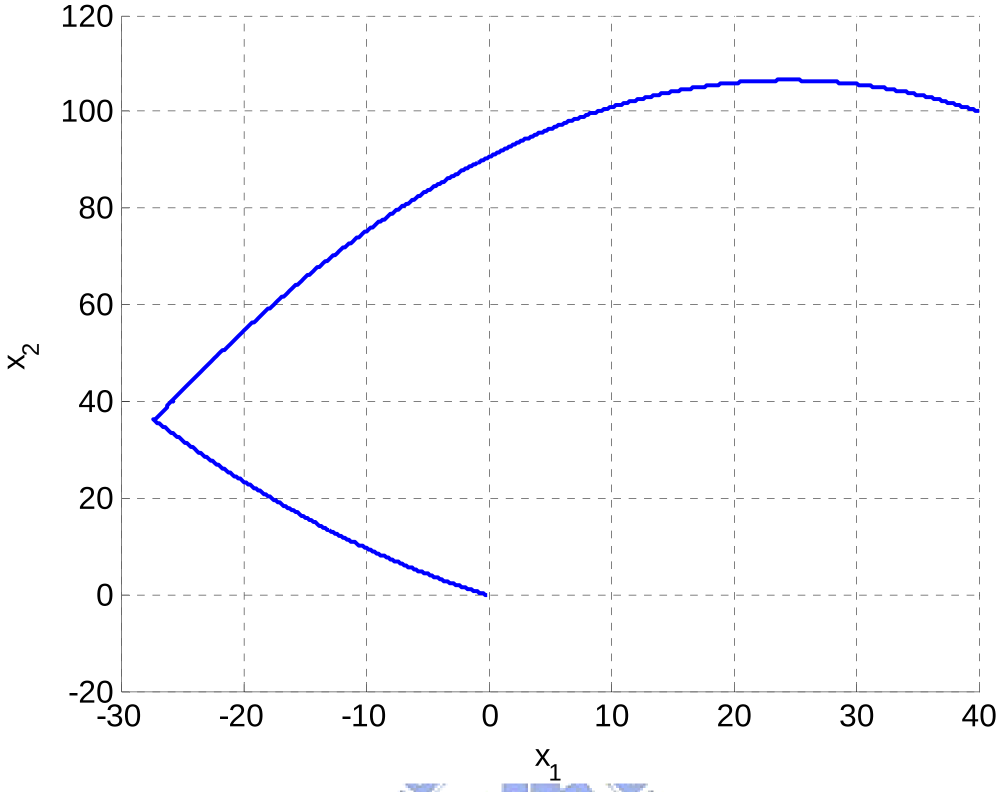

Fig. 28 Time-optimal trajectory in phase plane (Case II).

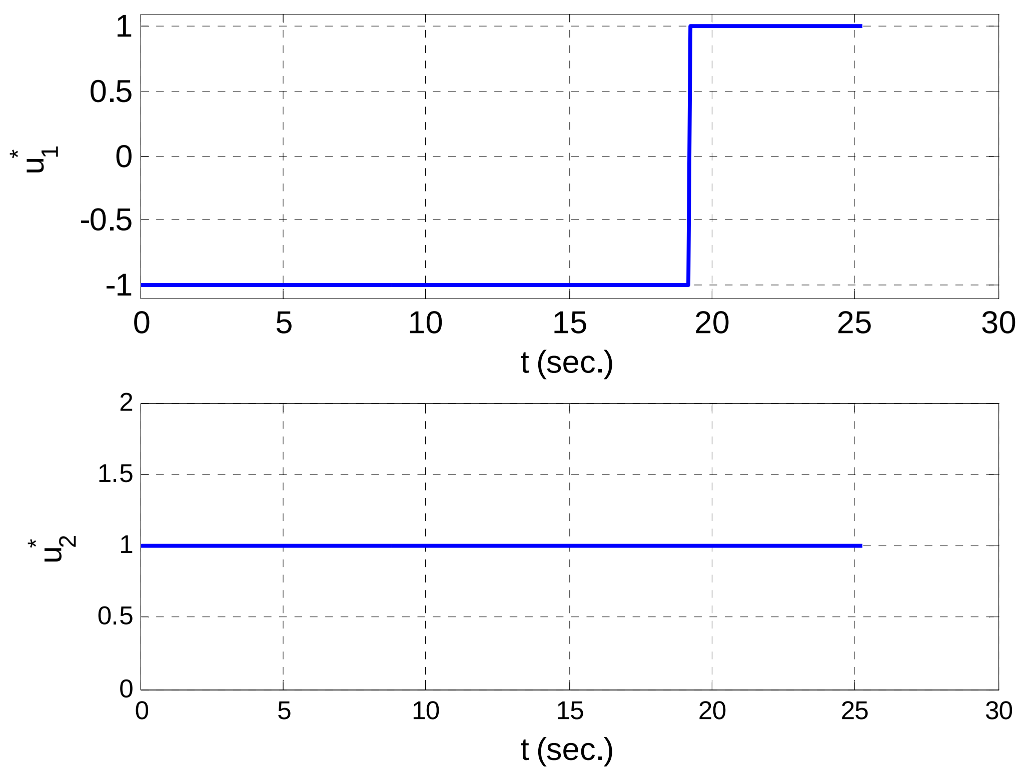

Fig. 29 The corresponded time-optimal control input (Case II).

Chapter 4
The Maximal Convergence Rate of T-S Fuzzy Control

The time-optimal control problem of T-S fuzzy model was discussed in previously section. The time-optimal control is a bang–bang control and implemented successfully by reachable set. If the system is not accessible, the number of switching can not be found and the computation cost is too much under this situation. Fast response is always a considered property in this dissertation. A notion directly relates to fast response is the convergence rate of the state trajectories. For a linear system, the convergence rate is determined by the real part of the pole which is closest to the imaginary axis. In this section, we will give a controller design of T-S fuzzy model on maximal convergence rate by the introduced level set function. The result of maximizing the convergence rate is characterized from the maximal invariant ellipsoid. The controller is also bang-bang within both the initial states and target states are belong to level set.

4.1 Problem Formulation

Consider a nonlinear system (1) with zero input. The ellipsoid Ω is invariant for the system if all the trajectories starting from it will stay inside of it. It is contractive invariant if

$$\dot{V}(x) = 2x^T Pf(x) < 0 .$$

The objective is to find a control law with constrained input such that convergence rate is maximal. To obtain a control law, problem is turn out that $-\dot{V}(x)$ is maximized at each x. Then the overall convergence rate of the system on $\Omega(x,\rho)$ can be defined as

$$\inf\left\{-\frac{\dot{V}(x)}{V(x)}\right\}.$$

Note that $V(x)$ over a fixed time interval do not to be maximal. The most important consequence of the maximal convergence control is that it produces the maximal invariant ellipsoid of a given shape. It is easy to see that an ellipsoid can be made invariant if and only if the maximal $\dot{V}(x)$ on the boundary of the ellipsoid under the maximal convergence control is negative.

In conventional, T-S fuzzy controller design employs the parallel distributed compensation (PDC) via the Lyapunov technique [36]. The PDC is designed by locally feedback gain F_i as

Controller Rule i: IF $z_1(t)$ is $M_{i1}\cdots$ and $z_p(t)$ is M_{ip}, THEN

$$u=-F_i x$$

The entire PDC can be formulated as follows:

$$U(t)=\sum_{i=1}^{r}\mu_i(t)F_i x(t). \tag{50}$$

The entire feedback type of system (4) via PDC are given as following:

$$\dot{x}=\sum_{i=1}^{r}\mu_i(t)\sum_{j=1}^{r}\mu_j(t)\mu_j(t)\left(A_i x-B_i F_j\right)x(t) \tag{51}$$

In general, this type controller is difficult to solve since the coupling relation of $\left(A_i x-B_i F_j\right)$. For simplifying the design process, in here, we consider the system (4) with single controller input and make the following assumptions. For $i = 1, \ldots, r$, $B_i = B$ and the state feedback controller is given as

$$U = -Fx \tag{52}$$

where F is denoted as the state feedback gain, therefore the feedback T-S fuzzy model can be rewritten as

$$\dot{x} = \sum_{i=1}^{r} \mu_i(t)(A_i - BF)x \tag{53}$$

for any $x \in \mathbb{R}^n \setminus \{0\}$.

4.2 On Maximum the Convergence Rate

Consider the feedback fuzzy model (53) under the constraint that $U \leq 1$, we have following definitions.

Definition 14. A function $V(x)$ is a Lyapunov function, the level set of T-S fuzzy model (53) is given as

$$\Omega(x, \alpha) = \left\{ x \in \mathbb{R}^n \middle| V(x) = x^T P x \leq \alpha \right\} \tag{54}$$

where P is a positive-definite matrix and α is a positive number.

Definition 15. The convergence rate of level set Ω can be given as

$$\gamma := \frac{1}{2} \inf \left\{ -\frac{\dot{V}(x)}{V(x)} \middle| x \in \Omega(x,\ \alpha) \setminus \{0\} \right\}. \tag{55}$$

In here, we give the controller design for maximizing the convergence rate.

The following lemma will illustrate the level set Ω found by Linear Matrix Inequalities (LMIs) [36].

Lemma 1. *Consider a T-S fuzzy model (53) with zero input if* $P > 0$, $\alpha \geq 0$ *and*

$$\begin{bmatrix} A_i^T P + PA_i + \alpha P & PB \\ B^T P & -\alpha I \end{bmatrix} \le 0\,, \quad i = 1, \ldots, r\,, \tag{56}$$

then the ellipsoid Ω *is level set of the form* $x^T Px \le \alpha$ *.*

Theorem 11. *Consider* $V = x^T Px$ *to be a Lyapunov function for fuzzy model (53), if there exists a* $P > 0$ *,* $\alpha \ge 0$ *such that (56) is satisfied. Then there exists a feedback control* $U = -SGN(B^T Px)$ *such that the closed-loop system is asymptotically stable on maximizing the convergence rate.*

Proof:

Let $V(x) = x^T Px$ and $P > 0$. For a positive number α , the level set associated with $V(x)$ is ellipsoid,

$$\Omega(x,\alpha) = \left\{ x \in \mathbb{R}^n \,\middle|\, V(x) = x^T Px \le \alpha \right\}$$

Along the trajectory of the system (53),

$$\dot{V}(x) = x^T \left(A_i^T P + PA_i \right) x + 2x^T PB \cdot U < 0\,, \tag{57}$$

$\forall x \in \Omega(x,\alpha) \setminus \{0\}$. From Definition 15, the controller is minimizing (57), we have

$$U = -SGN(B^T Px)$$

where $SGN(\cdot)$ is sign function. It is clear that the maximal convergence control produces the maximal invariant ellipsoid of a given ellipsoid $\Omega(x,\alpha)$. **Q.E.D.**

Remark 12. The system will have no solution if $x = 0$. This is due to the switching plane $B^T Px = 0$. When the system state close to the switching plane, it is easy to have the chattering.

Remark 13. It becomes obvious that the maximal convergence control is also a bang-bang control.

4.3 Illustrative Examples

In this section, we demonstrate the application of the proposed maximal convergence rate for T-S fuzzy model.

Example 6.

Consider a nonlinear mass-spring-damper mechanical system that can be formulated as

$$M\ddot{x}+g\left(x,\dot{x}\right)+f\left(x\right)=\phi\left(\dot{x}\right)u \tag{58}$$

where M is the mass and u is the control force. $f\left(x\right)$ is the nonlinear or uncertain term of the spring system, $g\left(x,\dot{x}\right)$ is the nonlinear or uncertain term with respect to the damper, and $\phi\left(\dot{x}\right)$ is the nonlinear term with respect to the input term. We use the following mass-spring-damper and fuzzy model formulated in [15]:

$$\ddot{x}=-0.1\dot{x}^3-0.02x-0.67x^3+u$$

where the control input is constrained in 1, $x\in\left[-1.5 \quad 1.5\right]$, and $\dot{x}\in\left[-1.5 \quad 1.5\right]$. Let $X\left(t\right)=\left[\dot{x}\left(t\right) \; x\left(t\right)\right]^T$. The system can be described as the following T-S type fuzzy model [15]:

Rule i: IF $x\left(t\right)$ is M_{i1} and $\dot{x}\left(t\right)$ is M_{i2} THEN

$$\dot{X}(t)=A_iX\left(t\right)+B_iu\left(t\right), \qquad i=1,\ldots,4$$

where membership functions are chosen as $M_{11}=M_{21}=1-x^2\left(t\right)/2.25$, $M_{31}=M_{41}=x^2\left(t\right)/2.25$, $M_{12}=M_{32}=1-\dot{x}^2\left(t\right)/2.25$, $M_{22}=M_{42}=\dot{x}^2\left(t\right)/2.25$, and the consequent parts are chosen as

$$A_1 = \begin{bmatrix} 0 & -0.02 \\ 1 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1.0 \\ 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -0.225 & -0.02 \\ 1 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1.0 \\ 0 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & -1.5275 \\ 1 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1.0 \\ 0 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} -0.225 & -1.5275 \\ 1 & 0 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 1.0 \\ 0 \end{bmatrix}.$$

In this example, the system is not solvable therefore there are no information about the number of switching. In this situation, the numerical reachable set is difficult obtained and computation cost is high. We design the controller by purposed controller on maximal convergency rate. With all the ellipsoids satisfying the set invariance condition in Lemma 1, we have

$$P = \begin{bmatrix} 0.0252 & -0.0131 \\ -0.0131 & 0.0179 \end{bmatrix}.$$

and $\alpha = 2$. The ellipsoids as the largest level set are depicted in Fig 30.

Case I

In the case, the saturation control in [46] is introduced to compare our results. The initial point is $x = [1,\ -0.5]^T$. In Fig. 31, the states are converged by saturation control over 35 (sec.). The saturation control input is depicted in Fig. 32.

Case II

Let the initial point as Case I, the maximal convergency rate control is considered in this Case. The states converge at 1.2 (sec.) and depicted in Fig. 33. The corresponded control input is depicted in Fig. 34. The convergence rate of states is expected faster then Case I. We can conclude that the system has faster response by the maximal convergency rate control. Obviously, the sign function is sensitivity when the states approach the original. This phenomenon is called chartering. Since that the sign function is sensitivity when the states approach the original (switching plane). To overcome this phenomenon, we combine two approach in Case I and II and demonstrated in the following.

Case III

In this case, the mixed control is applied for overcoming the chartering phenomenon. At first, the he maximal convergency rate control is adopted for fast response and then the saturation control is applied when the states approach the switching plane. In this case, we consider the following control strategy:

$$U(t)=\begin{cases}-SAT\left(B^{T}PX\right), & \|\mathrm{x}\|<0.01\\ -SGN\left(B^{T}PX\right), & other.\end{cases}$$

The trajectory is depicted in Fig. 35 and control input is depicted in Fig. 36. We can conclude that the system has fast property by the maximal convergence rate control and smooth when approach the switching plane.

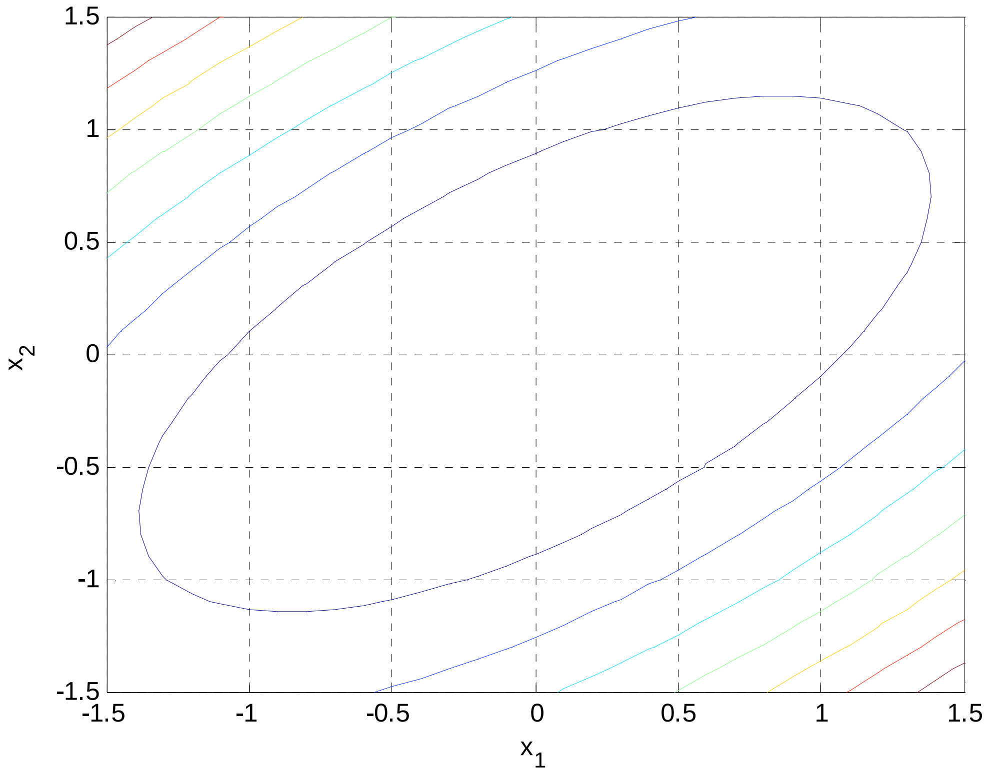

Fig. 30 The level set of Example 6.

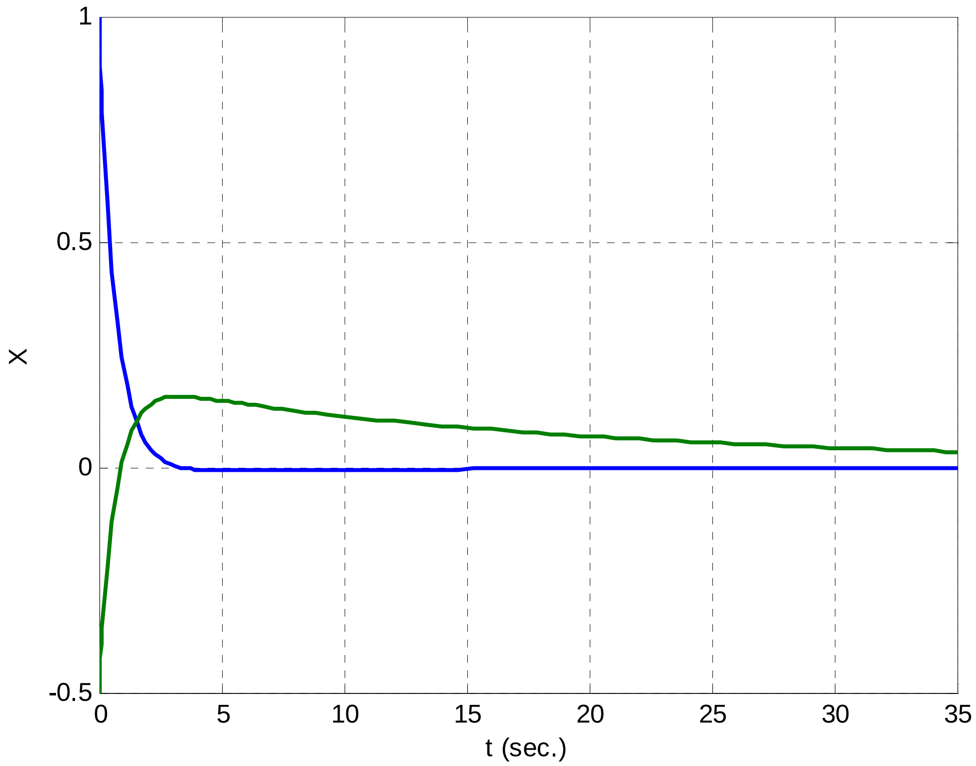

Fig. 31 The trajectory in phase plane (Case I).

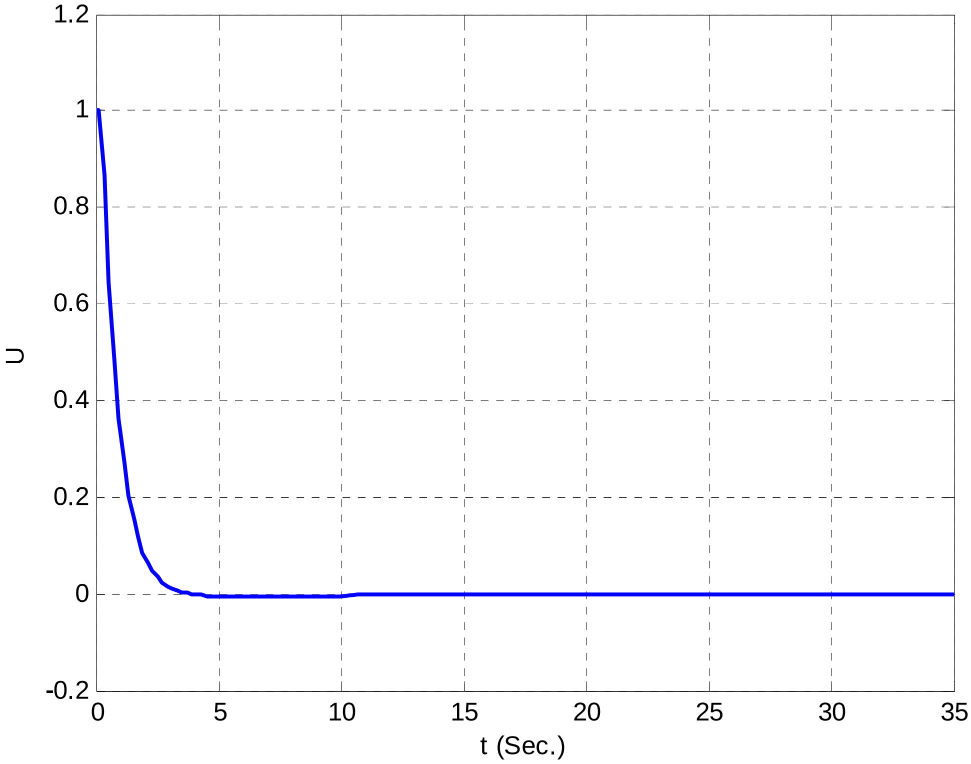

Fig. 32　Corresponded control input (Case I).

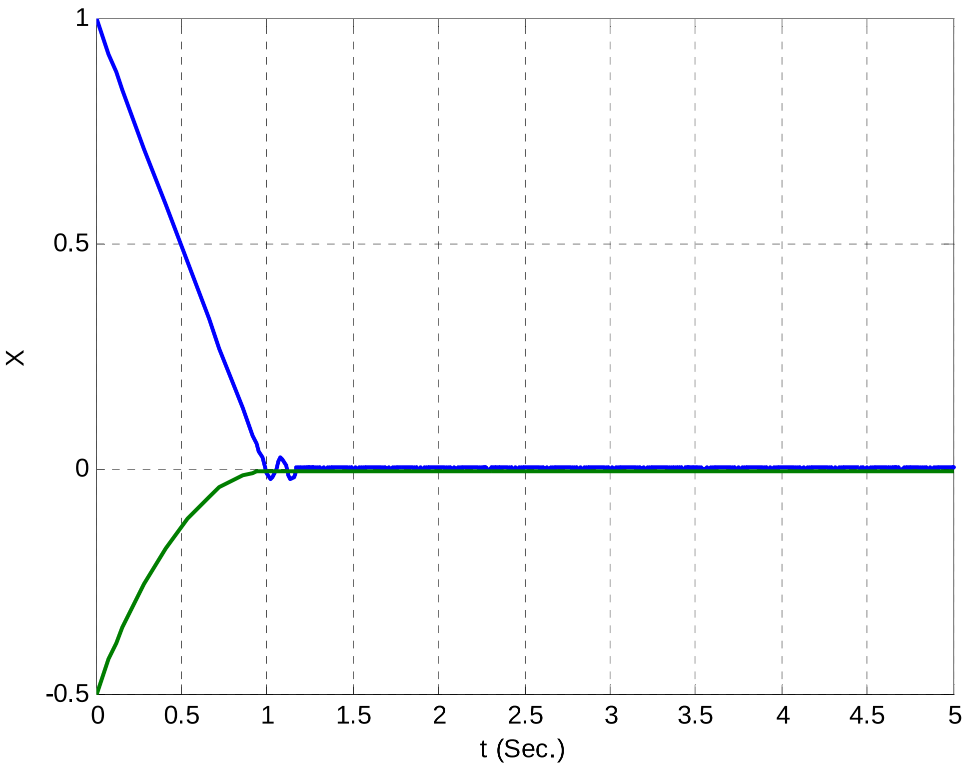

Fig. 33　The trajectory in phase plane (Case II).

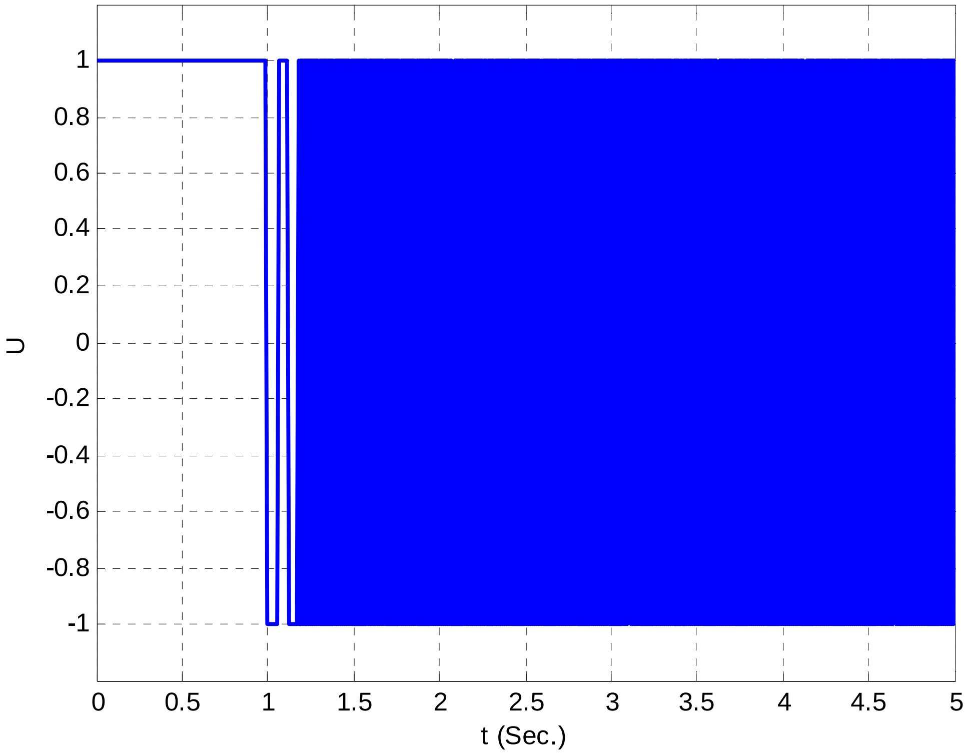

Fig. 34　Corresponded control input (Case II).

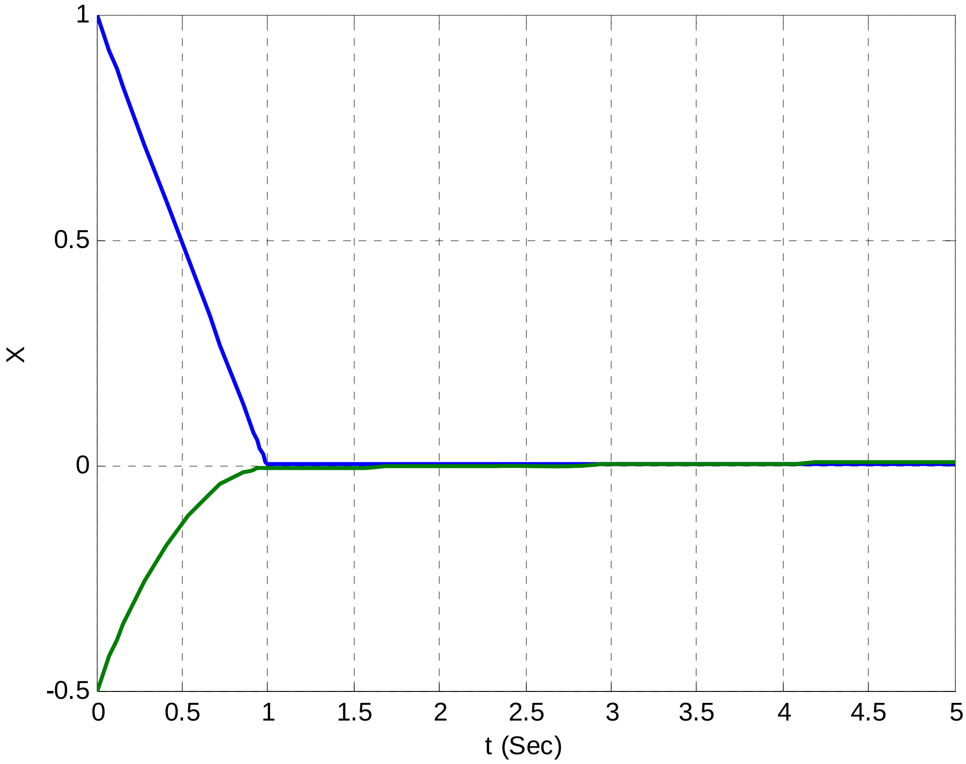

Fig. 35　The trajectory in phase plane (Case III).

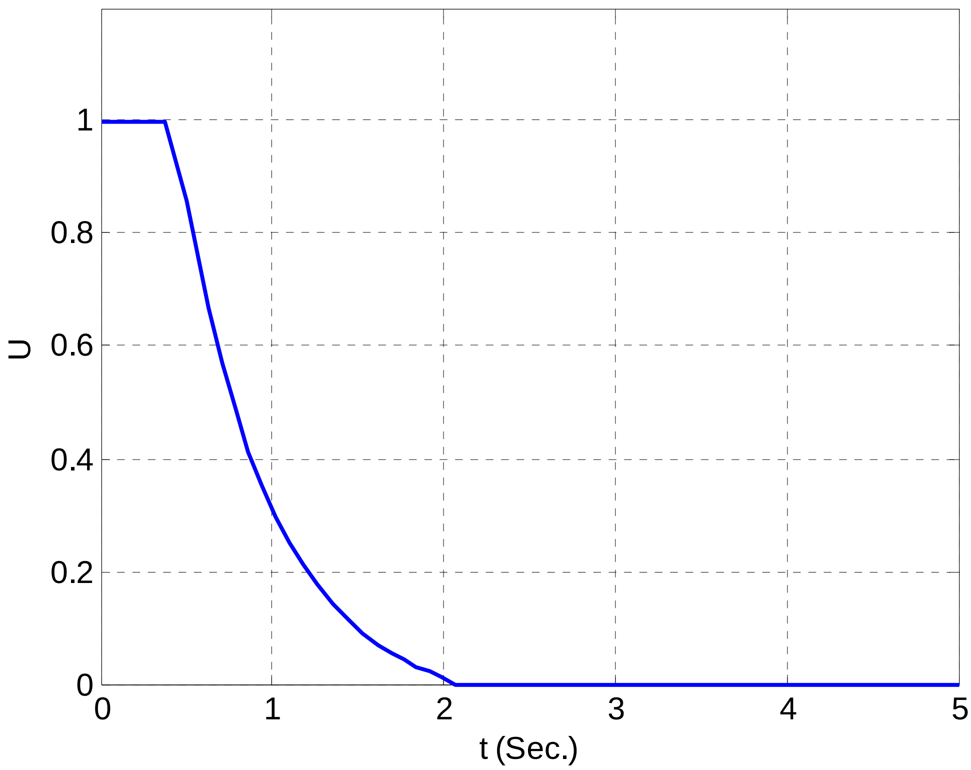

Fig. 36 Corresponded control input (Case III).

Chapter 5
Conclusion

This dissertation presents a new design of time-optimal controller for controllable Takagi-Sugeno (T-S) fuzzy model in which the maximum principle is applied. In particular, the subsystems of T-S fuzzy model are blended by a set of firing strengths, which leads it to a class of nonlinear system. First, we proposed the proof of the existence of optimal control in T-S fuzzy model, which can be addressed as the compactness of reachable set. The generalized rank condition of accessible Lie algebra is also applied for the proof of the existence of optimal controller for T-S fuzzy model. This also results in the controllability of the T-S fuzzy model. According to the maximum principle, the time-optimal control of T-S fuzzy model is *bang-bang* which is determined by switching function. By investigating the singular structure of the switching functions of the controllable T-S fuzzy model, we can yield the conditions for the existence, i.e., *if the extremal* is *normal then there exists the time-optimal controller for the T-S fuzzy model.* In other words, the time-optimal control of controllable T-S fuzzy model is *bang-bang* with finite number of switching over all trajectories for all t. The bounded number of switching is related to the polynomial degree of switching function which is obtained by introducing solvable Lie algebra. Several examples are fully illustrated to show the conditions for the existence of time-optimal controller with their optimal trajectories found by numerical simulation. Further, the feedback controller design of T-S fuzzy model on maximal convergence rate is introduced by level set function. The result of maximizing the convergence rate is characterized from the maximal invariant ellipsoid. The controller is also bang-bang with a simple switching strategy. To handle the chartering phenomenon, a two stages control of saturation and maximizing the convergence rate is also demonstrated. Numerical simulations show the system response is fast and control input is smooth.

References

[1] K. Tanaka and M. Sano, "A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck-trailer" *IEEE Transactions on Fuzzy Systems*, vol. 2, pp.119-134, May 1994.

[2] K. Tanaka, T. Ikeda, and H. O. Wang, "Robust stabilization of a class of uncertain nonlinear system via fuzzy control: Quadratic stabilizability, H^{∞} control theory and linear matrix inequalities," *IEEE Transactions on Fuzzy Systems*, vol. 4, pp. 1-13, Feb. 1996.

[3] A. E. Gegov and P. M. Frank, "Hierarchical fuzzy control of multivariable systems," *Fuzzy Sets and Systems*, vol. 72, no. 3 pp. 299-310, 1995.

[4] C. Lee, "Fuzzy logic in control systems: Fuzzy logic controller Part 1, Part 2," *IEEE Transaction on Systems, Man, and Cybernetics, Part B*, vol. 20, pp. 404-435, 1990.

[5] J. I. Horiuchi and M. Kishimoto, "Application of fuzzy control to industrial bioprocesses in Japan," *Fuzzy Sets and Systems*, vol. 128, no. 1 pp. 117-124, 2002.

[6] C. C. Hsiao, S. F. Su, T. T. Lee, and C. C. Chuang, "Hybrid compensation control for affine TSK fuzzy control systems," *IEEE Transaction on Systems, Man, and Cybernetics, Part B*, 2004.

[7] Y.G. Leu, T.T. Lee and W.Y. Wang, "Observer-based adaptive fuzzy-neural control for unknown nonlinear dynamical systems," *IEEE Transaction on Systems, Man, and Cybernetics, Part B*, vol. 29, no.5, and pp.583-591, Oct. 1999.

[8] H. O. Wang, K. Tanaka, Michael F. Griffin, "An approach to fuzzy control of

nonlinear systems: Stability and design issues," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 1 pp. 14-23, Feb. 1996.

[9] T. Takagi and M. Sugeno, "Fuzzy identification of systems and it's Applications to modeling and control," *IEEE Transactions on System Man and Cybernetics*, vol. 15, pp. 116-132 no. 1, 1985.

[10] S. Sugeno and G. T. Kang, "Structure identification of fuzzy model," *Fuzzy Sets and Systems*, vol. 28, no. 10, pp. 15-33, 1988.

[11] T. Takagi and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets and Systems*, vol. 45, no. 2, pp. 135-156, 1993.

[12] M. C. M. Teixerira and S.H. Zak, "Stabilization controller design for uncertain nonlinear systems using fuzzy *models*," *IEEE Transactions on Fuzzy Systems*, vol. 7, pp. 133-142, Apr. 1999.

[13] S. G. Cao, N. W. Rees, and G. Feng, "H-infinity control of nonlinear continuous-time system based on dynamical fuzzy models," *Int. J. Syst. Sci.*, vol. 27, pp. 821-830, 1996.

[14] K. Tanaka, T. Ikeda, and H. O. Wang, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs," *IEEE Transactions on Fuzzy Systems*, vol. 6, pp. 250-265, 1998.

[15] B.S. Chen, C.S. Tseng, H.J. Uang, "Mixed H_2/H_∞ fuzzy output feedback control design for nonlinear dynamic systems: An LMI approach," *IEEE Transactions on Fuzzy Systems*, vol. 8, pp. 249-265, 2000.

[16] H. J. Lee, J.B. Park, and G. Chen, "Robust fuzzy control of nonlinear system with parametric uncertainties," *IEEE Transactions on Fuzzy Systems*, vol. 9, pp. 369-379, 2001.

[17] G. Feng, "Approaches to quadratic stabilization of uncertain continuous time fuzzy dynamic systems," *IEEE Transactions on Circuits Systems Part I*, vol. 48,

pp. 369-379, 2001.

[18] K. R. Lee, E.T. Jeung, and H. B. Park, "Robust fuzzy H_∞ control for uncertain nonlinear systems via state feedback: An LMI approach," *Fuzzy Sets and Systems*, vol. 120, pp. 123-134, 2001.

[19] Piotr Kulczycki, "Fuzzy controller for mechanical systems," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 5, pp. 645-652, Oct 2000.

[20] Pontryagin, L. S., V. Boltyanskii, T. Gamkrelideze, and E. Mishchenko, "*The mathematical theory of optimal processes*," Interscience Publishers, Inc., New York, 1962.

[21] Michael Athans and Peter L. Palb, *Optimal Control*, McGraw-hill Inc., 1966

[22] Henry Hermes and Joseph P. Lasalle, "*Functional Analysis and Time Optimal Control*," Academic Press, 1969.

[23] Jack Macki and Aaron Strauss, *Introduction to Optimal Control Theory*, Springer-Verlag, New York, 1982.

[24] Donald E. Kirk, *Optimal Control Theory: An Introduction*, Prentice-Hall, 1970.

[25] Bronislaw Jakubczyk and Witold Respondek, *Geometry of Feedback and Optimal Control*, Marcel Dekker Inc., 1998.

[26] H. J. Sussmann, *Differential Geometric Control Theory*, Birkhäuser, Boston, MA, 1983.

[27] H. J. Sussmann, *The Maximum Principle of Optimal Control Theory*, J. Baillieul, J.C. Willems(Eds), Springer-Verlag, pp. 140-198, 1998.

[28] H. J. Sussmann and V. Jurdjevic, "Controllability of nonlinear system," *Journal of Diff. Equations*, 12,95, 1972.

[29] E.D. Sontag and H. J. Sussmann, "Time-optimal control of manipulators," *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 1692-1697, 1986.

[30] E.D. Sontag, "Remarks on the time-optimal control of a class of Hamiltonian

systems," *IEEE Conf. on Decision and Control*, pp. 317-221, 1989.

[31] M. Chyba, N.E. Leonard and E.D. Sontag, "Singular trajectories in multi-input time-optimal problems: Application to controlled mechanical," *Journal on Dynamics and Control Systems*, vol. 9, pp. 73-88, 2003.

[32] A.D. Lewis, *The Geometry of the Maximum Principle for Affine Connection Control Systems*, Preprint, 2000.

[33] A. A. Agrachev, *Mathematical Control Theory,* A. A. Agrachev (Eds), Steklov Mathematical Institue, Moscow and SISSA, Trieste.

[34] R. Vinter, *Optimal Control*, Birkhäuser, Boston, MA, 2000.

[35] R. W. Brockett, "Nonlinear systems and differential geometry," *Proc. IEEE,* vol. 64, no. 1, pp. 61-71, 1976.

[36] Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory,*" SIAM, Philadelphia, PA, 1994.

[37] J.-P. Aubin and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory,* Springer-Verlag, Berlin, 1984.

[38] Hans Samelson, *Note on Lie Algebras,* Van Nostrand Reinhold, New York, 1969

[39] Olver, P.J., *Application of Lie Groups to Differential Equations*," 2nd ed., Springer-Verlag, 1993

[40] Lastman, G. J., "A Shooting Method For Solving Two-Point Boundary-Value Problems Arising From Nonsingular Bang--bang Optimal Control Processes," *International Journal of Control*, vol. 27, pp. 513-524, 1978.

[41] Zhendong Sun1, S.S. Gee and T.H. Lee, "Controllability and reachability criteria for switched linear systems," *Automatica,* 38, pp. 775-786, 2002.

[42] Guangming Xie, Dazhong Zheng and Long Wang, "Controllability of switched linear systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 4, pp. 1401-1405, 2002.

[43] Daizhan Cheng, "Controllability of switched bilinear systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 4, pp. 511-515, 2005.

[44] Alberto Bemporad, Giancarlo Ferrari-Trecate, and Manfred Morari, "Observability and controllability of piecewise affine and hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 4, pp. 1864-1876, 2000.

[45] L. Habets and J.H. van Schuppen, "A controllability result for piecewise-linear hybrid systems," *Proceedings of European Control Conference*, 2001.

[46] Tingshu Hu and Zongli Lin, *Control Systems with Actuator Saturation*, Birkhäuser, Boston, MA, 2001

Vita

Name: Pao-Tsun Lin

Personal:

Place of Birth: Hsinchu County, Taiwan, R.O.C.

Day of Birth: January 14, 1976

Gender: Male

Education background:

Degree	Date	School
B.S. E.E.	1997/9 - 1999/6	National Taiwan University of Science and Technology, Department of Electronic Engineering
M.S. E.E.	1999/9 - 2001/6	National Taiwan University of Science and Technology, Department of Electronic Engineering
Ph.D. E.C.E.	2001/9 - 2008/7	National Chiao Tung University, Department of Electrical and Control Engineering

Advisor(s):

M.S. -

Dr. Shun-Feng Su, National Taiwan University of Science and Technology

Ph.D. –

Dr. Chi-Hsu Wang, National Chiao Tung University

Dr. Tsu-Tian Lee, National Taipei University of Technology

Publication List

Accepted Journal Paper:

[1] Pao-Tsun Lin, Chi-Hsu Wang and Tsu-Tian Lee, "Time-Optimal Control of T-S Fuzzy Models via Lie Algebra" accepted to be published in *IEEE Transactions on Fuzzy Systems*.

International Conference Papers:

[1] Pao-Tsun Lin, T. T. Lee and Chi-Hsu Wang," Accessibility of T-S Fuzzy Models via Lie Algebra," accepted to be published in *2008 National Symposium on System Science and Engineering*.

[2] Pao-Tsun Lin, T. T. Lee and Chi-Hsu Wang," Analysis of Time-Optimal Problem in T-S Fuzzy Model via Lie Algebra," *Proc. of The 26th IASTED International Conference on Modelling, Identification, and Control*, pp. 749-755 2007

[3] Pao-Tsun Lin, Shun-Feng Su, and Tsu-Tian Lee, "Time-optimal fuzzy control based on computation method," in *Proc. of 2005 IEEE International Conference on Systems, Man and Cybernetics*, Vol. 1. pp. 600- 605, Oct. 2005.

[4] Pao-Tsun Lin, Shun-Feng Su, and Tsu-Tian Lee, "Support vector regression performance analysis and systematic parameter selection," in *Proc. of 2005 IEEE International Joint Conference on Neural Network*, Vol. 2. pp. 877- 882, Aug. 2005.

[5] Pao-Tsun Lin, Shun-Feng Su, and Tsu-Tian Lee, "Time-optimal control via fuzzy approach," in *Proc. of 2004 IEEE International Conference on Systems, Man and Cybernetics*, Vol. 4. pp. 3817- 3821, Oct. 2004.