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Two theorems of generalized unsynchronization for coupled chaotic systems

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Abstract

Generalized chaos synchronization has been widely studied and many control methods have been presented, but up to now no criterion has been given for generalized unsynchronization. The generalized unsynchronization means that the state variables of two coupled chaotic systems cannot approach generalized synchronization. In this paper, we propose two theorems which give the criteria of generalized unsynchronization for two different chaotic dynamic systems with whatever large strength of linear coupling. Two simulated examples are also given.

MSC: 34C15; 34C28; 37D45

Keywords: Generalized unsynchronization; Chaos; Synchronization; Coupled chaotic systems

1. Introduction

In recent years, synchronization in chaotic dynamic system has been a very interesting problem and has been widely studied [1–5]. Besides, generalized synchronization also has been investigated in various fields [6–17]. Generalized synchronization means that there is a functional relation between the states of the driving system and the response system. In Section 2, we propose two theorems which give the criteria of generalized unsynchronization for two different chaotic dynamic systems with whatever large strength of linear coupling. In Section 3, the Chen system and a new chaotic system which we proposed are presented as a simulated example for the first theorem [18]. The Rössler system with corresponding new chaotic system proposed are presented as simulated examples for the second theorem [19]. In Section 4, conclusions are given.

2. Two theorems of generalized unsynchronizability

Consider the following nonautonomous systems

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \tag{1}$$

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where $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{f} : \Omega_1 \subset \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$. Eq. (1) is considered as a master system. A slave system is given by

$$\dot{\mathbf{y}} = \mathbf{g}(t, \mathbf{y}), \tag{2}$$

where $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{g} : \Omega_2 \subset \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$. Both $\mathbf{f}$ and $\mathbf{g}$ satisfy Lipschitz condition. Ω_1, Ω_2 are domains containing the origin. Assume that the solutions of Eqs. (1) and (2) have bounds then they must exist for infinite time.

Now we consider the following unidirectional nonautonomous coupled system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{g}(t, \mathbf{y}) + \mathbf{U}(t, \mathbf{x}, \mathbf{y}), \end{aligned} \tag{3}$$

where $\mathbf{U}(t, \mathbf{x}, \mathbf{y})$ is a coupled term.

Definition. The system (3) is generalized synchronized if there is a continuous function $H(\mathbf{x})$ and let error $\mathbf{e} = \mathbf{y} - H(\mathbf{x})$ s.t. $\lim_{t\to\infty} \|\mathbf{e}\| = 0$. But, if no positive constant C can be found such that $\mathbf{e} \to 0$ as $t \to \infty$ for all $\|\mathbf{e}(t_0)\| < C$, system (3) is generalized unsynchronizable [20].

In order to discuss the generalized synchronization of $\mathbf{x}$ and $\mathbf{y}$, define $\mathbf{z} = H(\mathbf{x})$ and error $\mathbf{e} = \mathbf{y} - \mathbf{z}$. Error equation can be written as

$$\dot{\mathbf{e}} = \dot{\mathbf{y}} - \dot{\mathbf{z}} = \mathbf{g}(t, \mathbf{e} + \mathbf{z}) - \frac{\partial H}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}) + \mathbf{U}(t, \mathbf{z}, \mathbf{e} + \mathbf{z}). \tag{4}$$

Now the first theorem will be given for a special case of Eq. (3). Consider unidirectional coupled nonautonomous systems as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{g}(t, \mathbf{y}) + \Gamma(\mathbf{z} - \mathbf{y}), \end{aligned} \tag{5}$$

where $\mathbf{f}$ and $\mathbf{g}$ satisfy Lipschitz condition, and the Lipschitz constant of $\mathbf{g}$ is L. $\Gamma \in M_{n\times n}$is a constant diagonal matrix with positive entries which represents the strength of the linear coupling term $\mathbf{z} - \mathbf{y}$. Since $\mathbf{e} = \mathbf{y} - H(\mathbf{x}) = \mathbf{y} - \mathbf{z}$, the error dynamic equation can be obtained as

$$\dot{\mathbf{e}} = \dot{\mathbf{y}} - \dot{\mathbf{z}} = \mathbf{g}(t, \mathbf{e} + \mathbf{z}) - \frac{\partial H}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}) - \Gamma \mathbf{e}. \tag{6}$$

Let $\mathbf{h}(t, \mathbf{z}) = \frac{\partial H}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x})$, system (6) can be written as

$$\dot{\mathbf{e}} = \dot{\mathbf{y}} - \dot{\mathbf{z}} = \mathbf{g}(t, \mathbf{e} + \mathbf{z}) - \mathbf{h}(t, \mathbf{z}) - \Gamma \mathbf{e} \tag{7}$$

which is a nonautonomous system of differential equations for state $\mathbf{e}$. Now we give a theorem of unsynchronizability:

Theorem 1. *Two different dynamic systems in Eq. (5) are unsynchronizable for however large coupling strength Γ with positive entries, if $h_i(t, \mathbf{z}) \le g_i(t, \mathbf{z})$ $(i = 1, \ldots, n)$ in $\Omega_1 \cap \Omega_2$, and $\|\mathbf{h}(t, \mathbf{z}) - \mathbf{g}(t, \mathbf{z})\| > 0$ except at the origin, for any solution $\mathbf{z}(t)$.*

Proof. Choose a Lyapunov function $V(\mathbf{e}) = e_1 e_2 \cdots e_n$ which is positive in quadrant $e_1 > 0, e_2 > 0, \ldots, e_n > 0$, then $\dot{V}$ along any state trajectory of system (6) becomes [21]:

$$\begin{aligned} \dot{V} &= e_2 e_3 \cdots e_n \dot{e}_1 + e_1 e_3 \cdots e_n \dot{e}_2 + \cdots + e_1 e_2 \cdots e_{n-1} \dot{e}_n \\ &= e_2 e_3 \cdots e_n [g_1(t, \mathbf{e} + \mathbf{z}) - h_1(t, \mathbf{z}) - \Gamma_1 e_1] + e_1 e_3 \cdots e_n [g_2(t, \mathbf{e} + \mathbf{z}) - h_2(t, \mathbf{z}) - \Gamma_2 e_2] \cdots \\ &\quad + e_1 e_2 \cdots e_{n-1} [g_n(t, \mathbf{e} + \mathbf{z}) - h_n(t, \mathbf{z}) - \Gamma_n e_n] \\ &= e_2 e_3 \cdots e_n [g_1(t, \mathbf{e} + \mathbf{z}) - g_1(t, \mathbf{z}) + g_1(t, \mathbf{z}) - h_1(t, \mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\quad + e_1 e_2 \cdots e_{n-1} [g_n(t, \mathbf{e} + \mathbf{z}) - g_n(t, \mathbf{z}) + g_n(t, \mathbf{z}) - h_n(t, \mathbf{z}) - \Gamma_n e_n]. \end{aligned}$$

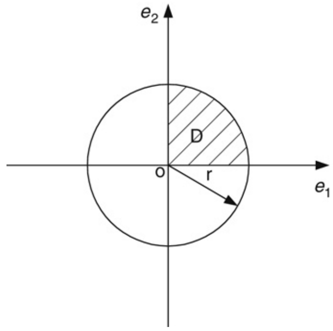

Fig. 1. D region for $n = 2$.

When $e_1 > 0, e_2 > 0, \ldots, e_n > 0$, we have

$$\begin{aligned}\dot{V} &\leq e_2 e_3 \cdots e_n[|g_1(t, \mathbf{e}+\mathbf{z}) - g_1(t, \mathbf{z})| + g_1(t, \mathbf{z}) - h_1(t, \mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\quad + e_1 e_2 \cdots e_{n-1}[|g_n(t, \mathbf{e}+\mathbf{z}) - g_n(t, \mathbf{z})| + g_n(t, \mathbf{z}) - h_n(t, \mathbf{z}) - \Gamma_n e_n] \\ &\leq e_2 e_3 \cdots e_n[L\,\|\mathbf{e}\| + g_1(t, \mathbf{z}) - h_1(t, \mathbf{z}) - \Gamma_1 e_1] + \cdots, \end{aligned} \tag{8}$$

where $|g_1(t, \mathbf{e}+\mathbf{z}) - g_1(t, \mathbf{z})| \leq L\,\|\mathbf{e}\|$ follows the Lipschitz condition. When $\|\mathbf{e}\| \gg 1$, the terms of lower degree of error components $e_2 e_3 \cdots e_n[g_1(t, \mathbf{z}) - h_1(t, \mathbf{z})]$, $e_1 e_3 \cdots e_n[g_2(t, \mathbf{z}) - h_2(t, \mathbf{z})], \ldots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \leq e_2 e_3 \cdots e_n[L\,\|\mathbf{e}\| - \Gamma_1 e_1] + e_1 e_3 \cdots e_n[L\,\|\mathbf{e}\| - \Gamma_2 e_2] + \cdots. \tag{9}$$

For sufficiently large Γ_i, $\dot{V}$ can be negative in the quadrant $e_1 > 0, e_2 > 0, \ldots, e_n > 0$. So the state point tends to decrease $\|\mathbf{e}(t)\|$ with time when $\|\mathbf{e}_0\|$ is sufficiently large. When $\|\mathbf{e}\| \ll 1$, the proof is as follows. Now when $e_1 > 0, e_2 > 0, \ldots, e_n > 0$, $\dot{V}$ is expressed as

$$\begin{aligned}\dot{V} &\geq e_2 e_3 \cdots e_n[-|g_1(t, \mathbf{e}+\mathbf{z}) - g_1(t, \mathbf{z})| + g_1(t, \mathbf{z}) - h_1(t, \mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\geq e_2 e_3 \cdots e_n[-L\,\|\mathbf{e}\| + g_1(t, \mathbf{z}) - h_1(t, \mathbf{z}) - \Gamma_1 e_1] + \cdots. \end{aligned} \tag{10}$$

When $\|\mathbf{e}\| \ll 1$, the terms of higher degree $e_2 e_3 \cdots e_n[-L\,\|\mathbf{e}\| - \Gamma_1 e_1], \cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \geq e_2 e_3 \cdots e_n[g_1(t, \mathbf{z}) - h_1(t, \mathbf{z})] + e_1 e_3 \cdots e_n[g_2(t, \mathbf{z}) - h_2(t, \mathbf{z})] + \cdots. \tag{11}$$

By the condition $h_i(t, \mathbf{z}) \leq g_i(t, \mathbf{z}) \quad (i = 1, \ldots, n)$ in$\Omega_1 \cap \Omega_2$, $\|\mathbf{h}(t, \mathbf{z}) - \mathbf{g}(t, \mathbf{z})\| > 0$, $f_i(t, \mathbf{z}) = g_i(t, \mathbf{z})(i = 1, \ldots, n)$ do not occur simultaneously. Therefore the right-hand side of above inequality is positive, i.e. $\dot{V}$ is positive in region D of Fig. 1, which is the quadrant $e_1 > 0, e_2 > 0, \ldots, e_n > 0$ of the neighborhood of the origin.

Choose $r > 0$ such that for the ball $B_r = \{\mathbf{e} \in R^n |\, \|\mathbf{e}\| \leq r\}$, we have

$$D = \{\mathbf{e} \in B_r | V(\mathbf{e}) > 0\} \tag{12}$$

of which the boundary is the surface $V(\mathbf{e}) = 0$ and the sphere $\|\mathbf{e}\| = r$. Since $V(\mathbf{0}) = 0$, the origin lies on the boundary of D inside B_r. The point $\mathbf{e}_0$ is in the interior of D and $V(\mathbf{e}_0) = b > 0$. Now we prove that the trajectory $\mathbf{e}(t)$ started at $\mathbf{e}(0) = \mathbf{e}_0$ must leave the set D, i.e. the trajectory must leave the neighborhood of the origin, $\mathbf{e}$ cannot approach zero. To see this point, notice that as long as $\mathbf{e}(t)$ is inside D, $V(\mathbf{e}(t)) \geq b$ since $\dot{V}(\mathbf{e}) > 0$ in D. Let

$$\beta = \min\{\dot{V}(\mathbf{e}) | \mathbf{e} \in D \text{ and } V(\mathbf{e}) \geq b\} \tag{13}$$

which exists since the continuous function $\dot{V}(\mathbf{e})$ has a minimum over the compact set $\{\mathbf{e} \in D \text{ and } V(\mathbf{e}) \geq b\} = \{\mathbf{e} \in B_r, \text{and} V(\mathbf{e}) \geq b\}$ [22]. Then, $\beta > 0$ and

$$V(\mathbf{e}(t)) = V(\mathbf{e}_0) + \int_0^t \dot{V}(\mathbf{e}(s))\mathrm{d}s \geq b + \int_0^t \beta \mathrm{d}s = b + \beta t. \tag{14}$$

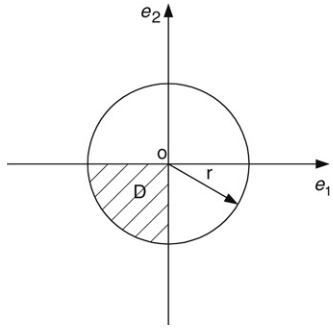

Fig. 2. D region for $n = 2$.

This inequality shows that $\mathbf{e}(t)$ cannot stay forever in D because $V(\mathbf{e})$ is bounded on D. Now, $\mathbf{e}(t)$ cannot leave D through the surface $V(\mathbf{e}) = 0$ since $V(\mathbf{e}(t)) \geq b$. Hence, it must leave D through the sphere $\|\mathbf{e}\| = r$, i.e. it must leave the neighborhood of the origin, $\mathbf{e}$ can never approach zero. Two different dynamic systems in Eq.(5) are unsynchronizable for however large Γ.

Theorem 2. *Two different dynamic systems in Eq. (5) is unsynchronizable for however large coupling strength Γ, if $h_i(t,\mathbf{z}) \geq g_i(t,\mathbf{z})(i = 1,\ldots,n)$ in $\Omega_1 \cap \Omega_2$, and $\|\mathbf{h}(t,\mathbf{z}) - \mathbf{g}(t,\mathbf{z})\| > 0$ except at the origin, for any solution $\mathbf{z}(t)$.*

Proof. Choose a Lyapunov function $V(\mathbf{e}) = e_1 e_2 \cdots e_n$, then $\dot{V}$ along any state trajectory of system (6) becomes:

Case 1. When n is odd, $V(\mathbf{e})$ is negative in the quadrant $e_1 < 0, e_2 < 0, \ldots e_n < 0$.

$$\begin{aligned}\dot{V} &= e_2 e_3 \cdots e_n \dot{e}_1 + e_1 e_3 \cdots e_n \dot{e}_2 + \cdots + e_1 e_2 \cdots e_{n-1} \dot{e}_n \\ &= e_2 e_3 \cdots e_n [g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z}) + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\quad + e_1 e_2 \cdots e_{n-1}[g_n(t,\mathbf{e}+\mathbf{z}) - g_n(t,\mathbf{z}) + g_n(t,\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n].\end{aligned}$$

When $e_1 < 0, e_2 < 0, \ldots e_n < 0$, we have

$$\begin{aligned}\dot{V} &\geq e_2 e_3 \cdots e_n [-|g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\quad + e_1 e_2 \cdots e_{n-1}[-|g_n(t,\mathbf{e}+\mathbf{z}) - g_n(t,\mathbf{z})| + g_n(t,\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n] \\ &\geq e_2 e_3 \cdots e_n [-L\|\mathbf{e}\| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots,\end{aligned} \tag{15}$$

where $|g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| \leq L\|\mathbf{e}\|$ follows the Lipschitz condition. When $\|\mathbf{e}\| \gg 1$, the terms of lower degree of error components $e_2 e_3 \cdots e_n[g_1(t,\mathbf{z}) - h_1(t,\mathbf{z})]$, $e_1 e_3 \cdots e_n[g_2(t,\mathbf{z}) - h_2(t,\mathbf{z})], \ldots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\begin{aligned}\dot{V} &\geq e_2 e_3 \cdots e_n[-L\|\mathbf{e}\| - \Gamma_1 e_1] + e_1 e_3 \cdots e_n[-L\|\mathbf{e}\| - \Gamma_2 e_2] + \cdots \\ &= -e_2 e_3 \cdots e_n[L\|\mathbf{e}\| + \Gamma_1 e_1] - e_1 e_3 \cdots e_n[L\|\mathbf{e}\| + \Gamma_2 e_2] + \cdots.\end{aligned} \tag{16}$$

For sufficiently large Γ_i, $\dot{V}$ can be positive in the quadrant $e_1 < 0, e_2 < 0, \ldots, e_n < 0$. So the state point tends to decrease $\|\mathbf{e}(t)\|$ with time when $\|\mathbf{e}_0\|$ is sufficiently large. When $\|\mathbf{e}\| \ll 1$, the proof is as follows. Now when $e_1 < 0, e_2 < 0, \ldots, e_n < 0$, $\dot{V}$ is expressed as

$$\begin{aligned}\dot{V} &\leq e_2 e_3 \cdots e_n[|g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\leq e_2 e_3 \cdots e_n[L\|\mathbf{e}\| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots.\end{aligned} \tag{17}$$

When $\|\mathbf{e}\| \ll 1$, the terms of higher degree $e_2 e_3 \cdots e_n[L\|\mathbf{e}\| - \Gamma_1 e_1], \ldots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \leq e_2 e_3 \cdots e_n[g_1(t,\mathbf{z}) - h_1(t,\mathbf{z})] + e_1 e_3 \cdots e_n[g_2(t,\mathbf{z}) - h_2(t,\mathbf{z})] + \cdots. \tag{18}$$

By the condition $\|\mathbf{h}(t,\mathbf{z}) - \mathbf{g}(t,\mathbf{z})\| > 0$, $h_i(t,\mathbf{z}) = g_i(t,\mathbf{z})(i = 1,\ldots,n)$ do not occur simultaneously. Therefore the right-hand side of above inequality is negative, i.e. $\dot{V}$ is negative in region D of Fig. 2, which is the quadrant $e_1 < 0$, $e_2 < 0, \ldots, e_n < 0$ of the neighborhood of the origin.

Choose $r > 0$ such that for the ball $B_r = \{\mathbf{e} \in R^n \mid \|\mathbf{e}\| \le r\}$, we have

$$D = \{\mathbf{e} \in B_r \mid V(\mathbf{e}) < 0\}. \tag{19}$$

By the similar reasoning as that in the latter part of the proof for Theorem 1, we can prove that the state trajectory started from D must leave the neighborhood of the origin, $\mathbf{e}$ can never approach zero. Two different dynamic systems in Eq. (5) are unsynchronizable for however large Γ.

Case 2. When n is even, $V(\mathbf{e})$ is positive in the quadrant $e_1 < 0, e_2 < 0, \ldots, e_n < 0$.

$$\begin{aligned}\dot{V} &= e_2 e_3 \cdots e_n \dot{e}_1 + e_1 e_3 \cdots e_n \dot{e}_2 + \cdots + e_1 e_2 \cdots e_{n-1} \dot{e}_n \\ &= e_2 e_3 \cdots e_n [g_1(t, \mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z}) + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\quad + e_1 e_2 \cdots e_{n-1} [g_n(t, \mathbf{e}+\mathbf{z}) - g_n(t,\mathbf{z}) + g_n(t,\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n].\end{aligned}$$

When $e_1 < 0, e_2 < 0, \ldots, e_n < 0$, we have

$$\begin{aligned}\dot{V} &\le e_2 e_3 \cdots e_n [-|g_1(t, \mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\quad + e_1 e_2 \cdots e_{n-1} [-|g_n(t, \mathbf{e}+\mathbf{z}) - g_n(t,\mathbf{z})| + g_n(t,\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n] \\ &\le e_2 e_3 \cdots e_n [-L\|\mathbf{e}\| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots,\end{aligned} \tag{20}$$

where $|g_1(t, \mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| \le L\|\mathbf{e}\|$ follows the Lipschitz condition. When $\|\mathbf{e}\| \gg 1$, the terms of lower degree of error components $e_2 e_3 \cdots e_n [g_1(t,\mathbf{z}) - h_1(t,\mathbf{z})]$, $e_1 e_3 \cdots e_n [g_2(t,\mathbf{z}) - h_2(t,\mathbf{z})], \ldots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\begin{aligned}\dot{V} &\le e_2 e_3 \cdots e_n [-L\|\mathbf{e}\| - \Gamma_1 e_1] + e_1 e_3 \cdots e_n [-L\|\mathbf{e}\| - \Gamma_2 e_2] + \cdots \\ &= -e_2 e_3 \cdots e_n [L\|\mathbf{e}\| + \Gamma_1 e_1] - e_1 e_3 \cdots e_n [L\|\mathbf{e}\| + \Gamma_2 e_2] + \cdots.\end{aligned} \tag{21}$$

For sufficiently large Γ_i, $\dot{V}$ can be negative in the quadrant $e_1 < 0, e_2 < 0, \ldots, e_n < 0$. So the state point tends to decrease $\|\mathbf{e}(t)\|$ with time when $\|\mathbf{e}_0\|$ is sufficiently large. When $\|\mathbf{e}\| \ll 1$, the proof is as follows. Now when $e_1 < 0$, $e_2 < 0, \ldots, e_n < 0$, $\dot{V}$ is expressed as

$$\begin{aligned}\dot{V} &\ge e_2 e_3 \cdots e_n [|g_1(t, \mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\ge e_2 e_3 \cdots e_n [L\|\mathbf{e}\| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots.\end{aligned} \tag{22}$$

When $\|\mathbf{e}\| \ll 1$, the terms of higher degree $e_2 e_3 \cdots e_n [L\|\mathbf{e}\| - \Gamma_1 e_1], \cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \ge e_2 e_3 \cdots e_n [g_1(t,\mathbf{z}) - h_1(t,\mathbf{z})] + e_1 e_3 \cdots e_n [g_2(t,\mathbf{z}) - h_2(t,\mathbf{z})] + \cdots \tag{23}$$

By the condition $\|\mathbf{h}(t,\mathbf{z}) - \mathbf{g}(t,\mathbf{z})\| > 0$, $h_i(t,\mathbf{z}) = g_i(t,\mathbf{z})(i = 1, \ldots, n)$ do not occur simultaneously. Therefore the right-hand side of above inequality is positive, i.e. $\dot{V}$ is positive in region D of Fig. 2 which is the quadrant $e_1 < 0$, $e_2 < 0, \ldots, e_n < 0$ of the neighborhood of the origin.

By the same reasoning as that in the latter part of the proof for Theorem 1, we can prove that the state trajectory started from the neighborhood of the origin in the quadrant $e_1 < 0, e_2 < 0, \ldots, e_n < 0$ must leave the neighborhood and can never approach zero. Two different dynamic systems in Eq. (5) are unsynchronizable for however large Γ_i.

3. Simulated examples

An example for the first theorem is Chen system with a new chaotic system proposed. Consider the following unidirectional coupled systems:

$$\begin{aligned}\dot{x} &= a(y - x) \\ \dot{y} &= (c - a)x - xz + cy \\ \dot{z} &= xy - bz\end{aligned} \tag{24a}$$

$$\begin{aligned}\dot{\tilde{x}} &= a(\tilde{y} - \tilde{x}) + \sin^2 \tilde{x} - \gamma e_1 \\ \dot{\tilde{y}} &= (c - a)\tilde{x} - \tilde{x}\tilde{z} + c\tilde{y} + \tilde{x}^2 - \gamma e_2 \\ \dot{\tilde{z}} &= \tilde{x}\tilde{y} - b\tilde{z} + \tilde{x}^2 - \gamma e_3\end{aligned} \tag{24b}$$

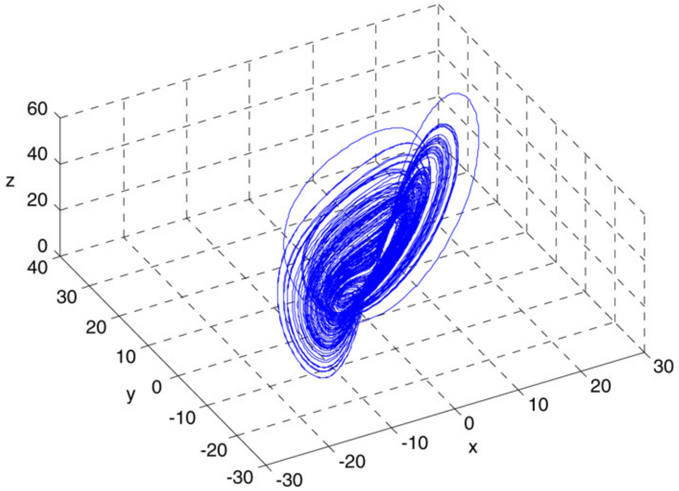

Fig. 3. Chaotic attractor for the Chen system (24a), with $a = 35$, $b = 3$ and $c = 28$, initial condition (0.5, 1, 5).

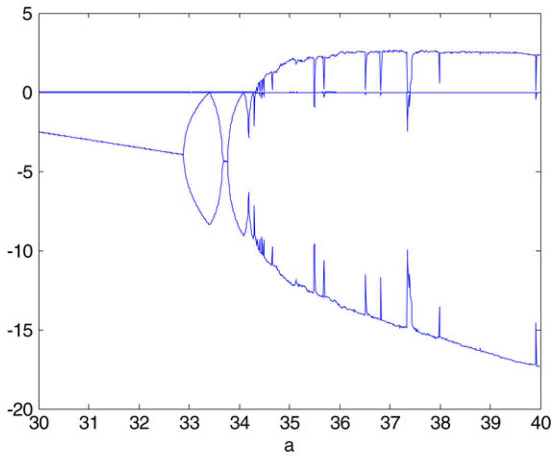

Fig. 4. Lyapunov exponent for the Chen system (24a), with $b = 3$ and $c = 28$, initial condition (0.5, 1, 5).

$$\mathbf{e} = \mathbf{y} - \mathbf{z}, \qquad \mathbf{z} = H(\mathbf{x}) = \boldsymbol{A}\mathbf{x} + \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix},$$

where $\gamma = 1000$ which is sufficiently large. Eq. (24a) is the Chen system and Eq. (24b) without coupling is a new chaotic system which we proposed. The chaotic attractor and Lyapunov exponent diagrams for systems (24a) and (24b) without coupling term are shown in Figs. 3–6. For initial states (0.5, 1,5), (30, 20, 18) and system parameters $a = 35, b = 3$ and $c = 28$, three state errors and error versus time are shown in Figs. 7–9. Fig. 8 shows that errors decreases with time when error is large, but one can clearly find in Fig. 9 that the errors cannot approach zero as time evolves.

An example for the second theorem is the Rössler system with a new chaotic system proposed. Consider the following unidirectional coupled systems with linear coupling in the form of Eq. (5):

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \tag{25a}$$

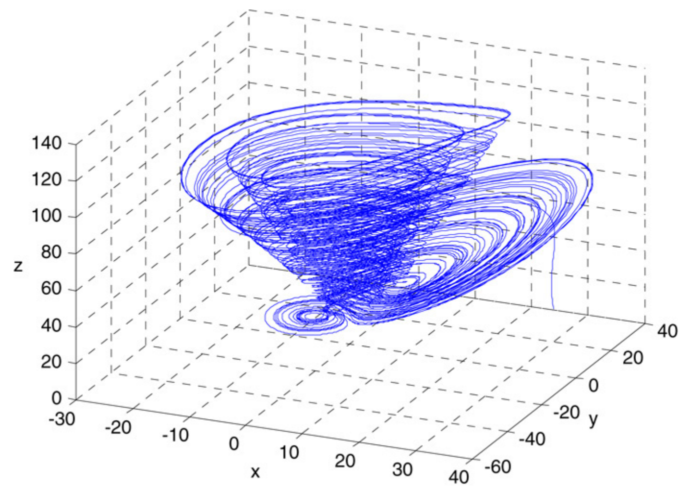

Fig. 5. Chaotic attractor for the chaotic system (24b), with $a = 35$, $b = 3$ and $c = 28$, initial condition (30, 20, 18).

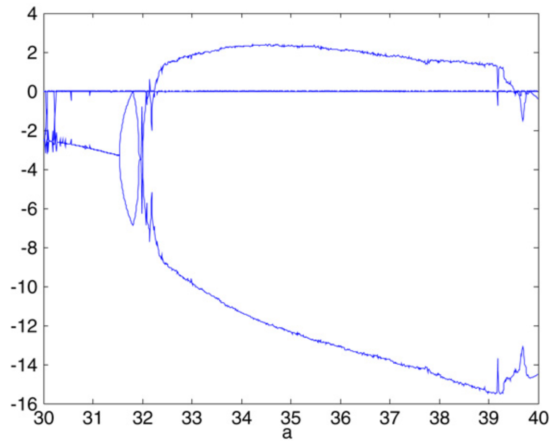

Fig. 6. Lyapunov exponent for the chaotic system (24b), with $b = 3$ and $c = 28$, initial condition (30, 20, 18).

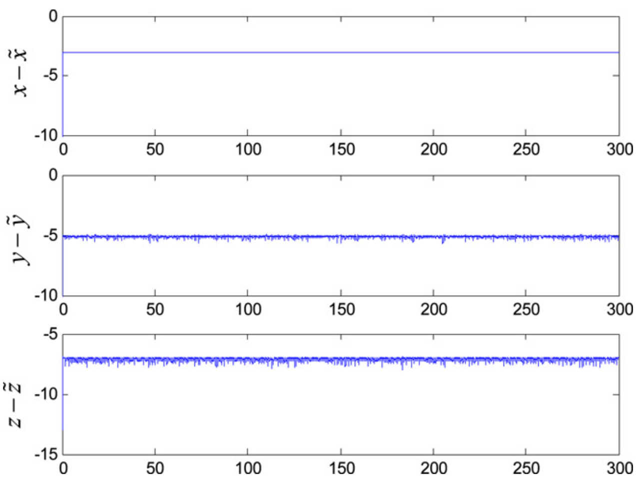

Fig. 7. State errors versus time for unidirectional coupled systems (24), with $a = 35$, $b = 3$ and $c = 28$, initial conditions (0.5, 1, 5), (30, 20, 18).

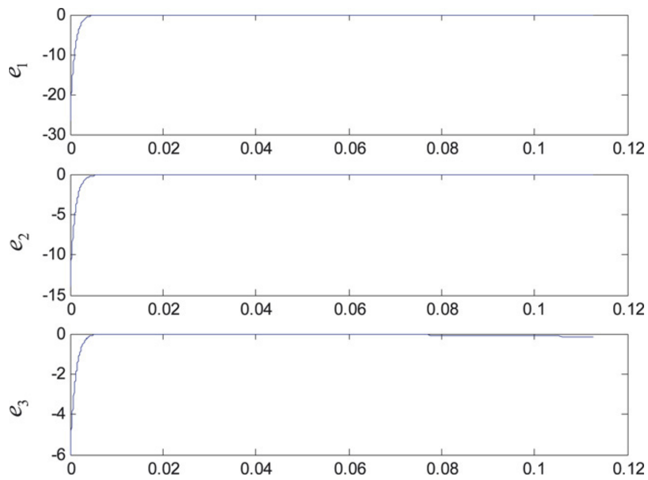

Fig. 8. Error versus time for unidirectional coupled systems (24), with $a = 35$, $b = 3$ and $c = 28$, initial conditions (0.5, 1,5), (30, 20, 18).

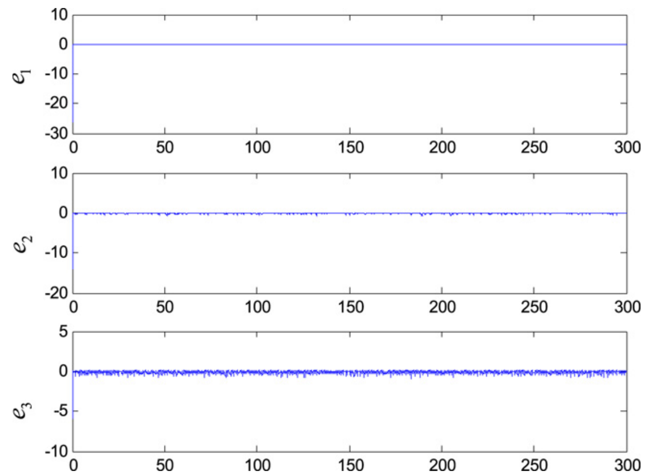

Fig. 9. Error versus time for unidirectional coupled systems (24), with $a = 35$, $b = 3$ and $c = 28$, initial conditions (0.5, 1, 5), (30, 20, 18).

$$\begin{aligned} \dot{\tilde{x}} &= -\tilde{y} - \tilde{z} - \sin^2 y - \gamma e_1 \\ \dot{\tilde{y}} &= \tilde{x} + a\tilde{y} - \sin^2 y - \gamma e_2 \\ \dot{\tilde{z}} &= b + \tilde{z}(\tilde{x} - c) - \sin^2 \tilde{z} - \gamma e_3 \\ \mathbf{e} &= \mathbf{y} - \mathbf{z}, \qquad \mathbf{z} = H(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b} \\ \mathbf{A} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \end{aligned} \tag{25b}$$

where $\gamma = 300$. The Lyapunov exponent diagrams for systems (25a) and (25b) without coupling term are shown in Figs. 10 and 11. For initial states (20, 10, 25), (2.5, 2, 2.5) and system parameter $a = 0.2$, $b = 0.2$ and $c = 5.7$, three state errors and errors versus time are shown in Figs. 12–14. Fig. 13 shows that errors decreases with time when error is large, but one can clearly find in Fig. 14 that the errors cannot approach zero as time evolves.

4. Conclusions

In this paper, two theorems are proposed. They give the criteria of generalized unsynchronization for two different chaotic dynamic systems with whatever large strength of linear coupling. The Chen system and the Rössler system

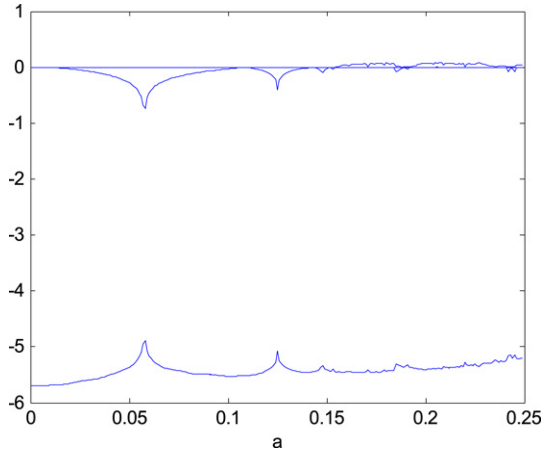

Fig. 10. Lyapunov exponent for the Rössler system (25a), with $b = 0.2$ and $c = 5.7$, initial condition (20, 10, 25).

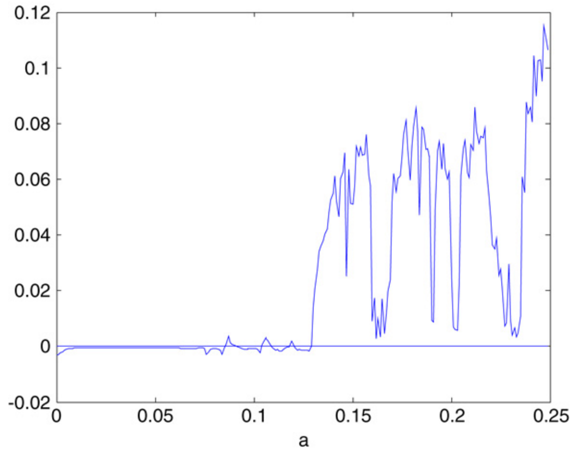

Fig. 11. Lyapunov exponent for the chaotic system (25b), with $b = 0.2$ and $c = 5.7$, initial condition (2.5, 2, 2.5).

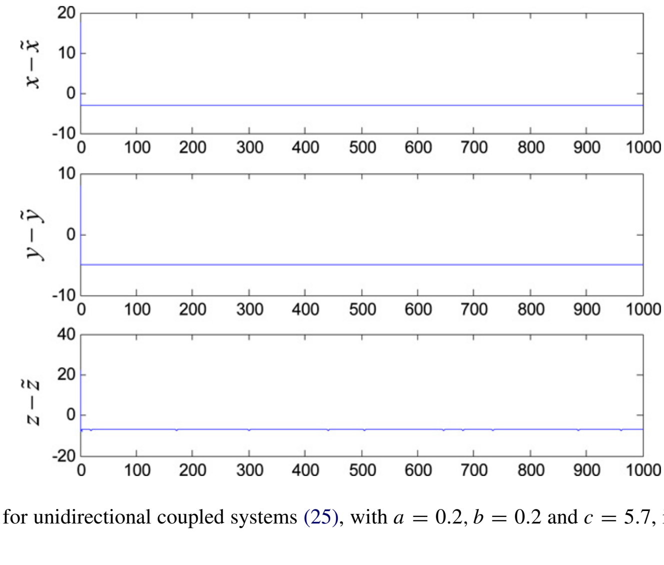

Fig. 12. State errors versus time for unidirectional coupled systems (25), with $a = 0.2$, $b = 0.2$ and $c = 5.7$, initial conditions (20, 10, 25), (2.5, 2, 2.5).

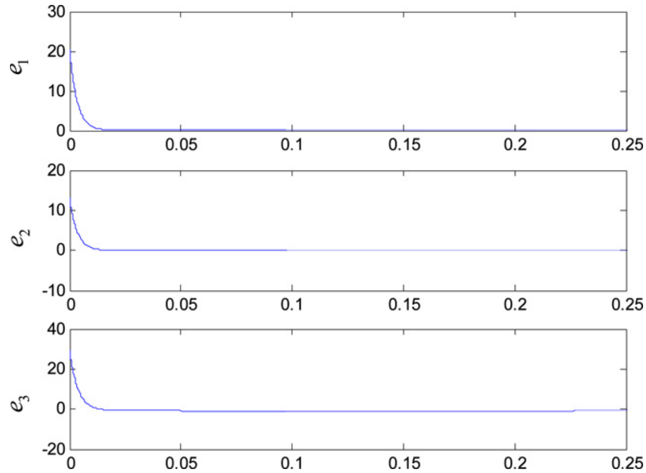

Fig. 13. Error versus time for unidirectional coupled systems (25), with $a = 0.2$, $b = 0.2$ and $c = 5.7$, initial conditions (20, 10, 25), (2.5, 2, 2.5).

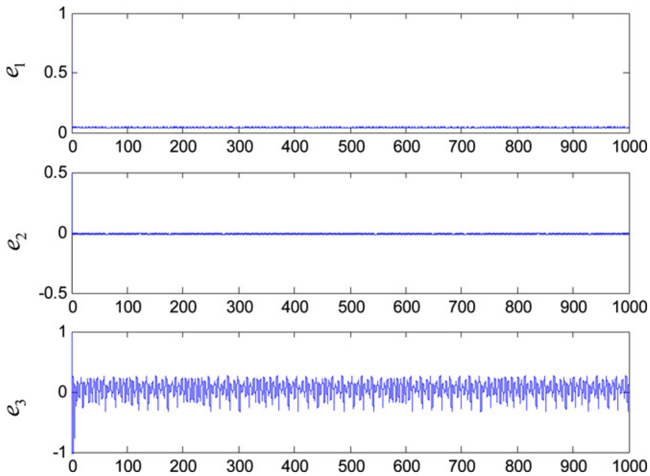

Fig. 14. Errors versus time for unidirectional coupled systems (25), with $a = 0.2$, $b = 0.2$ and $c = 5.7$, initial conditions (20, 10, 25), (2.5, 2, 2.5).

with two corresponding new chaotic systems proposed are used as simulation examples which effectively confirm the theorems.

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References

[1] L.M. Pecora, T.L. Carroll, Synchronization in chaotic systems, Phys. Rev. Lett. 64 (1990) 821.
[2] R. He, P.G. Vaidya, Analysis and synthesis of synchronous periodic and chaotic systems, Phys. Rev. A 46 (1992) 7387.
[3] M. Ding, Ott, Enhancing synchronism of chaotic systems, Phys. Rev. E 49 (1994) 945.
[4] K. Murali, M. Lakshmanan, Drive-response scenario of chaos synchronization in identical nonlinear systems, Phys. Rev. E 49 (1994) 4882.
[5] L. Kocarev, U. Parlitz, R. Brown, Robust synchronization of chaotic systems, Phys. Rev. E 61 (2000) 3716.
[6] T.-L. Liao, S.-H. Tsai, Adaptive synchronization of chaotic systems and its application to secure communications, Chaos Solitons Fractals 11 (2000) 1387.
[7] S. Chen, J. Lu, Chaos synchronization of an uncertain unified chaotic system via adaptive control, Chaos Solitons Fractals 14 (2002) 643.
[8] M. di Bernardo, A purely adaptive controller to synchronize and control chaotic systems, Phys. Lett. A 214 (1996) 139.

[9] J.-Q. Fang, Y.-G. Hong, G.-R. Chen, Switching manifold approach to chaos synchronization, Phys. Rev. E 59 (1999) R2523.
[10] X.-H. Yu, Y.-X. Song, Chaos synchronization via controlling partial state of chaotic systems, Internat. J. Bifur. Chaos 11 (2001) 1737.
[11] Zheng-Ming Ge, W.-Y. Leu, Anti-control of chaos of two-degrees-of-freedom loudspeaker system and chaos synchronization of different order systems, Chaos Solitons Fractals 20 (2004) 503.
[12] Zheng-Ming Ge, C.-C. Chen, Phase synchronization of coupled chaotic multiple time scales systems, Chaos Solitons Fractals 20 (2004) 639.
[13] Zheng-Ming Ge, C.-M. Chang, Chaos synchronization and parameters identification of single time scale brushless DC motors, Chaos Solitons Fractals 20 (2004) 883.
[14] Zheng-Ming Ge, W.-Y. Leu, Chaos synchronization and parameter identification for loudspeaker systems, Chaos Solitons Fractals 21 (2004) 1231.
[15] Y. Yu, S. Zhang, The synchronization of linearly bidirectional coupled chaotic systems, Chaos Solitons Fractals 22 (2004) 189.
[16] C. Li, X. Liao, Complete and lag synchronization of hyperchaotic systems using small impulses, Chaos Solitons Fractals 22 (2004) 857.
[17] Zheng-Ming Ge, Yen-Sheng Chen, Synchronization of unidirectional coupled chaotic systems via partial stability, Chaos Solitons Fractals 21 (2004) 101.
[18] G Chen, T Ueta, Yet another chaotic attractor, Internat. J. Bifur. Chaos 9 (1999) 1465.
[19] O.E. Rössler, An equation for continuous chaos, Phys. Lett. A 57 (1976) 397.
[20] Zheng-Ming Ge, Pu-Chien Tsen, The theorems of unsynchronizability and synchronization for coupled chaotic systems, Int. J. Nonlinear Sci. Numer. Simul. 8 (2007) 101.
[21] Zheng-Ming Ge, Developing Theory of Motion Stability, Gau Lih Book Company, Taipei, 2001.
[22] T.M. Apostol, Mathematical Analysis, Addison-Wesley, Reading, MA, 1957.