國立交通大學

機械工程學系

博士論文

藉由純誤差動態方程與精巧李亞普諾夫函數所達成之新渾沌系統廣義同步，非完整系統之渾沌現象，不同渾沌系統之變尺度時間非等時交織同步與不同渾沌系統之雙交織同步

Generalized Synchronization of New Chaotic Systems by Pure Error Dynamics and Elaborate Lyapunov Function, Chaos of Nonholonomic Systems, Non-Simultaneous Symplectic Synchronization of Different Chaotic Systems with Variable Scale Time, and Double Symplectic Synchronization of Different Chaotic Systems

研 究 生：張晉銘

指導教授：戈正銘　教授

中華民國九十八年六月

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研 究 生：張晉銘　　Student：Ching-Ming Chang

指導教授：戈正銘　　Advisor：Zheng-Ming Ge

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學生：張晉銘　　　　　　　　　　　　　　　指導老師：戈正銘 教授

國立交通大學機械工程學系

摘要

本論文探討藉由純誤差動態方程與精巧李亞普諾夫函數所達成之新渾沌系統廣義同步，非完整系統之渾沌現象，不同渾沌系統之變尺度時間非等時交織同步與不同渾沌系統之雙交織同步。

首先，提出以兩組不同的帶阻尼非線性 Mathieu 系統之相互線性耦合而構成的新渾沌系統，並研究其規律與渾沌動態行為。

其次，利用純誤差動態方程、精巧李亞普諾夫函數與精巧非對角化李亞普諾夫函數來達成廣義同步。採用不需要數值模擬輔助的純誤差動態方程來實現廣義同步，以取代目前常用的含主從狀態變量的混合誤差動態方程。此外還採用精巧李亞普諾夫函數與精巧非對角化李亞普諾夫函數，以取代目前廣泛使用，一成不變，且大幅地削弱李亞普諾夫直接法威力的平方和李亞普諾夫函數。在數值模擬中以新渾沌系統為範例。

接著，首次完整地確認了非完整系統的渾沌現象，包括具有外加非完整約束的非完整系統，如目標為直線振動的追蹤問題與目標為繞圓周旋轉的追蹤問題，及具有外加非線性非完整約束的非線性非完整系統，如速度大小保持不變的問題。渾沌的研究範圍首次被拓展至非完整系統與非線性非完整系統。系統的動態方程式是藉由基本非完整形式的拉格朗日方程與非線性非完整形式的拉格朗日方程而導出。透過所有的渾沌數值判據，包括最可靠的李雅普諾夫指數、相位圖、

龐卡萊圖與分歧圖，首次完整地證明了渾沌現象存在於非完整系統與非線性非完整系統。並更進一步地發現費根堡常數法則在非線性非完整系統中依然適用。

此外，提出了一種新型的渾沌同步，稱為“非等時交織同步”，可表達為 $\mathbf{y}(t)=\mathbf{F}(\mathbf{x}(\tau),\mathbf{y}(t),t)$，其中 τ 是時間 t 之給定函數，即所謂的變尺度時間。它是廣義同步的延伸，之所以命名為“非等時交織同步”是因為 $\mathbf{y}(t)$ 扮演著交織的角色，且 $\mathbf{x}(\tau)$ 與 $\mathbf{y}(t)$ 分別在不同的時刻 $\tau(t)$ 及 t 達成同步。當非等時交織同步應用在秘密通訊時，由於函數關係比傳統廣義同步的函數關係更為複雜，且對於攻擊者而言，除了破解系統的模型和複雜的函數關係，還多了破解變尺度時間 $\tau(t)$ 的困難，因此傳送訊號時採用非等時交織同步會比採用傳統廣義同步更難被偵測到，可用來加強秘密通訊的安全。在此以非線性控制與適應控制的方法來實現非等時交織同步。使用適應控制時，估測到的 Lipschitz 常數遠小於使用非線性控制所求得的 Lipschitz 常數，這使得控制器的增益大幅減少，換句話說，控制器的成本隨之而降。採用的控制方法可有效地適用於自治與非自治渾沌系統，無論 $\mathbf{x}(\tau)$ 與 $\mathbf{y}(t)$ 系統的維度相同與否。

並更進一步地提出一種新型的渾沌同步，稱為“雙交織同步”，可表達為 $\mathbf{G}(\mathbf{x},\mathbf{y})=\mathbf{F}(\mathbf{x},\mathbf{y},t)$。它是交織同步 $\mathbf{y}=\mathbf{F}(\mathbf{x},\mathbf{y},t)$ 的延伸，因為交織函數同時出現在等式的左右兩邊，故命名為“雙交織同步”。利用其同步形式的複雜性，可用來加強秘密通訊的安全。基於 Barbalat 引理，提出以主動控制達到雙交織同步的方法，並成功地應用在自治與非自治渾沌系統。

Generalized Synchronization of New Chaotic Systems by Pure Error Dynamics and Elaborate Lyapunov Function, Chaos of Nonholonomic Systems, Non-Simultaneous Symplectic Synchronization of Different Chaotic Systems with Variable Scale Time, and Double Symplectic Synchronization of Chaotic Systems

Student：Ching-Ming Chang Advisor：Zheng-Ming Ge

Department of Mechanical Engineering
National Chiao Tung University

Abstract

Generalized synchronization of new chaotic systems by pure error dynamics and elaborate Lyapunov function, chaos of nonholonomic systems, non-simultaneous symplectic synchronization of different chaotic systems with variable scale time, and double symplectic synchronization of different chaotic systems are studied in this thesis.

Firstly, the new chaotic systems constructed by mutual linear coupling of two non-identical nonlinear damped Mathieu systems are introduced, and the regular and chaotic dynamics of the new chaotic systems are studied.

Then, by applying pure error dynamics, elaborate Lyapunov function, and elaborate nondiagonal Lyapunov function, the generalized synchronization is obtained. Instead of current mixed error dynamics in which master state variables and slave

state variables are presented, the generalized synchronization can be obtained by pure error dynamics without auxiliary numerical simulation. The elaborate Lyapunov function and the elaborate nondiagonal Lyapunov function are applied rather than the current monotonous square sum Lyapunov function, deeply weakening the powerfulness of Lyapunov direct method. New chaotic systems are used as examples with numerical simulations.

Chaos of nonholonomic systems with external nonholonomic constraint, the straightly oscillating target pursuit problem, or the circularly rotating target pursuit problem, and chaos of nonlinear nonholonomic system with external nonlinear nonholonomic constraint, the magnitude of velocity keeping constant, is completely identified for the first time. The scope of chaos study is firstly extended to nonholonomic systems and nonlinear nonholonomic system. By applying the fundamental nonholonomic form of Lagrange's equations and the nonlinear nonholonomic form of Lagrange's equations, the dynamic equations are expressed. The existence of chaos in nonholonomic systems and nonlinear nonholonomic systems are firstly completely identified by all numerical criteria of chaos, i.e. the most reliable Lyapunov exponents, phase portraits, Poincaré maps and bifurcation diagrams. Furthermore, it is found that the Feigenbaum number rule still holds for nonlinear nonholonomic system.

We propose a new type of synchronization, "non-simultaneous symplectic synchronization", $\mathbf{y}(t)=\mathbf{F}(\mathbf{x}(\tau),\mathbf{y}(t),t)$, where τ is a given function of time t, so-called variable scale time. It is an extension of generalized synchronization and it is called "non-simultaneous symplectic synchronization" due to $\mathbf{y}(t)$ plays the "interwined" role and the synchronization is achieved at "different time" for $\mathbf{x}(\tau)$ and $\mathbf{y}(t)$. When applying the non-simultaneous symplectic synchronization in secret communication, since the functional relation of the non-simultaneous symplectic synchronization is more complex than that of the traditional generalized synchronization, and cracking the variable scale time τ is an extra task for the attackers in addition to cracking the system model and cracking the functional relation, the message is harder to be detected by applying the non-simultaneous symplectic synchronization than by applying traditional generalized synchronization. Therefore, the non-simultaneous symplectic synchronization may be applied to increase the security of secret communication. Nonlinear control and adaptive control are applied to obtain the non-simultaneous symplectic synchronization. The estimated Lipschitz constant obtained by applying adaptive control is much less than the Lipschitz constant obtained by applying nonlinear control. This result in the reduction of the gain of the controller, i.e. the cost of controller is reduced. The proposed scheme is effective and feasible for both autonomous and nonautonomous chaotic systems, whether the dimensions of $\mathbf{x}(\tau)$ and $\mathbf{y}(t)$ systems are the same or not.

Furthermore, a new type of synchronization, "double symplectic synchronization", $\mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{F}(\mathbf{x},\mathbf{y},t)$, is proposed in this thesis. It is an extension of symplectic synchronization, $\mathbf{y} = \mathbf{F}(\mathbf{x},\mathbf{y},t)$. Since the symplectic functions are presented at both the right hand side and the left hand side of the equality, it is called "double symplectic synchronization". The double symplectic synchronization may be applied to increase the security of secret communication due to the complexity of its synchronization form. By applying active control, the scheme of double symplectic synchronization is derived based on Barbalat's lemma, and it is applied successfully to both autonomous and nonautonomous chaotic systems.

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Chapter 1
Introduction

Chaotic phenomena have been observed in physics, chemistry, physiology, and many disciplines [1-3]. In contrast with the famous chaotic systems, such as Lorenz system, Duffing system, and Rössler system, nonlinear Mathieu system is less mentioned [4-9]. However, nonlinear Mathieu system is important and can be applied in analysis of the resonant micro electro mechanical systems [10-12]. In this thesis, the new autonomous and new nonautonomous chaotic systems constructed by mutual linear coupling of two non-identical nonlinear damped Mathieu systems are studied.

Chaos synchronization has been widely applied in secure communication [13, 14], biological systems [15, 16], and many other fields [17, 18]. The generalized synchronization is a complex type of chaos synchronization and gives rise to extensive investigations recently [19-26]. The mixed error dynamics and the plain square sum Lyapunov function are currently applied in studying the generalized synchronization, but there are some shortcomings and restrictions in them. The auxiliary numerical simulation is unavoidable for current mixed error dynamics in which master state variables and slave state variables are presented while their maximum values must be determined by simulation [27-31]. However, the pure error dynamics can be analyzed theoretically without additional numerical simulation. Besides, monotonous and self-limited square sum Lyapunov function, $V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^T\mathbf{e}$, is always used in most literatures [32-37], but the Lyapunov function can be chosen in a variety of elaborate and ingenious forms for different systems. Restricting Lyapunov function to square sum makes a long river brooklike, deeply weakens the powerfulness of Lyapunov direct method. Instead of current plain square sum

Lyapunov function, the elaborate Lyapunov function is applied in this thesis. A systematic method of designing Lyapunov function is proposed based on the Lyapunov direct method [38]. The generalized synchronization is achieved for both new autonomous and nonautonomous chaotic systems by applying this technique.

Since Hertz [39] distinguished nonholonomic system from holonomic system in 1894, the study of nonholonomic system [40, 41] has been developed over one hundred years. A great number of studies in this field are connected with the extension of the developed analytical methods for holonomic system and for the systems with nonholonomic constraints. At present the dynamics of nonholonomic system has many applications in the problems of modern technology, such as the pursuit problems, the motion of automobiles, landing devices of airplanes, railway wheels, etc. However, the complete study of chaos in nonholonomic systems remains deficient. As far as we know, the only paper studies the chaos of nonholonomic system with an external constraint is Ref. [42], in which the chaotic phenomena of rattleback dynamics are studied. But in this paper, only Poincaré maps are given. As it is well-known, the only Poincaré map can not identify the existence of chaos reliably.

The moving target pursuit problem [43] is a typical example of nonholonomic system. In this thesis, chaos of nonholonomic systems with external nonholonomic constraint for two types of pursuit problems, a straightly oscillating target, and a circularly rotating target, is studied by applying the fundamental nonholonomic form of Lagrange's equations [44, 45]. Moreover, chaos of nonholonomic system with external nonlinear nonholonomic constraint, the magnitude of velocity keeping constant, is studied in this thesis by applying the nonlinear nonholonomic form of Lagrange's equations. All numerical criteria of chaos, i.e. the most reliable Lyapunov exponents [46], phase portraits, Poincaré maps and bifurcation diagrams are firstly wholly given to identify the existence of chaos of nonholonomic and nonlinear nonholonomic systems. Furthermore, it is found that the Feigenbaum number rule [47] still holds for nonlinear nonholonomic system.

There are various types of synchronization, such as complete synchronization [48], generalized synchronization [49], phase synchronization [50], lag synchronization [51], and so on. Among these types of synchronization, generalized synchronization is one of the most interesting topics. Generalized synchronization refers to a functional relation between the state vectors of master and slave, i.e. $\mathbf{y} = \mathbf{F}(\mathbf{x}, t)$, where $\mathbf{x}$ and $\mathbf{y}$ are the state vectors of master and slave. In the work of Ref. [52], the generalized synchronization is extended to a more general form, $\mathbf{y} = \mathbf{F}(\mathbf{x}, \mathbf{y}, t)$, where the "slave" $\mathbf{y}$ is not a traditional pure slave obeying the "master" $\mathbf{x}$ completely but plays a role to determine the final desired state of the "slave". Since the "slave" $\mathbf{y}$ plays an "interwined" role, this type of synchronization is called "symplectic synchronization"[1], the master is called "partner A", and the slave is called "partner B". In this thesis, we propose two types of new chaos synchronization, "non-simultaneous symplectic synchronization" and "double symplectic synchronization".

We propose the "non-simultaneous symplectic synchronization", $\mathbf{y}(t) = \mathbf{F}(\mathbf{x}(\tau), \mathbf{y}(t), t)$, where τ is a given function of time t, so-called variable scale time. The synchronization is achieved at "different time" for "partner A" $\mathbf{x}(\tau)$ and "partner B" $\mathbf{y}(t)$, therefore we call this type of synchronization "non-simultaneous symplectic synchronization". When $\tau = t$, non-simultaneous symplectic synchronization reduces to symplectic synchronization. When applying the non-simultaneous symplectic synchronization in secret communication, since the functional relation of the non-simultaneous symplectic synchronization is more

[1] The term ''**symplectic**'' comes from the Greek for ''interwined''. H. Weyl first introduced the term in 1939 in his book "The Classical Groups" (p. 165 in both the first edition, 1939, and second edition, 1946, Princeton University Press).

complex than that of the traditional generalized synchronization, and cracking the variable scale time τ is an extra task for the attackers in addition to cracking the system model and cracking the functional relation, the message is harder to be detected by applying the non-simultaneous symplectic synchronization than by applying traditional generalized synchronization. Therefore, the non-simultaneous symplectic synchronization may be applied to increase the security of secret communication. In order to achieve non-simultaneous symplectic synchronization, nonlinear control [53] and adaptive control are applied. In the work of Ref. [53], the induced matrix norm and the Lipschitz constant are obtained by auxiliary numerical simulation. However, they can be estimated theoretically by using the property of induced matrix norms [54a] and by applying adaptive control. Furthermore, in our case, non-simultaneous symplectic synchronization, the complexity of the functional relation $\mathbf{F}(\mathbf{x}(\tau),\mathbf{y}(t),t)$ is greater than that studied in Ref. [53], thus the Lipschitz constant may be enormous. However, by applying adaptive control, the estimated Lipschitz constant is much less than the Lipschitz constant obtained by applying nonlinear control. This result in the reduction of the gain of the controller, i.e. the cost of controller is reduced. The proposed scheme is effective and feasible for both autonomous and nonautonomous chaotic systems, whether the dimensions of $\mathbf{x}(\tau)$ and $\mathbf{y}(t)$ systems are the same or not.

Finally, the "double symplectic synchronization", $\mathbf{G}(\mathbf{x},\mathbf{y})=\mathbf{F}(\mathbf{x},\mathbf{y},t)$, is proposed. Since the symplectic functions are presented at both the right hand side and the left hand side of the equality, it is called "double symplectic synchronization". It is an extension of symplectic synchronization, $\mathbf{y}=\mathbf{F}(\mathbf{x},\mathbf{y},t)$. When $\mathbf{G}(\mathbf{x},\mathbf{y})=\mathbf{y}$, the double symplectic synchronization is reduced to the symplectic synchronization. Due to the complexity of the form of the double symplectic synchronization, it may be applied to increase the security of secret communication. The double symplectic synchronization is obtained by applying active control. A scheme of synchronization is derived based on Barbalat's lemma [54b], and it is effective and feasible for both autonomous and nonautonomous chaotic systems.

The contents of this thesis are as follows. Chapter 2 contains the dynamics of new autonomous and nonautonomous chaotic systems. The system models are described and the numerical results of regular and chaotic behaviors are presented. In Chapter 3, generalized synchronization of new chaotic systems is achieved by applying pure error dynamics and elaborate Lyapunov function. The methods of designing Lyapunov function are presented, and both new autonomous and new nonautonomous chaotic systems are illustrated in examples. By applying pure error dynamics and elaborate nondiagonal Lyapunov function, nonlinear generalized synchronization of new chaotic systems is obtained in Chapter 4. We propose the methods of designing Lyapunov function, and illustrate them by both new autonomous and new nonautonomous chaotic systems in examples. In Chapter 5, the dynamics of nonholonomic systems is studied by applying the fundamental nonholonomic form of Lagrange's equations. Two types of external nonholonomic constraints are studied for moving target pursuit problems: a straightly oscillating target and a circularly rotating target. Numerical results show that chaos exists in each case. By applying the nonlinear nonholonomic form of Lagrange's equations, the dynamics of nonlinear nonholonomic system is studied in Chapter 6. We investigate external nonlinear nonholonomic constraint: the magnitude of velocity keeping constant. Chaos is proved to exist in each case by numerical results. Furthermore, Feigenbaum number rule still holds for nonlinear nonholonomic system. In Chapter 7, the non-simultaneous symplectic synchronization is proposed, and it is achieved by applying adaptive control. The synchronization scheme is presented, and chaotic systems with the same or different dimensions are illustrated in examples. We investigate the double symplectic synchronization by applying active control in Chapter 8. The synchronization scheme is derived, and both autonomous and nonautonomous chaotic systems are illustrated in examples. Finally, the conclusions of the whole thesis are drawn in Chapter 9.

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Chapter 2

Regular and Chaotic Dynamics of New Chaotic Systems

The nonlinear Mathieu system [1-6] is important and can be applied in analysis of the resonant micro electro mechanical systems [7-9]. In this Chapter, we propose new autonomous and new nonautonomous chaotic systems constructed by mutual linear coupling of two non-identical nonlinear damped Mathieu systems.

Consider two non-identical nonlinear damped Mathieu systems [5, 6] described by

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -a(1+\sin\omega t)x_1-(1+\sin\omega t)x_1^3-ax_2,\end{aligned}\tag{2.1}$$

$$\begin{aligned}\dot{x}_3 &= x_4,\\ \dot{x}_4 &= -(1+\sin\omega t)x_3-a(1+\sin\omega t)x_3^3-ax_4,\end{aligned}\tag{2.2}$$

where a and ω are constants.

A new autonomous chaotic system can be constructed by mutual linear coupling of two non-identical nonlinear damped Mathieu systems, Eq. (2.1) and Eq. (2.2). The term $\sin\omega t$ of one Mathieu system is replaced by one state of the other Mathieu system, and linear coupling terms are added to each other:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -a(1+x_4)x_1-(1+x_4)x_1^3-ax_2+bx_3,\\ \dot{x}_3 &= x_4,\\ \dot{x}_4 &= -(1+x_2)x_3-a(1+x_2)x_3^3-ax_4+bx_1.\end{aligned}\tag{2.3}$$

The parameters in simulation are $a=0.5, b=1\sim1.254,$ and the initial condition is $x_1(0)=0.1, x_2(0)=0.1, x_3(0)=0.2, x_4(0)=0.2.$ The phase portraits, Poincaré maps, bifurcation diagram, and Lyapunov exponents of the new autonomous chaotic system are shown in Fig. 2.1-2.3. It can be observed that the motion is period 1 for $b = 1.1$, period 4 for $b = 1.243$, and period 8 for $b = 1.246$. For $b = 1.24$, the motion is chaotic.

A new nonautonomous chaotic system can also be constructed by mutual linear coupling of two non-identical nonlinear damped Mathieu systems, Eq. (2.1) and Eq. (2.2). The terms $\sin\omega t$ of each Mathieu system are preserved, and linear coupling terms are added to each other:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + bx_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -(1+\sin\omega t)x_3 - a(1+\sin\omega t)x_3^3 - ax_4 + bx_1.
\end{aligned} \tag{2.4}$$

The parameters in simulation are $a = 0.5, b = 0.9 \sim 1, \omega = 1,$ and the initial condition is $x_1(0) = 0.1, x_2(0) = 0.1, x_3(0) = 0.2, x_4(0) = 0.2.$ The phase portraits, Poincaré maps, bifurcation diagram, and Lyapunov exponents of the new nonautonomous chaotic system are shown in Fig. 2.4-2.6. It can be observed that the motion is period 1 for $b = 0.9$, period 2 for $b = 0.93$, and period 4 for $b = 0.934$. For $b = 1$, the motion is chaotic.

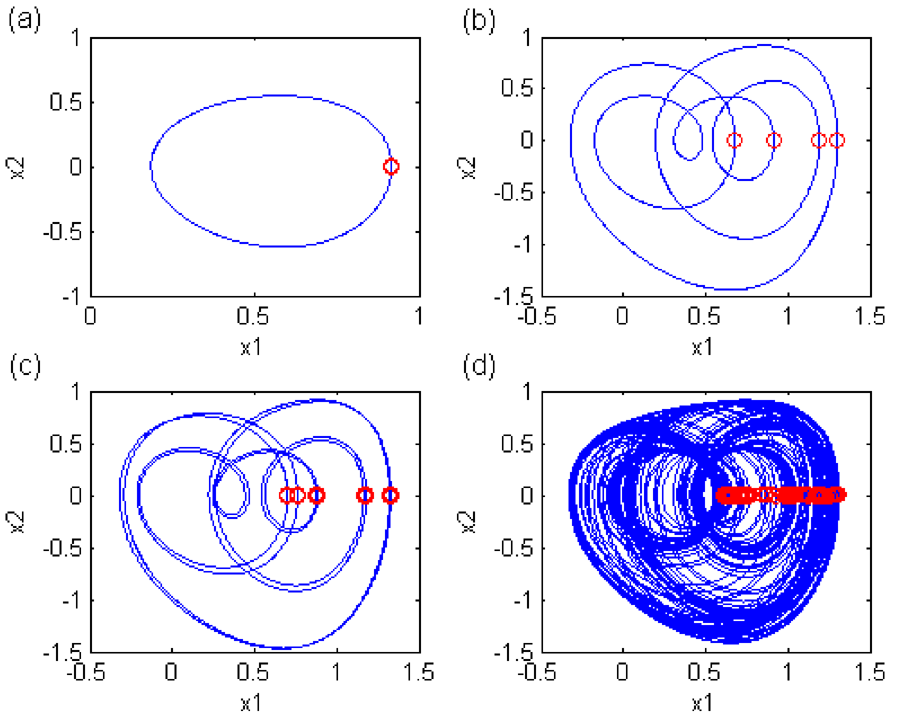

Fig. 2.1 Phase portraits and Poincaré maps of the new autonomous chaotic system: (a) period 1 for $b=1.1$, (b) period 4 for $b=1.243$, (c) period 8 for $b=1.246$, (d) chaotic for $b=1.24$.

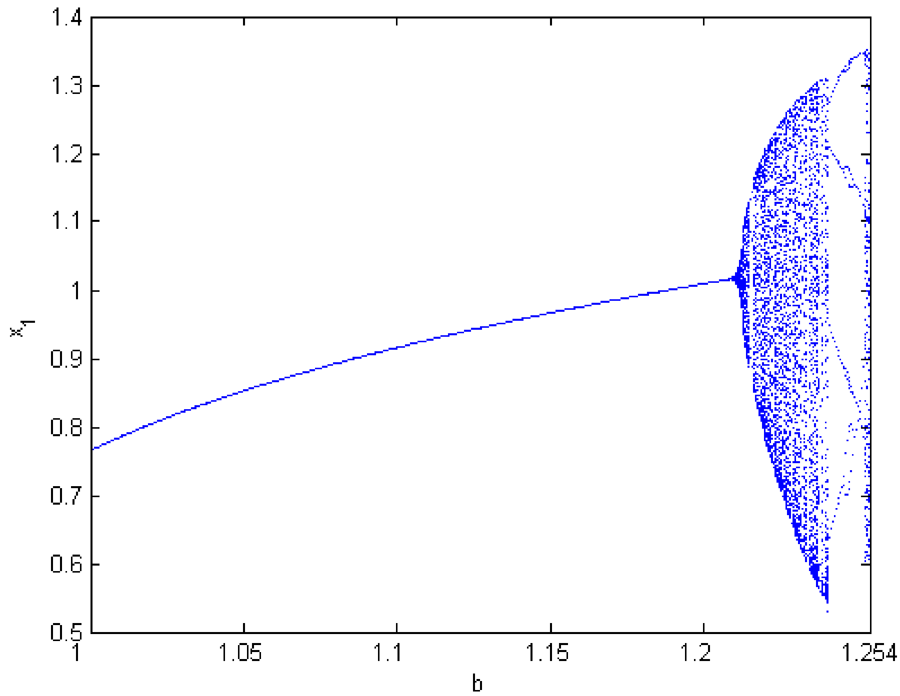

Fig. 2.2 Bifurcation diagram of the new autonomous chaotic system.

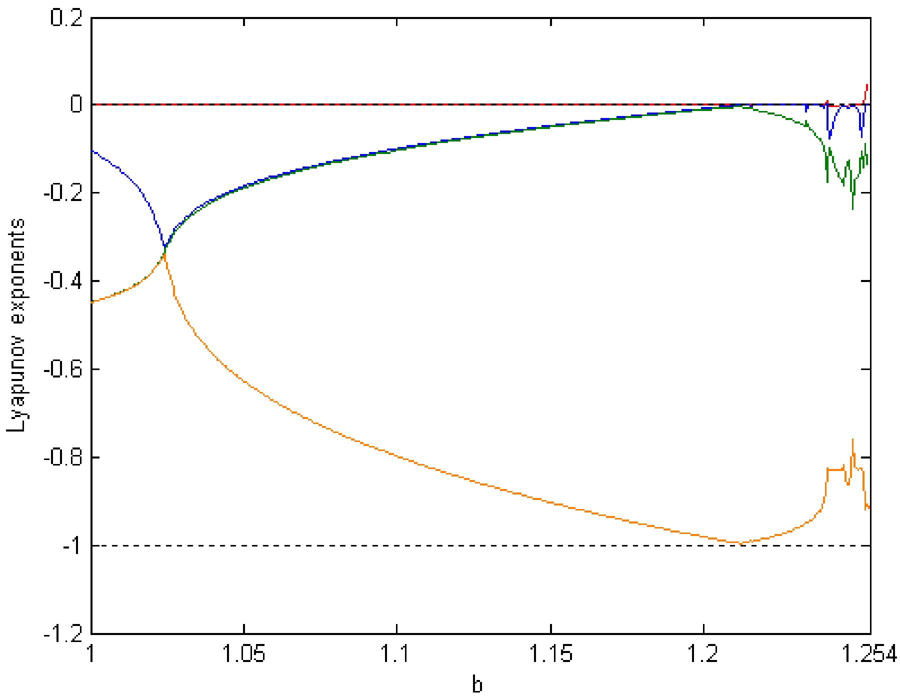

Fig. 2.3 Lyapunov exponents of the new autonomous chaotic system, where the sum of Lyapunov exponents is represented as a doted line at -1.

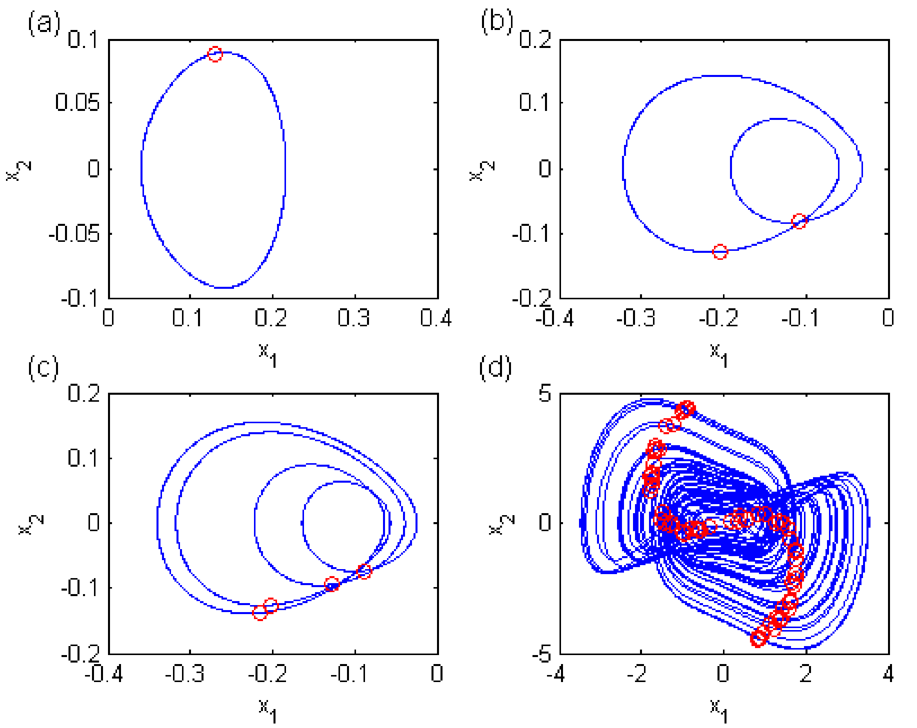

Fig. 2.4 Phase portraits and Poincaré maps of the new nonautonomous chaotic system: (a) period 1 for $b = 0.9$, (b) period 2 for $b = 0.93$, (c) period 4 for $b = 0.934$, (d) chaotic for $b = 1$.

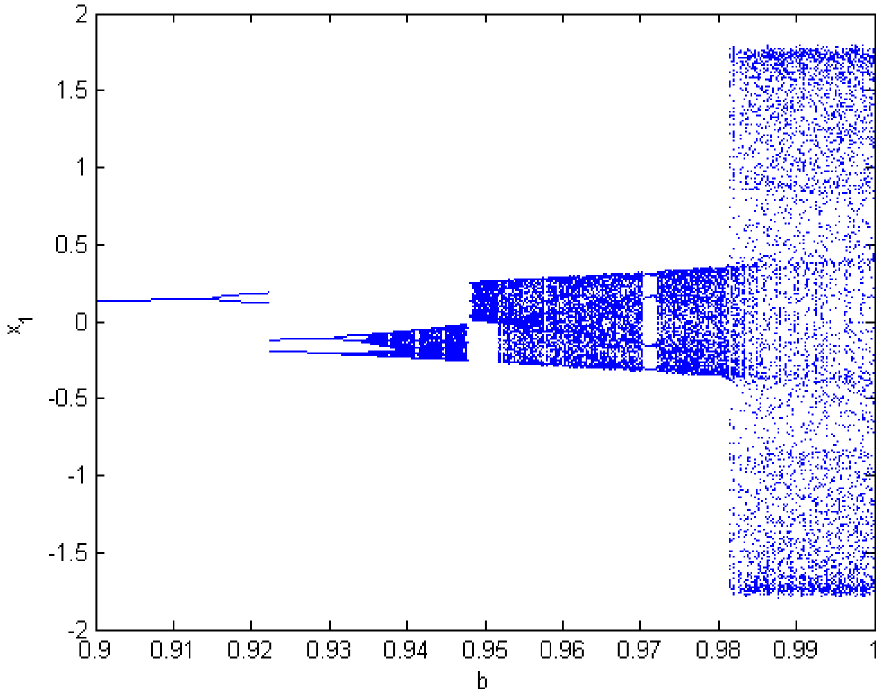

Fig 2.5 Bifurcation diagram of the new nonautonomous chaotic system.

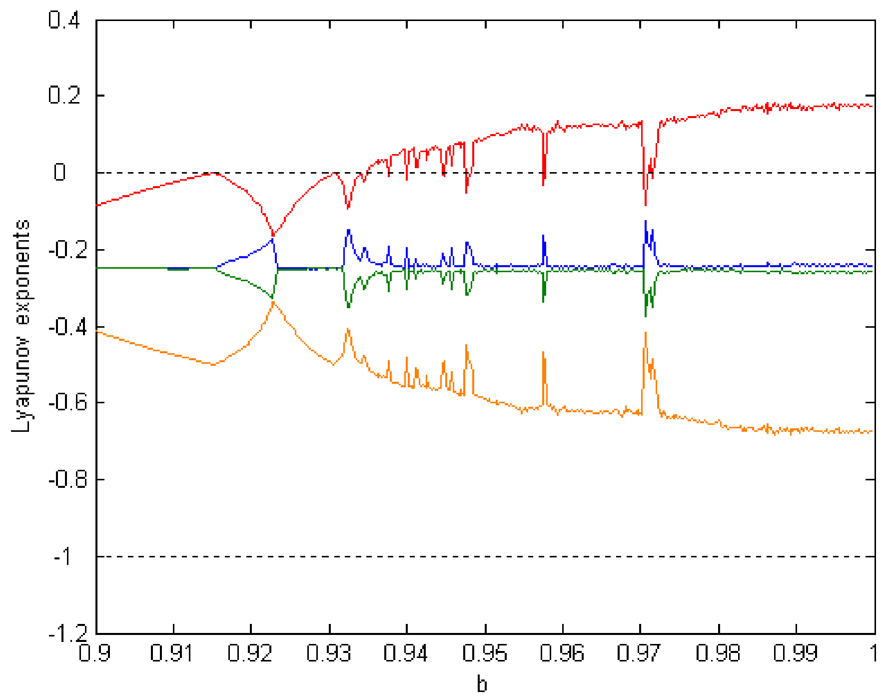

Fig. 2.6 Lyapunov exponents of the new nonautonomous chaotic system, where the sum of Lyapunov exponents is represented as a doted line at -1.

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Chapter 3

Generalized Synchronization of New Chaotic Systems by Pure Error Dynamics and Elaborate Lyapunov Function

3.1 Preliminaries

In this Chapter, the generalized synchronization is studied by applying pure error dynamics and elaborate Lyapunov function. The pure error dynamics can be analyzed theoretically without auxiliary numerical simulation, whereas the aid of additional numerical simulation is unavoidable for current mixed error dynamics in which master state variables and slave state variables are presented while their maximum values must be determined by simulation [1-5]. Besides, the elaborate Lyapunov function is applied rather than current plain square sum Lyapunov function, $V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^T\mathbf{e}$, which is currently used [6-11] for convenience. However, Lyapunov function can be chosen in a variety of forms for different systems. Restricting Lyapunov function to square sum makes a long river brooklike, deeply weakens the powerfulness of Lyapunov direct method. Based on Lyapunov direct method [12], the generalized synchronization is achieved and a systematic method of designing Lyapunov function is proposed.

3.2 Design of Lyapunov Function

Consider the master and slave nonlinear dynamic systems described by

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \tag{3.1}$$

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) + \mathbf{u}(t, \mathbf{x}, \mathbf{y}), \tag{3.2}$$

where $\mathbf{x}, \mathbf{y} \in R^n$ are master and slave state vectors, $\mathbf{f}: R_+ \times R^n \to R^n$ is a nonlinear vector function, and $\mathbf{u}$: $R_+ \times R^n \times R^n \to R^n$ is controller vector.

Generalized synchronization means that there is a functional relation $\mathbf{y} = \mathbf{g}(t, \mathbf{x})$ between master and slave states as time goes to infinity, where $\mathbf{g}: R_+ \times R^n \to R^n$ is a continuously differentiable vector function. Define $\mathbf{e} = \mathbf{y} - \mathbf{g}(t, \mathbf{x})$ as generalized synchronization error vector, and the error dynamics can be obtained:

$$\begin{aligned}\dot{\mathbf{e}} &= \dot{\mathbf{y}} - \dot{\mathbf{g}}(t, \mathbf{x}) \\ &= \dot{\mathbf{y}} - \frac{\partial \mathbf{g}(t, \mathbf{x})}{\partial \mathbf{x}} \dot{\mathbf{x}} - \frac{\partial \mathbf{g}(t, \mathbf{x})}{\partial t} \\ &= \mathbf{f}(t, \mathbf{y}) - \frac{\partial \mathbf{g}(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}) - \frac{\partial \mathbf{g}(t, \mathbf{x})}{\partial t} + \mathbf{u}(t, \mathbf{x}, \mathbf{y}).\end{aligned} \tag{3.3}$$

Based on Lyapunov direct method [12], the scheme of generalized synchronization and the procedure of designing Lyapunov function are described as follows:

Step 1. Construct a Lyapunov function

$$V(t, \mathbf{e}) = \frac{1}{2}\mathbf{e}^T \mathbf{\Lambda}(t) \mathbf{e} = \frac{1}{2}\lambda_{11}(t) e_1^2 + \frac{1}{2}\lambda_{22}(t) e_2^2 + \cdots + \frac{1}{2}\lambda_{nn}(t) e_n^2, \tag{3.4}$$

where $\mathbf{\Lambda}(t) = [\lambda_{ii}(t)] \in R^{n \times n}$ is an unknown continuously differentiable positive definite diagonal matrix to be designed. Its derivative is

$$\begin{aligned}\dot{V}(t, \mathbf{e}) &= \dot{\mathbf{e}}^T \mathbf{\Lambda}(t) \mathbf{e} + \frac{1}{2}\mathbf{e}^T \dot{\mathbf{\Lambda}}(t) \mathbf{e} \\ &= \lambda_{11}(t) e_1 \dot{e}_1 + \lambda_{22}(t) e_2 \dot{e}_2 + \cdots + \lambda_{nn}(t) e_n \dot{e}_n \\ &\quad + \frac{1}{2}\dot{\lambda}_{11}(t) e_1^2 + \frac{1}{2}\dot{\lambda}_{22}(t) e_2^2 + \cdots + \frac{1}{2}\dot{\lambda}_{nn}(t) e_n^2.\end{aligned} \tag{3.5}$$

Step 2. Eq. (3.5) can be rewritten in the following form:

$$\begin{aligned}\dot{V}(t, \mathbf{e}) &= G_1(\lambda_{11}, \dot{\lambda}_{11}) e_1^2 + G_2(\lambda_{22}, \dot{\lambda}_{22}) e_2^2 + \cdots + G_n(\lambda_{nn}, \dot{\lambda}_{nn}) e_n^2 \\ &\quad + [H_1(\lambda_{11}, \cdots, \lambda_{nn}, x_1, \cdots, x_n, y_1, \cdots, y_n, t) + \lambda_{11} u_1] e_1 \\ &\quad + [H_2(\lambda_{11}, \cdots, \lambda_{nn}, x_1, \cdots, x_n, y_1, \cdots, y_n, t) + \lambda_{22} u_2] e_2 \\ &\quad + \cdots + [H_n(\lambda_{11}, \cdots, \lambda_{nn}, x_1, \cdots, x_n, y_1, \cdots, y_n, t) + \lambda_{nn} u_n] e_n,\end{aligned} \tag{3.6}$$

where $G_i(\lambda_{ii},\dot{\lambda}_{ii})$ and $H_i(\lambda_{11},\cdots,\lambda_{nn},x_1,\cdots,x_n,y_1,\cdots,y_n,t)$ $(i=1,2,\cdots,n)$ are continuous differentiable functions, u_i $(i=1,2,\cdots,n)$ are controllers to be determined.

Step 3. Eq. (3.6) may be classified as two general forms: (1) All $G_i(\lambda_{ii},\dot{\lambda}_{ii})$ depend on $\lambda_{ii}(t)$ and $\dot{\lambda}_{ii}(t)$, (2) Some of $G_j(\lambda_{jj},\dot{\lambda}_{jj})$ depend on $\lambda_{jj}(t)$ and $\dot{\lambda}_{jj}(t)$, the remaining $G_k(\lambda_{kk},\dot{\lambda}_{kk})$ depend only on $\dot{\lambda}_{kk}(t)$.

Form (1): All $G_i(\lambda_{ii},\dot{\lambda}_{ii})$ depend on $\lambda_{ii}(t)$ and $\dot{\lambda}_{ii}(t)$.

Step 4. Design the controllers u_i such that

$$H_i(\lambda_{11},\cdots,\lambda_{nn},x_1,\cdots,x_n,y_1,\cdots,y_n,t)+\lambda_{ii}u_i=0 \quad (i=1,2,\cdots,n), \tag{3.7}$$

i.e. current mixed error dynamics has been replaced by pure error dynamics. By Eq. (3.7), we design the controllers u_i such that λ_{ii} $(i=1,2,\cdots,n)$ are linear function of each other with positive coefficients.

Step 5. Design $\lambda_{11}(t),\lambda_{22}(t),\cdots,\lambda_{nn}(t)$ such that

$$\forall t\geq 0,\quad 0<\lambda_{mii}\leq\lambda_{ii}(t)\leq\lambda_{Mii} \quad (i=1,2,\cdots,n), \tag{3.8}$$

where λ_{mii}, λ_{Mii} are positive constants, and

$$\forall t\geq 0,\quad G_i(\lambda_{ii},\dot{\lambda}_{ii})<0 \quad (i=1,2,\cdots,n), \tag{3.9}$$

then the Lyapunov function can be obtained and the generalized synchronization is achieved according to Lyapunov direct method.

Form (2): Some of $G_j(\lambda_{jj},\dot{\lambda}_{jj})$ depend on $\lambda_{jj}(t)$ and $\dot{\lambda}_{jj}(t)$, and the remaining $G_k(\lambda_{kk},\dot{\lambda}_{kk})$ depend only on $\dot{\lambda}_{kk}(t)$.

Step 4. Assume

$$\forall k, \quad \lambda_{kk}(t) = 1, \tag{3.10}$$

$$\forall k, \quad H_k(\lambda_{11}, \cdots, \lambda_{nn}, x_1, \cdots, x_n, y_1, \cdots, y_n, t) + \lambda_{kk}(t) u_k = -e_k, \tag{3.11}$$

$$\forall j, \quad H_j(\lambda_{11}, \cdots, \lambda_{nn}, x_1, \cdots, x_n, y_1, \cdots, y_n, t) + \lambda_{jj}(t) u_j = 0, \tag{3.12}$$

i.e. current mixed error dynamics has been replaced by pure error dynamics, and appropriately design the controllers $u_i \quad (i = 1, 2, \cdots, n)$ and $\lambda_{jj}(t)$ such that

$$\forall t \geq 0, \quad 0 < \lambda_{m\,jj} \leq \lambda_{jj}(t) \leq \lambda_{M\,jj}, \tag{3.13}$$

where $\lambda_{m\,jj}$, $\lambda_{M\,jj}$ are positive constants, and

$$\forall t \geq 0, \quad G_j(\lambda_{jj}, \dot{\lambda}_{jj}) < 0, \tag{3.14}$$

then the Lyapunov function can be obtained and the generalized synchronization is achieved according to Lyapunov direct method.

3.3 Example for New Autonomous Chaotic Systems

In the following two Sections, the functional relation between master and slave states is $y_i = g_i(t, x_i) = \alpha(t) x_i + \beta(t) \quad (i = 1, 2, \cdots, n)$.

The new autonomous chaotic system is constructed by mutual linear coupling of two non-identical nonlinear damped Mathieu systems, and the master and slave new autonomous chaotic systems can be described by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a(1 + x_4) x_1 - (1 + x_4) x_1^3 - a x_2 + b x_3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -(1 + x_2) x_3 - a(1 + x_2) x_3^3 - a x_4 + b x_1, \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
\dot{y}_1 &= y_2 + u_1, \\
\dot{y}_2 &= -a(1+y_4)y_1 - (1+y_4)y_1^3 - ay_2 + by_3 + u_2, \\
\dot{y}_3 &= y_4 + u_3, \\
\dot{y}_4 &= -(1+y_2)y_3 - a(1+y_2)y_3^3 - ay_4 + by_1 + u_4.
\end{aligned}
\tag{3.16}
$$

The parameters in simulation are $a = 0.5$, $b = 1.24$, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $x_4(0) = 0.2$, $y_1(0) = 0.3$, $y_2(0) = 0.3$, $y_3(0) = 0.4$, $y_4(0) = 0.4$. The phase portraits of the master new autonomous chaotic system are shown in Fig. 3.1.

Let $e_i = y_i - \alpha(t)x_i - \beta(t) \quad (i = 1, \cdots, 4)$, then the error dynamics can be obtained:

$$
\begin{aligned}
\dot{e}_1 &= e_2 - \dot{\alpha}(t)x_1 + \beta(t) - \dot{\beta}(t) + u_1, \\
\dot{e}_2 &= -ae_1 - ae_2 + be_3 - a(y_1y_4 - \alpha(t)x_1x_4) - [(1+y_4)y_1^3 - \alpha(t)(1+x_4)x_1^3] \\
&\quad - \dot{\alpha}(t)x_2 + (b-2a)\beta(t) - \dot{\beta}(t) + u_2, \\
\dot{e}_3 &= e_4 - \dot{\alpha}(t)x_3 + \beta(t) - \dot{\beta}(t) + u_3, \\
\dot{e}_4 &= -e_3 - ae_4 + be_1 - (y_2y_3 - \alpha(t)x_2x_3) - a[(1+y_2)y_3^3 - \alpha(t)(1+x_2)x_3^3] \\
&\quad - \dot{\alpha}(t)x_4 + (b-a-1)\beta(t) - \dot{\beta}(t) + u_4.
\end{aligned}
\tag{3.17}
$$

Step 1. Construct a Lyapunov function

$$
V(t,\mathbf{e}) = \frac{1}{2}\mathbf{e}^T\mathbf{\Lambda}(t)\mathbf{e} = \frac{1}{2}\lambda_{11}(t)e_1^2 + \frac{1}{2}\lambda_{22}(t)e_2^2 + \frac{1}{2}\lambda_{33}(t)e_3^2 + \frac{1}{2}\lambda_{44}(t)e_4^2 .
\tag{3.18}
$$

Its derivative is

$$
\begin{aligned}
\dot{V}(t,\mathbf{e}) &= \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2 + \lambda_{11}(t)e_1\dot{e}_1 + \frac{1}{2}\dot{\lambda}_{22}(t)e_2^2 + \lambda_{22}(t)e_2\dot{e}_2 \\
&\quad + \frac{1}{2}\dot{\lambda}_{33}(t)e_3^2 + \lambda_{33}(t)e_3\dot{e}_3 + \frac{1}{2}\dot{\lambda}_{44}(t)e_4^2 + \lambda_{44}(t)e_4\dot{e}_4.
\end{aligned}
\tag{3.19}
$$

Step 2. Eq. (3.19) can be rewritten in the following form

$$
\begin{aligned}
\dot{V}(t,\mathbf{e}) = & G_1(\lambda_{11},\dot{\lambda}_{11})e_1^2 + G_2(\lambda_{22},\dot{\lambda}_{22})e_2^2 + G_3(\lambda_{33},\dot{\lambda}_{33})e_3^2 + G_4(\lambda_{44},\dot{\lambda}_{44})e_4^2 \\
& + [H_1(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t) + \lambda_{11}u_1]e_1 \\
& + [H_2(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t) + \lambda_{22}u_2]e_2 \\
& + [H_3(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t) + \lambda_{33}u_3]e_3 \\
& + [H_4(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t) + \lambda_{44}u_4]e_4,
\end{aligned}
\tag{3.20}
$$

where

$$
\begin{aligned}
& G_1(\lambda_{11},\dot{\lambda}_{11}) = \frac{1}{2}\dot{\lambda}_{11}(t) - \lambda_{11}(t), \quad G_2(\lambda_{22},\dot{\lambda}_{22}) = \frac{1}{2}\dot{\lambda}_{22}(t) - a\lambda_{22}(t), \\
& G_3(\lambda_{33},\dot{\lambda}_{33}) = \frac{1}{2}\dot{\lambda}_{33}(t) - \lambda_{33}(t), \quad G_4(\lambda_{44},\dot{\lambda}_{44}) = \frac{1}{2}\dot{\lambda}_{44}(t) - a\lambda_{44}(t), \\
& H_1(\lambda_{11},\cdots,t) = \lambda_{11}(t)[-\dot{\alpha}(t)x_1 + \beta(t) - \dot{\beta}(t) + e_1] + b\lambda_{44}(t)e_4, \\
& H_2(\lambda_{11},\cdots,t) = \lambda_{11}(t)e_1 + \lambda_{22}(t)[-ae_1 - a(y_4y_1 - \alpha(t)x_4x_1) - ((1+y_4)y_1^3 \\
& \qquad - \alpha(t)(1+x_4)x_1^3) - \dot{\alpha}(t)x_2 + (b-2a)\beta(t) - \dot{\beta}(t)], \\
& H_3(\lambda_{11},\cdots,t) = b\lambda_{22}(t)e_2 + \lambda_{33}(t)[-\dot{\alpha}(t)x_3 + \beta(t) - \dot{\beta}(t) + e_3], \\
& H_4(\lambda_{11},\cdots,t) = \lambda_{33}(t)e_3 + \lambda_{44}(t)[-e_3 - (y_2y_3 - \alpha(t)x_2x_3) - a((1+y_2)y_3^3 \\
& \qquad - \alpha(t)(1+x_2)x_3^3) - \dot{\alpha}(t)x_4 + (b-a-1)\beta(t) - \dot{\beta}(t)].
\end{aligned}
\tag{3.21}
$$

Step 3. Since all $G_i(\lambda_{ii},\dot{\lambda}_{ii})$ depend on $\lambda_{ii}(t)$ and $\dot{\lambda}_{ii}(t)$ $(i=1,\cdots,4)$, Eq. (3.20) can be classified as form (1).

Step 4. Design the controllers

$$
\begin{aligned}
u_1 &= -y_1 - by_4 + (\alpha(t)+\dot{\alpha}(t))x_1 + b\alpha(t)x_4 + b\beta(t) + \dot{\beta}(t), \\
u_2 &= a(y_1y_4 - \alpha(t)x_1x_4) + (1+y_4)y_1^3 - \alpha(t)(1+x_4)x_1^3 \\
&\quad + \dot{\alpha}(t)x_2 - (b-2a)\beta(t) + \dot{\beta}(t), \\
u_3 &= -by_2 - y_3 + (\alpha(t)+\dot{\alpha}(t))x_3 + b\alpha(t)x_2 + b\beta(t) + \dot{\beta}(t), \\
u_4 &= y_2y_3 - \alpha(t)x_2x_3 + a(1+y_2)y_3^3 - \alpha(t)(1+x_2)x_3^3 \\
&\quad + (1-\frac{1}{a})y_3 - (1-\frac{1}{a})\alpha(t)x_3 + \dot{\alpha}(t)x_4 - (b-a-\frac{1}{a})\beta(t) + \dot{\beta}(t),
\end{aligned}
\tag{3.22}
$$

such that

$$
H_i(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t) + \lambda_{ii}(t)u_i = 0 \quad (i=1,\cdots,4), \tag{3.23}
$$

and λ_{ii} $(i=1,\cdots,4)$ are linear function of each other with positive coefficients:

$$\lambda_{11}(t)=\lambda_{44}(t),\quad \lambda_{22}(t)=\frac{1}{a}\lambda_{11}(t),\quad \lambda_{33}(t)=\frac{1}{a}\lambda_{11}(t). \tag{3.24}$$

Now, the mixed error dynamics is replaced by pure error dynamics:

$$\dot{V}(t,\mathbf{e})=G_1(\lambda_{11},\dot{\lambda}_{11})e_1^2+G_2(\lambda_{22},\dot{\lambda}_{22})e_2^2+G_3(\lambda_{33},\dot{\lambda}_{33})e_3^2+G_4(\lambda_{44},\dot{\lambda}_{44})e_4^2. \tag{3.25}$$

Step 5. Design

$$\lambda_{11}(t)=\frac{1}{1+e^{-t}},\lambda_{22}(t)=\frac{1}{a(1+e^{-t})},\lambda_{33}(t)=\frac{1}{a(1+e^{-t})},\lambda_{44}(t)=\frac{1}{1+e^{-t}}, \tag{3.26}$$

such that

$$\begin{aligned}
&\forall t\ge 0,\quad 0<\lambda_{m11}(t)=\frac{1}{2}\le\lambda_{11}(t)\le\lambda_{M11}(t)=1,\\
&\forall t\ge 0,\quad 0<\lambda_{m22}(t)=\frac{1}{2a}\le\lambda_{22}(t)\le\lambda_{M22}(t)=\frac{1}{a},\\
&\forall t\ge 0,\quad 0<\lambda_{m33}(t)=\frac{1}{2a}\le\lambda_{33}(t)\le\lambda_{M33}(t)=\frac{1}{a},\\
&\forall t\ge 0,\quad 0<\lambda_{m44}(t)=\frac{1}{2}\le\lambda_{44}(t)\le\lambda_{M44}(t)=1,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
&\forall t\ge 0,\quad G_1(\lambda_{11},\dot{\lambda}_{11})=\frac{1}{2}\dot{\lambda}_{11}(t)-\lambda_{11}(t)=\frac{-2-e^{-t}}{2(1+e^{-t})^2}<0,\\
&\forall t\ge 0,\quad G_2(\lambda_{22},\dot{\lambda}_{22})=\frac{1}{2}\dot{\lambda}_{22}(t)-a\lambda_{22}(t)=\frac{-2a+(1-2a)e^{-t}}{2a(1+e^{-t})^2}=\frac{-1}{(1+e^{-t})^2}<0,\\
&\forall t\ge 0,\quad G_3(\lambda_{33},\dot{\lambda}_{33})=\frac{1}{2}\dot{\lambda}_{33}(t)-\lambda_{33}(t)=\frac{-2-e^{-t}}{2a(1+e^{-t})^2}=\frac{-2-e^{-t}}{(1+e^{-t})^2}<0,\\
&\forall t\ge 0,\quad G_4(\lambda_{44},\dot{\lambda}_{44})=\frac{1}{2}\dot{\lambda}_{44}(t)-\lambda_{44}(t)=\frac{-2a+(1-2a)e^{-t}}{2(1+e^{-t})^2}=\frac{-1}{2(1+e^{-t})^2}<0,
\end{aligned} \tag{3.28}$$

then the Lyapunov function can be obtained

$$V(t,\mathbf{e})=\frac{1}{2(1+e^{-t})}e_1^2+\frac{1}{2a(1+e^{-t})}e_2^2+\frac{1}{2a(1+e^{-t})}e_3^2+\frac{1}{2(1+e^{-t})}e_4^2, \tag{3.29}$$

and

$$\dot{V}(t,\mathbf{e})=-\frac{2+e^{-t}}{2(1+e^{-t})^2}e_1^2-\frac{1}{(1+e^{-t})^2}e_2^2-\frac{2+e^{-t}}{(1+e^{-t})^2}e_3^2-\frac{1}{2(1+e^{-t})^2}e_4^2. \tag{3.30}$$

Since Lyapunov global asymptotical stability theorem is satisfied, the global generalized synchronization is achieved. $\alpha(t)=\sin\omega t$, $\beta(t)=\cos\omega t$, $\omega=1$ are chosen in simulation, and the results are shown in Fig. 3.2-3.3.

3.4 Example for New Nonautonomous Chaotic Systems

The new nonautonomous chaotic system is constructed by mutual linear coupling of two non-identical nonlinear damped Mathieu systems, and the master and slave new nonautonomous chaotic systems can be described by

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + bx_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -(1+\sin\omega t)x_3 - a(1+\sin\omega t)x_3^3 - ax_4 + bx_1,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\dot{y}_1 &= y_2 + u_1, \\
\dot{y}_2 &= -a(1+\sin\omega t)y_1 - (1+\sin\omega t)y_1^3 - ay_2 + by_3 + u_2, \\
\dot{y}_3 &= y_4 + u_3, \\
\dot{y}_4 &= -(1+\sin\omega t)y_3 - a(1+\sin\omega t)y_3^3 - ay_4 + by_1 + u_4,
\end{aligned} \tag{3.32}$$

The parameters in simulation are $a=0.5$, $b=1$, $\omega=1$, and the initial condition is $x_1(0)=0.1$, $x_2(0)=0.1$, $x_3(0)=0.2$, $x_4(0)=0.2$, $y_1(0)=0.3$, $y_2(0)=0.3$, $y_3(0)=0.4$, $y_4(0)=0.4$. The phase portraits of the master new nonautonomous chaotic system are shown in Fig. 3.4.

Let $e_i = y_i - \alpha(t)x_i - \beta(t)$ $(i=1,\cdots,4)$, then the error dynamics can be obtained:

$$\begin{aligned}
\dot{e}_1 &= e_2 - \dot{\alpha}(t)x_1 + \beta(t) - \dot{\beta}(t) + u_1, \\
\dot{e}_2 &= -a(1+\sin\omega t)e_1 - ae_2 + be_3 - (1+\sin\omega t)(y_1^3 - \alpha(t)x_1^3) - \dot{\alpha}(t)x_2 \\
&\quad + (-a(1+\sin\omega t) - a + b)\beta(t) - \dot{\beta}(t) + u_2, \\
\dot{e}_3 &= e_4 - \dot{\alpha}(t)x_3 + \beta(t) - \dot{\beta}(t) + u_3, \\
\dot{e}_4 &= -(1+\sin\omega t)e_3 - ae_4 + be_1 - a(1+\sin\omega t)(y_3^3 - \alpha(t)x_3^3) - \dot{\alpha}(t)x_4 \\
&\quad + (-(1+\sin\omega t) - a + b)\beta(t) - \dot{\beta}(t) + u_4,
\end{aligned} \tag{3.33}$$

Step 1. Construct a Lyapunov function

$$V(t,\mathbf{e})=\frac{1}{2}\mathbf{e}^T\mathbf{\Lambda}(t)\mathbf{e}=\frac{1}{2}\lambda_{11}(t)e_1^2+\frac{1}{2}\lambda_{22}(t)e_2^2+\frac{1}{2}\lambda_{33}(t)e_3^2+\frac{1}{2}\lambda_{44}(t)e_4^2. \tag{3.34}$$

Its derivative is

$$\begin{aligned}\dot{V}(t,\mathbf{e})=&\frac{1}{2}\dot{\lambda}_{11}(t)e_1^2+\lambda_{11}(t)e_1\dot{e}_1+\frac{1}{2}\dot{\lambda}_{22}(t)e_2^2+\lambda_{22}(t)e_2\dot{e}_2\\&+\frac{1}{2}\dot{\lambda}_{33}(t)e_3^2+\lambda_{33}(t)e_3\dot{e}_3+\frac{1}{2}\dot{\lambda}_{44}(t)e_4^2+\lambda_{44}(t)e_4\dot{e}_4.\end{aligned} \tag{3.35}$$

Step 2. Eq. (3.35) can be rewritten in the following form

$$\begin{aligned}\dot{V}(t,\mathbf{e})=&G_1(\lambda_{11},\dot{\lambda}_{11})e_1^2+G_2(\lambda_{22},\dot{\lambda}_{22})e_2^2+G_3(\lambda_{33},\dot{\lambda}_{33})e_3^2+G_4(\lambda_{44},\dot{\lambda}_{44})e_4^2\\&+[H_1(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t)+\lambda_{11}u_1]e_1\\&+[H_2(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t)+\lambda_{22}u_2]e_2\\&+[H_3(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t)+\lambda_{33}u_3]e_3\\&+[H_4(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t)+\lambda_{44}u_4]e_4,\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}&G_1(\lambda_{11},\dot{\lambda}_{11})=\frac{1}{2}\dot{\lambda}_{11}(t),\quad G_2(\lambda_{22},\dot{\lambda}_{22})=\frac{1}{2}\dot{\lambda}_{22}(t)-a\lambda_{22}(t),\\&G_3(\lambda_{33},\dot{\lambda}_{33})=\frac{1}{2}\dot{\lambda}_{33}(t),\quad G_4(\lambda_{44},\dot{\lambda}_{44})=\frac{1}{2}\dot{\lambda}_{44}(t)-a\lambda_{44}(t),\\&H_1(\lambda_{11},\cdots,t)=\lambda_{11}(t)[-\dot{\alpha}(t)x_1+\beta(t)-\dot{\beta}(t)]+b\lambda_{44}(t)e_4,\\&H_2(\lambda_{11},\cdots,t)=\lambda_{11}(t)e_1+\lambda_{22}(t)[-a(1+\sin\omega t)e_1-(1+\sin\omega t)(y_1^3-\alpha(t)x_1^3)\\&\qquad-\dot{\alpha}(t)x_2+(-a(1+\sin\omega t)-a+b)\beta(t)-\dot{\beta}(t)],\\&H_3(\lambda_{11},\cdots,t)=b\lambda_{22}(t)e_2+\lambda_{33}(t)[-\dot{\alpha}(t)x_3+\beta(t)-\dot{\beta}(t)],\\&H_4(\lambda_{11},\cdots,t)=\lambda_{33}(t)e_3+\lambda_{44}(t)[-(1+\sin\omega t)e_3-a(1+\sin\omega t)(y_3^3-\alpha(t)x_3^3)\\&\qquad-\dot{\alpha}(t)x_4+(-(1+\sin\omega t)-a+b)\beta(t)-\dot{\beta}(t)].\end{aligned} \tag{3.37}$$

Step 3. Since some of $G_j(\lambda_{jj},\dot{\lambda}_{jj})$ depend on $\lambda_{jj}(t)$ and $\dot{\lambda}_{jj}(t)$ $(j=2,4)$, the remaining $G_k(\lambda_{kk},\dot{\lambda}_{kk})$ depend only on $\dot{\lambda}_{kk}(t)$ $(k=1,3)$, Eq. (3.37) can be classified as form (2).

Step 4. Assume

$$\lambda_{11}(t)=1, \quad \lambda_{33}(t)=1, \tag{3.38}$$

$$\begin{aligned} &H_1(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t)+\lambda_{11}(t)u_1=-e_1,\\ &H_3(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t)+\lambda_{33}(t)u_3=-e_3, \end{aligned} \tag{3.39}$$

$$\begin{aligned} &H_2(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t)+\lambda_{22}(t)u_2=0,\\ &H_4(\lambda_{11},\cdots,\lambda_{44},x_1,\cdots,x_4,y_1,\cdots,y_4,t)+\lambda_{44}(t)u_4=0, \end{aligned} \tag{3.40}$$

and appropriately design the controllers $u_i \quad (i=1,\cdots,4)$ and $\lambda_{22}(t)$, $\lambda_{44}(t)$

$$\begin{aligned} u_1 &= -y_1-\frac{b}{2+\sin\omega t}y_4+(\alpha(t)+\dot{\alpha}(t))x_1+\frac{b\alpha(t)}{2+\sin\omega t}x_4+\frac{b\beta(t)}{2+\sin\omega t}+\dot{\beta}(t),\\ u_2 &= -ay_1+a\alpha(t)x_1+\dot{\alpha}(t)x_2+(1+\sin\omega t)(y_1^3-\alpha(t)x_1^3)\\ &\quad +(a\sin\omega t+3a-b)\beta(t)+\dot{\beta}(t),\\ u_3 &= -y_3-\frac{b}{2+\sin\omega t}y_2+(\alpha(t)+\dot{\alpha}(t))x_3+\frac{b\alpha(t)}{2+\sin\omega t}x_2+\frac{b\beta(t)}{2+\sin\omega t}+\dot{\beta}(t),\\ u_4 &= -y_3+\alpha(t)x_3+\dot{\alpha}(t)x_4+a(1+\sin\omega t)(y_3^3-\alpha(t)x_3^3)\\ &\quad +(\sin\omega t+a-b+2)\beta(t)+\dot{\beta}(t), \end{aligned} \tag{3.41}$$

$$\lambda_{22}(t)=\frac{1}{a(2+\sin\omega t)}, \quad \lambda_{44}(t)=\frac{1}{2+\sin\omega t}, \tag{3.42}$$

such that

$$\begin{aligned} &\forall t\ge 0, \quad 0<\lambda_{m22}=\frac{1}{3a}\le\lambda_{22}(t)\le\lambda_{M22}=\frac{1}{a},\\ &\forall t\ge 0, \quad 0<\lambda_{m44}=\frac{1}{3}\le\lambda_{44}(t)\le\lambda_{M44}=1, \end{aligned} \tag{3.43}$$

$$\begin{aligned} \forall t\ge 0, \quad G_2(\lambda_{22},\dot{\lambda}_{22}) &= \frac{1}{2}\dot{\lambda}_{22}(t)-a\lambda_{22}(t)\\ &= \frac{-(4a+2a\sin\omega t+\omega\cos\omega t)}{2a(2+\sin\omega t)^2}=\frac{-(2+\sin t+\cos t)}{(2+\sin t)^2}<0,\\ \forall t\ge 0, \quad G_4(\lambda_{44},\dot{\lambda}_{44}) &= \frac{1}{2}\dot{\lambda}_{44}(t)-a\lambda_{44}(t)\\ &= \frac{-(4a+2a\sin\omega t+\omega\cos\omega t)}{2(2+\sin\omega t)^2}=\frac{-(2+\sin t+\cos t)}{2(2+\sin t)^2}<0. \end{aligned} \tag{3.44}$$

Now, the mixed error dynamics is replaced by pure error dynamics:

$$\begin{aligned} \dot{V}(t,\mathbf{e}) &= [G_1(\lambda_{11},\dot{\lambda}_{11})-\lambda_{11}]e_1^2+G_2(\lambda_{22},\dot{\lambda}_{22})e_2^2\\ &\quad +[G_3(\lambda_{33},\dot{\lambda}_{33})-\lambda_{33}]e_3^2+G_4(\lambda_{44},\dot{\lambda}_{44})e_4^2. \end{aligned} \tag{3.45}$$

Then the Lyapunov function can be obtained

$$V(t,\mathbf{e}) = \frac{1}{2}e_1^2 + \frac{1}{2a(2+\sin\omega t)}e_2^2 + \frac{1}{2}e_3^2 + \frac{1}{2(2+\sin\omega t)}e_4^2, \tag{3.46}$$

and

$$\dot{V}(t,\mathbf{e}) = -e_1^2 - \frac{2+\sin t+\cos t}{(2+\sin t)^2}e_2^2 - e_3^2 - \frac{2+\sin t+\cos t}{2(2+\sin t)^2}e_4^2. \tag{3.47}$$

Since Lyapunov global asymptotical stability theorem is satisfied, the global generalized synchronization is achieved. $\alpha(t)=\sin\omega t$, $\beta(t)=\cos\omega t$, $\omega=1$ are chosen in simulation, and the results are shown in Fig. 3.5-3.6.

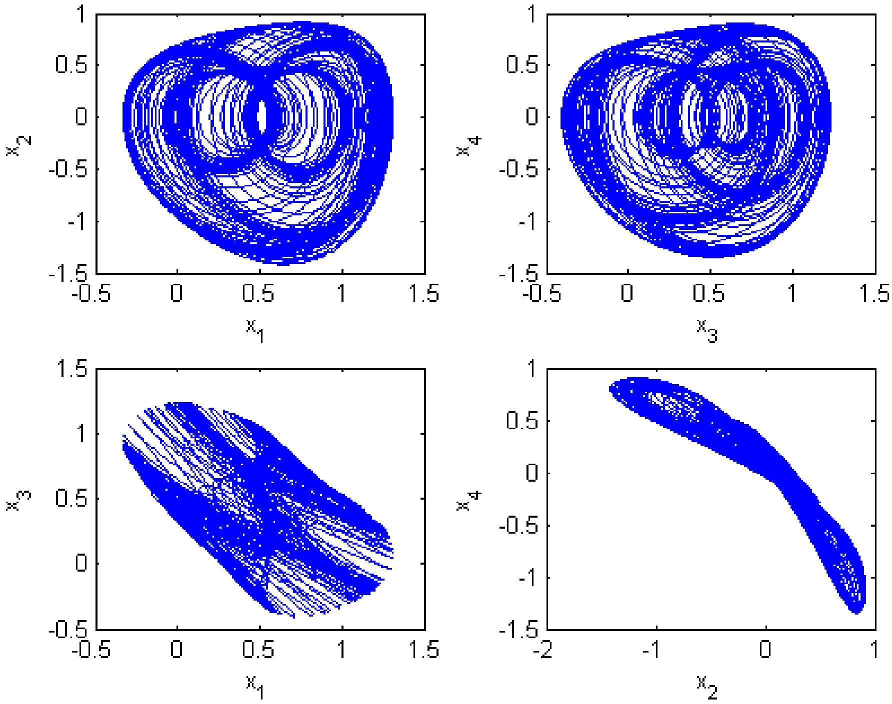

Fig. 3.1 Phase portraits of the master new autonomous chaotic system.

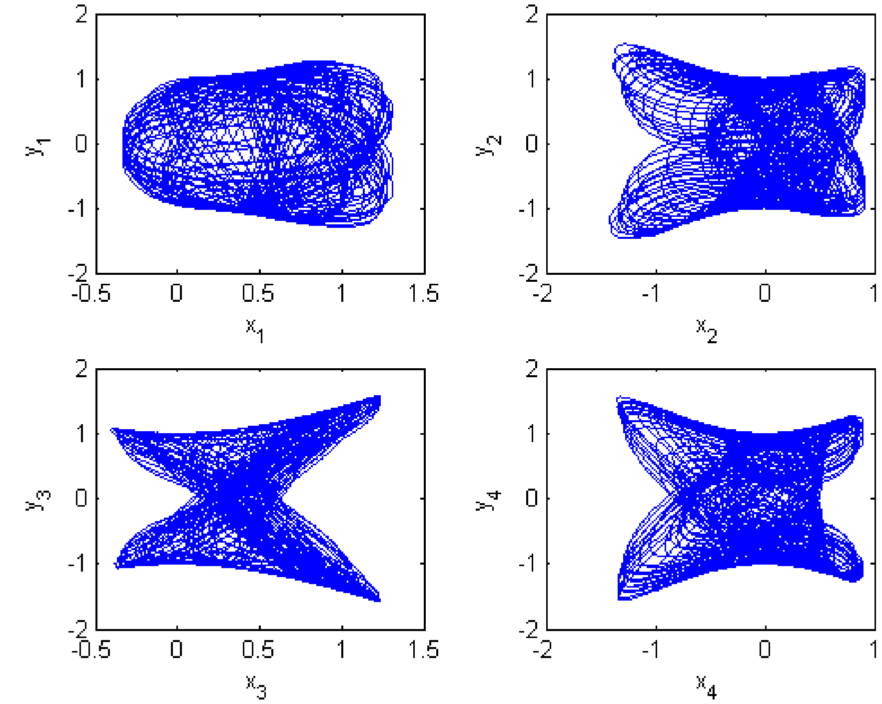

Fig. 3.2 Phase portraits of x_i to y_i ($i = 1, \cdots, 4$) for Section 3.3 when the generalized synchronization is obtained.

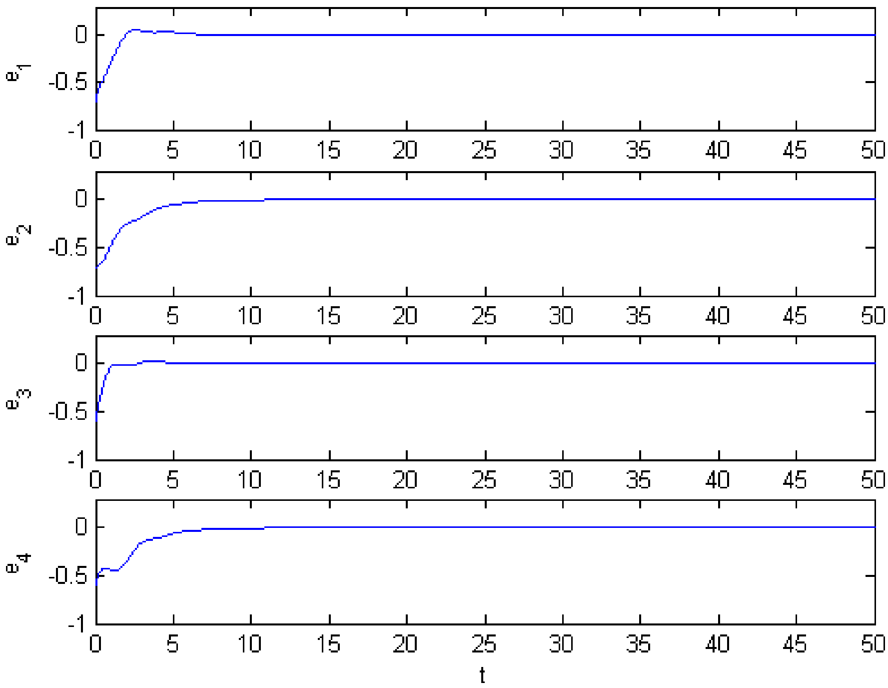

Fig. 3.3 Time histories of the state errors for Section 3.3.

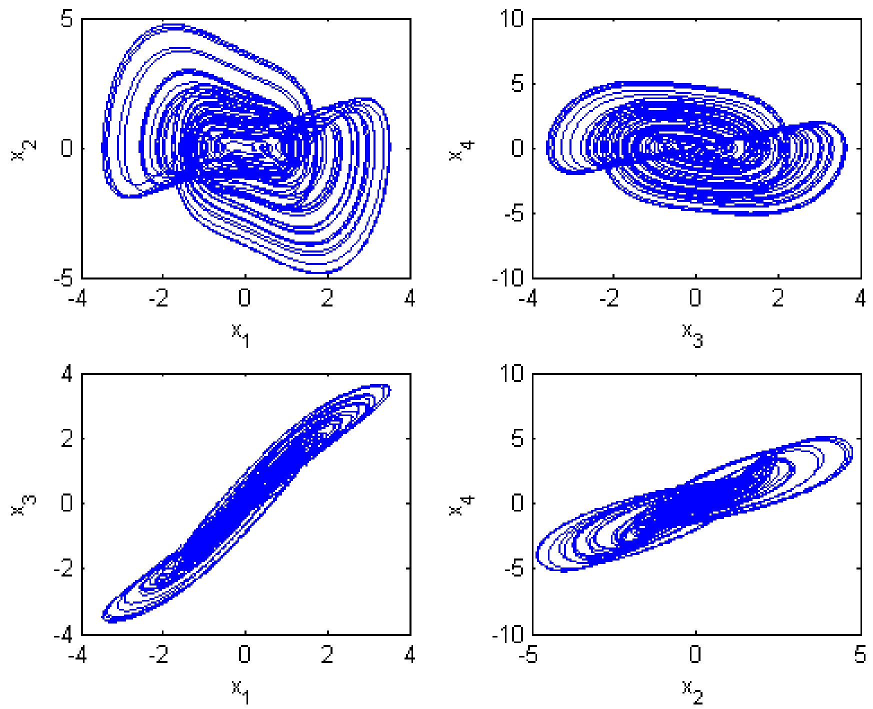

Fig. 3.4 Phase portraits of the master new nonautonomous chaotic system.

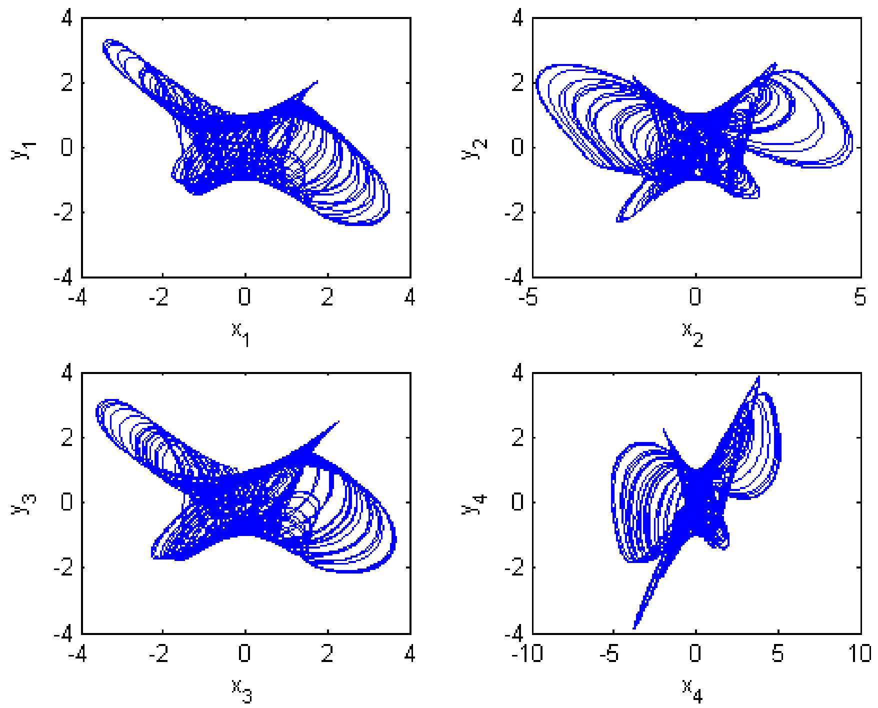

Fig. 3.5 Phase portraits of x_i to y_i ($i = 1, \cdots, 4$) for Section 3.4 when the generalized synchronization is obtained.

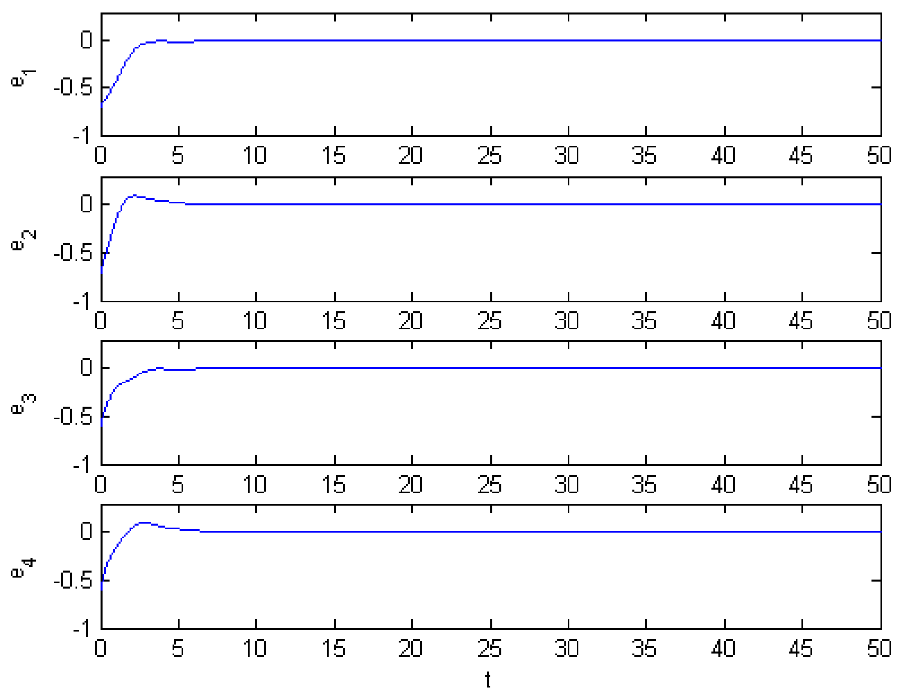

Fig. 3.6 Time histories of the state errors for Section 3.4.

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Chapter 4

Nonlinear Generalized Synchronization of New Chaotic Systems by Pure Error Dynamics and Elaborate Nondiagonal Lyapunov Function

4.1 Preliminaries

By applying pure error dynamics and elaborate nondiagonal Lyapunov function, the nonlinear generalized synchronization is studied in this Chapter. In stead of current plain square sum Lyapunov function [1-6], the elaborate nondiagonal Lyapunov function is applied in this study. A systematic method of designing Lyapunov function is proposed based on Lyapunov direct method [7], and the nonlinear generalized synchronization is achieved by applying this technique.

4.2 Design of Lyapunov Function

Consider the master and slave nonlinear dynamic systems described by

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \tag{4.1}$$

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) + \mathbf{u}(t, \mathbf{x}, \mathbf{y}), \tag{4.2}$$

where $\mathbf{x}, \mathbf{y} \in R^n$ are master and slave state vectors, $\mathbf{f}: R_+ \times R^n \to R^n$ is a nonlinear vector function, and $\mathbf{u}$: $R_+ \times R^n \times R^n \to R^n$ is controller vector.

Generalized synchronization means that there is a functional relation $\mathbf{y} = \mathbf{g}(\mathbf{x})$ between master and slave states as time goes to infinity, where $\mathbf{g}: R^n \to R^n$ is a continuously differentiable nonlinear vector function. Define $\mathbf{e} = \mathbf{y} - \mathbf{g}(\mathbf{x})$ as generalized synchronization error vector, and the error dynamics can be obtained:

$$\dot{\mathbf{e}} = \dot{\mathbf{y}} - \dot{\mathbf{g}}(\mathbf{x}) = \dot{\mathbf{y}} - \frac{d\mathbf{g}(\mathbf{x})}{d\mathbf{x}}\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{y}) - \frac{d\mathbf{g}(\mathbf{x})}{d\mathbf{x}}\mathbf{f}(t,\mathbf{x}) + \mathbf{u}(t,\mathbf{x},\mathbf{y}) . \tag{4.3}$$

Eq. (4.3) can be rewritten in the following form:

$$\dot{\mathbf{e}} = \mathbf{p}(t,\mathbf{e}) + \mathbf{q}(t,\mathbf{x},\mathbf{y}) + \mathbf{u}(t,\mathbf{x},\mathbf{y}) , \tag{4.4}$$

where $\mathbf{p}: R_+ \times R^n \to R^n$ and $\mathbf{q}: R_+ \times R^n \times R^n \to R^n$ are continuous vector functions represent the error variable terms and the state variable terms in the error dynamics respectively.

In order to transform current mixed error dynamics into pure error dynamics, the controller vector is chosen as

$$\mathbf{u}(t,\mathbf{x},\mathbf{y}) = -\mathbf{q}(t,\mathbf{x},\mathbf{y}) + \mathbf{v}(t,\mathbf{e}) , \tag{4.5}$$

where $\mathbf{v}: R_+ \times R^n \to R^n$ is a continuous vector function.

Now the pure error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(t,\mathbf{e}) + \mathbf{v}(t,\mathbf{e}) , \tag{4.6}$$

Based on Lyapunov direct method [7], the scheme of nonlinear generalized synchronization and the procedure of designing elaborate nondiagonal Lyapunov function are described as follows:

Step 1. Construct a Lyapunov function

$$\begin{aligned} V(t,\mathbf{e}) &= \sum_{i=1}^{n} \frac{1}{2}\mathbf{e}_i^T \mathbf{\Lambda}_i(t)\mathbf{e}_i \\ &= [\frac{1}{2}\lambda_{11}(t)e_1^2 + \lambda_{12}e_1e_2 + \frac{1}{2}\lambda_{22}(t)e_2^2] + \cdots + [\frac{1}{2}\lambda_{nn}(t)e_n^2 + \lambda_{n1}e_ne_1 + \frac{1}{2}\lambda_{11}(t)e_1^2], \end{aligned} \tag{4.7}$$

where $\mathbf{e}_i = \begin{bmatrix} e_i \\ e_{i+1} \end{bmatrix}$ $(i = 1, 2, \cdots, n-1)$, $\mathbf{e}_n = \begin{bmatrix} e_n \\ e_1 \end{bmatrix}$, $\mathbf{\Lambda}_i(t) = \begin{bmatrix} \lambda_{ii}(t) & \lambda_{ii+1} \\ \lambda_{ii+1} & \lambda_{i+1i+1}(t) \end{bmatrix}$ $(i = 1, 2, \cdots, n-1)$, $\mathbf{\Lambda}_n(t) = \begin{bmatrix} \lambda_{nn}(t) & \lambda_{n1} \\ \lambda_{n1} & \lambda_{11}(t) \end{bmatrix}$, and $\mathbf{\Lambda}_i(t) \in R^{2\times 2}$ $(i = 1, 2, \cdots, n)$ are unknown continuously differentiable positive definite matrices to be designed and $\mathbf{\Lambda}_i(t)$, $\mathbf{\Lambda}_n(t)$ are nondiagonal. According to Sylvester's criterion, $\mathbf{\Lambda}_i(t)$ have to be chosen that

$$\begin{aligned} \forall t \geq 0, \quad & \lambda_{ii}(t) > 0, \lambda_{ii}(t)\lambda_{i+1i+1}(t) - \lambda_{ii+1}^2 > 0 \quad (i = 1, 2, \cdots, n-1), \\ & \lambda_{nn}(t) > 0, \lambda_{nn}(t)\lambda_{11}(t) - \lambda_{n1}^2 > 0 \quad (i = n), \end{aligned} \tag{4.8}$$

and

$$\forall t \geq 0, \quad 0 < \lambda_{mii} \leq \lambda_{ii}(t) \leq \lambda_{Mii} \quad (i = 1, 2, \cdots, n), \tag{4.9}$$

where $\lambda_{mii}, \lambda_{Mii}$ are positive constants.

Step 2. The derivative of Lyapunov function is

$$\begin{aligned} \dot{V}(t, \mathbf{e}) &= \sum_{i=1}^{n} [\dot{\mathbf{e}}_i^T \mathbf{\Lambda}_i(t)\mathbf{e}_i + \frac{1}{2}\mathbf{e}_i^T \dot{\mathbf{\Lambda}}_i(t)\mathbf{e}_i] \\ &= [\lambda_{11}(t)e_1\dot{e}_1 + \lambda_{12}\dot{e}_1 e_2 + \lambda_{12}e_1\dot{e}_2 + \lambda_{22}(t)e_2\dot{e}_2 + \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2 + \frac{1}{2}\dot{\lambda}_{22}(t)e_2^2] \\ &\quad + \cdots + [\lambda_{nn}(t)e_n\dot{e}_n + \lambda_{n1}\dot{e}_n e_1 + \lambda_{n1}e_n\dot{e}_1 + \lambda_{11}(t)e_1\dot{e}_1 + \frac{1}{2}\dot{\lambda}_{nn}(t)e_n^2 + \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2], \end{aligned} \tag{4.10}$$

Eq. (4.10) can be rewritten in the following form:

$$\begin{aligned} \dot{V}(t, \mathbf{e}) = & F_1(\dot{\lambda}_{11}, \lambda_{11}, \cdots, \lambda_{nn}, \lambda_{12}, \cdots, \lambda_{n1}, t)e_1^2 \\ & + \cdots + F_n(\dot{\lambda}_{nn}, \lambda_{11}, \cdots, \lambda_{nn}, \lambda_{12}, \cdots, \lambda_{n1}, t)e_n^2 \\ & + G_1(\lambda_{11}, \cdots, \lambda_{nn}, \lambda_{12}, \cdots, \lambda_{n1}, t)e_1 e_2 \\ & + \cdots + G_m(\lambda_{11}, \cdots, \lambda_{nn}, \lambda_{12}, \cdots, \lambda_{n1}, t)e_{n-1}e_n \\ & + (2\lambda_{11}v_1 + \lambda_{12}v_2 + \lambda_{n1}v_n)e_1 \\ & + \cdots + (2\lambda_{nn}v_n + \lambda_{n1}v_1 + \lambda_{n-1n}v_{n-1})e_n, \end{aligned} \tag{4.11}$$

where $F_i(\dot{\lambda}_{ii}, \lambda_{11}, \cdots, \lambda_{nn}, \lambda_{12}, \cdots, \lambda_{n1}, t) \quad (i = 1,2, \cdots, n)$, $G_j(\lambda_{11}, \cdots, \lambda_{nn}, \lambda_{12}, \cdots, \lambda_{n1}, t)$ $(j = 1, 2, \cdots, m, \quad m = \dfrac{n(n-1)}{2})$ are continuous differentiable functions, and $v_i \quad (i = 1, 2, \cdots, n)$ are controllers to be determined.

Step 3. Appropriately design the controllers v_i such that Eq. (4.11) can be reduced to

$$\begin{aligned}\dot{V}(t,\mathbf{e}) &= \hat{F}_1(\dot{\lambda}_{11},\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)e_1^2 \\ &+\cdots+\hat{F}_n(\dot{\lambda}_{nn},\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)e_n^2 \\ &+\hat{G}_1(\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)e_1e_2 \\ &+\cdots+\hat{G}_m(\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)e_{n-1}e_n,\end{aligned} \tag{4.12}$$

where $\hat{F}_i(\dot{\lambda}_{ii},\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)$ $(i=1,2,\cdots,n)$ and $\hat{G}_j(\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)$ $(j=1,2,\cdots,m,\quad m=\dfrac{n(n-1)}{2})$ are continuous differentiable functions.

Step 4. Assume

$$\forall j,\quad \hat{G}_j(\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)=0, \tag{4.13}$$

then the relationship between λ_{ij} can be obtained.

Step 5. Use the results of Step 4 to check if

$$\forall t\geq 0,\quad \hat{F}_i(\dot{\lambda}_{ii},\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)<0\quad (i=1,2,\cdots,n). \tag{4.14}$$

Step 6. If Eq. (4.14) can be satisfied, the conditions derived from Eq. (4.14) can be obtained. If Eq. (4.14) can not be satisfied, i.e.

$$\begin{aligned}\forall t\geq 0,\quad &\hat{F}_j(\dot{\lambda}_{jj},\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)\geq 0, \\ &\hat{F}_k(\dot{\lambda}_{kk},\lambda_{11},\cdots,\lambda_{nn},\lambda_{12},\cdots,\lambda_{n1},t)<0,\end{aligned} \tag{4.15}$$

return to Step 3 and modify the controllers v_j by addition of k_je_j, where k_j are constant gains to be determined. Repeat Step 4 and Step 5, then the conditions guarantee the validity of Eq. (4.14) can be assured.

Step 7. Appropriately design k_j and $\lambda_{ij}(t)$ such that each condition derived from the above procedure holds. Finally the elaborate nondiagonal Lyapunov function can be obtained and the generalized synchronization is achieved according to Lyapunov direct method.

4.3 Example for New Autonomous Chaotic Systems

In the following two Sections, the nonlinear functional relation between master and slave states is $y_i = g_i(x_i) = \alpha x_i^2 + \beta x_i + \gamma \quad (i = 1, 2, \cdots, n)$. The master and slave new autonomous chaotic systems can be described by Eq.(3.15) and Eq. (3.16), respectively. The parameters and the initial conditions of the master and slave systems are the same as shown in Section 3.3.

Let $e_i = y_i - \alpha x_i^2 - \beta x_i - \gamma \quad (i = 1, \cdots, 4)$, then the error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(\mathbf{e}) + \mathbf{q}(\mathbf{x}, \mathbf{y}) + \mathbf{u}(\mathbf{x}, \mathbf{y}), \tag{4.16}$$

where

$$\begin{aligned}
&\mathbf{p}(\mathbf{e}) = \left[p_1(\mathbf{e}) \quad p_2(\mathbf{e}) \quad p_3(\mathbf{e}) \quad p_4(\mathbf{e}) \right]^T, \\
&\mathbf{q}(\mathbf{x}, \mathbf{y}) = \left[q_1(\mathbf{x}, \mathbf{y}) \quad q_2(\mathbf{x}, \mathbf{y}) \quad q_3(\mathbf{x}, \mathbf{y}) \quad q_4(\mathbf{x}, \mathbf{y}) \right]^T, \\
&p_1(\mathbf{e}) = e_2, \quad p_2(\mathbf{e}) = -ae_1 - ae_2 + be_3, \quad p_3(\mathbf{e}) = e_4, \quad p_4(\mathbf{e}) = -e_3 - ae_4 + be_1, \\
&q_1(\mathbf{x}, \mathbf{y}) = \alpha x_2^2 - 2\alpha x_1 x_2 + \gamma, \\
&q_2(\mathbf{x}, \mathbf{y}) = -\alpha(ax_1^2 - ax_2^2 - bx_3^2) - a(y_4 y_1 - \beta x_4 x_1) + (b - 2a)\gamma \\
&\qquad -[(1 + y_4)y_1^3 - \beta(1 + x_4)x_1^3] + 2\alpha x_2[a(1 + x_4)x_1 + (1 + x_4)x_1^3 - bx_3], \\
&q_3(\mathbf{x}, \mathbf{y}) = \alpha x_4^2 - 2\alpha x_3 x_4 + \gamma, \\
&q_4(\mathbf{x}, \mathbf{y}) = -\alpha(x_3^2 - ax_4^2 - bx_1^2) - (y_2 y_3 - \beta x_2 x_3) + (b - a - 1)\gamma \\
&\qquad - a[(1 + y_2)y_3^3 - \beta(1 + x_2)x_3^3] + 2\alpha x_4[(1 + x_2)x_3 + a(1 + x_2)x_3^3 - bx_1].
\end{aligned} \tag{4.17}$$

In order to transform current mixed error dynamics into pure error dynamics, the controller vector is chosen as

$$\mathbf{u}(\mathbf{x}, \mathbf{y}) = -\mathbf{q}(\mathbf{x}, \mathbf{y}) + \mathbf{v}(\mathbf{e}). \tag{4.18}$$

Now the pure error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(\mathbf{e}) + \mathbf{v}(\mathbf{e}). \tag{4.19}$$

Step 1. Construct a Lyapunov function

$$\begin{aligned} V(\mathbf{e}) &= \sum_{i=1}^{4} \frac{1}{2} \mathbf{e}_i^T \mathbf{\Lambda}_i \mathbf{e}_i \\ &= [\frac{1}{2}\lambda_{11}e_1^2 + \lambda_{12}e_1e_2 + \frac{1}{2}\lambda_{22}e_2^2] + \cdots + [\frac{1}{2}\lambda_{44}e_4^2 + \lambda_{41}e_4e_1 + \frac{1}{2}\lambda_{11}e_1^2], \end{aligned} \tag{4.20}$$

where $\mathbf{e}_i = \begin{bmatrix} e_i \\ e_{i+1} \end{bmatrix}$ $(i = 1, \cdots, 3)$, $\mathbf{e}_4 = \begin{bmatrix} e_4 \\ e_1 \end{bmatrix}$, $\mathbf{\Lambda}_i = \begin{bmatrix} \lambda_{ii} & \lambda_{ii+1} \\ \lambda_{ii+1} & \lambda_{i+1i+1} \end{bmatrix}$ $(i = 1, \cdots, 3)$, $\mathbf{\Lambda}_4 = \begin{bmatrix} \lambda_{44} & \lambda_{41} \\ \lambda_{41} & \lambda_{11} \end{bmatrix}$, and $\mathbf{\Lambda}_i$ $(i = 1, \cdots, 4)$ are unknown continuously differentiable positive definite nondiagonal matrices to be designed. According to Sylvester's criterion, $\mathbf{\Lambda}_i$ have to be chosen that

$$\begin{aligned} &\lambda_{11} > 0, \lambda_{11}\lambda_{22} - \lambda_{12}^2 > 0, \\ &\lambda_{22} > 0, \lambda_{22}\lambda_{33} - \lambda_{23}^2 > 0, \\ &\lambda_{33} > 0, \lambda_{33}\lambda_{44} - \lambda_{34}^2 > 0, \\ &\lambda_{44} > 0, \lambda_{44}\lambda_{11} - \lambda_{41}^2 > 0. \end{aligned} \tag{4.21}$$

Step 2. The derivative of Lyapunov function is

$$\begin{aligned} \dot{V}(\mathbf{e}) &= \sum_{i=1}^{4} \dot{\mathbf{e}}_i^T \mathbf{\Lambda}_i \mathbf{e}_i \\ &= [\lambda_{11}e_1\dot{e}_1 + \lambda_{12}\dot{e}_1e_2 + \lambda_{12}e_1\dot{e}_2 + \lambda_{22}e_2\dot{e}_2] \\ &\quad + \cdots + [\lambda_{44}e_4\dot{e}_4 + \lambda_{41}\dot{e}_4e_1 + \lambda_{41}e_4\dot{e}_1 + \lambda_{11}e_1\dot{e}_1]. \end{aligned} \tag{4.22}$$

Eq. (4.22) can be rewritten in the following form:

$$\begin{aligned} \dot{V}(\mathbf{e}) &= F_1(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_1^2 + F_2(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_2^2 \\ &\quad + F_3(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_3^2 + F_4(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_4^2 \\ &\quad + G_1(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_1e_2 + G_2(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_1e_3 \\ &\quad + G_3(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_1e_4 + G_4(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_2e_3 \\ &\quad + G_5(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_2e_4 + G_6(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41})e_3e_4 \\ &\quad + (2\lambda_{11}v_1 + \lambda_{12}v_2 + \lambda_{41}v_4)e_1 + (2\lambda_{22}v_2 + \lambda_{23}v_3 + \lambda_{12}v_1)e_2 \\ &\quad + (2\lambda_{33}v_3 + \lambda_{34}v_4 + \lambda_{23}v_2)e_3 + (2\lambda_{44}v_4 + \lambda_{41}v_1 + \lambda_{34}v_3)e_4, \end{aligned} \tag{4.23}$$

where

$$
\begin{aligned}
&F_1(\lambda_{11},\cdots,\lambda_{41}) = -a\lambda_{12} + b\lambda_{41}, \quad F_2(\lambda_{11},\cdots,\lambda_{41}) = \lambda_{12} - 2a\lambda_{22}, \\
&F_3(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{23} - \lambda_{34}, \quad F_4(\lambda_{11},\cdots,\lambda_{41}) = \lambda_{34} - 2a\lambda_{44}, \\
&G_1(\lambda_{11},\cdots,\lambda_{41}) = 2\lambda_{11} - a\lambda_{12} - 2a\lambda_{22}, \quad G_2(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{12} - a\lambda_{23} + b\lambda_{34} - \lambda_{41}, \\
&G_3(\lambda_{11},\cdots,\lambda_{41}) = 2b\lambda_{44} - a\lambda_{41}, \quad G_4(\lambda_{11},\cdots,\lambda_{41}) = 2b\lambda_{22} - a\lambda_{23}, \\
&G_5(\lambda_{11},\cdots,\lambda_{41}) = \lambda_{23} + \lambda_{41}, \quad G_6(\lambda_{11},\cdots,\lambda_{41}) = 2\lambda_{33} - a\lambda_{34} - 2\lambda_{44}.
\end{aligned} \tag{4.24}
$$

Step 3. Design the controllers

$$
v_1 = -e_2, \quad v_2 = ae_1, \quad v_3 = -e_4, \quad v_4 = e_3, \tag{4.25}
$$

such that Eq. (4.23) can be reduced to

$$
\begin{aligned}
\dot{V}(\mathbf{e}) = {} & \hat{F}_1(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_1^2 + \hat{F}_2(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_2^2 \\
& + \hat{F}_3(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_3^2 + \hat{F}_4(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_4^2 \\
& + \hat{G}_1(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_1e_2 + \hat{G}_2(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_1e_3 \\
& + \hat{G}_3(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_1e_4 + \hat{G}_4(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_2e_3 \\
& + \hat{G}_5(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_2e_4 + \hat{G}_6(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41})e_3e_4,
\end{aligned} \tag{4.26}
$$

where

$$
\begin{aligned}
&\hat{F}_1(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{41}, \quad \hat{F}_2(\lambda_{11},\cdots,\lambda_{41}) = -2a\lambda_{22}, \\
&\hat{F}_3(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{23}, \quad \hat{F}_4(\lambda_{11},\cdots,\lambda_{41}) = -2a\lambda_{44}, \\
&\hat{G}_1(\lambda_{11},\cdots,\lambda_{41}) = -a\lambda_{12}, \quad \hat{G}_2(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{12} + b\lambda_{34}, \\
&\hat{G}_3(\lambda_{11},\cdots,\lambda_{41}) = 2b\lambda_{44} - a\lambda_{41}, \quad \hat{G}_4(\lambda_{11},\cdots,\lambda_{41}) = 2b\lambda_{22} - a\lambda_{23}, \\
&\hat{G}_5(\lambda_{11},\cdots,\lambda_{41}) = 0, \quad \hat{G}_6(\lambda_{11},\cdots,\lambda_{41}) = -a\lambda_{34}.
\end{aligned} \tag{4.27}
$$

Step 4. Assume

$$
\forall j, \quad \hat{G}_j(\lambda_{11},\cdots,\lambda_{41}) = 0, \tag{4.28}
$$

then the relationship between λ_{ij} can be obtained:

$$
\lambda_{12} = 0, \quad \lambda_{23} = \frac{b}{2a}\lambda_{22}, \quad \lambda_{34} = 0, \quad \lambda_{41} = \frac{b}{2a}\lambda_{44}. \tag{4.29}
$$

Step 5. Use the results of Step 4 to check if

$$
\hat{F}_i(\lambda_{11},\cdots,\lambda_{41}) < 0 \quad (i = 1,\cdots,4). \tag{4.30}
$$

It can be obtained that

$$\hat{F}_1(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{41} > 0, \quad \hat{F}_2(\lambda_{11},\cdots,\lambda_{41}) = -2a\lambda_{22} < 0,$$
$$\hat{F}_3(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{23} > 0, \quad \hat{F}_4(\lambda_{11},\cdots,\lambda_{41}) = -2a\lambda_{44} < 0. \tag{4.31}$$

Step 6. Since Eq. (4.30) is not satisfied, i.e.

$$\hat{F}_j(\lambda_{11},\cdots,\lambda_{41}) \geq 0 \quad (j = 1,3),$$
$$\hat{F}_k(\lambda_{11},\cdots,\lambda_{41}) < 0 \quad (k = 2,4), \tag{4.32}$$

return to Step 3 and modify the controllers v_1 and v_3 by addition of $k_1 e_1$ and $k_3 e_3$ respectively, where k_1 and k_3 are constant gains to be determined. Because $\dot{V}$ has been modified, Eq. (4.27) becomes

$$\begin{aligned}
&\hat{F}_1(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{41} + 2k_1\lambda_{11}, \quad \hat{F}_2(\lambda_{11},\cdots,\lambda_{41}) = -2a\lambda_{22},\\
&\hat{F}_3(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{23} + 2k_3\lambda_{33}, \quad \hat{F}_4(\lambda_{11},\cdots,\lambda_{41}) = -2a\lambda_{44},\\
&\hat{G}_1(\lambda_{11},\cdots,\lambda_{41}) = (k_1 - a)\lambda_{12}, \quad \hat{G}_2(\lambda_{11},\cdots,\lambda_{41}) = b\lambda_{12} + b\lambda_{34},\\
&\hat{G}_3(\lambda_{11},\cdots,\lambda_{41}) = 2b\lambda_{44} + (k_1 - a)\lambda_{41}, \quad \hat{G}_4(\lambda_{11},\cdots,\lambda_{41}) = 2b\lambda_{22} + (k_3 - a)\lambda_{23},\\
&\hat{G}_5(\lambda_{11},\cdots,\lambda_{41}) = 0, \quad \hat{G}_6(\lambda_{11},\cdots,\lambda_{41}) = (k_3 - a)\lambda_{34}.
\end{aligned} \tag{4.33}$$

Repeat Step 4 and Step 5, then the relationship between λ_{ij} becomes

$$\lambda_{12} = 0, \quad \lambda_{22} = \frac{a-k_3}{2b}\lambda_{23}, \quad \lambda_{34} = 0, \quad \lambda_{44} = \frac{a-k_1}{2b}\lambda_{41}, \tag{4.34}$$

and Eq. (4.30) can be satisfied if

$$\lambda_{41} < \frac{-2k_1}{b}\lambda_{11}, \quad \lambda_{23} < \frac{-2k_3}{b}\lambda_{33}. \tag{4.35}$$

Step 7. The conditions derived from the above procedure can be summed up as follows:

$$\lambda_{12} = 0, \lambda_{34} = 0, \tag{4.36}$$

$$\lambda_{11} > 0, \lambda_{44} > 0, \lambda_{44}\lambda_{11} - \lambda_{41}^2 > 0, \lambda_{44} = \frac{a-k_1}{2b}\lambda_{41}, \lambda_{41} < \frac{-2k_1}{b}\lambda_{11}, \tag{4.37}$$

$$\lambda_{22} > 0, \lambda_{33} > 0, \lambda_{22}\lambda_{33} - \lambda_{23}^2 > 0, \lambda_{22} = \frac{a-k_3}{2b}\lambda_{23}, \lambda_{23} < \frac{-2k_3}{b}\lambda_{33}. \tag{4.38}$$

Design

$$\begin{aligned} & k_1 = -a, \quad k_3 = -a, \\ & \lambda_{11} = b, \lambda_{22} = \frac{a^2}{2b}, \lambda_{33} = b, \lambda_{44} = \frac{a^2}{2b}, \\ & \lambda_{12} = 0, \lambda_{23} = \frac{a}{2}, \lambda_{34} = 0, \lambda_{41} = \frac{a}{2}, \end{aligned} \tag{4.39}$$

such that each condition holds. Then the elaborate nondiagonal Lyapunov function can be obtained

$$V(\mathbf{e}) = \frac{a^2}{2b}e_2^2 + \frac{a}{2}e_2e_3 + be_3^2 + \frac{a^2}{2b}e_4^2 + \frac{a}{2}e_4e_1 + be_1^2, \tag{4.40}$$

and

$$\dot{V}(\mathbf{e}) = -\frac{3ab}{2}e_1^2 - \frac{a^3}{b}e_2^2 - \frac{3ab}{2}e_3^2 - \frac{a^3}{b}e_4^2. \tag{4.41}$$

Since Lyapunov global asymptotical stability theorem is satisfied, the global generalized synchronization is achieved. $\alpha = 1$, $\beta = 2$, $\gamma = 3$ are chosen in simulation, and the results are shown in Fig. 4.1-4.2.

4.4 Example for New Nonautonomous Chaotic Systems

The master and slave new nonautonomous chaotic systems can be described by Eq. (3.31) and Eq. (3.32), respectively. The parameters and the initial conditions of the master and slave systems are the same as shown in Section 3.4.

Let $e_i = y_i - \alpha x_i^2 - \beta x_i - \gamma \quad (i = 1, \cdots, 4)$, then the error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(t, \mathbf{e}) + \mathbf{q}(t, \mathbf{x}, \mathbf{y}) + \mathbf{u}(t, \mathbf{x}, \mathbf{y}), \tag{4.42}$$

where

$$\mathbf{p}(t,\mathbf{e}) = \begin{bmatrix} p_1(t,\mathbf{e}) & p_2(t,\mathbf{e}) & p_3(t,\mathbf{e}) & p_4(t,\mathbf{e}) \end{bmatrix}^T,$$
$$\mathbf{q}(t,\mathbf{x},\mathbf{y}) = \begin{bmatrix} q_1(t,\mathbf{x},\mathbf{y}) & q_2(t,\mathbf{x},\mathbf{y}) & q_3(t,\mathbf{x},\mathbf{y}) & q_4(t,\mathbf{x},\mathbf{y}) \end{bmatrix}^T,$$
$$\begin{aligned}
&p_1(t,\mathbf{e}) = e_2, \quad p_2(t,\mathbf{e}) = -a(1+\sin\omega t)e_1 - ae_2 + be_3, \\
&p_3(t,\mathbf{e}) = e_4, \quad p_4(t,\mathbf{e}) = -(1+\sin\omega t)e_3 - ae_4 + be_1, \\
&q_1(t,\mathbf{x},\mathbf{y}) = \alpha x_2^2 - 2\alpha x_1 x_2 + \gamma, \\
&q_2(t,\mathbf{x},\mathbf{y}) = -\alpha[a(1+\sin\omega t)x_1^2 - ax_2^2 - bx_3^2] + 2\alpha(1+\sin\omega t)x_1^3 x_2 \\
&\qquad + 2\alpha[a(1+\sin\omega t)x_1x_2 - bx_2x_3] - (1+\sin\omega t)(y_1^3 - \beta x_1^3) \\
&\qquad - \gamma[a(1+\sin\omega t) + a - b], \\
&q_3(t,\mathbf{x},\mathbf{y}) = \alpha x_4^2 - 2\alpha x_3 x_4 + \gamma, \\
&q_4(t,\mathbf{x},\mathbf{y}) = -\alpha[(1+\sin\omega t)x_3^2 - ax_4^2 - bx_1^2] + 2\alpha a(1+\sin\omega t)x_3^3 x_4 \\
&\qquad + 2\alpha[(1+\sin\omega t)x_3x_4 - bx_1x_4] - a(1+\sin\omega t)(y_3^3 - \beta x_3^3) \\
&\qquad - \gamma[1+\sin\omega t + a - b].
\end{aligned} \tag{4.43}$$

In order to transform current mixed error dynamics into pure error dynamics, the controller vector is chosen as

$$\mathbf{u}(t,\mathbf{x},\mathbf{y}) = -\mathbf{q}(t,\mathbf{x},\mathbf{y}) + \mathbf{v}(\mathbf{e}). \tag{4.44}$$

Now the pure error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(t,\mathbf{e}) + \mathbf{v}(\mathbf{e}). \tag{4.45}$$

Step 1. Construct a Lyapunov function

$$\begin{aligned}
V(t,\mathbf{e}) &= \sum_{i=1}^{4} \frac{1}{2}\mathbf{e}_i^T \mathbf{\Lambda}_i(t)\mathbf{e}_i \\
&= [\frac{1}{2}\lambda_{11}(t)e_1^2 + \lambda_{12}e_1e_2 + \frac{1}{2}\lambda_{22}(t)e_2^2] + \cdots + [\frac{1}{2}\lambda_{44}(t)e_4^2 + \lambda_{41}e_4e_1 + \frac{1}{2}\lambda_{11}(t)e_1^2],
\end{aligned} \tag{4.46}$$

where $\mathbf{e}_i = \begin{bmatrix} e_i \\ e_{i+1} \end{bmatrix}$ $(i = 1,\cdots,3)$, $\mathbf{e}_4 = \begin{bmatrix} e_4 \\ e_1 \end{bmatrix}$, $\mathbf{\Lambda}_i(t) = \begin{bmatrix} \lambda_{ii}(t) & \lambda_{ii+1} \\ \lambda_{ii+1} & \lambda_{i+1i+1}(t) \end{bmatrix}$ $(i = 1,\cdots,3)$,

$\mathbf{\Lambda}_4(t) = \begin{bmatrix} \lambda_{44}(t) & \lambda_{41} \\ \lambda_{41} & \lambda_{11}(t) \end{bmatrix}$, and $\mathbf{\Lambda}_i(t)$ $(i = 1,\cdots,4)$ are unknown continuously differentiable positive definite nondiagonal matrices to be designed. According to Sylvester's criterion, $\mathbf{\Lambda}_i(t)$ have to be chosen that

$$
\begin{aligned}
&\lambda_{11}(t) > 0,\ \lambda_{11}(t)\lambda_{22}(t) - \lambda_{12}^2 > 0, \\
&\lambda_{22}(t) > 0,\ \lambda_{22}(t)\lambda_{33}(t) - \lambda_{23}^2 > 0, \\
&\lambda_{33}(t) > 0,\ \lambda_{33}(t)\lambda_{44}(t) - \lambda_{34}^2 > 0, \\
&\lambda_{44}(t) > 0,\ \lambda_{44}(t)\lambda_{11}(t) - \lambda_{41}^2 > 0,
\end{aligned}
\tag{4.47}
$$

and

$$
\begin{aligned}
&0 < \lambda_{m11} \le \lambda_{11}(t) \le \lambda_{M11}, \\
&0 < \lambda_{m22} \le \lambda_{22}(t) \le \lambda_{M22}, \\
&0 < \lambda_{m33} \le \lambda_{33}(t) \le \lambda_{M33}, \\
&0 < \lambda_{m44} \le \lambda_{44}(t) \le \lambda_{M44},
\end{aligned}
\tag{4.48}
$$

where $\lambda_{mii}, \lambda_{Mii}$ $(i = 1, \cdots, 4)$ are positive constants.

Step 2. The derivative of Lyapunov function is

$$
\begin{aligned}
\dot{V}(t,\mathbf{e}) &= \sum_{i=1}^{4} \dot{\mathbf{e}}_i^T \mathbf{\Lambda}_i(t)\mathbf{e}_i \\
&= [\lambda_{11}(t)e_1\dot{e}_1 + \lambda_{12}\dot{e}_1 e_2 + \lambda_{12}e_1\dot{e}_2 + \lambda_{22}(t)e_2\dot{e}_2 + \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2 + \frac{1}{2}\dot{\lambda}_{22}(t)e_2^2] \\
&\quad + \cdots + [\lambda_{44}(t)e_4\dot{e}_4 + \lambda_{41}\dot{e}_4 e_1 + \lambda_{41}e_4\dot{e}_1 + \lambda_{11}(t)e_1\dot{e}_1 + \frac{1}{2}\dot{\lambda}_{44}(t)e_4^2 + \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2].
\end{aligned}
\tag{4.49}
$$

Eq. (4.49) can be rewritten in the following form:

$$
\begin{aligned}
\dot{V}(t,\mathbf{e}) &= F_1(\dot{\lambda}_{11}, \lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_1^2 + F_2(\dot{\lambda}_{22}, \lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_2^2 \\
&\quad + F_3(\dot{\lambda}_{33}, \lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_3^2 + F_4(\dot{\lambda}_{44}, \lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_4^2 \\
&\quad + G_1(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_1 e_2 + G_2(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_1 e_3 \\
&\quad + G_3(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_1 e_4 + G_4(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_2 e_3 \\
&\quad + G_5(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_2 e_4 + G_6(\lambda_{11}, \cdots, \lambda_{44}, \lambda_{12}, \cdots, \lambda_{41}, t)e_3 e_4 \\
&\quad + (2\lambda_{11}v_1 + \lambda_{12}v_2 + \lambda_{41}v_4)e_1 + (2\lambda_{22}v_2 + \lambda_{23}v_3 + \lambda_{12}v_1)e_2 \\
&\quad + (2\lambda_{33}v_3 + \lambda_{34}v_4 + \lambda_{23}v_2)e_3 + (2\lambda_{44}v_4 + \lambda_{41}v_1 + \lambda_{34}v_3)e_4,
\end{aligned}
\tag{4.50}
$$

where

$$
\begin{aligned}
&F_1(\dot{\lambda}_{11},\cdots,t)=\dot{\lambda}_{11}-a(1+\sin\omega t)\lambda_{12}+b\lambda_{41},\quad F_2(\dot{\lambda}_{22},\cdots,t)=\dot{\lambda}_{22}-2a\lambda_{22}+\lambda_{12},\\
&F_3(\dot{\lambda}_{33},\cdots,t)=\dot{\lambda}_{33}+b\lambda_{23}-(1+\sin\omega t)\lambda_{34},\quad F_4(\dot{\lambda}_{44},\cdots,t)=\dot{\lambda}_{44}-2a\lambda_{44}+\lambda_{34},\\
&G_1(\lambda_{11},\cdots,t)=2\lambda_{11}-a\lambda_{12}-2a(1+\sin\omega t)\lambda_{22},\\
&G_2(\lambda_{11},\cdots,t)=b\lambda_{12}-a(1+\sin\omega t)\lambda_{23}+b\lambda_{34}-(1+\sin\omega t)\lambda_{41},\\
&G_3(\lambda_{11},\cdots,t)=2b\lambda_{44}-a\lambda_{41},\quad G_4(\lambda_{11},\cdots,t)=2b\lambda_{22}-a\lambda_{23},\\
&G_5(\lambda_{11},\cdots,t)=\lambda_{23}+\lambda_{41},\quad G_6(\lambda_{11},\cdots,t)=2\lambda_{33}-a\lambda_{34}-2(1+\sin\omega t)\lambda_{44}.
\end{aligned}
\tag{4.51}
$$

Step 3. Design the controllers

$$
v_1=ae_1,\quad v_2=-be_3-ae_1,\quad v_3=ae_3,\quad v_4=-be_1-e_3,
\tag{4.52}
$$

such that Eq. (4.50) can be reduced to

$$
\begin{aligned}
\dot{V}(t,\mathbf{e})=&\hat{F}_1(\dot{\lambda}_{11},\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_1^2+\hat{F}_2(\dot{\lambda}_{22},\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_2^2\\
&+\hat{F}_3(\dot{\lambda}_{33},\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_3^2+\hat{F}_4(\dot{\lambda}_{44},\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_4^2\\
&+\hat{G}_1(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_1e_2+\hat{G}_2(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_1e_3\\
&+\hat{G}_3(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_1e_4+\hat{G}_4(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_2e_3\\
&+\hat{G}_5(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_2e_4+\hat{G}_6(\lambda_{11},\cdots,\lambda_{44},\lambda_{12},\cdots,\lambda_{41},t)e_3e_4,
\end{aligned}
\tag{4.53}
$$

where

$$
\begin{aligned}
&\hat{F}_1(\dot{\lambda}_{11},\cdots,t)=\dot{\lambda}_{11}+2a\lambda_{11}-a(2+\sin\omega t)\lambda_{12},\quad \hat{F}_2(\dot{\lambda}_{22},\cdots,t)=\dot{\lambda}_{22}-2a\lambda_{22}+\lambda_{12},\\
&\hat{F}_3(\dot{\lambda}_{33},\cdots,t)=\dot{\lambda}_{33}+2a\lambda_{33}-(2+\sin\omega t)\lambda_{34},\quad \hat{F}_4(\dot{\lambda}_{44},\cdots,t)=\dot{\lambda}_{44}-2a\lambda_{44}+\lambda_{34},\\
&\hat{G}_1(\lambda_{11},\cdots,t)=2\lambda_{11}-2a(1+\sin\omega t)\lambda_{22},\\
&\hat{G}_2(\lambda_{11},\cdots,t)=-a(2+\sin\omega t)\lambda_{23}-(2+\sin\omega t)\lambda_{41},\\
&\hat{G}_3(\lambda_{11},\cdots,t)=0,\quad \hat{G}_4(\lambda_{11},\cdots,t)=0,\\
&\hat{G}_5(\lambda_{11},\cdots,t)=\lambda_{23}+\lambda_{41},\quad \hat{G}_6(\lambda_{11},\cdots,t)=2\lambda_{33}-2(2+\sin\omega t)\lambda_{44}.
\end{aligned}
\tag{4.54}
$$

Step 4. Assume

$$
\forall j,\quad \hat{G}_j(\lambda_{11},\cdots,t)=0,
\tag{4.55}
$$

then the relationship between λ_{ij} can be obtained:

$$
\lambda_{11}=a(2+\sin\omega t)\lambda_{22},\quad \lambda_{23}=0,\quad \lambda_{33}=(2+\sin\omega t)\lambda_{44},\quad \lambda_{41}=0.
\tag{4.56}
$$

Step 5. Use the results of Step 4 to check if

$$\forall t \ge 0, \quad \hat{F}_i(\dot{\lambda}_{ii}, \cdots, t) < 0 \quad (i = 1, \cdots, 4). \tag{4.57}$$

Assume

$$\lambda_{11} = c_1, \quad \lambda_{22} = \frac{c_1}{a(2+\sin\omega t)}, \quad \lambda_{33} = c_2, \quad \lambda_{44} = \frac{c_2}{2+\sin\omega t}, \tag{4.58}$$

where c_1 and c_2 are positive constants to be designed. Eq. (4.57) can be satisfied if the following conditions hold:

$$\begin{aligned} &\hat{F}_1(\dot{\lambda}_{11}, \cdots, t) = 2ac_1 - a(2+\sin\omega t)\lambda_{12} < 0, \\ &\hat{F}_2(\dot{\lambda}_{22}, \cdots, t) = \frac{-2c_1}{2+\sin\omega t} - \frac{c_1\omega\cos\omega t}{a(2+\sin\omega t)^2} + \lambda_{12} < 0, \end{aligned} \tag{4.59}$$

$$\begin{aligned} &\hat{F}_3(\dot{\lambda}_{33}, \cdots, t) = 2ac_2 - (2+\sin\omega t)\lambda_{34} < 0, \\ &\hat{F}_4(\dot{\lambda}_{44}, \cdots, t) = \frac{-2ac_2}{2+\sin\omega t} - \frac{c_2\omega\cos\omega t}{(2+\sin\omega t)^2} + \lambda_{34} < 0. \end{aligned} \tag{4.60}$$

However, both results of Eq. (4.59) and Eq. (4.60) show the contradiction: $\hat{F}_1 < 0$ and $\hat{F}_2 < 0$ can not hold in the same time, neither can $\hat{F}_3 < 0$ and $\hat{F}_4 < 0$. To simplify the following work, assume only $\hat{F}_2 < 0$ and $\hat{F}_4 < 0$ can hold.

Step 6. Since Eq. (4.57) is not satisfied, i.e.

$$\begin{aligned} \forall t \ge 0, \quad &\hat{F}_j(\lambda_{11}, \cdots, \lambda_{41}) \ge 0 \quad (j = 1, 3), \\ &\hat{F}_k(\lambda_{11}, \cdots, \lambda_{41}) < 0 \quad (k = 2, 4), \end{aligned} \tag{4.61}$$

return to Step 3 and modify the controllers v_1 and v_3 by addition of $k_1 e_1$ and $k_3 e_3$ respectively, where k_1 and k_3 are constant gains to be determined. Because $\dot{V}$ has been modified, Eq. (4.54) becomes

$$
\begin{aligned}
&\hat{F}_1(\dot{\lambda}_{11},\cdots,t)=\dot{\lambda}_{11}+2(a+k_1)\lambda_{11}-a(2+\sin\omega t)\lambda_{12},\\
&\hat{F}_2(\dot{\lambda}_{22},\cdots,t)=\dot{\lambda}_{22}-2a\lambda_{22}+\lambda_{12},\\
&\hat{F}_3(\dot{\lambda}_{33},\cdots,t)=\dot{\lambda}_{33}+2(a+k_3)\lambda_{33}-(2+\sin\omega t)\lambda_{34},\\
&\hat{F}_4(\dot{\lambda}_{44},\cdots,t)=\dot{\lambda}_{44}-2a\lambda_{44}+\lambda_{34},\\
&\hat{G}_1(\lambda_{11},\cdots,t)=2\lambda_{11}+k_1\lambda_{12}-2a(1+\sin\omega t)\lambda_{22},\\
&\hat{G}_2(\lambda_{11},\cdots,t)=-a(2+\sin\omega t)\lambda_{23}-(2+\sin\omega t)\lambda_{41},\\
&\hat{G}_3(\lambda_{11},\cdots,t)=k_1\lambda_{41},\quad \hat{G}_4(\lambda_{11},\cdots,t)=k_3\lambda_{23},\\
&\hat{G}_5(\lambda_{11},\cdots,t)=\lambda_{23}+\lambda_{41},\quad \hat{G}_6(\lambda_{11},\cdots,t)=2\lambda_{33}+k_3\lambda_{34}-2(2+\sin\omega t)\lambda_{44}.
\end{aligned}
\tag{4.62}
$$

Repeat Step 4 and Step 5, then the relationship between λ_{ij} becomes

$$
\lambda_{11}=a(2+\sin\omega t)\lambda_{22}-\frac{k_1}{2}\lambda_{12},\quad \lambda_{23}=0,\quad \lambda_{33}=(2+\sin\omega t)\lambda_{44}-\frac{k_3}{2}\lambda_{34},\quad \lambda_{41}=0\,. \tag{4.63}
$$

Assume

$$
\begin{aligned}
&\lambda_{11}=c_1-\frac{k_1}{2}c_3,\quad \lambda_{22}=\frac{c_1}{a(2+\sin\omega t)},\quad \lambda_{33}=c_2-\frac{k_3}{2}c_4,\quad \lambda_{44}=\frac{c_2}{2+\sin\omega t},\\
&\lambda_{12}=c_3,\quad \lambda_{34}=c_4,
\end{aligned}
\tag{4.64}
$$

where c_1, c_2, c_3, c_4 are constants to be designed, and c_1, c_2 are positive numbers. Eq. (4.57) can be satisfied if

$$
\begin{aligned}
&2(a+k_1)c_1<(k_1^2+ak_1+2a+a\sin\omega t)c_3,\\
&(4a+2a\sin\omega t+\omega\cos\omega t)c_1>a(2+\sin\omega t)^2c_3,\\
&2(a+k_3)c_2<(k_3^2+ak_3+2+\sin\omega t)c_4,\\
&(4a+2a\sin\omega t+\omega\cos\omega t)c_2>(2+\sin\omega t)^2c_4.
\end{aligned}
\tag{4.65}
$$

Step 7. The conditions derived from the above procedure can be summed up as follows:

$$
\lambda_{23}=0,\quad \lambda_{41}=0, \tag{4.66}
$$

$$
\begin{aligned}
&c_1 > 0, \quad c_1 > \frac{k_1}{2} c_3, \\
&2(a + k_1)c_1 < (k_1^2 + ak_1 + 2a + a\sin\omega t)c_3, \\
&(4a + 2a\sin\omega t + \omega\cos\omega t)c_1 > a(2 + \sin\omega t)^2 c_3, \\
&(2c_1 - k_1c_3)c_1 > c_3^2(4a + 2a\sin\omega t),
\end{aligned} \tag{4.67}
$$

$$
\begin{aligned}
&c_2 > 0, \quad c_2 > \frac{k_3}{2} c_4, \\
&2(a + k_3)c_2 < (k_3^2 + ak_3 + 2 + \sin\omega t)c_4, \\
&(4a + 2a\sin\omega t + \omega\cos\omega t)c_2 > (2 + \sin\omega t)^2 c_4, \\
&(2c_2 - k_3c_4)c_2 > c_4^2(4 + 2\sin\omega t).
\end{aligned} \tag{4.68}
$$

Design

$$
\begin{aligned}
&k_1 = -0.4,\ k_3 = -0.4,\ \lambda_{23} = 0,\ \lambda_{41} = 0, \\
&c_1 = 20, c_3 = 9 \quad \Rightarrow \lambda_{12} = 9, \lambda_{11} = 21.8, \lambda_{22} = \frac{20}{a(2 + \sin\omega t)}, \\
&c_2 = 50, c_4 = 11 \ \Rightarrow \lambda_{34} = 11, \lambda_{33} = 52.2, \lambda_{44} = \frac{50}{2 + \sin\omega t},
\end{aligned} \tag{4.69}
$$

such that each condition can be satisfied. Then the elaborate nondiagonal Lyapunov function can be obtained

$$
V(t, \mathbf{e}) = 21.8e_1^2 + 9e_1e_2 + \frac{20}{a(2 + \sin\omega t)} e_2^2 + 52.2e_3^2 + 11e_3e_4 + \frac{50}{2 + \sin\omega t} e_4^2, \tag{4.70}
$$

and

$$
\begin{aligned}
\dot{V}(t, \mathbf{e}) = &-(4.64 + 4.5\sin t)e_1^2 - \frac{44 + 4\sin t + 40\cos t - 9\sin^2 t}{(2 + \sin t)^2} e_2^2 \\
&-(11.56 + 11\sin t)e_3^2 - \frac{56 + 6\sin t + 50\cos t - 11\sin^2 t}{(2 + \sin t)^2} e_4^2.
\end{aligned} \tag{4.71}
$$

Since Lyapunov global asymptotical stability theorem is satisfied, the global generalized synchronization is achieved. $\alpha = 1$, $\beta = 2$, $\gamma = 3$ are chosen in simulation, and the results are shown in Fig. 4.3-4.4.

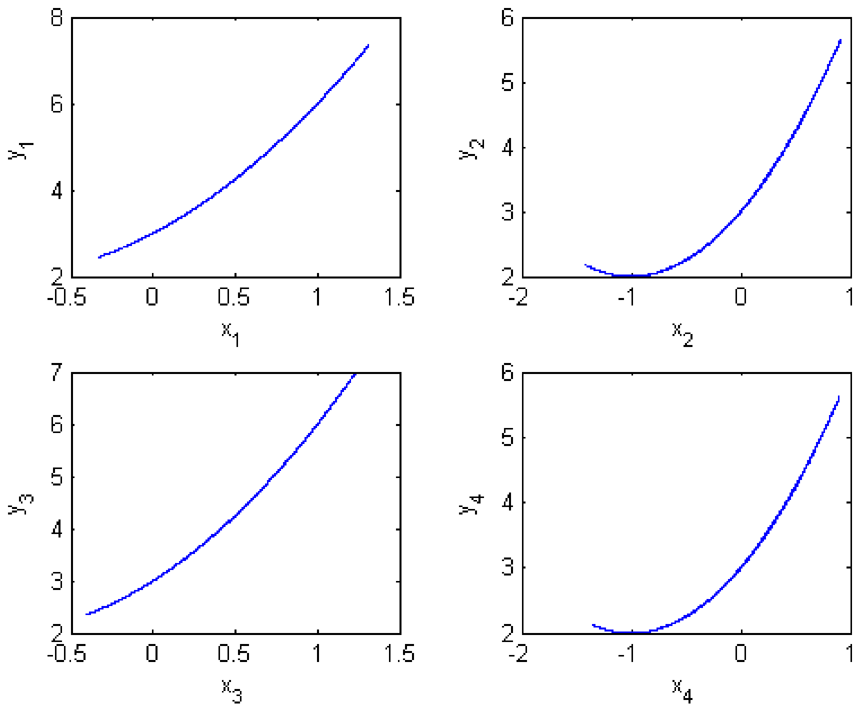

Fig. 4.1 Phase portraits of x_i to y_i $(i = 1, \cdots, 4)$ for Section 4.3 when the nonlinear generalized synchronization is obtained.

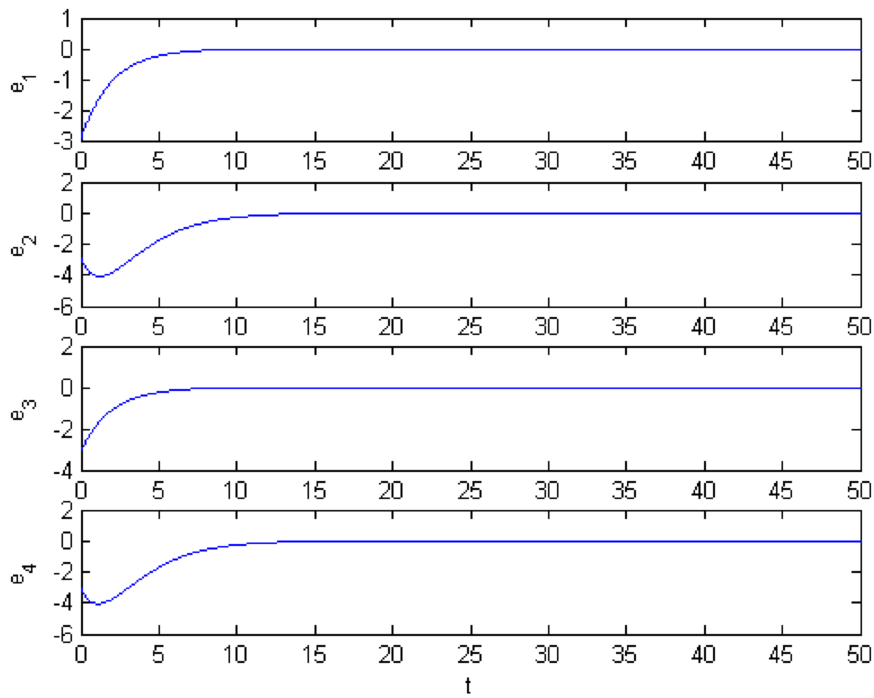

Fig. 4.2 Time histories of the state errors for Section 4.3.

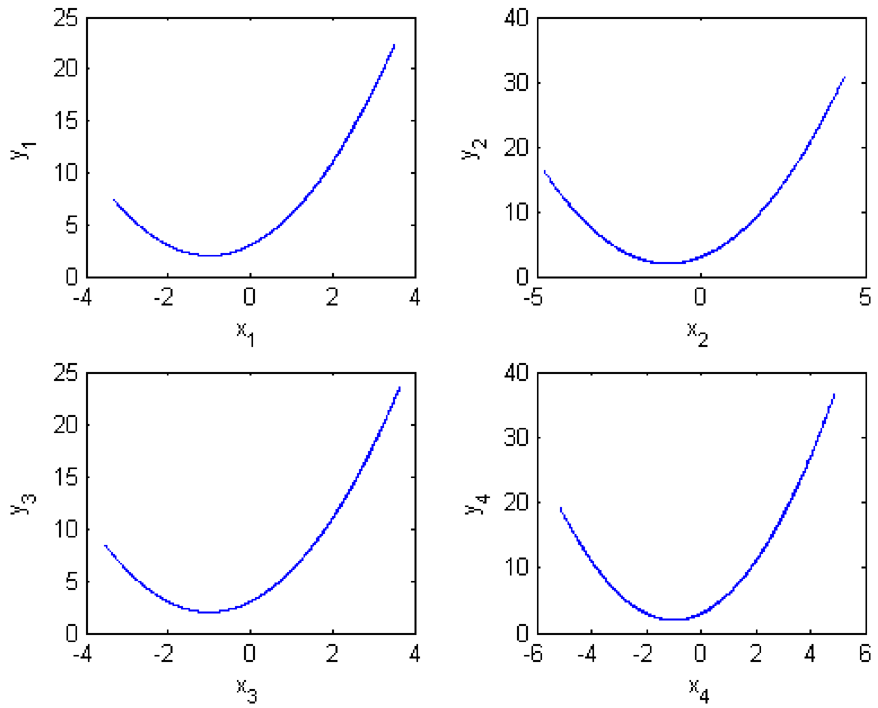

Fig. 4.3 Phase portraits of x_i to y_i $(i=1,\cdots,4)$ for Section 4.4 when the nonlinear generalized synchronization is obtained.

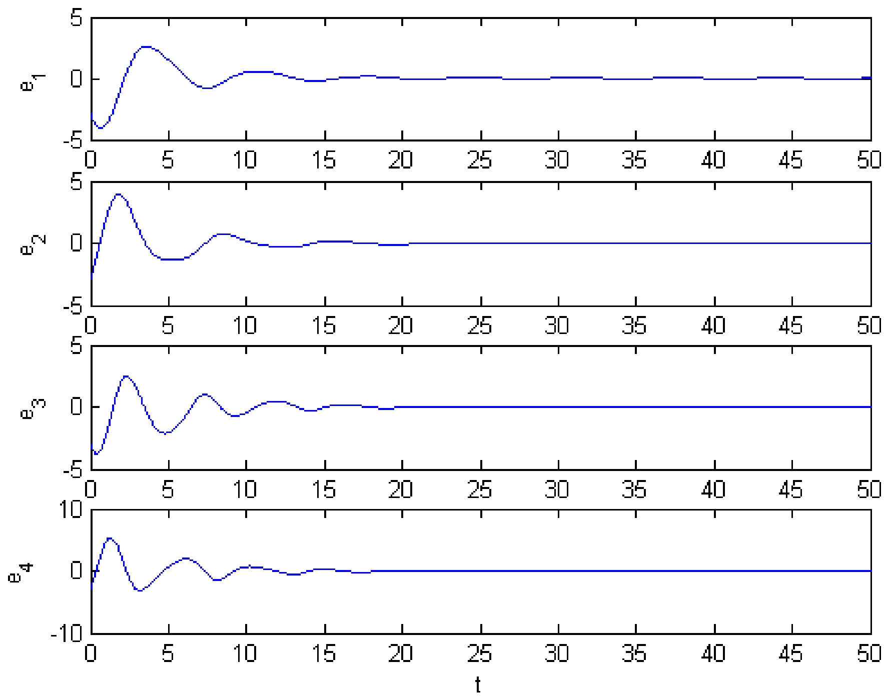

Fig. 4.4 Time histories of the state errors for Section 4.4.

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Chapter 5
Complete Identification of Chaos of Nonholonomic Systems

5.1 Preliminaries

The study of nonholonomic system [1, 2] has been developed over one hundred years since Hertz [3] distinguished nonholonomic system from holonomic system in 1894. There are a great number of studies in this field connected with the extension of the developed analytical methods for holonomic system and for the systems with nonholonomic constraints. Many applications of the dynamics of nonholonomic system can be found in the problems of modern technology, such as the pursuit problems, the motion of automobiles, landing devices of airplanes, railway wheels, etc. However, it is still deficient for the complete study of chaos in nonholonomic systems. As far as we know, the only paper which studies the chaos of nonholonomic system with an external linear nonholonomic constraint is Ref. [4], the chaotic phenomena of rattleback dynamics. But only Poincaré maps are given in this paper. It is well-known that only Poincaré map cannot identify the existence of chaos reliably.

Moving target pursuit problem [5] is a typical example of nonholonomic system. By applying the fundamental nonholonomic form of Lagrange's equations [6, 7], chaos of nonholonomic systems with external nonholonomic constraint for two types of pursuit problems, a straightly oscillating target, and a circularly rotating target, is studied in this Chapter. The existence of chaos of nonholonomic system is firstly completely identified by all numerical criteria of chaos, i.e. the most reliable Lyapunov exponents [8], phase portraits, Poincaré maps and bifurcation diagrams.

5.2 Straightly Oscillating Target Pursuit Problem

From Newton's law, the dynamic equations of a free particle with unit mass moving in a horizontal smooth (x_1, y_1) plane are

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1, \\
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1,
\end{aligned} \tag{5.1}$$

where $-a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1$ and $-(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1$ are x_1 and y_1 components of the forces applied on the particle, respectively, and a, b, ω are constants. Eq. (5.1) consists of two linearly coupled nonlinear damped Mathieu systems.

Now an external pursuit constraint

$$(f_2 + f\sin t - y_1)\dot{x}_1 - (f_1 - x_1)\dot{y}_1 = 0 \tag{5.2}$$

is added, the particle is no more free but a constrained particle. f_1, f_2, f are constants and $f_1 > x_1$, $f_2 + f\sin t > y_1$, $f \neq 0$. The constraint makes that the direction of velocity of the particle always points at the target oscillating in y_1 direction as shown in Fig. 5.1. The constraint can be expressed as a Pfaffian form:

$$A(x_1, y_1, t)dx_1 + B(x_1, y_1, t)dy_1 + C(x_1, y_1, t)dt = 0, \tag{5.3}$$

where $A = f_2 + f\sin t - y_1$, $B = -(f_1 - x_1)$, $C = 0$. It can be derived that

$$A(\frac{\partial B}{\partial t} - \frac{\partial C}{\partial y_1}) + B(\frac{\partial C}{\partial x_1} - \frac{\partial A}{\partial t}) + C(\frac{\partial A}{\partial y_1} - \frac{\partial B}{\partial x_1}) = f(f_1 - x_1)\cos t. \tag{5.4}$$

Since $A(\frac{\partial B}{\partial t} - \frac{\partial C}{\partial y_1}) + B(\frac{\partial C}{\partial x_1} - \frac{\partial A}{\partial t}) + C(\frac{\partial A}{\partial y_1} - \frac{\partial B}{\partial x_1})$ does not equal to zero identically [9], it is a nonholonomic constraint. Then the particle becomes a nonholonomic pursuit system, of which the dynamic equations can be obtained as follows.

According to d'Alembert principle [6]

$$\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_i}\right)-\frac{\partial T}{\partial q_i}-Q_i\right]\delta q_i=0 \tag{5.5}$$

with nonholonomic constraint equations

$$\sum_{i=1}^{n} a_{ji}\delta q_i = 0 \quad (j=1,\ldots,m), \tag{5.6}$$

we can use Lagrange multiplier method to obtain the fundamental nonholonomic form of Lagrange's equations [7]:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i}\right)-\frac{\partial T}{\partial q_i}=Q_i+\sum_{j=1}^{m}\lambda_j a_{ji} \quad (i=1,\ldots,n), \tag{5.7}$$

where q_i is the generalized coordinate, T is the kinetic energy of the system, Q_i is the generalized applied force, λ_j is the Lagrange multiplier, and a_{ji} is the coefficient of the virtual constraint function in Eq. (5.6). Together with m nonholonomic constraint equations, there are $(m+n)$ equations solving for n generalized coordinates and m Lagrange multipliers.

In our case, choose $(q_1, q_2) = (x_1, y_1)$. The kinetic energy is $T=\frac{1}{2}(\dot{x}_1^2+\dot{y}_1^2)$, the generalized applied forces are $Q_1=-a(1+\sin\omega t)x_1-(1+\sin\omega t)x_1^3-a\dot{x}_1+by_1$, $Q_2=-(1+\sin\omega t)y_1-a(1+\sin\omega t)y_1^3-a\dot{y}_1+bx_1$, and the coefficients of the virtual constraint equation are $a_{11}=f_2+f\sin t-y_1$, $a_{12}=-(f_1-x_1)$. Then the fundamental nonholonomic form of Lagrange's equations is obtained:

$$\ddot{x}_1=-a(1+\sin\omega t)x_1-(1+\sin\omega t)x_1^3-a\dot{x}_1+by_1+\lambda(f_2+f\sin t-y_1), \tag{5.8}$$

$$\ddot{y}_1=-(1+\sin\omega t)y_1-a(1+\sin\omega t)y_1^3-a\dot{y}_1+bx_1-\lambda(f_1-x_1). \tag{5.9}$$

Together with nonholonomic constraint equation

$$(f_2 + f\sin t - y_1)\dot{x}_1 - (f_1 - x_1)\dot{y}_1 = 0, \tag{5.10}$$

there are three equations solving two generalized coordinates and one Lagrange multiplier.

In order to solve λ, differentiate the nonholonomic constraint equation Eq. (5.10) with respect to time and get

$$(f_2 + f\sin t - y_1)\ddot{x}_1 - (f_1 - x_1)\ddot{y}_1 + (f\cos t)\dot{x}_1 = 0. \tag{5.11}$$

Substituting Eq. (5.8) and Eq. (5.9) into Eq. (5.11), λ can be solved:

$$\begin{aligned}\lambda = \{&(f_2 + f\sin t - y_1)[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1] \\ &-(f_1 - x_1)[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1] - (f\cos t)\dot{x}_1\} \\ &/[(f_1 - x_1)^2 + (f_2 + f\sin t - y_1)^2].\end{aligned} \tag{5.12}$$

Finally, the differential equations of nonholonomic system can be expressed as

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1 + \frac{h_1(x_1, x_2, y_1, y_2, t)}{(f_1 - x_1)^2 + (f_2 + f\sin t - y_1)^2}, \\ \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1 + \frac{h_2(x_1, x_2, y_1, y_2, t)}{(f_1 - x_1)^2 + (f_2 + f\sin t - y_1)^2},\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}h_1 = &(f_2 + f\sin t - y_1)^2[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1] \\ &-(f_1 - x_1)(f_2 + f\sin t - y_1)[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1] \\ &-(f\cos t)(f_2 + f\sin t - y_1)x_2, \\ h_2 = &-(f_1 - x_1)(f_2 + f\sin t - y_1)[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1] \\ &+(f_1 - x_1)^2[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1] \\ &+(f\cos t)(f_1 - x_1)x_2.\end{aligned} \tag{5.14}$$

The term $\dfrac{h_1(x_1, x_2, y_1, y_2, t)}{(f_1 - x_1)^2 + (f_2 + f\sin t - y_1)^2}$ is the x_1 component of the nonholonomic constraint force, and the term $\dfrac{h_2(x_1, x_2, y_1, y_2, t)}{(f_1 - x_1)^2 + (f_2 + f\sin t - y_1)^2}$ is the y_1 component of the nonholonomic constraint force. The parameters in simulation are $a = 0.5$,

$b = 0.3 \sim 2$, $\omega = 1$, $f_1 = 100$, $f_2 = 100$, $f = 5$, and the initial condition is $x_1(0) = 5$, $x_2(0) = 0.1$, $y_1(0) = 5$, $y_2(0) = 0.1$. The phase portraits, Poincaré maps, bifurcation diagram and Lyapunov exponents, for nonholonomic system are shown in Fig. 5.2-5.4. It can be observed that the motion is period 1 for $b = 0.4$ and $b = 0.7$. For $b = 1.0$ and $b = 1.8$, the motion is chaotic. All numerical criteria of chaos prove that the chaotic phenomena exist in nonholonomic system for pursuit problem with a straightly oscillating target.

5.3 Circularly Rotating Target Pursuit Problem

From Newton's law, the dynamic equations of a free particle with unit mass moving in a horizontal smooth (x_1, y_1) plane are

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1, \\
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1,
\end{aligned} \tag{5.15}$$

where $-a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1$ and $-(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1$ are x_1 and y_1 components of the forces applied on the particle, respectively, and a, b, ω are constants. Eq. (5.15) consists of two linearly coupled nonlinear Mathieu systems.

Now an external pursuit constraint

$$(f_2 + r\sin t - y_1)\dot{x}_1 - (f_1 + r\cos t - x_1)\dot{y}_1 = 0 \tag{5.16}$$

is added, the particle is no more free but a constrained particle. f_1, f_2, r are constants and $f_1 + r\cos t > x_1$, $f_2 + r\sin t > y_1$, $r \neq 0$. The constraint makes that the direction of the velocity of particle always points at the target rotating in (x_1, y_1) plane as shown in Fig. 5.5. The constraint can be expressed as a Pfaffian form:

$$A(x_1, y_1, t)dx_1 + B(x_1, y_1, t)dy_1 + C(x_1, y_1, t)dt = 0, \tag{5.17}$$

where $A = f_2 + r\sin t - y_1$, $B = -(f_1 + r\cos t - x_1)$, $C = 0$. It can be derived that

$$\begin{aligned} &A(\frac{\partial B}{\partial t} - \frac{\partial C}{\partial y_1}) + B(\frac{\partial C}{\partial x_1} - \frac{\partial A}{\partial t}) + C(\frac{\partial A}{\partial y_1} - \frac{\partial B}{\partial x_1}) \\ &= r[(f_2 + r\sin t - y_1)\sin t - (f_1 + r\cos t - x_1)\cos t]. \end{aligned} \tag{5.18}$$

Since $A(\frac{\partial B}{\partial t} - \frac{\partial C}{\partial y_1}) + B(\frac{\partial C}{\partial x_1} - \frac{\partial A}{\partial t}) + C(\frac{\partial A}{\partial y_1} - \frac{\partial B}{\partial x_1})$ does not equal zero identically [9], it is a nonholonomic constraint. Then the particle becomes a nonholonomic pursuit system, of which the dynamic equations can be obtained as follows.

According to d'Alembert principle [6] and nonholonomic constraint equations, we use Lagrange multiplier method to obtain the fundamental nonholonomic form of Lagrange's equations [7].

In our case, choose $(q_1, q_2) = (x_1, y_1)$. The kinetic energy is $T = \frac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2)$, the generalized applied forces are $Q_1 = -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - a\dot{x}_1 + by_1$, $Q_2 = -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - a\dot{y}_1 + bx_1$, and the coefficients of the virtual constraint equation are $a_{11} = f_2 + r\sin t - y_1$, $a_{12} = -(f_1 + r\cos t - x_1)$. Then the fundamental nonholonomic form of Lagrange's equations is obtained:

$$\ddot{x}_1 = -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - a\dot{x}_1 + by_1 + \lambda(f_2 + r\sin t - y_1), \tag{5.19}$$

$$\ddot{y}_1 = -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - a\dot{y}_1 + bx_1 - \lambda(f_1 + r\cos t - x_1). \tag{5.20}$$

Together with nonholonomic constraint equation

$$(f_2 + r\sin t - y_1)\dot{x}_1 - (f_1 + r\cos t - x_1)\dot{y}_1 = 0, \tag{5.21}$$

there are three equations solving two generalized coordinates and one Lagrange multiplier.

In order to solve λ, differentiate the nonholonomic constraint equation Eq.

(5.21) with respect to time and get

$$(f_2 + r\sin t - y_1)\ddot{x}_1 - (f_1 + r\cos t - x_1)\ddot{y}_1 + (r\cos t)\dot{x}_1 + (r\sin t)\dot{x}_3 = 0. \tag{5.22}$$

Substitute Eq. (5.19) and Eq. (5.20) into Eq. (5.22), and λ can be solved:

$$\begin{aligned}\lambda = \{&(f_2 + r\sin t - y_1)[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1]\\ &-(f_1 + r\cos t - x_1)[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1]\\ &-(r\cos t)\dot{x}_1 - (r\sin t)\dot{y}_1\}\\ &/[(f_1 + r\cos t - x_1)^2 + (f_2 + r\sin t - y_1)^2].\end{aligned} \tag{5.23}$$

Finally, the differential equations of nonholonomic system can be expressed as

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1\\ &\quad + \frac{h_1(x_1,x_2,y_1,y_2,t)}{(f_1 + r\cos t - x_1)^2 + (f_2 + r\sin t - y_1)^2},\\ \dot{y}_1 &= y_2,\\ \dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1\\ &\quad + \frac{h_2(x_1,x_2,y_1,y_2,t)}{(f_1 + r\cos t - x_1)^2 + (f_2 + r\sin t - y_1)^2},\end{aligned} \tag{5.24}$$

where

$$\begin{aligned}h_1 &= (f_2 + r\sin t - y_1)^2[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1]\\ &\quad -(f_1 + r\cos t - x_1)(f_2 + r\sin t - y_1)[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1]\\ &\quad - r(f_2 + r\sin t - y_1)(x_2\cos t + y_2\sin t),\\ h_2 &= -(f_1 + r\cos t - x_1)(f_2 + r\sin t - y_1)[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1]\\ &\quad +(f_1 + r\cos t - x_1)^2[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1]\\ &\quad + r(f_1 + r\cos t - x_1)(x_2\cos t + y_2\sin t).\end{aligned} \tag{5.25}$$

The term $\dfrac{h_1(x_1,x_2,y_1,y_2,t)}{(f_1 + r\cos t - x_1)^2 + (f_2 + r\sin t - y_1)^2}$ is the x_1 component of the nonholonomic constraint force, and the term $\dfrac{h_2(x_1,x_2,y_1,y_2,t)}{(f_1 + r\cos t - x_1)^2 + (f_2 + r\sin t - y_1)^2}$ is the y_1 component of the nonholonomic constraint force. The parameters in simulation are $a = 0.5$, $b = 0.1 \sim 0.85$, $\omega = 1$, $f_1 = 100$, $f_2 = 100$, $r = 5$, and the initial condition is $x_1(0)=5$, $x_2(0)=0.1$, $x_3(0)=5$, $x_4(0)=0.095$. The phase portraits, Poincaré maps, bifurcation diagram, and Lyapunov exponents for nonholonomic system are shown in Fig. 5.6-5.8. It can be observed that the motion is period 1 for $b=0.2$, and period 2 for $b=0.58$. For $b=0.78$ and $b=0.81$, the motion is chaotic. All numerical criteria of chaos prove that the chaotic phenomena exist in nonholonomic system for pursuit problem with a circularly rotating target.

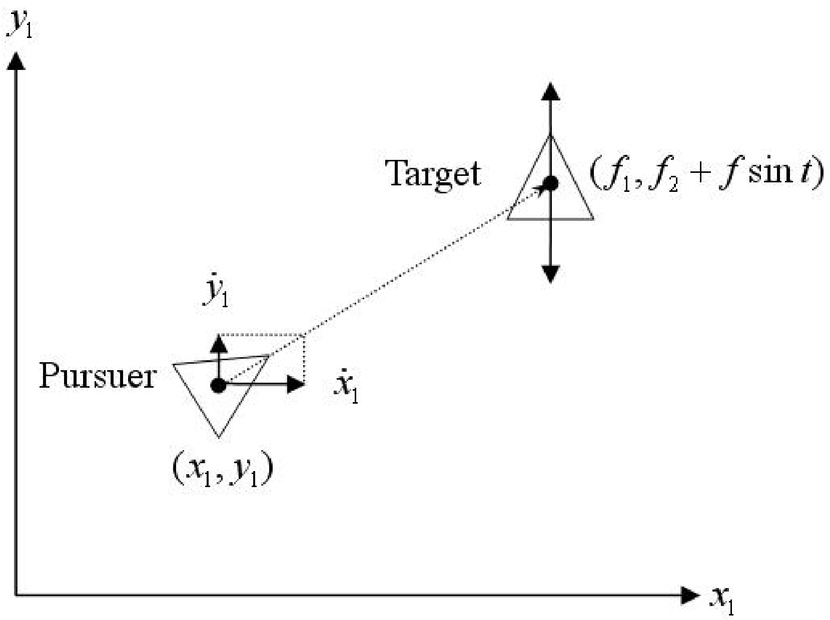

Fig. 5.1 A sketch of a pursuit problem of a straightly oscillating target in (x_1, y_1) plane.

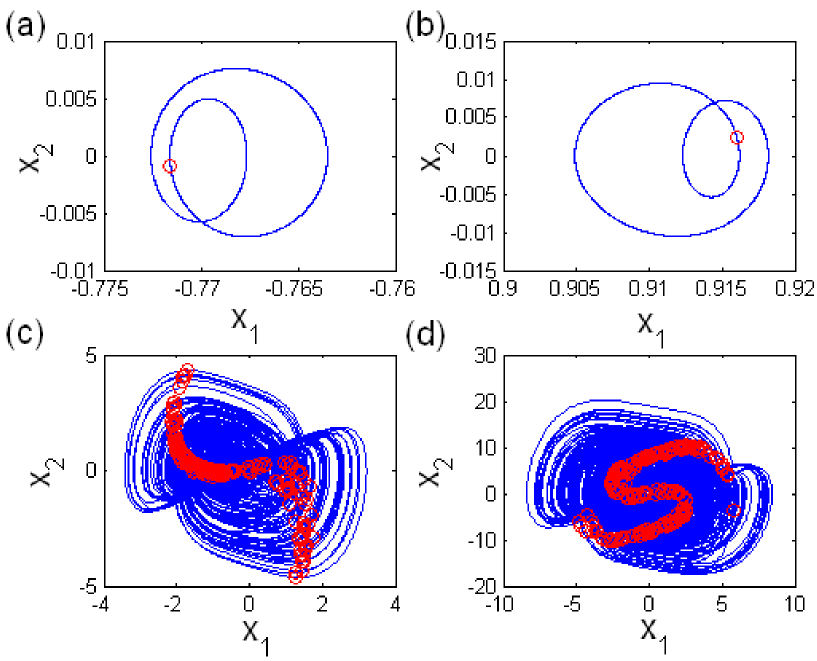

Fig. 5.2 Phase portraits and Poincaré maps for straightly oscillating target pursuit problem: (a) period 1 for $b = 0.4$, (b) period 1 for $b = 0.7$, (c) chaotic for $b = 1.0$, (d) chaotic for $b = 1.8$.

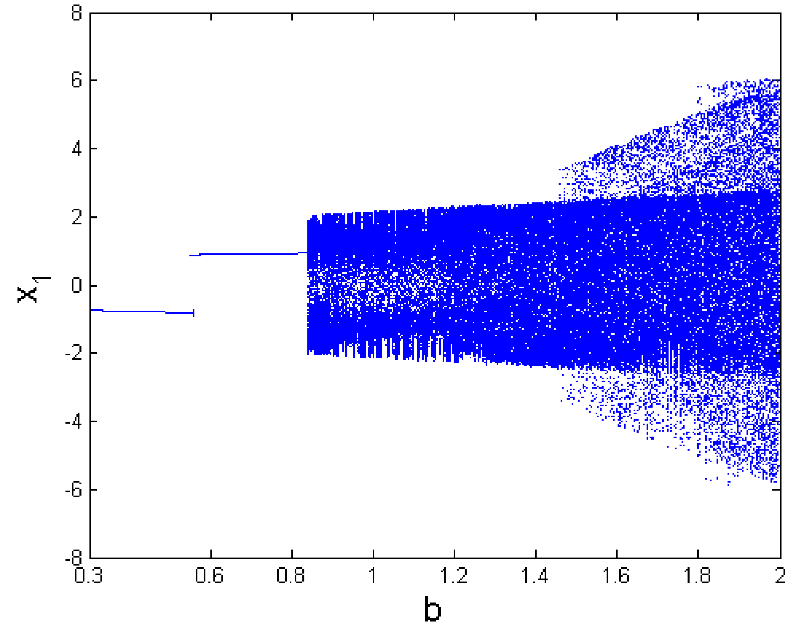

Fig. 5.3 Bifurcation diagram for straightly oscillating target pursuit problem.

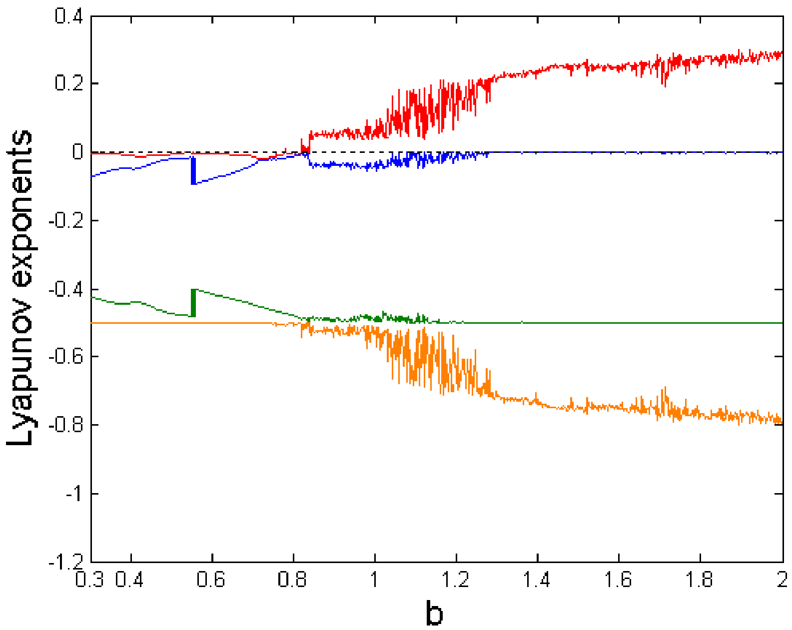

Fig. 5.4 Lyapunov exponents for straightly oscillating target pursuit problem.

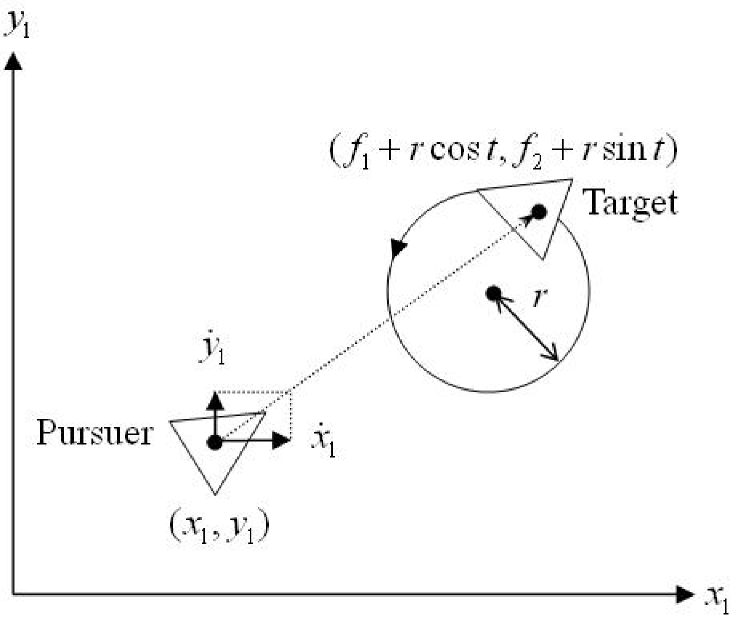

Fig. 5.5 A sketch of a pursuit problem of a circularly rotating target in (x_1, y_1) plane.

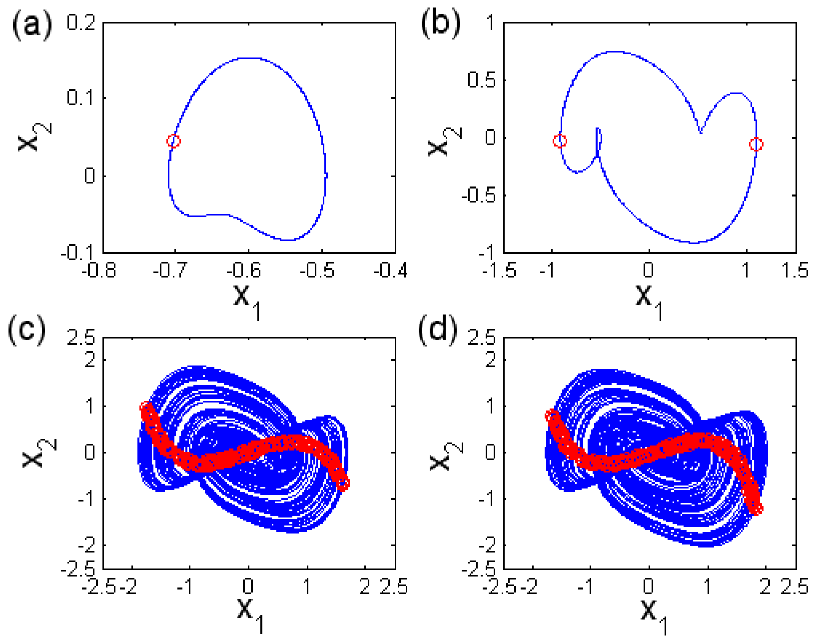

Fig. 5.6 Phase portraits and Poincaré maps for circularly rotating target pursuit problem: (a) period 1 for $b = 0.2$, (b) period 2 for $b = 0.58$, (c) chaotic for $b = 0.78$, (d) chaotic for $b = 0.81$.

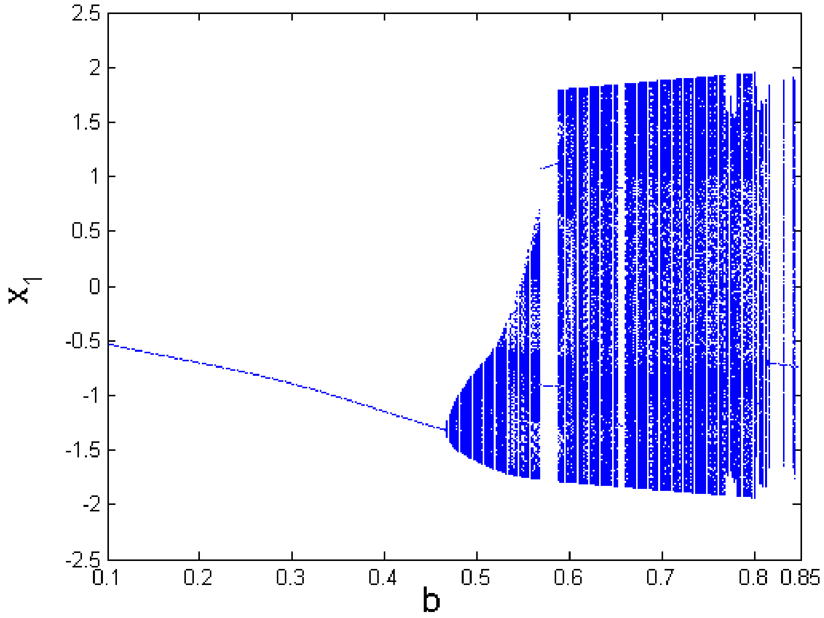

Fig. 5.7 Bifurcation diagram for circularly rotating target pursuit problem.

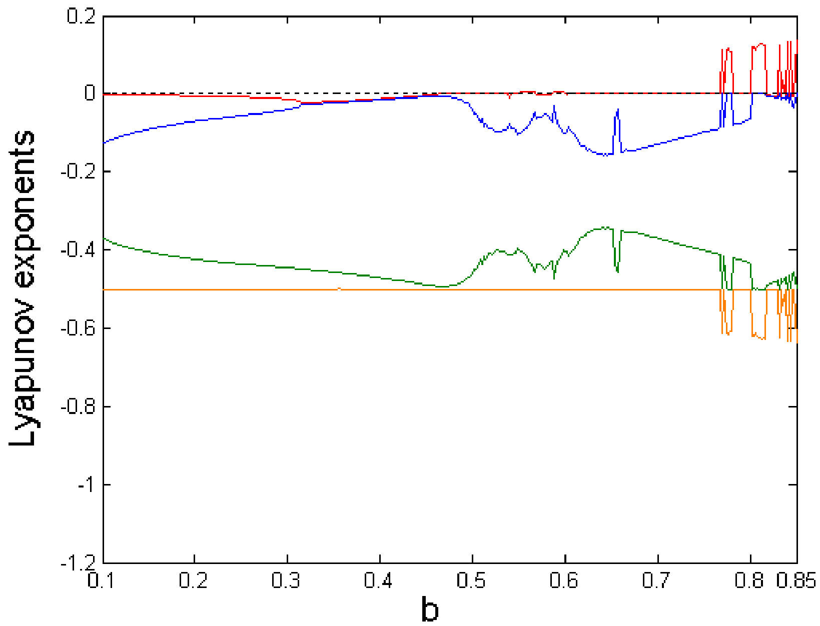

Fig. 5.8 Lyapunov exponents for circularly rotating target pursuit problem.

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Chapter 6

Complete Identification of Chaos of Nonlinear Nonholonomic Systems

6.1 Preliminaries

By applying the nonlinear nonholonomic form of Lagrange's equations, chaos of nonholonomic systems with external nonlinear nonholonomic constraint, the magnitude of velocity keeping constant, is studied in this Chapter. The existence of chaos of nonlinear nonholonomic system is firstly completely identified by all numerical criteria of chaos, i.e. the most reliable Lyapunov exponents [1], phase portraits, Poincaré maps and bifurcation diagrams. Furthermore, it is found that the Feigenbaum number rule [2] still holds for nonlinear nonholonomic system.

6.2 The Magnitude of Velocity Keeping Constant

From Newton's law, the dynamic equations of a free particle with unit mass moving in a horizontal smooth (x_1, y_1) plane are

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1, \\
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1.
\end{aligned} \tag{6.1}$$

where $-a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1$ and $-(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1$ are x_1 and y_1 components of the forces applied on the particle, respectively, and a, b, ω are constants. Eq. (6.1) consists of two linearly coupled nonlinear damped Mathieu systems.

Now an external nonlinear nonholonomic constraint

$$\dot{x}_1^2 + \dot{y}_1^2 = c \tag{6.2}$$

is added, the particle is no more free but a constrained particle. c is a constant, i.e. the constraint makes the magnitude of the velocity constant. Since the constraint is nonlinear nonholonomic, the particle becomes a nonlinear nonholonomic system, of which the dynamic equations can be obtained as follows.

From the general nonlinear nonholonomic constraint equations

$$f_j(q_i, \dot{q}_i, t) = 0 \quad (j = 1, \ldots, m; i = 1, \ldots, n), \tag{6.3}$$

the constraint conditions of the virtual velocities can be derived:

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial \dot{q}_i} \delta \dot{q}_i = 0 \quad (j = 1, \ldots, m). \tag{6.4}$$

According to Jourdain's principle [3]

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} - Q_i \right] \delta \dot{q}_i = 0 \tag{6.5}$$

and the constraint conditions of the virtual velocities, Lagrange multiplier method is used to obtain the nonlinear nonholonomic form of Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial \dot{q}_i} \quad (i = 1, \ldots, n), \tag{6.6}$$

where q_i is the generalized coordinate, T is the kinetic energy of the system, Q_i is the generalized applied force, λ_j is the Lagrange multiplier, and f_j is the nonholonomic constraint function in Eq. (6.3). Together with m nonholonomic constraint equations, there are $(m+n)$ equations solving for n generalized coordinates and m Lagrange multipliers.

In our case, choose $(q_1, q_2) = (x_1, y_1)$. The kinetic energy is $T = \frac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2)$, the generalized applied forces are $Q_1 = -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - a\dot{x}_1 + by_1$, $Q_2 = -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - a\dot{y}_1 + bx_1$, and the nonholonomic constraint equation is $f = \dot{x}_1^2 + \dot{y}_1^2 - c$. Then the nonlinear nonholonomic form of Lagrange's equations is obtained:

$$\ddot{x}_1 = -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - a\dot{x}_1 + by_1 + 2\lambda\dot{x}_1, \tag{6.7}$$

$$\ddot{y}_1 = -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - a\dot{y}_1 + bx_1 + 2\lambda\dot{y}_1. \tag{6.8}$$

Together with nonholonomic constraint equation

$$\dot{x}_1^2 + \dot{y}_1^2 = c, \tag{6.9}$$

there are three equations solving two generalized coordinates and one Lagrange multiplier. In order to solve λ, differentiate the nonholonomic constraint equation Eq. (6.9) with respect to time and get

$$\dot{x}_1\ddot{x}_1 + \dot{y}_1\ddot{y}_1 = 0. \tag{6.10}$$

Substituting Eq. (6.7) and Eq. (6.8) into Eq. (6.10), λ can be solved:

$$\begin{aligned}\lambda = \frac{a}{2} + \{&[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1]\dot{x}_1 \\ &+ [(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1]\dot{y}_1\}/2c.\end{aligned} \tag{6.11}$$

Finally, the differential equations of nonholonomic system can be expressed as

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 + by_1 + \frac{h_1(x_1, x_2, y_1, y_2, t)}{c}, \\ \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 + bx_1 + \frac{h_2(x_1, x_2, y_1, y_2, t)}{c},\end{aligned} \tag{6.12}$$

where

$$
\begin{aligned}
h_1 &= [a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1]x_2^2 \\
&\quad + [(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1]x_2 y_2, \\
h_2 &= [a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1]x_2 y_2 \\
&\quad + [(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1]y_2^2.
\end{aligned}
\tag{6.13}
$$

The parameters in simulation are $a = 0.5$, $b = 0.1 \sim 6$, $c = 1$, $\omega = 1$, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $y_1(0) = 0.1$, $y_2(0) = \sqrt{0.99}$. The phase portraits, Poincaré maps, Lyapunov exponents, and bifurcation diagram for nonholonomic system are shown in Fig. 6.1-6.4. It can be observed that the motion is period 1 for $b = 5.8$, period 2 for $b = 2.5$, period 4 for $b = 4.1$. For $b = 5.3$, the motion is chaotic. All numerical criteria of chaos prove that the chaotic phenomena exist in nonlinear nonholonomic system where the magnitude of velocity of the particle keeping constant.

From bifurcation diagram, Fig. 6.4, it shows that the period-doubling phenomenon occurs from $b = 6$ to $b = 5$. Take enlargement of Fig. 6.4, we can clearly observe the period-doubling phenomenon as shown in Fig. 6.5. Then the Feigenbaum number [2] can be calculated. Feigenbaum number δ is defined as:

$$
\delta = \lim_{k\to\infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}, \tag{6.14}
$$

where μ_k is the kth bifurcation point. The results are shown in Table 6.1, it can be found that the Feigenbaum number approaches the universal number $\delta = 4.6692016091029909\ldots$. This means that the Feigenbaum number still holds for nonlinear nonholonomic system where the magnitude of velocity keeping constant.

Table 6.1 Calculation of Feigenbaum number for system (6.12)

Period doubling	Bifurcation point	Feigenbaum number

Period 1 to Period 2	5.519779	
Period 2 to Period 4	5.495639	3.8986
Period 4 to Period 8	5.489447	4.5263
Period 8 to Period 16	5.488079	4.5600
Period 16 to Period 32	5.487779	4.6154
Period 32 to Period 64	5.487714	

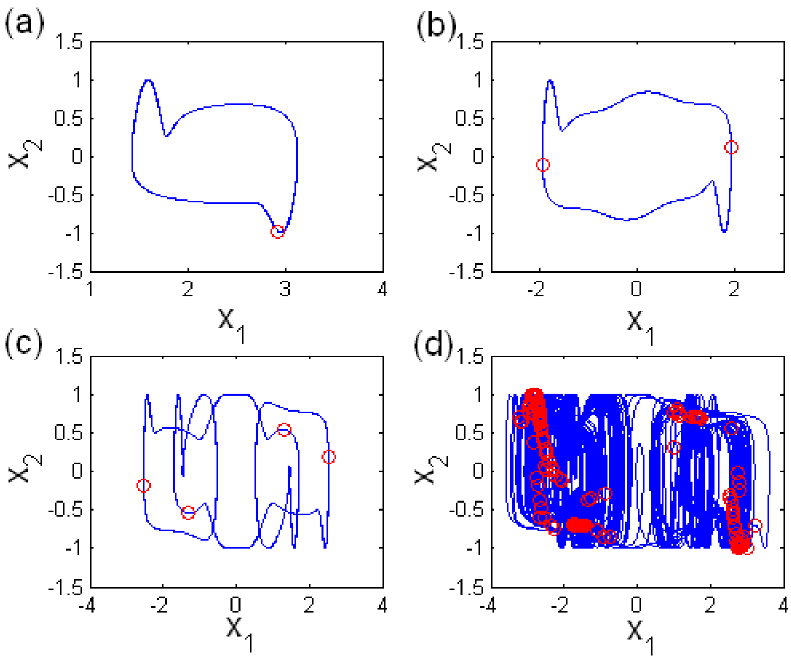

Fig. 6.1 Phase portraits and Poincaré maps for nonlinear nonholonomic system where the magnitude of velocity keeping constant: (a) period 1 for $b = 5.8$, (b) period 2 for $b = 2.5$, (c) period 4 for $b = 4.1$, (d) chaotic for $b = 5.3$.

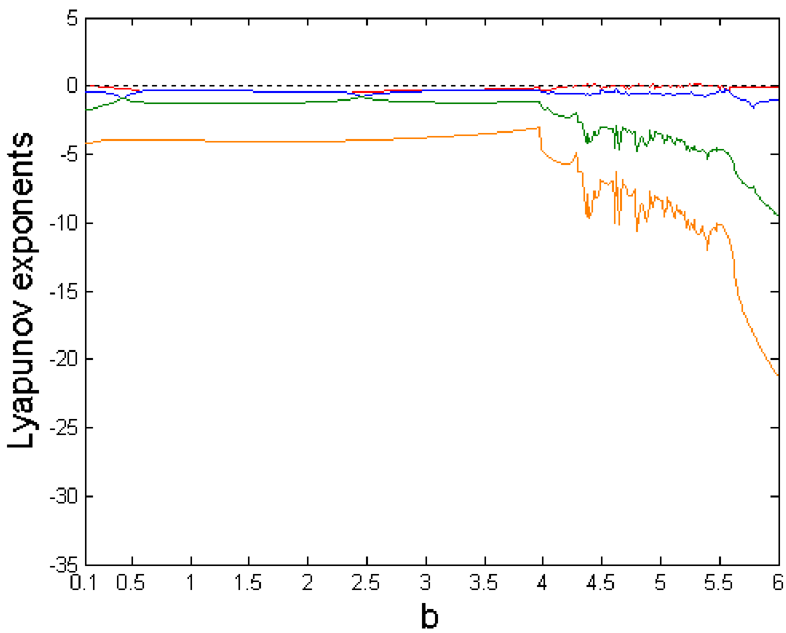

Fig. 6.2 Lyapunov exponents for nonlinear nonholonomic system where the magnitude of velocity keeping constant.

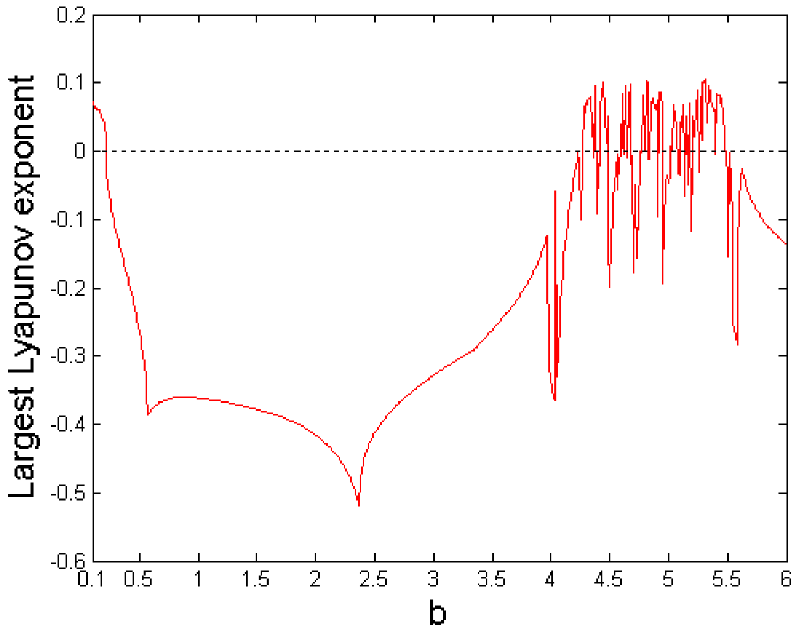

Fig. 6.3 Largest Lyapunov exponent for nonlinear nonholonomic system where the magnitude of velocity keeping constant.

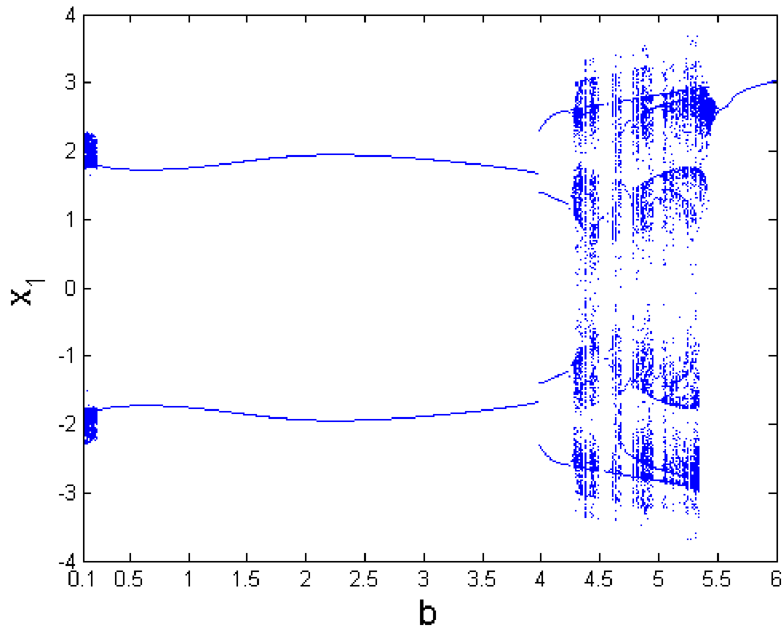

Fig. 6.4 Bifurcation diagram for nonlinear nonholonomic system where the magnitude of velocity keeping constant.

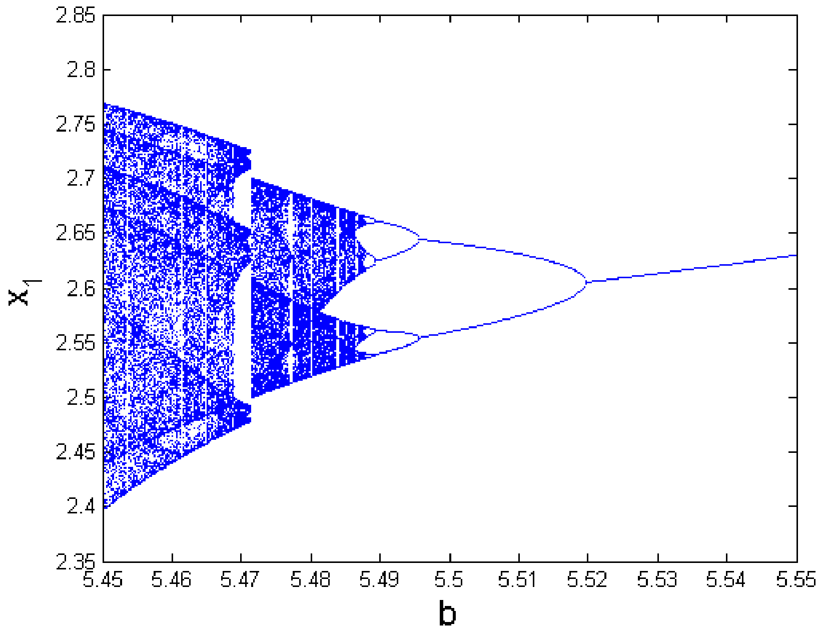

Fig. 6.5 Period-doubling phenomenon for nonlinear nonholonomic system where the magnitude of velocity keeping constant.

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Chapter 7

Non-simultaneous Symplectic Synchronization of Different Chaotic Systems with Variable Scale Time

7.1 Preliminaries

There are various types of synchronization, such as complete synchronization [1], generalized synchronization [2], phase synchronization [3], and lag synchronization [4], and so on. Among these types of synchronization, the generalized synchronization is one of the most interesting topics. Generalized synchronization refers to a functional relation between the state vectors of master and slave, i.e. $\mathbf{y}=\mathbf{F}(\mathbf{x},t)$, where $\mathbf{x}$ and $\mathbf{y}$ are the state vectors of master and slave. In the work of Ref. [5], the generalized synchronization is extended to a more general form, $\mathbf{y}=\mathbf{F}(\mathbf{x},\mathbf{y},t)$, where the "slave" $\mathbf{y}$ is not a traditional pure slave obeying the "master" $\mathbf{x}$ completely but plays a role to determine the final desired state of the "slave". Since the "slave" $\mathbf{y}$ plays an "interwined" role, this type of synchronization is called "symplectic synchronization"[1], the master is called "partner A", and the slave is called "partner B".

In this Chapter, we propose a new type of synchronization, $\mathbf{y}(t)=\mathbf{F}(\mathbf{x}(\tau),\mathbf{y}(t),t)$, where τ is a given function of time t, so-called variable scale time. The synchronization is achieved at "different time" for "partner A" $\mathbf{x}(\tau)$ and "partner B" $\mathbf{y}(t)$, therefore we call this type of synchronization "non-simultaneous symplectic synchronization". When $\tau=t$, non-simultaneous symplectic synchronization reduces to symplectic synchronization. The non-simultaneous symplectic synchronization may be applied to increase the security of secret communication since the functional

[1] The term ‘‘**symplectic**’’ comes from the Greek for ‘‘interwined’’. H. Weyl first introduced the term in 1939 in his book "The Classical Groups" (p. 165 in both the first edition, 1939, and second edition, 1946, Princeton University Press).

relation of non-simultaneous symplectic synchronization is more complex than that of traditional generalized synchronization, and cracking the variable scale time τ is an extra task for the attackers in addition to cracking the system model and cracking the functional relation. The aim of this Chapter is to achieve non-simultaneous symplectic synchronization by applying nonlinear control [6] and by applying adaptive control. In the work of Ref. [6], the induced matrix norm and the Lipschitz constant are obtained by auxiliary numerical simulation. However, they can be estimated theoretically by using the property of induced matrix norms [7a] and by applying adaptive control. Furthermore, in our case, non-simultaneous symplectic synchronization, the complexity of the functional relation $\mathbf{F}(\mathbf{x}(\tau), \mathbf{y}(t), t)$ is greater than that studied in Ref. [6], thus the Lipschitz constant may be enormous. However, by applying adaptive control, the estimated Lipschitz constant is much less than the Lipschitz constant obtained by applying nonlinear control. This result in the reduction of the gain of the controller, i.e. the cost of controller is reduced. The proposed scheme is effective and feasible for both autonomous and nonautonomous chaotic systems, whether the dimensions of $\mathbf{x}(\tau)$ and $\mathbf{y}(t)$ systems are the same or not.

7.2 Non-simultaneous Symplectic Synchronization Scheme

Consider two different nonlinear chaotic systems, partner A and partner B, described by

$$\frac{d\mathbf{x}(\tau)}{d\tau} = \mathbf{f}(\mathbf{x}(\tau), \tau), \tag{7.1}$$

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{C}(t)\mathbf{y}(t) + \mathbf{D}(t)\mathbf{g}(\mathbf{y}(t), t) + \mathbf{u}, \tag{7.2}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in R^n$ and $\mathbf{y} = [y_1, y_2, \ldots, y_m]^{\mathrm{T}} \in R^m$ are the state vectors of partner A and partner B, τ is a given function of time t, so-called variable scale time, $\mathbf{C} \in R^{m\times m}$ and $\mathbf{D} \in R^{m\times m}$ are system matrices, $\mathbf{f}$ and $\mathbf{g}$ are continuous nonlinear vector functions, and $\mathbf{u}$ is the controller. Function $\mathbf{g}(\mathbf{y},t)$ is usually globally Lipschitz continuous, i.e., function $\mathbf{g}(\mathbf{y},t)$ satisfies the inequality $\|\mathbf{g}(\mathbf{y}_1,t)-\mathbf{g}(\mathbf{y}_2,t)\| \le L\|\mathbf{y}_1-\mathbf{y}_2\|$ for $\mathbf{y}_1,\mathbf{y}_2 \in R^m$, $t \in R^+$. Our goal is to design the controller $\mathbf{u}$ such that the state vector $\mathbf{y}(t)$ of partner B asymptotically approaches $\mathbf{F}(\mathbf{x}(\tau),\mathbf{y}(t),t)$, where $\mathbf{F}$ is a continuous nonlinear vector function.

Property 7.1 [7a]: An $m\times n$ matrix A of real elements defines a linear mapping $y = Ax$ from R^n into R^m, and the induced p-norm of A for $p=1,\ 2,$ and ∞ is given by

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|, \quad \|A\|_2 = \left[\lambda_{\max}(A^{\mathrm{T}}A)\right]^{1/2}, \quad \|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|. \tag{7.3}$$

The useful property of induced matrix norms for real matrix A is as follow:

$$\|A\|_2 \le \sqrt{\|A\|_1 \|A\|_\infty}\ . \tag{7.4}$$

Theorem 7.1 (*nonlinear control* [6]): For chaotic systems "partner A" (7.1) and "partner B" (7.2) that are globally Lipschitz continuous, if the controller $\mathbf{u}$ is designed as

$$\begin{aligned}\mathbf{u} = (\mathbf{I}-\mathbf{D_y F})^{-1}[&\mathbf{D_x F f}(\mathbf{x}(\tau),\tau)\frac{d\tau}{dt} + \mathbf{D_y F}(\mathbf{C}(t)\mathbf{y}+\mathbf{D}(t)\mathbf{g}(\mathbf{y},t)) + \mathbf{D}_t\mathbf{F} \\ &-\mathbf{C}(t)\mathbf{F}-\mathbf{D}(t)\mathbf{g}(\mathbf{F},t)-\mathbf{K}(\mathbf{y}-\mathbf{F})],\end{aligned} \tag{7.5}$$

where $\mathbf{D_x F}$, $\mathbf{D_y F}$, $\mathbf{D}_t\mathbf{F}$ are the Jacobian matrices of $\mathbf{F}(\mathbf{x}(\tau),\mathbf{y}(t),t)$, $\mathbf{K} = \mathrm{diag}(k_1,k_2,\ldots,k_m)$, and satisfies

$$\frac{\min(k_i)}{L\|\mathbf{C}(t)\|+\|\mathbf{D}(t)\|} > 1, \tag{7.6}$$

then the non-simultaneous symplectic synchronization will be achieved.

Proof: Define the error vectors as

$$\mathbf{e} = \mathbf{y}(t) - \mathbf{F}(\mathbf{x}(\tau), \mathbf{y}(t), t) , \tag{7.7}$$

then the following error dynamics can be obtained by introducing the designed controller

$$\begin{aligned} \frac{d\mathbf{e}}{dt} &= \dot{\mathbf{e}} = \dot{\mathbf{y}} - \mathbf{D}_{\mathbf{x}}\mathbf{F}\frac{d\mathbf{x}(\tau)}{d\tau}\frac{d\tau}{dt} - \mathbf{D}_{\mathbf{y}}\mathbf{F}\dot{\mathbf{y}} - \mathbf{D}_t\mathbf{F} \\ &= \mathbf{C}(t)\mathbf{y} + \mathbf{D}(t)\mathbf{g}(\mathbf{y},t) - \mathbf{D}_{\mathbf{x}}\mathbf{F}\mathbf{f}(\mathbf{x}(\tau),\tau)\frac{d\tau}{dt} - \mathbf{D}_{\mathbf{y}}\mathbf{F}(\mathbf{C}(t)\mathbf{y} + \mathbf{D}(t)\mathbf{g}(\mathbf{y},t)) \\ &\quad - \mathbf{D}_t\mathbf{F} + (\mathbf{I} - \mathbf{D}_{\mathbf{y}}\mathbf{F})\mathbf{u} \\ &= \mathbf{C}(t)\mathbf{e} + \mathbf{D}(t)(\mathbf{g}(\mathbf{y},t) - \mathbf{g}(\mathbf{F},t)) - \mathbf{K}\mathbf{e}. \end{aligned} \tag{7.8}$$

Choose a non-negative Lyapunov function of the form

$$V(t) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e} . \tag{7.9}$$

Taking the time derivative of $V(t)$ along the trajectory of Eq. (7.8) and applying the Lipschitz condition, we have

$$\begin{aligned} \dot{V}(t) &= \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} \\ &= \mathbf{e}^{\mathrm{T}}\mathbf{C}(t)\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{D}(t)(\mathbf{g}(\mathbf{y},t) - \mathbf{g}(\mathbf{F},t)) - \mathbf{e}^{\mathrm{T}}\mathbf{K}\mathbf{e} \\ &\le \|\mathbf{C}(t)\| \cdot \|\mathbf{e}\|^2 + \|\mathbf{e}\| \cdot \|\mathbf{D}(t)\| \cdot \|\mathbf{g}(\mathbf{y},t) - \mathbf{g}(\mathbf{F},t)\| - \min(k_i)\|\mathbf{e}\|^2 \\ &\le \|\mathbf{C}(t)\| \cdot \|\mathbf{e}\|^2 + L\|\mathbf{D}(t)\| \cdot \|\mathbf{e}\|^2 - \min(k_i)\|\mathbf{e}\|^2 \\ &= (\|\mathbf{C}(t)\| + L\|\mathbf{D}(t)\| - \min(k_i))\|\mathbf{e}\|^2 . \end{aligned} \tag{7.10}$$

Let $M = \min(k_i) - \|\mathbf{C}(t)\| - L\|\mathbf{D}(t)\| > 0$, then $\dot{V}(t) \le -M\|\mathbf{e}\|^2 = -2MV(t)$. Therefore, it can be obtained that

$$V(t) \le V(0)\mathrm{e}^{-2Mt} \tag{7.11}$$

and $\lim_{t\to\infty}\int_0^t |V(\xi)|\, d\xi$ is bounded. Besides, $V(t)$ is uniformly continuous. According to Barbalat's lemma [7b], the conclusion can be drawn that $\lim_{t\to\infty} V(t) = 0$, i.e. $\lim_{t\to\infty}\|\mathbf{e}(t)\| = 0$. Thus, the non-simultaneous symplectic synchronization can be achieved asymptotically.

Theorem 7.2 (*adaptive control*): For chaotic systems "partner A" (7.1) and "partner B" (7.2) that are globally Lipschitz continuous, if the controller $\mathbf{u}$ and the updated law of $\hat{L}$ is designed as

$$\begin{aligned}\mathbf{u} = (\mathbf{I}-\mathbf{D}_{\mathrm{y}}\mathbf{F})^{-1}[&\mathbf{D}_{\mathrm{x}}\mathbf{F}\mathbf{f}(\mathbf{x}(\tau),\tau)\frac{d\tau}{dt} + \mathbf{D}_{\mathrm{y}}\mathbf{F}(\mathbf{C}(t)\mathbf{y}+\mathbf{D}(t)\mathbf{g}(\mathbf{y},t)) + \mathbf{D}_t\mathbf{F} \\ &-\mathbf{C}(t)\mathbf{F} - \mathbf{D}(t)\mathbf{g}(\mathbf{F},t) - (\mathbf{K}+\hat{L}\mathbf{D}(t))(\mathbf{y}-\mathbf{F})]\end{aligned} \tag{7.12}$$

and

$$\dot{\hat{L}} = \|\mathbf{D}(t)\|\cdot\|\mathbf{y}-\mathbf{F}\|^2, \tag{7.13}$$

where $\mathbf{D}_{\mathrm{x}}\mathbf{F}$, $\mathbf{D}_{\mathrm{y}}\mathbf{F}$, $\mathbf{D}_t\mathbf{F}$ are the Jacobian matrices of $\mathbf{F}(\mathbf{x}(\tau),\mathbf{y}(t),t)$, $\hat{L}$ is the estimate of Lipschitz constant L, $\mathbf{K} = \mathrm{diag}(k_1, k_2, \ldots, k_m)$, and satisfies

$$\frac{\min(k_i)}{\|\mathbf{C}(t)\|} > 1, \tag{7.14}$$

then the non-simultaneous symplectic synchronization will be achieved.

Proof: Define the error vectors as

$$\mathbf{e} = \mathbf{y}(t) - \mathbf{F}(\mathbf{x}(\tau),\mathbf{y}(t),t), \tag{7.15}$$

then the following error dynamics can be obtained by introducing the designed controller

$$\begin{aligned}\frac{d\mathbf{e}}{dt} = \dot{\mathbf{e}} &= \dot{\mathbf{y}} - \mathbf{D}_{\mathrm{x}}\mathbf{F}\frac{d\mathbf{x}(\tau)}{d\tau}\frac{d\tau}{dt} - \mathbf{D}_{\mathrm{y}}\mathbf{F}\dot{\mathbf{y}} - \mathbf{D}_t\mathbf{F} \\ &= \mathbf{C}(t)\mathbf{y} + \mathbf{D}(t)\mathbf{g}(\mathbf{y},t) - \mathbf{D}_{\mathrm{x}}\mathbf{F}\mathbf{f}(\mathbf{x}(\tau),\tau)\frac{d\tau}{dt} - \mathbf{D}_{\mathrm{y}}\mathbf{F}(\mathbf{C}(t)\mathbf{y}+\mathbf{D}(t)\mathbf{g}(\mathbf{y},t)) \\ &\quad - \mathbf{D}_t\mathbf{F} + (\mathbf{I}-\mathbf{D}_{\mathrm{y}}\mathbf{F})\mathbf{u} \\ &= \mathbf{C}(t)\mathbf{e} + \mathbf{D}(t)(\mathbf{g}(\mathbf{y},t) - \mathbf{g}(\mathbf{F},t)) - (\mathbf{K}+\hat{L}\mathbf{D}(t))\mathbf{e}.\end{aligned} \tag{7.16}$$

Choose a positive definite Lyapunov function of the form

$$V(\mathbf{e},\tilde{L}) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e} + \frac{1}{2}(\hat{L}-L)^2. \tag{7.17}$$

where $\tilde{L} = \hat{L} - L$. Taking the time derivative of $V(\mathbf{e}, \tilde{L})$ along the trajectory of Eq. (7.16) and applying the updated law of $\hat{L}$ and the Lipschitz condition, we have

$$\begin{aligned}
\dot{V}(\mathbf{e}, \tilde{L}) &= \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} + (\hat{L} - L)\dot{\hat{L}} \\
&= \mathbf{e}^{\mathrm{T}}\mathbf{C}(t)\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{D}(t)(\mathbf{g}(\mathbf{y},t) - \mathbf{g}(\mathbf{F},t)) - \mathbf{e}^{\mathrm{T}}(\mathbf{K} + \hat{L}\mathbf{D}(t))\mathbf{e} + (\hat{L} - L)\|\mathbf{D}(t)\| \cdot \|\mathbf{e}\|^2 \\
&\le \|\mathbf{C}(t)\| \cdot \|\mathbf{e}\|^2 + \|\mathbf{e}\| \cdot \|\mathbf{D}(t)\| \cdot \|\mathbf{g}(\mathbf{y},t) - \mathbf{g}(\mathbf{F},t)\| - \min(k_i)\|\mathbf{e}\|^2 - L\|\mathbf{D}(t)\| \cdot \|\mathbf{e}\|^2 \\
&\le \|\mathbf{C}(t)\| \cdot \|\mathbf{e}\|^2 + L\|\mathbf{D}(t)\| \cdot \|\mathbf{e}\|^2 - \min(k_i)\|\mathbf{e}\|^2 - L\|\mathbf{D}(t)\| \cdot \|\mathbf{e}\|^2 \\
&= (\|\mathbf{C}(t)\| - \min(k_i))\|\mathbf{e}\|^2 .
\end{aligned} \tag{7.18}$$

Let $M = \min(k_i) - \|\mathbf{C}(t)\| > 0$, then $\dot{V}(\mathbf{e}, \tilde{L})$ is negative semidefinite. According to Lyapunov stability theorem, $\lim_{t\to\infty} \mathbf{e}(t) = 0$. Therefore, the non-simultaneous symplectic synchronization can be achieved asymptotically.

7.3 Synchronization of Different Chaotic Systems with the Same Dimension

Consider the van der Pol system as partner A described by

$$\begin{aligned}
\frac{dx_1(\tau)}{d\tau} &= x_1(\tau) - \frac{1}{3}x_1^3(\tau) - x_2(\tau) + p + q\cos\Omega\tau, \\
\frac{dx_2(\tau)}{d\tau} &= \gamma x_1(\tau) + \alpha - \beta x_2(\tau),
\end{aligned} \tag{7.19}$$

where $\alpha = 0.7$, $\beta = 0.8$, $\gamma = 0.1$, $p = 0$, $q = 0.74$, $\Omega = 1$, and the initial condition is $x_1(0) = 1$, $x_2(0) = 1$. Define $\tau = ct + d\sin t$, then Eq. (7.19) can be rewritten as

$$\begin{aligned}
\frac{dx_1(\tau)}{dt} &= \frac{dx_1(\tau)}{d\tau}\frac{d\tau}{dt} = (c + d\cos t)(x_1(\tau) - \frac{1}{3}x_1^3(\tau) - x_2(\tau) + p + q\cos(c\Omega t + d\Omega\sin t)), \\
\frac{dx_2(\tau)}{dt} &= \frac{dx_2(\tau)}{d\tau}\frac{d\tau}{dt} = (c + d\cos t)(\gamma x_1(\tau) + \alpha - \beta x_2(\tau)),
\end{aligned} \tag{7.20}$$

where $c = 2$ and $d = 1$ are chosen in simulation. The chaotic attractor of the van der Pol system for $\tau = 2t + \sin t$ is shown in Fig. 7.1.

The controlled forced nonlinear damped Mathieu system is considered as partner B described by

$$\frac{dy_1(t)}{dt} = y_2(t) + u_1,$$
$$\frac{dy_2(t)}{dt} = -a(1+\sin\omega t)y_1(t) - (1+\sin\omega t)y_1^3(t) - ay_2(t) + b\sin\omega t + u_2, \tag{7.21}$$

where $a = 0.3$, $b = 1$, $\omega = 1$, $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ is the controller, and the initial condition is $y_1(0) = 0.01$, $y_2(0) = 0.01$. The chaotic attractor of uncontrolled forced nonlinear damped Mathieu system is shown in Fig. 7.2. Eq. (7.21) can be rewritten in the form of Eq. (7.2), where

$\mathbf{C}(t) = \begin{bmatrix} 0 & 1 \\ -a(1+\sin\omega t) & -a \end{bmatrix}$, $\mathbf{D}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and

$\mathbf{g}(\mathbf{y}, t) = \begin{bmatrix} 0 \\ -(1+\sin\omega t)y_1^3(t) + b\sin\omega t \end{bmatrix}$.

By applying Property 7.1, it can be derived that $\|\mathbf{C}(t)\|_1 = 1 + a$, $\|\mathbf{C}(t)\|_\infty = 1$, and $\|\mathbf{C}(t)\|_2 \le \sqrt{1+a} = \sqrt{1.3}$. Then $\|\mathbf{C}(t)\| = 1$ and $\|\mathbf{D}(t)\| = 1$ are obtained.

Define $\mathbf{F}(\mathbf{x}(\tau), \mathbf{y}(t), t) = \begin{bmatrix} x_1^2(\tau) - (\sin^2 t)y_1(t) \\ x_2^2(\tau) - (\cos^2 t)y_2(t) \end{bmatrix}$, and our goal is to achieve the non-simultaneous simplectic synchronization $\mathbf{y}(t) = \mathbf{F}(\mathbf{x}(\tau), \mathbf{y}(t), t)$.

Method 1:

According to Theorem 7.1, the inequality $\dfrac{\min(k_i)}{L\|\mathbf{C}(t)\| + \|\mathbf{D}(t)\|} > 1$ has to be satisfied. It can be obtained by numerical simulation that the Lipschitz constant $L = 45$, then $\min(k_i) > 46$. Thus we choose $\mathbf{K} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \begin{bmatrix} 47 & 0 \\ 0 & 48 \end{bmatrix}$ and design the controller as

$$u_1=(1+\cos^2 t)\{-x_2^2(\tau)+(\cos^2 t)y_2+2(c+d\cos t)x_1(\tau)[x_1(\tau)-x_1^3(\tau)/3-x_2(\tau)+p$$
$$+q\cos(c\Omega t+d\Omega\sin t)]-2(\sin t)(\cos t)y_1-(\sin^2 t)y_2+k_1[x_1^2(\tau)-(1+\sin^2 t)y_1]\}$$
$$/[1+\sin^2 t+\cos^2 t+(\sin^2 t)(\cos^2 t)],$$

$$u_2=(1+\sin^2 t)\{2\gamma(c+d\cos t)x_2(\tau)[x_1(\tau)+\alpha-\beta x_2(\tau)]+a[x_2^2(\tau)-(\cos^2 t)y_2]$$
$$+k_2[x_2^2(\tau)-(1+\cos^2 t)y_2]+2(\sin t)(\cos t)y_2+(1+\sin\omega t)[x_1^2(\tau)-(\sin^2 t)y_1]^3$$
$$+a(1+\sin\omega t)[x_1^2(\tau)-(\sin^2 t)y_1]+(\cos^2 t)[ay_2+(1+\sin\omega t)y_1^3+a(1+\sin\omega t)y_1$$
$$-b\sin\omega t]-b\sin\omega t\}/[1+\sin^2 t+\cos^2 t+(\sin^2 t)(\cos^2 t)].$$

When the non-simultaneous simplectic synchronization is achieved, the chaotic attractor of the controlled forced nonlinear damped Mathieu system and the time histories of the state errors are shown in Fig. 7.3 and Fig. 7.4, respectively.

Method 2:

According to Theorem 7.2, the inequality $\frac{\min(k_i)}{\|\mathbf{C}(t)\|}>1$ has to be satisfied. It can be obtained that $\min(k_i)>1$. Thus $\mathbf{K}=\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$ is chosen, and the updated law of $\hat{L}$ and the controller are designed as

$$\dot{\hat{L}}=[(1+\sin^2 t)y_1-x_1^2(\tau)]^2+[(1+\cos^2 t)y_2-x_2^2(\tau)]^2,$$

$$u_1=(1+\cos^2 t)\{-x_2^2(\tau)+(\cos^2 t)y_2+2(c+d\cos t)x_1(\tau)[x_1(\tau)-x_1^3(\tau)/3-x_2(\tau)+p$$
$$+q\cos(c\Omega t+d\Omega\sin t)]-2(\sin t)(\cos t)y_1-(\sin^2 t)y_2+k_1[x_1^2(\tau)-(1+\sin^2 t)y_1]\}$$
$$/[1+\sin^2 t+\cos^2 t+(\sin^2 t)(\cos^2 t)],$$

$$u_2=(1+\sin^2 t)\{2\gamma(c+d\cos t)x_2(\tau)[x_1(\tau)+\alpha-\beta x_2(\tau)]+a[x_2^2(\tau)-(\cos^2 t)y_2]$$
$$+(k_2+\hat{L})[x_2^2(\tau)-(1+\cos^2 t)y_2]+2(\sin t)(\cos t)y_2+(1+\sin\omega t)[x_1^2(\tau)-(\sin^2 t)y_1]^3$$
$$+a(1+\sin\omega t)[x_1^2(\tau)-(\sin^2 t)y_1]+(\cos^2 t)[ay_2+(1+\sin\omega t)y_1^3+a(1+\sin\omega t)y_1$$
$$-b\sin\omega t]-b\sin\omega t\}/[1+\sin^2 t+\cos^2 t+(\sin^2 t)(\cos^2 t)],$$

where the initial condition of $\hat{L}$ is $\hat{L}(0)=0$. When the non-simultaneous simplectic synchronization is achieved, the time histories of the state errors and the time histories of $\hat{L}$ are shown in Fig. 7.5 and Fig. 7.6, respectively. It can be observed that $\hat{L}$ approaches 0.4 asymptotically.

Compare the results between method 1 and method 2, it is found that the estimated Lipschitz constant $\hat{L}=0.4$ derived from method 2 is much less than the Lipschitz constant $L=45$ derived from method 1. In other words, by applying adaptive control, the gain of the controller is reduced, and the cost of controller is reduced.

7.4 Synchronization of Different Chaotic Systems with Different Dimensions

Consider the forced nonlinear damped Mathieu system as partner A described by

$$\begin{aligned}
\frac{dx_1(\tau)}{d\tau} &= x_2(\tau), \\
\frac{dx_2(\tau)}{d\tau} &= -a(1+\sin\omega\tau)x_1(\tau)-(1+\sin\omega\tau)x_1^3(\tau)-ax_2(\tau)+b\sin\omega\tau,
\end{aligned} \tag{7.22}$$

where $a=0.3$, $b=1$, $\omega=1$, and the initial condition is $x_1(0)=1$, $x_2(0)=1$. Define $\tau=ct$, then Eq. (7.22) can be rewritten as

$$\begin{aligned}
\frac{dx_1(\tau)}{dt} &= \frac{dx_1(\tau)}{d\tau}\frac{d\tau}{dt} = cx_2(\tau), \\
\frac{dx_2(\tau)}{dt} &= \frac{dx_2(\tau)}{d\tau}\frac{d\tau}{dt} = -ac(1+\sin\omega ct)x_1(\tau)-c(1+\sin\omega ct)x_1^3(\tau)-acx_2(\tau)+bc\sin\omega ct,
\end{aligned} \tag{7.23}$$

where $c=5$ is chosen in simulation. The chaotic attractor of the forced nonlinear damped Mathieu system for $\tau=5t$ is shown in Fig. 7.7.

The controlled Rössler system is considered as partner B described by

$$\frac{dy_1(t)}{dt} = -y_2(t) - y_3(t) + u_1,$$
$$\frac{dy_2(t)}{dt} = y_1(t) + \alpha y_2(t) + u_2, \qquad (7.24)$$
$$\frac{dy_3(t)}{dt} = \beta + y_3(t)(y_1(t) - \gamma) + u_3,$$

where $\alpha = 0.15$, $\beta = 0.2$, $\gamma = 10$, $\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ is the controller, and the initial condition is $y_1(0) = 0.01$, $y_2(0) = 0.01$, $y_3(0) = 0.01$. The chaotic attractor of uncontrolled Rössler system is shown in Fig. 7.8. Eq. (7.24) can be rewritten in the form of Eq. (7.2), where

$$\mathbf{C}(t) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \alpha & 0 \\ 0 & 0 & -\gamma \end{bmatrix}, \mathbf{D}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{g}(\mathbf{y}, t) = \begin{bmatrix} 0 \\ 0 \\ y_1(t)y_3(t) + \beta \end{bmatrix}.$$

By applying Property 7.1, it can be derived that $\|\mathbf{C}(t)\|_1 = 1 + \gamma$, $\|\mathbf{C}(t)\|_\infty = \gamma$, and $\|\mathbf{C}(t)\|_2 \le \sqrt{\gamma(1+\gamma)} = \sqrt{110}$. Then $\|\mathbf{C}(t)\| = 10$ and $\|\mathbf{D}(t)\| = 1$ are obtained.

Define $\mathbf{F}(\mathbf{x}(\tau), \mathbf{y}(t), t) = \begin{bmatrix} -x_1^2(\tau)y_1(t) - y_1^3(t) + \sin t \\ -x_2^2(\tau)y_2(t) - y_2^3(t) + \cos t \\ -x_1^2(\tau)y_3(t) - y_3^3(t) + \sin t \end{bmatrix}$, and our goal is to achieve the non-simultaneous simplectic synchronization $\mathbf{y}(t) = \mathbf{F}(\mathbf{x}(\tau), \mathbf{y}(t), t)$.

Method 1:

According to Theorem 7.1, the inequality $\dfrac{\min(k_i)}{L\|\mathbf{C}(t)\| + \|\mathbf{D}(t)\|} > 1$ has to be satisfied. It can be obtained by numerical simulation that the Lipschitz constant $L = 1550$, then $\min(k_i) > 1560$. Thus we choose

$$\mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} = \begin{bmatrix} 1570 & 0 & 0 \\ 0 & 1580 & 0 \\ 0 & 0 & 1590 \end{bmatrix}$$ and design the controller as

$$u_1 = \{-2cx_1(\tau)x_2(\tau)y_1 - x_1^2(\tau)y_3 - x_2^2(\tau)y_2 - y_2^3 - y_3^3 + [x_1^2(\tau) + 3y_1^2](y_2 + y_3)$$
$$- k_1[x_1^2(\tau)y_1 + y_1^3 + y_1 - \sin t] + \sin t + 2\cos t\} / (x_1^2(\tau) + 3y_1^2 + 1),$$

$$u_2 = \{x_1^2(\tau)y_1 + y_1^3 - [x_2^2(\tau) + 3y_2^2](y_1 + \alpha y_2) - k_2[x_2^2(\tau)y_2 + y_2^3 + y_2 - \cos t]$$
$$+ 2cx_2(\tau)y_2[(1 + \sin \omega ct)x_1^3(\tau) + a(1 + \sin \omega ct)x_1 + ax_2(\tau) - b\sin \omega ct]$$
$$+ \alpha[x_2^2(\tau)y_2 + y_2^3 - \cos t] - 2\sin t\} / (x_2^2(\tau) + 3y_2^2 + 1),$$

$$u_3 = \{-2cx_1(\tau)x_2(\tau)y_3 - [x_1^2(\tau) + 3y_3^2](y_1y_3 - \gamma y_3 + \beta) - k_3[x_1^2(\tau)y_3 + y_3^3 + y_3 - \sin t]$$
$$- \gamma[x_1^2(\tau)y_3 + y_3^3 - \sin t] - [x_1^2(\tau)y_1 + y_1^3 - \sin t][x_1^2(\tau)y_3 + y_3^3 - \sin t] + \cos t - \beta\}$$
$$/ (x_1^2(\tau) + 3y_3^2 + 1).$$

When the non-simultaneous simplectic synchronization is achieved, the chaotic attractor of the controlled Rössler system and the time histories of the state errors are shown in Fig. 7.9 and Fig. 7.10, respectively.

Method 2:

According to Theorem 7.2, the inequality $\frac{\min(k_i)}{\|\mathbf{C}(t)\|} > 1$ has to be satisfied. It can be obtained that $\min(k_i) > 10$. Thus $\mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 16 \end{bmatrix}$ is chosen, and the updated law of $\hat{L}$ and the controller are designed as

$$\dot{\hat{L}} = [y_1^3 + (1 + x_1^2(\tau))y_1 - \sin t]^2 + [y_2^3 + (1 + x_2^2(\tau))y_2 - \cos t]^2$$
$$+ [y_3^3 + (1 + x_1^2(\tau))y_3 - \sin t]^2,$$

$$u_1 = \{-2cx_1(\tau)x_2(\tau)y_1 - x_1^2(\tau)y_3 - x_2^2(\tau)y_2 - y_2^3 - y_3^3 + [x_1^2(\tau) + 3y_1^2](y_2 + y_3)$$
$$- k_1[x_1^2(\tau)y_1 + y_1^3 + y_1 - \sin t] + \sin t + 2\cos t\} / (x_1^2(\tau) + 3y_1^2 + 1),$$

$$u_2 = \{x_1^2(\tau)y_1 + y_1^3 - [x_2^2(\tau) + 3y_2^2](y_1 + \alpha y_2) - k_2[x_2^2(\tau)y_2 + y_2^3 + y_2 - \cos t]$$
$$+ 2cx_2(\tau)y_2[(1 + \sin \omega ct)x_1^3(\tau) + a(1 + \sin \omega ct)x_1 + ax_2(\tau) - b\sin \omega ct]$$
$$+ \alpha[x_2^2(\tau)y_2 + y_2^3 - \cos t] - 2\sin t\} / (x_2^2(\tau) + 3y_2^2 + 1),$$

$$u_3 = \{-2cx_1(\tau)x_2(\tau)y_3 - [x_1^2(\tau)+3y_3^2](y_1y_3 - \gamma y_3 + \beta) - (k_3+\hat{L})[x_1^2(\tau)y_3 + y_3^3 + y_3 - \sin t] - \gamma[x_1^2(\tau)y_3 + y_3^3 - \sin t] - [x_1^2(\tau)y_1 + y_1^3 - \sin t][x_1^2(\tau)y_3 + y_3^3 - \sin t] + \cos t - \beta\} / (x_1^2(\tau) + 3y_3^2 + 1).$$

where the initial condition of $\hat{L}$ is $\hat{L}(0) = 0$. When the non-simultaneous simplectic synchronization is achieved, the time histories of the state errors and the time histories of $\hat{L}$ are shown in Fig. 7.11 and Fig. 7.12, respectively. It can be observed that $\hat{L}$ approaches 0.035 asymptotically.

By comparing the results between method 1 and method 2, it is found that the estimated Lipschitz constant $\hat{L} = 0.035$ derived from method 2 is much less than the Lipschitz constant $L = 1550$ derived from method 1. It means that the gain of the controller is reduced, and the cost of controller is reduced by applying adaptive control.

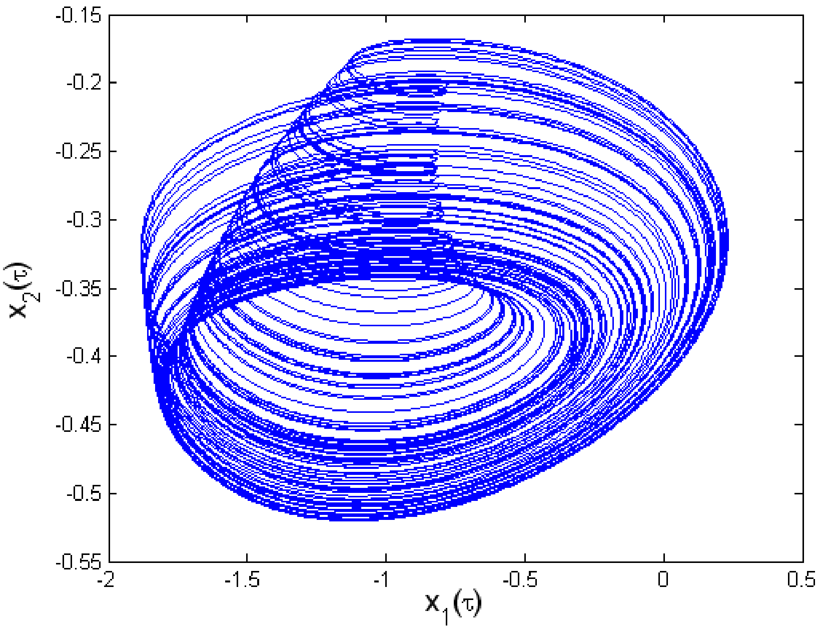

Fig. 7.1 The chaotic attractor of the van der Pol system for $\tau = 2t + \sin t$.

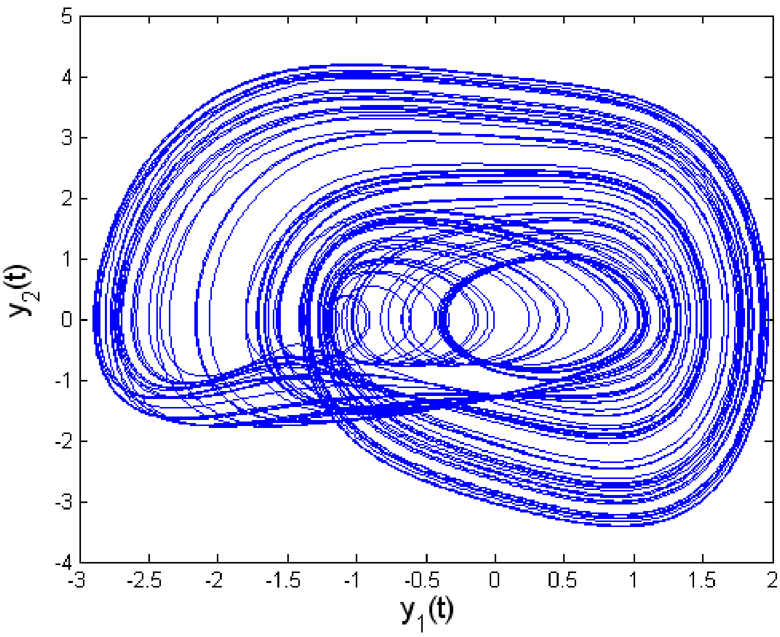

Fig. 7.2 The chaotic attractor of uncontrolled forced nonlinear damped Mathieu system.

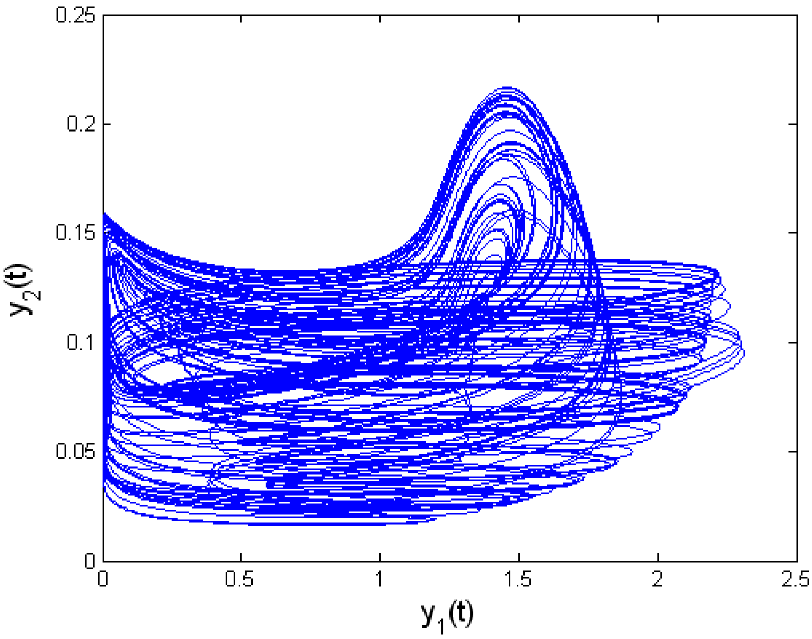

Fig. 7.3 The chaotic attractor of the controlled forced nonlinear damped Mathieu system.

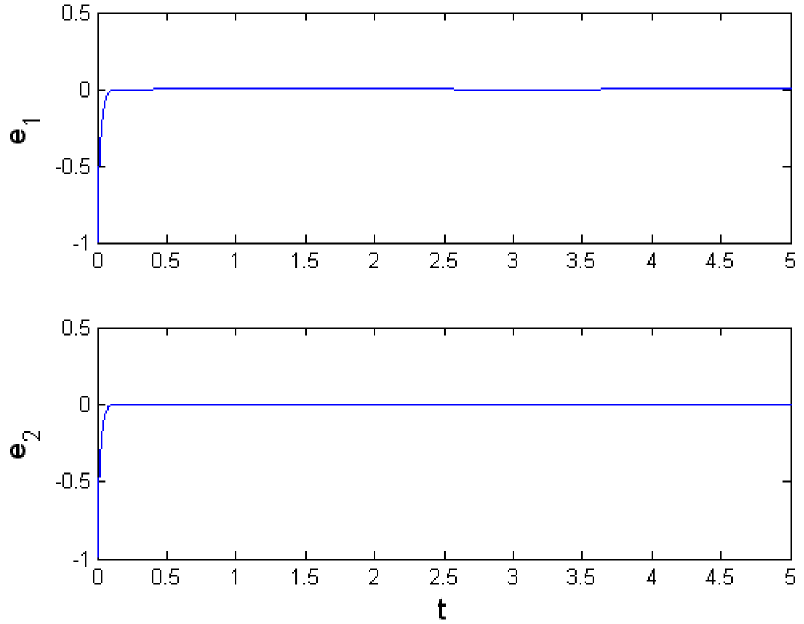

Fig. 7.4 Time histories of the state errors for Section 7.3 by applying method 1.

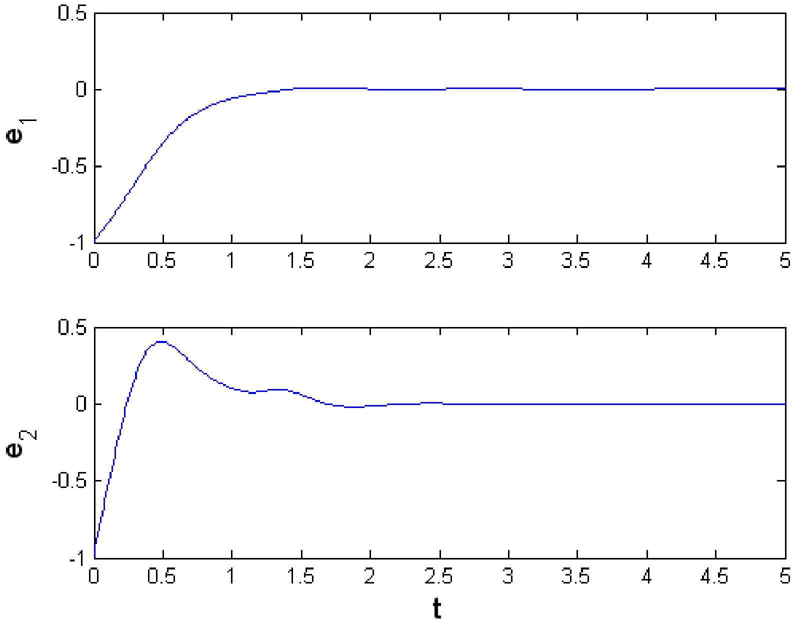

Fig. 7.5 Time histories of the state errors for Section 7.3 by applying method 2.

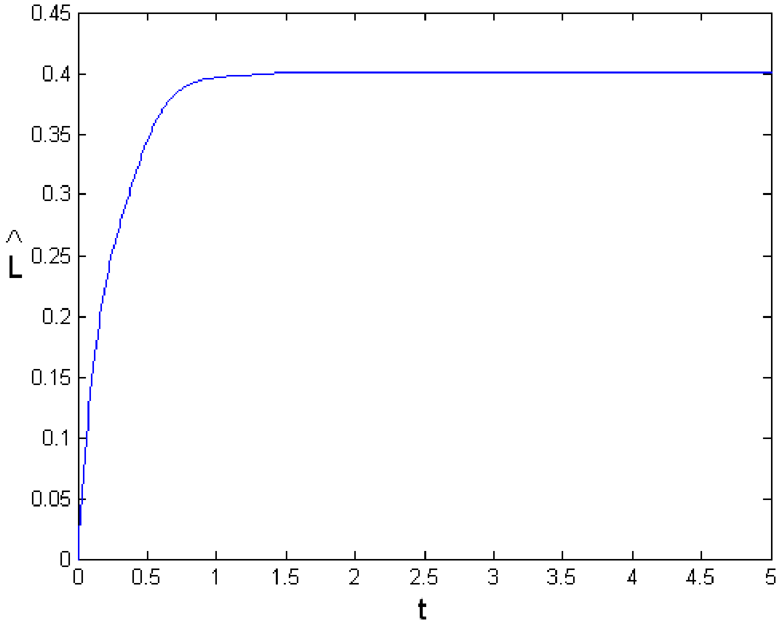

Fig. 7.6 Time histories of $\hat{L}$ for Section 7.3.

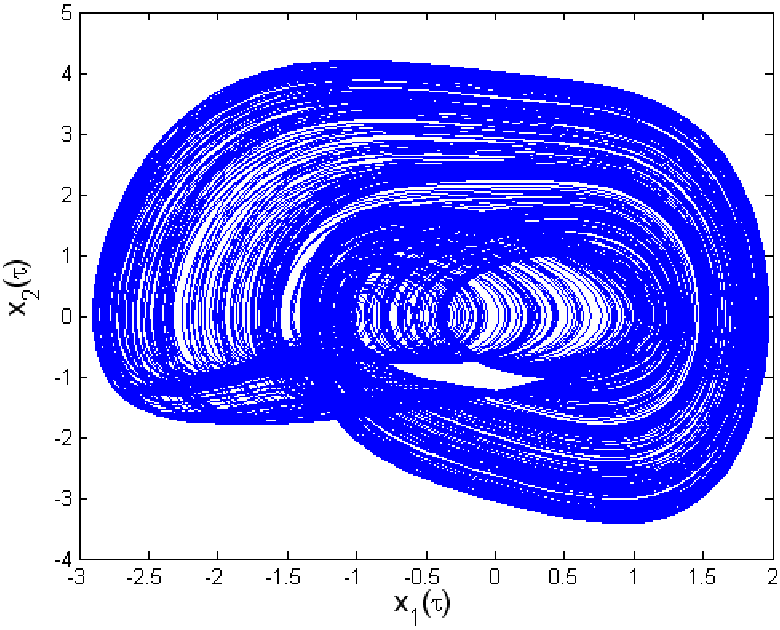

Fig. 7.7 The chaotic attractor of the forced nonlinear damped Mathieu system for $\tau = 5t$.

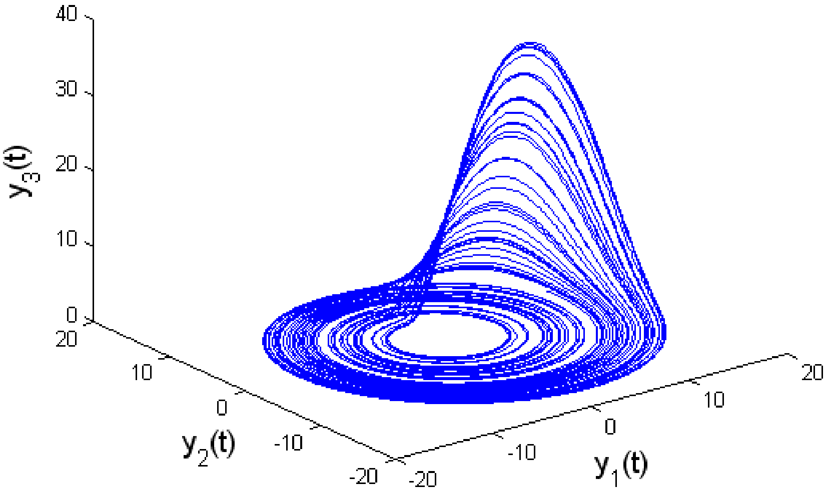

Fig. 7.8 The chaotic attractor of uncontrolled Rössler system.

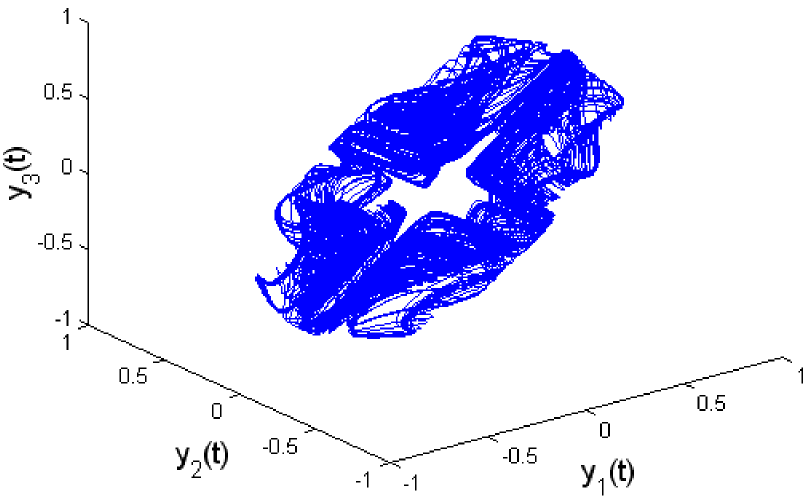

Fig. 7.9 The chaotic attractor of the controlled Rössler system.

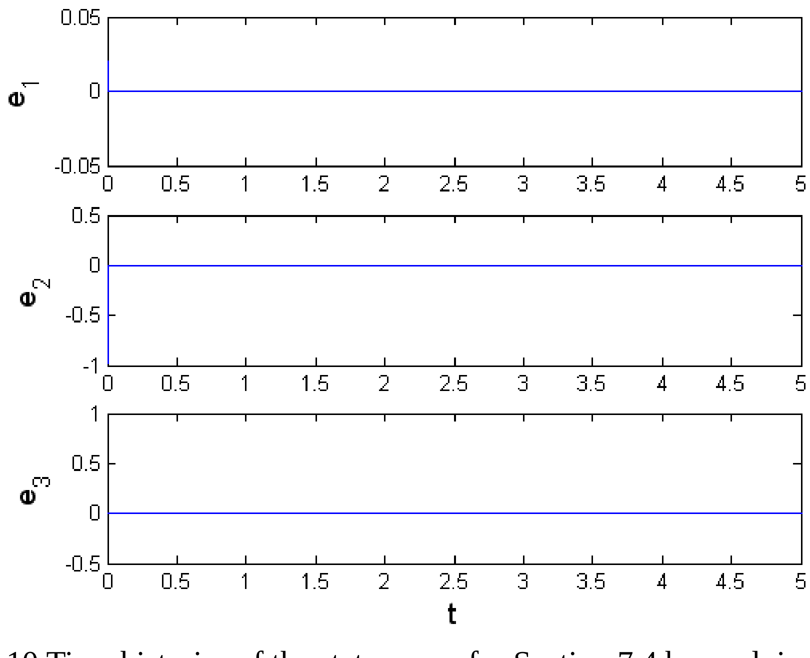

Fig. 7.10 Time histories of the state errors for Section 7.4 by applying method 1.

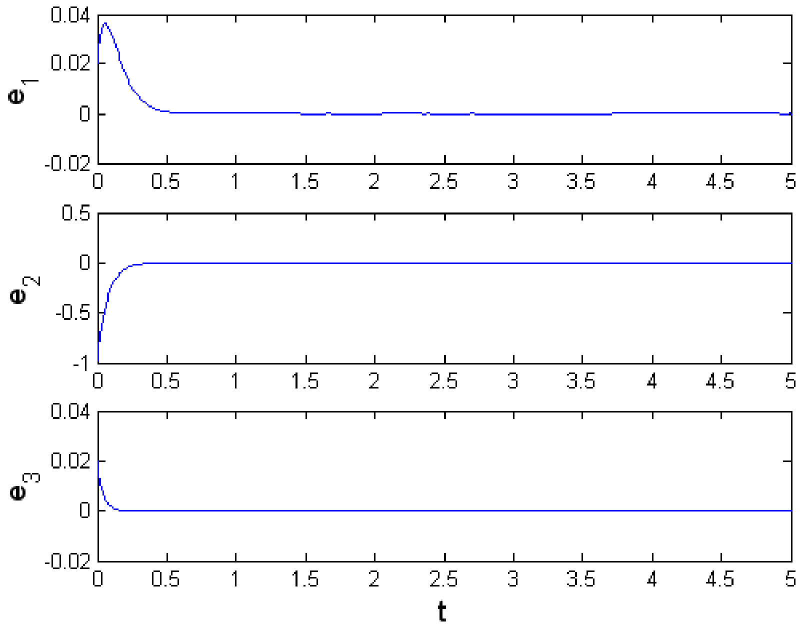

Fig. 7.11 Time histories of the state errors for Section 7.4 by applying method 2.

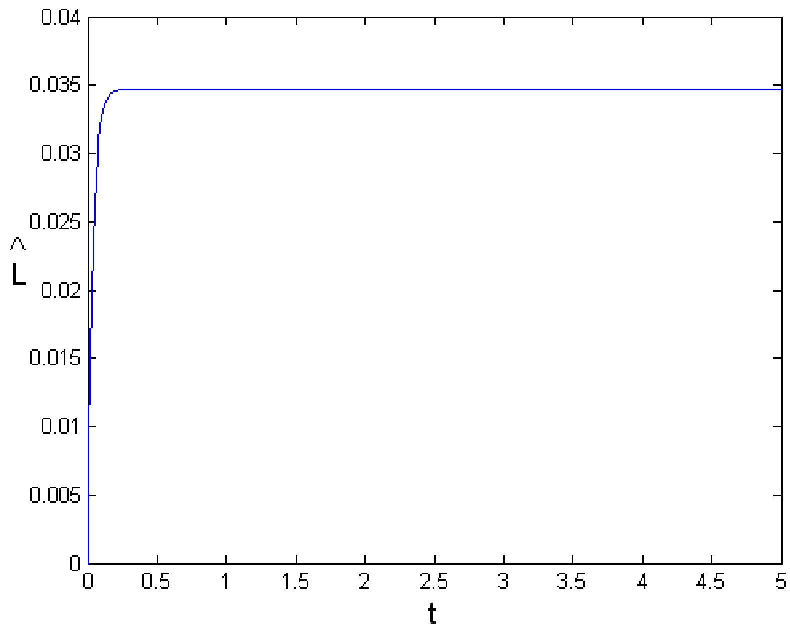

Fig. 7.12 Time histories of $\hat{L}$ for Section 7.4.

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Chapter 8

Double Symplectic Synchronization of Different Chaotic Systems by Active Control

8.1 Preliminaries

In this Chapter, we propose a new type of synchronization, "double symplectic synchronization", $\mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{F}(\mathbf{x},\mathbf{y},t)$. Since the symplectic functions are presented at both the right hand side and the left hand side of the equality, it is called "double symplectic synchronization". It is an extension of symplectic synchronization, $\mathbf{y} = \mathbf{F}(\mathbf{x},\mathbf{y},t)$. When $\mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{y}$, the double symplectic synchronization is reduced to the symplectic synchronization. Due to the complexity of the double symplectic synchronization, it may be applied to increase the security of secret communication. The double symplectic synchronization is obtained by applying active control. A scheme of synchronization is derived based on Barbalat's lemma [1a], and it is effective and feasible for both autonomous and nonautonomous chaotic systems.

8.2 Double Symplectic Synchronization Scheme

Consider two different nonlinear chaotic systems, partner A and partner B, described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t), \tag{8.1}$$

$$\dot{\mathbf{y}} = \mathbf{C}(t)\mathbf{y} + \mathbf{g}(\mathbf{y},t) + \mathbf{u}, \tag{8.2}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in R^n$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]^{\mathrm{T}} \in R^n$ are the state vectors of partner A and partner B, $\mathbf{C} \in R^{n\times n}$ is the system matrix, $\mathbf{f}$ and $\mathbf{g}$ are continuous nonlinear vector functions, and $\mathbf{u}$ is the controller. Our goal is to design the controller $\mathbf{u}$ such that $\mathbf{G}(\mathbf{x},\mathbf{y})$ approaches $\mathbf{F}(\mathbf{x},\mathbf{y},t)$ asymptotically, where $\mathbf{G}$ and $\mathbf{F}$ are continuous vector functions. For simplicity, we take $\mathbf{G}(\mathbf{x},\mathbf{y})=\mathbf{x}+\mathbf{y}$.

Property 8.1 [1b]: An $m\times n$ matrix A of real elements defines a linear mapping $y=Ax$ from R^n into R^m, and the induced p-norm of A for $p=1,\ 2,$ and ∞ is given by

$$\|A\|_1=\max_j\sum_{i=1}^{m}|a_{ij}|,\quad \|A\|_2=\left[\lambda_{\max}(A^{\mathrm{T}}A)\right]^{1/2},\quad \|A\|_\infty=\max_i\sum_{j=1}^{n}|a_{ij}|. \tag{8.3}$$

The useful property of induced matrix norms for real matrix A is as follow:

$$\|A\|_2\le\sqrt{\|A\|_1\|A\|_\infty}\ . \tag{8.4}$$

Theorem 8.1: For chaotic systems "partner A" (8.1) and "partner B" (8.2), if the controller $\mathbf{u}$ is designed as

$$\begin{aligned}\mathbf{u}=(\mathbf{I}-\mathbf{D_yF})^{-1}[&\mathbf{D_xFf}(\mathbf{x},t)+\mathbf{D_yF}(\mathbf{C}(t)\mathbf{y}+\mathbf{g}(\mathbf{y},t))+\mathbf{D}_t\mathbf{F}-\mathbf{f}(\mathbf{x},t)-\mathbf{g}(\mathbf{y},t)\\&+\mathbf{C}(t)(\mathbf{x}-\mathbf{F})-\mathbf{K}(\mathbf{x}+\mathbf{y}-\mathbf{F})],\end{aligned} \tag{8.5}$$

where $\mathbf{D_xF}$, $\mathbf{D_yF}$, $\mathbf{D}_t\mathbf{F}$ are the Jacobian matrices of $\mathbf{F}(\mathbf{x},\mathbf{y},t)$, $\mathbf{K}=\mathrm{diag}(k_1,k_2,\ldots,k_m)$, and satisfies

$$\frac{\min(k_i)}{\|\mathbf{C}(t)\|}>1, \tag{8.6}$$

then the double symplectic synchronization will be achieved.

Proof: Define the error vectors as

$$\mathbf{e} = \mathbf{x} + \mathbf{y} - \mathbf{F}(\mathbf{x}, \mathbf{y}, t), \tag{8.7}$$

then the following error dynamics can be obtained by introducing the designed controller

$$\begin{aligned} \frac{d\mathbf{e}}{dt} = \dot{\mathbf{e}} &= \dot{\mathbf{x}} + \dot{\mathbf{y}} - \mathbf{D}_{\mathbf{x}}\mathbf{F}\dot{\mathbf{x}} - \mathbf{D}_{\mathbf{y}}\mathbf{F}\dot{\mathbf{y}} - \mathbf{D}_t\mathbf{F} \\ &= \mathbf{f}(\mathbf{x}, t) + \mathbf{C}(t)\mathbf{y} + \mathbf{g}(\mathbf{y}, t) - \mathbf{D}_{\mathbf{x}}\mathbf{F}\mathbf{f}(\mathbf{x}, t) - \mathbf{D}_{\mathbf{y}}\mathbf{F}(\mathbf{C}(t)\mathbf{y} + \mathbf{g}(\mathbf{y}, t)) - \mathbf{D}_t\mathbf{F} \\ &\quad + (\mathbf{I} - \mathbf{D}_{\mathbf{y}}\mathbf{F})\mathbf{u} \\ &= (\mathbf{C}(t) - \mathbf{K})\mathbf{e}. \end{aligned} \tag{8.8}$$

Choose a non-negative Lyapunov function of the form

$$V(t) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}. \tag{8.9}$$

Taking the time derivative of $V(t)$ along the trajectory of Eq. (8.8), we have

$$\begin{aligned} \dot{V}(t) &= \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} \\ &= \mathbf{e}^{\mathrm{T}}\mathbf{C}(t)\mathbf{e} - \mathbf{e}^{\mathrm{T}}\mathbf{K}\mathbf{e} \\ &\le \|\mathbf{C}(t)\| \cdot \|\mathbf{e}\|^2 - \min(k_i)\|\mathbf{e}\|^2 \\ &= (\|\mathbf{C}(t)\| - \min(k_i))\|\mathbf{e}\|^2. \end{aligned} \tag{8.10}$$

Let $M = \min(k_i) - \|\mathbf{C}(t)\| > 0$, then $\dot{V}(t) \le -M\|\mathbf{e}\|^2 = -2MV(t)$. Therefore, it can be obtained that

$$V(t) \le V(0)\mathrm{e}^{-2Mt} \tag{8.11}$$

and $\lim_{t\to\infty}\int_0^t |V(\xi)|\,d\xi$ is bounded. Besides, $V(t)$ is uniformly continuous. According to Barbalat's lemma [1a], the conclusion can be drawn that $\lim_{t\to\infty} V(t) = 0$, i.e. $\lim_{t\to\infty}\|\mathbf{e}(t)\| = 0$. Thus, the double symplectic synchronization can be achieved asymptotically.

8.3 Synchronization of Different Nonautonomous Chaotic Systems

Consider the Duffing system as partner A described by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -p_2 x_1 - p_3 x_1^3 - p_1 x_2 + q\cos\Omega t, \end{aligned} \tag{8.12}$$

where $p_1 = 0.4$, $p_2 = -1.1$, $p_3 = 1$, $q = 2.1$, $\Omega = 1.8$, and the initial condition is $x_1(0) = 1$, $x_2(0) = 1$. The chaotic attractor of the Duffing system is shown in Fig. 8.1. Eq. (8.12) can be rewritten in the form of Eq. (8.1), where $\mathbf{f}(\mathbf{x},t) = \begin{bmatrix} x_2 \\ -p_2 x_1 - p_3 x_1^3 - p_1 x_2 + q\cos\Omega t \end{bmatrix}$.

The controlled forced nonlinear damped Mathieu system is considered as partner B described by

$$\begin{aligned} \dot{y}_1 &= y_2 + u_1, \\ \dot{y}_2 &= -a(1+\sin\omega t)y_1 - (1+\sin\omega t)y_1^3 - a y_2 + b\sin\omega t + u_2, \end{aligned} \tag{8.13}$$

where $a = 0.3$, $b = 1$, $\omega = 1$, $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ is the controller, and the initial condition is $y_1(0) = 0.01$, $y_2(0) = 0.01$. The chaotic attractor of uncontrolled forced nonlinear damped Mathieu system is shown in Fig. 8.2. Eq. (8.13) can be rewritten in the form of Eq. (8.2), where $\mathbf{C}(t) = \begin{bmatrix} 0 & 1 \\ -a(1+\sin\omega t) & -a \end{bmatrix}$ and $\mathbf{g}(\mathbf{y},t) = \begin{bmatrix} 0 \\ -(1+\sin\omega t)y_1^3(t) + b\sin\omega t \end{bmatrix}$. By applying Property 8.1, it is derived that $\|\mathbf{C}(t)\|_1 = 1 + a$, $\|\mathbf{C}(t)\|_\infty = 1$, and $\|\mathbf{C}(t)\|_2 \le \sqrt{1+a} = \sqrt{1.3}$. Then $\|\mathbf{C}(t)\| = 1$ is estimated.

Define $\mathbf{F}(\mathbf{x},\mathbf{y},t)=\begin{bmatrix} -x_1^2 y_1 + x_1 y_1^2 - y_1^3 + \sin t \\ -x_2^2 y_2 + x_2 y_2^2 - y_2^3 + \sin t \end{bmatrix}$, and our goal is to achieve the double simplectic synchronization $\mathbf{x}+\mathbf{y}=\mathbf{F}(\mathbf{x},\mathbf{y},t)$. According to Theorem 8.1, the inequality $\dfrac{\min(k_i)}{\|\mathbf{C}(t)\|}>1$ has to be satisfied. It can be obtained that $\min(k_i)>1$. Thus we choose $\mathbf{K}=\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and design the controller as

$$\begin{aligned} u_1 = \{ & y_2(x_2^2 - x_2 y_2 + y_2^2 - x_1^2 + 2x_1 y_1 - 3y_1^2) - x_2(2x_1 y_1 - y_1^2) + \cos t - \sin t \\ & - k_1(x_1 + y_1 + x_1^2 y_1 - x_1 y_1^2 + y_1^3 - \sin t)\} / (1 + x_1^2 - 2x_1 y_1 + 3y_1^2), \end{aligned}$$

$$\begin{aligned} u_2 = \{ & (x_2^2 - 2x_2 y_2 + 3y_2^2)[a(1+\sin \omega t) y_1 + (1+ sin\omega t) y_1^3 + a y_2 - b \sin \omega t] \\ & + (2x_2 y_2 - y_2^2 + 1)(p_2 x_1 + p_3 x_1^3 + p_1 x_2 - q \cos \Omega t) \\ & - a(1+\sin \omega t)(x_1 + x_1^2 y_1 - x_1 y_1^2 + y_1^3 - \sin t) \\ & - a(x_2 + x_2^2 y_2 - x_2 y_2^2 + y_2^3 - \sin t) + (1+\sin \omega t) y_1^3 - b \sin \omega t + \cos t \\ & - k_2(x_2 + y_2 + x_2^2 y_2 + y_2^3 - \sin t)\} / (1 + x_2^2 - 2x_2 y_2 + 3y_2^2). \end{aligned}$$

When the double simplectic synchronization is achieved, the chaotic attractor of the controlled forced nonlinear damped Mathieu system and the time histories of the state errors are shown in Fig. 8.3 and Fig. 8.4, respectively.

8.4 Synchronization of Different Autonomous Chaotic Systems

Consider the Lorenz system as partner A described by

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= -x_1 x_3 + r x_1 - x_2, \\ \dot{x}_3 &= x_1 x_2 - b x_3, \end{aligned} \tag{8.14}$$

where $\sigma=10$, $r=28$, $b=8/3$, and the initial condition is $x_1(0)=1$, $x_2(0)=1$, $x_3(0)=1$. Eq. (8.14) can be rewritten in the form of Eq. (8.1), where

$\mathbf{f}(\mathbf{x},t) = \begin{bmatrix} \sigma(x_2 - x_1) \\ -x_1x_3 + rx_1 - x_2 \\ x_1x_2 - bx_3 \end{bmatrix}$. The chaotic attractor of the Lorenz system is shown in Fig. 8.5.

The controlled Rössler system is considered as partner B described by

$$\begin{aligned} \dot{y}_1 &= -y_2 - y_3 + u_1, \\ \dot{y}_2 &= y_1 + \alpha y_2 + u_2, \\ \dot{y}_3 &= \beta + y_3(y_1 - \gamma) + u_3, \end{aligned} \tag{8.15}$$

where $\alpha = 0.15$, $\beta = 0.2$, $\gamma = 10$, $\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ is the controller, and the initial condition is $y_1(0) = 0.01$, $y_2(0) = 0.01$, $y_3(0) = 0.01$. The chaotic attractor of uncontrolled Rössler system is shown in Fig. 8.6. Eq. (8.15) can be rewritten in the form of Eq. (8.2), where $\mathbf{C}(t) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \alpha & 0 \\ 0 & 0 & -\gamma \end{bmatrix}$ and $\mathbf{g}(\mathbf{y},t) = \begin{bmatrix} 0 \\ 0 \\ y_1y_3 + \beta \end{bmatrix}$. By applying Property 8.1, it is derived that $\|\mathbf{C}(t)\|_1 = 1 + \gamma$, $\|\mathbf{C}(t)\|_\infty = \gamma$, and $\|\mathbf{C}(t)\|_2 \le \sqrt{\gamma(1+\gamma)} = \sqrt{110}$. Then $\|\mathbf{C}(t)\| = 10$ is estimated.

Define $\mathbf{F}(\mathbf{x},\mathbf{y},t) = \begin{bmatrix} -y_1 \sin^2 t/(1+x_1^2) \\ -y_2 \sin^2 t/(1+x_2^2) \\ -y_3 \sin^2 t/(1+x_3^2) \end{bmatrix}$, and our goal is to achieve the double simplectic synchronization $\mathbf{x} + \mathbf{y} = \mathbf{F}(\mathbf{x},\mathbf{y},t)$. According to Theorem 8.1, the inequality $\dfrac{\min(k_i)}{\|\mathbf{C}(t)\|} > 1$ has to be satisfied. It can be obtained that $\min(k_i) > 10$. Thus we choose $\mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 13 \end{bmatrix}$ and design the controller as

$$u_1 = \left\{ \sigma x_1 - (\sigma+1)x_2 - x_3 + \frac{(y_2+y_3)\sin^2 t - 2y_1 \sin t \cos t}{1+x_1^2} - \frac{y_2 \sin^2 t}{1+x_2^2} - \frac{y_3 \sin^2 t}{1+x_3^2} \right.$$

$$\left. + \frac{2\sigma x_1 (x_2 - x_1) y_1 \sin^2 t}{(1+x_1^2)^2} - k_1 (x_1 + y_1 + \frac{y_1 \sin^2 t}{1+x_1^2}) \right\} \Big/ \left(1 + \frac{\sin^2 t}{1+x_1^2}\right),$$

$$u_2 = \left\{ (1-r)x_1 + (\alpha+1)x_2 + x_1 x_3 + \frac{y_1 \sin^2 t}{1+x_1^2} - \frac{(y_1 + 2\alpha y_2)\sin^2 t + 2y_2 \sin t \cos t}{1+x_2^2} \right.$$

$$\left. + \frac{2x_2 (r x_1 - x_2 - x_1 x_3) y_2 \sin^2 t}{(1+x_2^2)^2} - k_2 (x_2 + y_2 + \frac{y_2 \sin^2 t}{1+x_2^2}) \right\} \Big/ \left(1 + \frac{\sin^2 t}{1+x_2^2}\right),$$

$$u_3 = \left\{ -x_1 x_2 + b x_3 - y_1 y_3 - \beta - \gamma x_3 + \frac{(2r y_3 - y_1 y_3 - \beta)\sin^2 t - 2y_3 \sin t \cos t}{1+x_3^2} \right.$$

$$\left. + \frac{2x_3 (x_1 x_2 - b x_3) y_3 \sin^2 t}{(1+x_3^2)^2} - k_3 (x_3 + y_3 + \frac{y_3 \sin^2 t}{1+x_3^2}) \right\} \Big/ \left(1 + \frac{\sin^2 t}{1+x_3^2}\right).$$

When the double simplectic synchronization is achieved, the chaotic attractor of the controlled Rössler system and the time histories of the state errors are shown in Fig. 8.7 and Fig. 8.8, respectively.

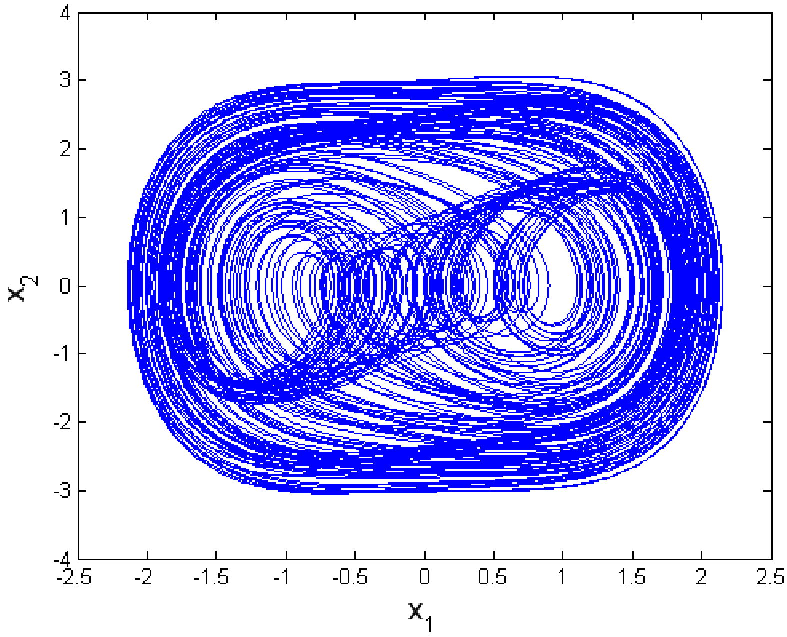

Fig. 8.1 The chaotic attractor of the Duffing system.

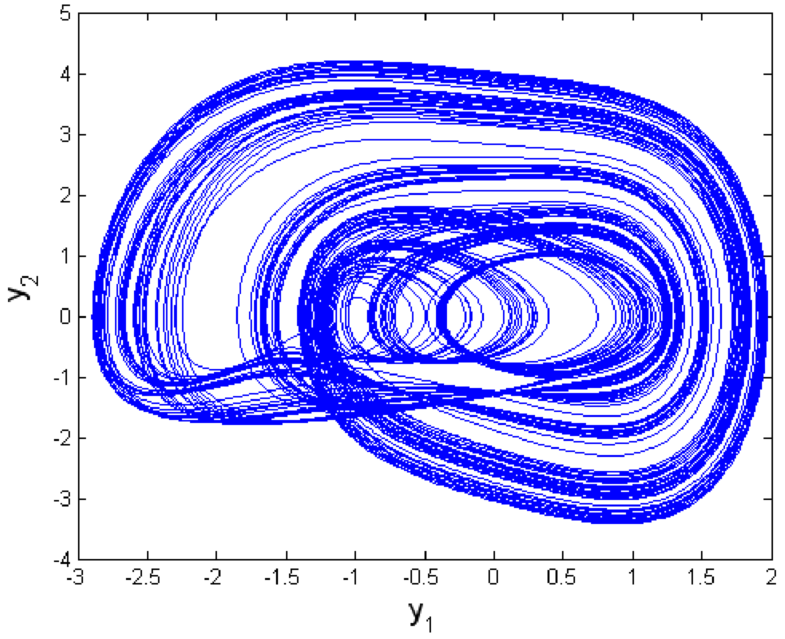

Fig. 8.2 The chaotic attractor of uncontrolled forced nonlinear damped Mathieu system.

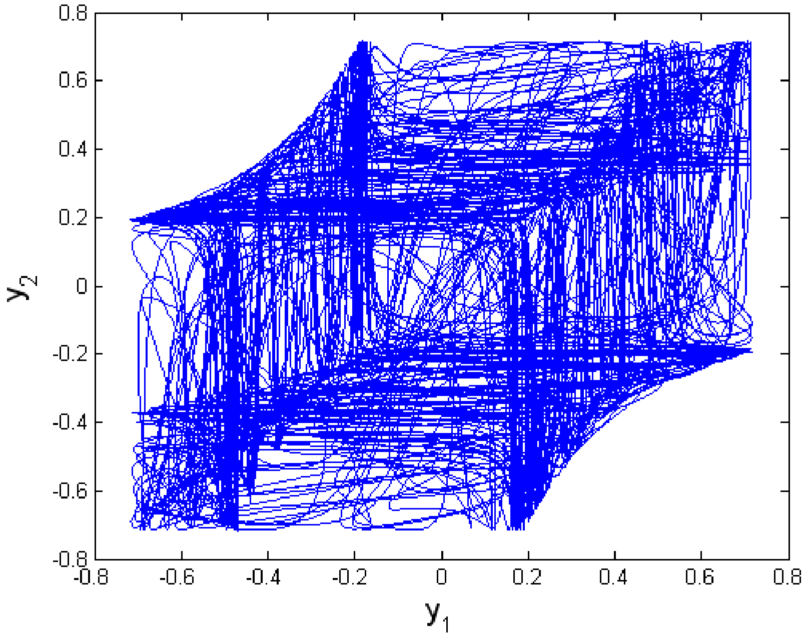

Fig. 8.3 The chaotic attractor of the controlled forced nonlinear damped Mathieu system.

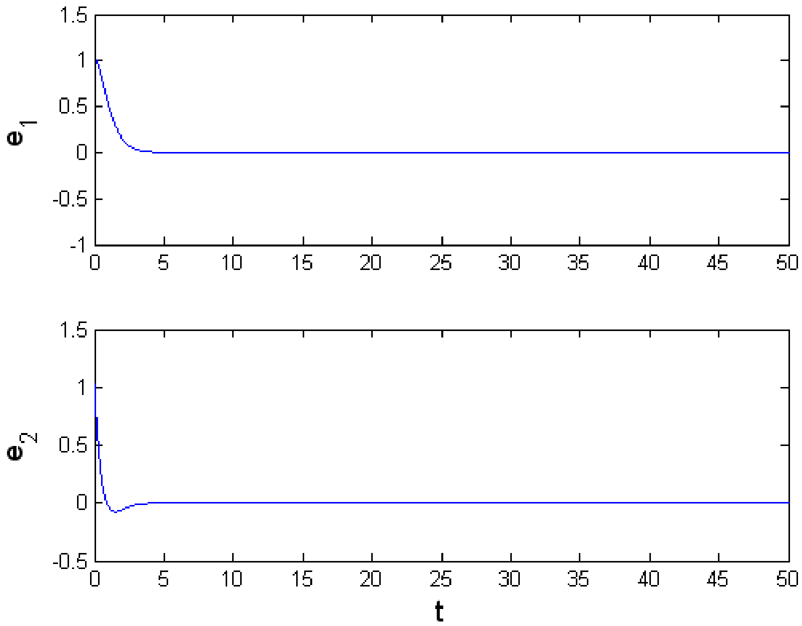

Fig. 8.4 Time histories of the state errors for Section 8.3.

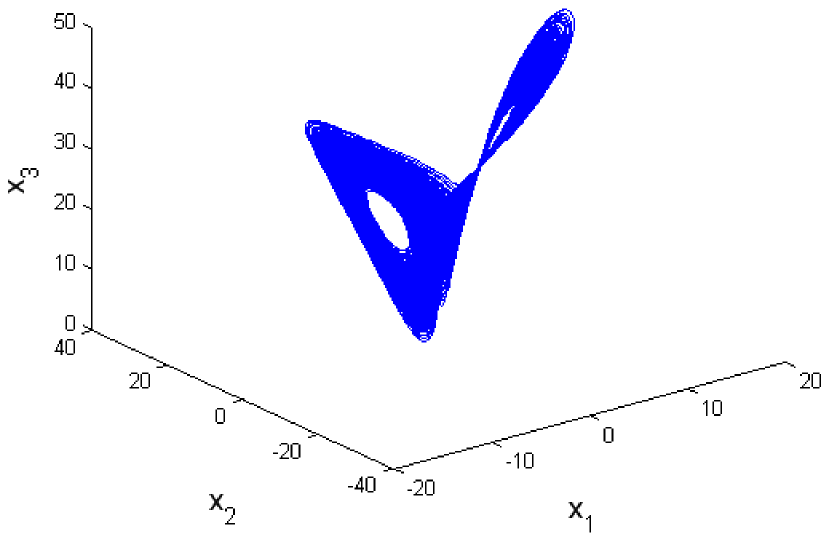

Fig. 8.5 The chaotic attractor of the Lorenz system.

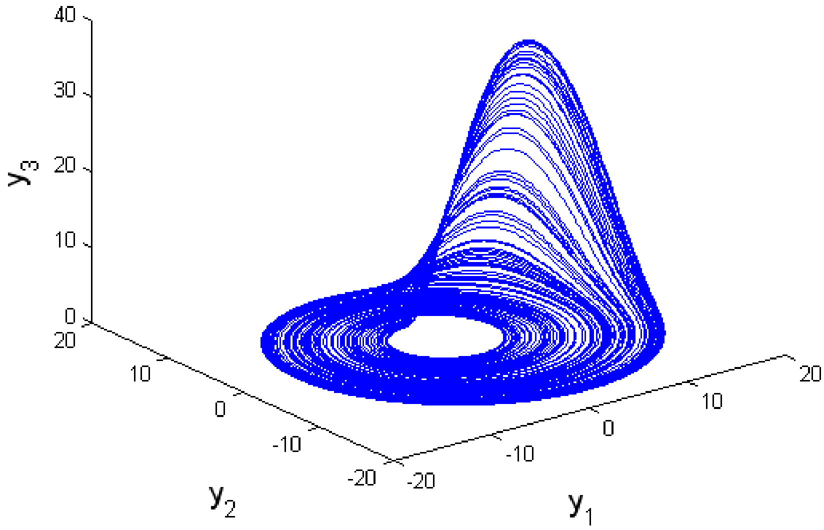

Fig. 8.6 The chaotic attractor of uncontrolled Rössler system.

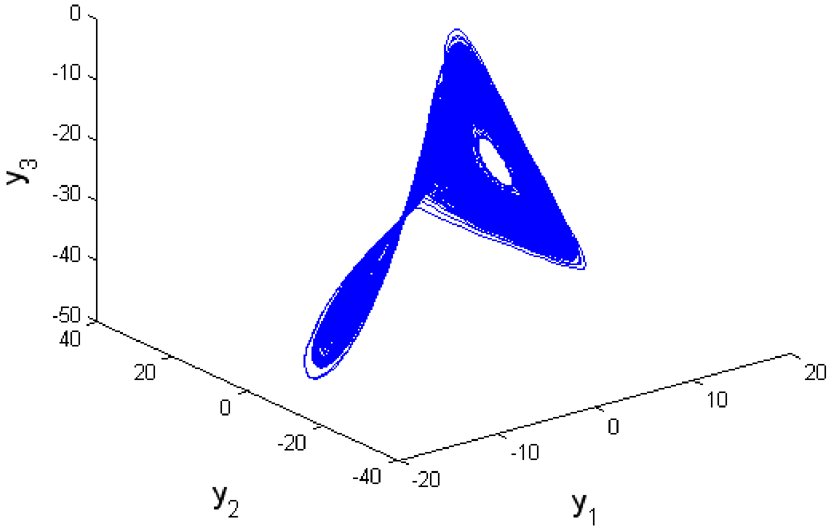

Fig. 8.7 The chaotic attractor of the controlled Rössler system.

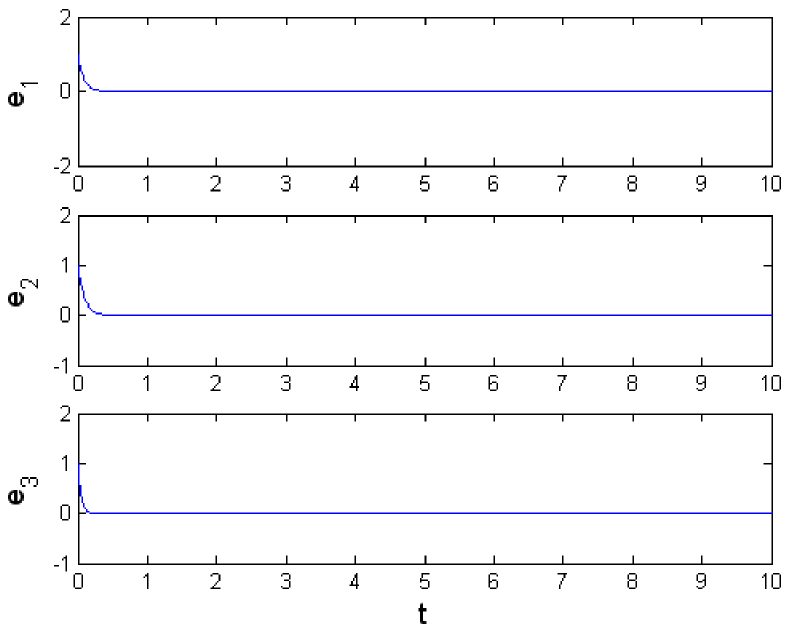

Fig. 8.8 Time histories of the state errors for Section 8.4.

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[1a] H. K. Khalil, *Nonlinear Systems*, Third Edition, Prentice Hall, New Jersey, 2002, p. 323.

[1b] H. K. Khalil, *Nonlinear Systems*, Third Edition, Prentice Hall, New Jersey, 2002, p. 648.

Chapter 9

Conclusions

In this thesis, the generalized synchronization of new chaotic systems by pure error dynamics and elaborate Lyapunov function, chaos of nonholonomic systems, non-simultaneous symplectic synchronization of different chaotic systems with variable scale time, and double symplectic synchronization of different chaotic systems are studied.

Chapter 2 contains the dynamics of new autonomous and new nonautonomous chaotic systems. The system model and the numerical results of regular and chaotic phenomena are presented.

In Chapter 3, the generalized synchronization is studied by applying pure error dynamics and elaborate Lyapunov function. In Chapter 4, by applying pure error dynamics and elaborate nondiagonal Lyapunov function, the nonlinear generalized synchronization is achieved. The methods give rigorous theories for generalized synchronization and nonlinear generalized synchronization and greatly extend the use of various forms of Lyapunov function while current method only gives semi-simulation theory for generalized synchronization, in which the maximum values of state variables must be given by simulation, and monotonous square sum Lyapunov function is used. By the systematic procedures, the complexity of designing suitable elaborate Lyapunov function and elaborate nondiagonal Lyapunov function is reduced greatly. The proposed methods are effectively applied to both new autonomous and new nonautonomous chaotic systems.

Complete identification of chaos in nonholonomic systems and nonlinear nonholonomic systems is firstly presented in Chapter 5 and Chapter 6. The scope of chaos study has been extended to nonholonomic systems and nonlinear nonholonomic system. By applying the fundamental nonholonomic form of Lagrange's equations, the chaos of two nonholonomic moving target pursuit systems is studied in Chapter 5, in which nonholonomic pursuit system with a straightly oscillating target and nonholonomic pursuit system with a circularly rotating target are investigated. In Chapter 6, chaos of nonlinear nonholonomic problem, the magnitude of velocity keeping constant, is studied by applying the nonlinear nonholonomic form of Lagrange's equations. Complete identification of chaotic phenomena is obtained in nonlinear nonholonomic system by Lyapunov exponents, phase portraits, Poincaré maps, and bifurcation diagrams. Futhermore, the Feigenbaum number rule still holds for nonlinear nonholonomic system.

In Chapter 7, the non-simultaneous symplectic synchronization with variable scale time, $\mathbf{y}(t) = \mathbf{F}(\mathbf{x}(\tau), \mathbf{y}(t), t)$, is studied. By applying adaptive control, the non-simultaneous symplectic synchronization is achieved and the estimated Lipschitz constant is much less than the Lipschitz constant obtained by applying nonlinear control. This result in the reduction of the gain of the controller, i.e. the cost of controller is reduced. The simulation results show that the proposed scheme is feasible for both autonomous and nonautonomous chaotic systems, whether the dimensions of $\mathbf{x}(\tau)$ and $\mathbf{y}(t)$ are the same or not. Furthermore, when applying the non-simultaneous symplectic synchronization in secret communication, since the functional relation of the non-simultaneous symplectic synchronization is more complex than that of traditional generalized synchronization, and cracking the variable scale time τ is an extra task for the attackers in addition to cracking the system model and cracking the functional relation, the non-simultaneous symplectic synchronization may be applied to increase the security of secret communication.

In Chapter 8, the double symplectic synchronization, $\mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}, t)$, is studied. It is an extension of symplectic synchronization, $\mathbf{y} = \mathbf{F}(\mathbf{x}, \mathbf{y}, t)$. By applying active control, the double symplectic synchronization is achieved. By simulation results, it is shown that the proposed scheme is effective and feasible for both autonomous and nonautonomous chaotic systems. Furthermore, the double symplectic synchronization may be applied to increase the security of secret communication due to the complexity of its synchronization form.

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* 以博士論文內容撰寫者