Chapter 1
Introduction

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with correspondence between Leibniz and L'Hospital in 1695, where the meaning of derivative of order one half was discussed [1-4]. Although fractional calculus has a 300-year-old history, its applications to physics and engineering are just a recent focus of interest [34-38]. It was found that many systems in interdisciplinary fields can be described by the fractional differential equations, such as viscoelastic systems, dielectric polarization [5], electrode electrolyte polarization [6], and electromagnetic waves [7]. More recently, many investigations are devoted to the control [8-12] and dynamics [13-25] of fractional order dynamical systems. In [13], it is shown that the fractional order Chua's circuit of order as low as 2.7 can produce a chaotic attractor. In [14], it is shown that nonautonomous Duffing systems of order less than 2 can still behave in a chaotic manner. In [15], chaotic behaviors of the fractional order "jerk" model is studied, in which chaotic attractor can be obtained with the system order as low as 2.1, and in [16] chaos control of this fractional order chaotic system is investigated. In [17], the fractional order Wien bridge oscillator is studied, where it is shown that limit cycle can be generated for any fractional order, with a proper value of the amplifier gain.

In 1990, the idea of synchronizing two identical chaotic systems with different initial conditions was introduced by Pecora and Carroll [38]. Since then, there has been particular interest in chaotic synchronization, due to many potential applications in secure communication, biological science, chemical reaction, social science, and many other fields. The concept of synchronization has been extended to the scope, such as complete synchronization (CS), phase synchronization (PS), lag synchronization (LS), anticipated synchronization (AS), and generalized synchronization (GS), etc [3-7, 37-55, 60-66]. However most of synchronizations can only be realized under the condition that there exists coupling between two chaotic systems. Sometimes, it is difficult even impossible to couple two chaotic systems such as in physical and electrical systems. In comparison with coupled chaotic systems, synchronization between the uncoupled chaotic systems has many advantages [45-46, 55-60]. In this thises, synchronization of two double Duffing systems whose corresponding parameters are driven by a chaotic signal of a third system is analyzed. The chaos synchronizations of two uncoupled double Duffing systems are obtained by replacing their corresponding parameters by the same function of chaotic state variables of a third chaotic system. It is noted that whether CS or AS appears depends on the initial conditions. Besides, CS and AS are also characterized by great sensitivity to initial conditions and on the strengths of the substituting chaotic variable. It is found that CS or AS alternatively occurs under certain conditions [5-7, 14, 19]

Then we focus on the synchronization and antisynchronization of two identical double Duffing systems whose corresponding parameters are replaced by a white noise, a Rayleigh noise, a Rician noise or a uniform noise respectively. It is noted that whether CS or AS appears depends on the driving strength [3, 6, 19-20, 66-67].

In practice, some or all of the system parameters are uncertain. Moreover, these parameters change from time to time. Many researchers solve this problem by adaptive synchronization [68-73]. In current scheme of adaptive synchronization, traditional Lyapunov asymptotical stability theorem and Babalat lemma are used to prove the errors of synchronizing states approach zero. But the question that why the estimated parameters also approach the uncertain values, has still remained without answer. By the pragmatical asymptotical stability theorem [74-75] and an assumption of equal probability for ergodic initial conditions, the answer can be given.

Among many kinds of synchronizations, the generalized synchronization is investigated [76-88]. It means there exists a given functional relationship between the states of the master and that of the slave $y = G(x)$, where x, y are the states vector of master system and slave system respectively. In this paper, a special kind of generalized synchronizations

$$y = G(x) = x + F(t) \tag{1}$$

is studied, where $F(t)$ is a given vector function of time which may take various forms, either regular or chaotic function of time. When $F(t) = 0$, it reduces to a complete synchronization [89-90].

As a numerical example, two identical double Duffing chaotic systems [91] and a double van der Pol chaotic system [92-93] are used as master system, slave system, and goal system, respectively. The goal system gives chaotic $F(t)$. Next, the robustness of the generalized synchronization is also studied [94-100].

This thesis is organized as follows. In Chapter 2 the fractional derivative and its approximation are introduced. And then gives the dynamic equation of double Duffing system. The system under study is described both in its integer and fractional forms. Numerical simulation results are presented. In Chapter 3, a brief description of synchronization scheme based on the substitution of the strengths of the mutual coupling term of two identical chaotic double Duffing systems by the chaotic variable of a third double Duffing system are presented. And numerical simulations are given for illustration. It is found that one can obtain CS or AS by adjusting the driving strength and initial conditions. In Chapter 4, chaos synchronization and antisynchronization are obtained by replacing two corresponding parameters of two uncoupled identical double Duffing chaotic dynamical systems by a white noise, a Rayleigh noise, a Rician noise or a uniform noise respectively. It is found that one can obtain CS or AS by adjusting the driving strength.

In Chapter 5, theoretical analyses of the pragmatical asymptotical stability are quoted [74-75]. Adaptive controllers are designed for the pragmatical generalized synchronization of two double Duffing chaotic oscillators with a double van der Pol chaotic system as a goal system. High robustness of the generalized synchronization is also obtained in Chapter 5.

Finally, conclusions follow sequentially in Chapter 6.

Chapter 2

Chaos in Integral Order and in Fractional Order Double Duffing Systems

2.1 Preliminaries

In this chapter, the fractional derivative and its approximation are introduced. And gives the dynamic equation of double Duffing system. The system under study is described both in its integer and fractional forms. Numerical simulation results are presented.

2.2 Fractional derivative and its approximation

Two commonly used definitions for the general fractional differintegral are the Grunwald definition and the Riemann-Liouville definition . The Riemann-Liouville definition of the fractional integral is given here as [26]

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(-q)} \int_0^t \frac{f(\tau)}{(t-\tau)^{q+1}} d\tau, q < 0 \tag{2.1}$$

where q can have noninteger values, and thus the name fractional differintegral. Notice that the definition is based on integration and more importantly is a convolution integral for $q < 0$. When $q > 0$, then the usual integer nth derivative must be taken of the fractional $(q-n)$th integral, and yields the fractional derivative of order q as

$$\frac{d^q f}{dt^q} = \frac{d^n}{dt^n}\left[\frac{d^{q-n} f}{dt^{q-n}}\right], \qquad q>0 \text{ and n an integer} >q \tag{2.2}$$

This appears so vastly different from the usual intuitive definition of derivative and integral that the reader must abandon the familiar concepts of slope and area and attempt to get some new insight. Fortunately, the basic engineering tool for analyzing linear systems, the Laplace transform, is still applicable and works as one would expect; that is,

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{q-1-k} f(t)}{dt^{q-1-k}}\right]_{t=0}, \text{ for all q} \tag{2.3}$$

where n is an integer such that $n - 1 < q < n$. If the initial conditions are considered to be zero, this formula reduces to the more expected and comforting form

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} \tag{2.4}$$

An efficient method is to approximate fractional operators by using standard integer order operators. In [26], an effective algorithm is developed to approximate fractional order transfer functions. Basically, the idea is to approximate the system behavior in the frequency domain. By utilizing frequency domain techniques based on Bode diagrams, one can obtain a linear approximation of fractional order integrator, the order of which depends on the desired bandwidth and discrepancy between the actual and the approximate magnitude Bode diagrams. In Table 1 of [13], approximations for $1/s^q$ with q=0.1~0.9 in steps 0.1 are given, with errors of approximately 2dB. These approximations are used in following simulations.

2.3 A fractional order double Duffing system

The famous Duffing system is

$$\ddot{x} + a\dot{x} + bx + cx^3 = d\cos\omega t \tag{2.5}$$

where a, b are constant parameters, $d\cos\omega t$ is an external excitation.

It can be written as two first order ordinary differential equations :

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -ay - bx - cx^3 + d\cos\omega t \end{cases} \tag{2.6}$$

Consider the following double Duffing system:

$$
\begin{cases}
\dfrac{dx}{dt} = y \\
\dfrac{dy}{dt} = -ay - bx - cx^3 + du \\
\dfrac{du}{dt} = v \\
\dfrac{dv}{dt} = -ev - gu - hu^3 + kx
\end{cases}
\tag{2.7}
$$

It consists of two Duffing systems in which two external excitations are replaced by two coupling terms. It is an autonomous system with four states where *a*, *b*, *c*, *d*, *e*, *g*, *h*, and *k* are constant parameters of the system.

Now, consider a fractional order Duffing system. Here, the conventional derivatives in Eq.(2.7) are replaced by the fractional derivatives as follows:

$$
\begin{cases}
\dfrac{dx^{q_1}}{dt} = y \\
\dfrac{dy^{q_2}}{dt} = -ay - bx - cx^3 + du \\
\dfrac{du^{q_3}}{dt} = v \\
\dfrac{dv^{q_4}}{dt} = -ev - gu - hu^3 + kx
\end{cases}
\tag{2.8}
$$

where system parameter *d* is allowed to be varied, and q_1, q_2, q_3 and q_4 are four fractional order numbers. Simulations are then performed using q_i ($i = 1 \sim 4$) varied from 0.1~0.9. We take $q_1 = q_2 = q_3 = q_4$. The approximations from Table 1 of [13] are used for the simulations of the appropriate q_i th integrals. When $q_i < 1$, then the approximations are used directly. It should further be noted that approximations used in the simulations for $1/s^{q_i}$, when $q_i > 1$, are obtained by using 1/*s* times the approximation for $1/s^{q_i - 1}$ from Table 1.

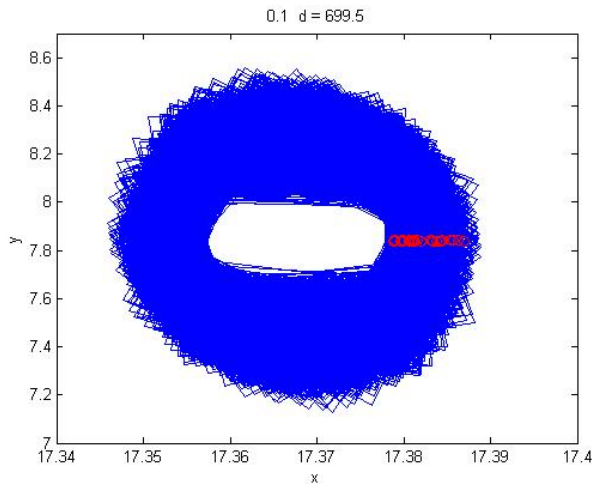
2.4 Simulation results

In this Section, all numerical simulations are run by block diagrams in Simulink environment, using ODE45 solver algorithm, where the fractional integrators have been approximated by linear time invariant transfer functions following the procedure in [13]. In so far as the attractor shape is concerned, both procedures gave very similar results. In numerical simulations, three parameters $a = 0.05$, $b = 1$, $c = 1$, $e = 0.0005$, $g = 1$, $h = 1$ and $k = 25$ are fixed and d is varied.

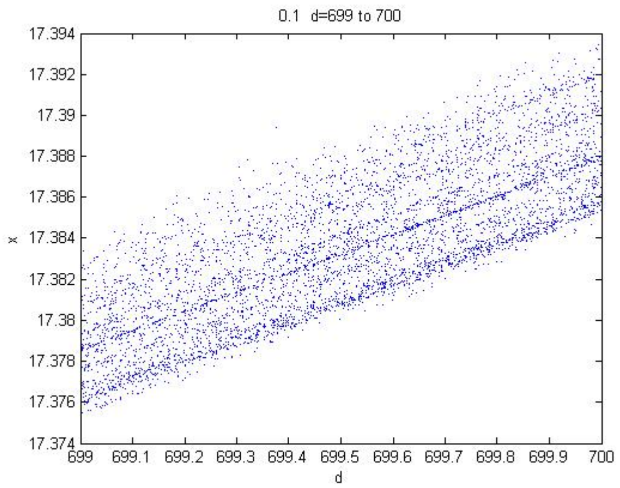
Table 1. Relation between orders of derivatives and existences of chaos.

Cases	Orders	Existence of chaos
1	$q_i = 0.1$	Yes
2	$q_i = 0.2$	Yes
3	$q_i = 0.3$	Yes
4	$q_i = 0.4$	Yes
5	$q_i = 0.5$	Yes
6	$q_i = 0.6$	Yes
7	$q_i = 0.7$	Yes
8	$q_i = 0.8$	No
9	$q_i = 0.9$	No
10	$q_i = 1$	Yes

For $q_i = 0.1 \sim q_i = 1$, chaotic behaviors are found.. The corresponding phase portraits, Poincaré maps and the bifurcation diagrams are shown in Fig. 2.1 ~ Fig. 2.8. It can be seen that when q_i is larger, the range of d state is also larger. when d is larger, the phase portrait move to the positive direction of x and y axis.

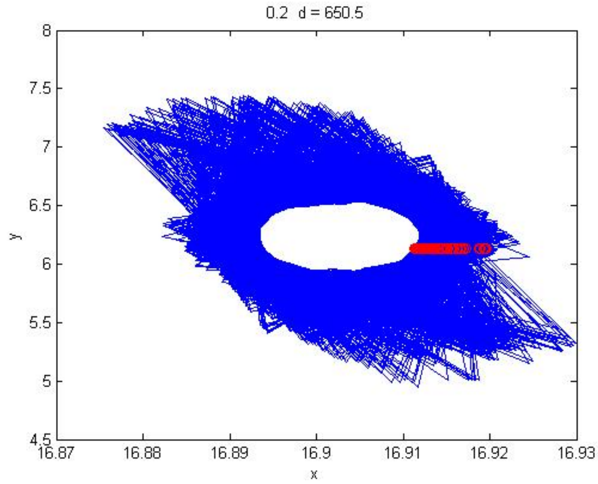


(a) d = 699.5

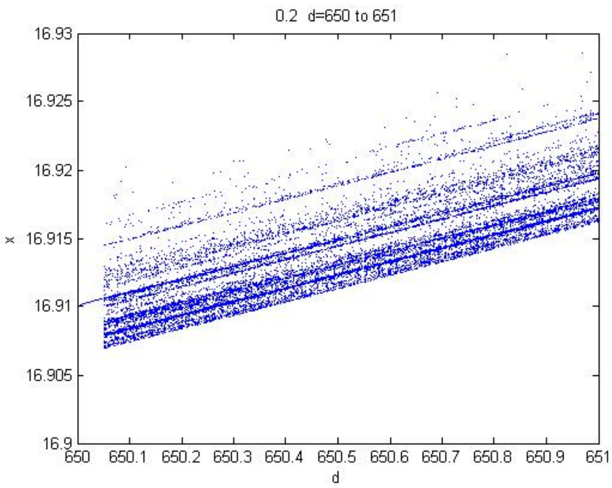


(b) d = 699 ~ 700

Fig. 1 The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order double Duffing system, *x* versus *y* and *d* versus q_i =0.1.

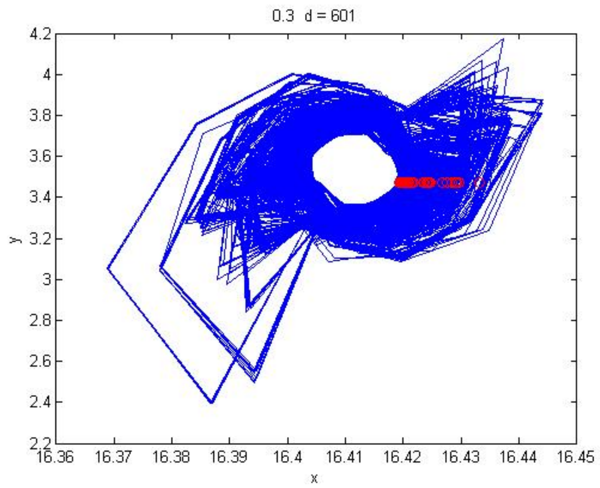


(a) d = 650.5

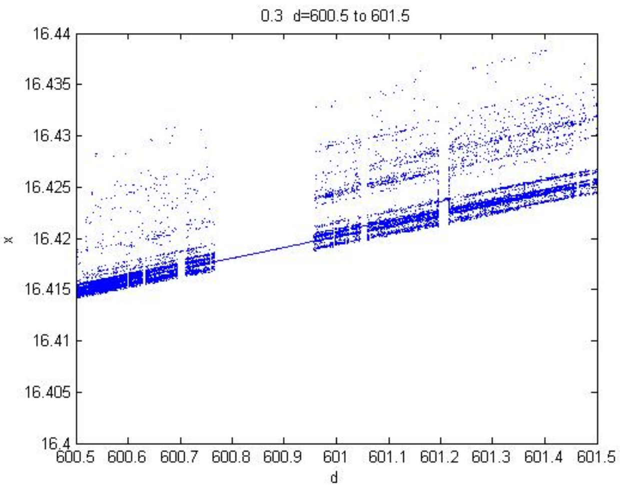


(b) d = 650 ~ 651

Fig. 2 The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order double Duffing system, *x* versus *y* and *d* versus q_i =0.2.

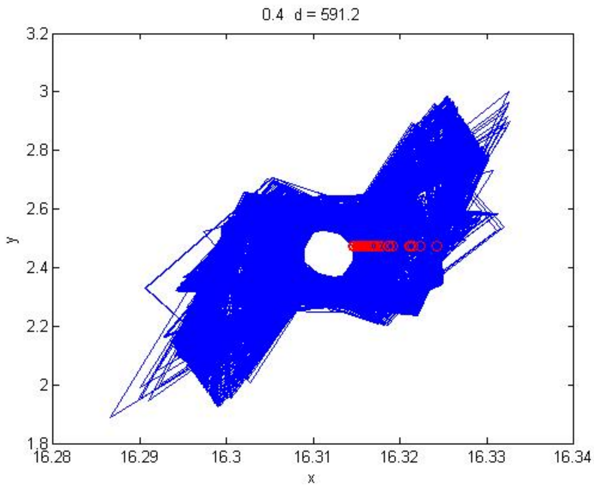


(a) d = 601

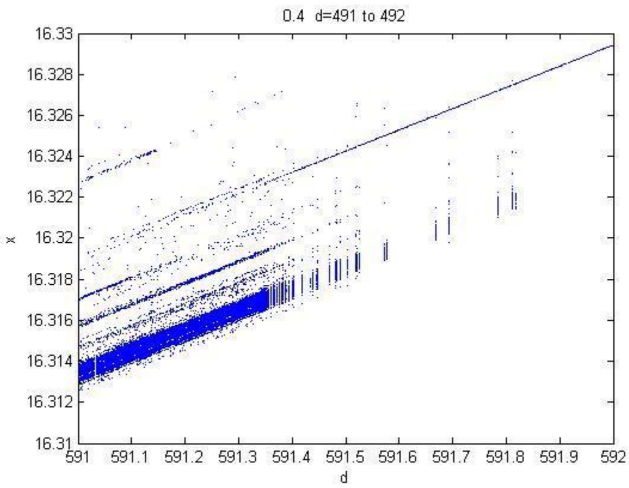


(b) d = 600.5 ~ 601.5

Fig. 3 The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order double Duffing system, *x* versus *y* and *d* versus q_i =0.3.

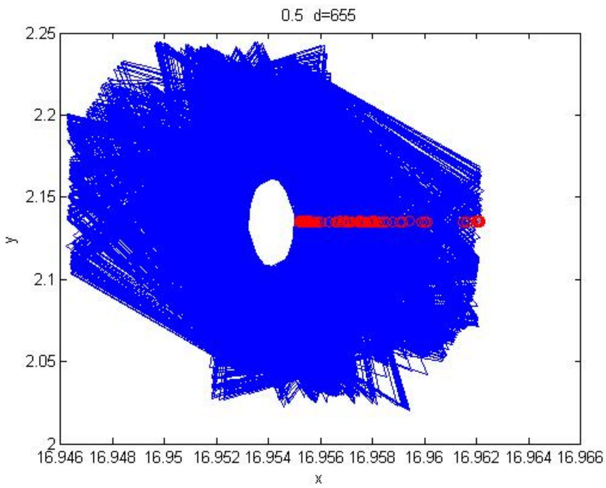


(a) d = 591.2

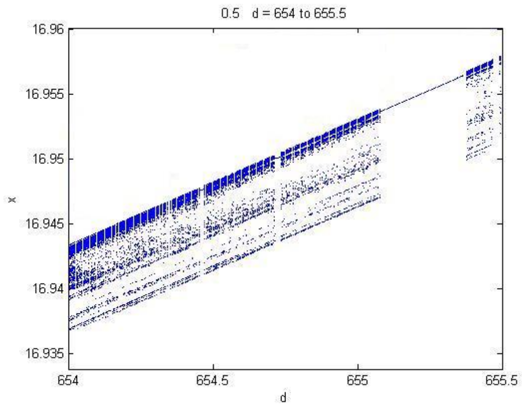


(b) d = 591 ~ 592

Fig. 4 The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order double Duffing system, *x* versus *y* and *d* versus q_i=0.4

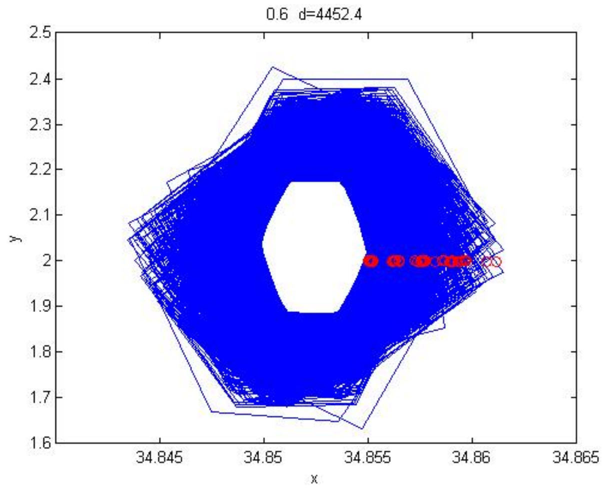


(a) d = 655

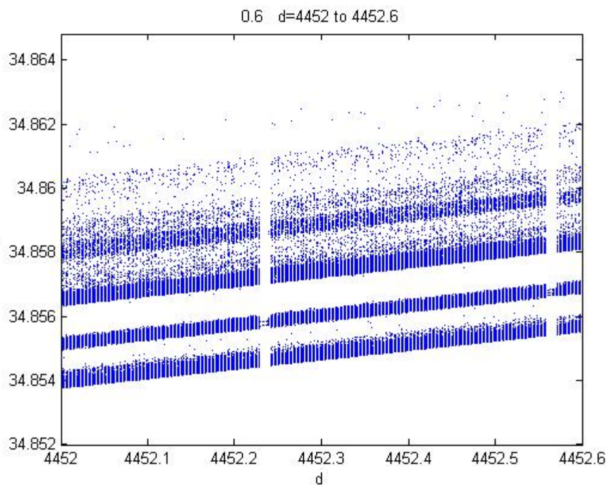


(b) d = 655

Fig. 5 The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order double Duffing system, x versus y and d versus q_i=0.5.

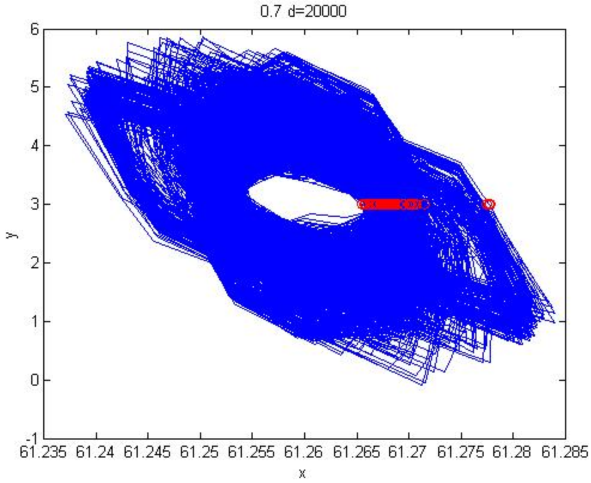


(a) d = 4452.4

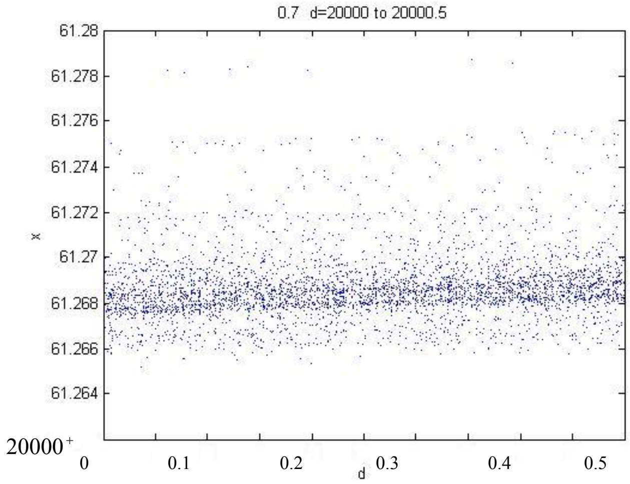


(b) d = 4452 ~ 4452.6

Fig. 2.6 The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order double Duffing system, x versus y and d , q_i =0.6.

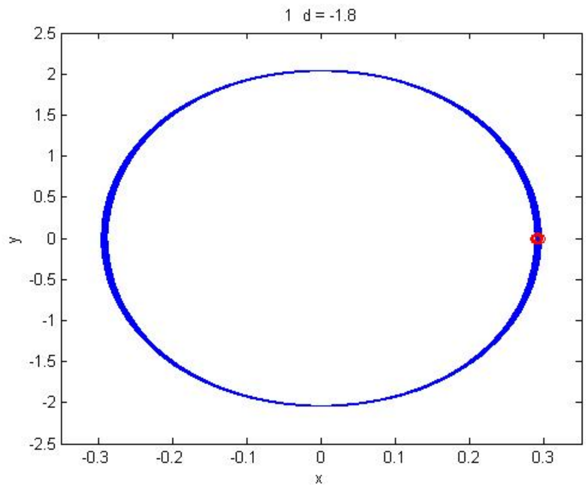


(a) d = 20000

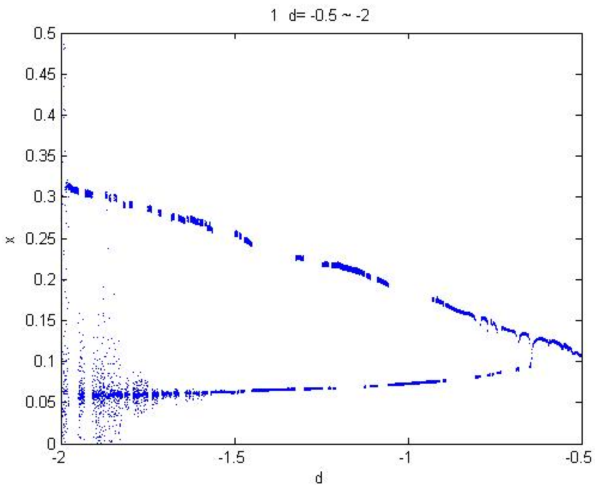


(b) d = 20000 ~ 20000.5

Fig. 2.7 The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order double Duffing system, *x* versus *y* and *d* , q_i =0.7.



(a) d = -1.8



(b) d = -2 ~ -0.5

Fig. 2.8 The phase portraits, Poincaré maps and the bifurcation diagram for the fractional order double Duffing system, x versus y and d, q_i =1.

Chapter 3

Chaos Synchronization of Double Duffing Systems with Parameters Excited by a Chaotic Signal

3.1 Preliminaries

A brief description of synchronization scheme based on the substitution of the strengths of the mutual coupling term of two identical chaotic double Duffing systems by a function of chaotic state variables of a third double Duffing system are presented. And numerical simulations are given for illustration. It is found that one can obtain CS or AS by adjusting the driving strength and initial conditions.

3.2 Synchronization of two double Duffing systems

It consists of two Duffing systems in which two external excitations are replaced by two coupling terms. It is an autonomous system with four states where *a*, *b*, *c*, *d*, *e*, *g*, *h*, and *k* are constant parameters of the systems. Two identical double Duffing systems to be synchronized are

$$\begin{cases} \dfrac{dx_1}{dt} = y_1 \\ \dfrac{dy_1}{dt} = -ay_1 - bx_1 - cx_1^3 + d_1u_1 \\ \dfrac{du_1}{dt} = v_1 \\ \dfrac{dv_1}{dt} = -ev_1 - gu_1 - hu_1^3 + kx_1 \end{cases} \tag{3.1}$$

$$
\begin{cases}
\dfrac{dx_2}{dt} = y_2 \\
\dfrac{dy_2}{dt} = -ay_2 - bx_2 - cx_2^{\ 3} + d_2 u_2 \\
\dfrac{du_2}{dt} = v_2 \\
\dfrac{dv_2}{dt} = -ev_2 - gu_2 - hu_2^{\ 3} + kx_2
\end{cases}
\tag{3.2}
$$

where *a, b, c, d, e, g, h* and *k* are positive scalars, and $d_1 = d_2$ are the control inputs to be designed. The third system is also a double Duffing system :

$$
\begin{cases}
\dfrac{dx_3}{dt} = y_3 \\
\dfrac{dy_3}{dt} = -ay_3 - bx_3 - cx_3^{\ 3} + du_3 \\
\dfrac{du_3}{dt} = v_3 \\
\dfrac{dv_3}{dt} = -ev_3 - gu_3 - hu_3^{\ 3} + kx_3
\end{cases}
\tag{3.3}
$$

In order to obtain the chaos synchronization of systems (3.1) and (3.2), two corresponding parameters $d_1 = d_2$ of them are replaced by a chaotic signal $px_3 + qy_3$ of the third system (3.3), where *p*, *q* are constant driving strengths. The error state variables are defined :

$$
\begin{cases}
e_1 = x_1 - x_2 \\
e_2 = y_1 - y_2 \\
e_3 = u_1 - u_2 \\
e_4 = v_1 - v_2
\end{cases}
\tag{3.4}
$$

Giving suitable values for *p*, *q* and initial conditions, the synchronization or anti-synchronization of system (3.1) and (3.2) can be obtained.

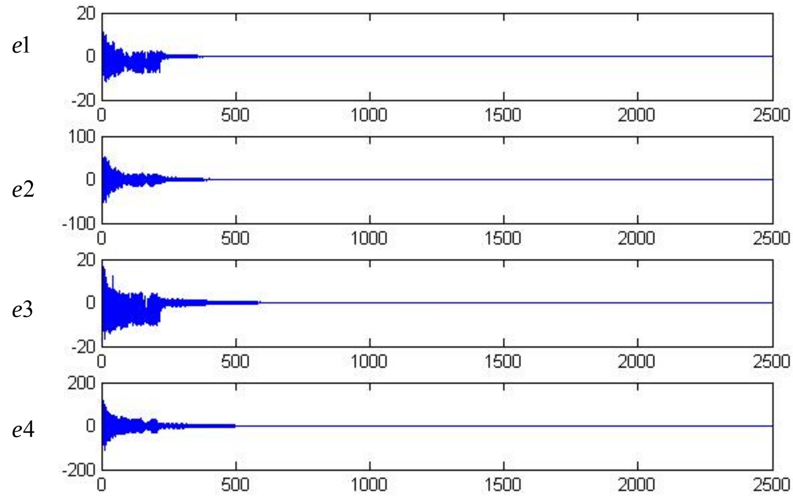
3.3 Numerical simulations

Matlab method is used to all of our simulations with time step 0.01. The parameters of two systems (3.1) and (3.2) are given as a = 0.5, b = 1, c = 3, d= -2, e = 5, g = 1, h = 2 , k=2 to ensure the chaotic behavior. To verify CS and AS, it is convenient to introduce the following coordinate transformation: $E_1 = (x_1 + x_2)$ and $e_1 = (x_1 - x_2)$ and the same transformation for *y, u* and v variables. Therefore, the new coordinate systems (E_1, E_2, E_3, E_4) and (e_1, e_2, e_3, e_4) represent the sum and difference motions of the original coordinate system, respectively. We can easily see that (e_1, e_2, e_3, e_4) subspace represents the CS case, and (E_1, E_2, E_3, E_4) subspace the AS one.

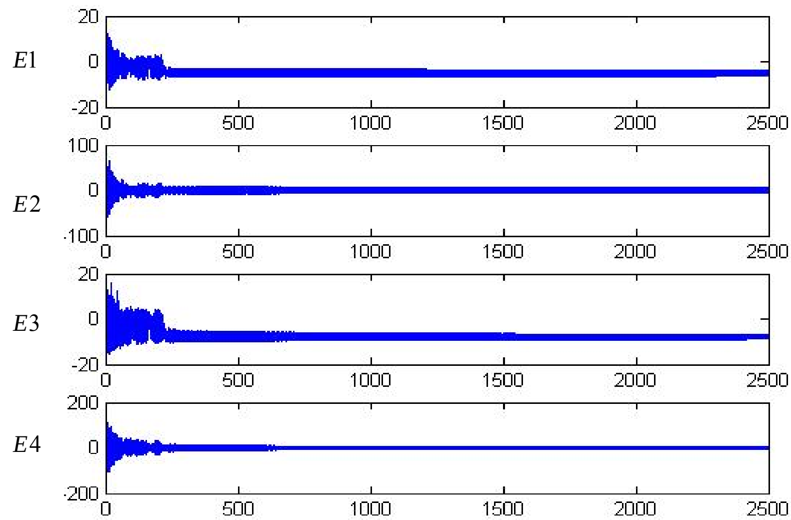
How the synchronization phenomena depend on the initial conditions will be studied. At the beginning, we choose (x_1, y_1, u_1, v_1) = (2, 5, 1, 0.3) and (x_2, y_2, u_2, v_2) = (-8, -9, 0, 5) as the initial conditions of system (3.1) and system (3.2). Let the driving strengths be p = 10, q = 8 and p = 10, q = 10. Fig. 3.1 and Fig.3. 2 show the time-series of AS and CS phenomena for different driving strengths, respectively. The simulation results are shown in Fig. 3.1 for case (a) and in Fig. 3.2 for case (b). These simulation results indicate that the final result develops to CS or AS depending sensitively on driving strength in spite of the identical initial conditions in both cases. For AS case (Fig. 3.1(a) and (b)), the sums of the variables converge to zero, while the differences remain chaotic. For CS case (Fig. 3.2(a) and (b)), on the other hand, e_1, e_2, e_3 and e_4 converge to zero, while E_1, E_2, E_3 and E_4 remain chaotic.

In order to know how these phenomena depend upon the initial conditions, different initial conditions are given for fixed driving strength. The results are shown in Figs. 3.3 and 3.4. Fig. 3.3(b) shows that E_1, E_2, E_3 and E_4 tend to zero. As shown in Fig. 3.3(a), e_1, e_2, e_3 and e_4 do not go to zero. Comparing Fig. 3.1 with Fig. 3.3, it is found that they have contrary behaviors. The only reason lies in the different initial conditions. Similar result also exists by comparing Fig. 3.2 with Fig. 3.4.

Besides, we also discover the alternative CS and AS. In Fig. 3.5, the system shows alternative switching between these two states where the initial condition $(x_1, y_1, u_1, v_1) = (2, 5, 1, 0.3)$, $(x_2, y_2, u_2, v_2) = (-8, -9, 0, 5)$, and $p = 12$, $q = 12$.

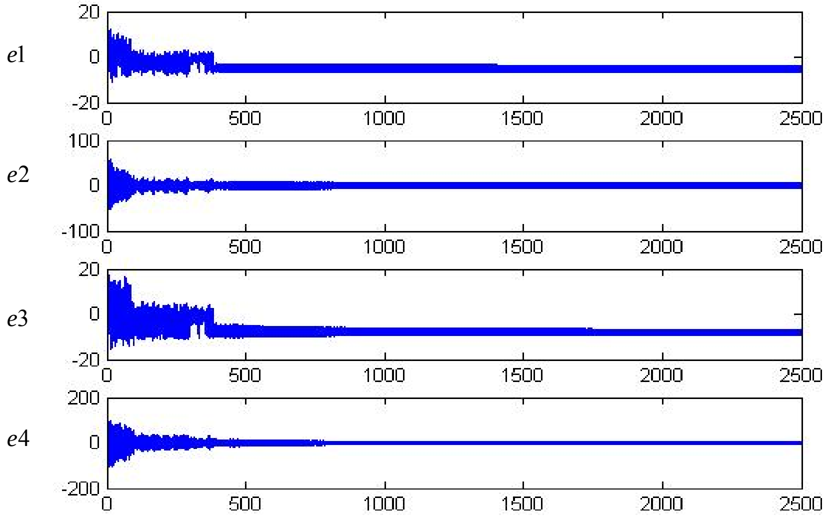


(a)

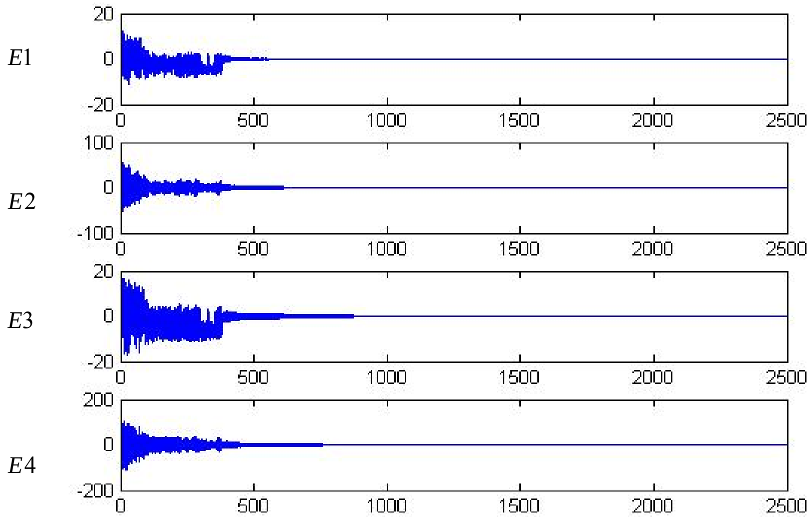


(b)

Fig. 3.1 CS and AS for initial condition $(x_2, y_2, u_2, v_2) = (-8, -9, 0, 5)$, and $p = 10$, $q = 8$. (a) e_1, e_2, e_3, e_4 (b) E_1, E_2, E_3, E_4.

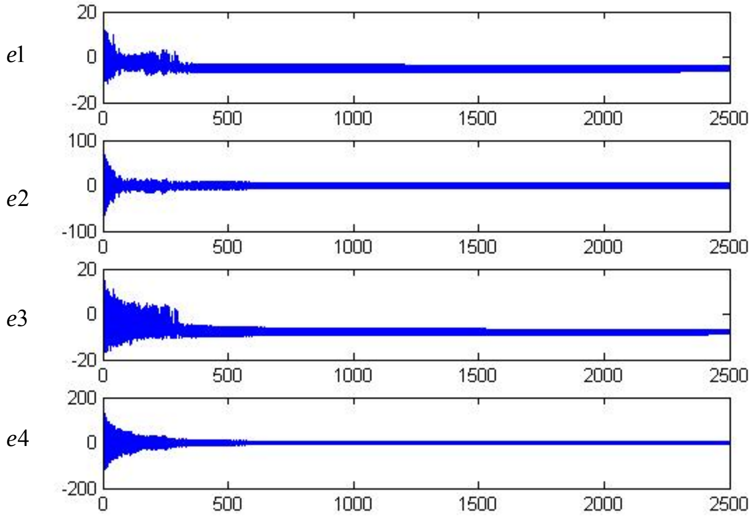


(a)

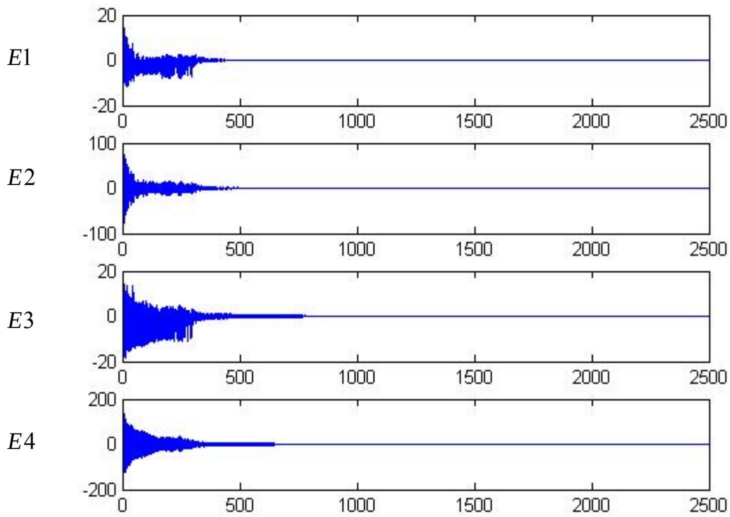


(b)

Fig. 3.2 CS and AS for initial condition $(x_2, y_2, u_2, v_2) = (-8, -9, 0, 5)$, and $p = 10$, $q = 10$. (a) e_1, e_2, e_3, e_4 (b) E_1, E_2, E_3, E_4.

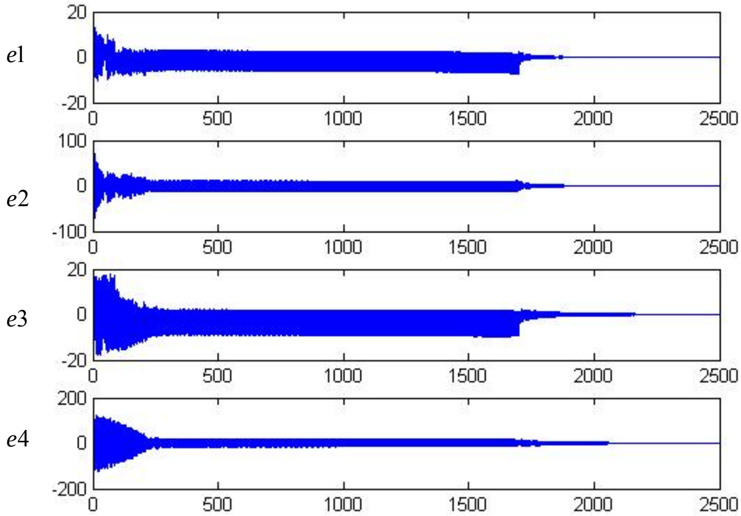


(a)

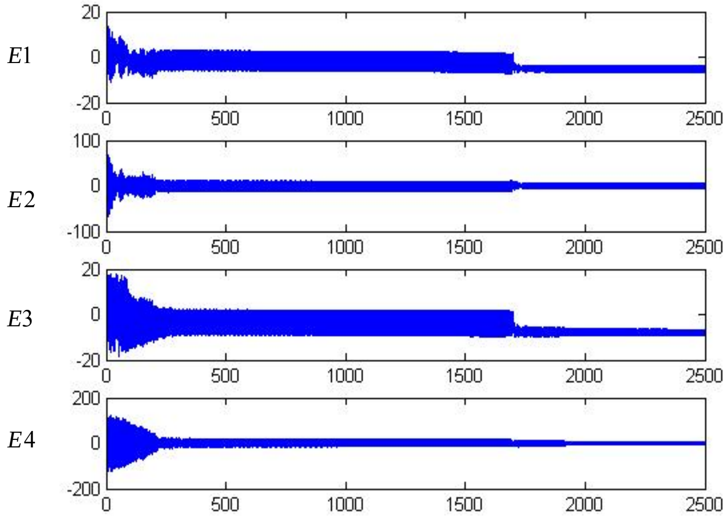


(b)

Fig. 3.3 CS and AS for initial condition $(x_2, y_2, u_2, v_2) = (9, 5, -7, 9)$, and $p = 10$, $q = 10$. (a) e_1, e_2, e_3, e_4 (b) E_1, E_2, E_3, E_4.

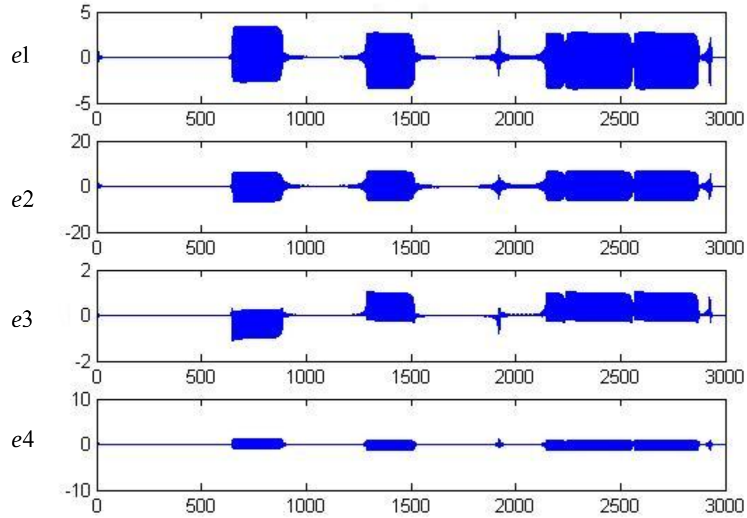


(a)

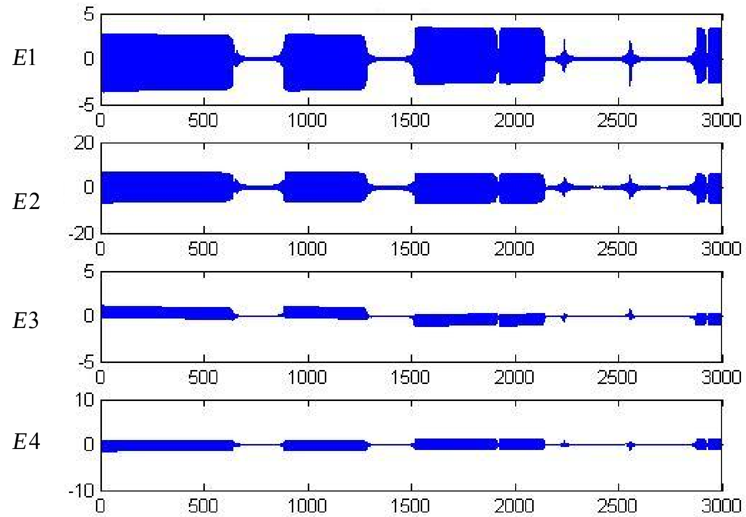


(b)

Fig. 3.4 CS and AS for initial condition $(x_2, y_2, u_2, v_2) = (-8, -9, 0, 5)$, and $p = 10$, $q = 13$. (a) $e_{1,}$ $e_{2,}$ $e_{3,}$ e_4 (b) $E_{1,}$ $E_{2,}$ $E_{3,}$ E_4.



(a)



(b)

Fig. 3.5 Alternative CS and AS for initial condition $(x_1, y_1, u_1, v_1) = (2, 5, 1, 0.3)$, $(x_2, y_2, u_2, v_2) = (-3, 5, 2, 9)$, and $p = 12$, $q = 12$. (a) e_1, e_2, e_3, e_4 (b) E_1, E_2, E_3, E_4.

Chapter 4

Uncoupled Chaos Synchronization and Antisynchronization of Double Duffing Systems by Noise Excited Parameters

4.1 Preliminaries

Chaos synchronization and antisynchronization by replacing two corresponding parameters of two uncoupled identical double Duffing chaotic dynamical systems by a white noise, a Rayleigh noise, a Rician noise or a uniform noise respectively. Numerical simulations are given to demonstrate the effectiveness of the proposed scheme.

4.2 Synchronization and antisynchronization of two double Duffing systems

In this Section, consider the following double Duffing system:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -ay - bx - cx^3 + du \\ \dfrac{du}{dt} = v \\ \dfrac{dv}{dt} = -ev - gu - hu^3 + kx \end{cases} \tag{4.1}$$

It consists of two Duffing systems in which two external excitations are replaced by two coupling terms. It is an autonomous system with four states where *a*, *b*, *c*, *d*, *e*, *g*, *h*, and *k* are constant parameters of the systems.

When $a = 0.05, b = 1, c = 3, d = 7, e = 0.05, g = 1, h = 3, k = -7$, the chaotic behavior is presented in Fig 1. Two identical double Duffing systems to be synchronized are

$$
\begin{cases}
\dfrac{dx_1}{dt} = y_1 \\
\dfrac{dy_1}{dt} = -ay_1 - bx_1 - cx_1{}^3 + du_1 \\
\dfrac{du_1}{dt} = v_1 \\
\dfrac{dv_1}{dt} = -ev_1 - gu_1 - hu_1{}^3 + kx_1
\end{cases}
\tag{4.2}
$$

$$
\begin{cases}
\dfrac{dx_2}{dt} = y_2 \\
\dfrac{dy_2}{dt} = -ay_2 - bx_2 - cx_2{}^3 + du_2 \\
\dfrac{du_2}{dt} = v_2 \\
\dfrac{dv_2}{dt} = -ev_2 - gu_2 - hu_2{}^3 + kx_2
\end{cases}
\tag{4.3}
$$

In order to obtain the chaos synchronization and antisynchronization of systems (4.2) and (4.3), the corresponding parameters *a, b, c, d, e, g, h, k* of two systems are replaced respectively by a noise signal $pf(x)$, where *p* is constant driving strength and $f(x)$ is the noise signal.

The error state variables are defined:

$$
\begin{cases}
e_1 = x_1 - x_2 \\
e_2 = y_1 - y_2 \\
e_3 = u_1 - u_2 \\
e_4 = v_1 - v_2
\end{cases}
\tag{4.4}
$$

Giving suitable values for *p* and initial conditions, the synchronization or anti-synchronization of systems (4.2) and (4.3) can be obtained.

4.3 Numerical simulations

Matlab method for simulations with time step 0.01. The parameters of systems (4.2) and (4.3) are given as $a = 0.05, b = 1, c = 3, d = 7, e = 0.05, g = 1, h = 3, k = -7$ to ensure the chaotic behavior. The chaotic phase portraits for double Duffing system is shown in Fig 4.1. To verify CS and AS, it is convenient to introduce the following coordinate transformation: $E_1 = (x_1 + x_2)$ and $e_1 = (x_1 - x_2)$ and the same transformation for *y, u* and v variables. Therefore, the new coordinate systems (E_1, E_2, E_3, E_4) and (e_1, e_2, e_3, e_4) represent the sum and difference motions of the original coordinate system, respectively. We can easily see that (e_1, e_2, e_3, e_4) subspace represents the CS case, and (E_1, E_2, E_3, E_4) subspace the AS one.

In order to obtain the chaos synchronization or antisynchronization of systems (4.2) and (4.3), we replace two corresponding parameters of two identical systems by the same noise signal as follows:

Case1: White noise

The probability density function of n-dimensional Gaussian noise is

$$f(x) = ((2\pi)^n \det K)^{-\frac{1}{2}} \exp(-(x-\mu)^T K^{-1}(x-\mu)/2)$$

where x is a length-n vector, K is the n-by-n covariance matrix, μ is the mean value vector, and the superscript T indicates matrix transpose. The Simulink Communications toolbox provides the Gaussian Noise Generator block. The initial seed, the mean value and the variance in the simulation must be specified. We take the initial seed 41, the mean value 1 and the variance 1 in the simulation.

With the strength $p = 1$ or -1, replace two corresponding parameters of the systems (4.2) and system (4.3) by a white noise signal. No AS or CS can be found.

Case2: Rayleigh noise

The Rayleigh probability density function is given by

$$f(x)=\begin{cases}\dfrac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} & x\geq 0\\ 0 & x<0\end{cases}$$

where σ^2 is known as the fading envelope of the Rayleigh distribution. The Simulink Communications toolbox provides the Rayleigh Noise Generator block. The initial seed and the sigma parameter in the simulation must be specified. We specify the initial seed 47 and the sigma parameter 5 in the simulation.

For this case, we replace two corresponding parameters of the systems (4.2) and system (4.3) by a Rayleigh noise chaotic signal. We only find AS or CS in replacement of the *a* parameters. In replacement of two *b, c, d, e, f, h, k* by a Rayleigh noise, neither CS nor AS is found. In Fig 4.2, $p=10$, two parameters *a* of systems (4.2) and (4.3) are replaced. In Fig 4.2 (a), CS is obtained. In Fig 4.2 (b) no AS is obtained. For $p=22$, CS and temporary AS are obtained as shown in Fig 4.3 (a), (b). For $p=0.18$, no CS is obtained, AS is obtained as shown in Fig 4.4 (a), (b). For $p=1$, no CS is obtained, AS is obtained as shown in Fig 4.5 (a), (b).

Case3: Rician noise

The Rician probability density function is given by

$$f(x)=\begin{cases}\dfrac{x}{\sigma^2}I_0\left(\dfrac{mx}{\sigma^2}\right)e^{-\frac{x^2+m^2}{2\sigma^2}} & x\geq 0\\ 0 & x<0\end{cases}$$

where σ is the standard deviation of the Gaussian distribution that underlies the Rician distribution noise, $m^2=m_I^2+m_Q^2$, where m_I and m_Q are the mean values of two independent Gaussian components, and I_0 is the modified 0th-order Bessel function of the first kind given by

$$I_0(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{y\cos t}\,dt$$

Note that m and σ are not the mean value and standard deviation for the Rician noise. The Simulink Communications toolbox provides the Rician Noise Generator block. The initial seed, Rician K-factor and the sigma parameter must be specified in the simulation. We specify the initial seed 59, Rician K-factor 10 and the sigma parameter 5 in the simulation.

For this case, we replace two corresponding parameters of the systems (4.2) and system (4.3) by using the Rician noise signal. We only find AS in replacement of two a parameters, with the strength $p = 1$, as shown in Fig. 4.6.

Case4: Uniform noise

The probability density function of uniform noise is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

The mean of this density function is given by $\mu = \dfrac{a+b}{2}$ and its variance by $\sigma^2 = \dfrac{(b-a)^2}{12}$.

The Simulink Communications toolbox provides the Uniform Noise Generator block. The initial seed, the noise lower bound and the noise upper bound must be specified in the simulation. We specify the initial seed 31, the noise lower bound 0 and the noise upper bound 5 in the simulation.

For this case, we replace two corresponding parameters of the systems (4.2) and system (4.3) by using the uinform noise signal. We only find AS in replacement of two parameters a, with the strength $p = 1$ as shown in Fig. 4.7.

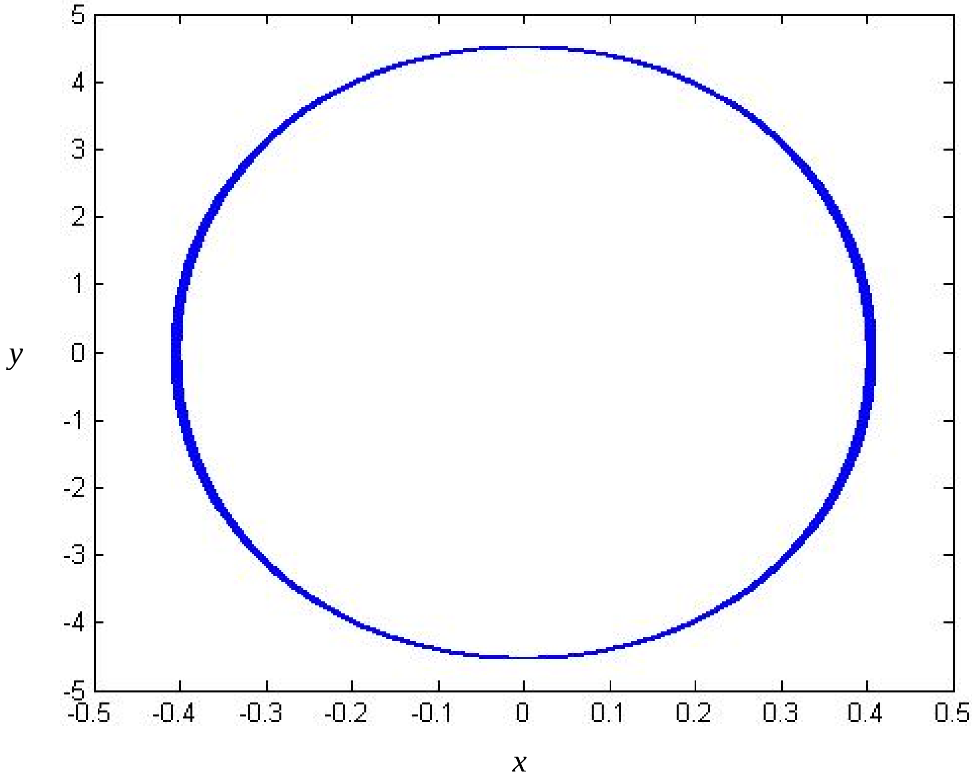
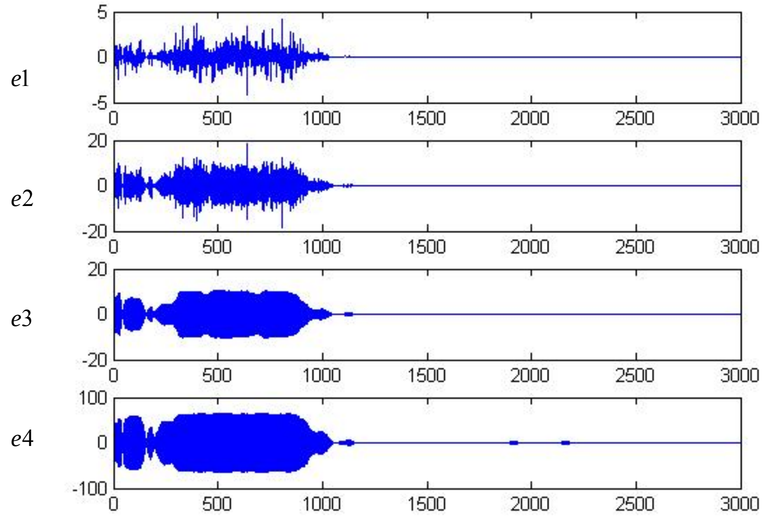
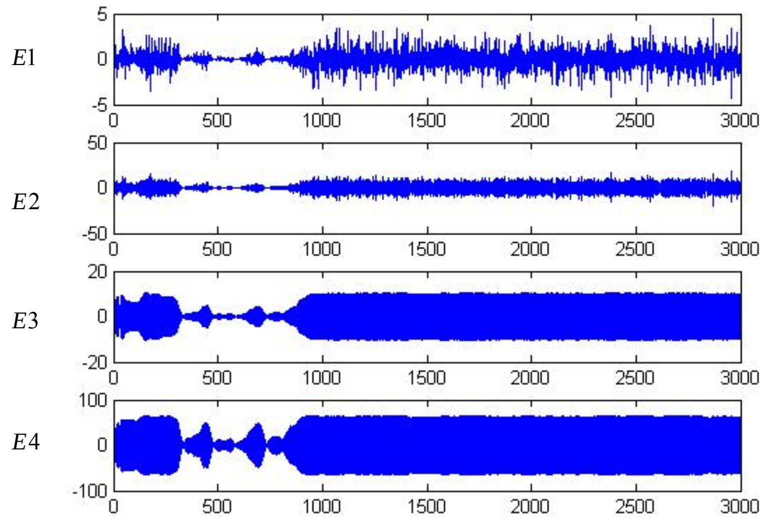


Fig. 4.1. The double Duffing chaotic behavior is presented with the system paraments $a = 0.05, b = 1, c = 3, d = 7, e = 0.05, g = 1, h = 3, k = -7$.

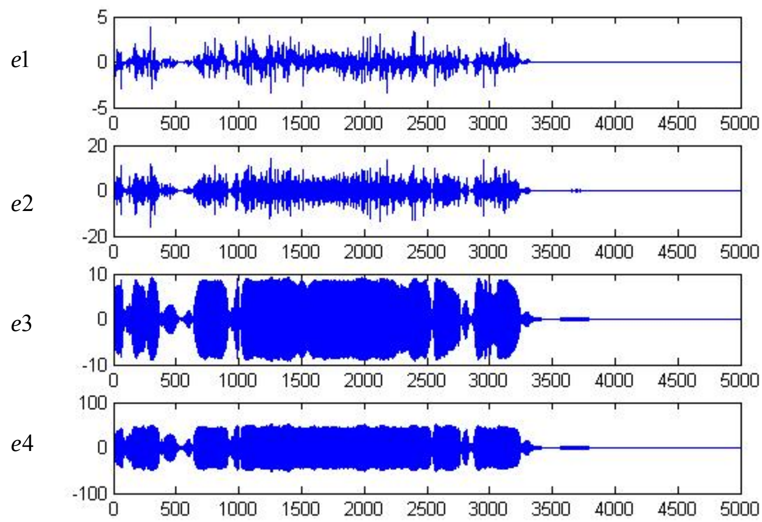


(a)

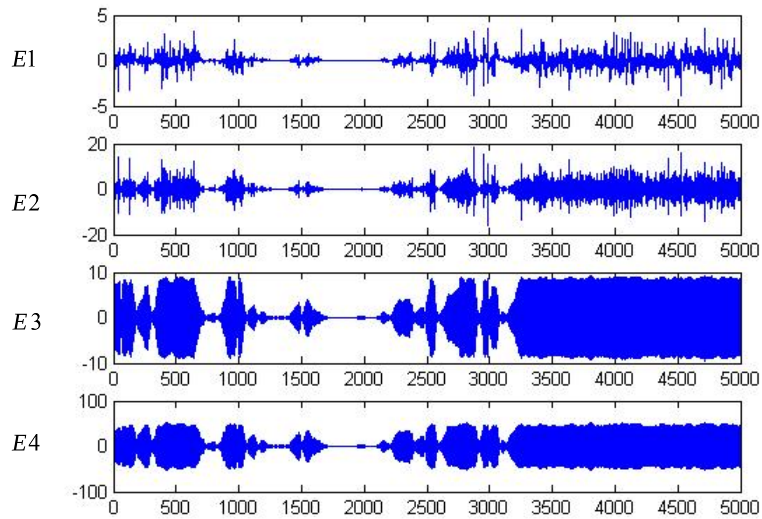


(b)

Fig. 4.2. Two paraments *a* of systems (4.2) and (4.3) are replaced by a Rayleigh noise, with the strength p = 10. (a) e_1, e_2, e_3, e_4, CS is obtained. (b) E_1, E_2, E_3, E_4, no AS is obtained.

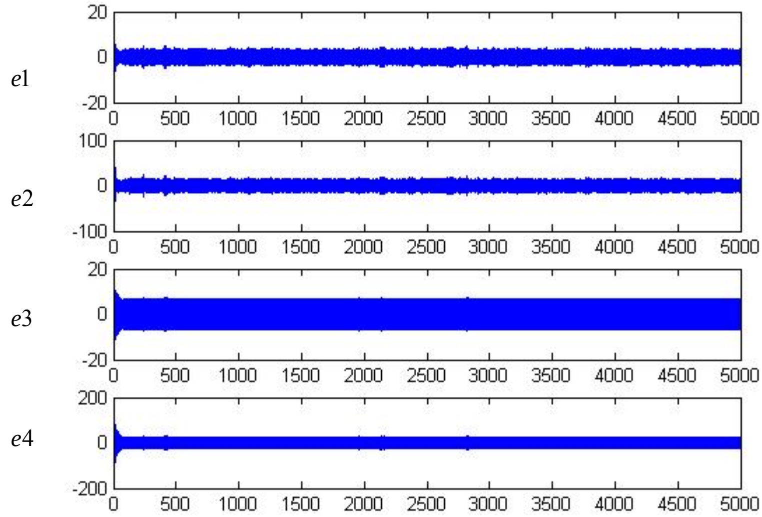


(a)

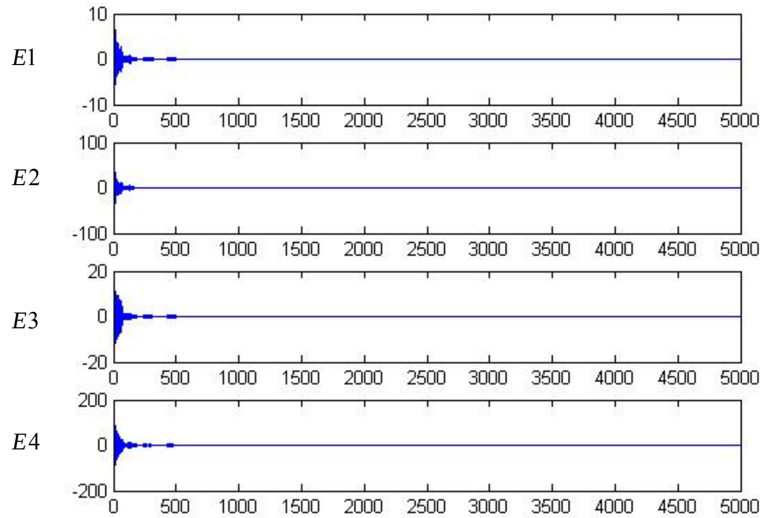


(b)

Fig. 4.3. Two paraments *a* of systems (4.2) and (4.3) are replaced by a Rayleigh noise, with the strength p = 22. (a) e_1, e_2, e_3, e_4, CS is obtained. (b) E_1, E_2, E_3, E_4, temporary AS is obtained.

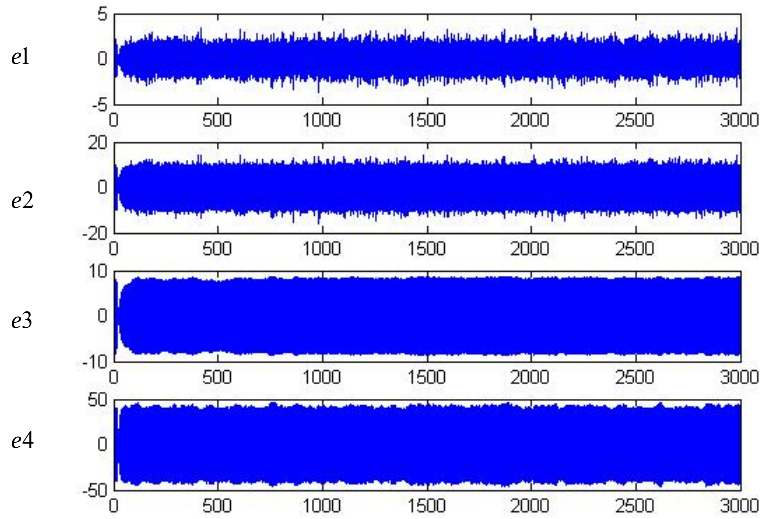


(a)

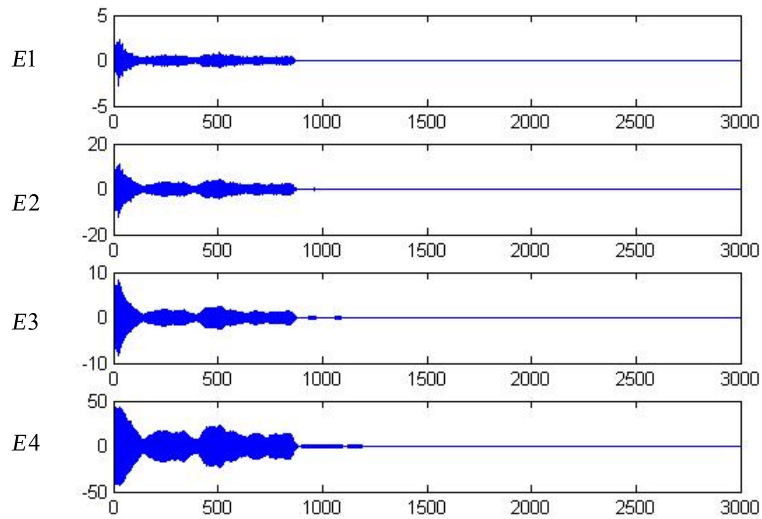


(b)

Fig. 4.4. Two paraments a of systems (4.2) and (4.3) are replaced by a Rayleigh noise, with the strength p = 0.18. (a) e_1, e_2, e_3, e_4, no CS is obtained. (b) E_1, E_2, E_3, E_4, AS is obtained.

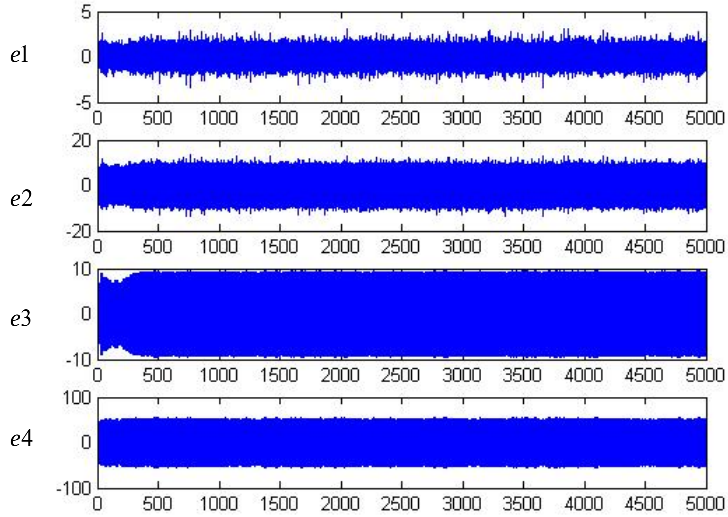


(a)

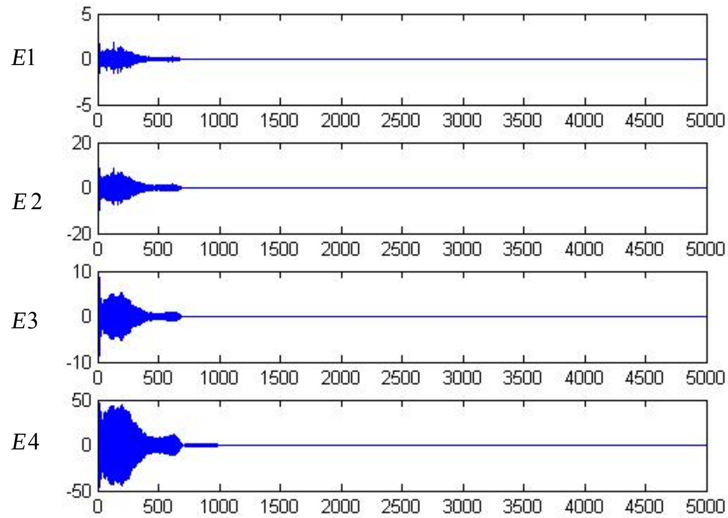


(b)

Fig. 4.5. Two paraments a of systems (4.2) and (4.3) are replaced by a Rayleigh noise, with the strength p = 1. (a) e_1, e_2, e_3, e_4, no CS is obtained. (b) E_1, E_2, E_3, E_4, AS is obtained.

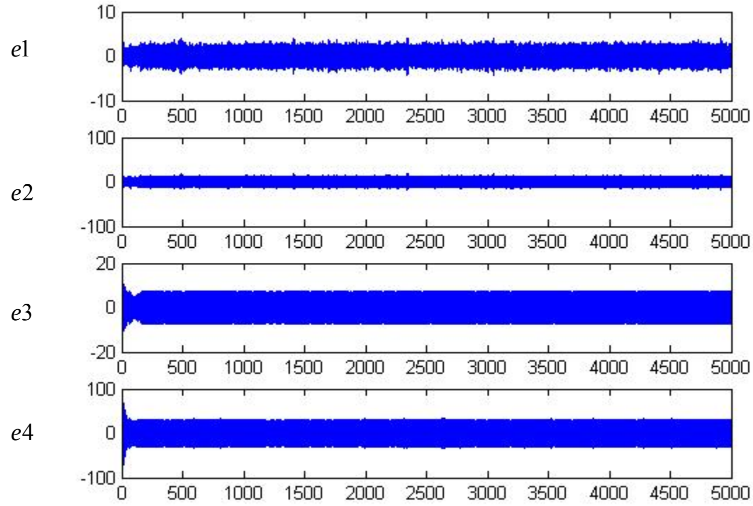


(a)

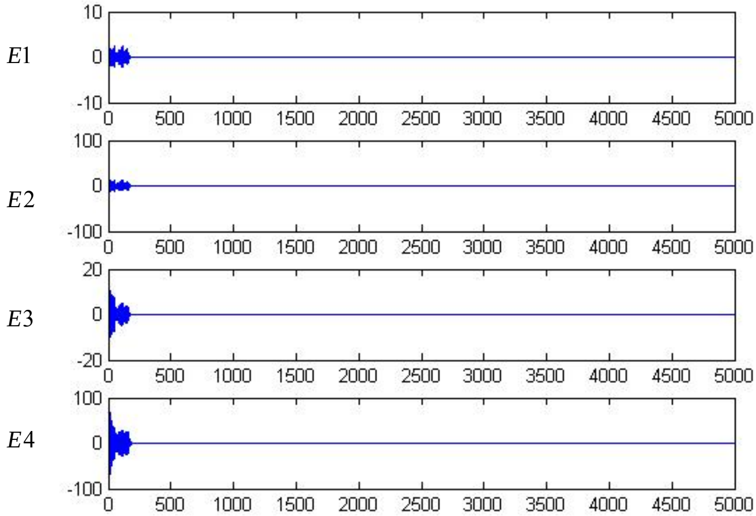


(b)

Fig. 4.6. Two paraments a of systems (4.2) and (4.3) are replaced by a Rician noise, with the strength p = 1. (a) $e_1, e_2, e_3, e_4,$ no CS is obtained. (b) $E_1, E_2, E_3, E_4,$ AS is obtained.



(a)



(b)

Fig. 4.7. Two paraments *a* of systems (4.2) and (4.3) are replaced by a Uniform noise, with the strength p = 1. (a) e_1, e_2, e_3, e_4, no CS is obtained. (b) E_1, E_2, E_3, E_4, AS is obtained.

Chapter 5

Highly Robust Pragmatical Generalized Synchronization of Double Duffing Systems with Uncertain Parameters via Adaptive Control

5.1 Preliminaries

A scheme is proposed to achieve generalized synchronization for two chaotic systems with uncertain parameters. By the pragmatical asymptotical stability theorem using the concept of probability, we can prove strictly that the common null solution of error dynamics and of parameter dynamics is actually asymptotically stable.

5.2 Pragmatical Generalized Synchronization Scheme by Adaptive Control

There are two identical nonlinear dynamical systems, and the master system controls the slave system. The master system is given by

$$\dot{x} = Ax + f(x, B) \tag{5.1}$$

where $x = [x_1, x_2, \cdots x_n]^T \in R^n$ denotes a state vector, A is an $n \times n$ uncertain constant coefficient matrix, f is a nonlinear vector function, and B is a vector of uncertain constant coefficients in f.

The slave system is given by

$$\dot{y} = \hat{A}y + f(y, \hat{B}) + u(t) \tag{5.2}$$

where $y = [y_1, y_2, \cdots y_n]^T \in R^n$ denotes a state vector, $\hat{A}$ is an $n \times n$ estimated coefficient matrix, $\hat{B}$ is a vector of estimated coefficients in f, and

$u(t)=[u_1(t),u_2(t),\cdots u_n(t)]^T \in R^n$ is a control input vector.

Our goal is to design a controller $u(t)$ so that the state vector of the slave system (5.2) asymptotically approaches the state vector of the master system (5.1) plus a given chaotic vector function $F(t)=[F_1(t),F_2(t),\cdots F_n(t)]^T$. This is a special kind of generalized synchronization:

$$y = G(x) = x + F(t). \tag{5.3}$$

The synchronization is accomplished when $t \to \infty$, the limit of the error vector $e(t)=[e_1,e_2,\cdots e_n]^T$ approaches zero:

$$\lim_{t\to\infty} e = 0 \tag{5.4}$$

where

$$e = x - y + F(t). \tag{5.5}$$

From Eq. (5.5) we have

$$\dot{e} = \dot{x} - \dot{y} + \dot{F}(t) \tag{5.6}$$

$$\dot{e} = Ax - \hat{A}y + f(x,B) - f(y,\hat{B}) + \dot{F}(t) - u(t). \tag{5.7}$$

A Lyapunov function $V(e,\widetilde{A}_c,\widetilde{B}_c)$ is chosen as a positive definite function

$$V(e,\widetilde{A}_c,\widetilde{B}_c) = \frac{1}{2}e^T e + \frac{1}{2}\widetilde{A}_c{}^T\widetilde{A}_c + \frac{1}{2}\widetilde{B}_c{}^T\widetilde{B}_c \tag{5.8}$$

where $\widetilde{A} = A - \hat{A}$, $\widetilde{B} = B - \hat{B}$, $\widetilde{A}_c$ and $\widetilde{B}_c$ are two column matrices whose elements are all the elements of matrix $\widetilde{A}$ and of matrix $\widetilde{B}$, respectively.

Its derivative along any solution of the differential equation system consisting of Eq. (5.7) and update parameter differential equations for $\widetilde{A}_c$ and $\widetilde{B}_c$ is

$$\therefore \dot{V}(e,\widetilde{A}_c,\widetilde{B}_c) = e^T[Ax - \hat{A}y + Bf(x) - \hat{B}f(y) + \dot{F}(t) - u(t)] + \widetilde{A}_c\dot{\widetilde{A}}_c + \widetilde{B}_c\dot{\widetilde{B}}_c \tag{5.9}$$

where $u(t)$, $\dot{\widetilde{A}}_c$ and $\dot{\widetilde{B}}_c$ are chosen so that $\dot{V} = e^T Ce$, C is a diagonal negative definite matrix, and $\dot{V}$ is a negative semi-definite function of e and parameter differences $\widetilde{A}_c$ and $\widetilde{B}_c$. In current scheme of adaptive synchronization [71-73], traditional Lyapunov asymptotical stability theorem and Babalat lemma are used to prove the error vector approaches zero, as time approaches infinity. But the question, why the estimated parameters also approach to the uncertain parameters, remains no answer. By pragmatical asymptotical stability theorem, the question can be answered strictly.

The stability for many problems in real dynamical systems is actual asymptotical stability, although may not be mathematical asymptotical stability. The mathematical asymptotical stability demands that trajectories from all initial states in the neighborhood of zero solution must approach the origin as $t \to \infty$. If there are only a small part or even a few of the initial states from which the trajectories do not approach the origin as $t \to \infty$, the zero solution is not mathematically asymptotically stable. However, when the probability of occurrence of an event is zero, it means the event does not occur actually. If the probability of occurrence of the event that the trajectries from the initial states are that they do not approach zero when $t \to \infty$, is zero, the stability of zero solution is actual asymptotical stability though it is not mathematical asymptotical stability. In order to analyze the asymptotical stability of the equilibrium point of such systems, the pragmatical asymptotical stability theorem is used.

Let *X* and *Y* be two manifolds of dimensions m and n $(m < n)$, respectively, and φ be a differentiable map from *X* to *Y*, then $\varphi(X)$ is a subset of Lebesque measure 0 of Y. For an autonomous system

$$\frac{dx}{dt} = f(x_1, x_2, \cdots x_n) \tag{5.10}$$

where $x = [x_1, x_2, \cdots x_n]^T$ is a state vector, the function $f = [f_1, f_2, \cdots f_n]^T$ is defined on $D \subset R^n$ and $\|x\| \leq H > 0$. Let $x = 0$ be an equilibrium point for the system (5.10).

Then

$$f(0) = 0 \tag{5.11}$$

Definition : The equilibrium point for the system (5.11) is pragmatically asymptotically stable provided that with initial points on *C* which is a subset of Lebesque measure 0 of *D*, the behaviors of the corresponding trajectories cannot be determined, while with initial points on $D - C$, the corresponding trajectories behave as that agree with traditional asymptotical stability.

Theorem : Let $V = [x_1, x_2, \cdots x_n]^T : D \to R_+$ be positive definite and analytic on *D*, such that the derivative of V through Eq. (5.10), $\dot{V}$ is negative semi-definite.

Let *X* be the m-manifold consisted of point set for which $\forall x \neq 0$, $\dot{V}(x) = 0$ and *D* is a n-manifold. If $m + 1 < n$, then the equilibrium point of the system is pragmatically asymptotically stable.

Proof : Since every point of *X* can be passed by a trajectory of Eq. (5.10), which is one dimensional, the collection of these trajectories, *C*, is a $(m+1)$-manifold [74-75]. If $(m+1) < n$, then the collection *C* is a subset of Lebesque measure 0 of *D*. By the above definition, the equilibrium point of the system is pragmatically asymptotically stable.

If an initial point is ergodicly chosen in *D*, *the probability of that the initial point falls on the collection C is zero. Here, equal probability is assumed for every point chosen as an initial point in the neighborhood of the equilibrium point.* Hence, the event that the initial point is chosen from collection *C does not occur actually*. Therefore, under the equal probability assumption, pragmatical asymptotical stability becomes actual asymptotical stability. When the initial point falls on $D - C$, $\dot{V}(x) < 0$, the corresponding trajectories behave as that agree with traditional asymptotical stability because by the existence and uniqueness of the solution of initial-value problem, these trajectories never meet *C*.

In Eq. (5.8) V is a positive definite function of n variables, i.e. p error state variables and $n-p=m$ differences between unknown and estimated parameters, while $\dot{V}=e^{T}Ce$ is a negative semi-definite function of n variables. Since the number of error state variables is always more than one, $p>1$, $(m+1)<n$ is always satisfied, by pragmatical asymptotical stability theorem we have

$$\lim_{t\to\infty} e = 0 \tag{5.12}$$

and the estimated parameters approach the uncertain parameters. The pargmatical generalized synchronizations is obtained. Therefore, *the equilibrium point of the system is pragmatically asymptotically stable. Under the equal probability assumption, it is actually asymptotically stable for both error state variables and parameter variables.*

5.3 Numerical Results of Pragmatical Generalized Chaos Synchronization of Two Double Duffing systems by Adaptive Control

5.3.1 Two double Duffing systems with double van der Pol system as goal system

The chaotic states of a goal system, a double van der Pol chaotic system, used as $F(t)$. For a double Duffing system [91], the following differential equations are used as master system:

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -ax_2 - bx_1 - cx_1{}^3 + dx_3 \\ \dfrac{dx_3}{dt} = x_4 \\ \dfrac{dx_4}{dt} = -fx_4 - gx_3 - hx_3{}^3 + kx_1 \end{cases} \tag{5.13}$$

It consists of two Duffing systems in which two external excitations are replaced by two coupling terms. It is an autonomous system with four states where *a*, *b*, *c*, *d*, *e*, *g*, *h*, and *k* are constant unknown parameters of the systems. The chaotic phase portraits for double Duffing system and double van der Pol system are shown in Fig.5.1

In numerical simulation, we take $a = 0.05, b = 1, c = 3, d = 7, f = 0.0005, g = 1$, $h = 3$ and $k = -7$. The slave system is described by

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = -\hat{a}y_2 - \hat{b}y_1 - \hat{c}{y_1}^3 + \hat{d}y_3 \\ \dfrac{dy_3}{dt} = y_4 \\ \dfrac{dy_4}{dt} = -\hat{f}y_4 - \hat{g}y_3 - \hat{h}{y_3}^3 + \hat{k}y_1 \end{cases} \tag{5.14}$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{f}$, $\hat{g}$, $\hat{h}$ and $\hat{k}$ are estimated parameters.

In order to lead (y_1, y_2, y_3, y_4) to $(x_1 + F_1(t), x_2 + F_2(t), x_3 + F_3(t), x_4 + F_4(t))$, we add u_1, u_2, u_3, and u_4 to each equation of Eq. (5.14), respectively:

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 + u_1 \\ \dfrac{dy_2}{dt} = -\hat{a}y_2 - \hat{b}y_1 - \hat{c}{y_1}^3 + \hat{d}y_3 + u_2 \\ \dfrac{dy_3}{dt} = y_4 + u_3 \\ \dfrac{dy_4}{dt} = -\hat{f}y_4 - \hat{g}y_3 - \hat{h}{y_3}^3 + \hat{k}y_1 + u_4 \end{cases} \tag{5.15}$$

Subtracting Eq. (5.15) from Eq. (5.13), we obtain an error dynamics. The initial values of the master system and the slave system are taken as $x_1(0) = 2$, $x_2(0) = 5$, $x_3(0) = 1$, $x_4(0) = 3, y_1(0) = 2.1, y_2(0) = 4.9, y_3(0) = 0.9$ and $y_4(0) = 3.1$, respectively.

The goal system for generalized synchronization is a double van der Pol chaotic system [93-94].

$$\begin{cases} \dfrac{dz_1}{dt} = z_2 \\ \dfrac{dz_2}{dt} = -z_1 + a_3(1 - b_3 z_1^2)z_2 + c_3 z_3 \\ \dfrac{dz_3}{dt} = z_4 \\ \dfrac{dz_4}{dt} = -z_3 + a_4(1 - b_4 z_3^2)z_4 + c_4 z_1 \end{cases} \tag{5.16}$$

where a_3=0.2, b_3=1, c_3= -0.01, a_4=-2,b_4=1, c_4=0.3, $z_1(0)$=3, $z_2(0)$=4, $z_3(0)$=3, and $z_4(0)$=4.

We have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + z_i) = 0 \qquad i = 1, 2, 3, 4 \tag{5.17}$$

where $\dot{e} = \dot{x} - \dot{y} + \dot{z}$, and

$$\begin{cases} \dot{e}_1 = x_2 - y_2 - u_1 + \dot{z}_1 \\ \dot{e}_2 = -ax_2 - bx_1 - cx_1^{\ 3} + dx_3 + \hat{a}y_2 + \hat{b}y_1 + \hat{c}y_1^{\ 3} - \hat{d}y_3 - u_2 + \dot{z}_2 \\ \dot{e}_3 = x_4 - y_4 - u_3 + \dot{z}_3 \\ \dot{e}_4 = -fx_4 - gx_3 - hx_3^{\ 3} + kx_1 + \hat{f}y_4 + \hat{g}y_3 + \hat{h}y_3^{\ 3} - \hat{k}y_1 - u_4 + \dot{z}_4 \end{cases} \tag{5.18}$$

where $e_1 = x_1 - y_1 + z_1$, $e_2 = x_2 - y_2 + z_2$, $e_3 = x_3 - y_3 + z_3$, and $e_4 = x_4 - y_4 + z_4$.

Choose a Lyapunov function in the form of a positive definite function:

$$V = \frac{1}{2}(e_1^{\ 2} + e_2^{\ 2} + e_3^{\ 2} + e_4^{\ 2} + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{f}^2 + \tilde{g}^2 + \tilde{h}^2 + \tilde{k}^2) \tag{5.19}$$

where $\tilde{a} = (a - \hat{a})$, $\tilde{b} = (b - \hat{b})$, $\tilde{c} = (c - \hat{c})$, $\tilde{d} = (d - \hat{d})$, $\tilde{f} = (f - \hat{f})$, $\tilde{g} = (g - \hat{g})$, $\tilde{h} = (h - \hat{h})$, $\tilde{k} = (k - \hat{k})$ and $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{f}$, $\hat{g}$, $\hat{h}$ and $\hat{k}$ are estimates of uncertain parameters a, b, c, d, f, g, h and k respectively. Its time derivative is

$$
\begin{aligned}
\dot{V} &= e_1(x_2 - y_2 - u_1 + z_2) \\
&+ e_2[-ax_2 - bx_1 - cx_1^{\ 3} + dx_3 + \hat{a}y_2 + \hat{b}y_1 + \hat{c}y_1^{\ 3} - \hat{d}y_3 - u_2 \\
&- z_1 + a_3(1 - b_3 z_1^2)z_2 + c_3 z_3] \\
&+ e_3(x_4 - y_4 - u_3 + z_4) \\
&+ e_4[-fx_4 - gx_3 - hx_3^{\ 3} + kx_1 + \hat{f}y_4 + \hat{g}y_3 + \hat{h}y_3^{\ 3} - \hat{k}y_1 - u_4 \\
&- z_3 + a_4(1 - b_4 z_3^2)z_4 + c_4 z_1] \\
&+ \tilde{a}(-\dot{\hat{a}}) + \tilde{b}(-\dot{\hat{b}}) + \tilde{c}(-\dot{\hat{c}}) + \tilde{d}(-\dot{\hat{d}}) + \tilde{f}(-\dot{\hat{f}}) + \tilde{g}(-\dot{\hat{g}}) + \tilde{h}(-\dot{\hat{h}}) + \tilde{k}(-\dot{\hat{k}})
\end{aligned}
\tag{5.20}
$$

Choose

$$
\begin{cases}
u_1 = x_2 - y_2 + z_2 + e_1 \\
u_2 = \hat{a}(y_2 - x_2) + \hat{b}(y_1 - x_1) + \hat{c}(y_1^{\ 3} - x_1^{\ 3}) + \hat{d}(x_3 - y_3) \\
+[-z_3 + a_4(1 - b_4 z_3^2)z_4 + c_4 z_1] + e_2(1 - \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2) \\
u_3 = x_4 - y_4 + z_4 + e_3 \\
u_4 = \hat{f}(y_4 - x_4) + \hat{g}(y_3 - x_3) + \hat{h}(y_3^{\ 3} - x_3^{\ 3}) + \hat{k}(x_1 - y_1) \\
+[-z_3 + a_4(1 - b_4 z_3^2)z_4 + c_4 z_1] + e_4(1 - \tilde{f}^2 + \tilde{g}^2 + \tilde{h}^2 + \tilde{k}^2)
\end{cases}
\tag{5.21}
$$

$$
\begin{cases}
\dot{\tilde{a}} = -\dot{\hat{a}} = x_2 e_2 - \tilde{a} e_2 \\
\dot{\tilde{b}} = -\dot{\hat{b}} = x_1 e_2 - \tilde{b} e_2 \\
\dot{\tilde{c}} = -\dot{\hat{c}} = x_1^{\ 3} e_2 - \tilde{c} e_2 \\
\dot{\tilde{d}} = -\dot{\hat{d}} = -x_3 e_2 - \tilde{d} e_2 \\
\dot{\tilde{f}} = -\dot{\hat{f}} = x_4 e_2 - \tilde{f} e_2 \\
\dot{\tilde{g}} = -\dot{\hat{g}} = x_3 e_2 - \tilde{g} e_2 \\
\dot{\tilde{h}} = -\dot{\hat{h}} = x_3^{\ 3} e_2 - \tilde{h} e_4 \\
\dot{\tilde{k}} = -\dot{\hat{k}} = -x_1 e_2 - \tilde{k} e_4
\end{cases}
\tag{5.22}
$$

The initial values of estimate for uncertain parameters are $\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = \hat{d}(0) = \hat{f}(0) = \hat{g}(0) = \hat{h}(0) = \hat{k}(0) = 0$. Substituting Eq. (5.21) and Eq. (5.22) into Eq. (5.20), we obtain

$$
\dot{V} = -e_1^{\ 2} - e_2^{\ 2} - e_3^{\ 2} - e_4^{\ 2} \le 0 \tag{5.23}
$$

which is negative semi-definite function of $e_1, e_2, e_3, e_4, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{f}, \tilde{g}, \tilde{h}, \tilde{k}$. The

Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (5.18) and parameter dynamics (5.22) is asymptotically stable. Now, D is an 8-manifold, n=8 and the number of error state variables p=4. When $e_1=e_2=e_3=e_4=0$ and $\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{f},\tilde{g},\tilde{h},\tilde{k}$ take arbitrary values, $\dot{V}=0$, so X is 4-manifold, $m=n-p=8-4=4$. $m+1<n$ is satisfied. By pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained.

The equilibrium point $e_1=e_2=e_3=e_4=\tilde{a}=\tilde{b}=\tilde{c}=\tilde{d}=\tilde{f}=\tilde{g}=\tilde{h}=\tilde{k}=0$ is pragmatically asymptotically stable. *Under the assumption of equal probability, it is actually asymptotically stable*. State errors versus time and the estimates of uncertain parameters are shown in Fig.5.2.

5.3.2 Robustness of the above generalized synchronization

For a double Duffing system, the following differential equations are used as master system:

$$\begin{cases} \dfrac{dx_1}{dt}=x_2 \\ \dfrac{dx_2}{dt}=-ax_2-bx_1-cx_1^{\ 3}+dx_3+\Delta f_1 \\ \dfrac{dx_3}{dt}=x_4 \\ \dfrac{dx_4}{dt}=-fx_4-gx_3-hx_3^{\ 3}+kx_1+\Delta f_1 \end{cases} \tag{5.24}$$

A slave system is described by

$$
\begin{cases}
\dfrac{dy_1}{dt} = y_2 + u_1 \\
\dfrac{dy_2}{dt} = -\hat{a}y_2 - \hat{b}y_1 - \hat{c}{y_1}^3 + \hat{d}y_3 + u_2 + \Delta f_2 \\
\dfrac{dy_3}{dt} = y_4 + u_3 \\
\dfrac{dy_4}{dt} = -\hat{f}y_4 - \hat{g}y_3 - \hat{h}{y_3}^3 + \hat{k}y_1 + u_4 + \Delta f_2
\end{cases} \tag{5.25}
$$

They are two double Duffing systems with disturbance $\Delta f_1(t,x,y,z)$ and $\Delta f_2(t,x,y,z)$ respectively.

In simulation, the parameters of the master system in chosen as $a = 0.05, b = 1, c = 3, d = 7, f = 0.0005, g = 1, h = 3, k = -7$. The initial values of the master system and the slave system are taken as $x_1(0)=2$, $x_2(0)=5$, $x_3(0)=1$, $x_4(0)=3$, $y_1(0)=2.1$, $y_2(0)=4.9$, $y_3(0)=0.9$, and $y_4(0)=3.1$, respectively.

Let

$$
\begin{array}{l}
\Delta f_1(t,x,y) = \alpha(x_i - y_i + z_i)\Gamma_1(t,x) \\
\Delta f_2(t,x,y) = \alpha(x_i - y_i + z_i)\Gamma_2(t,y)
\end{array} \quad i = 2, 4 \tag{5.26}
$$

where $\Gamma_1(t,x)$ is the Gaussian noise and $\Gamma_2(t,y)$ is the Rayleigh noise, α is the strength constant. Since both disturbances are the products of chaos and noise, they are highly perturbative.

The goal system for generalized synchronization is a double Van der Pol chaotic

$$
\begin{cases}
\dfrac{dz_1}{dt} = z_2 \\
\dfrac{dz_2}{dt} = -z_1 + a_3(1 - b_3 z_1^2)z_2 + c_3 z_3 \\
\dfrac{dz_3}{dt} = z_4 \\
\dfrac{dz_4}{dt} = -z_3 + a_4(1 - b_4 z_3^2)z_4 + c_4 z_1
\end{cases} \tag{5.27}
$$

We demand

$$
\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + z_i) = 0, \qquad i = 1, 2, 3, 4
$$

then

$$\dot{e} = \dot{x} - \dot{y} + \dot{z},$$

and

$$\begin{cases} \dot{e}_1 = x_2 - y_2 - u_1 + \dot{z}_1 \\ \dot{e}_2 = -ax_2 - bx_1 - cx_1^{\ 3} + dx_3 + \hat{a}y_2 + \hat{b}y_1 + \hat{c}y_1^{\ 3} - \hat{d}y_3 - u_2 + (\Delta f_1 - \Delta f_2) + \dot{z}_2 \\ \dot{e}_3 = x_4 - y_4 - u_3 + \dot{z}_3 \\ \dot{e}_4 = -fx_4 - gx_3 - hx_3^{\ 3} + kx_1 + \hat{f}y_4 + \hat{g}y_3 + \hat{h}y_3^{\ 3} - \hat{k}y_1 - u_4 + (\Delta f_1 - \Delta f_2) + \dot{z}_4 \end{cases} \tag{5.28}$$

where $e_1 = x_1 - y_1 + z_1$, $e_2 = x_2 - y_2 + z_2$, $e_3 = x_3 - y_3 + z_3$, and $e_4 = x_4 - y_4 + z_4$.

Choose a Lyapunov function in the form of a positive definite function:

$$V = \frac{1}{2}(e_1^{\ 2} + e_2^{\ 2} + e_3^{\ 2} + e_4^{\ 2} + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{f}^2 + \tilde{g}^2 + \tilde{h}^2 + \tilde{k}^2) \tag{5.29}$$

where $\tilde{a} = (a - \hat{a})$, $\tilde{b} = (b - \hat{b})$, $\tilde{c} = (c - \hat{c})$, $\tilde{d} = (d - \hat{d})$, $\tilde{f} = (f - \hat{f})$, $\tilde{g} = (g - \hat{g})$, $\tilde{h} = (h - \hat{h})$, $\tilde{k} = (k - \hat{k})$ and $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{f}$, $\hat{g}$, $\hat{h}$ and $\hat{k}$ are estimates of uncertain parameters a, b, c, d, f, g, h and k respectively.

Its time derivative is

$$\begin{aligned} \dot{V} &= e_1(x_2 - y_2 - u_1 + z_2) \\ &+ e_2[-ax_2 - bx_1 - cx_1^{\ 3} + dx_3 + \hat{a}y_2 + \hat{b}y_1 + \hat{c}y_1^{\ 3} - \hat{d}y_3 - u_2 \\ &- z_1 + a_3(1 - b_3 z_1^2)z_2 + c_3 z_3] \\ &+ e_3(x_4 - y_4 - u_3 + z_4) \\ &+ e_4[-fx_4 - gx_3 - hx_3^{\ 3} + kx_1 + \hat{f}y_4 + \hat{g}y_3 + \hat{h}y_3^{\ 3} - \hat{k}y_1 - u_4 \\ &- z_3 + a_4(1 - b_4 z_3^2)z_4 + c_4 z_1] \\ &+ \tilde{a}(-\dot{\hat{a}}) + \tilde{b}(-\dot{\hat{b}}) + \tilde{c}(-\dot{\hat{c}}) + \tilde{d}(-\dot{\hat{d}}) + \tilde{f}(-\dot{\hat{f}}) + \tilde{g}(-\dot{\hat{g}}) + \tilde{h}(-\dot{\hat{h}}) + \tilde{k}(-\dot{\hat{k}}) \end{aligned} \tag{5.30}$$

Choose

$$\begin{cases} u_1 = x_2 - y_2 + z_2 + e_1 \\ u_2 = \hat{a}(y_2 - x_2) + \hat{b}(y_1 - x_1) + \hat{c}(y_1^{\ 3} - x_1^{\ 3}) + \hat{d}(x_3 - y_3) \\ +[-z_3 + a_4(1 - b_4 z_3^2)z_4 + c_4 z_1] + e_2(1 - \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2) \\ u_3 = x_4 - y_4 + z_4 + e_3 \\ u_4 = \hat{f}(y_4 - x_4) + \hat{g}(y_3 - x_3) + \hat{h}(y_3^{\ 3} - x_3^{\ 3}) + \hat{k}(x_1 - y_1) \\ +[-z_3 + a_4(1 - b_4 z_3^2)z_4 + c_4 z_1] + e_4(1 - \tilde{f}^2 + \tilde{g}^2 + \tilde{h}^2 + \tilde{k}^2) \end{cases} \tag{5.31}$$

$$
\begin{cases}
\dot{\tilde{a}} = -\dot{\hat{a}} = x_2 e_2 - \tilde{a} e_2 \\
\dot{\tilde{b}} = -\dot{\hat{b}} = x_1 e_2 - \tilde{b} e_2 \\
\dot{\tilde{c}} = -\dot{\hat{c}} = x_1^{\,3} e_2 - \tilde{c} e_2 \\
\dot{\tilde{d}} = -\dot{\hat{d}} = -x_3 e_2 - \tilde{d} e_2 \\
\dot{\tilde{f}} = -\dot{\hat{f}} = x_4 e_2 - \tilde{f} e_2 \\
\dot{\tilde{g}} = -\dot{\hat{g}} = x_3 e_2 - \tilde{g} e_2 \\
\dot{\tilde{h}} = -\dot{\hat{h}} = x_3^{\,3} e_2 - \tilde{h} e_4 \\
\dot{\tilde{k}} = -\dot{\hat{k}} = -x_1 e_2 - \tilde{k} e_4
\end{cases}
\tag{5.32}
$$

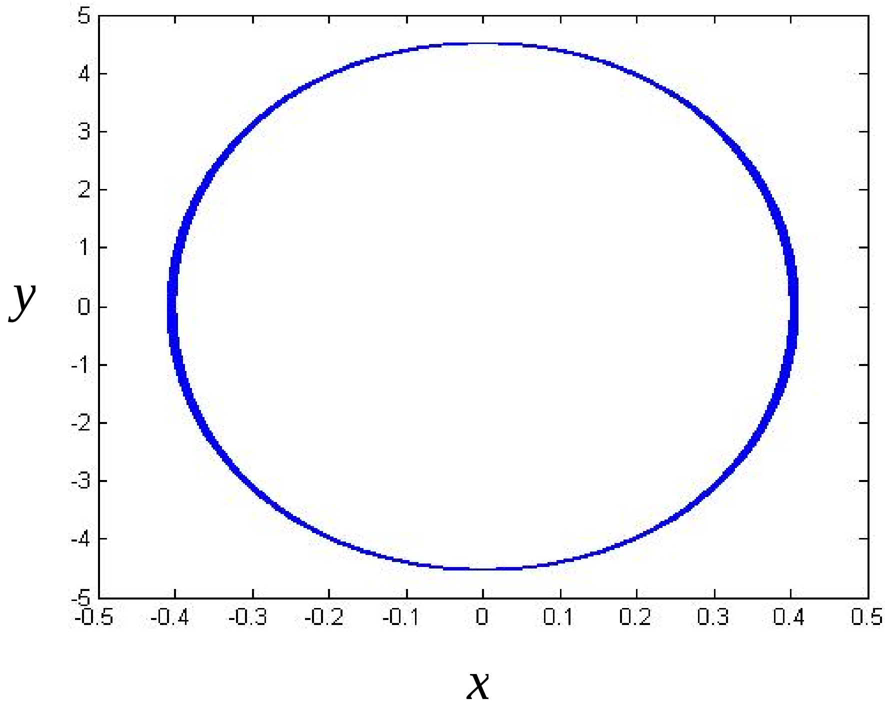
The initial values of estimate for uncertain parameters are $\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = \hat{d}(0)$ $= \hat{f}(0) = \hat{g}(0) = \hat{h}(0) = \hat{k}(0) = 0$.

Substituting Eq. (5.31) and Eq. (5.32) into Eq. (5.30), we obtain

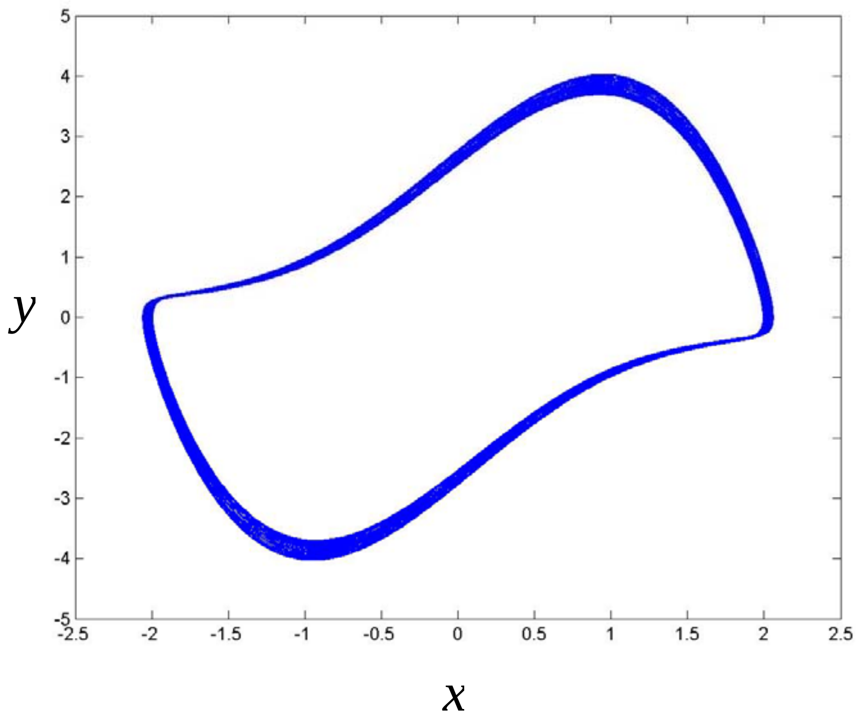
$$\dot{V} = -e_1^{\,2} - e_2^{\,2} - e_3^{\,2} - e_4^{\,2} \le 0 \tag{5.33}$$

which is negative semi-definite function of $e_1, e_2, e_3, e_4, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{f}, \tilde{g}, \tilde{h}, \tilde{k}$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (5.28) and parameter dynamics (5.32) is asymptotically stable. Now, D is an 8-manifold, n=8 and the number of error state variables p=4. When e_1=e_2=e_3= e_4=0 and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{f}, \tilde{g}, \tilde{h}, \tilde{k}$ take arbitrary values, $\dot{V} = 0$, so X is 4-manifold, $m = n - p = 8 - 4 = 4$. $m + 1 < n$ is satisfied. By pragmatical asymptotical stability theorem, when $\alpha = 0 \sim 12$, the error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained.

The equilibrium point $e_1 = e_2 = e_3 = e_4 = \tilde{a} = \tilde{b} = \tilde{c} = \tilde{d} = \tilde{f} = \tilde{g} = \tilde{h} = \tilde{k} = 0$ is pragmatically asymptotically stable. Under the assumption of equal probability, it is actually asymptotically stable. State errors versus time and the estimates of uncertain parameters with $\alpha = 11$ are shown in Fig.5.3. From Fig.5.3, the robustness of the generalized synchronization is very satisfactory when $\alpha \le 11$. i.e. when there are strong highly perturbative disturbances. The robustness obtained is very high.

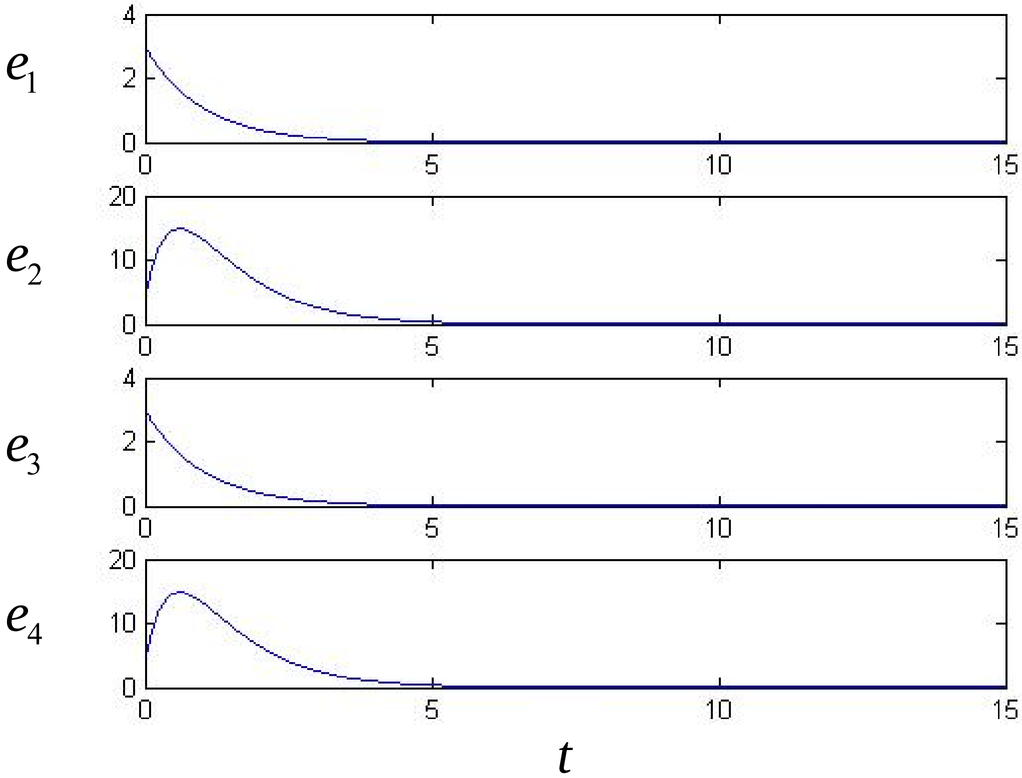


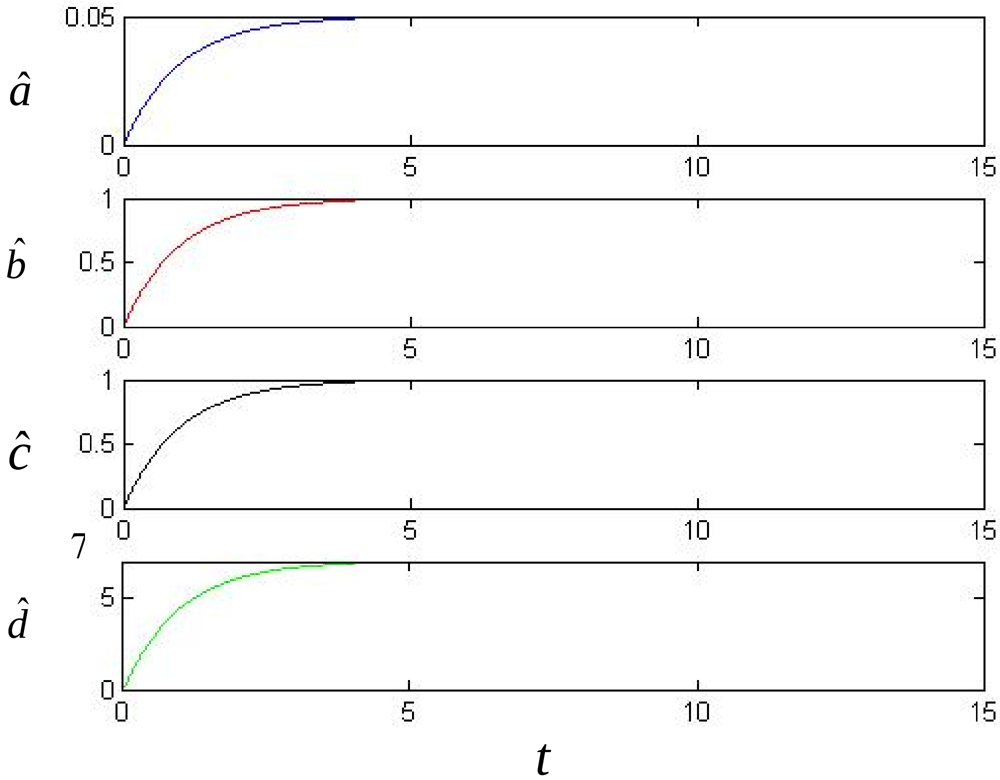
(a)



(b)

Fig 5.1 (a) The chaotic phase portrait for the double Duffing system,
(b) The chaotic phase portrait for the double van der Pol system.


e_1
e_2
e_3
e_4
t


$\hat{a}$
$\hat{b}$
$\hat{c}$
$\hat{d}$
t

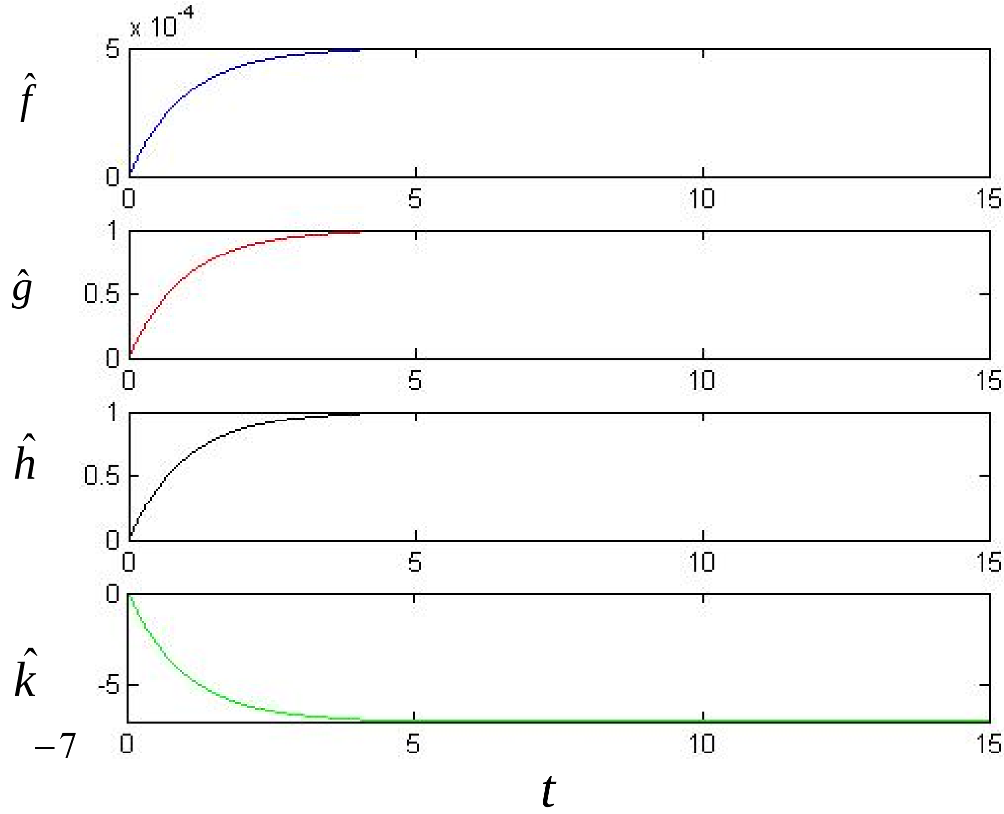
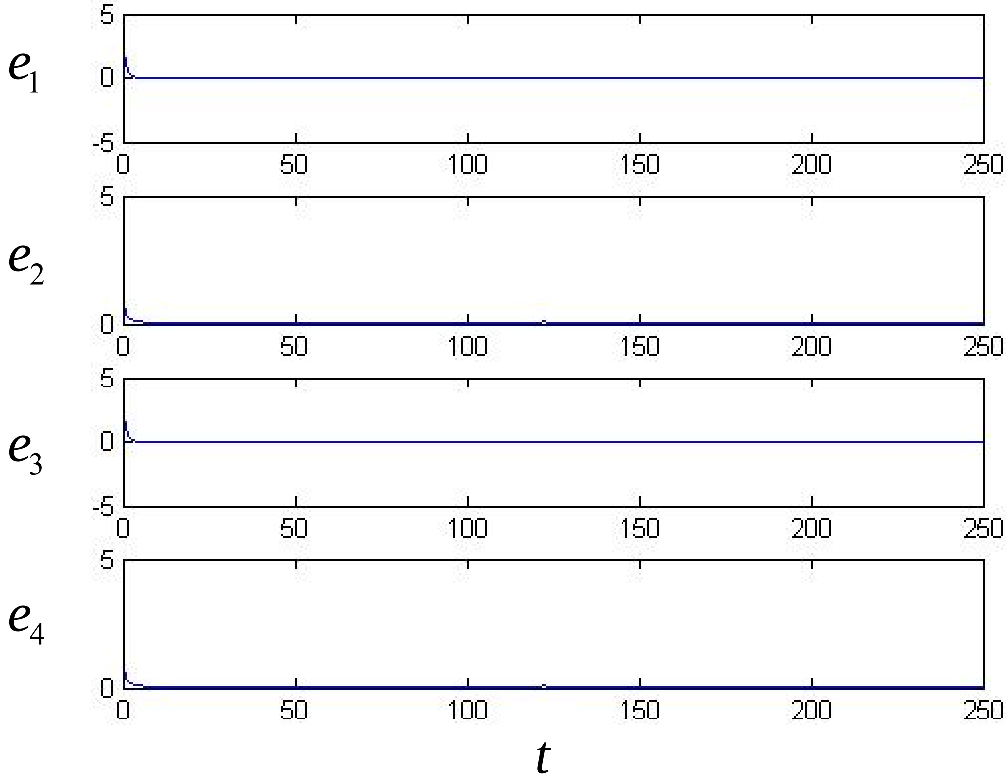
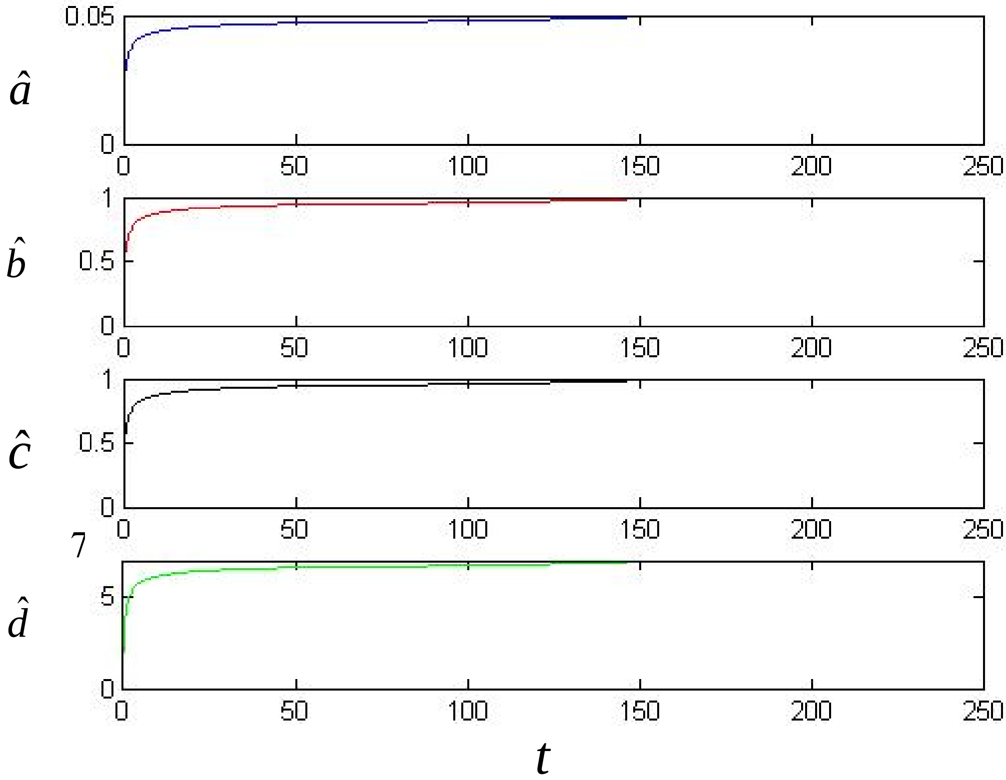


Fig 5.2 Time histories of state errors, $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{f}$, $\hat{g}$, $\hat{h}$ and $\hat{k}$ for Case 1 with $a = 0.05, b = 1, c = 3, d = 7, f = 0.0005, g = 1, h = 3, k = -7$.



e_1
e_2
e_3
e_4
t



$\hat{a}$
$\hat{b}$
$\hat{c}$
$\hat{d}$
t

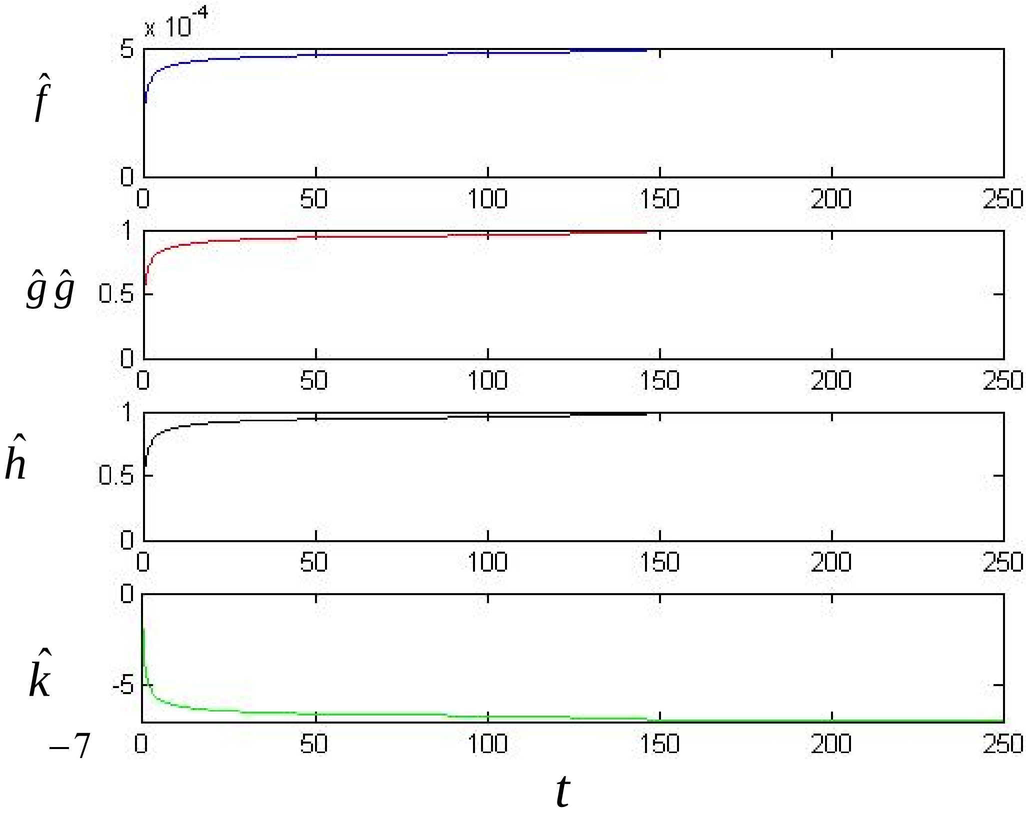


Fig 5.3 Time histories of state errors, $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{f}$, $\hat{g}$, $\hat{h}$ and $\hat{k}$ for Case 2 with $\alpha = 11$ and $a = 0.05, b = 1, c = 3, d = 7, f = 0.0005, g = 1, h = 3, k = -7$.

Chapter 6

Conclusions

Chaos synchronization is an important research topic in these years. In this thesis, chaos and chaos synchronization of double Duffing system are studied.

In Chapter 2, we have studied the chaos in the integral order and fractional order double Duffing system by phase portraits, Poincaré maps and bifurcation diagrams. The total orders of the system for the existence of chaos are 0.1 to 0.7 and 1.

In Chapter 3, parameter excited chaos synchronizations of two identical double Duffing systems are studied by adjusting the strengths of the substituting state variables. Numerical simulations are illustrated for CS or AS of which the occurrence depends on initial conditions and driving strength. Besides, alternative CS and AS is also discovered with same initial conditions and same driving strengths.

In Chapter 4, synchronization and antisynchronization scheme based on the substitution of the corresponding parameters in two identical chaotic double Duffing systems by a white noise, a Rayleigh noise, a Rician noise or a uniform noise respectively. For the white noise case, neither CS and AS are found. For the Rayleigh noise case, CS and AS are obtained for different noise strengths. For the Rician noise case and the uniform noise case, only AS is obtained. Numerical simulations show that whether CS or AS occurs is sensitive to the noise strength.

In Chapter 5, A new scheme to achieve the pragmatical generalized synchronization of adaptive control via the pragmatical asymptotical stability theorem is gavien. By the procedure of the proposed scheme, two double Duffing systems and a double van der Pol system are used as master system, slave system, and goal system ,respectively.

The validity of this approach is verified theoretically and numerically. Based on pragmatical asymptotical stability theorem, using this theorem, we can obtain the generalized synchronization of chaotic systems and prove that the estimated parameters approach the uncertain values.

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Appendix

Table 1. FRACTIONAL OPERATORS WITH APPROXIMATELY

2 db ERROR FROM $\omega = 10^{-2}$ to 10^{2} rad/sec

$\frac{1}{s^{0.1}} \approx \frac{220.4s^4 + 5004s^3 + 503s^2 + 234.5s + 0.484}{s^5 + 359.8s^4 + 5742s^3 + 4247s^2 + 147.7s + 0.2099}$
$\frac{1}{s^{0.2}} \approx \frac{60.95s^4 + 816.9s^3 + 582.8s^2 + 23.24s + 0.04934}{s^5 + 134s^4 + 956.5s^3 + 383.5s^2 + 8.953s + 0.01821}$
$\frac{1}{s^{0.3}} \approx \frac{23.76s^4 + 224.9s^3 + 129.1s^2 + 4.733s + 0.01052}{s^5 + 64.51s^4 + 252.2s^3 + 63.61s^2 + 1.104s + 0.002267}$
$\frac{1}{s^{0.4}} \approx \frac{25s^4 + 558.5s^3 + 664.2s^2 + 44.15s + 0.1562}{s^5 + 125.6s^4 + 840.6s^3 + 317.2s^2 + 7.428s + 0.02343}$
$\frac{1}{s^{0.5}} \approx \frac{15.97s^4 + 593.2s^3 + 1080s^2 + 135.4s + 1}{s^5 + 134.3s^4 + 1072s^3 + 543.4s^2 + 20.1s + 0.1259}$
$\frac{1}{s^{0.6}} \approx \frac{8.579s^4 + 255.6s^3 + 405.3s^2 + 35.93s + 0.1696}{s^5 + 94.22s^4 + 472.9s^3 + 134.8s^2 + 2.639s + 0.009882}$
$\frac{1}{s^{0.7}} \approx \frac{4.406s^4 + 177.6s^3 + 209.6s^2 + 9.179s + 0.0145}{s^5 + 88.12s^4 + 279.2s^3 + 33.3s^2 + 1.927s + 0.0002276}$
$\frac{1}{s^{0.8}} \approx \frac{5.235s^3 + 1453s^2 + 5306s + 254.9}{s^4 + 658.1s^3 + 5700s^2 + 658.2s + 1}$
$\frac{1}{s^{0.9}} \approx \frac{1.766s^2 + 38.27s + 4.914}{s^3 + 36.15s^2 + 7.789s + 0.01}$