Chapter 2

Chaos in Integral and Fractional Order Double Mackey-Glass Systems

In this Chapter, chaotic behaviors of integral and fractional order double Mackey-Glass time delay systems are studied by phase portraits and bifurcation diagrams. It is found that chaos exist both in integral order and fractional order double Mackey-Glass system.

2.1 Chaos in integral order double Mackey-Glass system

In 1977, the model of blood production is established by Mackey and Glass. It is given as:

$$\dot{x} = \frac{bx_\tau}{1+x_\tau^n} - rx \tag{2.1}$$

where $x_\tau \equiv x(t-\tau)$. The variable $x(t)$ represents the production of blood at time *t*, and $x(t-\tau)$ is the concentration when the request for more blood is made after τ seconds. The delay time τ may become excessively large in the patients of leukemia, and makes the blood oscillate. When τ is even larger, the concentration can vary chaotically.

In this thesis, we consider a new delay system which consists of two coupled Mackey-Glass systems, called double Mackey-Glass system:

$$\begin{cases} \dot{x}_1 = \dfrac{b_1 x_{1\tau_1}}{1+x_{1\tau_1}^n} - rx_1 \\ \dot{x}_2 = \dfrac{b_2 x_{2\tau_2}}{1+x_{2\tau_2}^n} - rx_2 - x_1 \end{cases} \tag{2.2}$$

where b_1, b_2, r and n are constants, and the delay times of x_1 and x_2 can be represented as τ_1 and τ_2, respectively. When the time delays are varied, the chaotic behaviors are also changed. The phase portraits and the bifurcation diagrams are showed in Fig. 2.1 and Fig. 2.2, where $b_1 = b_2 = 0.2$, $r = 0.1$, $n = 10$. Fig. 2.1 displays the chaos when τ_1 is varied while $\tau_2 = 20$ is fixed. Fig. 2.2 shows the existence of chaos when τ_1 and τ_2 $(\tau_1 = \tau_2 = \tau)$ are varied synchronously. Furthermore, we also observe that the chaos exists when the parameter b_1 is varied. The result is shown in Fig. 2.3.

2.2 Definition and approximation of fractional order operator

It is well known that the fractional derivatives/integrals have been defined in a variety of ways. According to the definition of Riemann-Liouville, it could be shown bellow:

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau \tag{2.3}$$

where q is a non-integer and n is an integer such that $n-1 < q < n$. This definition differs greatly from the definition of common differential, because the fractional differential is implemented via approximating fractional operators using the standard integer order operators. In this thesis, the approximation approach taken here is that of [81]**.** The basic idea is utilizing the frequency domain technique based on Bode diagrams. It is done by creating an approximation of which the Bode magnitude response is $20q$ dB/decade and which have a phase shift about $90q$ degrees over the required frequency band. If we choose the number of pole, the allowable error and initial breakpoint appropriately, then approximation of any desired accuracy over any frequency band can be achieved.

Although this appear is different from the usually intuitive definition, the basic engineering tool for analyzing linear systems, the Laplace transform, still works. The Laplace transformation of the Riemann-Liouville fractional derivative (2.3) is

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{q-1-k} f(t)}{dt^{q-1-k}}\right]_{t=0} \text{, for all q and } n-1 \le q < n \quad (2.4)$$

If the initial conditions are considered to be zero, this formula reduces to the more expected and comforting form

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} \quad (2.5)$$

Table in appendix of [82] gives approximations for $\frac{1}{s^q}$ with q=0.1~0.9 in steps 0.1. Each has the error of approximately 2 dB from $\omega = 10^{-2}$ to 10^2 rad/s. These approximations will be used in the following simulation.

The fractional order form of Double Mackey-Glass system can be rewritten from (2.2) as follow:

$$\begin{cases} \dfrac{d^{q_1} x_1}{dt^{q_1}} = \dfrac{b_1 x_{1\tau_1}}{1 + x_{1\tau_1}^n} - r x_1 \\ \dfrac{d^{q_2} x_2}{dt^{q_2}} = \dfrac{b_2 x_{2\tau_2}}{1 + x_{2\tau_2}^n} - r x_2 - x_1 \end{cases} \quad (2.6)$$

where q_1 and q_2 are noninteger values.

2.3 The results of numerical simulations for fractional order systems

The parameters are given: $b_1 = b_2 = b = 0.2$, $r = 0.1$, $n = 10$ and we vary the delay time to study the chaos behaviors for fractional order systems. Two cases are studied: (i) Only τ_1 is varied, $\tau_2 = 20$ is fixed (ii) τ_1, τ_2 are varied synchronously. All simulation is accomplished by Matlab with using the fractional operator in Table I of [82] in the Simulink environment. The results of simulations are summarized in Table I.

When we vary τ_1 only, the chaos exists with the order 0.9, 0.8 and 0.1. The phase portraits and the bifurcation diagrams are showed in Fig. 2.4~Fig. 2.6. Then we vary τ_1 and τ_2 simultaneously, the chaos can also be observed in the order 0.9, 0.8

and 0.1. The phase portraits and the bifurcation diagrams are showed in Fig. 2.7~ Fig. 2.9. There are period 1, period 2 and chaotic behavior in the systems with order 0.9 and 0.8. However, for the order 0.1 only chaotic motions are found. For other orders, no chaos is found. Most of the phase portraits of order 0.7~0.2 are curviform. A few of them have some loops. No chaotic behaviors are found in these orders. In Fig. 2.10~Fig. 2.12, we pick some of them to show the situation.

Table I. Relation between orders of derivatives and existence of chaos.

Only τ_1 is varied, $\tau_2 = 20$				τ_1 and τ_2 are varied synchronously($\tau_1 = \tau_2 = \tau$)			
Case	Order	Existence of Chaos	Fig. No	Case	Order	Existence of Chaos	Fig. No
1	0.9	Yes	2.4	10	0.9	Yes	2.7
2	0.8	Yes	2.5	11	0.8	Yes	2.8
3	0.7	No		12	0.7	No	
4	0.6	No		13	0.6	No	
5	0.5	No		14	0.5	No	2.12
6	0.4	No		15	0.4	No	
7	0.3	No	2.10	16	0.3	No	
8	0.2	No	2.11	17	0.2	No	
9	0.1	Yes	2.6	18	0.1	Yes	2.9