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Synchronization of chaotic systems with uncertain chaotic parameters by linear coupling and pragmatical adaptive tracking

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We study the synchronization of general chaotic systems which satisfy the Lipschitz condition only, with uncertain **chaotic** parameters by linear coupling and pragmatical adaptive tracking. The uncertain parameters of a system vary with time due to aging, environment, and disturbances. A sufficient condition is given for the asymptotical stability of common zero solution of error dynamics and parameter update dynamics by the Ge–Yu–Chen pragmatical asymptotical stability theorem based on equal probability assumption. Numerical results are studied for a Lorenz system and a quantum cellular neural network oscillator to show the effectiveness of the proposed synchronization strategy. © *2008 American Institute of Physics*. [DOI: 10.1063/1.3049320]

Theoretical and experimental investigations have shown that synchronization, in particular chaos synchronization, has great potential in a large amount of application areas ranging from secure communications to modeling brain activity. In this paper, we introduce a synchronization of chaotic systems with uncertain chaotic parameters by linear coupling and pragmatical adaptive tracking. Based on pragmatical stability theorem and Lipschitz condition, some less conservative conditions for determining linear coupling synchronization of general chaotic systems are obtained. Two examples are simulated to illustrate the validity of the theoretical analysis.

I. INTRODUCTION

The idea of synchronizing two identical chaotic systems with different initial conditions was introduced by Pecora and Carroll.[1] Since then there has been particular interest in chaotic synchronization, due to many potential applications in secure communication,[2] chemical and biological systems.[3,4] There are many control methods to synchronize chaotic systems, such as, linear coupling, for which the implementation is rather easy, adaptive control, impulsive control, sliding mode control, and other methods.[5] Most of them are based on the exact knowledge of the system structure and parameters. But in practice, some or all of the system parameters are uncertain. Moreover these parameters may change from time to time and become chaotic because of chaotic disturbances. For uncertain parameters, a lot of works have proceeded to solve this problem by adaptive synchronization.[6–12] In the current scheme of adaptive synchronization,[13–15] the traditional Lyapunov stability theorem and Barbalat lemma are used to prove that the error vector approaches zero as time approaches infinity. But the question, why the estimated parameters also approach the uncertain parameters, has remained without answer. From the Ge–Yu–Chen (GYC) pragmatical asymptotical stability theorem,[16–18] the question is strictly answered. In this paper, the synchronization of general chaotic systems which satisfy the Lipschitz condition only, with unknown parameters which are altered under some **chaotic** disturbances, by linear coupling and pragmatical adaptive tracking, is studied first.

As numerical examples, the Lorenz system and recently developed quantum cellular neural network (Quantum-CNN) chaotic oscillator are used. Pragmatical adaptive tracking is used to track **chaotic** parameters in unidirectional coupled systems. Two Lorenz systems and two Quantum-CNN systems by pragmatical adaptive tracking are given as simulation examples. Quantum-CNN oscillator equations are derived from a Schrödinger equation taking into account quantum dots cellular automata structures to which in the last decade a wide interest has been devoted with particular attention towards quantum computing.[19–21]

This paper is organized as follows: In Sec. II, by pragmatical asymptotical stability theorem and by using Lipschitz conditions, theoretical analysis of synchronization is given. In Sec. III linear feedback controllers are used. By pragmatical adaptive tracking, chaos synchronization of two Lorenz systems and of two Quantum-CNN oscillator systems are achieved by numerical simulations. Conclusions are given in Sec. IV. GYC pragmatical asymptotical stability theorem is presented in the Appendix. Intuitively this theorem is different from traditional Lyapunov stability theorem at that when the points in the neighborhood of zero solution initiating trajectories not approaching zero with time are "not too many," i.e., in a subset of Lebesque measure 0 in mathematical language,[22] we can neglect their existence, i.e., the zero solution is actually asymptotically stable.

II. STRATEGY OF THE CHAOTIC SYNCHRONIZATION

Consider a nonautonomous system in the form as follows:

$$\dot{x} = F[t,x,B(t)]. \tag{1}$$

The slave system is given by

$$\dot{y} = F[t,y,\hat{B}(t)] + \hat{K}(x-y), \tag{2}$$

where $x=[x_1,x_2,\ldots,x_n]^T \in R^n$, $y=[y_1,y_2,\ldots,y_n]^T \in R^n$, and $B=[B_1,B_2,\ldots,B_M]^T \in R^M$ is a vector of uncertain **chaotic** coefficients in F, $\hat{B}=[\hat{B}_1,\hat{B}_2,\ldots,\hat{B}_M]^T \in R^M$ is a vector of estimated coefficients in F, $F:\Omega \subset R_+ \times R^n \times R^M \to R^n$ satisfies Lipschitz conditions $\|F(t,x_{\mathrm{I}},B)-F(t,x_{\mathrm{II}},B)\| \leqslant G\|x_{\mathrm{I}}-x_{\mathrm{II}}\|$, where x_{I} and x_{II} are two neighbor state vectors, and $\|F(t,x,B)-F(t,x,\hat{B})\| \leqslant G\|B-\hat{B}\|$ in Ω with Lipschitz constant G. $\hat{K}=\mathrm{diag}[\hat{K}_1,\ldots,\hat{K}_i,\ldots,\hat{K}_n]$ is a constant matrix. $\hat{K}(x-y)$ is the estimated linear coupling term. Ω is the domain containing the origin. For given $(t_0,x_0,y_0,B_0) \in \Omega$, the solutions $[x^T(t,t_0,x_0,B_0),y^T(t,t_0,x_0,y_0,B_0)]^T$ of Eqs. (1) and (2) exist for $t \geqslant t_0$.

If the synchronization can be accomplished when $t \to \infty$, the limit of the error vector $e(t)=[e_1,e_2,\ldots,e_n]^T$ must approach zero,

$$\lim_{t\to\infty} e = 0, \tag{3}$$

where

$$e = x - y. \tag{4}$$

From Eqs. (1), (2), and (4), we have

$$\dot{e} = \dot{x} - \dot{y}, \tag{5}$$

$$\dot{e} = F(t,x,B) - F(t,x-e,\hat{B}) - \hat{K}(x-y). \tag{6}$$

A Lyapunov function $V(e,\tilde{B},\tilde{G})$ is chosen as a positive definite function

$$V(e,\tilde{B},\tilde{G}) = \tfrac{1}{2}e^Te + \tfrac{1}{2}\tilde{B}^T\tilde{B} + \tfrac{1}{2}\tilde{G}^2, \tag{7}$$

where $\tilde{G}=G-\hat{G}$; $\hat{G}$ is the estimated Lipschitz constant, $\tilde{B}=B-\hat{B}$.

When $M=n$, the time derivative of V along any solution of the differential equation system consisting of Eq. (6) and update differential equations for $\tilde{B}$ and $\tilde{G}$ is

$$\begin{aligned}\dot{V}(e,\tilde{B},\tilde{G}) &= e^T[F(t,x,B) - F(t,x-e,B) + F(t,x-e,B) \\ &\quad - F(t,x-e,\hat{B}) - \hat{K}e] + \tilde{B}^T\dot{\tilde{B}} + \tilde{G}\dot{\tilde{G}} \\ &= e^T[F(t,x,B) - F(t,x-e,B) - \hat{K}e] + \tilde{G}\dot{\tilde{G}} \\ &\quad + e^T[F(t,x-e,B) - F(t,x-e,\hat{B})] + \tilde{B}^T\dot{\tilde{B}}.\end{aligned} \tag{8}$$

From the Lipschitz condition,

$$\begin{aligned}\dot{V}(e,\tilde{B},\tilde{G}) &\leqslant G\|e\|^2 - e^T\hat{K}e + \tilde{G}\dot{\tilde{G}} + e^T[F(t,x-e,B) \\ &\quad - F(t,x-e,\hat{B})] + \tilde{B}^T\dot{\tilde{B}}.\end{aligned} \tag{9}$$

Since

$$\begin{aligned}&e^T[F(t,x-e,B) - F(t,x-e,\hat{B})] \\ &\quad \leqslant |e_1|\cdot|F_1(t,x-e,B) - F_1(t,x-e,\hat{B})| \\ &\qquad + \cdots + |e_n|\cdot|F_n(t,x-e,B) - F_n(t,x-e,\hat{B})|\end{aligned} \tag{10}$$

by Schwarz inequality and Lipschitz condition, it is obtained that

$$\begin{aligned}&|e_1|\cdot|F_1(t,x-e,B) - F_1(t,x-e,\hat{B})| \\ &\quad + \cdots + |e_n|\cdot|F_n(t,x-e,B) - F_n(t,x-e,\hat{B})| \\ &\leqslant \|e\|\cdot\|F(t,x-e,B) - F(t,x-e,\hat{B})\| \leqslant G\|e\|\cdot\|\tilde{B}\|.\end{aligned} \tag{11}$$

Therefore,

$$\begin{aligned}\dot{V}(e,\tilde{B},\tilde{G}) &\leqslant G\|e\|^2 - e^T\hat{K}e + \tilde{G}\dot{\tilde{G}} + G\|e\|\cdot\|\tilde{B}\| + \tilde{B}_1\dot{\tilde{B}}_1 \\ &\quad + \cdots + \tilde{B}_n\dot{\tilde{B}}_n.\end{aligned} \tag{12}$$

Choosing

$$\dot{\tilde{G}} = -e^Te, \quad \hat{K} = \mathrm{diag}[\hat{G}+G] \tag{13}$$

and choosing

$$\dot{\tilde{B}}_1 = -G\tilde{B}_1\|e\|/\|\tilde{B}\|, \cdots, \dot{\tilde{B}}_n = -G\tilde{B}_n\|e\|/\|\tilde{B}\|, \tag{14}$$

we have

$$\begin{aligned}\tilde{B}^T\dot{\tilde{B}} &= -G(\tilde{B}_1^2 + \cdots \tilde{B}_n^2)\|e\|/\|\tilde{B}\| \\ &= -G\|\tilde{B}\|^2\cdot\|e\|/\|\tilde{B}\| \\ &= -G\|e\|\cdot\|\tilde{B}\|.\end{aligned} \tag{15}$$

Introducing Eqs. (15) and (13) in Eq. (12), we get

$$\begin{aligned}\dot{V}(e,\tilde{B},\tilde{G}) &\leqslant G\|e\|^2 - \mathrm{diag}[\hat{G}+G]\|e\|^2 - \tilde{G}\|e\|^2 \\ &\quad + G\|e\|\cdot\|\tilde{B}\| - G\|e\|\cdot\|\tilde{B}\| \\ &= -G\|e\|^2 \\ &= -G(e_1^2 + \cdots + e_n^2),\end{aligned} \tag{16}$$

$\dot{V}$ is a negative semidefinite function of $e,\tilde{B},\tilde{G}$. By GYC pragmatical asymptotical stability theorem (see Appendix), the solution $e=0$, $\tilde{B}=0$, $\tilde{G}=0$ is asymptotically stable, which means that the two coupled systems are synchronized even if different initial conditions are used and the estimation of the parameters is not exact.

When $M \neq n$, all the other terms in Eq. (9) are kept unchanged, and only the last two terms will be reduced as follows. When $M > n$, we put

$$e^T = [e_1,\ldots,e_n,e_{n+1},\ldots,e_M]^T, \tag{17}$$

where $e_{n+1}=e_{n+2}=\cdots=e_M=0$. Then we have

$$e^T[F(t,x-e,B)-F(t,x-e,\hat{B})] \leq |e_1|\cdot|F_1(t,x-e,B)-F_1(t,x-e,\hat{B})|+\cdots+|e_n|\cdot|F_n(t,x-e,B)-F_n(t,x-e,\hat{B})| + |e_{n+1}|\cdot|F_{n+1}(t,x-e,B)-F_{n+1}(t,x-e,\hat{B})|+\cdots+|e_M|\cdot|F_M(t,x-e,B)-F_M(t,x-e,\hat{B})| \leq G\|e\|\cdot\|\tilde{B}\|. \quad (18)$$

In Eq. (18), the last term is obtained by the Schwarz inequality. Similar to Eqs. (14) and (15) in which n is substituted by M, we choose $\tilde{B}_1\cdots\tilde{B}_M$, then

$$\tilde{B}^T\dot{\tilde{B}} = -G\|e\|\cdot\|\tilde{B}\| \quad (19)$$

is obtained.

Introducing Eqs. (18) and (19) in Eq. (9), we can also get, lastly,

$$\dot{V}(e,\tilde{B},\tilde{G}) \leq -G(e_1^2+\ldots+e_n^2). \quad (20)$$

By the same reasoning as when $M=n$, the solution $e=0$, $\tilde{B}=0$, $\tilde{G}=0$ is asymptotically stable.

When $M<n$, we put

$$F_i(t,x-e,B)-F_i(t,x-e,\hat{B})=0, \quad i=M+1,\ldots,n \quad (21)$$

since $B_{M+1},\cdots B_n$ does not exist,

$$\tilde{B}_{M+1}=\cdots=\tilde{B}_n=0, \quad (22)$$

$$\|\tilde{B}\|^2=\tilde{B}_1^2+\cdots+\tilde{B}_M^2+\tilde{B}_{M+1}^2+\cdots+\tilde{B}_n^2. \quad (23)$$

Then by the Schwarz inequality, we can obtain the same result as Eq. (18) except that n and M are exchanged. Similarly, choose

$$\dot{\tilde{B}}_1=-G\tilde{B}_1\|e\|/\|\tilde{B}\|,\ldots,\dot{\tilde{B}}_M=-G\tilde{B}_M\|e\|/\|\tilde{B}\|,$$
$$\dot{\tilde{B}}_{M+1}=-G\tilde{B}_{M+1}\|e\|/\|\tilde{B}\|,\ldots,\dot{\tilde{B}}_n=-G\tilde{B}_n\|e\|/\|\tilde{B}\|, \quad (24)$$

$$\tilde{B}^T\dot{\tilde{B}}=-G(\tilde{B}_1^2+\cdots\tilde{B}_n^2)\|e\|/\|\tilde{B}\| = -G\|\tilde{B}\|^2\|e\|/\|\tilde{B}\| = -G\|e\|\cdot\|\tilde{B}\|. \quad (25)$$

Introducing Eq. (18) in which n and M are exchanged and Eq. (25) in Eq. (9), we can also get lastly

$$\dot{V}(e,\tilde{B},\tilde{G}) \leq -G(e_1^2+\cdots+e_n^2) = -Ge^Te. \quad (26)$$

By the same reasoning as the case $M=n$, the solution $e=0$, $\tilde{B}=0$, $\tilde{G}=0$ is asymptotically stable.

Remark. In the current scheme of adaptive synchronization,[13–15] traditional Lyapunov stability theorem and Barbalat lemma are used to prove the error vector approaches zero, as time approaches infinity. But the question, why the estimated parameters also approach uncertain parameters, remains no answer. By GYC pragmatical asymptotical stability theorem, the question can be answered strictly. Moreover, the asymptotical stability is global, see the Appendix.

III. NUMERICAL SIMULATIONS

Case I: Chaotic parameters for the Lorenz system, $M<n(2<3)$.

The master Lorenz system with uncertain chaotic parameters is

$$\begin{aligned}\dot{x}_1&=-A_1(t)(x_1-x_2),\\ \dot{x}_2&=A_2(t)x_1-x_2-x_1x_3,\\ \dot{x}_3&=x_1x_2-A_3(t)x_3,\end{aligned} \quad (27)$$

where $A_1(t)$ and $A_2(t)$ are uncertain parameters, $A_3(t)$ is the given parameter. In simulation, we take

$$\begin{aligned}A_1(t)&=\sigma(1+d_1z_1),\\ A_2(t)&=\gamma(1+d_2z_2),\\ A_3(t)&=b(1+d_3z_3),\end{aligned} \quad (28)$$

where d_1, d_2, and d_3 are positive constants.

The chaotic signals z_1,z_2,z_3, are the states of

$$\begin{aligned}\dot{z}_1&=-\sigma_1(z_1-z_2),\\ \dot{z}_2&=\gamma_1z_1-z_2-z_1z_3,\\ \dot{z}_3&=z_1z_2-b_1z_3,\end{aligned} \quad (29)$$

where $\sigma_1=8$, $\gamma_1=27$, $b_1=3.2$, and $[z_0^T]^T=[222]^T$.

From Eq. (2), the slave Lorenz system is

$$\begin{aligned}\dot{y}_1&=-\hat{A}_1(t)(y_1-y_2)+(\hat{G}+G)(x_1-y_1),\\ \dot{y}_2&=\hat{A}_2(t)y_1-y_2-y_1y_3+(\hat{G}+G)(x_2-y_2),\\ \dot{y}_3&=y_1y_2-A_3(t)y_3+(\hat{G}+G)(x_3-y_3),\end{aligned} \quad (30)$$

where $\hat{A}_1(t)$ and $\hat{A}_2(t)$ are estimated parameters. The initial condition be $[x_0^T\ y_0^T\ \hat{A}_0^T\ \hat{G}_0]^T=[111\ 000\ 00\ 0]^T$.

Subtracting Eq. (30) from Eq. (27), we obtain an error dynamics,

$$\begin{aligned}\dot{e}_1&=-A_1(t)(x_1-x_2)+\hat{A}_1(t)(y_1-y_2)-(\hat{G}+G)(x_1-y_1),\\ \dot{e}_2&=A_2(t)x_1-x_2-x_1x_3-\hat{A}_2(t)y_1+y_2+y_1y_3\\ &\quad-(\hat{G}+G)(x_2-y_2),\\ \dot{e}_3&=x_1x_2-A_3(t)x_3-y_1y_2+A_3(t)y_3-(\hat{G}+G)(x_3-y_3),\end{aligned} \quad (31)$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2$, $e_3=x_3-y_3$.

Our aim is

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i) = 0, \quad i = 1,2,3. \tag{32}$$

Let the adaptive law be

$$\dot{\tilde{G}} = \dot{G} - \dot{\hat{G}} = -\dot{\hat{G}} = -e^T e. \tag{33}$$

Since G is constant, $\dot{G}=0$. Define

$$\tilde{A}(t) = [\tilde{A}_1(t)\tilde{A}_2(t)]^T, \tag{34}$$

$$\tilde{A}_1(t) = A_1(t) - \hat{A}_1(t), \quad \tilde{A}_2(t) = A_2(t) - \hat{A}_2(t), \tag{35}$$

then

$$\dot{\tilde{A}}_1(t) = \sigma d_1 \dot{z}_1 - \dot{\hat{A}}_1(t), \quad \dot{\tilde{A}}_2(t) = \gamma d_2 \dot{z}_2 - \dot{\hat{A}}_2(t). \tag{36}$$

Choose $\dot{\tilde{A}}_1(t)$ and $\dot{\tilde{A}}_2(t)$ as

$$\dot{\tilde{A}}_1 = -G\tilde{A}_1\|e\|/\|\tilde{A}\|, \quad \dot{\tilde{A}}_2 = -G\tilde{A}_2\|e\|/\|\tilde{A}\|. \tag{37}$$

A Lyapunov function is given in the form of the positive definite function,

$$V(e_1,e_2,e_3,\tilde{A}_1,\tilde{A}_2,\tilde{G}) = \tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2 + \tilde{A}_1^2 + \tilde{A}_2^2 + \tilde{G}^2). \tag{38}$$

Its time derivative along any solution of Eqs. (31), (32), and (37) is

$$\dot{V} = e_1[-A_1(t)(x_1 - x_2) + \hat{A}_1(t)(y_1 - y_2) - (\hat{G}+G)(x_1 - y_1)] + e_2[A_2(t)x_1 - x_2 - x_1x_3 - \hat{A}_2(t)y_1 + y_2 + y_1y_3 - (\hat{G}+G)(x_2 - y_2)]$$
$$+ e_3[x_1x_2 - A_3(t)x_3 - y_1y_2 + A_3(t)y_3 - (\hat{G}+G)(x_3 - y_3)] + \tilde{A}_1\dot{\tilde{A}}_1 + \tilde{A}_2\dot{\tilde{A}}_2 - \tilde{G}\dot{\hat{G}},$$

$$\dot{V} = e_1[-A_1(t)(x_1 - x_2) + A_1(t)(y_1 - y_2) - (\hat{G}+G)(x_1 - y_1)] + e_2[A_2(t)x_1 - x_2 - x_1x_3 - A_2(t)y_1 + y_2 + y_1y_3 - (\hat{G}+G)(x_2 - y_2)]$$
$$+ e_3[x_1x_2 - A_3(t)x_3 - y_1y_2 + A_3(t)y_3 - (\hat{G}+G)(x_3 - y_3)] + \tilde{A}_1(y_1 - y_2)e_1 - \tilde{A}_2 y_1 e_2 - G\|e\|(\tilde{A}_1^2 + \tilde{A}_2^2)/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}},$$

$$\dot{V} \leqslant G\|e\|^2 - (\hat{G}+G)\|e\|^2 + G\|e\|\|\tilde{A}\| - G\|e\|\|\tilde{A}\|^2/\|\tilde{A}\| - \tilde{G}\dot{\hat{G}}.$$

$\dot{V}$ can be rewritten as

$$\dot{V}(e_i) \leqslant -G\|e\|^2. \tag{39}$$

$\dot{V}$ is a negative semidefinite function of $e,\tilde{A},\tilde{G}$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (31), adaptive laws (33), and parameter dynamics (37) is asymptotically stable. Now, D is a 3-manifold, $n=6$ and the number of error state variables $p=3$. When $e_i=0$, $(i=1,2,3)$ and $\tilde{A}_j$, $\tilde{G}$, $(j=1,2)$ take arbitrary values, $\dot{V}=0$, so X is a 3-manifold, $m=n-p=6-3=3$. $m+1<n$ is satisfied. By GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The pragmatical generalized synchronization is obtained. The equilibrium point $e_i=\tilde{A}_j=\tilde{G}=0$ ($i=1,2,3$; $j=1,2$) is asymptotically stable. Moreover, the result is global asymptotically stable (see Appendix). The numerical results of the time series of states, state errors, parameters, and estimated Lipschitz constant $\hat{G}$ are shown in Figs. 1 and 2. The chaos synchronization is accomplished at 0.6 s. $\hat{G}$ approaches constant near 0.5 s. The coupling strength required is $K=2G=39.34$.

Case II: Chaotic parameters for the Quantum-CNN system, $M=n(4=4)$.

For a two-cell Quantum-CNN, the following differential equations are obtained:[8–20]

$$\begin{aligned}
\dot{x}_1 &= -2a_1\sqrt{1-x_1^2}\,\sin x_2,\\
\dot{x}_2 &= -\omega_1(x_1 - x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2,\\
\dot{x}_3 &= -2a_2\sqrt{1-x_3^2}\,\sin x_4,\\
\dot{x}_4 &= -\omega_2(x_3 - x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4,
\end{aligned} \tag{40}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase dis-

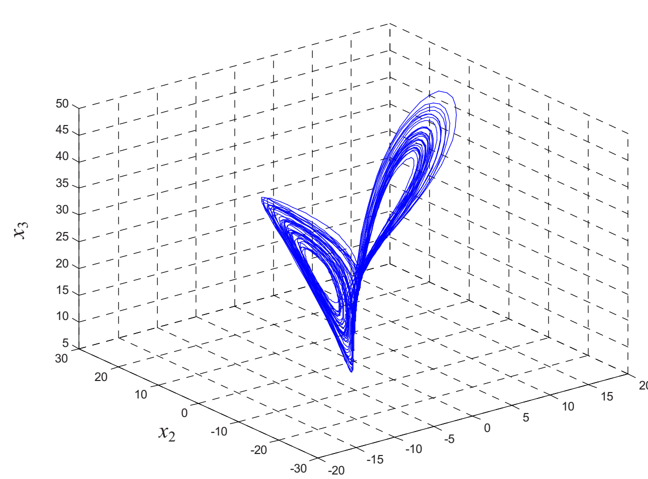

FIG. 1. (Color online) Phase portrait for the Lorenz system with $\sigma=10$, $\gamma=28$, $b=8/3$.

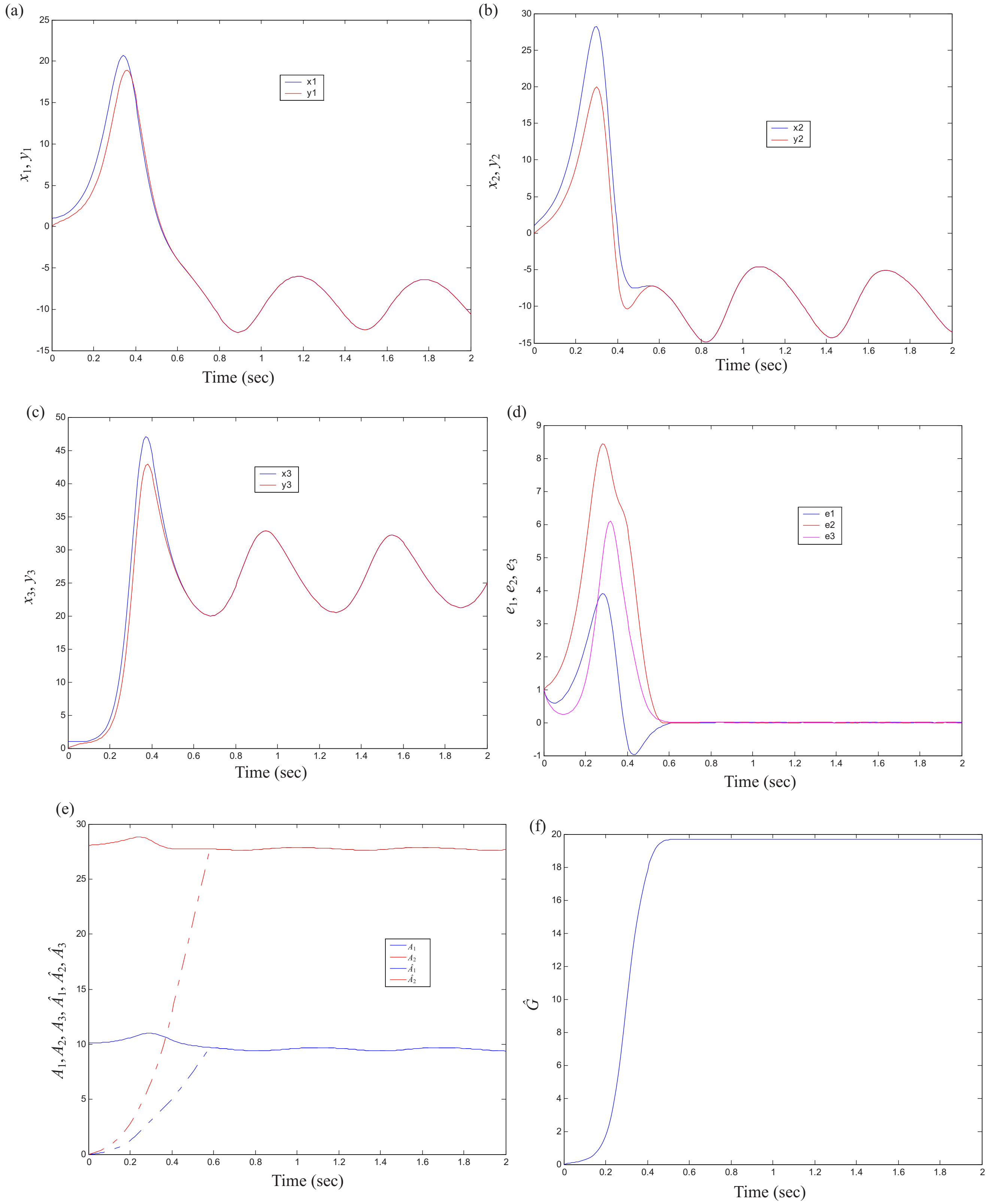

FIG. 2. (Color online) Time series of states, state errors, $A_1,A_2,A_3,\hat{A}_1,\hat{A}_2,\hat{A}_3$ and estimated Lipschitz constant $\hat{G}$ for Case I.

placements, a_1 and a_2 are proportional to the interdot energy inside each cell and ω_1 and ω_2 are parameters that weigh effects on the cell of the difference of the polarization of neighboring cells, like the cloning templates in traditional CNNs. When $a_1=6.8$, $a_2=4.3$, $\omega_1=4.7$, $\omega_2=3.9$, the system is chaotic as shown in Fig. 3.

The master Quantum-CNN system with uncertain chaotic parameters is

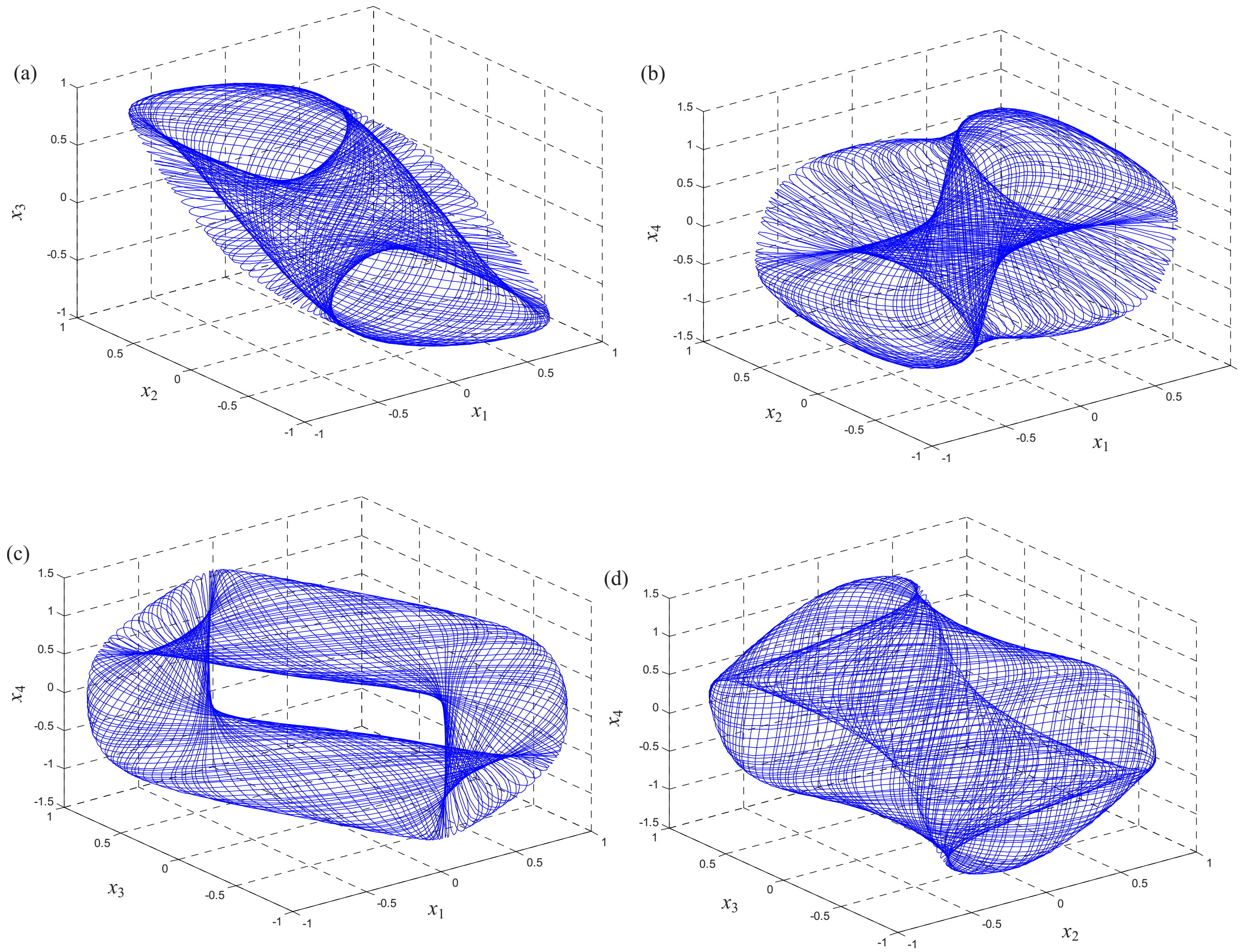

FIG. 3. (Color online) Projections of phase portraits for chaotic system (40).

$$\begin{aligned}
\dot{x}_1 &= -2A_1(t)\sqrt{1-x_1^2}\sin x_2,\\
\dot{x}_2 &= -A_3(t)(x_1-x_3)+2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2,\\
\dot{x}_3 &= -2A_2(t)\sqrt{1-x_3^2}\sin x_4,\\
\dot{x}_4 &= -A_4(t)(x_3-x_1)+2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4,
\end{aligned} \tag{41}$$

where $A_1(t)$, $A_2(t)$, $A_3(t)$, and $A_4(t)$ are uncertain parameters (see Fig. 4). In simulation, we take

$$\begin{aligned}
A_1(t) &= a_1(1+d_1z_1), \quad A_2(t) = a_2(1+d_2z_2),\\
A_3(t) &= \omega_1(1+d_3z_3), \quad A_4(t) = \omega_2(1+d_4z_4),
\end{aligned} \tag{42}$$

where d_1, d_2, and d_3 are positive constants. Take $d_1=0.039$, $d_2=0.043$, $d_3=0.045$, and $d_4=0.038$. This system is chaotic as shown in Fig. 5.

The chaotic signals z_1,z_2,z_3,z_4 are the states of

$$\begin{aligned}
\dot{z}_1 &= -2a_{21}\sqrt{1-z_1^2}\sin z_2,\\
\dot{z}_2 &= -\omega_{21}(z_1-z_3)+2a_{21}\frac{z_1}{\sqrt{1-z_1^2}}\cos z_2,\\
\dot{z}_3 &= -2a_{22}\sqrt{1-z_3^2}\sin z_4,\\
\dot{z}_4 &= -\omega_{22}(z_3-z_1)+2a_{22}\frac{z_3}{\sqrt{1-z_3^2}}\cos z_4,
\end{aligned} \tag{43}$$

where $a_{21}=5.2$, $a_{22}=4.2$, $\omega_{21}=4.7$, and $\omega_{22}=3.5$.

From Eq. (2), the slave Quantum-CNN system is

$$\begin{aligned}
\dot{y}_1 &= -2\hat{a}_1\sqrt{1-y_1^2}\sin y_2+(\hat{G}+G)(x_1-y_1),\\
\dot{y}_2 &= -\hat{\omega}_1(y_1-y_3)+2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2+(\hat{G}+G)(x_2-y_2),\\
\dot{y}_3 &= -2\hat{a}_2\sqrt{1-y_3^2}\sin y_4+(\hat{G}+G)(x_3-y_3),\\
\dot{y}_4 &= -\hat{\omega}_2(y_3-y_1)+2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4+(\hat{G}+G)(x_4-y_4).
\end{aligned} \tag{44}$$

Subtracting Eq. (44) from Eq. (41), we obtain an error dynamics. The initial values are taken as $x_1(0)=0.8$, $x_2(0)=-0.77$, $x_3(0)=-0.72$, $x_4(0)=0.57$, $y_1(0)=-0.2$, $y_2(0)=0.41$,

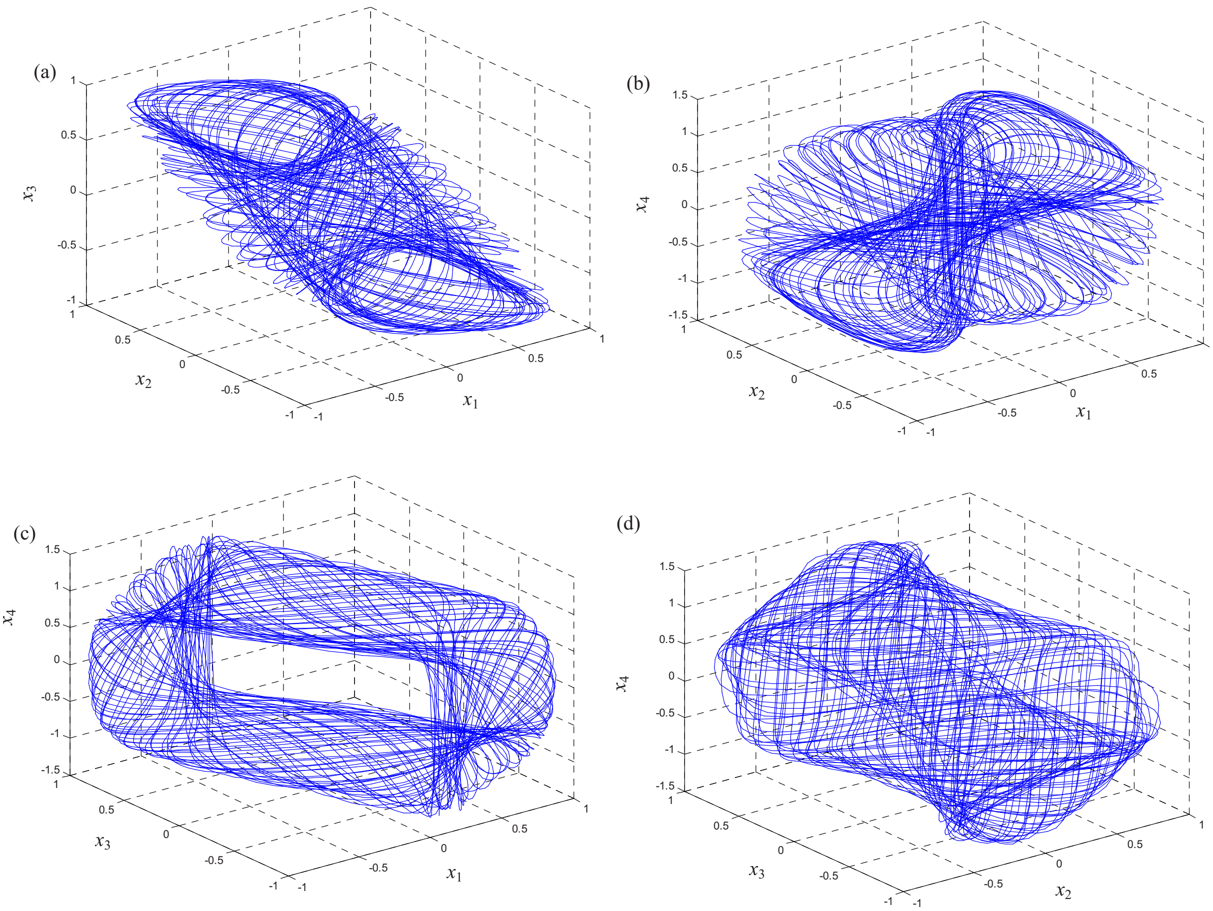

FIG. 4. (Color online) Projections of phase portraits for chaotic system (41).

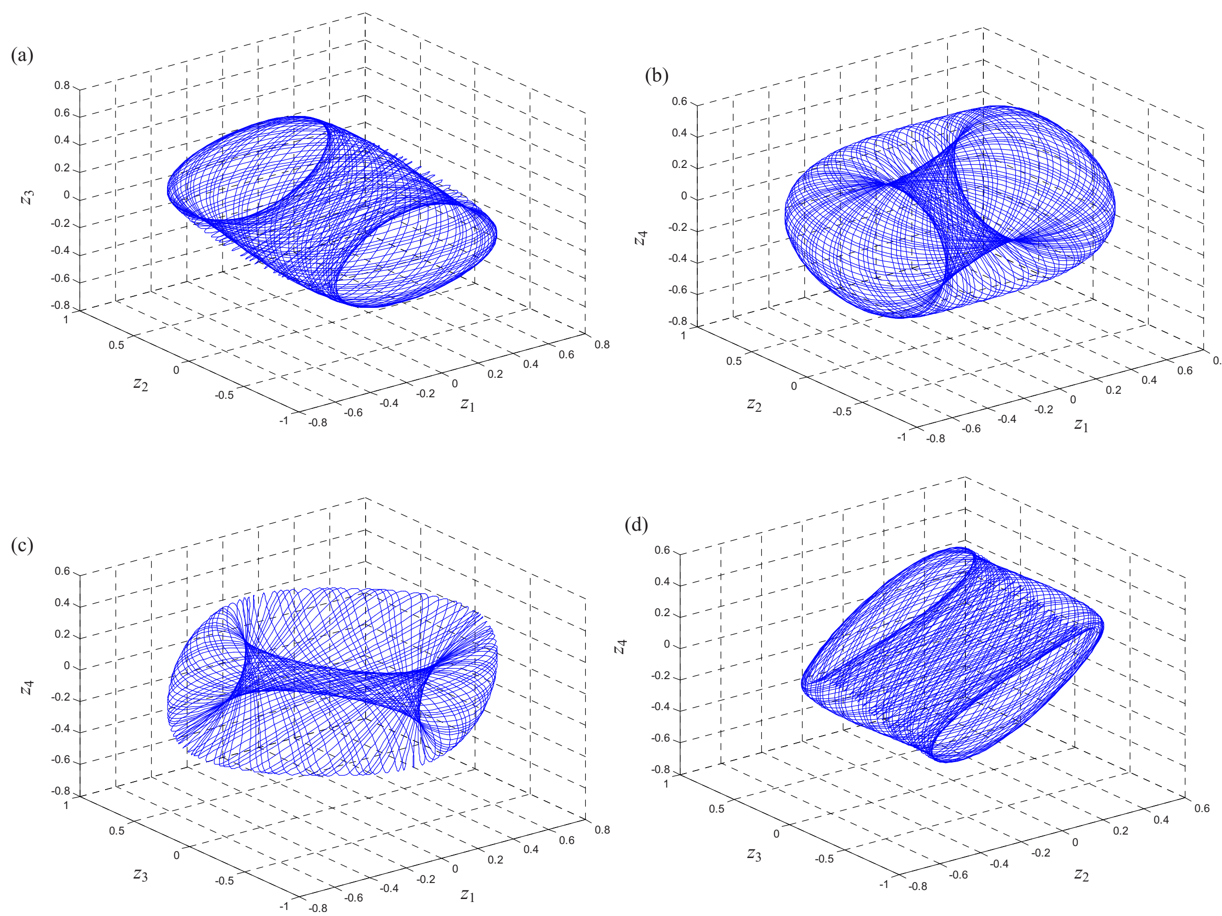

FIG. 5. (Color online) Projections of phase portraits for chaotic system (42).

$y_3(0)=0.25$, $y_4(0)=-0.81$, $z_1(0)=0.5$, $z_2(0)=-0.3$, $z_3(0)=0.1$, $z_4(0)=0.2$, and $[\hat{a}_{10}\hat{a}_{20}\hat{\omega}_{10}\hat{\omega}_{20}\hat{G}_0]^T=[00\ 00\ 0]^T$. The error dynamics is

$$\dot{e}_1=-2A_1(t)\sqrt{1-x_1^2}\sin x_2+2\hat{a}_1\sqrt{1-y_1^2}\sin y_2-(\hat{G}+G)e_1,$$
$$\dot{e}_2=-A_3(t)(x_1-x_3)+2A_1(t)\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2+\hat{\omega}_1(y_1-y_3)-2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2-(\hat{G}+G)e_2,$$
$$\dot{e}_3=-2A_2(t)\sqrt{1-x_3^2}\sin x_4+2\hat{a}_2\sqrt{1-y_3^2}\sin y_4-(\hat{G}+G)e_3, \tag{45}$$
$$\dot{e}_4=-A_4(t)(x_3-x_1)+2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4+\hat{\omega}_2(y_3-y_1)-2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4-(\hat{G}+G)e_4,$$

where $e_1=x_1-y_1$, $e_2=x_2-y_2$, $e_3=x_3-y_3$, $e_4-x_4-y_4$.

Our aim is

$$\lim_{t\to\infty} e_i=\lim_{t\to\infty}(x_i-y_i)=0,\quad i=1,2,3,4. \tag{46}$$

Let the adaptive law be

$$\dot{\tilde{G}}=\dot{G}-\dot{\hat{G}}=-\dot{\hat{G}}=-e^Te. \tag{47}$$

Since G is constant, $\dot{G}=0$. Define

$$\tilde{A}(t)=[\tilde{a}_1(t)\ \tilde{a}_2(t)\ \tilde{\omega}_1(t)\ \tilde{\omega}_2(t)]^T, \tag{48}$$

$$\tilde{a}_1=A_1(t)-\hat{a}_1,\quad \tilde{a}_2=A_2(t)-\hat{a}_2,\qquad \tilde{\omega}_1=A_3(t)-\hat{\omega}_1,\quad \tilde{\omega}_2=A_4(t)-\hat{\omega}_2, \tag{49}$$

$$\dot{\tilde{a}}_1=a_1d_1\dot{z}_1-\dot{\hat{a}}_1,\quad \dot{\tilde{a}}_2=a_2d_2\dot{z}_2-\dot{\hat{a}}_2,\qquad \dot{\tilde{\omega}}_1=\omega_1d_3\dot{z}_3-\dot{\hat{\omega}}_1,\quad \dot{\tilde{\omega}}_2=\omega_2d_4\dot{z}_4-\dot{\hat{\omega}}_2. \tag{50}$$

Choose $\dot{\tilde{a}}_1$, $\dot{\tilde{a}}_2$, $\dot{\tilde{\omega}}_1$, and $\dot{\tilde{\omega}}_2$ as

$$\dot{\tilde{a}}_1=-G\tilde{a}_1\|e\|/\|\tilde{A}(t)\|,\quad \dot{\tilde{\omega}}_1=-G\tilde{\omega}_1\|e\|/\|\tilde{A}(t)\|,\qquad \dot{\tilde{a}}_2=-G\tilde{a}_2\|e\|/\|\tilde{A}(t)\|,\quad \dot{\tilde{\omega}}_2=-G\tilde{\omega}_2\|e\|/\|\tilde{A}(t)\|. \tag{51}$$

A Lyapunov function is given in the form of a positive definite function,

$$V(e_1,e_2,e_3,e_4,\tilde{a}_1,\tilde{a}_2,\tilde{\omega}_1,\tilde{\omega}_2,\tilde{G})=\tfrac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2+\tilde{a}_1^2+\tilde{a}_2^2+\tilde{\omega}_1^2+\tilde{\omega}_2^2+\tilde{G}^2). \tag{52}$$

Its time derivative along any solution of Eqs. (45), (47), and (51) is

$$\dot{V}=e_1[-2A_1(t)\sqrt{1-x_1^2}\sin x_2+2\hat{a}_1\sqrt{1-y_1^2}\sin y_2-(\hat{G}+G)e_1]+e_2\left[-A_3(t)(x_1-x_3)+2A_1(t)\times\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2+\hat{\omega}_1(y_1-y_3)-2\hat{a}_1\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2-(\hat{G}+G)e_2\right]+e_3[-2A_2(t)\sqrt{1-x_3^2}\sin x_4+2\hat{a}_2\sqrt{1-y_3^2}\sin y_4-(\hat{G}+G)e_3]+e_4\left[-A_4(t)(x_3-x_1)+2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4+\hat{\omega}_2(y_3-y_1)-2\hat{a}_2\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4-(\hat{G}+G)e_4\right]+\tilde{a}_1\dot{\tilde{a}}_1+\tilde{a}_2\dot{\tilde{a}}_2+\tilde{\omega}_1\dot{\tilde{\omega}}_1+\tilde{\omega}_2\dot{\tilde{\omega}}_2-\tilde{G}\dot{\hat{G}},$$

$$\dot{V}=e_1[-2A_1(t)\sqrt{1-x_1^2}\sin x_2+2A_1(t)\sqrt{1-y_1^2}\sin y_2-(\hat{G}+G)e_1]+e_2\left[-A_3(t)(x_1-x_3)+2A_1(t)\times\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2+A_3(t)(y_1-y_3)-2A_1(t)\frac{y_1}{\sqrt{1-y_1^2}}\cos y_2-(\hat{G}+G)e_2\right]+e_3[-2A_2(t)\sqrt{1-x_3^2}\sin x_4+2A_2(t)\sqrt{1-y_3^2}\sin y_4-(\hat{G}+G)e_3]+e_4\left[-A_4(t)(x_3-x_1)+2A_2(t)\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4+A_4(t)(y_3-y_1)-2A_2(t)\frac{y_3}{\sqrt{1-y_3^2}}\cos y_4-(\hat{G}+G)e_4\right]+\tilde{a}_1\left[2\sqrt{1-y_1^2}\sin y_2e_1-\frac{2y_1}{\sqrt{1-y_1^2}}\cos y_2e_2\right]+\tilde{\omega}_1[(y_1-y_3)e_2]+\tilde{a}_2\left[2\sqrt{1-y_3^2}\sin y_4e_3-\frac{2y_3}{\sqrt{1-y_3^2}}\cos y_4e_4\right]+\tilde{\omega}_2[(y_3-y_1)e_4]-G\|e\|(\tilde{a}_1^2+\tilde{a}_2^2+\tilde{\omega}_1+\tilde{\omega}_2)/\|\tilde{A}\|-\tilde{G}\dot{\hat{G}},$$

$$\dot{V}\leqslant G\|e\|^2-(\hat{G}+G)\|e\|^2+G\|e\|\|\tilde{A}\|-G\|e\|\|\tilde{A}\|^2/\|\tilde{A}\|-\tilde{G}\dot{\hat{G}}.$$

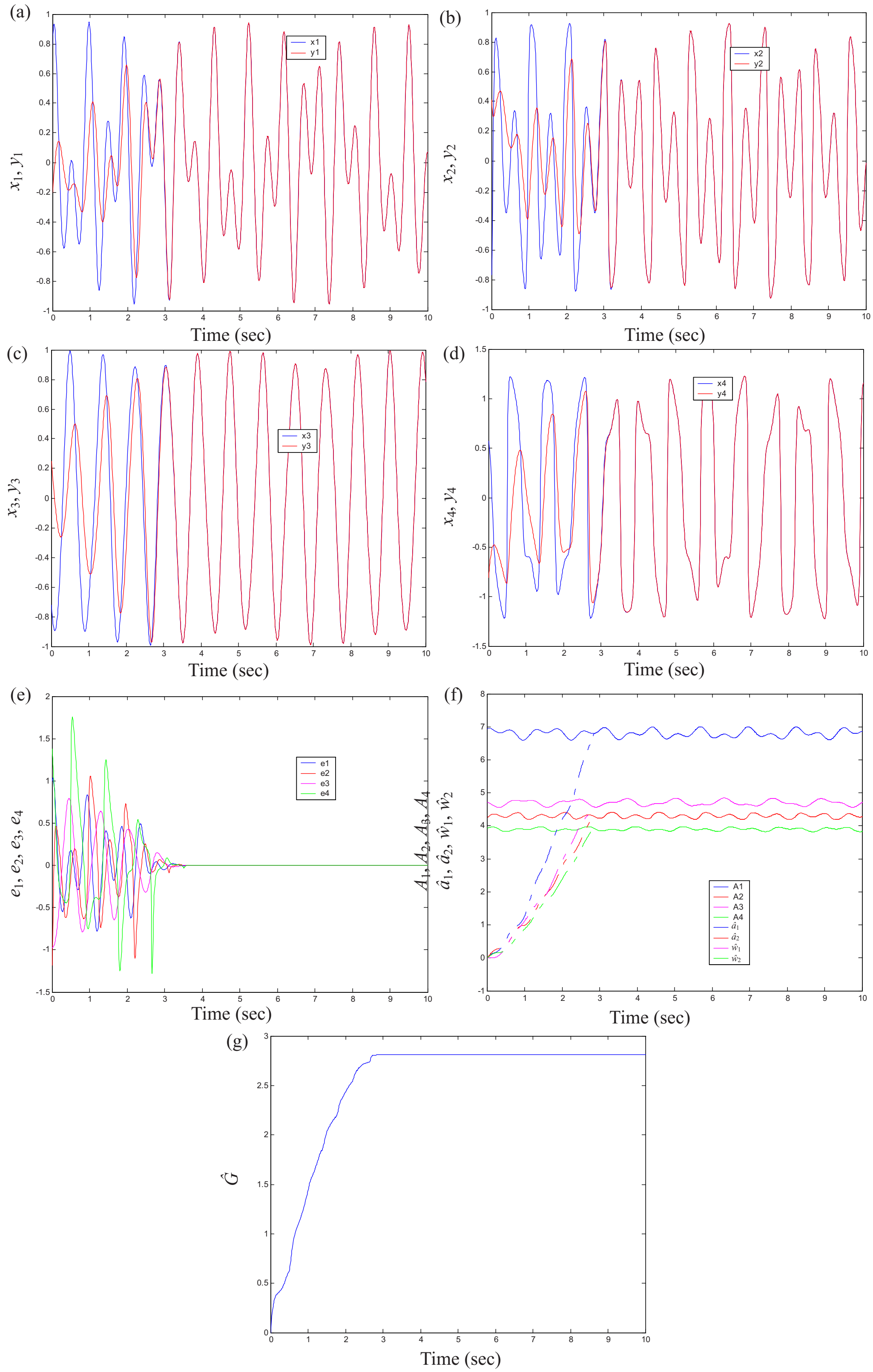

FIG. 6. (Color online) Time series of states, state errors, $A_1, A_2, A_3, A_4, \hat{a}_1, \hat{a}_2, \hat{w}_1, \hat{w}_2$ and estimated Lipschitz constant $\hat{G}$ for Case II.

$\dot{V}$ can be rewritten as

$$\dot{V} \leq -G(e_1^2 + e_2^2 + e_3^2 + e_4^2). \quad (53)$$

$\dot{V}$ is a negative semidefinite function of $e, \tilde{a}, \tilde{\omega}, \tilde{G}$. The Lyapunov asymptotical stability theorem is not satisfied. We cannot obtain that the common origin of error dynamics (45), adaptive laws (47), and parameter dynamics (51) is asymptotically stable. Now, D is a 5-manifold, $n=9$ and the number of error state variables $p=4$. When $e_i=0$, $(i=1,2,3,4)$, and $\tilde{a}_j, \tilde{\omega}_j, \tilde{G}$, $(i=1,2,3,4;\ j=1,2)$ take arbitrary values, $\dot{V}=0$, so

X is a 5-manifold, $m=n-p=9-4=5$. $m+1<n$ is satisfied. From the GYC pragmatical asymptotical stability theorem, error vector e approaches zero and the estimated parameters also approach the uncertain parameters. The equilibrium point $e_i=\tilde{a}_j=\tilde{\omega}_j=\tilde{G}=0$ $(i=1,2,3,4;\ j=1,2)$ is asymptotically stable. Moreover, the result is global asymptotically stable (see Appendix). The numerical results of the time series of states, state errors, parameters and estimated Lipschitz constant $\hat{G}$ are shown in Fig. 6. The chaos synchronization is accomplished near 3 s. $\hat{G}$ approaches constant also near 3 s. The coupling strength required is $K=2G=5.62$.

IV. CONCLUSIONS

Using the Lipschitz condition, the synchronization of Lorenz chaotic systems and of Quantum-CNN chaotic oscillator systems with uncertain **chaotic** parameters by linear coupling and pragmatical adaptive tracking are accomplished by the GYC pragmatical asymptotical stability theorem. Tracking uncertain **chaotic** parameters is first studied in this paper. This is of practical interest, because system parameters may be varied chaotically due to aging, environment, and disturbances. Two Lorenz systems are synchronized for chaotic parameters $M<n$. Two Quantum-CNN systems are synchronized for chaotic parameters $M=n$. The simulation results imply that this scheme is very effective. By GYC pragmatical asymptotical stability theorem, the question, why the estimated parameters approach the uncertain parameters, has been strictly answered and verified by numerical simulations.

ACKNOWLEDGMENTS

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APPENDIX: GYC PRAGMATICAL ASYMPTOTICAL STABILITY THEOREM

The stability for many problems in real dynamical systems is actual asymptotical stability, although it may not be mathematical asymptotical stability. The mathematical asymptotical stability demands that trajectories from all initial states in the neighborhood of zero solution must approach the origin as $t\to\infty$. If there is only a small part or even a few of the initial states from which the trajectories do not approach the origin as $t\to\infty$, the zero solution is not mathematically asymptotically stable. If the probability of occurrence of the event that the trajectories from the initial states are that they do not approach zero when $t\to\infty$, i.e., these trajectories are not asymptotical stable for zero solution, is zero, the stability of zero solution is actual asymptotical stability though it is not mathematical asymptotical stability. In order to analyze the asymptotical stability of the equilibrium point of such systems, the pragmatical asymptotical stability theorem is used. The conditions for pragmatical asymptotical stability are more slack than that for traditional Lyapunov asymptotical stability.

Let X and Y be two manifolds of dimensions m and n $(m<n)$, respectively, and φ be a differentiable map from X to Y; then $\varphi(X)$ is a subset of the Lebesque measure 0 of Y.[22] For an autonomous system

$$\dot{x}=f(x_1,\ldots,x_n), \tag{A1}$$

where $x=[x_1,\ldots,x_n]^T$ is a state vector, the function $f=[f_1,\ldots,f_n]^T$ is defined on $D\subset R^n$, an n-manifold.

Let $x=0$ be an equilibrium point for the system (A1). Then

$$f(0)=0. \tag{A2}$$

For a nonautonomous system,

$$\dot{x}=f(x_1,\ldots,x_{n+1}), \tag{A3}$$

where $x=[x_1,\ldots,x_{n+1}]^T$, the function $f=[f_1,\ldots,f_n]^T$ is defined on $D\subset R^n\times R_+$, here $t=x_{n+1}\subset R_+$. The equilibrium point is

$$f(0,x_{n+1})=0. \tag{A4}$$

Definition. The equilibrium point for the system is pragmatically asymptotically stable provided that with initial points on C which is a subset of the Lebesque measure 0 of D, the behaviors of the corresponding trajectories cannot be determined, while with initial points on $D-C$, the corresponding trajectories behave as those that agree with traditional asymptotical stability.

Theorem: Let $V=[x_1,x_2,\ldots,x_n]^T$: $D\to R_+$ be positive definite and analytic on D, where $x_1,x_2,\ldots,x_n$ are all space coordinates such that the derivative of V through Eqs. (A1) or (A3), $\dot{V}$, is negative semidefinite of $[x_1,x_2,\ldots,x_n]^T$.

For an autonomous system, let X be the m-manifold consisting of a point set for which $\forall x\neq 0$, $\dot{V}(x)=0$ and D is an m-manifold. If $m+1<n$, then the equilibrium point of the system is pragmatically asymptotically stable.

For a nonautonomous system, let X be the $m+1$-manifold consisting of the point set for which $\forall x\neq 0$, $\dot{V}(x_1,x_2,\ldots,x_n)=0$, and D is an $n+1$-manifold. If $m+1+1<n+1$, i.e., $m+1<n$, then the equilibrium point of the system is pragmatically asymptotically stable. Therefore, for both autonomous and nonautonomous systems, the formula $m+1<n$ is universal. So the following proof is only for an autonomous system. The proof for the nonautonomous system is similar.

Proof: Since every point of X can be passed by a trajectory of Eq. (A1), which is one-dimensional, the collection of these trajectories, C, is an $(m+1)$-manifold.[16,17]

If $m+1<n$, then the collection C is a subset of Lebesque measure 0 of D. By the above definition, the equilibrium point of the system is pragmatically asymptotically stable.

If an initial point is ergodicly chosen in D, *the probability of that the initial point falls on the collection C is zero. Here, equal probability is assumed for every point chosen as an initial point in the neighborhood of the equilibrium point.*

Hence, the event that the initial point is chosen from collection C *does not actually occur*. Therefore, under the equal probability assumption, pragmatical asymptotical stability becomes actual asymptotical stability. When the initial point falls on $D-C$, $\dot{V}(x)<0$, the corresponding trajectories behave as that agree with traditional asymptotical stability because by the existence and uniqueness of the solution of the initial-value problem, these trajectories never meet C.

The Lyapunov function is a positive definite function of n variables, i.e., p error state variables and $n-p=m$ differences between unknown and estimated parameters, while $\dot{V}=e^TCe$ is a negative semidefinite function of n variables. Since the number of error state variables is always more than one, $p>1$, $m+1<n$ is always satisfied, by pragmatical asymptotical stability theorem we have

$$\lim_{t\to\infty} e = 0 \tag{A5}$$

and the estimated parameters approach the uncertain parameters. Therefore, the equilibrium point of the system is *pragmatically asymptotically stable. Under the equal probability assumption, it is actually asymptotically stable for both error state variables and parameter variables*.

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